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Scattering of Antiplane Surface Waves by an Embedded Crack in a Layered Elastic Solid

by

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A Thesis Submitted
in Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy

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Houston, Texas
January, 1996
Scattering of Antiplane Surface Waves by an Embedded Crack in a Layered Elastic Solid

AbdRahman D. Ad-Doheyan

Abstract

The scattering of an incident antiplane surface wave (Love-wave) by an embedded crack normal to the free surface in a layered half-space is investigated in this study. The crack is first assumed to be entirely contained in the surface layer, then the case in which the crack breaks through the interface is considered.

The total displacement and stress fields are analyzed as the superposition of the incident fields in an uncracked half-space and the scattered fields in the cracked half-space. A general solution for the scattered displacement and stress fields in the cracked half-space is obtained by using Fourier sine and cosine transforms techniques. The mixed displacement and stress boundary value problem is reduced to a singular integral equation for the density of displacement discontinuity across the crack faces (dislocation density). The singular integral equation is approximated by a linear system of equations by using a Gaussian method. Further, the amplitudes of the reflected and transmitted displacement fields in the cracked half-space at some distance away from the crack plane are evaluated. It is shown that these displacement fields are the superposition of a finite number of Love-wave modes.

In the case when the embedded crack breaks through the interface, the dislocation density function is shown to be discontinuous across the interface between the two
solids, and the magnitude of the discontinuity is related to the ratio of the shear moduli.

The numerical results for the reflection coefficients of the first three modes as well as for the transmitted coefficient of the first mode are presented for three different layer-embedded cracks and for four different interface-breaking cracks. These coefficients depend strongly on the position of the upper tip and the width of the crack. The results, when the upper tip is very close to the free surface, are compared with those for the surface-breaking crack configurations that are available in the literature. Good agreement is observed.
Acknowledgments

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Dedication

In the name of Allah Ar-Arham the most compassionate.

“O’ my lord advance me in knowledge”

Quran 20, 114.

This work is dedicated humbly to the pleasure of Allah (SWT) and his prophet and messenger Mohammed (PBUH). It is in appreciation to all prophets and messengers sent by Allah to mankind. It is dedicated to all victims of disease, aggression and injustice.

This thesis is dedicated to the memory of my father Doheyan Al-Abd-AlAziz.
Chapter 1

Introduction

1.1 Motivation

The studies presented in this dissertation are motivated by the need to gain an improved understanding of the scattering of an incident surface antiplane wave (Love wave) by an embedded breaking-crack normal to the free surface of a layered half-space. From the point of view of quantitative non-destructive evaluation of materials, the reflected and transmitted fields of the incident Love wave are of great interest and can be used to detect the position and to determine the size of the embedded crack. The earth crust and a large class of composite materials can be modeled by the cracked layered solid used in these studies.

The first part of this dissertation is concerned with the evaluation of the reflection and transmission fields of an incident antiplane surface wave (Love wave) scattered by an embedded crack in a layered elastic solid. The crack is normal to the free surface and is assumed to be contained entirely in the surface layer.

The second part of this dissertation is concerned with the case of an embedded interface-breaking crack in a layered elastic solid. The reflection and transmission fields of the incident antiplane surface wave (Love wave) scattered by the crack are evaluated.
1.2 Background and Significance

The problems of reflection and transmission of an elastic wave by cracks is of fundamental importance in quantitative non-destructive evaluation (QNDE) of structural elements and seismology. The scattered field, which carries a great deal of information about the scattering obstacle, is of interest to solve the inverse problem of detecting a crack and determining its size, shape, and orientation [1]. These problems have been the subject of various studies. In these studies, the solutions are generally obtained by using integral-transform (Fourier transform) techniques or boundary element (Green's function) methods. Using Fourier transform techniques, the problems can be reduced to systems of singular integral equations of the Cauchy type, which are evaluated numerically using reliable and efficient methods of the Gaussian type. The Green's function method is based on taking the limit of the solution of the equation of motion at a point away from the crack as this point approaches the crack face. The limiting process is carried out to satisfy the boundary conditions. Angel [9] showed that the Green's function method can also yield a system of singular integral equations after careful investigation of the limiting process involved in imposing the boundary conditions. In this dissertation, we use Fourier transform techniques to obtain systems of singular integrals of the Cauchy type for the problem of a layer-embedded crack and that of an interface-breaking crack. Next, we shall briefly discuss the important contributions in the area of scattering of elastic waves by surface-breaking cracks and embedded cracks in homogeneous elastic solids and layered elastic solids.

1.2.1 Scattering by Cracks in Elastic Solids

Over the last decade, the problems of reflection and transmission of a plane surface Rayleigh waves and of body waves by a surface-breaking crack or an embedded crack
in *homogeneous* elastic solids in two or three dimensions have been considered in numerous studies. In these studies, the incident angle of the Rayleigh waves is arbitrary, and the cracks are oriented in any directions with respect to the free surface.

Achenbach and Brind [3] consider the scattering of surface waves (Rayleigh waves) by a sub-surface crack. They use the boundary element (Green's function) method to reduce the boundary-value problem of the scattered field to an uncoupled system of integral equations, which are solved numerically. They extend their work [4] to investigate the scattering of body waves (transverse and longitudinal waves). They show that at a large distance from the crack face the scattered field consists of outgoing Rayleigh waves and cylindrical body waves for the two types of incident body waves.

Mendelsohn *et al.* [2] investigate the scattering of incident Rayleigh surface waves and body waves by a surface-breaking crack normal to the free surface. The angle of incidence of the surface Rayleigh waves is normal to the horizontal axis of the crack. They use the Fourier transform techniques to reduce the formulation of the scattering problem to a system of singular integral equations. Angel and Achenbach [5] consider the 3-dimensional problem corresponding to the work presented in [2]. They use integral-transform techniques, and study the effect of the angle of incidence of the Rayleigh waves on the scattered field. They find that if the angle of incidence exceeds a certain critical angle, which depends on the material properties of the solid, no mechanical energy is radiated into the solid by body waves.

Angel and Achenbach [6, 7] investigate the scattering of longitudinal and transverse plane waves with normal and oblique incidence, respectively, by a periodic array of cracks. They use Fourier series expansions to reduce the mixed-boundary value problem for a typical strip of the solid to a singular integral equation for the dislocation density across the crack faces (the slope of the crack-face displacement). They observe good agreement with two approximation methods at low frequencies. They
show that the reflected and transmitted far-field displacements are the superposition of a finite number of homogeneous plane body-wave modes.

Keer, Lin, and Achenbach [10] consider the plane strain problem for a crack parallel to the free surface in a homogeneous half space. They use the Green's function method to reduce the formulation to a system of coupled singular integral equations. They observe a resonance effect for a wide crack at a close distance from the free surface. Using the methodology presented by Neerhoff [12], Hijden and Neerhoff [14] discuss the plane strain problem for an arbitrarily oriented embedded crack.

Achenbach et al. [18] investigate the radiated wave motion for the scattering of Rayleigh surface waves by a surface-breaking crack of finite dimensions. The far-field surface radiated wave motion is evaluated in all directions.

Achenbach et al. [19] present an approximate approach to obtain the scattering of Rayleigh surface waves by a surface-breaking crack. The method of analysis used in this approach is based on the concepts of elastodynamic ray theory. They observe good agreement with exact numerical results in the high-frequency range.

1.2.2 Scattering by Cracks in Layered Elastic Solids

The surface wave motion in a layered half space is, in general, dispersive and decays exponentially in the direction of increasing depth. Antiplane surface motions in a layered half space are called Love waves. The explanation of this motion was given at the beginning of this century by A.E.H. Love [17]. He showed that these waves are essentially horizontally polarized shear waves trapped in a superficial layer of a half-space and propagated with multiple total reflections between the boundaries of the layer. The Love wave characteristics are similar to those of the secondary seismic (shear) waves. The other type of surface motion in a layered half space is the plane
generalized Rayleigh wave motion. This type of Rayleigh wave is dispersive, unlike the ordinary Rayleigh wave in a homogeneous half space. Although layered elastic solids can be used to model a large class of elastic composite materials and the earth crust, few works have been published in the area of scattering of elastic waves in cracked solids of this type. General solutions for the reflection and transmission of surface waves are very difficult to evaluate analytically and numerically, because the propagation of elastic waves becomes dispersive. In this section, we shall highlight the major works in this area.

Keer and Luong [11] study the propagation of an antiplane wave in a cracked layer bonded between two half spaces of a different material. They use integral transform techniques to formulate the problem in terms of a singular integral equation. They conclude that the diffracted waves can be detected at a large distance from the crack. Taking into account only the fundamental mode, the ratio of the reflection coefficient to the transmission coefficient is computed.

Neerhoff [12] investigates the scattering of an antiplane (Love) wave by a crack lying in the interface of a layered half-space. He derives an integral equation for the unknown jump in the particle displacement across the crack. Then, he uses the Green's function method to obtain a system of linear equations for the expanded unknown, which is solved numerically. This work is extended by Neerhoff and Hijden [15] to study the diffraction of SII-waves by an arbitrary oriented crack. In that work, the methodology of Neerhoff [12] is used.

Yang and Bogy [13] extend the work presented early in reference [10] to the case of layered elastic solids, and assume that the crack lies in the interface. They use the Green's function method to reduce the formulation of the problem to a system of singular integral equations. They notice a resonance effect of the same type as in reference [10].
Angel [8] investigates the scattering of Love waves by a surface-breaking crack normal to the free surface of a layered elastic solid. In that work, the mixed boundary value problem is reduced to a singular integral equation of the Cauchy type, using Fourier transform techniques. He shows that the reflected and transmitted displacement far-fields are the superposition of a finite number of Love wave modes. Several sharp resonances are observed, which can be associated with a particular Love wave mode. He extends this work to investigate the scattering of Love waves by a surface crack normal to the free surface and breaking through the interface of the two solids [16]. He shows that, at the interface, the slope of the crack face displacement is discontinuous and the magnitude of the discontinuity is related to the ratio of the shear moduli. These two works are the basis of this dissertation. We extend these works to investigate the scattering of Love waves by two types of embedded cracks (a layer-embedded crack and a crack breaking through the interface of the two solids).

1.3 Objectives

The major goals of our research are:

1. To investigate the reflection and transmission of an antiplane surface wave (Love wave) scattered by an embedded crack normal to the free surface in a layered elastic solid.

2. To obtain numerical results for the far-field displacement of the scattered field in the layered elastic solid.

3. To discuss the effect of the size and the position of the embedded crack on the reflection and the transmission of the incident Love wave.
1.4 Scope of Study

This thesis investigates the scattering of an incident surface antiplane wave (Love wave) by an embedded crack normal to the free surface of a layered half-space. A general solution for the scattered displacement and stress fields in the cracked half-space is obtained by using Fourier sine and cosine transform techniques. The mixed displacement and stress boundary value problem is reduced to a singular integral equation for the slope of the crack-face displacement (dislocation density). The singular integral equation is approximated by a linear system of equations of order \( n \) by using the method of Erdogan and Gupta [20]. The displacements for both the transmitted and the reflected far-fields are evaluated. It is shown that these displacement fields are the superposition of a finite number of Love-wave modes.

The geometrical description and the mathematical formulation of the problem for a layer-embedded crack are presented in chapter 2. The total displacement and stress fields are analyzed as the superposition of the incident field in the uncracked half-space and the scattered field in the cracked half-space. Further, we evaluate the amplitude of the reflected and transmitted displacement fields in the cracked half-space at some distance away from the crack plane. The discussion of the numerical results for the reflection coefficients of the first three modes as well as for the transmission coefficient of the first mode are presented for three different layer-embedded cracks.

In a subsequent investigation, we consider the effect of an embedded crack breaking through the interface between the two solids on the scattering of a Love wave. The mathematical formulation of this problem is discussed briefly in chapter 3. The slope of the crack-face displacement is shown to be discontinuous across the interface between the two solids, and the magnitude of the discontinuity is related to the ratio of the shear moduli. Numerical results for four different interface-breaking cracks are presented.
The results of this work are expected to provide guidelines for solving the inverse problem of detecting the location and determining the size of an embedded crack in a layered elastic solid. Further related research areas are recommended in Chapter 4.
References


Chapter 2

Scattering of Antiplane Surface Waves by a Layer-Embedded Crack

2.1 Introduction

The scattering of an incident Love wave by a layer-embedded crack normal to the free surface is investigated in this chapter. The crack is assumed to be entirely contained in the layer of a single-layered solid that is perfectly bonded to a half-space made of a different material. The position and the width of the crack vary along the vertical axis. The mixed displacement and stress boundary value problem is reduced to a singular integral equation. The displacements for both the transmitted and the reflected far-fields in the layer and the half-space are evaluated.

The geometrical description and the mathematical formulation of the problem are presented in section 2.2. The total displacement and stress fields are analyzed as the superposition of the incident fields in an uncracked half-space and the scattered fields in the cracked half-space. Only a quarter-space region is considered here, because of the symmetry of the problem with respect to the vertical axis.

In section 2.3, a general solution for the scattered displacement and stress fields in the cracked half-space is obtained, by using Fourier sine and cosine transform techniques. A singular integral equation for the slope of the crack-face displacement (dislocation density) is derived from the mixed-boundary value problem.
The singular integral equation is approximated by a linear system of equations of order \( n \) by using the method of Erdogan and Gupta [1]. The values of the dislocation density \( a(\nu) \) at \( n \) points along the crack face are obtained in section 2.4 as the solution of that linear system.

The scattered displacement fields are presented in terms of the dislocation density in section 2.5. The relation between the scattered far-field displacements in the cracked half-space and those of the incident wave is discussed. The solution of the linear system of section 2.4 is used to evaluate the amplitude of the reflected and transmitted displacement fields in the cracked half-space at some distance away from the crack plane.

The discussion of the numerical results for the reflection coefficients of the first three modes as well as for the transmitted coefficient of the first mode is presented in section 2.6. Both moduli and phases are plotted for various crack positions and widths.

This work is based on the work by Angel [2], who investigated the scattering of an incident Love wave by a surface-breaking crack normal to the free surface. The mixed-boundary value problem is reduced to a singular integral equation of the first kind for the dislocation density \( a(\nu) \) across the crack faces by using Fourier transform techniques, and the equation is solved numerically using the method of Erdogan and Gupta [1]. Moreover, the far-field displacements are evaluated at some distance from the crack, where they are shown to be the superposition of a finite number of Love wave modes. In this chapter, we extend the work of Angel in order to investigate the interaction of an incident Love wave with a layer-embedded crack, and we study the effect of this crack on the reflected and transmitted far-field displacements.
2.2 Mathematical Formulation

The mathematical formulation for the scattering of an incident Love wave by a layer-embedded crack normal to the free surface is presented in this section. The geometry of the problem is illustrated in Figure (2.1).

![Diagram of Love wave incidence on a layer-breaking crack](image)

**Figure 2.1:** Incidence of a Love wave on a layer-breaking crack of length $(d - d_1)$. 
The cracked layer analyzed here is perfectly bonded to a half-space made of a
different material. Both materials are linearly elastic, homogeneous, and isotropic.
The surface layer has thickness $h$ and contains a crack normal to the free surface
$x_2 = 0$. The crack lies in the plane $x_1 = 0$ and extends to infinity in the $\pm x_3$
directions. The upper and lower crack tips are located at distances $d_1$ and $d$ from the
free surface, respectively. Both $d_1$ and $d$ are less than the layer thickness $h$. Thus, the
crack has length $d - d_1$.

The mass density, the shear modulus, and the slowness of transverse waves in the
layer and in the half-space are denoted, respectively, by $\rho, \mu, s_T$ and $\rho\,'/\mu\,', s_T\,'$. The
slownesses $s_T$ and $s_T\,'$ are given by

\[ s_T = \frac{\rho}{\mu}, \quad s_T\, = \frac{\rho\,'}{\mu\,'}. \]  \hfill (2.1a)

We assume that the slowness $s_T\,'$ of transverse waves in the half-space is less than that
in the layer $s_T$. Thus, one has

\[ s_T\,' < s_T. \] \hfill (2.1b)

We define the ratio $\epsilon$ of the slownesses and the ratio $m$ of the shear moduli by the
following equations

\[ \epsilon = s_T\,' / s_T, \quad m = \mu/\mu\,'. \] \hfill (2.1c)

One can infer from (2.1b) and (2.1c) that the ratio of the slownesses $\epsilon$ is less than
unity. A time-harmonic antiplane surface wave (Love wave) is incident on the crack.
The amplitude, the frequency, and the slowness of this wave are denoted by $u_o$, $\omega$, and $s$. The time factor, $\exp(-i\omega t)$, which is common to all field variables in a steady-
state regime, is omitted in this analysis. The displacements generated by an incident
Love wave can be expressed in the form

\[ u^{'in}_3(x_1, x_2) = u_o \left\{ \begin{array}{c} \cos[\omega s_T \alpha(\eta) x_2] \\ \cos[\omega s_T \alpha(\eta) h - \omega s_T(x_2 - h)\alpha(\eta)] \end{array} \right\} e^{i\omega x_1}, \quad 0 \leq x_2 \leq h, \]

\[ \cos[\omega s_T \alpha(\eta) h - \omega s_T(x_2 - h)\alpha(\eta)] \right\} e^{i\omega x_1}, \quad h \leq x_2. \]  \hfill (2.2)
where \( \eta, \alpha(u), \) and \( \alpha'(u) \) are defined as follows

\[
\eta = s/s_r, \quad \alpha(u) = (1 - u^2)^{1/2}, \quad \alpha'(u) = (u^2 - \varepsilon^2)^{1/2}. \tag{2.3}
\]

The value of \( \eta \) lies in the interval \([\varepsilon, 1]\), and is a real root of the following Love wave frequency equation

\[
\tan [h \omega s_r \alpha(\eta)] - m \frac{\alpha'(\eta)}{\alpha(\eta)} = 0. \tag{2.4}
\]

In order to evaluate the roots of the above equation, we define the following auxiliary variable \( w \) as a function of the parameter \( \eta \)

\[
w = \frac{2}{\pi} h \omega s_r \alpha(\eta). \tag{2.5}
\]

Using the above definition, one can write equation (2.4) in the form

\[
\tan \left( \frac{\pi}{2} w \right) - m \frac{w}{w} \left( 1 - \varepsilon^2 \right) \left( \frac{2}{\pi} h \omega s_r \right)^2 - w^2 \right)^{1/2} = 0. \tag{2.6}
\]

There is at least one real root that satisfies equation (2.6), and it lies in the interval \((0, 1)\) for all positive frequencies. The number of roots depends on the frequency of the incident Love wave, and it is given by the unique integer \( k \) that satisfies the inequality

\[
k - 1 \leq \frac{\omega}{\pi} h s_r \alpha(\varepsilon) < k. \tag{2.7}
\]

The root of equation (2.6) of order \( l \) \((1 \leq l \leq k)\) lies in the interval \([2l - 2, 2l - 1]\). The stress generated in the plane of the crack \((x_1 = 0)\) by the displacement field of equation (2.2) is given by

\[
\sigma_{13}^{in}(0, x_2) = i \mu \nu_o s \omega \cos \left[ \omega s_r \alpha(\eta) x_2 \right], \quad 0 \leq x_2 \leq h. \tag{2.8}
\]

The linearity of the governing equations allows us to decompose the total field in the solid into the sum of the incident field in an uncracked solid and the scattered field in the cracked solid. Since the total surface tractions vanish on the crack faces, the scattered field can be taken as being generated by crack-face tractions that are opposite to the tractions of the incident field at a location coincident with the crack faces. The scattered field is physically antisymmetric with respect to the plane of the crack \((x_1 = 0)\). Thus, the displacement \(u_3\) and stress component \(\sigma_{23}\) are odd functions of \(x_1\), and the other stress component \(\sigma_{13}\) is an even function of \(x_1\). Therefore, we consider only the quarter-space region defined by \(x_1 > 0\) and \(x_2 > 0\). The mixed stress and displacement boundary conditions for the scattered field are given by the equations

\[
\sigma_{23} = 0, \quad x_1 \geq 0, \quad x_2 = 0, \quad (2.9a)
\]

\[
\sigma_{23} = \sigma_{23}', \quad x_1 \geq 0, \quad x_2 = h, \quad (2.9b)
\]

\[
u_3 = u_3', \quad x_1 \geq 0, \quad x_2 = h, \quad (2.9c)
\]

\[
\sigma_{13} = -\sigma_{13}^{in}, \quad x_1 = 0, \quad d_1 < x_2 < d, \quad (2.9d)
\]

\[
u_3 = 0, \quad x_1 = 0, \quad \begin{cases} 0 \leq x_2 \leq d_1, \\ d \leq x_2 \leq h, \end{cases} \quad (2.9e)
\]

\[
u_3' = 0, \quad x_1 = 0, \quad h \leq x_2, \quad (2.9f)
\]

where the primed quantities refer to the underlying half-space, and the unprimed ones to the layer. The incident stress \(\sigma_{13}^{in}\) is defined by equation (2.8).

In the next section, we obtain a general form of the scattered displacements and their associated stresses for the cracked layered half-space by using the Fourier sine and cosine transforms, and we reduce the mixed-boundary value problem to a singular integral equation.
2.3 Singular Integral Equation

In this section, the formulation of the mixed-boundary value problem is reduced to a singular integral equation by using Fourier transform techniques. The Fourier sine and cosine transforms of a function \( f(x_1, x_2) \), relative to the variable \( x_1 \), are defined as follows

\[
\hat{f}(\zeta, x_2) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x_1, x_2) \left\{ \frac{\sin}{\cos} \right\} (\zeta x_1) \, dx_1.
\] (2.10)

The corresponding inverse Fourier sine and cosine transforms are defined by the following equations

\[
f(x_1, x_2) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\zeta, x_2) \left\{ \frac{\sin}{\cos} \right\} (\zeta x_1) \, d\zeta.
\] (2.11)

The equations of motion for the steady-state scattered displacement fields in both the layer and half-space take the form

\[
u_{3,11} + u_{3,22} + (\omega s_y)^2 u_3 = 0, \quad 0 \leq x_2 \leq h,
\] (2.12a)

\[
u'_3,11 + u'_3,22 + (\omega s'_y)^2 u'_3 = 0, \quad h \leq x_2.
\] (2.12b)

A homogeneous second-order ordinary differential equation can be obtained by taking the Fourier sine-transform of equations (2.12) relative to the variable \( x_1 \). The bounded solution in the region \( x_2 \geq 0 \) is

\[
\hat{u}_3 = B e^{-\beta(\zeta) x_2} + C e^{\beta(\zeta) x_2}, \quad 0 \leq x_2 \leq h,
\] (2.13a)

\[
\hat{u}'_3 = A e^{-\beta'(\zeta) x_2}, \quad h \leq x_2.
\] (2.13b)

Another solution can be obtained by taking alternatively the Fourier cosine-transform of equations (2.12) relative to the variable \( x_2 \). The bounded solution in the region \( x_1 \geq 0 \) is

\[
\hat{u}_3 = E e^{-\beta(\zeta) x_1}, \quad 0 \leq x_2 \leq h.
\] (2.14)
The general solution of equations (2.12) for the quarter-space \( x_1 > 0 \) and \( x_2 > 0 \) is the superposition of the two Fourier transform solutions of equations (2.13a), (2.13b), and (2.14). Thus the scattered displacement fields \( u_3 \) and \( u'_3 \), and their associated shear stresses \( \sigma_{13} \), \( \sigma_{23} \), \( \sigma'_{13} \), and \( \sigma'_{23} \), take the form

\[
\sqrt{\frac{\pi}{2}} u_3 = \int_0^\infty \left[ B e^{-\beta(\zeta)x_2} + C e^{\beta(\zeta)x_2} \right] \sin(\zeta x_1) d\zeta \\
+ \int_0^\infty E e^{-\beta(\zeta)x_1} \cos(\zeta x_2) d\zeta, \tag{2.15a}
\]

\[
\sqrt{\frac{\pi}{2}} \sigma_{13} = \mu \int_0^\infty \zeta \left[ B e^{-\beta(\zeta)x_2} + C e^{\beta(\zeta)x_2} \right] \cos(\zeta x_1) d\zeta \\
- \mu \int_0^\infty \beta(\zeta) E e^{-\beta(\zeta)x_1} \cos(\zeta x_2) d\zeta, \tag{2.15b}
\]

\[
\sqrt{\frac{\pi}{2}} \sigma_{23} = -\mu \int_0^\infty \beta(\zeta) \left[ B e^{-\beta(\zeta)x_2} - C e^{\beta(\zeta)x_2} \right] \sin(\zeta x_1) d\zeta \\
- \mu \int_0^\infty \zeta E e^{-\beta(\zeta)x_1} \sin(\zeta x_2) d\zeta, \tag{2.15c}
\]

\[
\sqrt{\frac{\pi}{2}} u'_3 = \int_0^\infty A e^{-\beta'(\zeta)x_2} \sin(\zeta x_1) d\zeta, \tag{2.15d}
\]

\[
\sqrt{\frac{\pi}{2}} \sigma'_{13} = \mu' \int_0^\infty \zeta A e^{-\beta'(\zeta)x_2} \cos(\zeta x_1) d\zeta, \tag{2.15e}
\]

\[
\sqrt{\frac{\pi}{2}} \sigma'_{23} = -\mu' \int_0^\infty \beta'(\zeta) A e^{-\beta'(\zeta)x_2} \sin(\zeta x_1) d\zeta, \tag{2.15f}
\]

where the primed quantities refer to the underlying half-space, and the unprimed ones to the cracked layer. The quantities \( \beta(\zeta) \) and \( \beta'(\zeta) \) are defined by

\[
\beta^2(\zeta) = \zeta^2 - (\omega s_\tau)^2, \quad \beta^2(\zeta) = \zeta^2 - (\omega s'_\tau)^2. \tag{2.16a}
\]

The quantities \( A, B, C, \) and \( E \) are functions of \( \zeta \), and can be determined from the boundary conditions (2.9). Notice that the quantities \( \beta \) and \( \beta' \) may take real or purely imaginary values for real values of \( \zeta \). The crack face disturbances of the scattered field must propagate away from the crack in order to satisfy the radiation
condition (Lamb [3]). This condition is satisfied by taking negative values for the $\beta$ and $\beta'$ imaginary quantities such that

$$\text{Im}(\beta(\zeta)) \leq 0, \quad \text{Im}(\beta'(\zeta)) \leq 0.$$  

(2.16b)

Observe that the above two conditions are consistent with the choice of the time factor $\exp(-i\omega t)$.

