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RICE UNIVERSITY

Representation and Synthesis of Random Fields: ARMA, Galerkin, and Wavelet Procedures

by

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A Thesis Submitted in Partial Fulfillment of the Requirement for the Degree Doctor of Philosophy

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Abstract

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The dissertation considers methods of representation and synthesis of random fields and examines variance reduction techniques in conjunction with reliability analysis of engineering systems. The dissertation presents new approaches to the scale type method, the ARMA method, and the covariance method for random field simulation.

The scale type method is formulated by using the wavelet representation of random fields. In this regard it is shown that a large class of random fields is amenable to a simplified representation.

Also this dissertation presents a new efficient two-stage procedure for ARMA approximation of target stochastic processes. It is shown that this method yields quite low order ARMA models and reduces the requisite numerical computations for synthesizing samples of stochastic processes.

Criteria are established for efficient representation of random fields by a small number of random variables in conjunction with the covariance method of random field simulation.

A variance reduction method is developed by extending the Galerkin projection to stochastic mechanics problems. This method improves the Monte Carlo based reliability analysis of engineering systems with random properties.
The methods developed in this dissertation aim to expedite the application of stochastic mechanics concepts to design applications.
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Special thanks are expressed to my research advisor and thesis committee chairman, Dr. "Pol" D. Spanos, whose visionary instruction, broad-minded guidance, and friendly encouragement have led to the completion of this thesis; his teaching emphasizes self-motivation and creativity.

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To my mother, Frieda Aronovna Kachko
Chapter 1. Introduction

1.1 Motivation

The development of comprehensive mathematical models describing randomness is a useful part of science and engineering research.

Historically, the first physical models and theories were deterministic. This point of view of the first scientists was, to a great extent, supported by the capability of mechanicians to predict and describe the behavior of, sometimes, quite complicated systems. Simple extension of this view to the micro- and macro-world was the weakness of these philosophical concepts. At the end the nineteen century diverse scientific investigations questioned the foundation of human understanding of the nature. A new theory describing physical reality as a random world of probabilities immersed from the ruins of old physical theories. Only probabilities of events can be estimated, and yet, this theory exists in unison with the intuitive deterministic perception of the human brain. For instance, the uncertainty principle of quantum mechanics clearly indicates the inadmissibility of the deterministic models.

However, a more important source of uncertainties in engineering applications is related to the general lack of knowledge about systems and processes. Thus, civil engineering systems can be adequately interpreted as deterministic from the quantum mechanics point of view. Nevertheless, the complete description of civil engineering structures may require to obtain and process an enormous amount of information; this will always bring the probabilistic aspect into engineering design. For example, the mechanical properties of a concrete beam can be represented as a realization of a random field without collecting all information about the components of this composite material.
In spite of the importance of probabilistic phenomena, engineering design codes remain quasi-deterministic. The use of the safety and load factors in conjunction with deterministic analysis accounts for the effect of variation about the mean value of the design parameters. In this manner, probabilistic aspects of systems are estimated indirectly with a high level of approximations. However, only a complete and rigorous probabilistic approach can provide reliable results for problems where the level of uncertainty is high or a high level of precision is needed. Nowadays, probabilistic analysis and design incorporating elements of stochastic mechanics and random vibration are becoming increasingly popular for offshore, earthquake, environmental, aerospace, marine, and energy engineering. A serious drawback of probabilistic design is the significant increase in the requisite computational resources in comparison with the conventional deterministic analysis. However, the advent of computational systems has accelerated the application of probabilistic analysis and design methods.

Probabilistic design requires estimating the probability of failure of engineering systems. This probability ought to satisfy some criteria determined by a design engineer based on requirements of safety and functionality. Schematically, the determination of the probability of failure can be divided into two steps. First, the sources of uncertainties are identified and their mathematical models are described. Second, based on this mathematical modeling the probability for a system to be in the unsafe or technically unacceptable region is estimated. Note that the probability of failure must be quite small; this fact amplifies the problems associated with the estimation of tails of probability distributions. In this regard some methods of stochastic mechanics and the theory of random vibration can provide, based on the corresponding quantitative description of the system and load uncertainties, reliable estimates of some moments, or often even probability distribution functions, of the response of engineering systems.

The most straightforward and relatively easy to implement method of reliability is the
method of repeated simulations or the Monte Carlo method. The term “Monte Carlo” was introduced by Neumann and Ulman during the second World War for a classified project related to the development of the atomic bomb. This name contains an obvious reference to the famous gambling casinos of the city of Monte Carlo in Monaco. However, as early as 1908 the famous statistician Student used the Monte Carlo method to find probability distributions. The Monte Carlo simulation is a statistical sampling experiment; it includes sampling stochastic variates from specified probability distributions. Recently, the Monte Carlo method has been applied for solving complicated deterministic problems such as evaluating complex multidimensional integrals and solving system of differential or integral equations (Rubinstein 1981).

1.2 Objectives

This dissertation intends to further develop algorithms and methods of the theory of reliability and stochastic mechanics. Two problems are discussed in this regard. First, methods for random field simulation are considered and new synthesis algorithms are developed. Second, variance reduction techniques are examined for stochastic mechanics problems.

The development of reliable algorithms for synthesizing multi-dimensional random fields occupies a significant portion of the presented research effort. A wavelet based algorithm for random field simulation is proposed. It is shown that a large class of random fields is amenable to a simplified representation in this case. This approach lends itself, readily, into the theory of linear scale dynamic systems. Note that the expansion of stochastic fields in terms of the Haar basis yields the Local Average Subdivision method for random field simulation. The proposed method is equally well suited for synthesizing stationary and non-stationary random fields. Clearly, the techniques proposed in this study
can be used to advance wavelet based computational methods for civil, electrical, mechanical and environmental applications.

The auto-regressive moving average (ARMA) method is a well known and commonly used method for random field synthesis. However, the currently used techniques of ARMA modeling do not provide for an optimal, from the engineering standpoint, approximation of target spectra. This dissertation introduces an efficient rectification of the two-stage ARMA modeling procedures. It is shown that a few addition numerical operations yield a low-order ARMA model which ensures the numerical expediency of simulating stochastic processes.

The proposed frequency domain analysis of the Loeve-Karhunen expansion of a random field discusses some of the conditions under which the random field can be efficiently approximated by a small number of random variables. The introduced representation of random fields by using a few variables motivates a variance reduction method for the Monte Carlo simulation.

Galerkin procedures in dealing with random processes and systems have become increasingly popular. This dissertation shows that the stratified sampling method of reliability is equivalent to the Galerkin projection with a very special choice of basis functions. Moreover, the orthogonality of the Galerkin projection establishes a certain optimality with respect to the generalized sampling procedure for solving stochastic mechanics problems. The proposed approach can improve Monte Carlo based reliability analyses of a broad class of random systems.

1.3 Thesis Organization

There are nine chapters in the dissertation, including the introduction, summary, and
two appendices. Chapter 1 establishes the objectives of the undertaken research. It also provides mathematical concepts which are used in subsequent chapters. Though this dissertation is not self contained in terms of mathematical notation and requires reader’s familiarity with the theory of random vibrations and stochastic mechanics, it establishes a common background for the requisite mathematical formalism. The wavelet theory is a relatively new development of applied mathematics, and wavelets may not be familiar to the probabilistic engineering community. Therefore, this dissertation contains a comprehensive review of the wavelet theory aiming at popularizing its application for reliability analysis of random processes and signals.

The Monte Carlo method is introduced in Chapter 2 and several methods for synthesizing homogeneous random fields are discussed. This chapter establishes the theoretical foundation for every method discussed in the dissertation and estimates its computational efficiency. In this regard, several new theoretical and computational aspects which, to the author’s knowledge, have not been reported in the literature are considered. This chapter shows the necessity for further advances in the development of efficient numerical procedures for random field simulation.

Chapters 3, 4, and 5 develop theoretical and numerical aspects of the representation and synthesis of random fields by using wavelet bases. The local and multi-scale structure of wavelets make them a pertinent part of several efficient computing systems. Similarly, these properties are utilized in Chapter 3 to develop procedures for describing covariance matrices of the wavelet coefficients. The structure of these matrices is also analyzed and it is shown that a large class of random fields is amenable to a simplified representation. This analysis motivates several simple parametric models of stochastic processes which are developed in Chapter 4. These parametric models provide efficient synthesis schemes for simulating stochastic processes. The efficiency and the error of the introduced synthesis procedures are estimated. It is shown that the proposed method compares quite favorably,
in terms of the efficiency and the approximation ability, with previously developed methods for random field synthesis. Further, Chapter 5 generalizes the findings of the previous two chapters for the multi-dimensional case.

A new method for a two stage ARMA modeling is introduced in Chapter 6 in conjunction with synthesis of multi-variate stationary stochastic processes. It is shown that the developed method approximates spectral characteristics quite well by utilizing fewer model parameters than existing techniques.

The well known covariance method of random field simulation is examined in Chapter 7. It is shown that this method is quite efficient from the computational standpoint if the corresponding covariance matrix has just a few significant eigenvalues. Several criteria are established in this regard. The scale of fluctuation and the rate of decay of the spectral density function are shown to affect significantly the computational efficiency of this method. Based on these findings the requisite mesh size of discretization in conjunction with the stochastic finite element method is estimated.

Chapter 8 develops a numerical method for solving stochastic mechanics problems by representing the solution using a small number of random parameters as it is discussed in Chapter 7. In essence, the method is a Galerkin approximation in the sample space. The associated projection of the solution into the space of simple random variables reduces the stochastic mechanic problem to a set of deterministic problems. Alternatively, this method can be viewed as a modified - for computational efficiency - stratified sampling method, a well known variance reduction procedure. Several examples which deal with the determination of the natural frequencies and of the seismic response of a beam with random rigidity are considered.

Finally, Chapter 9 summarizes the findings of this dissertation in a concise form and reviews the developed computational methods.
1.4 Mathematical Models and Notations

Stochastic mechanics problems, which are discussed in this dissertation, are not of the conventional engineering nature. This study involves rather abstract mathematical concepts and notions. Nevertheless, the degree of abstraction and complexity are kept to a necessary minimum to avoid obscuring the engineering aspects of stochastic problems. Material of this section is given without proofs or exhaustive mathematical rigor. It should be considered, mainly, as a reference to some notions of measurable, Banach, and Hilbert space language. A complete and rigorous development of these mathematical concepts can be found in Courant and Hilbert (1953), Kolmogorov and Fomin (1961), Wong (1971), and Pfeiffer (1990).

1.4.1 Preliminaries and Calculus

The concept of the measurable space is basic to the definition of the Lebesgue integral and probability. The measurable space (Kolmogorov and Fomin 1961, Pfeiffer 1990) is a triple \((\Omega, G, \mu)\), where \(\Omega\) denotes the sample space of events, \(G\) is a sigma algebra of sets in \(\Omega\), and \(\mu\) is a measure, or a positive sigma-additive function defined on the elements of \(G\). The Lebesgue integral is associated with the operator of mathematical expectation which quantifies probability analysis with quantitative descriptions. The Lebesgue integral can be defined for measurable spaces where any alternative definitions of the integral, for example the Riemann integral, do not make sense. The Lebesgue integral is, defined, first for a simple function \(f = \sum t_i I_{(A_i)}\), where \(I_A\) is the "set indicator" function, \(I_{(A)}(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \not\in A \end{cases}\). Specifically

\[
\int f d\mu = \sum_{i} t_i \mu(A_i) .
\]

(1.1)
For a general measurable function $f$ the Lebesgue integral is defined as the limit of integrals of simple functions $f_n$ such that $f_n \to f$ uniformly. That is,

$$\int_\Omega f \, d\mu = \lim_{n \to \infty} \int_\Omega f_n \, d\mu.$$  \hspace{1cm} (1.2)

If $\mu$ is the Borel measure defined on the sigma-algebra generated by the open intervals on $\mathbb{R}$, then the Lebesgue integral is commonly denoted as $\int_\Omega f \, dx$.

When $f$ is a measurable function of several variables, $\int |f| \, dx dy < \infty$, Fubini’s theorem simplifies the evaluation of multi-dimensional integrals.

$$\int f \, dx dy = \int (\int f \, dx) \, dy = \int (\int f \, dy) \, dx.$$ \hspace{1cm} (1.3)

The theory of the Banach and Hilbert spaces is important for engineering applications. A complete linear space with a norm is called the Banach space. A set $L'$ of the Banach space $L$, $L' \subseteq L$, is called a subspace if it constitutes the Banach space itself. A set of functions, such that $\|f\|_{L^1} = \int_{\mathbb{R}} |f| \, dx < \infty$ is a paradigm of the Banach space. This Banach space is denoted as $L^1(\mathbb{R})$. A system of linear independent elements generating the Banach space is called a (Hammel) basis. However, the Banach space is more useful for quantitative analysis if it contains a countable, everywhere dense, set. Any element of such Banach space, also called separable, can be represented as the limit of a sequence of elements of this countable set.

The separable Banach space is called the Hilbert space if its norm is generated by the scalar product, $\|f\|^2 = \langle f, f \rangle$. The space $L^2(\mathbb{R})$, is a commonly used example of the Hilbert space. In this case, the scalar product is defined as

$$\langle f, g \rangle = \int_{\mathbb{R}} f \overline{g} \, dx.$$ \hspace{1cm} (1.4)
Also, the notion of an angle between two elements of the Hilbert space can be generated by the scalar product. In particular, elements \( f, g \in H \) are orthogonal if \( \langle f, g \rangle = 0 \). The Hilbert space always has a countable orthogonal basis.