We define a function $a(v)$ such that the displacement in the plane of the crack ($x_1 = 0$) takes the following form

$$u_3(0^+, x_2) = \begin{cases} 
0, & 0 \leq x_2 \leq d_1, \\
\int_{x_2}^{d} a(v) dv, & d_1 \leq x_2 < d, \\
0, & d \leq x_2 \leq h,
\end{cases}$$

(2.17)

together with

$$\int_{d_1}^{d} a(v) dv = 0.$$  

(2.18)

For every integrable function $a(v)$ over the interval $[d_1, d]$, the scattered displacement field $u_3(0^+, x_2)$ is continuous over the interval $[0, h]$ and the displacement boundary condition (2.9e) is satisfied. Notice that the scattered displacement $u_3(x_1, x_2)$ may be discontinuous across the crack faces, since it is specified as an odd function of $x_1$. This result, together with the form of the function $a(v)$ in equations (2.17) and (2.18), shows that the function $a(v)$ is a density of displacement discontinuity across the crack faces ($x_1 = 0$, $d_1 \leq x_2 \leq d$). The function $a(v)$ is also called dislocation density.

Next, we evaluate the quantities $A, B, C,$ and $E$ in equations (2.15 a-f) in terms of the dislocation density $a(v)$. The stress boundary condition (2.9a), together with equation (2.15c), implies that

$$B = C.$$  

(2.19)
Using equation (2.17) for the displacement on the crack face together with equation (2.15a), taking the Fourier cosine-transform of the resultant equation, and interchanging the order of the integrations, one finds that

\[ E = \sqrt{\frac{2}{\pi}} \int_{d_1}^{d} \left( \int_{d}^{v} \cos(\zeta x_2) \, dx_2 \right) a(v) \, dv. \]  

(2.20)

The inner integral of equation (2.20) over the interval \([d, v]\) is evaluated. Using equation (2.18), the quantity \(E\) can be written as the following single integral of the dislocation density \(a(v)\)

\[ E = \frac{1}{\zeta} \sqrt{\frac{2}{\pi}} \int_{d_1}^{d} a(v) \sin(\zeta v) \, dv. \]  

(2.21)

Next, we consider the boundary conditions (2.9b, c), which specify that the displacement and the stress are continuous along the interface between the layer and the half-space \((x_2 = h)\). These boundary conditions, together with equation (2.19), yield the following two integral equations

\[ \int_{0}^{\infty} \left[ Ae^{-\beta(\zeta) h} - B(e^{-\beta(\zeta) h} + e^{\beta(\zeta) h}) \right] \sin(\zeta x_1) \, d\zeta \]

\[ = \int_{0}^{\infty} E e^{-\beta(\zeta) x_1} \cos(\zeta h) \, d\zeta, \]  

(2.22a)

\[ \int_{0}^{\infty} \left[ \beta'(\zeta) m A e^{-\beta(\zeta) h} - \beta(\zeta) B(e^{-\beta(\zeta) h} - e^{\beta(\zeta) h}) \right] \sin(\zeta x_1) \, d\zeta \]

\[ = \int_{0}^{\infty} \zeta E e^{-\beta(\zeta) x_1} \sin(\zeta h) \, d\zeta. \]  

(2.22b)

Substituting equation (2.21) in equations (2.22a) and (2.22b), and taking the inverse sine-transform of the resultant equations, we obtain the following linear system of two equations for the two unknown quantities \(A\) and \(B\)

\[ e^{-\beta(\delta) h} A - \left[ e^{-\beta(\delta) h} + e^{\beta(\delta) h} \right] B = \int_{0}^{\infty} I(x_1) \sin(\delta x_1) \, dx_1, \]  

(2.23a)

\[ m \beta'(\delta) e^{-\beta(\delta) h} A - \beta(\delta) \left[ e^{-\beta(\delta) h} - e^{\beta(\delta) h} \right] B = \int_{0}^{\infty} I'(x_1) \sin(\delta x_1) \, dx_1. \]  

(2.23b)
The functions $I(x_1)$ and $I'(x_1)$ in equations (2.23a) and (2.23b) are defined

\[
I(x_1) = \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_{d_1}^d a(v) \left( \int_0^\infty \frac{e^{-\beta(\zeta)} x_1}{\zeta} \left( \sin[\zeta(h + v)] - \sin[\zeta(h - v)] \right) d\zeta \right) dv. \quad (2.23c)
\]

\[
I'(x_1) = \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \int_{d_1}^d a(v) \left( \int_0^\infty e^{-\beta(\zeta)} x_1 \left( \cos[\zeta(h + v)] - \cos[\zeta(h - v)] \right) d\zeta \right) dv. \quad (2.23d)
\]

The indefinite inner integrals in the above two equations can be evaluated over the intervals \([0, \infty)\) by using an integral formula in Gradshteyn and Ryzhik [4] (equation 3.914, page 517). Substituting these results in equations (2.23a, b), interchanging the order of integrations, and using an integral formula in Erdelyi [5] (equation 43, page 112), the linear system of equations (2.23a) and (2.23b) for the two unknowns $A$ and $B$ can be written in the form

\[
e^{-\beta'(\zeta) h} A - \left( e^{-\beta(\zeta) h} + e^{\beta(\zeta) h} \right) B = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\zeta}{\beta'(\zeta)} \int_{d_1}^d a(v) \left[ e^{-(h-v)\beta(\zeta)} - e^{-(h+v)\beta(\zeta)} \right] dv, \quad (2.24a)
\]

\[
m\beta'(\zeta) e^{-\beta'(\zeta) h} A - \beta(\zeta)(e^{-\beta(\zeta) h} - e^{\beta(\zeta) h}) B = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\zeta}{\beta(\zeta)} \int_{d_1}^d a(v) \left[ e^{-(h-v)\beta(\zeta)} - e^{-(h+v)\beta(\zeta)} \right] dv. \quad (2.24b)
\]

Solving this linear system of two equations yields the following integral expressions for the quantities $A$ and $B$

\[
A = \sqrt{\frac{2}{\pi}} \frac{\zeta}{\beta(\zeta) \Delta_0(\zeta)} e^{\beta(\zeta) h} \bar{T}(\zeta), \quad (2.24c)
\]

\[
B = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\zeta}{\beta'(\zeta) \Delta_0(\zeta)} e^{-\beta'(\zeta) h} \left[ \beta(\zeta) - m \beta'(\zeta) \right] \bar{T}(\zeta), \quad (2.24d)
\]

where $\Delta_0(\zeta)$ and the integral $\bar{T}(\zeta)$ are given by the following equations
\[ \Delta_0(\zeta) = e^{-\beta(\zeta)\delta} \left[ -\beta(\zeta) \left( e^{-\beta(\zeta)\delta} - e^{\beta(\zeta)\delta} \right) + m\beta'(\zeta) \left( e^{-\beta(\zeta)\delta} + e^{\beta(\zeta)\delta} \right) \right], \quad (2.24c) \]

\[ \overline{T}(\zeta) = \int_{d_1}^d a(v) \left[ e^{-(h-v)\beta(\zeta)} - e^{-(h+v)\beta(\zeta)} \right] dv. \quad (2.24f) \]

Next, we consider the boundary condition (2.9d), which specifies that the scattered shear stress along the crack \((x_1 = 0, d_1 < x_2 < d)\) is equal in magnitude and opposite in sign to the stress induced by the incident wave. The scattered shear stress on the plane of the crack at \(x_1 = 0\) is given by equation (2.15b) as follows

\[ \sqrt{\frac{\pi}{2}} \sigma_{13}(0, x_2) = \mu \int_0^\infty \frac{\zeta B(e^{-\beta(\zeta)x_2} + e^{\beta(\zeta)x_2})}{\zeta} \, d\zeta \]

\[ -\mu \int_0^\infty E\beta(\zeta) \cos(\zeta x_2) \, d\zeta. \quad (2.25) \]

Using the expressions (2.24d) and (2.21) for \(B\) and \(E\) in terms of the dislocation density \(a(v)\), one finds that the scattered stress specified in the above equation takes the form

\[ \frac{\pi}{\mu} \sigma_{13}(0, x_2) = \int_{d_1}^d a(v) K(v, x_2) \, dv \]

\[ -2 \int_0^\infty \frac{\beta(\zeta)}{\zeta} \left[ \int_{d_1}^d a(v) \sin(\zeta v) \, dv \right] \cos(\zeta x_2) \, d\zeta, \quad (2.26) \]

where the integral \(K(v, x_2)\) is defined by

\[ K(v, x_2) = \int_0^\infty \frac{\zeta}{\beta^2(\zeta)} \frac{K_1(\zeta, v, x_2)}{\Delta(\zeta)} \, d\zeta, \quad (2.27) \]

\[ K_1(\zeta, v, x_2) = \zeta \left[ e^{-(h-v)\beta(\zeta)} - e^{-(h+v)\beta(\zeta)} \right] \]

\[ \times \left( \beta(\zeta) - m\beta'(\zeta) \right) \left[ e^{-\beta(\zeta)x_2} + e^{\beta(\zeta)x_2} \right], \quad (2.28) \]

\[ \Delta(\zeta) = -\beta(\zeta) \left( e^{-\beta(\zeta)\delta} - e^{\beta(\zeta)\delta} \right) + m\beta'(\zeta) \left( e^{-\beta(\zeta)\delta} + e^{\beta(\zeta)\delta} \right). \quad (2.29) \]

Notice that the function \(\beta(\zeta)/\zeta\) in the second integral of equation (2.26) behaves as \(1 + 0(\zeta^{-2})\) as \(\zeta \to \infty\). Thus, the integral of this function over the interval
\([0, \infty)\) is not bounded. Therefore, special care is required to interchange the last two integrations in equation (2.26). We first write the last term in the right-hand side of equation (2.26) as the sum of two integrals \(I_1(x_2)\) and \(I_2(x_2)\). The first integral, \(I_1(x_2)\), is written as

\[
I_1(x_2) = - \int_{d_1}^{d} a(v) H_1(v, x_2) \, dv, 
\]

(2.30a)

where \(H_1(v, x_2)\) is given by the following integral

\[
H_1(v, x_2) = 2 \int_0^\infty \left[ \frac{\beta(\zeta)}{\zeta} - 1 \right] \sin(\zeta v) \cos(\zeta x_2) \, d\zeta. 
\]

(2.30b)

The second integral, \(I_2(x_2)\), has the form

\[
I_2(x_2) = -2 \int_0^\infty \left[ \int_{d_1}^{d} a(v) \sin(\zeta v) \, dv \right] \cos(\zeta x_2) \, d\zeta. 
\]

(2.31)

The order of the integrations in \(I_2(x_2)\) can be interchanged with some care. To perform this interchange, it is convenient to write

\[
I_2(x_2) = \lim_{X \to \infty} J_0(x_2), 
\]

where \(X\) is some large positive real number, and the quantity \(J_0(x_2)\) is defined by the following integral

\[
J_0(x_2) = - \int_{d_1}^{d} a(v) \int_0^X [\sin(\zeta(v + x_2)) + \sin(\zeta(v - x_2))] \, d\zeta \, dv. 
\]

(2.33a)

Evaluating the inner integral over the interval \([0, X]\) in the right-hand side of the above equation, one finds that the integral \(J_0(x_2)\) can be written as the sum of two integrals such that

\[
J_0(x_2) = \int_{d_1}^{d} \frac{a(v)}{v + x_2} \left[ \cos(X(v + x_2)) - 1 \right] \, dv \\
+ \int_{d_1}^{d} \frac{a(v)}{v - x_2} \left[ \cos(X(v - x_2)) - 1 \right] \, dv. 
\]

(2.33b)
Notice that the portion of the first integral that contains the trigonometric function, in the right-hand side of the above equation, vanishes in the limit as $X \to \infty$ by the Riemann-Lebesgue lemma (Churchill and Brown [6]). The second integral $J_0(x_2)$ in the right-hand side of equation (2.33b) has a simple pole when the variable $x_2$ lies in the interval $[d_1, d]$, and it is written as follows

\[
J_0(x_2) = a(x_2) \int_{d_1}^{d} \frac{\cos(X(v - x_2)) - 1}{v - x_2} \, dv - \int_{d_1}^{d} \frac{a(v) - a(x_2)}{v - x_2} \, dv \\
+ \int_{d_1}^{d} \frac{a(v) - a(x_2)}{v - x_2} \cos(X(v - x_2)) \, dv.
\] (2.34)

The third integral in the right-hand side of the above equation vanishes in the limit as $X \to \infty$ (by the Riemann-Lebesgue lemma). Further, the first integral of equation (2.34) can be evaluated by using the Riemann-Lebesgue lemma and the property that the integrand is an odd function of the variable $(v - x_2)$. Thus, one has

\[
\lim_{X \to \infty} a(x_2) \int_{d_1}^{d} \frac{\cos(X(v - x_2)) - 1}{v - x_2} \, dv = -\log |d - x_2| + \log |x_2 - d_1| \\
= -\int_{d_1}^{d} \frac{1}{v - x_2} \, dv,
\] (2.35)

where the last integral is interpreted in the Cauchy principal-value sense if the variable $x_2$ lies in the interval $[d_1, d]$.

Substituting equations (2.34) and (2.35) into (2.33b), and taking the limit of the resultant equation as $X \to \infty$, one finds that the integral $I_2(x_2)$ in equation (2.32) is given by

\[
I_2(x_2) = -\int_{d_1}^{d} a(v) \left[ \frac{1}{v + x_2} + \frac{1}{v - x_2} \right] \, dv.
\] (2.36)

Considering equations (2.26-30), (2.36), and the incident stress of equation (2.3), we find that the boundary condition (2.9d) along the crack faces yields the following singular integral equation for the dislocation density $a(v)$.
\[
\int_{d_1}^d a(v) \left[ \frac{1}{v + x_2} + \frac{1}{v - x_2} + H_1(v, x_2) - K(v, x_2) \right] dv \\
= i\pi u_0 s_\omega \cos [\omega s_\tau \alpha(\eta) x_2], \quad d_1 < x_2 < d, \quad (2.37)
\]

where \( K(v, x_2) \) and \( H_1(v, x_2) \) are defined in equations (2.27) and (2.30b), respectively. The integral in the left-hand side of equation (2.37) is defined only in the Cauchy principal-value sense.

Next, we write the singular integral equation (2.37) in dimensionless form. For that purpose, we need to define the following dimensionless quantities

\[
\bar{h} = \frac{h}{d - d_1}, \quad \bar{d} = \frac{d + d_1}{2(d - d_1)}, \quad \bar{u}_0 = \frac{u_0}{d - d_1}, \\
\bar{v} = (d - d_1) \bar{v}, \quad x_2 = (d - d_1) \bar{\tau}, \quad \zeta = \omega s_\tau u, \\
\bar{\nu} = \frac{\nu}{2} + \bar{d}, \quad \bar{\tau} = \frac{\tau}{2} + \bar{d}, \quad \bar{s}_\tau = \omega s_\tau (d - d_1), \\
\bar{\beta}_2(u) = u^2 - 1, \quad \bar{\beta}'(u) = u^2 + e^2, \quad \bar{a}(\nu) = a( (d - d_1) \bar{\nu} ), \\
\bar{\omega} = \frac{h}{\lambda_\tau} = h \omega s_\tau / (2\pi) = \bar{h} \bar{s}_\tau / (2\pi), \quad \lambda_\tau = \frac{2\pi}{\omega s_\tau}. \quad (2.38)
\]

These quantities are chosen such that the interval of integration of the singular integral equation (2.37) becomes \([-1, 1]\) instead of \([d_1, d]\).

Using the dimensionless notations in equation (2.38), the singular integral equation (2.37) and the condition (2.18) take the following form

\[
\int_{-1}^1 \bar{a}(\nu) \left[ \frac{1}{\nu - \tau} + \frac{1}{\nu + \tau + 4\bar{d}} + \frac{1}{2} \bar{H}_1(\nu, \tau) - \frac{1}{2} \bar{K}(\nu, \tau) \right] d\nu \\
= i\bar{u}_0 \bar{g}(\tau), \quad -1 < \tau < 1, \quad (2.39a)
\]

\[
\int_{-1}^1 \bar{a}(\nu) d\nu = 0. \quad (2.39b)
\]
The dimensionless functions \( \bar{g}(\tau) \), \( \bar{H}(\nu, \tau) \), and \( \bar{K}(\nu, \tau) \) are defined by

\[
\bar{g}(\tau) = \pi \eta \bar{s}_r \cos (\bar{s}_r \alpha(\eta)^\tau),
\]

\[
\bar{H}(\nu, \tau) = 2 \bar{s}_r \int_0^\infty \left[ \frac{\bar{\beta}(u)}{u} - 1 \right] \sin (\bar{s}_r \bar{\nu} u) \cos (\bar{s}_r \tau u) \, du,
\]

\[
\bar{K}(\nu, \tau) = \bar{s}_r \int_0^\infty \frac{u}{\beta^2(u)} \frac{\bar{K}_1(u, \nu, \tau)}{\Delta(u)} \, du,
\]

where the functions \( \bar{K}_1(u, \nu, \tau) \) and \( \Delta(u) \) are the dimensionless form of equations (2.28) and (2.29), and are such that

\[
\bar{K}_1(u, \nu, \tau) = u \left[ e^{-\bar{s}_r (\bar{h} - \bar{\nu}) \bar{\beta}(u)} - e^{-\bar{s}_r (\bar{h} + \bar{\nu}) \bar{\beta}(u)} \right] \times \left( \bar{\beta}(u) - m \bar{\beta}'(u) \right) \left[ e^{-\bar{s}_r \bar{\beta}(u)^\tau} + e^{\bar{s}_r \bar{\beta}(u)^\tau} \right],
\]

\[
\Delta(u) = -\bar{\beta}(u) \left[ e^{-\bar{s}_r \bar{h} \bar{\beta}(u)} - e^{\bar{s}_r \bar{h} \bar{\beta}(u)} \right]
+ m \bar{\beta}'(u) \left[ e^{-\bar{s}_r \bar{h} \bar{\beta}(u)} + e^{\bar{s}_r \bar{h} \bar{\beta}(u)} \right].
\]

The integral in equation (2.39a) is defined only in the Cauchy principal-value sense. Notice that the integral defining the function \( \bar{K}(\nu, \tau) \) over the interval \( (0, \infty) \) in equation (2.42) is not a regular integral, since the integrand has simple poles in the interval \( [\epsilon, 1] \). We write the function \( \bar{K}(\nu, \tau) \) as the sum of three integrals \( \bar{K}_a(\nu, \tau), \bar{K}_b(\nu, \tau), \) and \( \bar{K}_c(\nu, \tau) \), which are evaluated over the intervals \( [0, \epsilon] \), \( [\epsilon, 1] \), and \( [1, \infty) \), respectively. The first integral \( \bar{K}_a(\nu, \tau) \) is evaluated over the interval \( [0, \epsilon] \), and can be written as

\[
\bar{K}_a(\nu, \tau) = -\bar{s}_r \int_0^\epsilon \frac{u}{\alpha^2(u)} \frac{\bar{K}_{a1}(u, \nu, \tau)}{\Delta_a(u)} \, du.
\]

In the interval \( [0, \epsilon] \), one has (using the definition (2.3))

\[
\bar{\beta}(u) = -i(1 - u^2)^{1/2} = -i \alpha(u),
\]

\[
\bar{\beta}'(u) = -i(\epsilon^2 - u^2)^{1/2} = -i \alpha'(u).
\]
The functions \( \overline{K}_{a1}(u, \nu, \tau) \) and \( \overline{\Delta}_a(u) \) of equation (2.45) are defined as follows

\[
\overline{K}_{a1}(u, \nu, \tau) = u \left[ e^{i\tilde{s}_\tau \alpha(u)} - e^{i\tilde{s}_\tau \beta(u)} \right] \\
\quad \times \left( \alpha(u) - m\alpha'(u) \right) \left[ e^{-i\tilde{s}_\tau \cdot \alpha(u)} + e^{i\tilde{s}_\tau \cdot \alpha(u)} \right],
\]
(2.47)

\[
\overline{\Delta}_a(u) = \alpha(u) \left[ e^{-i\tilde{s}_\tau \cdot \alpha(u)} - e^{i\tilde{s}_\tau \cdot \alpha(u)} \right] \\
\quad + m\alpha'(u) \left[ e^{-i\tilde{s}_\tau \cdot \alpha(u)} + e^{i\tilde{s}_\tau \cdot \alpha(u)} \right].
\]
(2.48)

Observe that the integrand in the right-hand side of equation (2.45) is bounded over the interval of integration \((0, \epsilon)\), and the integral \( \overline{K}_a(\nu, \tau) \) has a complex value.

The second integral \( \overline{K}_b(\nu, \tau) \) is evaluated over the interval \([\epsilon, 1]\), and can be written as

\[
\overline{K}_b(\nu, \tau) = \frac{\tilde{s}_\tau}{2} \int_\epsilon^1 \frac{u}{\alpha^2(u)} \overline{K}_{b1}(u, \nu, \tau) \overline{\Delta}_b(u) du.
\]
(2.49)

In the interval \([\epsilon, 1]\), one has (using the definition (2.3))

\[
\bar{\beta}(u) = -i(1 - u^2)^{1/2} = -i\alpha(u),
\]
(2.50a)

\[
\bar{\beta}'(u) = (u^2 - \epsilon^2)^{1/2} = \alpha'(u).
\]
(2.50b)

The functions \( \overline{K}_{b1}(u, \nu, \tau) \) and \( \overline{\Delta}_b(u) \) in the right-hand side of equation (2.49) take the following form

\[
\overline{K}_{b1}(u, \nu, \tau) = u \left[ e^{i\tilde{s}_\tau \alpha(u)} - e^{i\tilde{s}_\tau \beta(u)} \right] \\
\quad \times \left( i\alpha(u) + m\alpha'(u) \right) \left[ e^{-i\tilde{s}_\tau \cdot \alpha(u)} + e^{i\tilde{s}_\tau \cdot \alpha(u)} \right],
\]
(2.51a)

\[
\overline{\Delta}_b(u) = -\alpha(u) \sin(\tilde{s}_\tau \cdot \alpha(u)) + m\alpha'(u) \cos(\tilde{s}_\tau \cdot \alpha(u)).
\]
(2.51b)

Notice that the equation \( \overline{\Delta}_b(u) = 0 \) is identical to equation (2.4) since \( \tilde{h}\tilde{s}_\tau = h\omega\tilde{s}_\tau \), and they have the same set of \( k \) simple roots, where the integer \( k \) is determined by equation (2.7). Thus, the integrand in the right-hand side of equation (2.49) has \( k \)
simple poles \( \eta_l (l = 1, \ldots, k) \) such that \( \epsilon \leq \eta_k < \ldots < \eta_1 < 1 \). It follows that the integral \( \overline{K}_b(\nu, \tau) \) defined in equation (2.49) must be interpreted as follows (Lamb [3])

\[
\overline{K}_b(\nu, \tau) = \frac{i \pi \bar{s}_b}{2} \sum_{l=1}^{k} \frac{\eta_l}{\alpha^2(\eta_l)} \frac{\overline{K}_{b1}(\eta_l, \nu, \tau)}{\overline{\Delta}'_b(\eta_l)}
+ \frac{\bar{s}_b}{2} \int_{\epsilon}^{1} \frac{u}{\alpha^2(u)} \frac{\overline{K}_{b1}(u, \nu, \tau)}{\overline{\Delta}_b(u)} \, du,
\]

where \( \overline{\Delta}'_b(u) \) is the derivative of \( \overline{\Delta}_b(u) \), which is defined by equation (2.51b), and the value of \( \overline{\Delta}'_b(\eta_l) \) is given by the following equation

\[
\overline{\Delta}'_b(\eta_l) = \frac{\eta_l \cos \left( \bar{s}_b \bar{h} \alpha(\eta_l) \right)}{\alpha^2(\eta_l) \alpha'(\eta_l)} \, Q_l.
\]

In (2.53a), \( Q_l \) is given by the following equation

\[
Q_l = m \left( 1 - \epsilon^2 \right) + \bar{s}_b \bar{h} \alpha'(\eta_l) \left( 1 - (m \epsilon)^2 + (m \eta_l)^2 - \eta_l^2 \right).
\]

The circle through the integral sign in the right-hand side of equation (2.52) indicates that the integral through each pole \( \eta_l (l = 1, \ldots, k) \) is performed in the principal-value sense. This integral is evaluated numerically in the form

\[
I_\circ = \int_{\epsilon}^{1} \frac{1}{\sqrt{1-u}} \left[ \frac{u}{\alpha_b(u)} \frac{\overline{K}_{b1}(u, \nu, \tau)}{\overline{\Delta}_b(u)} - \sum_{l=1}^{k} \frac{\eta_l}{\alpha_b(\eta_l)} \frac{\overline{K}_{b1}(\eta_l, \nu, \tau)}{(u - \eta_l) \overline{\Delta}'_b(\eta_l)} \right] \, du
+ \sum_{l=1}^{k} \frac{\eta_l}{\alpha_b(\eta_l)} \left( \frac{\overline{K}_{b1}(\eta_l, \nu, \tau)}{\overline{\Delta}'_b(\eta_l)} \right) J_l,
\]

where \( \alpha_b(u) \) and the integral \( J_l \) are defined by

\[
\alpha_b(u) = \sqrt{1-u} (1+u),
\]

\[
J_l = \int_{\epsilon}^{1} \frac{du}{\sqrt{1-u} (u - \eta_l)}.
\]
The integral $J_l$ in equation (2.56) is evaluated through the pole $\eta_l$ ($l = 1, \ldots, k$) in the principal-value sense. Changing variables, and evaluating the integral in equation (2.56) through each pole, one finds the following result

$$J_l = \frac{-1}{\sqrt{1 - \eta_l}} \log \frac{\eta_l - \epsilon}{\left( \sqrt{1 - \epsilon} + \sqrt{1 - \eta_l} \right)}.$$  

(2.57)

Using the above result of $J_l$, substituting equation (2.54) into equation (2.52), and changing integration variables, one can write the integral $\overline{K}_b(\nu, \tau)$ in the following form

$$\overline{K}_b(\nu, \tau) = \begin{array}{c}
\bar{s}_r \int_0^\infty \left[ \frac{\bar{v}}{\alpha_b(\bar{v})} \frac{D_{\bar{v}}(\bar{v}, \nu, \tau)}{D_b(\bar{v})} + \sum_{i=1}^k \frac{\eta_l}{\alpha_b(\eta_l)} \frac{D_{\eta_l}(\eta_l, \nu, \tau)}{D_b(\eta_l)} \right] d\bar{v} \\
+ \begin{array}{c}
\frac{\bar{s}_r}{2} \sum_{i=1}^k \frac{\eta_l}{\alpha^2(\eta_l)} \frac{D_{\eta_l}(\eta_l, \nu, \tau)}{D_b(\eta_l)} \end{array}
\left( i\pi - \log \frac{\eta_l - \epsilon}{\sqrt{1 - \epsilon} + \sqrt{1 - \eta_l}} \right),
\end{array}$$

(2.58)

where the functions $D_{\bar{v}}(\bar{v}, \nu, \tau)$, $D_b(u)$, and $D_b'(\eta)$ are defined in equations (2.51a), (2.51b), and (2.53). The constant $\bar{\epsilon}$ and the variable $\bar{v}$ are expressed by the following equations

$$\bar{\epsilon} = \sqrt{1 - \epsilon}, \quad \bar{v} = 1 - v^2.$$  

(2.59)

The integrand in the right-hand side of equation (2.58) is bounded in the interval of integration $[0, \epsilon]$, and the integral $\overline{K}_b(\nu, \tau)$ has a complex value.