Let \( M \) be a closed subspace of the Hilbert space \( H \). Any element \( f \in H \) can be uniquely represented as a sum \( f = h + h' \), where \( h \in M \) and \( h' \in M^\perp \perp M \). Then, \( M^\perp \) is a subspace orthogonal to \( M \). That is, any element of \( M^\perp \) is orthogonal to any element of \( M \). In this case, \( H \) is called the “direct sum” of two subspaces \( M \) and \( M^\perp \) denoted as

\[
H = M \oplus M^\perp. \tag{1.5}
\]

The following notation for the Fourier transform (for the one-dimensional case) is used in this dissertation

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx. \tag{1.6}
\]

With this normalization, the inverse Fourier transform is given by

\[
f(x) = \hat{f}(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi, \tag{1.7}
\]

and

\[
\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \tag{1.8}
\]

where the scalar product is defined as in the \( L^2(R) \) case.

Now, some important functional spaces, which are used to establish some properties of
wavelets are listed:

The symbol \( C([a, b]) \) denotes the Banach space of continuous functions with the norm defined as \( \|f\| = \max_{x \in [a, b]} |f(x)| \).

The symbol \( C^n([a, b]) \), \( n \) is integer, denotes the Banach space of \( n \) times continuously differentiable functions with the norm defined as \( \|f\| = \sum_{k=0}^{n} \max_{x \in [a, b]} |f^{(k)}(x)| \).

The symbol \( L^p([a, b]) \), \( 1 \leq p < \infty \) denotes the Banach space of functions which are completed with respect to the norm \( \|f\|_p = \left( \int_a^b |f|^p \, dx \right)^{1/p} \). As \( p \to \infty \), this norm tends to \( \|f\|_\infty = \text{ess sup}_{[a, b]} |f| \). The corresponding space is denoted as \( L^\infty([a, b]) \). Note that \( p = 2 \) corresponds to the Hilbert space \( L^2([a, b]) \).

The Sobolev space \( W^{n,p}([a, b]) \) is the Banach space of functions, completed with respect to the norm \( \|f\| = \left( \sum_{k=0}^{n} \int_a^b |f^{(k)}|^p \, dx \right)^{1/p} \). For \( p=2 \), this space is the Hilbert space with a scalar product defined as \( \langle f, g \rangle = \left( \sum_{k=0}^{n} \int_a^b f^{(k)} g^{(k)} \, dx \right) \).

The space \( C^n([a, b]) \) can be generalized for the non-integer index \( n \). This generalization is called the Holder space \( C^s([a, b]) \), \( s \) is real. For \( 0 < s < 1 \), the Holder space is defined as

\[
C^s([a, b]) = \left\{ f \in L^\infty; \sup_{h, x \in [a, b]} \frac{|f(x + h) - f(s)|}{h^s} < \infty \right\}.
\]

For \( s = n + s' \), \( n \) is integer and \( 0 < s' < 1 \), the Holder space is defined as

\[
C^s([a, b]) = \left\{ f \in C^n; \frac{d^n f}{dx^n} \in C^{s'} \right\}.
\]

An efficient and handy apparatus is needed to fructify the quantitative mathematical analysis by using the structure of the Banach and Hilbert spaces. Note that the Hilbert space \( H \) is separable and, therefore, contains a dense, countable set of elements. This fact simplifies the description of the elements of the space \( H \) and provides an efficient analysis.
tool: the analysis of elements of $H$ is reduced to the analysis of an ordered countable set. The problem can be simplified further, if upon careful examination, one can extract a countable system of basic elements \( \{ \psi_j, \ j = 1, \ldots \} \) which span the entire Hilbert (Banach) space. However, this system of basic elements has to satisfy additional requirements to be a useful vehicle of mathematical and engineering analysis. Namely, it must provide an efficient and stable mechanism of representing and reconstructing elements of $H$. Elements of the Hilbert space $H$ can be "characterized" by coefficients $c_j = \langle f, \psi_j \rangle$ if $\langle f_1, \psi_j \rangle = \langle f_2, \psi_j \rangle$ for all $j$ implies $f_1 \equiv f_2$. This condition can be equivalently reformulated as $\langle f, \psi_j \rangle = 0$ for all $j \Rightarrow f \equiv 0$. It is implicitly assumed that the norm of the sequence $\{ c_j, \ j = 1, \ldots \}$ is small whenever the norm of $f$ is small

$$
\sum_{j=1}^{\infty} |c_j|^2 = \sum_{j=1}^{\infty} |\langle f, \psi_j \rangle|^2 \leq B\|f\|^2, \quad (1.9)
$$

where $B$ is a constant. However, functions $\psi_j$ must provide more than just the ability to characterize; the element $f$ must be reconstructed in a numerically stable way from the coefficients $c_j$. Stable reconstruction means that if the norm of $\{ c_j, \ j = 1, \ldots \}$ is small, then $\|f\|^2$ must be also small. In particular, a constant $\alpha < \infty$ can be found so that

$$
\sum_{j=1}^{\infty} |\langle f, \psi_j \rangle|^2 \leq 1 \text{ implies } \|f\|^2 \leq \alpha. \quad (1.10)
$$

This condition is rearranged as

$$
A\|f\|^2 \leq \sum_{j=1}^{\infty} |\langle f, \psi_j \rangle|^2, \quad (1.10)
$$

where $A = 1/\alpha$. A family of functions $\{ \psi_j, \ j = 1, \ldots \}$ in the Hilbert space $H$ is called a frame if for all $f$ in $H$ there exist $0 < A, B < \infty$ satisfying Eqs. 1.9 and 1.10. The constants $A$ and $B$ are called the frame bounds. If the two frame bounds are equal, $A = B$, the frame is called tight.
Frames were introduced by Duffin and Schaeffer (1952) in the context of non-harmonic analysis. They are now used extensively for signal and image processing. For example, a frame can be generated by oversampling bandlimited functions. Note that elements of a frame are not linear independent and provide a redundant analysis. However, this redundancy can be quite useful; it increases algorithmic stability and reduces errors. Recently, a new class of frames generated by "wavelet atoms" has been introduced.

Next, a frame can be even more efficient if it is not redundant. Such frame is called a Riesz, or unconditional, basis. In this case the functions \( \{ \psi_j, \ j = 1, \ldots \} \) are linear independent and satisfy Eqs. 1.9 and 1.10. Note that if \( \{ \psi_j, \ j = 1, \ldots \} \) is a tight frame with the frame bound \( A = B = 1 \) and \( \|\psi_j\| = 1 \) for all \( j \), then \( \{ \psi_j, \ j = 1, \ldots \} \) constitute an orthogonal basis.

1.4.2 Elements of the Theory of Probability and Random Processes

The theory of probability is developed based on the concepts of the theory of measurable spaces, \( (\Omega, G, \mu) \). The measure \( \mu \) is called the probability measure if \( \mu(\Omega) = 1 \). The measurable function \( \eta \) defined on \( \Omega \) is called a random variable. Then, the random variable \( \eta \) generates another probability measure \( P \) on the Borel sigma-algebra of \( \mathbb{R} \) such that \( P([\neg \infty, \infty]) = 1 \). The joint probability function of the random vector \( \eta = (\eta_1, \eta_2, \ldots, \eta_n)' \) is defined as

\[
F_\eta(x) = P(\eta(\omega) \leq x), \ \omega \in \Omega, \quad (1.11)
\]

where \( \eta \leq x \) means that each element of the vector \( \eta \) is not greater than the corresponding element of \( x \). The joint density function is associated with the distribution function and is defined as
\[ f_{\eta}(x) = \frac{\partial^n F_{\eta}(x)}{\partial x_1 \partial x_2 \ldots \partial x_n}, \]  \hspace{1cm} (1.12)

if this derivative is defined at least in the generalized sense. The components of the random vector \( \eta \) are called statistically independent if

\[ F_{\eta}(x) = F_{\eta_1}(x_1)F_{\eta_2}(x_2)\ldots F_{\eta_n}(x_n) \]  \hspace{1cm} (1.13)

Two random variables \( \eta_1 \) and \( \eta_2 \) are said to be almost surely equal if \( P(\eta_1(\omega) \neq \eta_2(\omega)) = 0 \).

The operator of mathematical expectation \( E[\ ] \) is defined as the Lebesgue integral with measure \( \mu \)

\[ E[g(\eta)] = \int g(\eta(\omega))d\mu(\omega) = \int g(x)dF(x) \]  \hspace{1cm} (1.14)

Often the probability distribution functions of random variables are not available or it is difficult to evaluate them reliably. However, they can be characterized by corresponding probabilistic moments. Selecting \( g(x) = x \) in Eq. 1.14, yields the mean value \( \mu \) of the random vector \( \eta \). The covariance matrix is defined as

\[ V = E[(\eta - \mu)(\eta - \mu)^t] \]  \hspace{1cm} (1.15)

The diagonal elements of the covariance matrix are variances of the components of the random vector \( \eta \).

Several modes of convergence of a sequence of random variables are defined. The sequence of random variables \( \xi_1, \xi_2, \ldots \) is said to converge almost surely to the random variable \( \xi \) if \( P(\lim_{n \to \infty} \xi_n = \xi) = 1 \). The sequence of random variables \( \xi_1, \xi_2, \ldots \) is said to
converge in the mean square sense to the random variable \( \xi \) if \( \lim_{n \to \infty} E[(\xi_n - \xi)^2] = 0 \).

Finally, the sequence of random variables \( \xi_1, \xi_2, \ldots \) is said to converge in distribution to the random variable \( \xi \) if \( \lim_{n \to \infty} F_{\xi_n}(x) = F_{\xi}(x) \) for all \( x \in \mathbb{R} \).

In many practical applications, random variables have probability distributions which can be closely approximated by the ideal form known as the "normal" or "Gaussian" distribution. The theoretical basis for the widespread of the Gaussian assumption is the Central Limit Theorem which states that if \( \xi_1, \xi_2, \ldots, \xi_n \) are \( n \) independent random variables, with arbitrary distribution, the sum \( \eta = \xi_1 + \xi_2 + \cdots + \xi_n \) tends to the Gaussian random variable in probability. In practice, the random variables arise from a large number of unrelated effects and, thus, all together they yield a Gaussian distribution by virtue of the Central Limit Theorem. The distribution density function of the Gaussian random vector \( \eta \) is

\[
f(x) = \frac{1}{(2\pi)^{n/2} |det(V)|^{1/2}} \exp\left(-\frac{1}{2}(x-m)'V^{-1}(x-m)\right). \tag{1.16}
\]

Note that only the second order moments (the mean value and the covariance matrix) describe the Gaussian distribution completely. A linear combination of Gaussian random variables is a Gaussian random variable.

A stochastic process \( \eta(t) \) is a parameterized family of random variables with the parameter belonging to an index set. With this definition the random vector can be treated as a discrete stochastic process with a finite index set, \( t \in \{1, 2, \ldots, n\} \). If the index set is a continuous uncountable interval the stochastic process is called continuous. Stochastic processes which are used in engineering are separable processes. That is, they can be completely characterized by, at most, a countable set of multi-dimensional probability density functions.
\[ f_\eta(x_1, t_1), \ldots, f_\eta(x_1, t_1; x_2, t_2), \ldots, f_\eta(x_1, t_1; x_2, t_2; x_3, t_3), \ldots \] (1.17)

Gaussian processes are very important for practical applications. In this case, the corresponding distribution functions of Eq. 1.17 are Gaussian. A Gaussian process can be characterized by the mean value \( m_\eta(t) \) and the second order moments, also called the covariance function

\[ C_\eta(t_1, t_2) = E[\eta(t_1)\eta(t_2)] - m_\eta(t_1)m_\eta(t_2) \] (1.18)

The function

\[ R_\eta(t_1, t_2) = E[\eta(t_1)\eta(t_2)] = C_\eta(t_1, t_2) + m_\eta(t_1)m_\eta(t_2) \] (1.19)

is called the auto-correlation function. Note that the covariance and auto-correlation functions coincide for processes with a zero mean. Also, any linear operator transforms a Gaussian stochastic process into another Gaussian stochastic process; this simplifies the numerical calculations associated with analysis of Gaussian variables and processes.

Stationary processes have a simplified probability structure and are often used to model random phenomena in engineering applications. The process \( \eta(t) \) is stationary if two processes \( \eta(t) \) and \( \eta(t+\tau) \) have the same probability description, for any value of \( \tau \). It follows that the \( n \)-th order density function is such that

\[ f_\eta(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = f_\eta(x_1, 0; x_2, t_2-t_1; \ldots; x_n, t_n-t_1). \] (1.20)

A somewhat less stringent and more practical definition of stationary processes provides only for the "time independence" of the second order moments. That is, the process \( \eta(t) \) is called a wide sense stationary process if \( m_\eta(t) = \text{const} \) and \( R_\eta(t_1, t_2) = R_\eta(t_2-t_1) \).

Note that the wide sense stationary Gaussian process is also stationary in the strict sense.
The auto-correlation function cannot be arbitrary since it is the mathematical expectation of a product of two stochastic processes. In particular, it must be "positive definite". Thus, if $R_\eta(t)$ is the auto-correlation function of the wide sense stationary process $\eta(t)$, it must have the following spectral representation

$$R_\eta(t) = \int e^{i\omega t} F(d\omega)$$

(1.21)

where $F$ is the spectral measure of the process $\eta(x)$. The spectral density function $S(\lambda)$ is well defined if the measure $F$ is continuous with respect to the Lebegue measure. Specifically,

$$F(d\lambda) = S(\lambda) d\lambda .$$

(1.22)

The spectral density function and the auto-correlation function are related through the Wiener-Khintchine equations

$$R_\eta(t) = \int e^{i\omega t} S(\omega) d\omega ,$$

(1.23)

$$S(\omega) = \frac{1}{2\pi} \int e^{-i\omega t} R_\eta(t) dt .$$

(1.24)

The introduced spectral representation of the auto-correlation function motivates the corresponding spectral representation of the wide sense stationary stochastic processes. Specifically,

$$\eta(t) = \int e^{i\omega t} d\hat{Y}(\omega) ,$$

(1.25)
where \( \hat{X}(\omega) \) is the spectral process with orthogonal increment

\[
E[d\hat{X}(\lambda)d\hat{X}(\mu)] = \delta_{\lambda\mu}F(d\lambda) .
\]  

(1.26)

Eq. 1.25 provides an efficient representation of the stationary process by using another simpler stochastic process with orthogonal increment. If the spectral density function \( S(\lambda) \) is well defined, Eq. 1.26 can be rewritten as

\[
Y(x) = \int_{-\infty}^{\infty} e^{ix\omega}H(\omega)d\hat{W}(\omega) ,
\]  

(1.27)

where

\[
S(\omega) = |H(\omega)|^2 ,
\]  

(1.28)

and

\[
E[d\hat{W}(\lambda)d\hat{W}(\mu)] = \delta_{\lambda\mu}d\lambda .
\]  

(1.29)

\( \hat{W}(\omega) \) is often called an integrated white noise process; the Wiener process (Wong 1971) is an example.

Note that if \( Y(x) \) is the second order stationary stochastic process and \( R(0) < \infty \), then,

\[
\int_{-\infty}^{\infty} |H(\omega)|^2d\omega = \int_{-\infty}^{\infty} S(\omega)d\omega = R(0) < \infty .
\]  

(1.30)

That is, \( H(\omega) \in L^2(R) \). The complex valued function \( H(\omega) \) defined by Eq. 1.28 is not unique. Indeed, if \( H(\omega) \) satisfies Eq. 1.28 then so does any function
\[ \tilde{H}(\omega) = g(\omega)H(\omega) , \quad (1.31) \]

where \(|g(\omega)| = 1\).

Define function \(h(x)\) as the inverse Fourier transform of \(H(\omega)\) multiplied by a constant factor

\[ h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega x} d\omega . \quad (1.32) \]

Based on Eq. 1.8 the function \(h(x)\) can be shown to be also in \(L^2(\mathbb{R})\).

The stochastic analog of Eq. 1.8 can be established rigorously for some stochastic processes by using some quite tedious mathematical arguments (Wong 1971). Specifically,

\[ \int_{-\infty}^{\infty} H(\omega) d\tilde{W}(\omega) = \int_{-\infty}^{\infty} h(-x) dW(x) , \quad (1.33) \]

where \(W(x)\) is a process with orthogonal increment such that

\[ E[dW(x)dW(y)] = 2\pi \delta_{xy} dx . \quad (1.34) \]

Eq. 1.27 can be equivalently rewritten in the time domain by using Eq. 1.33. That is,

\[ Y(x) = \int_{-\infty}^{\infty} h(x-\xi) dW(\xi) = \int_{-\infty}^{\infty} h(x-\xi) w(\xi) d\xi , \quad (1.35) \]

where \(w(\xi)\) is the unit power white noise process

\[ E[w(\xi)w(\xi + x)] = 2\pi \delta(x) , \quad S_w(\omega) = 1 . \quad (1.36) \]
Eq. 1.35 shows that any stationary second-order process can be expressed as a filtered white noise process. Moreover, the function \( h(x) \) can be associated with the response of a linear system to impulse loading. Therefore, any stationary process can be treated as the steady-state response of a linear dynamic system driven by the white noise. In this regard, the most advantageous and physically meaningful representation of a random process is associated with a causal system. That is, the factor \( g(\omega) \) in Eq. 1.31 must be selected so that the impulse response function \( h(x) \) is equal to zero for \( x < 0 \). In this case, Eq. 1.35 can be rewritten as

\[
Y(x) = \int_{-\infty}^{x} h(x - \xi)w(\xi)d\xi .
\]  

(1.37)

Note that \( h(x) \) is a square integrable function and, therefore, tends to zero as \( x \to \infty \). The lower limit of the integral in Eq. 1.37 can be approximated by a, maybe, large but finite number

\[
Y(x) = \int_{x-T}^{x} h(x - \xi)w(\xi)d\xi .
\]  

(1.38)

Markovian random processes have drawn significant attention in engineering. Any second order stationary stochastic process can be well approximated by a component of the Markovian vector-process. The corresponding spectral density function is approximated by a proper rational function of \( \omega^2 \). The factorization of the spectral density function can be performed so that

\[
H(\omega) = \frac{N(s)}{D(s)} ,
\]  

(1.39)

where \( s = i\omega \), \( N \) and \( D \) are polynomials of order \( l_1 \) and \( l_2 \), respectively, \( l_1 \leq l_2 \), and all
zeros of $D$ are located in the left half of the complex plane $s = q + ip$. Then, Eqs. 1.35 and 1.39 show that the stochastic process $Y(x)$ can be viewed as the steady-state response of a linear system governed by the following differential equation

$$
D \left( \frac{d}{dx} \right) Y(x) = N \left( \frac{d}{dx} \right) \nu(x).
$$

(1.40)

Finally, comparing Eqs. 1.27 and 1.35 one finds

$$
Y(x) = \int_{-\infty}^{\infty} e^{i x \omega} H(\omega) d\dot{W}(\omega) = \int_{-\infty}^{\infty} h(x - \xi) dW(\xi)
$$

(1.41)

The two equalities of Eq. 1.41 signify two equivalent representations of stochastic fields in the frequency and time domains. Discretization of these equations yields two methods of synthesizing stochastic fields; the frequency domain integral produces the Spectral method, and the spatial/time domain integral yields the ARMA method.
1.5 Theory of Wavelets

The goal of the theory of wavelets is to provide a coherent set of concepts, methods, and algorithms that are well adapted to a variety of non-stationary signals and that are also suitable for numerical computations. This theory is well developed and several books have appeared recently with a comprehensive discussion of this topic (Chui 1992a, Daubechies 1992). Only some aspects of the wavelet theory which are most important for the subsequent material of the dissertation are reviewed in this section.

In this regard, historical perspectives of application of wavelets in engineering are discussed first by showing the generalization of the Haar basis (Haar 1910) in the modern theory of the multiresolution analysis (Mallat 1989a,b). The general framework of the multiresolution analysis motivated I. Daubechies (1988, 1990) to construct wavelet bases which are orthogonal, have local support, and some other useful properties. The Daubechies wavelets are used in sequent chapters of the dissertation for developing a new class of algorithm for synthesizing random fields. Further, the wavelet decomposition and reconstruction algorithms are formulated based on the theory of multiresolution analysis and pyramid filtering with exact reconstruction. These algorithms make wavelets especially attractive for numerical computations, since any function can be expended in a wavelet basis in $O(N)$ operations without performing numerical integrations. Finally, approximation properties of several wavelet bases are considered and application of the wavelet theory in engineering analysis and computational systems is discussed.

1.5.1 Theory of Wavelets: Historical Prospective and Preliminary Information

Historically, the elementary concept of wavelets appeared in several branches of
engineering. Nevertheless, neither of these separate intuitive efforts was finalized as a part of a coherent theory.

The Fourier transform is an efficient tool of mathematics and signal analysis. However, it has several essential drawbacks. On one hand, the Fourier series and integral are defined for square integrable functions and can not adequately represent other functional spaces, for example the Holder spaces of functions having local irregularities. On the other hand, the Fourier series/integral represents the global frequency contents of a function/signal. That is, the entire signal in the time domain must be acquired prior to the Fourier analysis. Also, changing the signal in a small interval of the time domain affects the entire spectrum. A more delicate mathematical tool which would conquer the difficulties associated with the Fourier integral is required for local analysis of functions and analysis of transient processes in engineering. These needs stimulated research in building the atomic decomposition of functional spaces which often is encountered in engineering literature as the “decomposition of signals in time-frequency atoms”.

The atomic decomposition is extraction of simple constituents that make up a complicated mixture. However, unlike the physical atoms with a prescribed structure, atoms extracted from signals are subjective and can be conveniently selected by an engineer. This leaves additional margin to adopt a particular form of atoms which simplifies signal analysis and reduces numerical computations. These “atoms” can be of different origins. For example, the short-time Fourier transform can be viewed as a decomposition of signals in terms of the “time-frequency atoms”. Recently, the theory of “time-scale atoms”, also often called wavelets, has been developed.

In this regard it is pointed out that the first orthonormal system of “localized” functions \( \{ \psi_{jk}(x), j = 1, 2, \ldots \} \) defined on [0,1] resembling wavelets was constructed by Haar (1910). In this case, for any function \( f(x) \) continuous on [0,1], the Haar series
\[ S_n(f)(x) = \langle f, \psi_0 \rangle \psi_0(x) + \langle f, \psi_1 \rangle \psi_1(x) + \cdots + \langle f, \psi_n \rangle \psi_n(x) \]  

(1.42)

converges to \( f(x) \) uniformly on the interval \([0,1]\). For \( n = 2^{-j} + k \geq 1 \), \( j = 0, -1, -2, \ldots \) and \( 0 \leq k < 2^{-j} \), the basis functions \( \psi_n(x) \) are defined as

\[ \psi_n(x) = \psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j} x - k), \]  

(1.43)

where

\[ \psi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1/2 \\
-1 & \text{if } 1/2 \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \]  

(1.44)

Also, the function

\[ \psi_0(x) = \phi(x) = \begin{cases} 
1, & 0 \leq x < 1 \\
0, & \text{otherwise}
\end{cases} \]  

(1.45)

is augmented to complete the Haar basis. It is straightforward to show that this basis is, indeed, orthogonal. Indeed, the support of \( (\psi_{j,k}) = [2^{-j}k, 2^{-j}(k + 1)] \) and, therefore, any two Haar wavelets of the same scale (same value of \( j \)) never overlap. Otherwise, overlapping of two basis functions can occur in such a way that the wavelet with the smaller support lies entirely within the region where the other wavelet is constant as it is shown in Figure 1.1. Then, the inner product of any two wavelets from different scales is also zero.

Any function of \( L^2([0,1]) \) can be arbitrarily closely approximated by a function which has a constant value on the intervals \([2^{-J_0}k, 2^{-J_0}(k + 1)]\); it is just necessary to select a sufficiently large value for \( J_0 \). This fact pointed out by Daubechies (1992)
provides a proof that the Haar wavelets \( \{ \psi_{j, n}(x) \} \) do constitute a basis of \( L^2([0, 1]) \). Let \( f_0 \) be such an approximation. Thus,

\[
f(x) = f_0(x) = \sum_k c_{-J_0, k} \phi_{-J_0, k},
\]

where

\[
\phi_{j, n}(x) = 2^{-j/2} \phi(2^{-j} x - k)
\]

\[
c_{-J_0, k} = \int f(x) \phi_{-J_0, k}(x) dx \approx 2^{-J_0/2} f(2^{-J_0} k).
\]

Now the function \( f_0 \) can be split into two components \( f_1 \), and \( \delta_1 \), where \( f_1 \) is a piecewise constant function over the intervals \( [2^{-J_0 + 1} k, 2^{-J_0 + 1} (k + 1)] \) and is defined similarly to \( f_0 \). Namely,

\[
f_1(x) = \sum_k c_{-J_0 + 1, k} \phi_{-J_0 + 1, k},
\]

where

\[
c_{-J_0 + 1, k} = (c_{-J_0, 2k - 1} + c_{-J_0, 2k})/\sqrt{2}.
\]

Alternatively, \( \delta_1 \) accumulates information lost in the transition from a fine scale to the coarser scale; it is a piece-wise constant function with the same stepwidth as \( f_0 \)

\[
\delta_1(x) = \sum_k d_{-J_0 + 1, k} \psi_{-J_0 + 1, k},
\]

where

\[
d_{-J_0 + 1, k} = (c_{-J_0, 2k - 1} - c_{-J_0, 2k})/\sqrt{2}.
\]
The procedure outlined above is schematically shown in Figure 1.2. Further, the same decomposition can be applied to $f_1$ increasing the stepwidth twice again

$$f_1(x) = f_2(x) + \sum_k d_{-J_0 + 2, k} \Psi_{-J_0 + 2, k}.$$  \hfill (1.53)

Finally, after $J_0$ steps of averaging an expansion of $f_0$ by using the Haar wavelets can be found. Since $f_0$ is an arbitrary piece-wise constant function, the Haar wavelets comprise an orthogonal basis for $L^2([0, 1])$. This proof of the completeness of the Haar basis was proposed by Daubechies (1992) as an illustration of the multiresolution analysis; it implicitly uses the multiresolution approach by projecting the function $f(x)$ on coarser and less detailed scales.

The Haar basis has been criticized from several perspectives. On one hand, the Haar "atoms" which are used to represent a continuous function $f(x)$ are not themselves continuous. More regular functions, for example $C^1$ functions, can not be generated by using the Haar basis. On the other hand, the discontinuous Haar basis has quite poor frequency localization for signal processing applications.