The last part of the integral $\overline{K}(\nu, \tau)$ defined in equation (2.12) is $\overline{K}_c(\nu, \tau)$, which is evaluated over the interval $[1, \infty)$. Changing integration variables, the bounded integral $\overline{K}_c(\nu, \tau)$ takes the following form

$$\overline{K}_c(\nu, \tau) = 2\bar{s}_r \int_0^\infty \frac{\tilde{v}}{\beta_c(\tilde{v})} \overline{K}_1(\tilde{v}, \nu, \tau) d\tilde{v},$$

(2.60)

where $\overline{K}_1(v, \nu, \tau)$ and $\overline{\Delta}(v)$ are defined by equations (2.43) and (2.44) with the following expressions for $\overline{\beta}(u)$ and $\overline{\beta}'(u)$.
\[
\overline{\beta}(u) = (u^2 - 1)^{1/2}, \quad \overline{\beta}'(u) = (u^2 - \epsilon^2)^{1/2} = \alpha'(u).
\]

(2.61)

The function \(\overline{\beta}_c(\tilde{v})\) and the variable \(\tilde{v}\) in the right-hand side of equation (2.60) are defined by

\[
\overline{\beta}_c(v) = \sqrt{v - 1}(v + 1), \quad \tilde{v} = v^2 + 1.
\]

(2.62)

The integrand of equation (2.60) is bounded in the interval of integration, and the integral \(\overline{K}_c(\nu, \tau)\) has a real value. The sum of the three functions \(\overline{K}_a(\nu, \tau), \overline{K}_b(\nu, \tau),\) and \(\overline{K}_c(\nu, \tau)\) which are defined by Equations (2.45), (2.58), and (2.60) is equal to the function \(\overline{K}(\nu, \tau)\) of (2.42).

The function \(\overline{H}_1(\nu, \tau)\) that is defined in equation (2.41) can be written as the sum of two integrals \(\overline{H}_{a1}(\nu, \tau)\) and \(\overline{H}_{b1}(\nu, \tau)\). The first integral \(\overline{H}_{a1}(\nu, \tau)\) is evaluated over the interval \([0, 1]\), and has a complex value, while the other one \(\overline{H}_{b1}(\nu, \tau)\) is evaluated over the interval \([1, \infty)\), and has a real value. These two integrals can be written as follows

\[
\overline{H}_{a1}(\nu, \tau) = -2\overline{s}_\tau \int_0^1 \left(1 + i \frac{\alpha(u)}{u}\right) \sin(\overline{s}_\tau \tilde{v} u) \cos(\overline{s}_\tau \overline{\tau} u) \, du,
\]

(2.63a)

\[
\overline{H}_{b1}(\nu, \tau) = -2\overline{s}_\tau \int_1^\infty \frac{1}{\overline{\Delta}_h(u)} \sin(\overline{s}_\tau \tilde{v} u) \cos(\overline{s}_\tau \overline{\tau} u) \, du,
\]

(2.63b)

where \(i\) is the imaginary unit, and \(\overline{\Delta}_h(u)\) has the following form

\[
\overline{\Delta}_h(u) = u \left[u + (u^2 - 1)^{1/2}\right].
\]

(2.64)

The functions \(\overline{H}_1(\nu, \tau)\) and \(\overline{K}(\nu, \tau)\) of the singular integral equation (2.39a) are now fully specified by equations (2.41) through (2.64). In the next section, the method of Erdogan and Gupta [1] is implemented to evaluate the dislocation density \(\tilde{\alpha}(\nu)\) across the crack faces.
2.4 Method of Solution

In this section, the singular integral equation (2.39a) is solved numerically by using the method of Erdogan and Gupta [1]. We first recall that the dislocation density \( \tilde{a}(\nu) \) has a square-root singularity at the crack tips \( \nu = \pm 1 \). Therefore, the function \( \tilde{a}(\nu) \) can be written over the interval \([-1, 1] \) in the form

\[
\tilde{a}(\nu) = \frac{\Phi(\nu)}{(1 - \nu^2)^{1/2}}, \quad -1 < \nu < 1. \tag{2.65}
\]

The function \( \Phi(\nu) \) is bounded in the interval \([-1, 1] \). Using the above definition of the dislocation density \( \tilde{a}(\nu) \), and equations (2.6) and (2.8) of Erdogan and Gupta [1] for a number \( n \) of quadrature points, one finds that the singular integral equation (2.39a) and the condition (2.39b) can be approximated by the following linear system of order \( n \)

\[
\sum_{j=1}^{n} \Phi(\nu_j) \left[ \frac{1}{\nu_j - \tau_r} + \frac{1}{\nu_j + \tau_r + 4d} + \frac{1}{2} \tilde{H}_1(\nu_j, \tau_r) - \frac{1}{2} K(\nu_j, \tau_r) \right] = \frac{i \tilde{g}(\tau_r)}{\pi}, \quad r = 1, \ldots, (n - 1), \tag{2.66}
\]

together with,

\[
\sum_{j=1}^{n} \Phi(\nu_j) = 0. \tag{2.67}
\]

The functions \( \tilde{H}_1(\nu, \tau), K(\nu, \tau) \), and \( \tilde{g}(\tau) \) are defined in equations (2.40)-(2.42), and the quantities \( \nu_j \) and \( \tau_r \) have the following values

\[
\nu_j = \cos \left( \pi [2j - 1]/2n \right), \tag{2.68}
\]

\[
\tau_r = \cos \left( \pi r/n \right). \tag{2.69}
\]

Observing that the coefficients, the unknowns \( \Phi(\nu_j) \), and the right-hand side terms of the linear system of equation (2.66) are complex numbers, we define the real and imaginary parts of these quantities by the following equations
\[ Q_R(\nu_j, \tau_r) = \text{Re} [Q(\nu_j, \tau_r)], \quad Q_I(\nu_j, \tau_r) = \text{Im} [Q(\nu_j, \tau_r)], \quad (2.70) \]
\[ \Phi_R(\nu_j) = \text{Re} [\Phi(\nu_j)], \quad \Phi_I(\nu_j) = \text{Im} [\Phi(\nu_j)], \quad (2.71) \]
\[ G_R(\tau_r) = 0, \quad G_I(\tau_r) = \frac{\tilde{u}_0 n}{\pi} \tilde{g}(\tau_r), \quad (2.72) \]

where \( Q(\nu_j, \tau_r) \) is defined by the following equation
\[ Q(\nu_j, \tau_r) = \frac{1}{\nu_j - \tau_r} + \frac{1}{\nu_j + \tau_r + 4d} + \frac{1}{2} H_1(\nu_j, \tau_r) - \frac{1}{2} K(\nu_j, \tau_r). \quad (2.73) \]

Using these notations, the complex-valued linear system of equations (2.66) and (2.67) can be written as a real-valued linear system such that
\[ \sum_{j=1}^{n} [Q_R(\nu_j, \tau_r)\Phi_R(\nu_j) - Q_I(\nu_j, \tau_r)\Phi_I(\nu_j)] = 0, \quad (2.74) \]
\[ \sum_{j=1}^{n} [Q_I(\nu_j, \tau_r)\Phi_R(\nu_j) + Q_R(\nu_j, \tau_r)\Phi_I(\nu_j)] = G_I(\tau_r), \quad r = 1, \ldots, (n - 1), \quad (2.75) \]

Together with
\[ \sum_{j=1}^{n} \Phi_R(\nu_j) = 0, \quad \sum_{j=1}^{n} \Phi_I(\nu_j) = 0. \quad (2.76) \]

The real linear system of equations (2.74)-(2.76) can be written in the following matrix form:

\[
\begin{pmatrix}
(n - 1) \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
Q_R(\nu_j, \tau_r) & -Q_I(\nu_j, \tau_r) & \\
\end{pmatrix}
\begin{pmatrix}
\Phi_R(\nu_j) \\
\Phi_I(\nu_j) \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
G_I(\tau_r) \\
0 \\
\end{pmatrix}
\]

\[
(2.77)
\]
where \( j = 1, \ldots, n \), and \( r = 1, \ldots, (n - 1) \). Equations (2.74)-(2.76) represent a linear system of order \( 2n \) for the \( 2n \) dislocation density real and imaginary unknowns \( \Phi_n(v_j) \) and \( \Phi_j(v_j) \). The system is solved numerically.

The region of admissible values for the two geometric ratios \( \bar{d} \) and \( \bar{h} \), which are defined by equation (2.38), is illustrated in Figure (2.2). Since the two crack tips must be contained inside the layer, one has

\[
\frac{d + d_1}{2} + \frac{d - d_1}{2} < h, \quad \frac{d + d_1}{2} - \frac{d - d_1}{2} > 0, \quad d - d_1 < h. \tag{2.78a}
\]

Dividing through by the crack thickness \( d - d_1 \), one infers from (2.78a) that

\[
0.5 < \bar{d} < \bar{h} - 0.5, \quad \bar{h} > 1.0. \tag{2.78b}
\]

Thus, the parameters \( \bar{h} \) and \( \bar{d} \) are subjected to the constraints (2.78b).

One can see from Figure (2.2) that \( \bar{h} \) and \( \bar{d} \) can be chosen inside a triangular-shaped region bounded from below by the horizontal line \( \bar{d} = 0.5 \) and from above by the inclined line of equation \( \bar{d} = \bar{h} - 0.5 \). When the value of \( \bar{d} \) is close to 0.5, the upper crack tip is close to the free surface. On the other hand, if this value is close to \( \bar{h} - 0.5 \), then the lower crack tip approaches the interface between the two solids.

When the value of \( \bar{h} \) is close to 1.0, the crack width is very large and approaches the layer thickness. Further, the inclined line of equation \( \bar{d} = 0.5 \bar{h} \) in Figure (2.2) represents the cracks that lie in the middle of the surface layer. If the values of \( \bar{d} \) and \( \bar{h} \) are chosen above this line then the crack is longer below the medium line of the layer than above, and if they are chosen on the other side of the line then the crack is longer above the medium line of the layer than below.

The value of six dimensionless parameters are required to solve the linear system (2.77). These are: the ratio of the slownesses, \( \epsilon = s'_r / s_r \); the ratio of the shear moduli, \( m = \mu' / \mu \); the ratio of the crack middle point depth to the crack width.
\( \bar{d} = (d + d_1)/(2(d - d_1)) \); the ratio of the thickness of the surface layer to the crack width, \( \bar{h} = h/(d - d_1) \); the amplitude of the incident wave, \( \bar{u}_0 = u_0/(d - d_1) \); the frequency of the incident wave, \( \bar{\omega} = \omega/\lambda_r \), where \( \lambda_r = 2\pi/\omega s_r \) is the wavelength of transverse waves in the layer. All the other dimensionless quantities can be written in terms of these six basic parameters.

Figure 2.2: Admissible values for the geometric parameters \( \bar{d} \) and \( \bar{h} \).
In the next section, the formulas of the scattered displacement fields in the cracked layer and the half-space are derived in terms of the dislocation density \( a(v) \). The amplitudes of the reflected and transmitted far-field displacements are obtained.

### 2.5 Far-Field Displacement

Equations (2.15a), (2.15d), and (2.17) are used here to obtain the scattered displacement field for the cracked half-space in terms of the dislocation density \( a(v) \). The reflected and transmitted displacement fields at a distance away from the crack plane are obtained as a sum of integrals of the dislocation density \( a(v) \). The results of the previous section for the dislocation density \( a(v) \) at \( n \) points along the crack face and the method of Erdogan and Gupta [1] are used in this section to evaluate the amplitude moduli and phases of both the reflected and transmitted far-field displacements.

We first obtain the scattered displacement field for both the cracked layer and the half-space in terms of the dislocation density \( a(v) \), using equations (2.15a), (2.15d), (2.19), (2.21), (2.21c) and (2.24d), as follows

\[
\pi u_3(x_1, x_2) = \int_{d_1}^{d} a(v) [G_1(v, x_1, x_2) + 2G_2(x_1, x_2)] dv, \tag{2.79}
\]

\[
\pi u'_3(x_1, x_2) = 2 \int_{d_1}^{d} a(v) G'(v, x_1, x_2) dv, \tag{2.80}
\]

where the integrals \( G_1, G_2, \) and \( G' \) are expressed by the following equations

\[
G_1(v, x_1, x_2) = \int_{0}^\infty \frac{1}{\beta^2(\zeta)} \frac{K_1(\zeta, v, x_2)}{\Delta(\zeta)} \sin(\zeta x_1) d\zeta, \tag{2.81}
\]

\[
G_2(v, x_1, x_2) = \int_{0}^\infty \frac{\sin(\zeta v)}{\zeta} \cos(\zeta x_2) e^{-\beta(\zeta) x_1} d\zeta, \tag{2.82}
\]

\[
G'(v, x_1, x_2) = \int_{0}^\infty \frac{\zeta}{\beta(\zeta)} \frac{e^{-\beta(\zeta) x_2}}{\Delta(\zeta)} \left[ e^{\nu \beta(\zeta)} - e^{-\nu \beta(\zeta)} \right] \sin(\zeta x_1) d\zeta. \tag{2.83}
\]
The functions $K_1$, $\Delta$, $\beta$ and $\beta'$ are defined in the previous sections by equations (2.28), (2.29), and (2.16). The dimensionless form of the coordinates, $x$ and $y$, as well as of the scattered displacement field, $\tilde{u}_3(x, y)$, are defined such that
\begin{align*}
x &= x_1/(d - d_1), \quad y = x_2/(d - d_1), \\
\tilde{u}_3(x, y) &= u_3(x_1, x_2)/(d - d_1).
\end{align*}
(2.84)
(2.85)

Using these notations, with the other dimensionless quantities already defined in equations (2.38), the scattered dimensionless displacement field takes the following integral form
\begin{equation}
\pi \tilde{u}_3(x, y) = \int_{-1}^{1} \tilde{a}(\nu) \begin{pmatrix} \overline{G}_1(\nu, x, y) + \overline{G}_2(\nu, x, y) \\ \overline{G}'(\nu, x, y) \end{pmatrix} d\nu, \quad 0 \leq y \leq \tilde{h}, \quad \tilde{h} \leq y, 
\end{equation}
(2.86)
where the dislocation density $\tilde{a}(\nu)$ as well as the dimensionless variable $\nu$ are defined in equation (2.38). The functions $\overline{G}'(\nu, x, y)$, $\overline{G}_1(\nu, x, y)$, and $\overline{G}_2(\nu, x, y)$ are defined by the following equations
\begin{align*}
\overline{G}'(\nu, x, y) &= \int_0^\infty \frac{u}{\beta^2(u)} \sin(\tilde{s}_T u \tilde{v}) e^{-(\tilde{s}_T \nu - \tilde{h})} \frac{\varphi_1(\varphi(u))}{\Delta(u)} \sin(\tilde{s}_T u x) du, \\
\overline{G}_1(\nu, x, y) &= \frac{1}{2} \int_0^\infty \frac{1}{\beta^2(u)} \overline{F}_1(u, \nu, u) \frac{\varphi_1(\varphi(u))}{\Delta(u)} \sin(\tilde{s}_T u x) du, \\
\overline{G}_2(\nu, x, y) &= \int_0^\infty \frac{1}{u} \sin(\tilde{s}_T u \tilde{v}) \cos(\tilde{s}_T u y) e^{-(\tilde{s}_T \nu - \tilde{h})} \overline{\beta}(u) du.
\end{align*}
(2.87)
(2.88)
(2.89)

The functions $\overline{\beta}(u)$, $\beta'(u)$, as well as the dimensionless quantities used in equations (2.87)-(2.89) are defined by equations (2.38). The function $\overline{\Delta}(u)$ is defined by equation (2.44) and the function $\overline{F}_1(u, \nu, u)$ is given by
\begin{equation}
\overline{F}_1(u, \nu, u) = u \left[ e^{-\tilde{s}_T (\tilde{h} - \tilde{v})} \overline{\beta}(u) - e^{-\tilde{s}_T (\tilde{h} + \tilde{v})} \overline{\beta}(u) \right] \left[ \overline{\beta}(u) - m \overline{\beta}'(u) \right] \left[ e^{\tilde{s}_T y \overline{\beta}(u)} + e^{-\tilde{s}_T y \overline{\beta}(u)} \right].
\end{equation}
(2.90)
The functions $\overline{G}_1(\nu, x, y)$ and $\overline{G}'(\nu, x, y)$ contain integrals of the type of equation (2.42), and the portions $\overline{G}_{16}(\nu, x, y)$ and $\overline{G}'_b(\nu, x, y)$ of these integrals over the interval $[\epsilon, 1]$ should be interpreted as in equation (2.52). This decomposition allows us to split the scattered displacement field into two parts. The first part oscillates, without decay, as a function of $x$, and it is contained only in the integrals $\overline{G}_{16}(\nu, x, y)$ and $\overline{G}'_b(\nu, x, y)$. The second part decays as rapidly as $|x|^{-\frac{1}{2}}$ as $|x| \to \infty$, and it is contained in the three parts of the integrals $\overline{G}_1(\nu, x, y)$ and $\overline{G}'(\nu, x, y)$ over the intervals $[0, \epsilon]$, $[\epsilon, 1]$, and $[1, \infty)$. We first write the integrals $\overline{G}_{16}(\nu, x, y)$ and $\overline{G}'_b(\nu, x, y)$ over the interval $[\epsilon, 1]$ as follows

$$\overline{G}_{16}(\nu, x, y) = i\pi \sum_{l=1}^{k} \frac{1}{4\alpha^2(\eta_l)} \frac{\overline{F}_{b1}(\eta_l, \nu, y)}{\Delta_b(\eta_l)} \sin(\overline{s}_t\eta_l x) + \frac{1}{2} \overline{J}_{G_1}(\nu, x, y), \quad \epsilon \leq y \leq \overline{h}, \quad (2.91)$$

$$\overline{G}'_b(\nu, x, y) = i\pi \sum_{l=1}^{k} \frac{1}{-2i\alpha(\eta_l)} \frac{\overline{F}'_b(\eta_l, \nu, y)}{\Delta'_b(\eta_l)} \sin(\overline{s}_t\eta_l x) + \overline{J}_{G'}(\nu, x, y), \quad \overline{h} \leq y, \quad (2.92)$$

where $\Delta'_b(u)$ is the derivative of $\Delta_b(u)$, and the value of $\Delta'_b(\eta_l)$ is expressed by equations (2.53). The integrals $\overline{J}_{G_1}(\nu, x, y)$ and $\overline{J}_{G'}(\nu, x, y)$ as well as the functions $\overline{F}_{b1}(u, \nu, y)$ and $\overline{F}'_b(u, \nu, y)$ are given by

$$\overline{J}_{G_1}(\nu, x, y) = \int_{-\epsilon}^{1} \frac{1}{2\alpha^2(u)} \frac{\overline{F}_{b1}(u, \nu, y)}{\Delta_b(u)} \sin(\overline{s}_t u x) \, du, \quad (2.93)$$

$$\overline{J}_{G'}(\nu, x, y) = \int_{-\epsilon}^{1} \frac{1}{-2i\alpha(u)} \frac{\overline{F}'_b(u, \nu, y)}{\Delta'_b(u)} \sin(\overline{s}_t u x) \, du, \quad (2.94)$$

$$\overline{F}_{b1}(u, \nu, y) = u \left[ e^{i\overline{s}_t(\overline{h}-\overline{\nu})\alpha(u)} - e^{i\overline{s}_t(\overline{h}+\overline{\nu})\alpha(u)} \right]$$

$$\times (i\alpha(u) + m\alpha'(u)) \left[ e^{-i\overline{s}_t y\alpha(u)} + e^{i\overline{s}_t y\alpha(u)} \right],$$

$$\overline{F}'_b(u, \nu, y) = u e^{-i\overline{s}_t h\alpha(u)} e^{-\overline{s}_t (y-h)\alpha'(u)} \left[ e^{i\overline{s}_t (\overline{h}-\overline{\nu})\alpha(u)} - e^{i\overline{s}_t (\overline{h}+\overline{\nu})\alpha(u)} \right], \quad (2.95)$$

$$\overline{F}'_b(u, \nu, y)$$
where $\overline{\Delta_b}(u)$ is given by equation (2.51b). The circle through the integral sign in the right-hand sides of equations (2.93) and (2.94) indicates that the integrals $\mathcal{J}_{G_1}(\nu, x, y)$ and $\mathcal{J}_{G_2}(\nu, x, y)$ are performed in the principal-value sense through each of the poles $\eta_l (l = 1, \ldots, k)$. These two integrals should be interpreted as in equation (2.54) in order to make the integrands bounded when the variable of integration reaches one of the poles $\eta_l (l = 1, \ldots, k)$. Thus, the functions $\mathcal{J}_{G_1}(\nu, x, y)$ and $\mathcal{J}_{G_2}(\nu, x, y)$ take the following form

\[
\mathcal{J}_{G_1}(\nu, x, y) = \int_c^1 \left[ \frac{1}{2\alpha^2(u)} \frac{\mathcal{F}_{b1}(u, \nu, y)}{\Delta_b(u)} - \sum_{\eta_l=1}^k \frac{1}{2\alpha^2(\eta_l)} \frac{\mathcal{F}_{b1}(\eta_l, \nu, y)}{(u - \eta_l)\Delta'_b(\eta_l)} \right] 
\times \sin(\tilde{s}_\tau ux) \, du + \sum_{\eta_l=1}^k \frac{1}{2\alpha^2(\eta_l)} \frac{\mathcal{F}_{b1}(\eta_l, \nu, y)}{\Delta'_b(\eta_l)} I_l(x), \quad (2.97)
\]

\[
\mathcal{J}_{G_2}(\nu, x, y) = \int_c^1 \left[ \frac{1}{-2i\alpha(u)} \frac{\mathcal{F}_{b2}(u, \nu, y)}{\Delta_b(u)} - \sum_{\eta_l=1}^k \frac{1}{-2i\alpha(\eta_l)} \frac{\mathcal{F}_{b2}(\eta_l, \nu, y)}{(u - \eta_l)\Delta'_b(\eta_l)} \right] 
\times \sin(\tilde{s}_\tau ux) \, du + \sum_{\eta_l=1}^k \frac{1}{-2i\alpha(\eta_l)} \frac{\mathcal{F}_{b2}(\eta_l, \nu, y)}{\Delta'_b(\eta_l)} I_l(x), \quad (2.98)
\]

where the integral $I_l(x)$ can be written as follows

\[
I_l(x) = \int_c^1 \frac{\sin(\tilde{s}_\tau ux)}{(u - \eta_l)} \, du. \quad (2.99)
\]

Using the Riemann-Lebesgue lemma, the integrals in the right-hand side of equations (2.97) and (2.98) vanish as $x \rightarrow \infty$. The integral $I_l(x)$ of equation (2.99) is performed in the principal-value sense across each of the poles $\eta_l (l = 1, \ldots, k)$. Evaluating the cross-pole integrals of equation (2.99), changing variables, and using the Riemann-Lebesgue lemma, the integral $I_l(x)$ yields the following result as $x \rightarrow \infty$

\[
I_l(x) = \pi \cos(\tilde{s}_\tau \eta_l x). \quad (2.100)
\]

Observe that the relation between the functions $\alpha(\eta_l)$ and $\alpha'(\eta_l)$ can be obtained by equating the value of equation (2.51b) at the poles $\eta_l (l = 1, \ldots, k)$ with zero. This relation is
\[ m \alpha'(\eta) = \tan \left( \tilde{s}_\tau \tilde{h} \alpha(\eta) \right) \alpha(\eta). \]  
(2.101a)

Observe now that the function \( \overline{F}_b'(\eta, \nu, y) \) can be written as

\[ \overline{F}_b'(\eta, \nu, y) = -2i \eta \sin(\tilde{s}_\tau \tilde{\nu} \alpha(\eta)) e^{-\tilde{s}_\tau (y-h) \alpha'(\eta)}. \]  
(2.101b)

Using equations (2.95) and (2.101a), one has

\[ \overline{F}_{b1}(\eta, \nu, y) = 4 \eta \frac{\sin(\tilde{s}_\tau \tilde{\nu} \alpha(\eta)) \cos(\tilde{s}_\tau y \alpha(\eta))}{\cos(\tilde{s}_\tau \tilde{h} \alpha(\eta))} \alpha(\eta). \]  
(2.101c)

Using equations (2.101b) and (2.101c) together with the result of equation (2.100), and substituting the integrals \( \overline{J}_{G_1}(\nu, x, y) \) and \( \overline{J}_{G_2}(\nu, x, y) \) of equations (2.97) and (2.98) into equations (2.91) and (2.92), the integrals \( \overline{G}_{1b}(\nu, x, y) \) and \( \overline{G}'_b(\nu, x, y) \) can be written in the following form as \( x \to \infty \)

\[ \overline{G}_{1b}(\nu, x, y) = \pi \sum_{i=1}^{k} \frac{\eta_i}{\alpha(\eta)} \frac{\sin(\tilde{s}_\tau \tilde{\nu} \alpha(\eta)) \cos(\tilde{s}_\tau y \alpha(\eta))}{\cos(\tilde{s}_\tau \tilde{h} \alpha(\eta))} e^{i \tilde{s}_\tau \eta_i x}, \quad 0 \leq y \leq \bar{h}, \]  
(2.102)

\[ \overline{G}'_b(\nu, x, y) = \pi \sum_{i=1}^{k} \frac{\eta_i}{\alpha(\eta)} \frac{\sin(\tilde{s}_\tau \tilde{\nu} \alpha(\eta)) \cos(\tilde{s}_\tau \tilde{h} \alpha(\eta))}{\cos(\tilde{s}_\tau \tilde{h} \alpha(\eta))} e^{-\tilde{s}_\tau (y-h) \alpha'(\eta)} \ e^{i \tilde{s}_\tau \eta_i x}, \quad h \leq y. \]  
(2.103)

The value of the derivative \( \Delta'_b(\eta) \) is given by equation (2.53). Substituting the result of the integrals \( \overline{G}_{1b}(\nu, x, y) \) and \( \overline{G}'_b(\nu, x, y) \) of equations (2.102) and (2.103) in equation (2.86), the scattered displacement field in the cracked half-space as \( x \to \infty \) takes the simple following form

\[ u_3^\infty(x, y) = \sum_{i=1}^{k} U_i \left\{ \frac{\cos(\tilde{s}_\tau y \alpha(\eta))}{\cos(\tilde{s}_\tau \tilde{h} \alpha(\eta)) e^{-\tilde{s}_\tau (y-h) \alpha'(\eta)}} \right\} e^{i \tilde{s}_\tau \eta_i x}, \quad 0 \leq y \leq \bar{h}, \]  
\[ \bar{h} \leq y, \]  
(2.104)

where the amplitude of the scattered displacement far-field \( U_i \) is defined by the following equation

\[ U_i = \frac{\eta_i}{\alpha(\eta)} \frac{1}{\Delta_b(\eta) \cos(\tilde{s}_\tau \tilde{h} \alpha(\eta))} \int_{-1}^{1} \tilde{a}(\nu) \sin(\tilde{s}_\tau \tilde{\nu} \alpha(\eta)) \ d\nu, \]  
(2.105)
and the dislocation density $\bar{a}(\nu)$ has the form

$$\bar{a}(\nu) = \frac{\Phi(\nu)}{\sqrt{1 - \nu^2}}.$$  

(2.106)

Using equations (2.6) and (2.8) of Erdogan and Gupta [1] and the above expression of the dislocation density $\bar{a}(\nu)$, the integral $I_\Phi$ in the right-hand side of equation (2.105) can be written as follows

$$I_\Phi = \frac{\pi}{n} \sum_{j=1}^{n} \Phi(t_j) \sin \left( \bar{s}_r \bar{l}_j \alpha(\eta_j) \right).$$  

(2.107)

Substituting the value of the above integral in equation (2.105), the amplitude $U_l$ can be expressed in the form

$$U_l = \frac{\pi}{n} \frac{\eta_l}{\alpha(\eta_l)} \frac{1}{\sum_{j=1}^{n} \cos(\bar{s}_r \bar{l}_j \alpha(\eta_j))} \sum_{j=1}^{n} \Phi(t_j) \sin \left( \bar{s}_r \bar{l}_j \alpha(\eta_j) \right),$$  

(2.108)

where the values of $t_j$ and $\bar{l}_j$ are given by

$$t_j = \cos \left( \frac{\pi(2j - 1)}{2n} \right), \quad \bar{l}_j = \frac{t_j}{2} + \bar{d}.$$  

(2.109)

The values of the real and imaginary parts of the function $\Phi(t_j), (j = 1, \ldots, n)$, at $n$ points along the crack face were computed in the previous section using the linear system of equations (2.74) through (2.76).