Faber and Schauder replaced the Haar basis by "primitives" based on the idea of approximating a graph of a continuous function with inscribed polygonal lines. In particular, they proposed the, so-called, Schauder basis which contained a set of functions

$$\{ \Delta_{j, k}(x) = \Delta(2^j x - k) \}_{j, k > 0},$$

where

$$\Delta(x) = \begin{cases} 
2x, & 0 \leq x \leq 1/2 \\
2-2x, & 1/2 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}.$$  \hfill (1.54)

completed with $x$ and $I$. The continuity of the function in Eq. 1.54 and the multiresolution structure of the Schauder basis were suitable for representation of a significantly larger
family of functional spaces, for example $C^r$ for $r < 1$. Thus, the Schauder basis is useful to show that sample trajectories of the Brownian motion belong almost surely to $C^r$ for $r < 1/2$, see also Section 1.5.5.

In 1927 Philip Franklin improved the Schauder basis by using the Gram-Schmidt orthogonalization procedure in $L^2([0, 1])$. The Franklin system constructed in this manner had the advantages of both the Haar and Schauder bases. The most significant drawback of the Franklin system relates to difficulties in deriving implicit expressions for the basis functions which can not be determined from a single function $\psi$ by translation and dilation. However, it has been shown, recently, that a representative $f_n(x)$, $n = 2^{-j} + k \geq 1$, of the Franklin system can be bounded as

$$|f_n(x)| \leq C 2^{j/2} \exp(-\gamma |2^j x - k|),$$

$$\left| \frac{d}{dx} f_n(x) \right| \leq C 2^{3j/2} \exp(-\gamma |2^j x - k|),$$

where $\gamma$ and $C$ are some constants. Thus, everything works for the Franklin system as if

$$f_n(x) = 2^{j/2} \psi(2^j x - k),$$

where $\psi$ is a Lipschitz continuous function with an exponential decay.

Calderon (1964) developed a principally new approach to the "atomic decomposition" introducing the, so-called, "Calderon’s identity". A double-indexed family of wavelets can be generated from a single function $\psi \in L^2(R)$ by dilation and translation

$$\psi_{a, b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right).$$

where the constant factor in the right hand side of Eq. 1.57 ensures the equality $\|\psi_{a, b}(x)\|_{L^2} = \|\psi(x)\|_{L^2}$. The wavelet function $\psi$ must satisfy the admissibility condition which in the one-dimensional case is written as
$$C_\psi = 2\pi \int |\xi|^{-1} |\hat{\psi}(\xi)|^2 d\xi < \infty.$$  

(1.58)

If $\psi(x)$ is an integrable function, Eq. 1.58 is equivalent to

$$\Psi(0) = 0.$$  

(1.59)

If $\|\psi(x)\| = 1$, the continuous wavelet transform is defined as

$$W_f(a, b) = \int f(x) |a|^{-1/2} \Psi\left(\frac{x-b}{a}\right) dx = |a|^{1/2} Q_a(f),$$  

(1.60)

where $Q_a$ is an operator of convolution with $|a|^{-1} \Psi(-x/a)$. Then, the identity operator can be decomposed as (Calderon 1964)

$$I = \frac{1}{C_\psi} \int |a|^{-1} Q_a Q_a^* da,$$  

(1.61)

where $Q_a^*$ is an operator adjoint to $Q_a$ and is expressed as convolution with $|a|^{-1} \Psi(x/a)$. Grossmann and Morlet (1984) rediscovered this identity 20 years after Calderon. However, they gave it a new interpretation of "the coherent state of quantum physics". Eq. 1.61 can be rewritten in the following equivalent form

$$\int \int W_{a,b}(f) \overline{W_{a,b}(g)} \frac{dadb}{a^2} = C_\psi \langle f, g \rangle.$$  

(1.62)

The proof of Eq. 1.62 is straightforward. Indeed, substituting Eq. 1.60 into Eq. 1.62 and shifting to the Fourier domain, the left hand side of Eq. 1.62 is equal to

$$\int \int (\hat{f}(\xi) e^{-ib\xi} \overline{\hat{\psi}(a\xi)} d\xi) (\int \overline{\hat{g}(\xi')} e^{ib\xi'} \hat{\psi}(a\xi') d\xi') \frac{dadb}{|a|} =$$  

(1.63)
\[ 2\pi \int |\psi(a_\xi)|^2 \frac{da}{a} \int \hat{f}(\xi) \overline{\hat{\psi}(\xi)} d\xi = C_\psi \langle f, g \rangle. \]

Eq. 1.62 yields the inverse continuous wavelet transform which must be interpreted in the weak \( L^2(R) \) sense

\[ f(x) = \frac{1}{C_\psi} \int \int W_{a, b}(f) \psi_{a, b}(x) \frac{dadb}{a^2}. \quad (1.64) \]

Formula 1.64 states that any function can be represented as a superposition of wavelets; this is a quite astonishing result. Indeed, if \( \psi(x), f(x) \in L^1(R) \) then, upon integrating, the right hand side of Eq. 1.64 will be nullified because of Eq. 1.59, whereas the function \( f(x) \) can be arbitrary and, in particular, have a non-zero integral. This apparent paradox can be resolved by noting that Eq. 1.64 is valid only in the weak \( L^2(R) \) sense and not in the \( L^1(R) \) sense.

An efficient numerical implementation of Eqs. 1.60 and 1.64 by using digital computers requires an appropriate discretization of the corresponding integrals (Riou and Duhamel 1992). A natural choice of the discrete parameters is

\[ a = a^j_0, \quad b = nb_0 a^j_0, \quad (1.65) \]

where \( a_0 > 1 \), and \( b_0 > 0 \). The discretization introduced by Eq. 1.65 generates a set of functions

\[ \psi_{j, n}(x) = a_0^{-j/2} \psi(a_0^{-j} x - nb_0). \quad (1.66) \]

Substituting Eq. 1.66 into Eq. 1.60 yields

\[ d_{j, n} = W_f(a^j_0, nb_0 a^j_0) = \langle f, \psi_{j, n} \rangle. \quad (1.67) \]
Several questions must be addressed to assess the usefulness of discrete wavelets (Daubechies 1992):

1) Can the wavelet coefficients $d_{j,n}$ completely characterize the function $f(x)$? That is, can $f(x)$ be reconstructed from the wavelet coefficients in a numerically stable way?

2) Can $f(x)$ be written as a superposition of the wavelet atoms $\psi_{j,n}(x)$ or any other "conjugate" atoms $\tilde{\psi}_{j,n}(x)$?

3) Can the wavelet atoms $\psi_{j,n}(x)$ constitute a Riesz basis or an orthogonal basis?

The theoretical study of these questions is not completed yet. Some conditions for generating wavelet frames and unconditional (Riesz) bases are given by Daubechies (1992), Grochenig (1993) and Chui (1992a).

It is difficult to determine the appropriate rate of the discretization in the time and scale domains so that the discrete wavelets comprise a frame or a basis. The wavelet atoms given by Eq. 1.66 have very special time-frequency localization properties (Daubechies 1990, 1992). Indeed, if the function $\psi(x)$ is localized around $t=0$ and $\omega=\omega_0$, then the function $\psi_{j,n}(x)$ is localized around $t = nb_0a_0^j$ and $\omega = \omega_0/a_0^j$. The set of wavelet atoms covers the time-frequency plane in the manner shown in Figure 1.3(a) where the black dots indicate the centers of localization. The wavelet parameters $a$ and $b$ must be sampled adequately and must cover the time-frequency plane sufficiently densely to form a wavelet basis or a frame. For example, $a_0 = 2$ is a common choice that corresponds to the octave-type discretization of the frequency domain. That is, wavelets from a given scale cover an octave in the frequency plane. If for $a_0 = 2$ there exists $b_0$ such that the family of wavelets $\psi_{j,n}(x)$ forms a frame or a Riesz basis, then this family is called a dyadic wavelet. In this case, by increasing the value of $b_0$ the set of functions $\psi_{j,n}(x)$ can be transformed from being a quite redundant frame through forming a Riesz basis to being inadequate to represent functions in $L^2(R)$. An explicit relationship between the
parameters $a_0$ and $b_0$ which is similar to the sampling theorem for the short-time Fourier transform has not been found. The absence of a physically meaningful notion of "sampling time" and "sampling frequency" complicates the interpretation of results of the wavelet analysis and obstructs the construction of wavelet bases.

The wavelet bases of the early 80's had quite complicated structure. Later, Mallat and Meyer developed the theory of the multi-resolution analysis which simplified the construction of wavelet bases. Using the similarities between the Meyer wavelet basis (Daubechies 1992) and the quadrature mirror filtering Mallat 1989(a,b) viewed wavelets as a method to describe mathematically the increment of information needed to go from a coarse approximation to a higher resolution. The multiresolution analysis provided a skeleton for generating wavelet bases which in addition to good localization had some other useful properties such as smoothness, fast decays at infinity, and vanishing moments. These additional properties ensured the success of wavelets far beyond the scope of their original application to signal and image processing and time-frequency analysis of transient processes. Wavelets have recently become a foundation for some efficient computational systems. In particular, this dissertation develops a numerical method for the representation and synthesis of random fields based on the local and multi-scale properties of the Daubechies wavelets.

Using Mallat's construction Daubechies (1988,1990) completed the Haar's work and developed orthogonal dyadic wavelet bases for $L^2(R)$ which have the following properties:

1) $a_0 = 2$, $b_0 = 1$;

2) the support of $\psi(x)$ is finite and equal to the interval $[0,2M-1]$;

3) $\psi(x)$ has $M$ zero moments. That is,
\[ \int \psi(x)dx = \ldots = \int x^{M-1} \psi(x)dx = 0; \]  
(1.68)

4) \( \psi(x) \) is at least \( M \) times continuously differentiable, where \( \gamma \) is a constant.

Another origin of wavelets is the time-frequency analysis of transient signals and processes. Gabor (1946) incorporated a time-localizing window to improve the Fourier analysis of non-stationary signals. The continuous "short-time Fourier transform", or the "Gabor transform", and its inverse are defined as follows

\[ F(\omega, b) = \int e^{-i\omega x} f(x) g(x-b) dx, \]  
(1.69)

\[ f(x) = \frac{1}{2\pi\|g\|^2} \int e^{i\omega x} F(\omega, b) \hat{g}(x-b) d\omega db. \]  
(1.70)

Eqs. 27 and 28 can be also discretized; this generates the following time-frequency atoms

\[ \varphi_{j,n} = e^{-i\omega_0 x} g(x-nt_0). \]  
(1.71)

If \( g(x) \) satisfies some non-restrictive conditions (Daubechies 1992), these atoms comprise a frame when \( \omega_0 t_0 < 2\pi \) and a basis when \( \omega_0 t_0 = 2\pi \). Originally, Gabor (1946) proposed to use the Gaussian window

\[ g(x) = \pi^{-1/4} e^{-x^2/2} \]  
(1.72)

in this regard. The usefulness of this window has been extensively discussed in the literature. It minimizes the indeterminacy of the signal frequency and time contents due to the uncertainty principle. That is, it exhibits the best time-frequency localization among all other windows (Chui 1992a) for some measures of signal concentration. Gabor (1946) proposed to use this window with \( \omega_0 t_0 = 2\pi \) for communication purposes. Unfortu-
nately, the Gabor basis is not an unconditional (Riesz) basis (Bacry et al. 1975) as the result of the "infinite" regularity for both $g(x)$ and $\hat{g}(\omega)$. Then, it does not provide stable reconstruction of signals and is not well suited for signal analysis and synthesis applications. Bastiaans (1980) constructed a dual basis to the Gabor basis which, however, is defined only in the sense of distributions.

The time-frequency atoms are localized around $(nt_0, j\omega_0)$ of the time frequency plane and cover it as shown in Figure 1.3(b) where the black dots indicate the centers of localization. The general idea how the short time Fourier transform works can be captured from Figure 1.4 where three time frequency atoms built using Eq. 1.71 are schematically shown. These atoms have the same envelope window and, for the given time moment, are designed to evaluate the respective frequency components of the signal by using the integral projection of Eq. 1.69. Note that the number of extrema within the support of these atoms is determined by the frequency parameter. Thus, if the intermediate frequency atom behaves satisfactory with respect to the ability to detect the presence of the corresponding frequency components for the selected time moment, the high frequency atom has too many extrema within its support; this provides good frequency detection over the time resolution. The high frequency atom can be modified by sacrificing some frequency resolution to improve its time resolution. Similarly, the low frequency atom can fail to evaluate the frequency components of a signal at the given time moment. Its superior time resolution can be "properly" adjusted in this regard.

These arguments show that for some applications it might be useful to change the time and frequency resolution along the time-frequency plane so that the high frequency components have better time resolution whereas the low frequency components have better frequency resolution. This modification can be implemented by changing the width of the window function so that all atoms have an equal number of extrema within their supports, see Figure 1.5. That is, a parameter dependent window can be introduced in Eq.
1.69 by the equation

\[ g_\omega(x) = |\omega|^{1/2} g(\omega x). \]  \hspace{1cm} (1.73)

Substituting Eq. 1.73 into Eq. 1.69 yields the, so called, constant-Q short-time analysis (Pimonow 1962, Flanagan, 1972, Levshin et.al. 1972, Youngberg and Boll, 1978)

\[ F(\omega, b) = |\omega|^{1/2} \int e^{-i\omega x} f(x) g(\omega(x - b)) dx. \]  \hspace{1cm} (1.74)

Alternatively, the width of the window can be an optimally selected, data dependent, function of both time and frequency (Jones and Parks 1990).

It can be shown that the constant-Q short-time analysis is a special case of the continuous wavelet transform. Indeed, changing variables \( \omega \to a = 1/\omega \) and introducing

\[ \psi(x) = e^{ix} g(x), \]  \hspace{1cm} (1.75)

Eq. 1.74 can be rewritten as

\[ F\left(\frac{1}{a}, b\right) = \frac{1}{|a|^{1/2}} e^{-ib/a} \int f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx. \]  \hspace{1cm} (1.76)

Eq. 1.76 coincides with Eq. 1.60 up to a constant factor of the unit module.

Figure 1.3 emphasizes the difference between the wavelet and short-time Fourier transforms; see Meyer (1993) and Flandrin (1989a) for further discussion of the historical perspective of the wavelet theory.
1.5.2 Multiresolution Analysis

The multiresolution analysis of $L^2(R)$, the space of square integrable functions, is a sequence of successive approximating spaces $V_j$. These spaces must satisfy the following properties

(1) $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$;

(2) $\text{clos}_{L^2} \bigcup_{j \in Z} V_j = L^2$;

(3) $\bigcap_{j \in Z} V_j = \{0\}$;

(4) If $f(x) \in V_0$, then $f(x-n) \in V_0$ and $f(2x) \in V_{-1}$.

(5) There exist a function $\phi(x) \in V_0$ such that $\{\phi_{j,n}; n \in Z\}$ is a Riesz basis of $V_0$,

where

$$\phi_{j,n} = 2^{-j/2} \phi(2^{-j}x - n + 1).$$ \hspace{1cm} (1.77)

Eq. 1.77 and the above definition show that the entire space $V_j$ is created by shifting a single function which is stretched (or dilated) to reveal local details of a signal. This construction leads to the important notion of a scale. The scale is given by the parameter $j$ and quantifies the measure of stretching $\phi(x)$ to capture local signal characteristics. This notion is associated with the commonly used scales of road maps; given a “real” object, scale of a map shows the ratio of the object length on the map to its real length. Alternatively, Cohen (1993) has used an operator formalism to define the scale by utilizing the Mellin transform. This quite abstract approach yields exactly the same results as the “physical” definition of the scale.

The function $\phi(x)$ is often called the scale function. Indeed, the set $\{\phi_{j,n}; n \in Z\}$ spans the $j$-th scale of the multiresolution analysis and stores all the details of a signal from the $j$-th scale. Orthogonal wavelets provide the most efficient algorithmic scheme
and are associated with the set of functions \( \{ \phi_{0,n}; n \in \mathbb{Z} \} \) that forms an orthonormal basis of \( V_0 \). Note that any wavelet basis developed within the framework of the multiresolution analysis can be readily orthogonalized (Daubechies 1992, Chui 1992a).

Further, the space \( W_j \) is defined as the orthogonal compliment of \( V_j \) in \( V_{j-1} \). That is,

\[
V_{j-1} = V_j \oplus W_j. \tag{1.78}
\]

Then, Eq. 1.78 and the definition of the multiresolution analysis show that the spaces \( W_j, j = \ldots, -1, 0, 1, \ldots \) must satisfy the following equations

\[
W_j \perp W_{j'}, \text{ if } j \neq j'; \tag{1.79}
\]

\[
V_j = V_j \oplus \left( \bigoplus_{k=0}^{j-1} W_{j-k} \right); \tag{1.80}
\]

\[
L^2 = \bigoplus_{j \in \mathbb{Z}} W_j. \tag{1.81}
\]

The space \( W_j \) contains information lost in the projection of a function into the space \( V_j \) in comparison with its projection into \( V_{j-1} \). The useful principle of the multiresolution analysis is the fact that there exists a function \( \psi(x) \in W_0 \) such that \( \{ \psi_{0,n}; n \in \mathbb{Z} \} \) is an orthogonal basis of \( W_0 \), where

\[
\psi_{j,n} = 2^{-j/2} \psi(2^{-j}x - n + 1). \tag{1.82}
\]

Eq. 1.81 yields that the set \( \{ \psi_{j,n}; j, n \in \mathbb{Z} \} \) forms an orthogonal wavelet basis of \( L^2(R) \).

The function \( \psi(x) \) is called the wavelet function. The properties of \( \psi(x) \) and \( \phi(x) \) are closely related and are addressed in the sequel. Note that the Haar basis discussed in Section 1.5.1 satisfies all of the preceding conditions and represents the simplest orthogonal wavelets with a multiresolution structure.

Since \( \phi \subset V_0 \subset V_{-1}, \psi \subset W_0 \subset V_{-1}, \) and \( \{ \phi_{-1,n} \} \) form an orthogonal basis of \( V_{-1} \),
the following multi-scale representation of the functions $\phi(x)$ and $\psi(x)$ can be readily established

$$\phi(x) = \sqrt{2} \sum_k h_{k+1} \phi(2x - k), \quad (1.83)$$

$$\psi(x) = \sqrt{2} \sum_{k=0}^\infty g_{k+1} \phi(2x - k). \quad (1.84)$$

Eqs. 1.83 and 1.84 can be equivalently rewritten using the Fourier transformation, Eq. 1.6. Specifically,

$$\hat{\phi}(\xi) = m_0(\xi/2) \hat{\phi}(\xi/2), \quad (1.85)$$

and

$$\hat{\psi}(\xi) = m_f(\xi/2) \hat{\phi}(\xi/2), \quad (1.86)$$

where the $2\pi$-periodic functions $m_0(\xi)$ and $m_f(\xi)$ are defined as

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_k h_{k+1} e^{-i k \xi}, \quad (1.87)$$

and

$$m_f(\xi) = \frac{1}{\sqrt{2}} \sum_k g_{k+1} e^{-i k \xi}. \quad (1.88)$$

The scale function $\phi(x)$ is normalized so that

$$\int_{-\infty}^{\infty} \phi(x) dx = 1, \quad \hat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) dx = \frac{1}{\sqrt{2\pi}}. \quad (1.89)$$
Eqs. 1.85 and 1.89 yield the following constraint on the set of parameters \( \{ h_k \} \). Specifically,

\[
m_0(0) = \frac{1}{\sqrt{2}} \sum_k h_{k+1} = 1. \tag{1.90}
\]

The orthogonality of \( \{ \phi_{0,n} | n \in \mathbb{Z} \} \) requires several additional constraints on \( m_0(\xi) \) and \( \{ h_k \} \). Indeed, the equation

\[
\delta_{k,0} = \int_{-\infty}^{\infty} \phi(x) \bar{\phi}(x-k) dx = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 e^{ik\xi} d\xi = \int_{0}^{2\pi} \left( \sum_l |\hat{\phi}(\xi + 2\pi l)|^2 \right) e^{ik\xi} d\xi,
\]

(1.91)

where \( \delta_{k,0} \) is the Kronecker delta, yields

\[
\sum_l |\hat{\phi}(\xi + 2\pi l)|^2 = \frac{1}{2\pi}. \tag{1.92}
\]

Substituting Eq. 1.85 into Eq. 1.92 leads to

\[
\sum_l |m_0(\xi + \pi l)|^2 |\hat{\phi}(\xi + \pi l)|^2 =
\]

\[
\sum_l \left( |m_0(\xi + 2\pi l)|^2 |\hat{\phi}(\xi + 2\pi l)|^2 + |m_0(\xi + (2l + 1)\pi)|^2 |\hat{\phi}(\xi + (2l + 1)\pi)|^2 \right) =
\]

\[
= \sum_l \left( |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \right) |\hat{\phi}(\xi + 2\pi l)|^2 = \frac{1}{2\pi}, \tag{1.93}
\]

where the periodicity of \( m_0(\xi) \) has been used. Finally, Eqs. 1.92 and 1.93 yield

\[
|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1. \tag{1.94}
\]

Eqs. 1.90 and 1.94 show that \( m_0(\xi) \) is, in fact, a low pass filter with module equal to one at
zero frequency and equal to zero at $\pi$. This filter can be designed to have some additional useful properties such as smoothness, assigned stop-band, etc., as long as it satisfies Eqs. 1.90 and 1.94. In signal and image processing literature, a filter with the frequency response that satisfies Eq. 1.94 is often called a “mirror quadrature” filter. Also, it is useful to rewrite Eq. 1.94 in terms of the parameters $\{h_k\}$ by substituting Eq. 1.87 into Eq. 1.94

$$
\sum_n h_n \overline{h_{n+2k}} = \delta_{k,0}.
$$

The wavelet function $\psi(x)$ can be defined by specifying the set of parameters $\{g_k\}$ or, equivalently, the function $m_f(\xi)$. This choice is not arbitrary since the wavelet function $\psi(x)$ must be orthogonal to $\psi(x-k)$ and $\phi(x-k)$ and the set $\{\psi_{0,n}, m \in \mathbb{Z}\}$ must constitute a basis for the space $W_0$. The right choice for $m_f(\xi)$ is

$$
m_f(\xi) = e^{ik\xi}m_0(\xi + \pi).
$$

Indeed,

$$
\int_{-\infty}^{\infty} \psi(x)\overline{\psi(x-k)} dx = \int_{-\infty}^{\infty} |\psi(\xi)|^2 e^{ik\xi} d\xi = \int_{0}^{2\pi} \left( \sum_l |\psi_l(\xi + 2\pi l)|^2 \right) e^{ik\xi} d\xi =
$$

$$
\int_{0}^{2\pi} \left( \left| m_0(\xi + \pi) \right|^2 \sum_l \left| \phi_l(\xi + 2\pi l) \right|^2 + \left| m_0(\xi/2) \right|^2 \sum_l \left| \phi_l(\xi/2 + \pi + 2\pi l) \right|^2 \right) e^{ik\xi} d\xi =
$$

$$
\frac{1}{2\pi} \int_{0}^{2\pi} \left( \left| m_0(\xi + \pi) \right|^2 + \left| m_0(\xi/2) \right|^2 \right) e^{ik\xi} d\xi = \frac{1}{2\pi} \int_{0}^{2\pi} e^{ik\xi} d\xi = \delta_{k,0}.
$$

Similarly,

$$
\int_{-\infty}^{\infty} \phi(x)\overline{\phi(x-k)} dx = \int_{-\infty}^{\infty} \frac{\psi(\xi)}{\psi_l(\xi/2)} e^{ik\xi} d\xi = \int_{0}^{2\pi} \left( \sum_l \overline{\psi_l(\xi + 2\pi l)} \phi_l(\xi + 2\pi l) \right) e^{ik\xi} d\xi =
$$

$$
\int_{0}^{2\pi} \left( \left| \psi_l(\xi + 2\pi l) \right|^2 \sum_l \left| \phi_l(\xi) \right|^2 + \left| \psi_l(\xi + 2\pi l) \right|^2 \sum_l \left| \phi_l(\xi + 2\pi l) \right|^2 \right) e^{ik\xi} d\xi =
$$

$$
\frac{1}{2\pi} \int_{0}^{2\pi} \left( \left| \psi_l(\xi) \right|^2 + \left| \psi_l(\xi + 2\pi l) \right|^2 \right) e^{ik\xi} d\xi = \frac{1}{2\pi} \int_{0}^{2\pi} e^{ik\xi} d\xi = \delta_{k,0}.
$$
\[ \int_0^{2\pi} \left( \sum_l m_0(\xi - 2\pi l) m_0(\xi) \phi_l(\xi - 2\pi l) \right)^2 e^{i\xi k} d\xi = \]  
\[ \int_0^{2\pi} \left( m_0(\xi) m_0(\xi) \right) e^{i\xi k} \left( \sum_l \phi_l(\xi - 2\pi l) \right)^2 - \sum_l \phi_l(\xi - 2\pi l) \right)^2 e^{i\xi k} d\xi = 0. \]  

Also, the set \( \{ \psi_{0,n} \, ; \, n \in \mathbb{Z} \} \), where \( \psi(x) \) is defined by Eqs. 1.86 and 1.96, generates a basis of \( W_0 \). That is, an arbitrary function from the space \( W_0 \) can be expressed as a linear combination of \( \{ \psi_{0,n} \, ; \, n \in \mathbb{Z} \} \). The proof of this assertion is also quite straightforward and can be found in Daubechies (1992).

Eq. 1.96 yields the following relationship

\[ g_{n+1} = (-1)^n h_{-n}. \]  

Note that Eq. 1.96 is not the only choice for \( m_f(\xi) \) which defines an orthogonal wavelet basis. In fact, the same is true for any

\[ \tilde{m}_f(\xi) = e^{i2\xi(n-1)} m_f(\xi), \quad n \in \mathbb{Z}, \]  

that corresponds to the shifted version of the original wavelet

\[ \tilde{\psi}(x) = \psi(x + n - 1). \]

Finally, the smoothness of the Fourier transform of orthonormal wavelets is discussed by noting that the high rate of decay at infinity of a function implies certain degree of smoothness of its Fourier transform and vice versa. However, the orthogonality conditions, Eqs. 1.97 and 1.98, imposed on the wavelet function yield an "extra regularity" of \( \psi(x) \) in the neighborhood of zero frequency. In particular, if \( \{ \psi_{j,n} \, ; \, j, n \in \mathbb{Z} \} \)
constitute an orthogonal set in $L^2(\mathbb{R})$ with $|\psi(x)| < C(1 + |x|)^{-m-1-\varepsilon}$, $\psi(x) \in C^m(\mathbb{R})$, and $\psi^{(l)}(x)$ is bounded for $l<m$, then

$$\int x^l \psi(x) dx = 0 \quad \text{for} \quad l = 0, 1, \ldots, m, \quad (1.102)$$

or equivalently

$$\frac{d^l}{d\xi^l} \psi \bigg|_{\xi = 0} = 0, \quad l = 0, 1, \ldots, m. \quad (1.103)$$

Eq. 1.103 yields

$$\frac{d^l m_0}{d\xi^l} \bigg|_{\xi = \pi} = 0, \quad l = 0, 1, \ldots, m \quad (1.104)$$

and

$$m_0(\xi) = \left(\frac{1 + e^{i\xi}}{2}\right)^{m+1} L(\xi), \quad (1.105)$$

where $L(\xi)$ is a $2\pi$-periodic function from $C^m$. Eq. 1.104 shows that the orthogonal wavelets must have several vanishing moments. However, this condition represents only the smallest number of the requisite vanishing moments. Wavelets with a large number of vanishing moments can be quite useful for some applications. In this regard, Daubechies (1988) discovered a procedure to generate wavelets which have support $[0,2M-I]$ and simultaneously $M$ vanishing moments. Also, Eqs. 1.94 and 1.104 are useful for studying the requisite smoothness of $m_0(\xi)$ at the origin.
1.5.3 Compactly Supported Orthogonal Wavelets

The Haar basis introduced in Section 1.5.1 is the simplest representative of orthogonal compactly supported wavelets. In this case, the wavelet and scale functions are defined as

\[ \phi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.106) \]

and

\[ \psi(x) = \phi(2x) - \phi(2x + 1) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2 \\ -1 & \text{if } 1/2 \leq x < 1 \\ 0, & \text{otherwise} \end{cases} . \quad (1.107) \]

Hence, the parameters \( h_n \) can be expressed as

\[ h_n = \sqrt{2} \int \phi(x) \phi(2x - n + 1) dx = \begin{cases} 1/\sqrt{2}, & \text{if } n=1,2 \\ 0, & \text{otherwise} \end{cases} . \quad (1.108) \]

Then, the function \( m_0(\xi) \) is equal to

\[ m_0(\xi) = \frac{1}{2} (1 + e^{-i\xi}) = e^{-i\xi/2} \cos \xi/2 . \quad (1.109) \]

The wavelets discovered by Daubechies (1988) are a natural generalization of the Haar basis (Haar 1910). These wavelets have the fundamental properties of the Haar wavelet: they constitute multiresolution analyses, they are compactly supported, and they form orthonormal bases of \( L^2(R) \).