Comparing the scattered displacement far-field (2.104) and that of the incident wave, which is defined in equation (2.2), we conclude that the scattered displacement field is the superposition of $k$ Love wave modes of the type of equation (2.2). Thus, the wave motion in the layer is dominated by these $k$ Love wave modes. This phenomenon occurs at a large distance from the plane of the crack ($|x| \to \infty$). By recalling the antisymmetry of the problem, the reflected and transmitted displacement fields in the cracked half-space can be defined by the following equations

$$u_3^{re}(x, y) = -u_3^{ci}(-x, y), \quad x < 0.$$  

(2.110a)

$$u_3^{tr}(x, y) = u_3^{in}(x, y) + u_3^{ci}(x, y), \quad x > 0.$$  

(2.110b)
where \( u_3^\infty(x, y) \) is defined by equations (2.104), and \( \bar{u}_3^{in}(x, y) \) is given by equation (2.2) and
\[
\bar{u}_3^{in}(x, y) = \frac{u_3^{in}((d - d_1)x, \ (d - d_1)y)}{d - d_1}.
\] (2.111)

The transmission coefficient \( T_1 \) and the reflection coefficients \( R_l \) \( (l = 1, \ldots, k) \) can be defined by the following equations
\[
T_1 = 1 + U_1/\bar{u}_0, \quad (2.112a)
\]
\[
R_l = U_l/\bar{u}_0, \quad l = 1, \ldots, k, \quad (2.112b)
\]
where the amplitudes \( U_l \), in general, are complex numbers, and are defined in equation (2.108). Thus, the moduli and phases of the reflected and transmitted Love wave displacements are different from those of the incident Love wave.

In the next section, we present the results for the scattered far-field displacements. We illustrate the dependence of the reflection and transmission coefficients \( T_1 \) and \( R_l \) on the frequency of the incident Love wave. In addition, we discuss these results in detail.

## 2.6 Discussion of Numerical Results

The values of the real and imaginary parts of \( \Phi(\nu) \), the bounded part of the dislocation density, are shown in Figures (2.3) and (2.4), respectively, for different numbers of quadrature points, \( n = 10, 15, 20, 30, \) and 40. The frequency \( \tilde{\omega} \) and amplitude \( \bar{u}_0 \) of the incident Love wave used in these plots are 2.6 and 1.0, respectively. The other geometric and material parameters of the cracked half-space which are used to plot these curves are: \( \tilde{\eta} = 2.0, \tilde{d} = 0.5001, \epsilon = 0.8, m = 1.75. \) These parameters are dimensionless as it was explained earlier. The total cpu time consumed to compute these results in a SPARC10 machine are: 19 seconds for \( n = 10, \) 44 seconds for \( n = 15, \) 77 seconds for \( n = 20, \) 175 seconds for \( n = 30, \) and 316 seconds for \( n = 40. \) These times are lower for an incident wave with a lower frequency. Figures 2.3 and
2.4 show that the most efficient plot with a good convergence is the one corresponding to \( n = 20 \), as it consumes one fourth of the cpu time taken by the plot of \( n = 40 \) and less than one half of that of \( n = 30 \). Observe from the figures that the absolute value of \( \Phi(\nu) \) tends to increase near \( \nu = \pm 1 \), but it remains bounded in the closed interval \([-1, +1]\).

The formulas of the scattered far-field displacement amplitudes, given in the previous section, are used here to evaluate the moduli and the phases of the first order transmission coefficient \( T_1 \), as well as of the first, second, and third order reflection coefficients \( R_l(l = 1, 2, 3) \) versus the dimensionless incident frequency \( \bar{\omega} \). These results are computed for the material parameters set of \( \epsilon = 0.8 \) and \( m = 1.75 \), and for three different cracks. The positions of these cracks are shown in Figures (2.5), (2.6), and (2.7). The width of the first and second crack is half the layer thickness. The first crack is at a very close distance from the free surface, and the second crack is located in the middle of the layer. The third crack occupies almost the entire thickness of the layer, and is located in the middle of the layer. The dimensionless parameters \( \bar{d} \) and \( \bar{h} \) corresponding to these cracks are presented in the following table.

<table>
<thead>
<tr>
<th></th>
<th>First Crack</th>
<th>Second Crack</th>
<th>Third Crack</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{d} )</td>
<td>0.5001</td>
<td>1.0000</td>
<td>0.5025</td>
</tr>
<tr>
<td>( \bar{h} )</td>
<td>2.0000</td>
<td>2.0000</td>
<td>1.0050</td>
</tr>
</tbody>
</table>

Table 2.1: Values of the geometric parameters \( \bar{d} \) and \( \bar{h} \) for the three layer-embedded cracks.
The values of \( \epsilon \) and \( m \) correspond approximately to those of the uniform, single layered, earth model of Neerhoff [7]. The number of quadrature points in equation (2.66) is \( n = 20 \). The range of the incident Love wave frequency \( \bar{\omega} \) is in the interval \([0, 3]\). With this choice of parameters, the Love wave mode of order \( k \) emerges at \( \bar{\omega} = 5(k - 1) / 6 \). In the frequency-range of interest, the relevant values are \( \bar{\omega} = 0.833, 1.667, \) and 2.5, which correspond to the emergence of the second, third, and fourth modes, respectively.

The moduli and phases plots of the transmission and reflection coefficients are classified in eight groups. Each group consists of three figures for the three cracks. Figures (2.8)-(2.19) show the moduli of the transmission and reflection coefficients versus the dimensionless incident frequency, and Figures (2.20)-(2.31) show their corresponding phases versus \( \bar{\omega} \). These four coefficients are defined by equations (2.112a) and (2.112b). More than 3000 computations were done to plot each of these figures. The range of frequencies \( \bar{\omega} \) was scanned thoroughly by using an increment of 1/1000 of a unit starting at the origin.

In the limit as \( \bar{\omega} \) approaches zero, the coefficient \( R_1 \) vanishes, whereas \( T_1 \) becomes equal to unity. This is a direct consequence of the vanishing of the stress in equation (2.8) when \( \omega = 0 \). Thus, only the incident field remains as the scattered field vanishes. This result is mentioned by Angel [2], and the physical interpretation of this phenomenon is that, for low frequencies (long wavelengths), the crack appears so small to the incident wave that it is ignored altogether.

The curves of the first order transmission coefficient modulus, \(| T_1 |\), and the first and second order reflection coefficients moduli, \(| R_1 |\) and \(| R_2 |\), appear as smooth oscillating functions of the incident frequency for the cracks of half-layer depth, and relatively smooth for the large crack. These curves oscillate two and a half times for the first two cracks, and about one and a half time for the third crack.
The four reflection and transmission coefficient moduli curves for the first crack, which is very close to the free surface, are in good agreement with those corresponding to these coefficients in the work presented by Angel [2]. The results of Angel are computed for the case of a surface-breaking crack with a half-layer depth, \( \varepsilon = 0.8 \), and \( m = 1.75 \). In contrast to the surface-breaking crack, we do not detect, even with a 0.001 frequency increment, any sharp peaks or dips in the range \( 0 \leq \tilde{\omega} \leq 3 \). This may be explained by the fact that the upper tip of the crack never touches the free surface, and there is always some solid material between the tip and the free surface. Therefore, the constructive and destructive interference phenomenon that might cause these sharp dips and peaks in the surface-breaking cracks do not seem to occur for layer-embedded cracks.

Figures (2.8) and (2.10) show that the values of the first order reflection coefficient moduli \( |R_1| \) of the first crack are much higher than those of the third crack for all frequencies, even though their upper tips are at the same close distance from the free surface. Thus, the amount of reflection increases as the length of the crack increases. Figures (2.11) and (2.12) show that the values of the first order transmission coefficient modulus \( |T_1| \) of the second crack, in the range \( \tilde{\omega} > 1 \), are much higher than those of the first crack, even though they have the same length. At the same time, for the same frequency range, Figures (2.8) and (2.9) show that the values of the first order reflection coefficient moduli \( |R_1| \) of these two cracks are much lower for the second crack than for the first one. Therefore, the distance between the upper tip of the crack and the free surface has a great influence on the transmission and reflection fields.

The phase of the first order reflection coefficient \( \varphi(R_1) \) has almost the same characteristics for the three cracks, and its values for the three cracks in two different ranges of the incident dimensionless frequency \( \tilde{\omega} \) are presented in the following table.
Table 2.2: Phase of the coefficient $R_1$ for the three layer-embedded cracks.

<table>
<thead>
<tr>
<th></th>
<th>$\varphi(R_1)$</th>
<th>$\bar{\omega}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>First crack</td>
<td>$[90^\circ, 160^\circ]$</td>
<td>$[0.001, 1.076]$</td>
</tr>
<tr>
<td></td>
<td>$[160^\circ, 175^\circ]$</td>
<td>$[1.077, 3.000]$</td>
</tr>
<tr>
<td>Second Crack</td>
<td>$[90^\circ, 160^\circ]$</td>
<td>$[0.001, 0.710]$</td>
</tr>
<tr>
<td></td>
<td>$[160^\circ, 177^\circ]$</td>
<td>$[0.711, 3.000]$</td>
</tr>
<tr>
<td>Third Crack</td>
<td>$[90^\circ, 160^\circ]$</td>
<td>$[0.001, 0.390]$</td>
</tr>
<tr>
<td></td>
<td>$[160^\circ, 178^\circ]$</td>
<td>$[0.391, 3.000]$</td>
</tr>
</tbody>
</table>

It can be seen from Table (2.2) and Figures (2.20)-(2.22) that the displacement of the first order reflected Love wave is always in opposite phase ($\varphi(R_1) = 180^\circ$) compared with that of the incident Love wave in the range $\bar{\omega} \geq 1.25$. This phenomenon, for high frequency, is apparently independent of the crack width or position. It is also shown that the change of $\varphi(R_1)$ from $90^\circ$ to $160^\circ$ is less rapid when the crack width is small, and the crack lies close to the interface between the two solids.

For the second crack, the phase of the first order transmitted and the second and third order reflected coefficients is almost in phase with the incident Love wave in the range $\bar{\omega} \geq 1.4$. This result and the curves of Figures (2.23) through (2.31) suggest that, for high frequency, these three coefficients are coming more in phase with the incident Love wave as the upper tip of the crack moves away from the free surface. It is observed also from Figure (2.26), for the phase of the coefficient $R_2$, that the values of $\varphi(R_2)$ oscillate sharply between $180^\circ$ and $-180^\circ$. The frequency increment over which the phase shift occurs is less than 0.001.
Figure 2.3: Bounded part of the dislocation density real values for different values of $n$: $\bar{\omega} = 2.6$, $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5001$. 
Figure 2.4: Bounded part of the dislocation density imaginary values for different values of n; $\bar{\omega} = 2.6$, $s'_T / s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5001$. 

\[\text{Im}[\Phi(\nu)]\]

\[\nu\]

\[n = 40\]
\[n = 30\]
\[n = 20\]
\[n = 15\]
\[n = 10\]
Figure 2.5: Position of the first layer-embedded crack.
Figure 2.6: Position of the second layer-embedded crack.
Figure 2.7: Position of the third layer-embedded crack.
Figure 2.8: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s_{T}'/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75, \bar{d} = 0.5001$. 
Figure 2.9: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 1.0$. 
Figure 2.10: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s_T'/s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.11: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8,
\tilde{h} = 2.0, \mu'/\mu = 1.75, \tilde{d} = 0.5001$. 
Figure 2.12: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 1.1$. 
Figure 2.13: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.14: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5001$. 
Figure 2.15: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T/s_T = 0.8$, $\tilde{h} = 2.0$, $\mu'/\mu = 1.75$, $\tilde{d} = 1.0$. 
Figure 2.16: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.17: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, \( \bar{d} = 0.5001 \).
Figure 2.18: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T/s_T = 0.8$, $\tilde{h} = 2.0$, $\mu'/\mu = 1.75$, $\tilde{d} = 1.0$. 
Figure 2.19: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.20: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s_T'/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5001$. 
Figure 2.21: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 1.0$. 
Figure 2.22: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.23: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 2.0$, $\mu' / \mu = 1.75$, $\bar{d} = 0.5001$. 
Figure 2.24: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 1.0$. 
Figure 2.25: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.26: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T / s_T = 0.8$, $h = 2.0$, $\mu' / \mu = 1.75$, $d = 0.5001$. 
Figure 2.27: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s_T' / s_T = 0.8$, $\bar{h} = 2.0$, $\mu' / \mu = 1.75$, $\bar{d} = 1.0$. 
Figure 2.28: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
Figure 2.29: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T/s_T = 0.8$, $\tilde{h} = 2$, $\mu'/\mu = 1.75$, $\tilde{d} = 0.5001$. 
Figure 2.30: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T/s_T = 0.8$, $\tilde{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 1.0$. 
Figure 2.31: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 1.005$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5025$. 
References


Chapter 3

Scattering of Antiplane Surface Waves by an Embedded Interface-Breaking Crack

3.1 Introduction

The scattering of an incident Love wave by a layer-embedded crack normal to the free surface was investigated in the previous chapter. In this chapter, we investigate the most general case which corresponds to a crack breaking through the layer and the half-space. The upper crack tip is assumed to be contained in the layer of a single-layered solid that is perfectly bonded to a half-space made of a different material. The lower tip lies below the interface of the two solids. The position and the width of the crack vary along the vertical axis. The mixed displacement and stress boundary value problem is reduced to a singular integral equation. The displacements for both the transmitted and the reflected far-fields in the layer and the half-space are evaluated.

The mathematical formulation of the problem is discussed briefly in section 3.2. In that section, the total displacement and stress fields are presented as the superposition of the incident fields in an uncracked half-space and the scattered fields in the cracked half-space.

In section 3.3, a general solution for the scattered displacement and the associated stress fields in the cracked layered half-space is obtained by using Fourier sine and cosine transform techniques. A singular integral equation for the density of the scat-
tered displacement discontinuity across the crack faces (*dislocation density function*) is derived.

The singular integral equation is approximated by a linear system of algebraic equations by using the method of Erdogan and Gupta [1]. Both portions of the crack, in the layer and the half-space, are made to be of equal length with respect to the interface between the two solids by using an appropriate change of integration variables. The values of the dislocation density function at equal numbers of points along the two portions of the crack (in the layer and the lower half-space) are obtained in section 3.4 as the solution of that linear system. The dislocation density function is shown to be discontinuous along the interface between the two solids, and the magnitude of the discontinuity is related to the ratio of the shear moduli.

The scattered displacement fields are presented in terms of the dislocation density in section 3.5. The relation between the scattered far-field displacements in the cracked half-space and those of the incident wave is discussed. The solution of the linear system of section 3.4 is used to evaluate the amplitude of the reflected and transmitted displacement fields in the cracked half-space at some distance away from the crack plane. It is shown that these displacements fields are the superposition of a finite number of Love wave modes.

The discussion of the numerical results for the reflection coefficients of the first three modes as well as for the transmission coefficient of the first mode are presented in section 3.6. Both the moduli and phases of these coefficients are plotted for four different cracks. Further, the convergence of the dislocation density function across the crack face for various numbers of points in the linear system of section 3.5 is discussed.

This work is based on the work by Angel [2], who investigated the scattering of an incident Love wave by a surface-breaking crack when the crack is normal to the free
surface and breaks through the lower half-space solid. The mixed-boundary value problem was reduced to a singular integral equation of the first kind for the slope of the crack-face displacement by using Fourier transform techniques, and the equation was solved numerically using the method of Erdogan and Gupta [1]. The formulas of the far-field scattered displacement of the cracked half-space were derived, and the wave motion at some distance from the plane of the crack was shown to be the superposition of a finite number of Love wave modes. Moreover, the discontinuity of the slope of the crack-face displacement across the interface of the two solids was investigated.

In this chapter, we extend the work of Angel in order to investigate the interaction of an incident Love wave with an embedded interface-breaking crack, and we study the effect of this crack on the reflected and transmitted far-field displacements.

3.2 Mathematical Formulation

The mathematical formulation for the scattering of an incident Love wave by an embedded interface-breaking crack normal to the free surface is presented briefly in this section. The reader may refer to section (2.2) for more detail.

The geometry of the problem is illustrated in Figure (3.1). The surface layer is perfectly bonded to a half-space made of a different material. Both materials are linearly elastic, homogeneous, and isotropic. The layered half-space contains an embedded crack normal to the free surface ($x_2 = 0$), which breaks through the two solids. The crack lies in the plane $x_1 = 0$ and extends to infinity in the $\pm x_3$ directions. The surface layer has thickness $h$ and contains the upper tip of the crack at a distance $d_1$ from the free surface. The lower crack tip is located in the lower half-space at a
distance $d$ from the free surface, which is greater than the layer thickness $h$. Thus, the crack has length $d - d_1$.

A time-harmonic antiplane surface wave (Love wave) is incident on the crack. The amplitude, the frequency, and the slowness of the wave are denoted by $u_0$, $\omega$, and $s$, respectively. The displacement generated by the wave is given in the previous chapter by equation (2.2).

Figure 3.1: Incidence of a Love wave in a layered half-space on an interface-breaking crack of length $(d - d_1)$. 
The stress generated by the incident wave in the layered half-space can be expressed by the following formulas

$$
\sigma_{13}^{in}(x_1, x_2) = i u_0 \omega s \left\{ \mu \cos(\omega s_{\tau} \alpha(\eta) x_2) \\
\mu' \cos(\omega s_{\tau} \alpha(\eta) h e^{-\omega s_{\tau}(x_2-h)\alpha'(\eta)}) \right\} e^{iu_0 x_1}, \quad 0 \leq x_2 \leq h, \quad h \leq x_2,
$$

(3.1)

where $\eta$, $\alpha(u)$, and $\alpha'(u)$ are defined as follows

$$
\eta = s / s_{\tau}, \quad \alpha(u) = (1 - u^2)^{1/2}, \quad \alpha'(u) = (u^2 - \epsilon^2)^{1/2}. \quad (3.2)
$$

The parameter $\epsilon$ is the ratio of the slowness $s_{\tau}$ of transverse waves in the layer to that in the half-space $s'_{\tau}$. The value of $\eta$ lies in the interval $[\epsilon, 1]$, and it is a real root of the Love wave frequency equation given by equation (2.4) of the previous chapter.

The number of roots depends on the frequency of the incident Love wave, and it is determined by the condition (2.7).

The total displacement and stress fields are analyzed as the superposition of the incident field in an uncracked half-space and the scattered field in the cracked half-space. Only a quarter-space region is considered here, because the scattered field is physically antisymmetric with respect to the plane of the crack ($x_1 = 0$). The mixed stress and displacement boundary conditions for the scattered field are given by the following equations

$$
\sigma_{23} = 0, \quad x_1 \geq 0, \quad x_2 = 0, \quad (3.3a)
$$

$$
\sigma_{23} = \sigma_{23}', \quad x_1 \geq 0, \quad x_2 = h, \quad (3.3b)
$$

$$
u_3 = \nu_3', \quad x_1 \geq 0, \quad x_2 = h, \quad (3.3c)
$$

$$\sigma_{13} = -\sigma_{13}^{in}, \quad x_1 = 0, \quad d_1 < x_2 \leq h, \quad (3.3d)
$$

$$\sigma_{13}' = -\sigma_{13}^{in}, \quad x_1 = 0, \quad h \leq x_2 < d, \quad (3.3e)
$$

$$u_3 = 0, \quad x_1 = 0, \quad 0 \leq x_2 \leq d_1, \quad (3.3f)
$$

$$u_3' = 0, \quad x_1 = 0, \quad d \leq x_2, \quad (3.3g)$$
where the primed quantities refer to the underlying half-space, and the unprimed ones to the layer. The incident stress $\sigma_{13}^{in}$ is defined by equation (3.1).

In the next section, we obtain a general form of the scattered displacements and their associated stresses for the cracked layered half-space by using the Fourier sine and cosine transforms, and we reduce the mixed-boundary value problem to a singular integral equation.

3.3 Singular Integral Equation

In this section, the formulation of the mixed-boundary value problem is reduced to a singular integral equation by using Fourier transform techniques. For steady-state antiplane motions, the scattered displacement equations of motion are expressed by the following equations

$$u_{3,11}(x_1, x_2) + u_{3,22}(x_1, x_2) + (\omega s) u_3(x_1, x_2) = 0, \quad 0 \leq x_2 \leq h, \quad (3.4a)$$

$$u'_{3,11}(x_1, x_2) + u'_{3,22}(x_1, x_2) + (\omega s') u'_3(x_1, x_2) = 0, \quad h \leq x_2, \quad (3.4b)$$

where $u_{3,ii}$ refers to the second derivative of the displacement with respect to $i$. Using the integral transforms defined by equations (2.10) and (2.11) of the previous chapter, the general solution of equations (3.4a) and (3.4b) for the quarter-space $x_1 > 0$ and $x_2 > 0$ may be written as follows

$$\sqrt{\frac{\pi}{2}} u_3(x_1, x_2) = \int_0^\infty \left[ B e^{-\beta(\zeta)x_2} + C e^{\beta(\zeta)x_2} \right] \sin(\zeta x_1) d\zeta$$

$$+ \int_0^\infty E e^{-\beta(\zeta)x_2} \cos(\zeta x_2) d\zeta, \quad (3.5a)$$

$$\sqrt{\frac{\pi}{2}} u'_3(x_1, x_2) = \int_0^\infty A e^{-\beta'(\zeta)x_2} \sin(\zeta x_1) d\zeta + \int_0^\infty D e^{-\beta'(\zeta)x_2} \cos(\zeta x_2) d\zeta. \quad (3.5b)$$

The associated shear stresses take the following form

$$\sqrt{\frac{\pi}{2}} \sigma_{13}(x_1, x_2) = \mu \int_0^\infty \zeta \left[ B e^{-\beta(\zeta)x_2} + C e^{\beta(\zeta)x_2} \right] \cos(\zeta x_1) d\zeta$$

$$- \mu \int_0^\infty \beta(\zeta) E e^{-\beta(\zeta)x_2} \cos(\zeta x_2) d\zeta, \quad (3.6a)$$

\[
\sqrt{\frac{\pi}{2}} \sigma_{23}(x_1, x_2) = -\mu \int_0^\infty \beta(\zeta) \left[ B e^{-\beta(\zeta)x_2} - C e^{\beta(\zeta)x_2} \right] \sin(\zeta x_1) \, d\zeta \\
- \mu \int_0^\infty \zeta E e^{-\beta(\zeta)x_1} \sin(\zeta x_2) \, d\zeta,
\]
(3.6b)
\[
\sqrt{\frac{\pi}{2}} \sigma'_{13}(x_1, x_2) = \mu' \int_0^\infty \zeta A e^{-\beta'(\zeta)x_2} \cos(\zeta x_1) \, d\zeta \\
- \mu' \int_0^\infty \beta'(\zeta) D e^{-\beta'(\zeta)x_1} \cos(\zeta x_2) \, d\zeta,
\]
(3.6c)
\[
\sqrt{\frac{\pi}{2}} \sigma'_{23}(x_1, x_2) = -\mu' \int_0^\infty \beta'(\zeta) A e^{-\beta'(\zeta)x_2} \sin(\zeta x_1) \, d\zeta \\
- \mu' \int_0^\infty \zeta D e^{-\beta'(\zeta)x_1} \sin(\zeta x_2) \, d\zeta,
\]
(3.6d)
where the primed quantities refer to the underlying cracked half-space, and the unprimed ones to the cracked layer. The quantities \( \beta \) and \( \beta' \) are defined by the following equations
\[
\beta^2(\zeta) = \zeta^2 - (\omega \sigma_\tau)^2, \quad \beta'^2(\zeta) = \zeta^2 - (\omega' \sigma_\tau')^2.
\]
(3.7)

One can observe from the above equation that the quantities \( \beta(\zeta) \) and \( \beta'(\zeta) \) may take real or purely imaginary values for real values of \( \zeta \). The crack face disturbances of the scattered field must propagate away from the crack in order to satisfy the radiation condition (Lamb [3]). This condition implies the following constraints, which are consistent with the choice of the time factor \( \exp(-i\omega t) \), for the imaginary quantities of \( \beta(\zeta) \) and \( \beta'(\zeta) \)
\[
\text{Im} \left( \beta(\zeta) \right) \leq 0, \quad \text{Im} \left( \beta'(\zeta) \right) \leq 0.
\]
(3.8)

The quantities \( A, B, C, D, \) and \( E \) are functions of \( \zeta \), and can be determined from the mixed displacement and stress boundary conditions (3.3). We define a functions \( a(v) \) over the interval \([d_1, h]\) and a function \( b(v) \) over the interval \([h, d]\) such that
\[
E(\zeta) = \frac{1}{\zeta} \sqrt{\frac{2}{\pi}} \left[ \int_{d_1}^h a(v) \sin(\zeta v) \, dv + \int_h^d b(v) \sin(\zeta v) \, dv \right],
\]
(3.9)
\[
D(\zeta) = \frac{1}{\zeta} \sqrt{\frac{2}{\pi}} \int_h^d b(v) \sin(\zeta v) \, dv,
\]
(3.10)
together with,
\[
\int_{d_1}^{d} a(v) \, dv + \int_{h}^{d} b(v) \, dv = 0. \tag{3.11}
\]

Using the fact that the functions \(a(v)\) and \(b(v)\) are independent of each other since they are defined in two different intervals, one can infer that the functions \(E(\zeta)\) and \(D(\zeta)\) are, in general, independent.