A compactly supported wavelet function which generates an orthogonal basis must satisfy Eqs. 1.94 and 1.105. However, these conditions are necessary but are not sufficient. Several equivalent sufficient conditions that ensure that wavelets form an orthogonal basis
have been recently found; see, for example, Cohen (1990). Several approaches are introduced to find orthogonal compactly supported wavelets. Thus, all orthogonal wavelets can have several alternative parameterizations of the parameters \((h_1, h_2, \ldots, h_{2M})\). For example, Pollen (1990) established a useful parametrization which in the case \(M=2\) can be written as

\[
\begin{align*}
    h_1 &= \frac{1 - \cos \theta + \sin \theta}{2\sqrt{2}} \\
    h_2 &= \frac{1 + \cos \theta + \sin \theta}{2\sqrt{2}} \\
    h_3 &= \frac{1 + \cos \theta - \sin \theta}{2\sqrt{2}} \\
    h_4 &= \frac{1 - \cos \theta - \sin \theta}{2\sqrt{2}}.
\end{align*}
\]

(1.110)

Tewfik et. al. (1992) utilized it implicitly to find the compactly supported wavelet function that leads to the best approximation of a signal on a given scale of the multiresolution analysis. However, this parametrization is not readily available for large \(M\) and can represent "bad", quite irregular, wavelet functions. Also, some additional useful properties, such as smoothness or vanishing moments, are difficult to study within this framework.

Alternatively, Daubechies (1988) has shown that compactly supported orthogonal wavelets can be constructed by using Eq. 1.105 with \(|L(\xi)|\) equal to

\[
|L(\xi)|^2 = P((\sin \xi / 2)^2),
\]

(1.111)

where \(P(x)\) is a polynomial

\[
P(x) = \sum_{k=0}^{m} \binom{m+k}{k} x^k + x^{m+1} R\left(\frac{1}{2} - x\right),
\]

(1.112)
and \( R(x) \) is an odd polynomial so that \( P(x) \geq 0 \) for \( x \in [0, 1] \). The most obvious choice \( R(x) \equiv 0 \) corresponds to orthogonal wavelets with the maximum number of vanishing moments for their support width. However, the expression in the right hand side of Eq. 1.111 must be factored to provide for the numerical values for the wavelet parameters \( h_k \). Several choices are available in this regard. Retaining zeros inside of the unit circle yields wavelets which are quite regular though asymmetric (Daubechies 1988, 1992). In this case, the filter \( m_0(\xi) \) is the minimal phase filter. Table 1.1 lists the coefficients \( h_k \) for several values of \( M=m+1 \). Smith and Barnwell (1986) explained the inherent asymmetric character of compactly supported wavelets by proving that the symmetry is not compatible with the exact reconstruction in subband coding; an orthonormal compactly supported wavelet basis can not be symmetric unless it is the Haar basis. Note that rearranging zeros in the factorization of Eq. 1.111, so that \( m_0(\xi) \) is closest to the linear phase filter among all filters with a given module, yields less asymmetric wavelets with the same number of vanishing moments. Some other variations of this scheme may include designing an appropriate polynomial \( R(x) \) for Eq. 1.112 so that the wavelet function has additional properties such as the maximum smoothness for the given support length, some vanishing moments for the scale function, etc.

The shift in Eq. 1.101 can be selected so that the wavelet function and the scale function have exactly the same support. This can be achieved by taking \( n=-N/2 \) in Eq. 1.100. Then, Eq. 1.99 must be replaced by the following formula

\[
g_n = (-1)^{n-1} h_{N-n+1},
\]

(1.113)

where \( N=2M \) denotes the width of the support of the orthogonal wavelet.

The closed-form formulae for \( \phi(x) \) and \( \psi(x) \) are not available. Several algorithms are proposed to evaluate these functions numerically. Thus, the Fourier transforms of \( \phi(x) \)
and $\psi(x)$ can be found by using Eqs. 1.85 and 1.86. Specifically,

$$
\hat{\phi}(\xi) = \hat{\phi}(0) \prod_{k=1}^{\infty} m_0(\xi/2^k) = \frac{1}{\sqrt{2\pi}} \prod_{k=1}^{\infty} m_0(\xi/2^k), \tag{1.114}
$$

$$
\hat{\psi}(\xi) = \hat{\phi}(0)m_f(\xi/2) \prod_{k=2}^{\infty} m_0(\xi/2^k) = \frac{1}{\sqrt{2\pi}} m_f(\xi/2) \prod_{k=2}^{\infty} m_0(\xi/2^k). \tag{1.115}
$$

Upon evaluating the Fourier transforms, the functions $\phi(x)$ and $\psi(x)$ can be found by applying consequently the inverse Fourier transformation which can be numerically evaluated with the fast Fourier transform (FFT) algorithm. Alternatively, $\phi(x)$ and $\psi(x)$ can be found by using the estimate

$$
\left| \phi(2^{-j}k) - 2^{j/2} \langle \phi^o, \phi_{-j, 2^{-j}k} \rangle \right| \leq C 2^{-j\alpha}, \tag{1.116}
$$

where $\alpha$ denotes the Hölder smoothness of the scale function $\phi(x)$; it is given in Table 1.2 for several small values of $M$. Eq. 1.116 leads to the exponentially converging algorithm

$$
c_{k}^{-j-1} = \sum_{n} h_{n-2n} c_{n}^{-j} = \langle \phi^o, \phi_{-j-1, k} \rangle, \quad c_{k}^{0} = \delta_{k, 0}, \tag{1.117}
$$

where the set $\{c_{n}^{-j}\}$ converges with an exponential rate, according to Eq. 1.116, to the scale function at the dyadic rationales. Upon evaluating the scale function, the corresponding wavelet function can be found by relying on Eq. 1.84.

Several scale and wavelet functions which are compactly supported and have the maximum number of vanishing moments for their support width with the external choice of the phase are shown in Figure 1.6 for several $M$. Figure 2.6 shows that smoothness of wavelets increases with the width of their support. In fact, $\phi \in C^{\mu M}$ asymptotically (Daubechies 1988) for large $M$, where
\[ \mu = 1 - \log 3 / (2 \log 2) = 0.2075. \]

A more delicate analysis of the wavelet regularity is feasible for small values of \( M \). Daubechies and Lagarias (1991) proved that the scale function is fractal so that, at least, \( \phi \in C^\alpha(R) \), where \( \alpha \) is given in Table 1.2 for some \( M \). The scale and wavelet functions with \( M=3 \) are already continuously differentiable, though their graphs shown in Figure 1.5 display "peaks". Some vanishing moments can be sacrificed to make wavelets smoother, as shown in Daubechies (1992, p.242).

The choice of a particular wavelet basis is determined by the properties warranted in applications. This dissertation focuses on methods of random field simulation using wavelet bases. In this regard, the most advantageous wavelet bases are those which have some vanishing moments and at the same time small support. These wavelets make the correlation structure of a large class of stochastic fields sparser, reduce numerical errors, and accelerate computations. Therefore, Chapters 3, 4, and 5 use wavelets with the maximum number of vanishing moments for random filed representation and synthesis.
Table 1.1: The filter coefficients $h_k$ for compactly supported wavelets with the highest number of vanishing moments compatible with their support width.

<table>
<thead>
<tr>
<th>$k$</th>
<th>coefficients $h_k$</th>
<th>$k$</th>
<th>coefficients $h_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$M=2$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$(1 + \sqrt{3})/(4\sqrt{2})$</td>
<td>1</td>
<td>.3326705529500825</td>
</tr>
<tr>
<td>2</td>
<td>$(3 + \sqrt{3})/(4\sqrt{2})$</td>
<td>2</td>
<td>.8068915093110924</td>
</tr>
<tr>
<td>3</td>
<td>$(3 - \sqrt{3})/(4\sqrt{2})$</td>
<td>3</td>
<td>.4598775021184914</td>
</tr>
<tr>
<td>4</td>
<td>$(1 - \sqrt{3})/(4\sqrt{2})$</td>
<td>4</td>
<td>-.1350110200102546</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>-.0854412738820267</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>.0352262918857095</td>
</tr>
<tr>
<td>$M=4$</td>
<td></td>
<td>1</td>
<td>.2303778133088964</td>
</tr>
<tr>
<td>2</td>
<td>.7148465705529154</td>
<td>2</td>
<td>.3965393914818912</td>
</tr>
<tr>
<td>3</td>
<td>.6308807679398587</td>
<td>3</td>
<td>.7291320908461957</td>
</tr>
<tr>
<td>4</td>
<td>-.0279837694168599</td>
<td>4</td>
<td>.4697822874051889</td>
</tr>
<tr>
<td>5</td>
<td>-.1870348117190931</td>
<td>5</td>
<td>-.1439060039285212</td>
</tr>
<tr>
<td>6</td>
<td>.0308413818355607</td>
<td>6</td>
<td>-.2240361849938412</td>
</tr>
<tr>
<td>7</td>
<td>.0328830116668852</td>
<td>7</td>
<td>.0713092192668272</td>
</tr>
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<td>8</td>
<td>-.0105974017850690</td>
<td>8</td>
<td>.0806126091510774</td>
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<td></td>
<td></td>
<td>9</td>
<td>-.038029369350104</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>-.0165745416306655</td>
</tr>
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<td></td>
<td></td>
<td>11</td>
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<td></td>
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<td>.0004295779729214</td>
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<td>13</td>
<td>-.0018016407040473</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14</td>
<td>.0003537137997745</td>
</tr>
</tbody>
</table>

Table 1.2: The Holder exponent for the scale function of wavelets with maximum number of vanishing moments for their support width

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.5500</td>
</tr>
<tr>
<td>3</td>
<td>1.0878</td>
</tr>
<tr>
<td>4</td>
<td>1.6179</td>
</tr>
</tbody>
</table>
1.5.4 Fast Wavelet Transform

Any function can be expanded in a wavelet basis in $O(N)$ operations without performing numerical integrations. This algorithm relies on the sub-band coding and reconstruction of signals and considers $m_0(\xi)$ and $m_f(\xi)$ as a low-pass filter and a high-pass filter, respectively (Rioual and Duhamel 1992, and Rioul 1993). The wavelet decomposition algorithm can be compared with the FFT algorithm that requires only $O(N \log N)$ operations to calculate the trigonometric expansion of a function.

Let $f_j$ be a projection of $f(x)$ into the space $V_j$ of the multiresolution analysis. Since $\text{clos}_{L^2(R)} \bigcup V_j = L^2(R)$ and $\cap V_j = \emptyset$,

$$\lim_{j \to -\infty} f_j = f, \quad \lim_{j \to \infty} f_j = 0,$$

(1.118)

where the limits are taking in the $L^2(R)$ sense. Therefore, $f_j$ can approximate $f(x)$ adequately if $J$ is sufficiently small. That is,

$$f = f_J = \sum_k c^J_k \phi_{j,k},$$

(1.119)

where

$$c^J_k = \int f \phi_{j,k} dx.$$  

(1.120)

As $V_j = V_{j+1} \oplus W_{j+1}$, the function $f_j$ can be decomposed into two functions defined in the coarser $(J+1)$-th scale

$$f_J = \delta_{J+1} + f_{J+1},$$

(1.121)

where $\delta_{J+1}$ is the projection of $f_J$ into the space $W_{J+1}$. That is,
\[ \delta_{J+1} = \sum_k d_k^{J+1} \psi_{J+1,k}, \]  

(1.122)

where

\[ d_k^j = \int f \psi_{j,k} \, dx. \]  

(1.123)

By repeating this procedure further the expansion of \( f(x) \) in the wavelet basis can be found

\[ f = f_J = \delta_{J+1} + f_{J+1} = \delta_{J+1} + \delta_{J+2} + \ldots + \delta_{J+L} + f_{J+L} = \delta_{J+1} + \delta_{J+2} + \ldots + \delta_{J+L}. \]  

(1.124)

Further, substituting Eqs. 1.83 and 1.83 into Eqs. 1.120 and 1.122, respectively, the wavelet and scale coefficients can be recursively determined as

\[ c_k^j = \sum_{l=0}^{2M-1} h_{l+1} c_{2k+l-1}^{j-1}, \]  

(1.125)

\[ d_k^j = \sum_{l=0}^{2M-1} g_{l+1} c_{2k+l-1}^{j-1}. \]  

(1.126)

Note that Eqs. 1.50 and 1.52 represent Eqs. 1.125 and 1.126 for the Haar basis. This recursive multi-scale procedure can be described as follows. From \( c_k^j \) compute \( c_k^{j+1} \) and \( d_k^{j+1} \) using Eqs. 1.125 and 1.126. Then, \( c_k^{j+2} \) and \( d_k^{j+2} \) can be computed again from \( c_k^{j+1} \), etc. Schematically, this procedure is shown in Figure 1.7. Note that Eqs. 1.125 and 1.126 represent moving average schemes sampled only at even integers.