The scattered displacement field along the crack \((x_1 = 0)\) can be written, according to (3.5a) and (3.5b), as follows
\[
u_3(0, x_2) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} E(\zeta) \cos(\zeta x_2) \, d\zeta, \quad 0 \leq x_2 \leq h, \tag{3.12}
\]
\[
u_3'(0, x_2) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} D(\zeta) \cos(\zeta x_2) \, d\zeta, \quad h \leq x_2. \tag{3.13}
\]

Substituting (3.9) into (3.12) and (3.10) into (3.13), interchanging the order of integration, and evaluating the infinite sine integrals, one finds that
\[
u_3(0, x_2) = \begin{cases} 
0, & 0 \leq x_2 \leq d_1, \\
\int_{x_2}^{d_1} a(v) \, dv + \int_{d_1}^{d} b(v) \, dv, & d_1 \leq x_2 \leq h, 
\end{cases} \tag{3.14}
\]
\[
u_3'(0, x_2) = \begin{cases} 
\int_{x_2}^{d} b(v) \, dv, & h \leq x_2 \leq d, \\
0, & d \leq x_2. 
\end{cases} \tag{3.15}
\]

Equations (3.14) and (3.15), together with (3.11), show that, for all integrable functions \(a(v)\) and \(b(v)\), the scattered displacement field is continuous in \([0, \infty)\), and the displacement boundary condition (3.3f) and (3.3g) are satisfied.

The boundary condition (3.3b) specifies that the stress component \(\sigma_{23}\) must be continuous along the interface \((x_2 = h)\). Thus, choosing \(x_1 = 0\), one has
\[
\sigma_{23} (0, h^-) = \sigma_{23}'(0, h^+). \tag{3.16}
\]

Differentiating (3.14) and (3.15) with respect to \(x_2\), and assuming that the functions
\(a(v)\) and \(b(v)\) are bounded at \(x_2 = h\), one infers from Hooke’s law and (3.16) the following linear relation

\[
a(h^-) = mb(h^+), \tag{3.17}
\]

where \(m\) is the shear moduli ratio given by (2.1c). The dislocation density functions \(a(v)\) and \(b(v)\) are not equal at the interface \(x_2 = h\), since the shear moduli ratio in general does not equal unity. Thus, the crack face displacement has a discontinuous slope at \(x_2 = h\). The boundary condition (3.3a) which specifies that the stress vanishes along the surface \(x_2 = 0\), together with (3.6b), implies that

\[
B = C. \tag{3.18}
\]

Next, we consider the boundary conditions (3.3b, c), which specify that the displacement and the stress are continuous along the interface of the two solids at \(x_2 = h\). These boundary conditions, together with equation (3.18), yield the following integral equations

\[
\int_0^\infty \left[ Ae^{-\beta(\zeta)h} - B (e^{-\beta(\zeta)h} + e^{\beta(\zeta)h}) \right] \sin(\zeta x_1) \, d\zeta = \int_0^\infty \left[ E e^{-\beta(\zeta)x_1} - D e^{-\beta(\zeta)x_1} \right] \cos(\zeta h) \, d\zeta, \tag{3.19a}
\]

\[
\int_0^\infty \left[ m \beta'(\zeta) A e^{-\beta(\zeta)h} - \beta(\zeta) B e^{-\beta(\zeta)h} + e^{\beta(\zeta)h} \right] \sin(\zeta x_1) \, d\zeta = \int_0^\infty \zeta \left[ E e^{-\beta(\zeta)x_1} - m D e^{-\beta(\zeta)x_1} \right] \sin(\zeta h) \, d\zeta. \tag{3.19b}
\]

Taking the inverse sine-transform of (3.19a) and (3.19b), we obtain the following linear system of equations for the two unknowns \(A\) and \(B\)

\[
e^{-\beta(\delta)h} A - (e^{-\beta(\delta)h} + e^{\beta(\delta)h}) B = \frac{2}{\pi} \int_0^\infty \left[ \int_0^\infty \left[ E e^{-\beta(\zeta)x_1} - D e^{-\beta(\zeta)x_1} \right] \cos(\zeta h) \, d\zeta \right] \sin(\delta x_1)\, dx_1, \tag{3.20a}
\]

\[
m \beta'(\delta) e^{-\beta(\delta)h} A - \beta(\delta) (e^{-\beta(\delta)h} - e^{\beta(\delta)h}) B = \frac{2}{\pi} \int_0^\infty \zeta \left[ E e^{-\beta(\zeta)x_1} - m D e^{-\beta(\zeta)x_1} \right] \sin(\zeta h) \, d\zeta \sin(\delta x_1)\, dx_1. \tag{3.20b}
\]
Substituting (3.9) and (3.10) in the above two equations, and using formulas in Gradshteyn and Ryzhik [4] (equation 3.914, page 517) and Erdelyi [5] (equation 4.3, page 112), one can evaluate the infinite inner integrals in the right-hand side of (3.20a) and (3.20b). Thus, the linear system of (3.20) can be written in terms of two finite integrals of the dislocation density functions such that

\[ e^{-\beta(\xi)h} A = (e^{-\beta(\xi)h} + e^{\beta(\xi)h}) B \]

\[ = \frac{1}{2} \sqrt{\frac{2}{\pi \beta^2(\xi)}} \int_{d_1}^{d} e(v) \left[ 1 - e^{-(h+v)\beta(\xi)} - \text{sgn}(h-v) \left[ 1 - e^{-(h-v)\beta(\xi)} \right] \right] dv \]

\[ - \frac{1}{2} \sqrt{\frac{2}{\pi \beta^2(\xi)}} \int_{d_1}^{d} e(v) H(v-h) \left[ 2 - e^{-(h+v)\beta(\xi)} - e^{-(v-h)\beta(\xi)} \right] dv, \tag{3.21a} \]

\[ m \beta'(\xi) e^{-\beta(\xi)h} A = \beta(\xi) (e^{-\beta(\xi)h} - e^{\beta(\xi)h}) B \]

\[ = \frac{1}{2} \sqrt{\frac{2}{\pi \beta'(\xi)}} \int_{d_1}^{d} e(v) \left[ e^{-(h-v)\beta(\xi)} - e^{-(h+v)\beta(\xi)} \right] dv \]

\[ - \frac{m}{2} \sqrt{\frac{2}{\pi \beta'(\xi)}} \int_{d_1}^{d} e(v) H(v-h) \left[ e^{-(v-h)\beta'(\xi)} - e^{-(v+h)\beta'(\xi)} \right] dv, \tag{3.21b} \]

where \( H \) and \( \text{sgn} \) denote the Heaviside step function and the sign function, respectively. The dislocation density function \( e(v) \) is defined such that

\[ e(v) = \begin{cases} a(v), & d_1 \leq v \leq h, \\ b(v), & h \leq v \leq d. \end{cases} \tag{3.22} \]

Solving the linear system of (3.21a, b) yields the following integral expressions for the quantities \( A(\xi) \) and \( B(\xi) \) in terms of the dislocation density function \( e(v) \)

\[ A(\xi) = \frac{1}{2} \sqrt{\frac{2}{\pi \Delta_0(\xi)}} \int_{d_1}^{d} e(v) A_1(\xi, v) dv, \tag{3.23a} \]

\[ B(\xi) = \frac{1}{2} \sqrt{\frac{2}{\pi \Delta_0(\xi)}} \int_{d_1}^{d} e(v) B_1(\xi, v) dv, \tag{3.23b} \]

where the functions \( \Delta_0(\xi), A_1(\xi, v), \) and \( B_1(\xi, v) \) are given by

\[ \Delta_0(\xi) = -\beta(\xi) \left( e^{-2\beta(\xi)h} - 1 \right) + m \beta'(\xi) (e^{-2\beta(\xi)h} + 1), \tag{3.24a} \]
\begin{align*}
A_1(\zeta, v) &= \frac{-\zeta}{\beta(\zeta)} (e^{-2\beta(\zeta)h} - 1) \left[ 1 - e^{-\beta(\zeta)(h+v)} - \text{sgn}(h - v) \left[ 1 - e^{-\beta(\zeta)(h-v)} \right] \right] \\
& \quad - m \frac{\zeta}{\beta'(\zeta)} (e^{-2\beta(\zeta)h} + 1) H(v - h) \left[ e^{-\beta'(\zeta)(h-v)} - e^{-\beta'(\zeta)(h+v)} \right] \\
& \quad + \frac{\zeta}{\beta'(\zeta)} (e^{-2\beta(\zeta)h} - 1) H(v - h) \left[ 2 - e^{-\beta'(\zeta)(v+h)} - e^{-\beta'(\zeta)(v-h)} \right] \\
& \quad + \frac{\zeta}{\beta'(\zeta)} (e^{-2\beta(\zeta)h} + 1) \left[ e^{-\beta(\zeta)(h-v)} - e^{-\beta(\zeta)(h+v)} \right], \quad (3.24b) \\
B_1(\zeta, v) &= -m \frac{\zeta \beta'(\zeta)}{\beta(\zeta)} \left[ 1 - e^{-\beta(\zeta)(h+v)} - \text{sgn}(h - v) \left[ 1 - e^{-\beta(\zeta)(h-v)} \right] \right] \\
& \quad + m \frac{2\zeta}{\beta'(\zeta)} H(v - h) \left[ 1 - e^{-\beta'(\zeta)(h-v)} \right] \\
& \quad + \frac{\zeta}{\beta'(\zeta)} \left[ e^{-\beta(\zeta)(h-v)} - e^{-\beta(\zeta)(h+v)} \right]. \quad (3.24c)
\end{align*}

Next, using (3.6a), (3.6c), and the expressions of \( A, B, D, \) and \( E \) in terms of the dislocation density \( e(v) \), the scattered stress along the crack face \( x_1 = 0 \) can be written in the following form

\begin{align*}
\pi \sigma_{13}(0, x_2) &= \mu \int_{d_1}^{d} e(v) K'(v, x_2) \, dv \\
& \quad - 2\mu \int_{0}^{\infty} \frac{\beta(\zeta)}{\zeta} \left[ \int_{d_1}^{d} e(v) \sin(\zeta v) \, dv \right] \cos(\zeta x_2) \, d\zeta, \quad (3.25a) \\
\pi \sigma'_{13}(0, x_2) &= \mu' \int_{d_1}^{d} e(v) K(v, x_2) \, dv \\
& \quad - 2\mu' \int_{0}^{\infty} \frac{\beta'(\zeta)}{\zeta} \left[ \int_{d_1}^{d} H(v - h) e(v) \sin(\zeta v) \, dv \right] \cos(\zeta x_2) \, d\zeta, \quad (3.25b)
\end{align*}

where \( H \) denotes the heaviside step function, and the functions \( K(v, x_2) \) and \( K'(v, x_2) \) take the following form

\begin{align*}
K(v, x_2) &= \int_{0}^{\infty} \zeta^2 \frac{e^{-\beta(\zeta)(x_2 - h)}}{\Delta_0(\zeta)} A_1(\zeta, v) \, d\zeta, \quad (3.26a) \\
K'(v, x_2) &= \int_{0}^{\infty} \zeta^2 \frac{e^{-\beta(\zeta)(h - x_2)} + e^{-\beta(\zeta)(x_2 + h)}}{\Delta_0(\zeta)} B_1(\zeta, v) \, d\zeta, \quad (3.26b)
\end{align*}

and the function \( \Delta_0(\zeta) \) is given by (3.24a).
Notice that the functions \((\beta(\zeta)/\zeta)\) and \((\beta'(\zeta)/\zeta)\) in the second integrals of the right-hand sides of equations (3.25a, b) behave as \((1 + 0(\zeta^{-2}))\) as \(\zeta \to \infty\). Thus, the integrals of these functions over the interval \([0, \infty)\) are not bounded. Therefore, the order of the second integrals in the right-hand sides of equations (3.25a, b) can only be interchanged with some care. We first write these integrals in terms of the dislocation density functions as follows

\[
I(x_2) = -2\mu \left[ \int_{d_1}^{d} e(v) H_1(v, x_2) dv + \int_{0}^{\infty} \left[ \int_{d_1}^{d} e(v) \sin(\zeta v) dv \right] \cos(\zeta x_2) d\zeta \right], \tag{3.27a}
\]

\[
I'(x_2) = -2\mu' \left[ \int_{d_1}^{d} e(v) H_2(v, x_2) dv + \int_{0}^{\infty} \left[ \int_{d_1}^{d} H(v - h) e(v) \sin(\zeta v) dv \right] \cos(\zeta x_2) d\zeta \right], \tag{3.27b}
\]

where the functions \(H_1(v, x_2)\) and \(H_2(v, x_2)\) are given by

\[
H_1(v, x_2) = \int_{0}^{\infty} \left[ \frac{\beta(\zeta)}{\zeta} - 1 \right] \sin(\zeta v) \cos(\zeta x_2) d\zeta, \tag{3.27c}
\]

\[
H_2(v, x_2) = H(v - h) \int_{0}^{\infty} \left[ \frac{\beta'(\zeta)}{\zeta} - 1 \right] \sin(\zeta v) \cos(\zeta x_2) d\zeta. \tag{3.27d}
\]

The second integrals in the right-hand sides of equations (3.27a, b) are similar to the one given by equation (2.31) of the previous chapter. Using the technique given by equations (2.30)-(2.35), together with the result of (2.36) of the previous chapter, these two integrals can be written in the form

\[
J(x_2) = \frac{1}{2} \left[ \int_{d_1}^{d} e(v) \frac{1}{v + x_2} dv + \int_{d_1}^{d} e(v) \frac{1}{v - x_2} dv \right], \tag{3.28a}
\]

\[
J'(x_2) = \frac{1}{2} \left[ \int_{d_1}^{d} e(v) \frac{H(v - h)}{v + x_2} dv + \int_{d_1}^{d} e(v) \frac{H(v - h)}{v - x_2} dv \right], \tag{3.28b}
\]

where the second integrals in the right-hand sides of the above two equations are meant to be principal-value integrals if \(d_1 < x_2 < d\) and Riemann integrals if \(0 < x_2 < d_1\) and \(d < x_2 < \infty\). Substituting equations (3.28a, b) in (3.27a, b), one can write the functions \(I(x_2)\) and \(I'(x_2)\) in the form
\[ I(x_2) = -\mu \int_{d_1}^{d} e(v) \left[ \frac{1}{v + x_2} + \frac{1}{v - x_2} + 2H_1(v, x_2) \right] dv, \quad (3.28c) \]

\[ I'(x_2) = -\mu' \int_{d_1}^{d} e(v) \left[ \frac{H(v - h)}{v + x_2} + \frac{H(v - h)}{v - x_2} + 2H_2(v, x_2) \right] dv. \quad (3.28d) \]

Using (3.25a, b), (3.27c, d), (3.28c, d), and (3.1), the two boundary conditions (3.3d, e), which specify that the incident and scattered stress of the cracked half-space are equal and have opposite signs along the crack faces, yield the following singular integral equations for the dislocation density function

\[ \int_{d_1}^{d} e(v) \left[ \frac{1}{v + x_2} + \frac{1}{v - x_2} + 2H_1(v, x_2) - K'(v, x_2) \right] dv = i\pi u_0 \omega \cos \left[ \omega \sigma \alpha(\eta)x_2 \right], \quad d_1 < x_2 < h, \quad (3.29a) \]

\[ \int_{d_1}^{d} e(v) \left[ \frac{H(v - h)}{v + x_2} + \frac{H(v - h)}{v - x_2} + 2H_2(v, x_2) - K(v, x_2) \right] dv = i\pi u_0 \omega \cos \left[ \omega \sigma \alpha(\eta)h \right] e^{-\omega \sigma \alpha(\eta)(x_2 - h)\alpha'(\eta)}, \quad h < x_2 < d, \quad (3.29b) \]

together with

\[ \int_{d_1}^{d} e(v) dv = 0, \quad (3.29c) \]

where \( H \) denotes the Heaviside step function. The functions \( K(v, x_2), K'(v, x_2) \), \( H_1(v, x_2), \) and \( H_2(v, x_2) \) are defined by equations (3.26a, b) and (3.27c, d), respectively. The integrals in the left-hand side of the system of equations (3.29a, b) are defined only in the Cauchy principal-value sense.

Next, dimensionless quantities are defined as follows

\[ \bar{h} = \frac{h}{d - d_1}, \quad \bar{d} = \frac{d + d_1}{2(d - d_1)}, \quad \bar{u}_0 = \frac{u_0}{d - d_1}, \quad \bar{h} = 2(h - \bar{d}), \]

\[ v = (d - d_1)\bar{\nu}, \quad x_2 = (d - d_1)\bar{\tau}, \quad \zeta = \omega \sigma \alpha u, \quad (3.29d) \]

\[ \bar{\omega} = \frac{h}{\lambda_T} = \frac{\lambda_T}{2\pi} = \frac{h \lambda_T}{2\pi}, \quad \lambda_T = \frac{2\pi}{\omega \sigma \alpha}. \]

Using the dimensionless notations given by equation (3.29d), the dimensionless form of the system of equations (3.29a) and (3.29b) and of the condition (3.29c) are
\[
\int_{-1}^{1} \tilde{e}(\nu) \left[ \frac{1}{\nu - \tau} + \frac{1}{\nu + \tau + 4d} + \frac{1}{2} \frac{\bar{H}_1(\nu, \tau)}{\nu} - \frac{1}{2} \frac{\bar{K}'(\nu, \tau)}{\nu} \right] \, d\nu
= i\tilde{u}_0 \tilde{g}_1(\tau), \quad -1 < \tau < \tilde{h}, \quad (3.30a)
\]

\[
\int_{-1}^{1} \tilde{e}(\nu) \left[ \frac{H(\nu - \tilde{h})}{\nu - \tau} + \frac{H(\nu + \tilde{h})}{\nu + \tau + 4\tilde{d}} + \frac{1}{2} \frac{\bar{H}_2(\nu, \tau)}{\nu} - \frac{1}{2} \frac{\bar{K}(\nu, \tau)}{\nu} \right] \, d\nu
= i\tilde{u}_0 \tilde{g}_2(\tau), \quad \tilde{h} < \tau < 1, \quad (3.30b)
\]

together with
\[
\int_{-1}^{1} \tilde{e}(\nu) \, d\nu = 0. \quad (3.30c)
\]

The singular integrals in equation (3.30a, b) are defined only in the Cauchy principal-
value sense. The dimensionless functions \( \tilde{g}_1(\tau), \tilde{g}_2(\tau), \bar{H}_1(\nu, \tau), \bar{H}_2(\nu, \tau), \bar{K}'(\nu, \tau), \)
and \( \bar{K}(\nu, \tau) \) are defined by the following equations

\[
\tilde{g}_1(\tau) = \pi \eta \tilde{s}_\tau \cos (\tilde{s}_\tau \alpha(\eta) \tilde{\tau}), \quad (3.31a)
\]

\[
\tilde{g}_2(\tau) = \pi \eta \tilde{s}_\tau \cos (\tilde{s}_\tau \alpha(\eta) \tilde{h}) e^{-\tilde{s}_\tau (\tilde{\tau} - \tilde{h}) \alpha'(u)}, \quad (3.31b)
\]

\[
\bar{H}_1(\nu, \tau) = \tilde{s}_\tau \int_0^\infty \left[ \frac{\bar{\beta}(u)}{u} - 1 \right] \sin (\tilde{s}_\tau \tilde{\nu} u) \cos (\tilde{s}_\tau \tilde{\tau} u) \, du, \quad (3.31c)
\]

\[
\bar{H}_2(\nu, \tau) = \tilde{s}_\tau \bar{H}(\nu - \tilde{h}) \int_0^\infty \left[ \frac{\bar{\beta}(u)}{u} - 1 \right] \sin (\tilde{s}_\tau \tilde{\nu} u) \cos (\tilde{s}_\tau \tilde{\tau} u) \, du, \quad (3.31d)
\]

\[
\bar{K}'(\nu, \tau) = \tilde{s}_\tau \int_0^\infty u \frac{P_1(\tilde{\tau}, u) + P_2(\tilde{\tau}, u)}{\Delta(u)} \bar{B}_1(u, \nu) \, du, \quad -1 < \tau < \tilde{h}, \quad (3.31e)
\]

\[
\bar{K}(\nu, \tau) = \tilde{s}_\tau \int_0^\infty u \frac{P_4(\tilde{\tau}, u)}{\Delta(u)} \bar{A}_1(u, \nu) \, du, \quad \tilde{h} < \tau < 1. \quad (3.31f)
\]

In equations (3.31), \( \alpha(u) \) and \( \alpha'(u) \) are given by (3.2), and dimensionless parameters
and functions are defined such that
\[
\tilde{\tau} = \frac{\tau}{2} + \tilde{d}, \quad \tilde{\nu} = \frac{\nu}{2} + \tilde{d},
\]
\[
\bar{\beta}(u) = \left( u^2 - 1 \right)^{1/2}, \quad \bar{\beta}'(u) = \left( u^2 - \epsilon^2 \right)^{1/2},
\]
\[
\tilde{e}(\nu) = e((d - d_1)\tilde{\nu}) = e(\nu), \quad \tilde{s}_\tau = \omega s_\tau (d - d_1).
\]
The functions $\overline{A}_1(u, \nu)$, $\overline{B}_1(u, \nu)$, and $\overline{\Delta}(u)$ are the dimensionless form of equations (3.24a, b, c), and they are given by

$$
\overline{A}_1(u, \nu) = \frac{u}{\overline{\beta}(u)} \left[ P_1(\overline{h}, u) + 1 \right] \left[ P_1(\overline{\nu}, u) + P_2(\overline{\nu}, u) \right]
- \frac{u}{\overline{\beta}(u)} \left( P_1(\overline{h}, u) - 1 \right) \left[ 1 - P_1(\overline{\nu}, u) - \text{sgn}(\overline{h} - \overline{\nu}) [1 - P_2(\overline{\nu}, u)] \right]
+ \frac{u \overline{\beta}(u)}{\overline{\beta}^2(u)} \left[ P_1(\overline{h}, u) - 1 \right] \text{H}(\overline{\nu} - \overline{h}) [2 - P_3(\overline{\nu}, u) - P_3(\overline{\nu}, u)]
- m \frac{u}{\overline{\beta}(u)} \left( P_1(\overline{h}, u) + 1 \right) \text{H}(\overline{\nu} - \overline{h}) [P_4(\overline{\nu}, u) - P_3(\overline{\nu}, u)],
$$

(3.33a)

$$
\overline{B}_1(u, \nu) = 2m \frac{u}{\overline{\beta}(u)} \text{H}(\overline{\nu} - \overline{h}) [1 - P_4(\overline{\nu}, u)] + \frac{u}{\overline{\beta}(u)} [P_2(\overline{\nu}, u) - P_1(\overline{\nu}, u)]
- m \frac{u \overline{\beta}(u)}{\overline{\beta}^2(u)} \left[ 1 - P_1(\overline{\nu}, u) - \text{sgn}(\overline{h} - \overline{\nu}) [1 - P_2(\overline{\nu}, u)] \right],
$$

(3.33b)

$$
\overline{\Delta}(u) = -\overline{\beta}(u) \left[ P_1(\overline{h}, u) - 1 \right] + m \overline{\beta}(u) \left[ P_1(\overline{h}, u) + 1 \right],
$$

(3.33c)

where the functions $P_1, P_2, P_3$, and $P_4$ are defined as follows

$$
P_1(\nu, u) = \exp \left[ -\bar{s}_r \left( \overline{h} + \nu \right) \overline{\beta}(u) \right],
$$

(3.34a)

$$
P_2(\nu, u) = \exp \left[ -\bar{s}_r \left| \overline{h} - \nu \right| \overline{\beta}(u) \right],
$$

(3.34b)

$$
P_3(\nu, u) = \exp \left[ -\bar{s}_r \left( \overline{h} + \nu \right) \overline{\beta}(u) \right],
$$

(3.34c)

$$
P_4(\nu, u) = \exp \left[ -\bar{s}_r \left| \overline{h} - \nu \right| \overline{\beta}(u) \right].
$$

(3.34d)

In the interval $[\epsilon, 1]$, one has (using definition (3.2))

$$
\overline{\beta}(u) = -i(1 - u^2)^{1/2} = -i\alpha(u), \quad \overline{\beta}'(u) = \alpha'(u).
$$

(3.34e)

Using equations (3.33c) and (3.34e), the function $\overline{\Delta}_h(u)$ is defined by

$$
\overline{\Delta}_h(u) = m\alpha'(u) \cos(\bar{s}_r \bar{h} \alpha(u)) - \alpha(u) \sin(\bar{s}_r \bar{h} \alpha(u)), \quad \epsilon \leq u \leq 1.
$$

(3.34f)
Notice that the integrals defining the functions $\overline{K}$ and $\overline{K}'$ over the interval $[0, \infty)$ in equations (3.31e, f) are not regular integrals, since the integrands have simple poles in the interval $[\epsilon, 1]$. Thus, using the fact that the equation $\overline{\Delta}_b(u) = 0$ is identical to equation (2.4) of the previous chapter and has the same number $k$ of roots, the integrals over the interval $[\epsilon, 1]$ can be interpreted as follows (Lamb [3])

\[
\overline{K}_b'(\nu, \tau) = \tilde{s}_T \oint_{\epsilon}^1 u \frac{\cos(\tilde{s}_T \tilde{\tau} \alpha(u))}{\overline{\Delta}_b(u)} B_{1b}(u, \nu) du \\
+ i\pi \tilde{s}_T \sum_{i=1}^k \eta_i \frac{\cos(\tilde{s}_T \tilde{\tau} \alpha(\eta_i))}{\overline{\Delta}_b(\eta_i)} \overline{B}_{1b}(\eta_i, \nu), \quad -1 < \tau < \hat{\tau}, \tag{3.35a}
\]

\[
\overline{K}_b(\nu, \tau) = \tilde{s}_T \oint_{\epsilon}^1 u \frac{P_4(\tilde{\tau}, u)}{\overline{\Delta}_b(u)} \overline{A}_{1b}(u, \nu) du \\
+ i\pi \tilde{s}_T \sum_{i=1}^k \eta_i \frac{P_4(\tilde{\tau}, \eta_i)}{\overline{\Delta}_b(\eta_i)} \overline{A}_{1b}(\eta_i, \nu), \quad \hat{\tau} < \tau < 1, \tag{3.35b}
\]

where the function $\overline{\Delta}_b(u)$ is given by (3.34f), and the functions $\overline{\Delta}_b(\eta_i)$, $\overline{B}_{1b}(u, \nu)$, and $\overline{A}_{1b}(u, \nu)$ are given in section A.1 of Appendix A. The circle through the integral sign in the right-hand side of equations (3.35a, b) indicates that the integral through each pole $\eta_i$ is performed in the principal-value sense. These integrals are evaluated numerically in the form

\[
I_1 = \tilde{s}_T \oint_{\epsilon}^1 \left[ u \frac{\cos(\tilde{s}_T \tilde{\tau} \alpha(u))}{\overline{\Delta}_b(u)} B_{1b}(u, \nu) - \sum_{i=1}^k \eta_i \frac{\sqrt{1 - \eta_i} \cos(\tilde{s}_T \tilde{\tau} \alpha(\eta_i))}{\sqrt{1 - u (u - \eta_i)} \overline{\Delta}_b(\eta_i)} B_{1b}(\eta_i, \nu) \right] du \\
+ \tilde{s}_T \sum_{i=1}^k \eta_i \frac{\sqrt{1 - \eta_i} \cos(\tilde{s}_T \tilde{\tau} \alpha(\eta_i))}{\overline{\Delta}_b(\eta_i)} \overline{B}_{1b}(\eta_i, \nu) J_{1l}, \tag{3.36a}
\]

\[
I_2 = \tilde{s}_T \oint_{\epsilon}^1 \left[ u \frac{P_4(\tilde{\tau}, u)}{\overline{\Delta}_b(u)} \overline{A}_{1b}(u, \nu) - \sum_{i=1}^k \eta_i \frac{\sqrt{\eta_i - \epsilon} P_4(\tilde{\tau}, \eta_i)}{\sqrt{u - \epsilon (u - \eta_i)} \overline{\Delta}_b(\eta_i)} \overline{A}_{1b}(\eta_i, \nu) \right] du \\
+ \tilde{s}_T \sum_{i=1}^k \eta_i \frac{\sqrt{\eta_i - \epsilon} P_4(\tilde{\tau}, \eta_i)}{\overline{\Delta}_b(\eta_i)} \overline{A}_{1b}(\eta_i, \nu) J_{2l}, \tag{3.36b}
\]

where the integrals $J_{1l}$ and $J_{2l}$ are defined by

\[
J_{1l} = \oint_{\epsilon}^1 \frac{du}{\sqrt{1 - u(u - \eta_i)}}, \quad J_{2l} = \oint_{\epsilon}^1 \frac{du}{\sqrt{u - \epsilon (u - \eta_i)}}. \tag{3.37}
\]
The above integrals \( J_{1l} \) and \( J_{2l} \) are evaluated through the pole \( \eta_l (l = 1, \ldots, k) \) in the principal-value sense. Changing variables, and evaluating the integrals in equations (3.37) through each pole, one finds that

\[
J_{1l} = -\frac{1}{\sqrt{1 - \eta_l}} \log \left( \frac{\eta_l - \epsilon}{\sqrt{1 - \epsilon} + \sqrt{1 - \eta_l}} \right)^2, \tag{3.38a}
\]

\[
J_{2l} = \frac{1}{\sqrt{\eta_l - \epsilon}} \log \left( \frac{1 - \eta_l}{\sqrt{1 - \epsilon} + \sqrt{\eta_l - \epsilon}} \right)^2. \tag{3.38b}
\]

Using the above result of \( J_{1l} \) and \( J_{2l} \), substituting equations (3.36a, b) into equations (3.35a, b), and changing integration variables, one can write the integrals \( \overline{K}'_{b}(\nu, \tau) \) and \( \overline{K}_b(\nu, \tau) \) in the following form

\[
\overline{K}'_{b}(\nu, \tau) = 2\bar{s}_r \int_0^\tau \left[ \frac{v}{\Delta_b(\bar{v})} \cos(\Theta(\bar{v})) \left( v \overline{B}_{1b}(\bar{v}, \nu) \right) \right.\left. - \sum_{i=1}^k \frac{\sqrt{1 - \eta_i} \cos(\Theta(\eta_i))}{(\bar{v} - \eta_i) \Delta_b(\eta_i)} \overline{B}_{1b}(\eta_i, \nu) \right] dv
+ \bar{s}_r \sum_{i=1}^k \frac{\cos(\Theta(\eta_i))}{\Delta_b(\eta_i)} \left[ i\pi - \log \frac{\eta_i - \epsilon}{\sqrt{1 - \epsilon} + \sqrt{1 - \eta_i}} \right] \overline{B}_{1b}(\eta_i, \nu), \tag{3.39a}
\]

\[
\overline{K}_b(\nu, \tau) = 2\bar{s}_r \int_0^\tau \left[ \frac{v}{\Delta_b(\bar{v})} \cos(\Theta(\bar{v})) \left( v \overline{A}_{1b}(\bar{v}, \nu) \right) \right.\left. - \sum_{i=1}^k \frac{\sqrt{1 - \eta_i} \cos(\Theta(\eta_i))}{(\bar{v} - \eta_i) \Delta_b(\eta_i)} \overline{A}_{1b}(\eta_i, \nu) \right] dv
+ \bar{s}_r \sum_{i=1}^k \frac{\cos(\Theta(\eta_i))}{\Delta_b(\eta_i)} \left[ i\pi + \log \frac{1 - \eta_i}{\sqrt{1 - \epsilon} + \sqrt{\eta_i - \epsilon}} \right] \overline{A}_{1b}(\eta_i, \nu), \tag{3.39b}
\]

where the parameter \( \bar{\epsilon} \), the variables \( \bar{v} \) and \( \hat{v} \), and the function \( \Theta(u) \) are

\[
\bar{\epsilon} = \sqrt{1 - \epsilon}, \quad \bar{v} = 1 - v^2, \quad \hat{v} = v^2 + \epsilon, \quad \Theta(u) = \bar{s}_r \bar{\tau} \alpha(u). \tag{3.40}
\]

The functions \( \Delta_b(\eta_i), \overline{B}_{1b}(u, \nu), \) and \( \overline{A}_{1b}(u, \nu) \) are given in (A.1) of Appendix A. The integrands in the right-hand sides of equations (3.39a, b) are bounded in the interval of integration \([0, \bar{\epsilon}]\), and the integrals \( \overline{K}'_{b}(\nu, \tau) \) and \( \overline{K}_b(\nu, \tau) \) have complex values. The other portions \( \overline{K}'_{c}(\nu, \tau), \overline{K}_c(\nu, \tau), \overline{K}_a(\nu, \tau), \) and \( \overline{K}_c(\nu, \tau) \) of the integrals in equations (3.31c, f), which are evaluated over the intervals \([0, \epsilon]\) and \([1, \infty)\) respectively, are given in (A.2) of Appendix A.
The integrals in equations (3.31c, d) which define the functions $\overline{H}_1(\nu, \tau)$ and $\overline{H}_2(\nu, \tau)$ are split over two intervals. The first integral of the function $\overline{H}_1(\nu, \tau)$ is evaluated over the interval $[0, 1]$ and has a complex value, while the second one is evaluated over the interval $[1, \infty)$ and has a real value. On the other hand, the first integral of the function $\overline{H}_2(\nu, \tau)$ is evaluated over the interval $[0, \epsilon]$ and has a complex value, while the other one is evaluated over the interval $[\epsilon, \infty)$ and has a real value. In the two intervals $[0, 1]$ and $[0, \epsilon]$ the following functions are defined by (using definition (3.2))

\[
\overline{\beta}(u) = -i \left(1 - u^2\right)^{1/2} = -i\alpha(u), \quad 0 \leq u \leq 1, \tag{3.41a}
\]

\[
\overline{\beta}'(u) = -i \left(\epsilon^2 - u^2\right)^{1/2} = -i\alpha'(u), \quad 0 \leq u \leq \epsilon. \tag{3.41b}
\]

Using the above relations, the functions $\overline{H}_1(\nu, \tau)$ and $\overline{H}_2(\nu, \tau)$ can be written as

\[
\overline{H}_1(\nu, \tau) = -\bar{s}_\tau \left[ \int_0^1 \left(1 + i \frac{\alpha(u)}{u}\right) \sin(\bar{s}_\tau u \bar{\nu}) \cos(\bar{s}_\tau u \bar{\tau}) du 
+ \int_1^\infty \frac{1}{\Delta_h(u)} \sin(\bar{s}_\tau u \bar{\nu}) \cos(\bar{s}_\tau u \bar{\tau}) du \right], \tag{3.42a}
\]

\[
\overline{H}_2(\nu, \tau) = -\bar{s}_\tau H(\bar{\nu} - \bar{h}) \left[ \int_0^{\epsilon} \left(1 + i \frac{\alpha'(u)}{u}\right) \sin(\bar{s}_\tau u \bar{\nu}) \cos(\bar{s}_\tau u \bar{\tau}) du 
+ \int_{\epsilon}^\infty \frac{\epsilon^2}{\Delta_h'(u)} \sin(\bar{s}_\tau u \bar{\nu}) \cos(\bar{s}_\tau u \bar{\tau}) du \right], \tag{3.42b}
\]

where $i$ is the imaginary unit, and the functions $\overline{\Delta}_h(u)$ and $\overline{\Delta}'_h(u)$ have the following form

\[
\overline{\Delta}_h(u) = u \left[u + \overline{\beta}(u)\right], \quad \overline{\Delta}'_h(u) = u \left[u + \overline{\beta}'(u)\right]. \tag{3.43}
\]

Now, all the functions that are used in the singular integrals of the system of equations (3.30a, b) are well defined by equations (3.31) through (3.43) and Appendix A. In the next section, the method of Erdogan and Gupta [1] is implemented to evaluate the dislocation density $c(\nu)$ across the crack faces.
3.4 Method of Solution

In this section, the singular integral (3.30a, b) is solved numerically by using the method of Erdogan and Gupta [1]. We first split each of these integrals at the point of the interface. Thus, the first integral is evaluated over the interval $[-1, \tilde{h}]$ and the second one over the interval $[\tilde{h}, 1]$. Then, the variables of these integrals as well as the dimensionless variable $y$ are changed according to

$$
\nu = \tilde{h} - (1 + \tilde{h})\zeta, \quad y = \tilde{h} - (1 + \tilde{h})z, \quad (-1 < \nu < \tilde{h}, \quad -1 < y < \tilde{h}), \quad (3.44a)
$$

$$
\nu = \tilde{h} + (1 - \tilde{h})\zeta, \quad y = \tilde{h} + (1 - \tilde{h})z, \quad (\tilde{h} < \nu < 1, \quad \tilde{h} < y < 1). \quad (3.44b)
$$

The intervals of these integrals in $\zeta$ are $[0, 1]$ and $[0, 1]$, respectively. Thus, using (3.44a, b), equations (3.30a, b, c) can be written in the form

$$
\int_0^1 \tilde{e}(\zeta_1) \left[ \frac{1}{z - \zeta} + \frac{1}{4\tilde{h}_1 - z - \zeta} + (1 + \tilde{h}) \left[ \bar{H}_1(\zeta_1, z_1) - \frac{1}{2} \bar{K}'(\zeta_1, z_1) \right] \right] d\zeta
$$

$$
+ \int_0^1 \tilde{e}(\zeta_2) \left[ \frac{1}{\zeta + \varsigma_1} + \frac{1}{4\tilde{h}_2 + \zeta - \varsigma_1} + (1 - \tilde{h}) \left[ \bar{H}_1(\zeta_2, z_1) - \frac{1}{2} \bar{K}'(\zeta_2, z_1) \right] \right] d\zeta
$$

$$
= i\tilde{u}_0\pi\tilde{s}_T\eta \cos \left[ s_T\alpha(\eta) \left( \frac{z_1}{2} + d \right) \right], \quad 0 < z < 1, \quad (3.45a)
$$

$$
\int_0^1 \tilde{e}(\zeta_2) \left[ \frac{1}{\zeta - z} + \frac{1}{4\tilde{h}_2 + z + \zeta} + (1 - \tilde{h}) \left[ \bar{H}_2(\zeta_2, z_2) - \frac{1}{2} \bar{K}(\zeta_2, z_2) \right] \right] d\zeta
$$

$$
- \frac{1}{2} \left( 1 + \tilde{h} \right) \int_0^1 \tilde{e}(\zeta_1) \left[ \bar{K}(\zeta_1, z_2) \right] d\zeta
$$

$$
= i\tilde{u}_0\pi\tilde{s}_T\eta \cos \left[ \tilde{s}_T\alpha(\eta)\tilde{h} \right] e^{-\tilde{s}_T((z_2/2)+d-k)}\alpha''(n), \quad 0 < z < 1, \quad (3.45b)
$$

together with

$$
c_1 \int_0^1 \tilde{e}(\zeta_1) \ d\zeta + \int_0^1 \tilde{e}(\zeta_2) \ d\zeta = 0. \quad (3.45c)
$$

The variables used in the above equations are defined as follows
\( \bar{h}_1 = \frac{\bar{h}}{1 + \bar{h}} \), \( \bar{h}_2 = \frac{\bar{h}}{1 - \bar{h}} \), \( c_1 = \frac{1 + \bar{h}}{1 - \bar{h}} \).

\( \zeta_1 = -(1 + \bar{h}) \zeta + \bar{h} \), \( \zeta_2 = (1 - \bar{h}) \zeta + \bar{h} \). \hspace{1cm} (3.46)

\( z_1 = -(1 + \bar{h}) z + \bar{h} \), \( z_2 = (1 - \bar{h}) z + \bar{h} \).

Recall that the dislocation density function \( \bar{e}(\nu) \) has a square-root singularity at the crack tips (\( \nu = \pm 1 \)). Thus, one can define the functions \( \Psi(\zeta) \) and \( \Phi(\zeta) \) by

\[
\bar{e} \left( \bar{h} - (1 + \bar{h}) \zeta \right) = \frac{\Psi(\zeta)}{(1 - \zeta^2)^{\frac{1}{2}}}, \quad 0 < \zeta < 1, \hspace{1cm} (3.47a)
\]

\[
\bar{e} \left( \bar{h} + (1 - \bar{h}) \zeta \right) = \frac{\Phi(\zeta)}{(1 - \zeta^2)^{\frac{1}{2}}}, \quad 0 < \zeta < 1. \hspace{1cm} (3.47b)
\]

The functions \( \Psi(\zeta) \) and \( \Phi(\zeta) \) are bounded in the interval \([0, 1] \). The discontinuity condition of equation (3.17) together with the definition of the dislocation density function \( \bar{e}(\nu) \) imply that

\[
\bar{e} \left( \bar{h}^- \right) = m \bar{e} \left( \bar{h}^+ \right). \hspace{1cm} (3.48)
\]

Using the above relation and the definitions (3.47) of the functions \( \Psi(\zeta) \) and \( \Phi(\zeta) \), one has the following condition

\[
\Psi(0) - m \Phi(0) = 0. \hspace{1cm} (3.49)
\]

Angel [2], [equation (3.13)], investigated the discontinuity of the dislocation density function \( e(\nu) \) across the interface of the two solids for the surface-breaking crack problem. He took the limits of the singular integral equations as the position variable approaches the interface from above and below. Using the dominant terms of the limit and the method of Muskhelishvili, he showed that the function \( e(\nu) \) does not have a power singularity near the interface. Moreover, based on the assumption that the function \( e(\nu) \) is bounded near the interface and on the results of the singular integral limits, he concluded that the discontinuity of \( e(\nu) \) is given by equation (3.48), and the singular integrals are consistent with the discontinuity of equation (3.17). In
this study, the discontinuity for the function \( e(\nu) \) at the interface is similar to that of the surface-breaking crack problem. Thus, we can assume that the function \( e(\nu) \) is bounded at the interface and satisfies the discontinuity condition given by equation (3.48).

Using the definitions of equation (3.47), and equations (3.6) and (3.8) of Erdogan and Gupta [1] for an odd number \( n \) of quadrature points, the general approximation forms of the singular integral of the dislocation density function \( \tilde{e}(\nu) \) are given as follows

\[
\int_0^1 \tilde{e}(\zeta_{1r}) F(\zeta_1, z_{1r}) d\zeta = \frac{\pi}{n} \sum_{j=1}^{n-1} \Psi(t_j) F(\zeta_{1j}, z_{1r}) + \frac{\pi}{2n} \Psi(0) F(0, z_{1r}), \quad (3.50a)
\]

\[
\int_0^1 \tilde{e}(\zeta_{2r}) F(\zeta_2, z_{2r}) d\zeta = \frac{\pi}{n} \sum_{j=1}^{n-1} \Phi(t_j) F(\zeta_{2j}, z_{2r}) + \frac{\pi}{2n} \Phi(0) F(0, z_{2r}), \quad (3.50b)
\]

where the parameters \( \zeta_{1j}, z_{1r}, \zeta_{2j}, \) and \( z_{2r} \) and the Gaussian points \( t_j \) and \( y_r \) are defined by the following equations

\[
\zeta_{1j} = -(1 + \hat{h}) t_j + \hat{h}, \quad \zeta_{2j} = (1 - \hat{h}) t_j + \hat{h}, \quad (3.51a)
\]

\[
z_{1r} = -(1 + \hat{h}) y_r + \hat{h}, \quad z_{2r} = (1 - \hat{h}) y_r + \hat{h}, \quad (3.51b)
\]

\[
t_j = \cos \left( \frac{\pi(2j - 1)}{2n} \right), \quad (3.51c)
\]

\[
y_r = \cos \left( \frac{\pi r}{n} \right), \quad r = 1, \ldots, (n - 1)/2. \quad (3.51d)
\]

The reason for splitting the integrals of (3.30a, b, c) is that the dislocation density functions \( \tilde{e}(\nu) \) is discontinuous at the interface of the two solids and the Gaussian formulas of Erdogan and Gupta [1] can be applied only to continuous function in the interval \([-1, 1]\). Thus, the formulas of equations (3.50a, b) are written in terms of \( \Psi(0) \) and \( \Phi(0) \) since these two quantities do not vanish in general, and the Gaussian formulas are used in the interval \([0, 1]\). The functions \( \Psi(0) \) and \( \Phi(0) \) in equations
(3.50a, b) are evaluated using the \((n-1)/2\) Lagrangian polynomials \(\Delta_k(x)\) (Hildebrand [6]) such that
\[
\Psi(0) = \sum_{k=1}^{n-1} \Psi(t_k) \Delta_k(0), \quad \Phi(0) = \sum_{k=1}^{n-1} \Phi(t_k) \Delta_k(0),
\]
(3.52)
where the functions \(\Delta_k(x)\) are defined by
\[
\Delta_k(x) = \frac{1}{D}(x - t_1) \cdots (x - t_{k-1})(x - t_k) \cdots (x - t_{(n-1)/2}),
\]
(3.53a)
\[
D = (t_k - t_1) \cdots (t_k - t_{k-1})(t_k - t_{k+1}) \cdots (t_k - t_{(n-1)/2}).
\]
(3.53b)

Thus, using (3.47) and (3.50) through (3.53), the dislocation density function \(\bar{\varepsilon}(r)\) of equations (3.45a, b, c) can be approximated in terms of \(\Psi(t_j)\) and \(\Phi(t_j)\). It follows that equations (3.45a, b) can be written in the form
\[
\sum_{j=1}^{n-1} \Psi(t_j) \left[ \frac{1}{y_r - t_j} + \frac{1}{y_r - t_j} + (1 + \tilde{h}) \left[ H_1(\zeta_1, z_1) - \frac{1}{2} K' \right] \right]
+ \sum_{j=1}^{n-1} \Phi(t_j) \left[ \frac{4h_1}{y_r(4h_1 - y_r)} + (1 + \tilde{h}) \left[ H_1(h^-, z_1) - \frac{1}{2} K' \right] \right]
= \frac{1}{2} \sum_{r=1}^{n-1} \left[ K'(\zeta_1, z_2) + \frac{\Delta_j(0)}{2} K'(h^-, z_2) \right]
\]
(3.54a)
\[
- \frac{1 + \tilde{h}}{2} \sum_{j=1}^{n-1} \Psi(t_j) \left[ \frac{1}{y_r - t_j} + \frac{1}{4h_2 + y_r} + (1 + \tilde{h}) \left[ H_2(\zeta_2, z_2) - \frac{1}{2} K(\zeta_2, z_2) \right] \right]
+ \sum_{j=1}^{n-1} \Phi(t_j) \left[ \frac{4h_2}{c_1y_r(4h_2 - c_1y_r)} + (1 + \tilde{h}) \left[ H_2(h^+, z_2) - \frac{1}{2} K(h^+, z_2) \right] \right]
= \frac{1}{2} \sum_{r=1}^{n-1} \left[ K'(\zeta_2, z_2) + \frac{\Delta_j(0)}{2} K'(h^+, z_2) \right]
\]
(3.54b)
Similarly, the equation of (3.45c) and the discontinuity conditions (3.49) can be approximated as follows

\[ \sum_{j=1}^{n+1} \left[ c_1 \tilde{\Psi}(t_j) + \tilde{\Phi}(t_j) \right] \left[ 1 + \frac{\Delta_j(0)}{2} \right] = 0, \quad (3.55a) \]

\[ \sum_{j=1}^{n+1} \Delta_j(0) \left[ \tilde{\Psi}(t_j) - m \tilde{\Phi}(t_j) \right] = 0, \quad (3.55b) \]

where \( m \) is the ratio of the shear moduli, and the parameter \( c_1 \) and the function \( \Delta_j(0) \) are given by equations (3.46) and (3.53a), respectively.

Observe that the coefficients and the unknowns \( \tilde{\Psi}(t_j) \) and \( \tilde{\Phi}(t_j) \) of the system (3.54a, b) are complex numbers, while the right-hand side terms have purely imaginary values. This system is solved numerically, subject to the constraints (3.55a, b). In order to use these two constraints, the first two equations of (3.54a, b), which correspond to \( r = 1 \), are eliminated and replaced by equations (3.55a, b). Thus, we solve a linear system of \((n-1)\) equations for the \((n-1)\) complex-valued unknowns \( \tilde{\Psi}(t_j) \) and \( \tilde{\Phi}(t_j) \). Examining the formula of the parameter \( y_r \) of equation (3.51d) shows that the two eliminated equations of the linear system of equations (3.54a, b) correspond to the positions of \( y_r \) closest to the lower and upper crack tips.

The value of six dimensionless parameters are required to solve this linear system. These are: the ratio of the slownesses, \( \epsilon = s_r'/s_r \); the ratio of the shear moduli, \( m = \mu'/\mu \); the ratio of the crack middle point depth to the crack width, \( \tilde{d} = (d + d_1)/[2(d - d_1)] \); the ratio of the thickness of the surface layer to the crack width, \( \tilde{h} = h/(d - d_1) \); the amplitude of the incident wave, \( \tilde{u}_0 = u_0/(d - d_1) \); the frequency of the incident wave, \( \tilde{\omega} = \omega/\omega_s \), where \( \lambda_r = 2\pi/\omega_s \) is the wavelength of transverse waves in the layer. All the other dimensionless quantities can be written in terms of these six basic parameters.
We write now that the upper and lower crack tips must be contained inside the layer and the half-space, respectively. Thus, the interface of the two solids lies in the crack interval, and the relation between the crack parameters \((d_1, h, d)\) is

\[
d_1 < h < d. \tag{3.56a}
\]

Dividing through by the crack thickness \((d - d_1)\), and subtracting the quantity \(\bar{d}\) of (3.29d), one infers from (3.56a) the following

\[-1 < \bar{h} < 1. \tag{3.56b}\]

Observe that the parameter \(\bar{h}\) may take positive or negative values. When the middle point of the crack lies in the interface of the two solids, the parameter \(\bar{h}\) vanishes. The condition (3.56b) is consistent with the choice of the limits of the parameter \(\tau\) of equations (3.30a, b), and yields the following relation between the crack parameters \(\bar{d}\) and \(\bar{h}\)

\[
\bar{d} - 0.5 < \bar{h} < \bar{d} + 0.5. \tag{3.56c}
\]

Using the definition of the parameter \(\bar{d}\), one can write the following

\[
\bar{d} = (d + d_1)/[2(d - d_1)] = 0.5 + d_1/(d - d_1). \tag{3.56d}
\]

Since, by definition, the quantity \((d - d_1) > 0\), the lower limit of the parameter \(\bar{d}\) can be written such that

\[
\bar{d} > 0.5. \tag{3.56e}
\]

The parameters \(\bar{h}\) and \(\bar{d}\) are subjected to the constraints (3.56c, e). The region of admissible values for these parameters is illustrated in Figure (3.2). One can see from that figure that the parameters \(\bar{h}\) and \(\bar{d}\) can be chosen inside a region bounded from the left by the inclined line of equation \(\bar{d} = \bar{h} + 0.5\), from below by the horizontal line of equation \(\bar{d} = 0.5\), and from the right by the inclined line of equation \(\bar{d} = \bar{h} - 0.5\). When the value of \(\bar{d}\) is close to 0.5, the upper crack tip is close to the free surface. On the other hand, if this value is close to \(\bar{h} + 0.5\), then the upper crack tip,
which lies in the surface layer, approaches the interface between the two solids. For small cracks, the parameters \( \bar{h} \) and \( \bar{d} \) take large values. Further, the inclined line of equation \( \bar{d} = \bar{h} \) in Figure (3.2) represents cracks with a middle point that lies in the interface of the two solids. If the values of \( \bar{d} \) and \( \bar{h} \) are chosen above this line then the crack is longer above the interface (in the layer) than below (in the half-space), and if they are chosen on the other side of the line then the crack is longer below the interface than above.

![Figure 3.2: Admissible values for the geometric parameters \( \bar{d} \) and \( \bar{h} \).](image_url)
3.5 Far-Field Displacement

Equations (3.15a), (3.15b), and (3.17) in section 3.3 of this chapter are used here to obtain the scattered displacement field for the cracked half-space in terms of the dislocation density $e(v)$. The reflected and transmitted displacement fields of the cracked layer as well as of the half-space at a distance away from the crack plane are obtained as a sum of integrals of the dislocation density $e(v)$. The results of the previous section for the dislocation density $e(v)$ at $n$ points along the crack face and the method of Erdogan and Gupta [1] are used in this section to evaluate the amplitude of both the reflected and transmitted far-field displacements.