The reconstruction algorithm can be developed by substituting Eq. 1.121 into Eq. 1.120. Specifically
\[ c_k^{j-1} = \langle f_{j-1}, \phi_{j-1,k} \rangle = \langle \sum_l c_l^j \phi_{j,l} + \sum_l d_l^j \psi_{j,l}, \phi_{j-1,k} \rangle = \sum_l (c_l^j h_{k-2l+2} + d_l^j g_{k-2l+2}) \cdot \]

This procedure schematically shown in Figure 1.8 can be described as follows. From \( c_k^{j+1} \) and \( d_k^{j+1} \) compute \( c_k^j \) using Eq. 1.127. Then, \( c_k^j \) and \( d_k^j \) are known and \( c_k^{j-1} \) can be computed, etc. Note that Eq. 1.127 represents two moving average schemes with the moving average coefficients sampled only at even integers. Thus, each of the arrows in Figure 1.8 can be viewed as a moving average algorithm preceded by an upsampling. In electrical engineering terms Eqs. 1.125, 1.126 and 1.127 are the analysis and synthesis steps of a sub-band scheme with exact reconstruction.

In this regard, Eqs. 1.87 and 1.88 can be viewed as the frequency response of the moving average filters associated with Eqs. 1.125 - 1.127. The module of this frequency response is plotted in Figure 1.9 for the Daubechies wavelets with the maximum number of vanishing moments for several values \( M \). As the value \( M \) becomes large these filters tend to the ideal low- and high-pass filters, respectively. Indeed, substituting Eqs. 1.111 and 1.112 into Eq. 1.105 yields (Saito and Belkin 1993)

\[ |m_0(\xi)|^2 = \frac{1}{2} + \frac{1}{2} \left[ \frac{(2M - 1)!}{(M - 1)!4^M - 1} \right]^2 \sum_{k=1}^{M} \frac{(-1)^{k-1} \cos(2k - 1)\xi}{(2k - 1)(M - k)!(M + k - 1)!}. \]

Therefore, as \( M \to \infty \),

\[ |m_0(\xi)|^2 \to \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cos(2k - 1)\xi}{(2k - 1)}. \]

Eq. 1.129 represents the Fourier series of the characteristic function of \([-\pi/2, \pi/2]\).

This tendency can be clearly observed in Figure 1.9. In the limit, the wavelet expansion of
a function is a collection of pieces which are band-limited in the frequency domain. These basis functions are often called the Shannon wavelets since the corresponding scale function is equal to \( \frac{\sin \pi x}{\pi x} \); this function relates directly to the sampling theorem (Aldroubi and Unser 1992, Walter 1994). These wavelets are also considered by Newland (1993a,b, 1994a,b) who defined them as harmonic wavelets. Unfortunately, the Haar and Shannon wavelets exhibit extremely slow decay in the frequency domain and the time domain, respectively. They provide quite poor time or frequency localization and are not efficient for computational algorithms. In this respect, the Daubechies wavelets provide the trade-off between the optimal time and frequency behavior.

Functions \( \phi_{j,k}, \phi_{j,l} \) have overlapping supports for \( k \neq l, M > 1 \), unless the Haar basis is considered. As a result, the pyramid structure of the wavelet decomposition and reconstruction algorithms spill out of the interval where the function \( f(x) \) is initially defined. This problem can be addressed either by making the original signal periodic and replacing the original wavelet basis with its periodized counterpart or by introducing boundary layers where only some wavelet parameters are handled precisely to eliminate the bias within the domain of interest.

The numerical expediency of the wavelet reconstruction and decomposition algorithms is addressed by Herley and Vatterli (1993). For example, the FFT algorithm can be incorporated to perform the MA convolutions of Eqs. 1.125-1.127.

Finally, sampling of continuous functions is discussed by Rioul and Duhamel (1992) in conjunction with the wavelet expansion. In particular, methods for converting an analog (digital) signal into the digital (analog) form are introduced and the corresponding errors are evaluated. Let the original analog input be related to a discrete sequence by a discrete-analog converter. Specifically,
\[ f(x) = \hat{f}(x) = \sum_{n} f(n\Delta x) \chi\left(\frac{x}{\Delta x} - n\right), \]  \hspace{1cm} (1.130)\]

where \( \Delta x = 2^{-J} \) is the sampling interval. The function \( \chi(x) \) is selected so that \( f(x) \) is well approximated by \( \hat{f}(x) \). Note that the discretization of Eq. 1.130 precedes the wavelet decomposition algorithm, and the choice of \( \chi(x) \) is determined by the structure of the signal and is completely independent of the wavelet function. Thus, if \( f(x) \) is band-limited, it is appropriate to take \( \chi(t) \) as \( \frac{\sin \omega x}{\omega x} \). Other approximation functions such as piecewise-constant or, more generally, spline approximations can be also considered.

Substituting Eq. 1.130 into Eq. 1.120 yields the following approximation of the scale coefficients on the finest scale

\[ \tilde{c}_k^{-J} = \int f \phi_{-J, k} dx = 2^{J/2} \int (\sum_{n} f(n2^{-J}) \chi(2^J x - n)) \phi(2^J x - k) dx = \]

\[ = 2^{-J/2} \sum_{n} f(n2^{-J}) t(k - n), \]  \hspace{1cm} (1.131)\]

where

\[ t(n) = \int \chi(x) \phi(x - n) dx. \]  \hspace{1cm} (1.132)\]

Eq. 1.130 is the only approximation introduced into the computational process. Thereafter, it is exact. Aldroudi and Unser (1992) considered a similar discretization of analog signals for non-orthogonal wavelet bases. Glowinski, et.al. (1990) utilized oversampled version of Eq. 1.130. That is,

\[ \hat{f}(x) = 2^{-J-s} \sum_{n} f(n2^{-J-s}) \delta(x - n2^{-J-s}). \]  \hspace{1cm} (1.133)\]

In this case Eq. 1.131 can be rewritten as
\[ \hat{\varepsilon}_k^{-J} = 2^{-s-J/2} \sum_n f(n 2^{-J-s}) \phi(2^{-s}(k-n)) . \tag{1.134} \]

If Daubechies wavelets are used there exist constants \( C_1 \) and \( C_2 \) such that

\[ \left| c_k^{-J} - \hat{\varepsilon}_k^{-J} \right| < C_1 2^{-3J/2 - s} + C_2 2^{-3J/2 - \lambda s} , \tag{1.135} \]

where \( \lambda = \min\{1, \alpha - 1\} \) and \( \alpha \) is the Holder exponent of the wavelet function given in Table 2.2. Belkin et. al. (1991) developed alternative quadrature formulae for fast evaluation of the coefficients \( c_k^J \) as it is discussed in Chapter 3 in the context of the determination of the correlation structure of stochastic processes in wavelet bases.

### 1.5.5 Approximation Properties of Wavelet Bases

This section discusses approximation of signals by retaining only a finite number of wavelet coefficients. The multiresolution analysis yields that any \( L^2(R) \) function can be uniquely represented by its wavelet series

\[ f(x) = \sum_{j, k = -\infty}^{\infty} d_k^j \psi_{j, k}(x) , \tag{1.136} \]

where the infinite sum converges in the \( L^2(R) \) sense. However, since computational resources are limited, Eq. 1.136 must be truncated, and information from only several scales can be retained. Only the wavelet approximation of a signal is available for analysis though the wavelet coefficients in Eq. 1.136 are assumed to be found exactly.

In this regard, it is pointed out that the approximation properties of wavelets depend on the difference between the signal and its projection on the \( f \)-th scale. This problem was discussed by Glowinski et.al. (1990), Meyer (1990), Daubechies (1992), Jaffard and
Laurencot (1992) who facilitated their studies by the multi-resolution and local structure of wavelets and developed several quite useful estimates. Though the multi-resolution analysis is formulated for the $L^2(R)$ space, the series in Eq. 95 is valid in a much broader sense. The local properties of functions, such as the smoothness and the differentiability, can be studied by exploring the magnitude of the wavelet coefficients from different scales. Thus, if $f(x)$ is Holder continuous at $x_0$ with the exponent $\alpha$, $0<\alpha<1$

$$ |f(x) - f(x_0)| < C|x - x_0|^\alpha, \quad (1.137) $$

then,

$$ \max_{k, \text{support}(\psi_{-j,k}) \ni x_0} d_k^j = O\left(2^{-\frac{j(\frac{1}{2} + \alpha)}{}}\right). \quad (1.138) $$

Vice versa, if Eq. 1.138 holds and if $f(x)$ is known to be in $C^\varepsilon$ for some $\varepsilon > 0$, then

$$ |f(x) - f(x_0)| < C|x - x_0|^\alpha \log \frac{1}{|x - x_0|}. \quad (1.139) $$

The fractal nature of sample realizations of the Brownian motion of Section 1.5.1 was determined by using Eqs. 1.138 and 1.139 in conjunction with the Schauder basis.

Meyer (1990) showed that functions from the space $W^{s,p}$ can be completely characterized by their wavelet coefficients. In particular,

$$ f \in W^{s,p} \iff \left(\sum_{j,k} |d_k^j|^2 \left(1 + 4^{-js}\right) 2^{-j} \chi_{[2^j k, 2^j (k+1)]}(x)\right)^{1/2} \in L^p, \quad 1 < p < \infty. \quad (1.140) $$

Eq. 1.140 can be readily used to find approximating properties of wavelets with respect to the $L^p$ norm. Note that the multi-resolution analysis implies that the difference
\[ \| f - f_J \|_{L^p} \]  \hspace{1cm} (1.141)

tends to zero for \( p = 2 \) as \( J \to -\infty \). However, Meyer’s Eq. 1.140 yields that this conclusion can be generalized for any \( p \) and the order of this approximation can be accordingly evaluated. Indeed, substituting Eq. 1.140 into Eq. 1.141 and performing some algebraic manipulations, one can derive (Jaffard and Laurencot 1992) that

\[ \| f - f_J \|_{L^p} \leq C 2^{J^p} \| f \|_{W^{\alpha,p}}. \]  \hspace{1cm} (1.142)

Suppose now that \( f \) is not homogeneously smooth, Eq. 1.142 is not satisfied everywhere, and in some region \( f \in W^{\alpha,p} \). Then, \( f(\mathbf{x}) \) can be approximated more precisely by using more wavelets in the region where it is not adequately smooth. Of course, it requires knowledge of the local structure of \( f(\mathbf{x}) \). For most applications, appropriate adaptive algorithms can be constructed in this regard.

Eq. 1.142 remains valid for any wavelet bases, orthogonal or non-orthogonal. In this regard, Glowinski et al. (1990) considered the property of the Daubechies wavelets which have \( M \) vanishing moments to approximate functions in the Sobolev norm \( W^{\alpha,2} \) and established several error estimates similar to Eq. 1.142. Since the Daubechies wavelets have several vanishing moments, the space \( W_J \) are orthogonal to the spaces of polynomial splines built by using the corresponding mesh. Then, if \( f, \phi, \psi \in W^{\alpha,2} \) the following inequality is readily obtained

\[ \| f - f_J \|_{W^{\alpha,2}} \leq C 2^{J(M-\alpha)} \| f \|_{W^{\alpha,2}}. \]  \hspace{1cm} (1.143)

A careful examination of Eq. 1.143 leads to two methods of reducing the approximation error. Specifically, one can take a smaller value of \( J \) or use wavelets with a larger number of vanishing moments.
These two methods are similar to the spline approximation in conjunction with the finite element method such as reducing the mesh of the discretization (h-adaptive method), or increasing the order of spline polynomials for the shape functions (p-adaptive method). Note that the support of the Daubechies wavelets is proportional to the number of vanishing moments. In this regard, it is noted that the method of reducing the approximation error must be selected upon carefully evaluating all related computational aspects.

1.5.6 Recent Development of the Wavelet Theory and Its Applications

Wavelets are a relatively recent development in applied mathematics. In the last ten years interest in them has grown at an explosive rate. The concept of wavelets combines ideas originated in signal processing (sub-band coding, pyramid algorithm), physics (coherent states), and pure mathematics (Calderon-Zygmund operators). As a consequence of this interdisciplinary origin, wavelets appeal to scientists and engineers of many different backgrounds.