We first obtain the scattered displacement field for both the layer and half-space in terms of the dislocation density $e(v)$ by substituting equations (3.9), (3.10), and the appropriate expressions of the parameters $A$ and $B$ of equations (3.23a, b) into (3.5a, b). The result is

$$u_3(x_1, x_2) = \frac{1}{\pi} \int_0^\infty \left[ \int_{d_1}^d e(v)B_1(\zeta, v) \, dv \right] \left( e^{-\overline{\beta}(\zeta)x_2} + e^{\overline{\beta}(\zeta)x_2} \right) \frac{\sin(\zeta x_1)}{\overline{\Delta}_0(\zeta)} e^{-\overline{\beta}(\zeta)h} \, d\zeta$$

$$+ \frac{2}{\pi} \int_0^\infty \frac{e^{-\overline{\beta}(\zeta)x_1}}{\zeta} \left[ \int_{d_1}^d e(v) \sin(\zeta v) \, dv \right] \cos(\zeta x_2) \, d\zeta, \quad 0 < x_2 < h, \quad (3.57a)$$

$$u'_3(x_1, x_2) = \frac{1}{\pi} \int_0^\infty \frac{e^{\overline{\beta}(\zeta)h}}{\overline{\Delta}_0(\zeta)} \left[ \int_{d_1}^d e(v)A_1(\zeta, v) \, dv \right] \sin(\zeta x_1)e^{-\overline{\beta}(\zeta)x_2} \, d\zeta$$

$$+ \frac{2}{\pi} \int_0^\infty \frac{e^{-\overline{\beta}(\zeta)x_1}}{\zeta} \left[ \int_{d_1}^d H(v - h)e(v) \sin(\zeta v) \, dv \right] \cos(\zeta x_2) \, d\zeta, \quad h < x_2, \quad (3.57b)$$

where $H$ denotes the Heaviside step function, and the functions $\beta(\zeta)$ and $\overline{\beta}(\zeta)$ are given by equation (3.7). The dimensionless functions $\overline{\Delta}_0(\zeta)$, $A_1(\zeta, v)$, and $B_1(\zeta, v)$ are given by equations (3.24a, b, c).

The dimensionless coordinates $x$ and $y$, the parameter $\tilde{v}$, as well as the scattered displacement fields $\tilde{u}(x, y)$ and $\tilde{u}'(x, y)$ are defined as follows
\[ x = \frac{x_1}{d - d_1}, \quad y = \frac{x_2}{d - d_1}, \quad \tilde{\nu} = \frac{\nu}{2} + \tilde{\tilde{\nu}}, \quad (3.58a) \]

\[ \tilde{u}(x, y) = \frac{u((d - d_1)x, (d - d_1)y)}{(d - d_1)} = \frac{u(x_1, x_2)}{(d - d_1)}, \quad (3.58b) \]

\[ \tilde{u}'(x, y) = \frac{u'(d - d_1)x, (d - d_1)y)}{(d - d_1)} = \frac{u'(x_1, x_2)}{(d - d_1)}, \quad (3.58c) \]

Using the above definitions together with the notations given by equation (3.29d), the dimensionless scattered displacement fields can be written as

\[ \tilde{u}_3(x, y) = \frac{1}{2\pi} \int_{-1}^{1} \bar{e}(\nu) \left[ \int_{0}^{\infty} \frac{e^{-\tilde{s}_\tau x\tilde{y}\tilde{y}(u)} + e^{\tilde{s}_\tau y\tilde{y}(u)}}{\tilde{\Delta}(u)} \bar{B}_1(u, \nu) \sin(\tilde{s}_\tau ux) e^{-\tilde{s}_\tau \tilde{y}\tilde{y}(u)} du \right] dv \]

\[ + \frac{1}{\pi} \int_{-1}^{1} \bar{e}(\nu) \left[ \int_{0}^{\infty} \frac{e^{-\tilde{s}_\tau x\tilde{y}\tilde{y}(u)}}{u} \sin(\tilde{s}_\tau ux) \cos(\tilde{s}_\tau uy) du \right] dv, \quad 0 < y < \bar{h}, \quad (3.59a) \]

\[ \tilde{u}'_3(x, y) = \frac{1}{2\pi} \int_{-1}^{1} \bar{e}(\nu) \left[ \int_{0}^{\infty} \frac{e^{\tilde{s}_\tau \tilde{y}\tilde{y}(u)}}{\tilde{\Delta}(u)} \tilde{A}_1(u, \nu) e^{-\tilde{s}_\tau y\tilde{y}(u)} \sin(\tilde{s}_\tau ux) du \right] dv \]

\[ + \frac{1}{\pi} \int_{-1}^{1} \bar{e}(\nu) H(\nu - \bar{h}) \left[ \int_{0}^{\infty} \frac{e^{-\tilde{s}_\tau x\tilde{y}\tilde{y}(u)}}{u} \sin(\tilde{s}_\tau ux) \cos(\tilde{s}_\tau uy) du \right] dv, \quad \bar{h} < y, \quad (3.59b) \]

where the dislocation density \( \bar{e}(\nu) \) as well as the functions \( \bar{\beta}(\nu) \) and \( \tilde{\beta}(\nu) \) are given by equation (3.32b, c). The dimensionless functions \( \tilde{\Delta}(u), \tilde{A}_1(u, \nu), \) and \( \bar{B}_1(u, \nu) \) are defined by equations (3.33a, b, c).

The inner integrals of the first term on the right-hand side of equations (3.59a, b) are similar to those of equation (3.31e, f). The portions of these integrals over the interval \([\epsilon, 1]\) should be interpreted as in equation (3.35a, b). Thus, these integrals will be split and evaluated over three different intervals \([0, \epsilon], [\epsilon, 1], \) and \([1, \infty)\). This decomposition allows us to split the scattered displacement fields into two major parts. The first part oscillates, without decay, as a function of \( x \), and it comes from the middle integrals, which are evaluated over the interval \([\epsilon, 1]\). The second part decays as rapidly as \( |x|^{-\frac{1}{2}} \) as \( |x| \to \infty \), and it is contained in the three parts of these
integrals, which are evaluated over the intervals \([0, \epsilon], [\epsilon, 1],\) and \([1, \infty),\) as well as in the second term of the right-hand side of equations (3.59a, b). In the interval \([\epsilon, 1],\) one has (using definition (3.2))
\[
\beta(u) = -i(1 - u^2)^{1/2} = -i\alpha(u), \quad \beta'(u) = \alpha'(u).
\] (3.60)

We first write the middle portions of the inner integrals of the first term on the right-hand side of equations (3.59a, b), which are evaluated over the interval \([\epsilon, 1],\) as follows
\[
I_1(\nu, x, y) = \int_{\epsilon}^{1} \frac{\cos(\tilde{s}_z y \alpha(u))}{\Delta_b(u)} B_{1b}(u, \nu) \sin(\tilde{s}_z u x) \, du,
\] (3.61a)
\[
I_2(\nu, x, y) = \int_{\epsilon}^{1} e^{-\tilde{s}_z (y-h) \alpha'(u)} \frac{\Delta_b(u)}{A_{1b}(u, \nu)} A_{1b}(u, \nu) \sin(\tilde{s}_z u x) \, du,
\] (3.61b)

where \(\Delta_b(u)\) is given by equation (3.34f). The integrals of equations (3.61a, b) are interpreted as in equation (3.35a, b) such that
\[
I_1(\nu, x, y) = i\pi \sum_{l=1}^{k} \frac{\cos(\tilde{s}_z y \alpha(\eta_l))}{\Delta_b(\eta_l)} B_{1b}(\eta_l, \nu) \sin(\tilde{s}_z \eta_l x) + \int_{\epsilon}^{1} \frac{\cos(\tilde{s}_z y \alpha(u))}{\Delta_b(u)} B_{1b}(u, \nu) \sin(\tilde{s}_z u x) \, du,
\] (3.62a)
\[
I_2(\nu, x, y) = i\pi \sum_{l=1}^{k} e^{-\tilde{s}_z (y-h) \alpha'(\eta_l)} \frac{\Delta_b(\eta_l)}{A_{1b}(\eta_l, \nu)} A_{1b}(\eta_l, \nu) \sin(\tilde{s}_z \eta_l x) + \int_{\epsilon}^{1} e^{-\tilde{s}_z (y-h) \alpha'(u)} \frac{\Delta_b(u)}{A_{1b}(u, \nu)} A_{1b}(u, \nu) \sin(\tilde{s}_z u x) \, du,
\] (3.62b)

where the functions \(\Delta_b(\eta_l), B_{1b}(u, \nu),\) and \(A_{1b}(u, \nu)\) are given in (A.1) of Appendix A. The circle through the integral sign in the right-hand sides of equations (3.62a, b) indicates that these integrals are performed in the principal-value sense through each of the poles \(\eta_l(l = 1, \ldots, k).\) These two integrals should be interpreted as in equation (3.36a, b) such that
\[ J_1(\nu, x, y) = \tilde{J}_1(\nu, x, y) + \sum_{l=1}^{k} \frac{\cos(\tilde{s}_r y \alpha(\eta_l))}{\Delta_b(\eta_l)} \overline{B}_{1b}(\eta_l, \nu) \tilde{J}_l(x), \]  
\[ J_2(\nu, x, y) = \tilde{J}_2(\nu, x, y) + \sum_{l=1}^{k} \frac{e^{-\tilde{s}_r (y-h) \alpha'(\eta_l)}}{\Delta_b(\eta_l)} \overline{A}_{1b}(\eta_l, \nu) \tilde{J}_l(x), \]  
where the functions \( \tilde{J}_l(x) \), \( \tilde{J}_1(\nu, x, y) \), and \( \tilde{J}_2(\nu, x, y) \) are

\[ \tilde{J}_l(x) = \int_{\epsilon}^{1} \frac{\sin(\tilde{s}_r u x)}{u - \eta_l} du, \]  
\[ \tilde{J}_1(\nu, x, y) = \int_{\epsilon}^{1} \left[ \frac{\cos(\tilde{s}_r y \alpha(u))}{\Delta_b(u)} \overline{B}_{1b}(u, \nu) \right. \]  
\[ - \sum_{l=1}^{k} \frac{\cos(\tilde{s}_r y \alpha(\eta_l))}{(u - \eta_l)\Delta_b(\eta_l)} \overline{B}_{1b}(\eta_l, \nu) \]  
\[ \left. \sin(\tilde{s}_r u x) du \right], \]  
\[ \tilde{J}_2(\nu, x, y) = \int_{\epsilon}^{1} \left[ \frac{e^{-\tilde{s}_r (y-h) \alpha'(u)}}{\Delta_b(u)} \overline{A}_{1b}(u, \nu) \right. \]  
\[ - \sum_{l=1}^{k} \frac{e^{-\tilde{s}_r (y-h) \alpha'(\eta_l)}}{(u - \eta_l)\Delta_b(\eta_l)} \overline{A}_{1b}(\eta_l, \nu) \]  
\[ \left. \sin(\tilde{s}_r u x) du \right]. \]

The integral \( \tilde{J}_l(x) \) of equation (3.64a) is performed in the principal-value sense across each of the poles \( \eta_l(l = 1, \ldots, k) \).

Notice that the integrals in the right-hand side of equations (3.64b, c) are regular integrals, since the functions inside the brackets are bounded over the interval \([\epsilon, 1]\). Using the Riemann-Lebesgue lemma, one can show that these two integrals vanish as \( x \to \infty \). Evaluating the cross-pole integrals of equation (3.64a), and changing variables, the function \( \tilde{J}_l(x) \) can be written in the form

\[ \tilde{J}_l(x) = \sin(\tilde{s}_r \eta_l x) \int_{-\epsilon_0}^{0} \frac{\cos(\tilde{s}_r v x)}{v} dv + \cos(\tilde{s}_r \eta_l x) \int_{-\epsilon_0}^{0} \frac{\sin(\tilde{s}_r v x)}{v} dv, \]

where the parameter \( \epsilon_0 \) has a very small positive value. The first integral in the right-hand side of equation (3.65a) has zero value over the interval \([-\epsilon_0, \epsilon_0]\), since the integrand is an odd function of the variable \( v \). Using the Riemann-Lebesgue lemma
and the above remark, the integral $\tilde{J}_1(x)$ of equation (3.64a) yields the following result as the variable $x \to \infty$

$$\tilde{J}_1(x) = \pi \cos(\bar{s}_y \eta_1 x).$$  \hspace{1cm} (3.65b)

Substituting (3.65b) in equations (3.63a, b), and using the facts that the functions $\tilde{J}_1$ and $\tilde{J}_2$ vanish as $x \to \infty$, one can express the functions $J_1$ and $J_2$ as follows

$$J_1(\nu, x, y) = \pi \sum_{i=1}^k \frac{\cos(\bar{s}_y \nu \alpha(\eta_i))}{\Delta_b(\eta_i)} \overline{B}_{1b}(\eta_i, \nu) \cos(\bar{s}_y \eta_i x),$$  \hspace{1cm} (3.66a)

$$J_2(\nu, x, y) = \pi \sum_{i=1}^k \frac{e^{-\bar{s}_y(y-\bar{h}) \alpha'(\eta_i)}}{\Delta_b(\eta_i)} \overline{A}_{1b}(\eta_i, \nu) \cos(\bar{s}_y \eta_i x).$$  \hspace{1cm} (3.66b)

Observe that the parameters $\eta_i$ are the roots of the equation $\Delta_b(u) = 0$. Thus, the relation between the functions $\alpha(\eta_i)$ and $\alpha'(\eta_i)$ can be written in the form

$$m \alpha'(\eta_i) = \tan(\bar{s}_y \bar{h} \alpha(\eta_i)) \alpha(\eta_i).$$  \hspace{1cm} (3.67a)

Using the above remark together with the definitions of functions $\overline{A}_{1b}(\nu, \eta_i)$ and $\overline{B}_{1b}(\nu, \eta_i)$, one can infer the following relationship

$$\overline{A}_{1b}(\eta_i, \nu) = \cos(\bar{s}_y \bar{h} \alpha(\eta_i)) \overline{B}_{1b}(\eta_i, \nu).$$  \hspace{1cm} (3.67b)

Substituting the expressions of equations (3.66a, b) in equations (3.62a, b) and using the relationship of equation (3.67b), the functions $I_1(\nu, x, y)$ and $I_2(\nu, x, y)$ can be expressed as follows in the limit as $x \to \infty$

$$I_1(\nu, x, y) = \pi \sum_{i=1}^k \frac{\cos(\bar{s}_y \nu \alpha(\eta_i))}{\Delta_b(\eta_i)} \overline{B}_{1b}(\eta_i, \nu) e^{i \bar{s}_y \eta_i x}, \hspace{1cm} 0 < y < \bar{h},$$  \hspace{1cm} (3.68a)

$$I_2(\nu, x, y) = \pi \sum_{i=1}^k \frac{e^{-\bar{s}_y(y-\bar{h}) \alpha'(\eta_i)}}{\Delta_b(\eta_i)} \overline{B}_{1b}(\eta_i, \nu) \cos(\bar{s}_y \bar{h} \alpha(\eta_i)) e^{i \bar{s}_y \eta_i x}, \hspace{0.5cm} \bar{h} < y. \hspace{1cm} (3.68b)$$

Using the result of equations (3.68a, b), the scattered displacement field in the cracked half-space as $x \to \infty$ takes the following form

$$u_3^{\infty}(x, y) = \sum_{i=1}^k U_i \cos(\bar{s}_y \nu \alpha(\eta_i)) e^{i \bar{s}_y \eta_i x}, \hspace{1cm} 0 \leq y \leq \bar{h},$$  \hspace{1cm} (3.69a)
\[ u_3^{\infty}(x, y) = \sum_{l=1}^{k} U_l \cos(\bar{s}_p \, \bar{\alpha}_l(\eta_l)) e^{-s_p(y-h)\alpha_l(\eta_l)} e^{i\bar{s}_p \eta_l x}, \quad \bar{h} \leq y, \]  

(3.69b)

where the amplitude of the scattered displacement far-field \( U_l \) is defined by the following equation

\[ U_l = \frac{1}{2\Delta_b(\eta_l)} \int_{-1}^{1} \tilde{G}(\nu) \overline{B}_{1b}(\eta_l, \nu) d\nu. \]  

(3.70)

The value of the derivative \( \Delta_b(\eta_l) \) and the function \( \overline{B}_{1b}(\eta_l, \nu) \) are given by (A.1) of Appendix A. In order to evaluate the integral in the right-hand side of equation (3.70), and because the function \( \tilde{G}(\nu) \) is not continuous in the range \([-1, 1]\), the method of section (3.4) is used here to split that integral at the interface by changing the integration variable such that

\[ \nu = \begin{cases} 
-(1+h) \zeta + h, & -1 < \nu < h, \\
(1-h) \zeta + h, & h < \nu < 1,
\end{cases} \]  

(3.71)

where the parameter \( h \) is given by (3.29d). Thus, the integral of (3.70) can be written in the form

\[ I_l = \left(1 + \frac{h}{1 - \zeta^2}\right) \int_{0}^{1} \tilde{G}(\zeta_1) \overline{B}_{1b}(\eta_l, \zeta_1) d\zeta + \left(1 - \frac{h}{1 - \zeta^2}\right) \int_{0}^{1} \tilde{G}(\zeta_2) \overline{B}_{1b}(\eta_l, \zeta_2) d\zeta. \]  

(3.72)

The variables \( \zeta_1 \) and \( \zeta_2 \) are given by equation (3.46), and the functions \( \tilde{G}(\zeta_1) \) and \( \tilde{G}(\zeta_2) \) have the following form

\[ \tilde{G}(\zeta_1) = \frac{\Psi(\zeta)}{(1 - \zeta^2)^{\frac{1}{2}}}, \quad \tilde{G}(\zeta_2) = \frac{\Phi(\zeta)}{(1 - \zeta^2)^{\frac{1}{2}}}, \quad 0 < \zeta < 1. \]  

(3.73)

Using equations (3.6) and (3.8) of Erdogan and Gupta [1] and the expressions of equation (3.73), the integrals in the right-hand side of equation (3.72) can be approximated as follows

\[ I_{1l} = \frac{n}{\pi} \sum_{j=1}^{n} \Psi(t_j) \overline{B}_{1b}(\eta_l, \zeta_j) + \frac{\pi}{2n} \Psi(0) \overline{B}_{1b}(\eta_l, \bar{h}), \]  

(3.74a)
\[ I_{I2} = \frac{\pi}{n} \sum_{j=1}^{n-1} \Phi(t_j) \overline{B_{1b}}(\eta_j, \zeta_{2j}) + \frac{\pi}{2n} \Phi(0) \overline{B_{1b}}(\eta_l, \hat{h}^+), \]  

(3.74b)

where the quantities \( \Psi(0) \) and \( \Phi(0) \) are evaluated using the \((n-1)/2 \) Lagrangian polynomials \( \Delta_k(x) \) (Hildebrand [6]) as in equation (3.52). The Gaussian points \( t_j \) and the parameters \( \zeta_{1j} \) and \( \zeta_{2j} \) are given by equations (3.51a, c).

Substituting equations (3.74a, b) in equation (3.72), the amplitude of the scattered displacement far-field \( U_l \) can be expressed as

\[
U_l = \frac{\pi}{2n} \frac{1 + \hat{h}}{\Delta_k(\eta_l)} \left[ \sum_{j=1}^{n-1} \Psi(t_j) \left[ \overline{B_{1b}}(\eta_j, \zeta_{1j}) + \frac{\Delta_j(0)}{2} \overline{B_{1b}}(\eta_l, \hat{h}^-) \right] 
+ \frac{1 - \hat{h}}{1 + \hat{h}} \sum_{j=1}^{n-1} \Phi(t_j) \left[ \overline{B_{1b}}(\eta_j, \zeta_{2j}) + \frac{\Delta_j(0)}{2} \overline{B_{1b}}(\eta_l, \hat{h}^+) \right] \right],
\]

(3.75)

where the function \( \Delta_j(0) \) is given by equation (3.53a). The complex values of functions \( \Psi(t_j) \) and \( \Phi(t_j) \) along the crack face were computed in the previous section using the linear system of equations (3.54a, b) and (3.55a, b).

The scattered displacement far-fields of equations (3.69a, b) are the superposition of \( k \) Love wave modes of the type of equation (2.2) of the previous chapter. Similar to those of the layer-breaking crack, the wave motions in the layer and the half space at a large distance from the plane of the crack \(|x| \to \infty \) are dominated by these \( k \) Love wave modes. By recalling the antisymmetry of the scattered field with respect to the plane \( x_1 = 0 \), the reflected and transmitted displacement fields in the cracked half-space can be defined by the following equations

\[
u^r_3(x, y) = -u_3^\infty(-x, y), \quad x < 0, \quad (3.76a)
\]

\[
u^r_3(x, y) = \tilde{u}_3^r(x, y) + u_3^\infty(x, y), \quad x > 0. \quad (3.76b)
\]
The displacement $u_3^\infty (x, y)$ is defined by equations (3.69a, b), and $\tilde{u}_3^{in}(x, y)$ is given by

$$
\tilde{u}_3^{in}(x, y) = \frac{u_3^{in} ((d - d_1)x, (d - d_1)y)}{d - d_1},
$$

(3.76c)

where the incident displacement $u_3^{in}(x_1, x_2)$ is defined by equation (2.2) of the previous chapter.

The first order reflection coefficient $R_1$, transmission coefficient $T_1$, and the higher-order reflection coefficients $R_l (l \geq 2)$ are defined by

$$
R_1 = \frac{U_1}{\bar{u}_0}, \quad T_1 = 1 + U_1/\bar{u}_0, \\
R_l = \frac{U_l}{\bar{u}_0}, \quad l = 2, \ldots, k.
$$

(3.77)

The amplitudes $U_l$ of equation (3.75), in general, are complex numbers. Thus, the moduli and phases of the reflected and transmitted Love wave displacements are different from those of the incident Love wave.

In the next section, we present the results of the bounded parts of the dislocation density functions as well as the scattered far-field displacements. In addition, we illustrate the dependence of the reflection and transmission coefficients $T_1$ and $R_l$ on the frequency of the incident Love wave.
3.6 Discussion of Numerical Results

The values of the real and imaginary parts of the bounded part of the dislocation density functions, \( \Psi(\nu) \) and \( \Phi(\nu) \), are plotted for different values of \( n \) in Figures (3.3), (3.4), (3.5), and (3.6). The numbers of quadrature points are \( n = 15, 25, 31, 45, \) and 55. For each value of the parameter \( n \), the linear system (3.54a)-(3.55b) is solved numerically for \( 2(n - 1) \) real-valued unknowns. These unknowns are the real and imaginary parts of \( \Psi(\nu) \) and \( \Phi(\nu) \). The variable \( \nu \) takes its values in the interval [0,1]. In Figures (3.3)-(3.6), the middle point of the crack lies in the interface and the values of \( \Psi(\nu) \) for \( 0 < \nu < 1 \) are shown in the interval \(-1 < \nu < 0\). The dimensionless amplitude, \( \hat{u}_0 \), of the incident Love wave used in these plots is 1.0, while the two values of the dimensionless frequency, \( \hat{\omega} \), of the incident wave are \( \hat{\omega} = 1.9 \) for Figures (3.3) and (3.5), and \( \hat{\omega} = 2.6 \) for Figures (3.4) and (3.6). The other geometric and material parameters of the cracked half-space used to plot these figures are: \( \bar{h} = 0.6667; \bar{d} = 0.6667; m = 1.75; \epsilon = 0.8 \). These parameters are dimensionless as it was explained earlier.

From Figures (3.3) through (3.6), one can see that the real parts of the functions \( \Psi(\nu) \) and \( \Phi(\nu) \) converge faster than their imaginary parts. These figures indicate that both the real and imaginary parts of \( \Psi(\nu) \) and \( \Phi(\nu) \) have good convergence all values of \( n \) for the dimensionless frequency \( \hat{\omega} \) in the range of \([0.01, 2.0]\). On the other hand, when \( \hat{\omega} > 2.0 \) the convergence of these plots requires \( n \geq 25 \). Notice from these figures that the absolute value of \( \Phi(\nu) \) and \( \Psi(\nu) \) remains bounded in the closed interval \([-1, 1]\). The total cpu time consumed in a SPARC10 machine to compute the dislocation density functions and the scattered far-field displacements for each value \( \hat{\omega} \) of the incident Love wave dimensionless frequency and for each number \( n \) of Gaussian quadrature points is presented in the following table.
<table>
<thead>
<tr>
<th>$\tilde{\omega}$</th>
<th>$n = 15$</th>
<th>$n = 25$</th>
<th>$n = 31$</th>
<th>$n = 45$</th>
<th>$n = 55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>2 : 47</td>
<td>8 : 55</td>
<td>14 : 00</td>
<td>28 : 47</td>
<td>41 : 34</td>
</tr>
<tr>
<td>2.6</td>
<td>4 : 09</td>
<td>12 : 41</td>
<td>20 : 08</td>
<td>40 : 23</td>
<td>66 : 12</td>
</tr>
</tbody>
</table>

Table 3.1: Total cpu time, in minutes, consumed to compute $\Psi(\nu)$ and $\Phi(\nu)$ by SPARC10 machine.

These times are lower for an incident wave with a lower frequency. Based on Figures (3.3) through (3.6) and on the results presented in Table (3.1), one can conclude that the most efficient plot with a good convergence is the one corresponding to $n = 31$, as it consumes one third of the cpu time needed for $n = 55$ and one half of that for $n = 45$.

The formulas of the scattered far-field displacement amplitudes of equation (3.75) are used here to evaluate the moduli and the phases of the first order transmission and reflection coefficients ($T_1$ and $R_1$) as well as of the higher order reflection coefficients ($R_l$, $l = 2, 3$), which are defined by equations (3.77). These results are plotted in Figures (3.9) through (3.40) versus the dimensionless incident frequency $\tilde{\omega}$ for four different cracks. The cracks lie in the surface layer and break through the half-space with different values of $d_1$ and $d$, as shown in Figures (3.7) and (3.8). The upper tips of the first and second cracks lie at a distance very close to the free surface, and the lower tip of the second crack is located just below the interface. The second crack and the upper part of the first crack occupy almost the entire thickness of the surface layer, while the third crack occupies $3/4$ of the thickness of the layer. The middle point of the third crack lies in the interface (as $d$ and $h$ are equal), and that of the first crack is located just below the interface (as $\tilde{d} = 0.999 \tilde{h}$). The middle point of
the fourth crack lies inside the half-space, and its upper part occupies the lower 1/8 of the thickness of the surface layer. The ratios of the middle point of the crack and the layer thickness to the crack width (\( \tilde{d} \) and \( \tilde{h} \)), as well as the ratio of the crack width to the thickness of the layer (\( \tilde{d} \)) corresponding to these cracks are presented in the following table.

<table>
<thead>
<tr>
<th></th>
<th>First Crack</th>
<th>Second Crack</th>
<th>Third Crack</th>
<th>Fourth Crack</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{d} )</td>
<td>0.5005</td>
<td>0.5009</td>
<td>0.6667</td>
<td>2.2500</td>
</tr>
<tr>
<td>( \tilde{h} )</td>
<td>0.5000</td>
<td>0.9000</td>
<td>0.6667</td>
<td>2.0000</td>
</tr>
<tr>
<td>( \tilde{d} )</td>
<td>2.0000</td>
<td>1.1111</td>
<td>1.5000</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

Table 3.2: Values of the geometric parameters \( \tilde{d} \) and \( \tilde{h} \) and the ratio \( \tilde{d} \) for the four interface-breaking cracks.

The values of the parameter \( \tilde{h} \) of the first and second cracks are identical to those of the two surface-breaking crack cases presented by Angel [2]. In order to compare the results of the first two embedded interface-breaking cracks with those of the surface-breaking cracks, we have placed their upper tips at a close distance from the free surface. This can be done by taking the value of the parameter \( \tilde{d} \), which controls the position of the crack in the layered half space, close to its lower limit 0.5. The parameter values of \( \epsilon = 0.8 \) and \( m = 1.75 \) were used to compute the plots of Figures (3.9) through (3.40). The values of \( \epsilon \) and \( m \) correspond approximately to those of the uniform, single layered, earth model of Neerhoff [7]. The range of
the incident Love wave dimensionless frequency $\tilde{\omega}$ lies in the interval $[0, 3.0]$, and is scanned thoroughly by using an increment of 0.01. Thus, 300 computations were done to plot these figures. For each frequency, the linear system of equations (3.55a)-(3.56b) is solved numerically with the number $n = 31$ of Gaussian quadrature points, and the amplitude of the scattered displacements $U_l$ is computed. With this choice of parameters, the Love wave mode of order $k$ emerges at $\tilde{\omega} = 5(k - 1)/6$. In the frequency-range of interest, the relevant values are $\tilde{\omega} = 0.833, 1.667, 2.5$, which correspond to the emergence of the second, third, and fourth modes, respectively.

The moduli and phases of the transmission and reflection coefficients are classified in eight groups. Each group consists of four figures for the four different cracks described in Table 3.2. Figures (3.9)-(3.24) show the moduli of the transmission and reflection coefficients versus the dimensionless frequency, $\tilde{\omega}$, and Figures (3.25)-(3.40) show their corresponding phases versus $\tilde{\omega}$.

In the limit as $\tilde{\omega}$ approaches zero, the coefficient $R_1$ vanishes, whereas $T_1$ becomes equal to unity. This is a direct consequence of the vanishing of the stress in equation (3.1) when $\omega = 0$. Thus, only the incident field remains as the scattered field vanishes. This result is mentioned by Angel [2], and the physical interpretation of this phenomenon is that for low frequencies (long wavelengths) the crack appears so small to the incident wave that it is ignored altogether.

Figures (3.9) and (3.10) show that the first order reflection coefficients moduli, $|R_1|$, of the first and second cracks have almost no oscillations. The values of $|R_1|$ are greater than 0.98 for frequencies $\tilde{\omega} > 0.8$ of the first crack, and for $\tilde{\omega} > 1.3$ of the second crack. Due to the fact that the first crack width is larger than that of the second crack, the values of $|R_1|$ of the first crack are greater and less oscillating than those of the second crack. On the other hand, Figures (3.13) and (3.14) show that the values of the first order transmission coefficient moduli, $|T_1|$, of the first
two cracks are less than 0.06 for frequencies \( \bar{\omega} > 1.39 \) of the first crack, and for \( \bar{\omega} > 1.44 \) of the second crack. Based on these results, one can conclude that these two cracks act nearly as perfect reflectors for the incident Love wave, and the amount of reflection increases as the crack length increases.

The transmission coefficient phases of the first two cracks, \( \varphi(T_1) \), of Figures (3.29) and (3.30) have almost the same characteristics. Their values are in the range \([65^\circ, 75^\circ]\) for frequencies \( \bar{\omega} > 0.5 \). These phases, which are out of phase with respect to the incident wave, may be explained by the fact that the cracks in these two cases occupy almost the entire layer except a very small interval between the upper tips and the free surface which allows, for high frequencies, a negligible amount of the incident wave to transmit. Thus, the reflection waves are dominant, and the phase \( \varphi(R_1) \) for \( \bar{\omega} > 0.5 \), as shown in Figures (3.25) and (3.26), is completely in opposite phase with respect to the incident Love wave.

The curves of the second and third order reflection coefficient moduli, \(| R_2 |\) and \(| R_3 |\), of the first two cracks versus frequency are given by Figures (3.17), (3.18), (3.21), and (3.22), respectively. No peaks are detected for \(| R_2 |\) and \(| R_3 |\) of the first crack. Two small peaks are detected for \(| R_2 |\) of the second crack at \( \bar{\omega} = 1.01 \) and 1.76, whereas the peak of \(| R_3 |\) occur at \( \bar{\omega} = 2.11 \). The peak values of \(| R_2 |\) and \(| R_3 |\) are less than 0.06, which are much smaller than those of \(| R_1 |\). The phases \( \varphi(R_2) \) and \( \varphi(R_3) \) of these two cracks are given by Figures (3.33), (3.34), (3.37), and (3.38). The values of these figures are in the interval \([60, 90]\) for \( \bar{\omega} > 1.5 \).

The reflection and transmission moduli of the first and second cracks, presented above, are in good agreement with those of the surface-breaking cracks with similar geometrical and material parameters in the work presented by Angel [2]. The results of Angel are computed for the case of a surface-breaking crack breaking through the interface with two values of depths, respectively double and 10/9 times the layer.
thickness. The reflection in the surface-breaking cracks is higher than that of the embedded cracks, and the surface cracks act as nearly perfect reflectors. This is because the upper tip of the embedded cracks never touches the free surface. Therefore, there is always some solid material between the tip and the free surface, which allows some of the incident Love wave to transmit.

The reflection and transmission moduli as well as the phases of the second crack of Figures (3.10, 14, 18, 22, 26, 30, 34, 36) are nearly identical to those of the third layer-embedded crack of Figures (2.10, 13, 16, 19, 22, 25, 28, 31), presented in the previous chapter. These two crack configurations have identical material parameters. The upper tips of these cracks lie at a very close distance from the free surface, and their lower tips are located just above the interface for the layer-embedded crack and just below the interface for the interface-breaking crack. The values of the reflection modulus $|R_1|$ of the interface-breaking crack are slightly higher than those of the layer-embedded crack. For high frequencies, $\bar{\omega} > 1.5$, the values of $|R_1|$ for these two cracks are nearly identical, and the breaking through the half space has little effect on the reflection of the incident Love wave.

It can be observed from Figure (3.11) that the reflection coefficient modulus $|R_1|$ of the third crack has two peaks at frequencies of $\bar{\omega} = 0.39$ and 0.71, and its values are less than 0.6 for $\bar{\omega} > 0.88$. Figure (3.15) shows that the values of the transmission coefficient modulus $|T_1|$ of the third crack is less than 0.46 for $\bar{\omega} > 0.40$. In contrast to the phase of the first order transmission coefficient of the first two cracks, the $\varphi(T_1)$ of the third crack, in Figure (3.31), is in phase with the incident Love wave for $\bar{\omega} > 1.5$. The other two reflection coefficient moduli $|R_2|$ and $|R_3|$ of this crack are illustrated in Figures (3.19) and (3.23). Their values are much higher than those of the first and second cracks. This may be explained by the fact that the phase
coefficients \(\varphi(R_2)\) and \(\varphi(R_3)\), as shown by figures (3.35) and (3.39), are in phase with the incident Love wave.

The values of \(\varphi(R_1)\) for the four cracks have almost the same characteristics, and they vary in the interval \([90^\circ, 180^\circ]\) for the range of the incident dimensionless frequency of \([0, 3.0]\). The values of \(\varphi(R_1)\) in two different ranges of \(\tilde{\omega}\) are presented in the following table.

<table>
<thead>
<tr>
<th>Crack No.</th>
<th>(\varphi(R_1))</th>
<th>(\tilde{\omega})</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Crack</td>
<td>([90^\circ, 170^\circ])</td>
<td>([0.01, 0.34])</td>
</tr>
<tr>
<td></td>
<td>([170^\circ, 178^\circ])</td>
<td>([0.35, 3.00])</td>
</tr>
<tr>
<td>Second Crack</td>
<td>([90^\circ, 170^\circ])</td>
<td>([0.01, 0.53])</td>
</tr>
<tr>
<td></td>
<td>([170^\circ, 178^\circ])</td>
<td>([0.54, 3.00])</td>
</tr>
<tr>
<td>Third Crack</td>
<td>([90^\circ, 160^\circ])</td>
<td>([0.01, 1.31])</td>
</tr>
<tr>
<td></td>
<td>([160^\circ, 178^\circ])</td>
<td>([1.32, 3.00])</td>
</tr>
<tr>
<td>Fourth Crack</td>
<td>([90^\circ, 148^\circ])</td>
<td>([0.01, 1.04])</td>
</tr>
<tr>
<td></td>
<td>([149^\circ, 158^\circ])</td>
<td>([1.05, 3.00])</td>
</tr>
</tbody>
</table>

**Table 3.3: Phase of the coefficient \(R_1\) for the four interface-breaking cracks; \(\epsilon = 0.8, m = 1.75\).**

It can be seen from Table (3.3) and Figures (3.25)-(3.28) that in general, for high frequencies, the displacement of the first order reflected Love wave is always in opposite phase \((\varphi(R_1) = 180^\circ)\) compared with that of the incident Love wave in the range \(\tilde{\omega} \geq 1.32\). This phenomenon, for high frequency, is apparently independent of
the crack width. In contrast to the layer-embedded crack, the position of the crack has little effect on \( \varphi (R_1) \). It is also seen that the change of \( \varphi (R_1) \) from 90° to 160° is less rapid when the crack width is small, and the upper tip of the crack lies away from the free surface. On the other hand, the phase of the displacement of the first order transmitted Love wave, \( \varphi (T_1) \), has a great deal of dependence on the position and the width of the crack for frequencies of \( \bar{\omega} \geq 1.5 \).

The three reflection and the transmission coefficient moduli of the fourth crack are illustrated in Figures (3.12), (3.20), (3.24), and (3.16), respectively. Due to the fact that most of the upper part of the surface layer is not cracked, the moduli of the three reflection coefficients are very low and almost negligible compared with the other results. The transmission coefficient modulus of the fourth crack is greater than 0.983 for \( \bar{\omega} > 1.5 \). Thus, for high frequencies most of the incident Love wave is transmitted through the large uncracked upper part of the layer, and almost nothing is reflected. Therefore, for \( \bar{\omega} > 1.5 \) the transmitted waves are dominant, and the phase \( \varphi (T_1) \), as shown by Figure (3.32), is completely in phase with that of the incident Love wave.
Figure 3.3: Bounded part of the dislocation density real values for different values of $n$; $\bar{\omega} = 1.9$, $s'/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $d = 0.6667$. 
Figure 3.4: Bounded part of the dislocation density real values for different values of $n$; $\bar{\omega} = 2.6$, $s_T'/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.5: Bounded part of the dislocation density imaginary values for different values of $n$; $\tilde{\omega} = 1.9$, $s'_T/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.6: Bounded part of the dislocation density imaginary values for different values of $n$; $\bar{\omega} = 2.6$, $s'_T/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.7: Position of the first and second interface-breaking cracks.
Figure 3.8: Position of the third and fourth interface-breaking cracks.
Figure 3.9: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.5$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5005$. 
Figure 3.10: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.9$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5009$. 
Figure 3.11: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.12: Modulus of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\hat{h} = 2.0$, $\mu'/\mu = 1.75$, $\tilde{d} = 2.25$. 
Figure 3.13: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 0.5$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5005$. 
Figure 3.14: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8$, $\tilde{h} = 0.9$, $\mu'/\mu = 1.75$, $\tilde{d} = 0.5009$. 
Figure 3.15: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8, \tilde{h} = 0.6667, \mu' / \mu = 1.75, \tilde{d} = 0.6667$. 
Figure 3.16: Modulus of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\hat{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 2.25$. 
Figure 3.17: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T/s_T = 0.8$, $\overline{h} = 0.5$, $\mu'/\mu = 1.75$, $\overline{d} = 0.5005$. 
Figure 3.18: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'/s_T = 0.8$, $\bar{n} = 0.9$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5009$. 
Figure 3.19: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.20: Modulus of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T/s_T = 0.8$, $h = 2.0$, $\mu'/\mu = 1.75$, $\tilde{d} = 2.25$. 
Figure 3.21: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 0.5$, $\mu' / \mu = 1.75$, $\bar{d} = 0.5005$. 
Figure 3.22: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s_T'/s_T = 0.8$, $\bar{h} = 0.9$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5009$. 
Figure 3.23: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s_T^\prime/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu^\prime/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.24: Modulus of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\tilde{n} = 2.0$, $\mu' / \mu = 1.75$, $\tilde{d} = 2.25$. 
Figure 3.25: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.5$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5005$. 
Figure 3.26: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 0.9$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5009$. 
Figure 3.27: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{n} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.28: Phase of coefficient $R_1$ for the reflected Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 2.25$. 
Figure 3.29: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 0.5$, $\mu' / \mu = 1.75$, $\bar{d} = 0.5005$. 
Figure 3.30: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.9$, $\mu'/\mu = 1.75$, $\bar{d} = 0.5009$. 
Figure 3.31: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T/s_T = 0.8$, $\bar{h} = 0.6667$, $\mu'/\mu = 1.75$, $\bar{d} = 0.6667$. 

Figure 3.32: Phase of coefficient $T_1$ for the transmitted Love wave of order one versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 2.0$, $\mu' / \mu = 1.75$, $\bar{d} = 2.25$. 
Figure 3.33: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'/s_T = 0.8$, $\tilde{h} = 0.5$, $\mu'/\mu = 1.75$, $\tilde{d} = 0.5005$. 
Figure 3.34: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s_T'/s_T = 0.8$, $\bar{h} = 0.9$, $\mu'/\mu = 1.75$, $\bar{d} = 0.509$. 
Figure 3.35: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'_T / s_T = 0.8, \, \bar{h} = 0.6667, \, \mu'/\mu = 1.75, \, \bar{d} = 0.6667.$
Figure 3.36: Phase of coefficient $R_2$ for the reflected Love wave of order two versus dimensionless frequency; $s'/s_T = 0.8$, $\tilde{h} = 2.0$, $\mu'/\mu = 1.75$, $\tilde{d} = 2.25$. 
Figure 3.37: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 0.5$, $\mu' / \mu = 1.75$, $\bar{d} = 0.5005$. 
Figure 3.38: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $h = 0.9$, $\mu'/\mu = 1.75$, $\bar{a} = 0.5009$. 
Figure 3.39: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\bar{h} = 0.6667$, $\mu' / \mu = 1.75$, $\bar{d} = 0.6667$. 
Figure 3.40: Phase of coefficient $R_3$ for the reflected Love wave of order three versus dimensionless frequency; $s'_T / s_T = 0.8$, $\tilde{h} = 2.0$, $\mu'/\mu = 1.75$, $\bar{d} = 2.25$. 
References


Chapter 4

Conclusion and Further Work

4.1 Conclusion

The conclusions of this work can be summarized as follows

1. The real and imaginary bounded parts of the dislocation density function across the crack faces converge very rapidly as the number $n$ of Gaussian quadrature points increases. The values of $n = 15$ for a layer-embedded crack and $n = 31$ for an interface-breaking crack yield convergent numerical results.

2. The ratio of the slowness of transverse waves in the layer to that in the half-space and the ratio of the shear modulus of the layer to that of the half-space correspond approximately to those of the uniform and single layered earth model of Neerhoff [1].

3. The transmission field for the layer-breaking and the interface-breaking cracks is dominant for frequencies with near zero values. This is a direct consequence of the vanishing of the incident-wave stress at zero frequency.

4. To compare our results with those of the surface-breaking cracks reported by Angel [2] and [3], we consider embedded cracks of the same length, and let the
upper tips of the embedded cracks lie at a very close distance from the free surface. Good agreement is observed.

5. The constructive and destructive interference phenomena that might cause the sharp dips and peaks in the surface-breaking cracks do not seem to occur in the layer-embedded cracks. This is because there is always some solid material between the upper tip and the free surface that allows part of the energy to transmit. In the case of a surface-breaking crack, the energy travels down the crack face.

6. The position of the layer-embedded crack and its size are considered to be the major factors affecting the reflection of incident Love waves. However, for an embedded crack breaking through the interface, the effect of the position of the upper crack tip on the reflection of the incident wave is considered to be much higher than that of the crack depth.

7. The amount of transmission of the incident Love wave is directly related to the distance between the free surface and the upper crack tip. For cracks that occupy almost the entire layer, a negligible amount of the incident Love wave will be transmitted through the small interval between the upper tip and the free surface. These cracks act nearly as perfect reflectors of the incident Love wave, and the reflection increases as the depth of the crack increases. Thus, for high frequency the reflected waves are dominant and the phase of the reflection coefficient of first order is completely in opposite phase with respect to the incident Love wave.

8. For high frequency, when the upper tip of an interface-breaking crack is coming close to the interface of the two solids, the transmitted waves are dominant and the phase of the transmission coefficient is closer to that of the incident Love
wave. Thus, most of the incident Love wave is transmitted through the large uncracked upper part of the layer, and almost nothing is reflected.

9. The discontinuity of the dislocation density function at the interface is found to be related to the ratio of the shear moduli of the two solids. Two equations of the approximated linear system are eliminated and replaced by the constraint of the discontinuity condition and that of the vanishing integral of the dislocation density function between the crack tips. These two eliminated equations correspond to the closest positions of the Gaussian quadrature points to the upper and lower crack tips.

4.2 Further Work

Various interesting areas of research related to the subject of this dissertation are recommended in this section as follows

1. The scattering of Love waves in a layered elastic solid by a layer-embedded crack parallel to the free surface or oriented in any direction with respect to the free surface.

2. The scattering of Love waves in a layered elastic solid by a layer-embedded or interface penny-shape crack.

3. The scattering of Love waves in a multi-layered elastic solid by a crack breaking through two or more interfaces.

4. The three dimensional scattering of obliquely-incident Love waves in a layered elastic solid by a crack normal to the free surface.
5. The scattering of in-plane surface waves (Rayleigh waves) in a layered elastic solid by a crack normal to the free surface.

The most interesting area of research that is directly related to the work of this dissertation and to the work of Angel [2, 3] is to construct the basic background to solve the inverse problem of detecting surface or sub-surface cracks in layered elastic solids and determining their position, size, and orientation.
References


Appendix A

A.1

In the interval \([\epsilon, 1]\) one has (using definition (3.2))

\[
\beta'(u) = -i(1 - u^2)^{1/2} = -i\alpha(u), \quad \beta''(u) = (u^2 - \epsilon^2)^{1/2} = \alpha'(u).
\]

(A.1)

Using (A.1), the functions \(\beta_{1b}(u, \nu), \bar{\alpha}_{1b}(u, \nu), \bar{\Delta}_b(u),\) and \(\bar{\Delta}'(\eta_1)\) in equations (3.35a, b) are defined by

\[
\beta_{1b}(u, \nu) = 2m \frac{u}{\alpha'(u)} H \left( \nu - \hat{h} \right) \left[ 1 - P_3(\nu, u) \right] + \frac{iu}{\alpha(u)} \left[ P_2(\nu, u) - P_1(\nu, u) \right]
\]
\[
+ \frac{m u \alpha'(u)}{\alpha^2(u)} \left[ 1 - P_1(\nu, u) - \text{sgn} \left( \hat{h} - \nu \right) \left[ 1 - P_2(\nu, u) \right] \right], \quad (A.2)
\]

\[
\bar{\alpha}_{1b}(u, \nu) = \frac{u}{\alpha(u)} \sin(\tilde{s}_\tau \hat{h} \alpha(u)) \left[ 1 - P_1(\nu, u) - \text{sgn} \left( \hat{h} - \nu \right) \left[ 1 - P_2(\nu, u) \right] \right]
\]
\[
+ \frac{u \alpha(u)}{\alpha'^2(u)} \sin(\tilde{s}_\tau \hat{h} \alpha(u)) H \left( \nu - \hat{h} \right) \left[ 2 - P_3(\nu, u) - P_4(\nu, u) \right]
\]
\[
+ \frac{iu}{\alpha(u)} \cos(\tilde{s}_\tau \hat{h} \alpha(u)) H \left( \nu - \hat{h} \right) \left[ P_3(\nu, u) - P_4(\nu, u) \right], \quad (A.3)
\]

\[
\bar{\Delta}_b(u) = m \alpha'(u) \cos(\tilde{s}_\tau \hat{h} \alpha(u)) - \alpha(u) \sin(\tilde{s}_\tau \hat{h} \alpha(u)),
\]

(1.4)

\[
\bar{\Delta}'(\eta_1) = \frac{\eta_1 \cos(\tilde{s}_\tau \hat{h} \alpha(\eta_1))}{\alpha(\eta_1) \alpha'(\eta_1)} \Delta_l. \quad (A.5)
\]

In (A.2)-(A.5), the functions \(P_3(\nu, u)\) and \(P_4(\nu, u)\) are given by equations (3.34). The parameter \(\Delta_l\), and the functions \(\bar{P}_1(\nu, u)\) and \(\bar{P}_2(\nu, u)\) can be written in the form

\[
\Delta_l = m(1 - \epsilon^2) + \tilde{s}_\tau \hat{h} \alpha'(\eta_1) \left[ 1 - (m c)^2 + (m \eta_1)^2 - \eta_1^2 \right], \quad (A.6)
\]

\[
\bar{P}_1(\nu, u) = \exp \left[ i \tilde{s}_\tau \left( \hat{h} + \nu \right) \alpha(u) \right], \quad \bar{P}_2(\nu, u) = \exp \left[ i \tilde{s}_\tau \left( \hat{h} - \nu \right) \alpha(u) \right]. \quad (A.7)
\]
A.2

In the interval $[0, \varepsilon]$, one has (using definition (3.2))
\[
\bar{\beta}(u) = -i(1 - u^2)^{1/2} = -iu(u), \quad \bar{\beta}'(u) = -i(\varepsilon^2 - u^2)^{1/2} = -i\tilde{\alpha}'(u). \quad (A.8)
\]

Using (A.9) and changing the integration variables, the portions of the integrals of equations (3.31e, f) over the interval $[0, \varepsilon]$ may be expressed in the form
\[
\bar{K}'_a(\nu, \tau) = s_\tau \int_0^\varepsilon u \frac{\bar{P}_1(\bar{\tau}, u) + \bar{P}_2(\bar{\tau}, u)}{\bar{\Delta}_a(u)} \bar{B}_{1a}(u, \nu) du, \quad -1 < \tau < \bar{h}, \quad (A.9)
\]
\[
\bar{K}_a(\nu, \tau) = 2s_\tau \int_0^\nu \frac{v}{\bar{\nu}} \frac{\bar{P}_4(\bar{\tau}, \bar{\nu})}{\bar{\Delta}_a(\bar{\nu})} \left[ v \bar{A}_{1a}(\bar{\nu}, \nu) \right] dv, \quad \bar{h} < \tau < 1, \quad (A.10)
\]

where the parameters in the above equations are
\[
\bar{\tau} = \frac{\tau}{2} + \bar{d}, \quad \bar{\nu} = \frac{\nu}{2} + \bar{d}, \quad \bar{\nu} = \varepsilon - \nu^2, \quad \bar{h} = 2 \left( \bar{h} - \bar{d} \right). \quad (A.11)
\]

The functions $\bar{B}_{1a}(u, \nu), \bar{A}_{1a}(u, \nu)$ and $\bar{\Delta}_a(u)$ in equations (A.9) and (A.10) have the following form

\[
\bar{B}_{1a}(u, \nu) = 2m \frac{u}{\bar{\alpha}(u)} \bar{H} \left( \bar{\nu} - \bar{h} \right) \left[ 1 - \bar{P}_4(\bar{\nu}, u) \right] + \frac{u}{\alpha(u)} \left[ \bar{P}_2(\bar{\nu}, u) - \bar{P}_1(\bar{\nu}, u) \right] \\
- m \frac{u}{\bar{\alpha}(u)} \left[ 1 - \bar{P}_1(\bar{\nu}, u) - \text{sgn} \left( \bar{h} - \bar{\nu} \right) \left[ 1 - \bar{P}_2(\bar{\nu}, u) \right] \right], \quad (A.12)
\]
\[
\bar{A}_{1a}(u, \nu) = -m \frac{u}{\bar{\alpha}(u)} \left( \bar{P}_1(\bar{h}, u) + 1 \right) \bar{H} \left( \bar{\nu} - \bar{h} \right) \left[ \bar{P}_4(\bar{\nu}, u) - \bar{P}_3(\bar{\nu}, u) \right] \\
- \frac{u}{\alpha(u)} \left( \bar{P}_1(\bar{h}, u) - 1 \right) \left[ 1 - \bar{P}_1(\bar{\nu}, u) - \text{sgn} \left( \bar{h} - \bar{\nu} \right) \left[ 1 - \bar{P}_2(\bar{\nu}, u) \right] \right] + \frac{u}{\alpha(u)} \left( \bar{P}_1(\bar{h}, u) + 1 \right) \left[ \bar{P}_2(\bar{\nu}, u) - \bar{P}_1(\bar{\nu}, u) \right], \quad (A.13)
\]
\[
\bar{\Delta}_a(u) = \alpha(u) \left( \bar{P}_1(\bar{h}, u) - 1 \right) - m \bar{\alpha'}(u) \left( \bar{P}_1(\bar{h}, u) + 1 \right). \quad (A.14)
\]
In equations (A.9)-(A.14), the functions \( \bar{P}_1(\bar{\nu}, u) \) and \( \bar{P}_2(\bar{\nu}, u) \) are given by equation (A.7), and the functions \( \bar{P}_3(\bar{\nu}, u) \) and \( \bar{P}_4(\bar{\nu}, u) \) are given by

\[
\bar{P}_3(\bar{\nu}, u) = \exp \left[ i \bar{\kappa} \left( \bar{h} + \bar{\nu} \right) \bar{\alpha}'(u) \right], \quad \bar{P}_4(\bar{\nu}, u) = \exp \left[ i \bar{\kappa} \left( \bar{h} - \bar{\nu} \right) \bar{\alpha}'(u) \right]. \tag{A.15}
\]

### A.3

The last portions of the integrals defining the functions \( \bar{K}'(\nu, \tau) \) and \( \bar{K}(\nu, \tau) \) of equations (3.31e, f), which are evaluated over the interval \([1, \infty)\), can be written in the form

\[
\bar{K}_c'(\nu, \tau) = 2 \bar{\kappa} \int_0^\infty \bar{\nu} \frac{P_1(\bar{\tau}, \bar{\nu}) + P_2(\bar{\tau}, \bar{\nu})}{\Delta(\bar{\nu})} \left( \bar{\nu} \bar{B}_1(\bar{\nu}, \nu) \right) d\bar{\nu}, \quad -1 < \tau < \bar{h}, \tag{A.16}
\]

\[
\bar{K}_c(\nu, \tau) = \bar{\kappa} \int_1^{\bar{h}} u \frac{P_3(\bar{\tau}, u)}{\Delta(u)} \bar{A}_1(u, \nu) du, \quad \bar{h} < \tau < 1, \tag{A.17}
\]

where the functions \( \bar{A}_1(u, \nu), \bar{B}_1(u, \nu) \) and \( \Delta(u) \) are given by equations (3.33a, b, c), and the functions \( P_1(\bar{\tau}, u), P_2(\bar{\tau}, u), \) and \( P_4(\bar{\tau}, u) \), are given by equation (3.34). The parameter \( \bar{\nu} \) is defined by

\[
\bar{\nu} = \nu^2 + 1. \tag{A.18}
\]

Notice that the integration variable of equation (A.16) which defines the function \( \bar{K}_c'(\nu, \tau) \) has been changed.