The development of the wavelet theory is driven by the needs of engineering applications. Thus, Grochenig (1993) examined the computational aspects of the frame algorithms and proposed some improvements in this respect. Similarly, the pyramid filtering procedure was studied by Herley and Vetterley (1993). Chui (1992a,b) considered cardinal spline wavelets which did not have finite supports. He showed that the infinite discrete convolutions associated with the wavelet reconstruction and decomposition algorithms can be readily truncated beyond a few elements; the related errors decrease exponentially in this case. New families of wavelets were developed recently. Walter (1992) extended wavelets for the representation of distributions (generalized functions). Chui (1992b) developed non-orthogonal spline wavelet bases and studied their properties. Similar studies were pursued by Unser et.al. (1993) with emphasis on the filter bank
theory. Application of wavelets to the solution of partial differential equation requires wavelets with additional restrictions; for example, they must satisfy certain boundary conditions. The related study was initiated by Auscher (1992a), Jaffard and Laurencot (1992) and others. Finally, several useful generalizations of the theory of multiresolution analysis and of the related construction of local and multi-scale bases are proposed in the literature. In this regard, Steffen et.al. (1993) developed orthogonal $K$-regular $M$-band wavelets that are a straightforward generalization of the Daubechies dyadic wavelets. Also, Auscher (1992b) generated wavelet bases having rational dilation factors. A brief review of the development and extension of the wavelet theory can be found in Daubechies (1992).

The research in the area of wavelet based signal and image processing continues to flourish with quite remarkable results. Wavelet packages were efficiently used for developing the theory of filter banks (Gopinath and Burrus 1992, Coifman et.al. 1992, Wickerhauser 1992). Filter banks are efficient convolution structures designed to convert time-division multiplexed data to frequency-division multiplexed data and vice versa. The convolution structure of filter banks can be viewed as special band-limited filters which decompose signal into time and frequency limited data. Further, Aldroudi and Unser (1992) developed an analog of the sampling theorem for multi-resolution analysis by generalizing the concept of band limited functions; the class of functions limited to a given wavelet scale was introduced. Moreover, they showed that cardinal spline wavelets asymptotically provide both the classical sampling theory of band limited functions and the Gabor transformation with the best time-frequency resolution.

Clippingdale and Wilson (1989) proposed a multiscale algorithm for filtering noisy images by using the two-dimensional Haar basis. Their estimates were robust and performed better that many of the existing techniques in terms of the computational efficiency, the signal to noise gain, and the subjective appearance. Extending this approach
Saito and Belkin (1993) suggested to use auto-correlation function of Daubechies wavelets. Though these functions do not form an orthonormal basis their properties can be quite useful for data processing and analysis. Further, the detection of image edge and the signal compression were addressed by Mallat (1989b). In fact, he introduced the multiresolution analysis, the cornerstone of the modern wavelet theory, specifically for image processing applications in conjunction with the quadrature mirror filtering. In subsequent papers he and his colleagues further substantiated the corresponding theory and applications. For example, Fronment and Mallat (1992) introduced a compact image coding algorithm that separates the edges from the texture information by analyzing the magnitude of the wavelet coefficients. Also, they proposed an iterative algorithm that reconstructs the whole image from important edges with a sufficiently small error within the image texture.

The magnitude analysis of the wavelet transform is a quite popular method of non-stationary signal analysis. It improves the "visible speech" aids for deaf, as well as provides tools for signal filtering and reconstruction (Irino and Kawahara 1992, Anderson 1993). Generally, the amplitude of the wavelet transformation of a signal, or the scalogram, can be considered as a representative of a new family of distributions revealing the time-frequency contents of signals in the manner shown in Figure 1.3(a). In this regard, Bentrand and Bentrand (1989) improved the traditional time-frequency analysis that fails to represent adequately broad band signals. In particular, they incorporated an affine smoothing of the time-frequency domain. A new distribution reflects, for example, the time-frequency contents of $\delta(t - t_0)$ quite adequately. Further, Flandrin (1989a) showed that this distribution represents the time-scale analysis related to the wavelet theory. This concept was further generalized by Rioul and Flandrin (1992) who constructed a family of time-scale distributions analogous to the Cohen' class of time-frequency distributions (Cohen 1966, 1989) by introducing the affine smoothing of the
Wigner-Ville distribution; they described some of the properties of the new time-scale distributions. Also, Katambe and Boundreaux-Bartels (1992) compared in detail the cross-terms of the time-frequency and time-scale distributions; the magnitude of these cross-terms is an essential characteristic in analyses of composite signals. They showed that in the general case the time-scale distributions are not better than the conventional time-frequency distributions and are advantageous only for quite special signals. Finally, Cohen (1993) generalized the concept of the distribution using operator formalism. He showed that the time-scale distribution can be introduced by utilizing the following definition of the scale operator

\[ C = \frac{1}{2} (TW + WT). \tag{1.144} \]

Here, \( T \) and \( W \) are the time and frequency operators, respectively, resulting in the multiplication by the corresponding physical quantity. This quite abstract formalism yields time-scale distributions proposed by Bentrand and Bentrand (1989) and Rioul and Flandrin (1992) as a particular case.

Note that Cohen (1993) considered the scale as an independent physical quantity. A transformation from the time domain to the scale domain was shown to be equivalent to the Mellin transform of functions. Recently, a new theory of scale dynamic processes was developed; see Basseville et. al. (1992a) for review and references. Scale dynamic systems can describe a large family of physical processes quite well. In this regard, it is pointed out that the wavelet reconstruction algorithm introduced in Section 1.5.4 can be viewed as a realization of a scale dynamic system when the scale coefficients of the finer scale are calculated by using a "dynamic" relationship between the scale coefficients on the coarser scale and the loading factors in the form of the wavelet coefficients. The wavelet based synthesis method developed in Chapters 4 and 5 can also be understood by invoking the
concepts of scale domain systems.

Finally, wavelets are successfully used for new numerical systems due to their quite exceptional property to generate local and smooth basis functions with some vanishing moments and a multiscale structure. Meyer (1989) investigated the capability of wavelets to compress some operators commonly used in numerical analysis. Using this approach Belkin et.al. (1991) developed a class of algorithms for numerically evaluating some integral operators; see also Alpert (1992), Alpert, et.al. (1990), Belkin (1993), Belkin, et.al. (1992). These algorithms require \( O(N) \) or \( O(N \log N) \) to multiply an \( N \times N \) non-Toeplitz matrix by a vector. Similar algorithms are also discussed in Section 3.3 in conjunction with the evaluation of covariance matrices for a class of stochastic processes.

The wavelet based finite element analysis has also attracted significant interest of scientists. In this case, the term "finite element" loses its physical meaning since the domain is not implicitly divided into a set of small "finite elements" with prescribed shape functions. The solution of a continuous mechanics problem is simply extended in a wavelet basis, and the Galerkin projection is used to minimize the error. This approach was discussed in Glowinski, et.al. (1990), Jaffard and Laurencot (1992), Liandrat, et.al. (1992), Meyer (1989), Perrier (1989), Tchamitchian (1989). They advocated the wavelet based Galerkin projection with the following arguments:

1) Due to the local structure and the quasi-differential nature, wavelets can effectively localize singularities. For example, the solution of the nonlinear Navier-Stokes equation may have areas of local disturbances. In this case, wavelets can be used as a mathematical microscope which is moved over the domain with different resolution. An efficient adaptive algorithm localizing the turbulence effect can be formulated.

2) Wavelet bases are used to discretize continuous differential operators so that the condition numbers of the associated matrices are size-independent upon preconditioning.
The condition number of matrices is an important factor which describes the stability of numerical algorithms for solving partial differential equations. Thus, the finite element method yields matrices with the condition number $\sim h^p$, where $p$ is the order of a differential equation and $h$ is the size of a discretization. The discretization of a continuous problem must be fine for an adequate approximation; this leads to selecting a small value for $h$. Also, the convergence of iterative methods is determined by the condition number of matrices. For example, in the conjugate gradient method, the number of iterations needed for a given accuracy is proportional to the condition number of a matrix. Several methods for "preconditioning" are developed to increase, as much as $O(h^{p/2})$, the condition number in conjunction with the finite element method. In this regard, the sensible study of the multi-scale structure of wavelets provides an effective preconditioning method which yields matrices of differential operators with the condition number independent of the "mesh" of the discretization.

3) There is a similarity between the wavelet and multigrid methods of solving differential equations. These two methods are closely related since both use the same ingredients: hierarchical bases and smoothing resolution algorithms. The difference lies in the way of using them. The multigrid method mixes decomposition on hierarchical bases and resolution, while for wavelets, these two parts are performed once for all (Jaffard and Laurencot 1992).

The complexity of numerical procedures for calculating matrix coefficients is a drawback of the wavelet based Galerkin projection. Also, the associated matrices are not as sparse as for the spline approximation. Moreover, only problems with a symmetric variational formulation can be addressed by using this projection method. However, the available theoretical and numerical results suggest that the advantages of incorporating wavelets can outweigh some of these subtleties.
Figure 1.1. Two Haar wavelets from different scales.
Figure 1.2. Decomposition of a piece-wise constant function using the Haar wavelet.

\[
\begin{align*}
\mathcal{H}_{[k/2]}^{-J_0+1} &= \frac{1}{2} f_0(2^{-J_0}2l) + \frac{1}{2} f_0(2^{-J_0}(2l+1)) \\
\mathcal{H}_{[k/2]+1}^{-J_0+1} &= \frac{1}{2} f_0(2^{-J_0}(2l+2)) + \frac{1}{2} f_0(2^{-J_0}(2l+3)) \\
\mathcal{H}_{[k/2]}^{-J_0+1} &= \frac{1}{2} f_0(2^{-J_0}2l) - \frac{1}{2} f_0(2^{-J_0}(2l+1)) \\
\mathcal{H}_{[k/2]+1}^{-J_0+1} &= \frac{1}{2} f_0(2^{-J_0}(2l+2)) - \frac{1}{2} f_0(2^{-J_0}(2l+3))
\end{align*}
\]
Figure 1.3. Time-frequency localization for
(a) the wavelet transform (b) the short-time Fourier transform.
Figure 1.4. Time-frequency atoms for a constant window function: 
a) signal, b) low frequency atom, 
c) intermediate frequency atom, d) high frequency atom.
Figure 1.5. Time-frequency atoms with a frequency dependent window:
a) signal, b) low frequency atom,
c) intermediate frequency atom, d) high frequency atom.
Figure 1.6. Orthogonal compactly supported scale and wavelet functions with the maximum number of vanishing moments.
Figure 1.7. Schematic representation of the wavelet decomposition algorithm.

Figure 1.8. Schematic representation of the wavelet reconstruction algorithm.
Figure 1.9. Absolute value of the frequency responses of filters associated with the Daubechies wavelets with the maximum number of vanishing moments.
Chapter 2. A Perspective on the Monte-Carlo Method

Modern engineering systems are becoming increasingly complex, involving state-of-the-art technology from both material sciences and control systems. This fact, along with the opportunity to use advanced computing systems in design practice, has intensified the interest in the analysis of environmental, structural, and mechanical systems the properties of which exhibit random fluctuations. Providing for these random effects at the modeling stage introduces more flexibility and realism in the resulting designs.

An adequate interpretation of the uncertainties in engineering systems requires development of quite sophisticated mathematical models. These complex engineering problems can not be analyzed with the relatively limited analytical methods available for practical implementation. Often, the straightforward Monte Carlo simulation is the most reliable method.

Monte Carlo simulation is a quite powerful, though computationally costly method of engineering analysis (Hammersley and Handscomb 1964, Rubinstein 1981). It involves a series of simulations of the random variates and consequent evaluation of the system response for particular realizations of the random variates by using deterministic methods. Averaging provides for the estimates of the corresponding probabilistic quantities of the response.

Often, the most adequate models of engineering uncertainties are manifested as stochastic processes and/or random fields. Then, the implementation of the Monte Carlo method requires obtaining of an ensemble of samples of the stochastic processes/fields. The numerical complexity of the Monte Carlo method is determined, to a great extent, by the expediency of the procedure used for the synthesizing random fields. In this regard, several methods for random field synthesis approximating only the second order moments
are reviewed in the sequel. Since the considered simulation methods are capable of approximating only the first and the second order probability moments, random fields are assumed to be zero mean and Gaussian, unless it is stated otherwise. Also, it is pointed out that a non-zero mean random field can be generated by adding a deterministic function to a zero mean random field. Non-Gaussian random fields can be simulated from Gaussian fields by using memoryless non-linear transformations (Yamazaki and Shinozuka 1988, Grigoriu and Ariarathnam 1988). However, in this case only univariate probability density functions are approximated reliably. Non-Gaussian synthesis procedures which approximate multi-variate probability density functions can be implemented numerically only for small-dimensional random vectors (Rubinstein 1981).

2.1 Generation of Random Variables

Random numbers are the fundamental elements of any simulation scheme. The necessity for rapid generation of a sequence of statistically independent and identically distributed random variates faces the fact that the modern computer systems are designed to process deterministic digital data. Any sequence generated by a computer can be only deterministic. Nevertheless, some less sophisticated algorithms transform digital numbers almost in an unpredictable way. On first appearance, the generated deterministic sequence look like a realization of a random variable. Moreover, statistical analysis of this sequence indicates that it does correspond to a realization of uncorrelated identically distributed random variables. The generated numbers are called pseudo-random numbers. The most commonly used methods for generating pseudo-random numbers yield a realization of uniformly distributed statistically independent random variables (Knuth 1969). Some algorithms for pseudo-random number generation and related statistical tests which can be used to assess the properties of random distributions are reviewed and discussed by

Nevertheless, uniformly distributed random variables are quite useful since any random variable $X$ with the cumulative distribution function $F$ can be generated from a pseudo-random number $Z$ uniformly distributed over the interval $[0, 1]$. Specifically,

\[ X = F^{-1}(Z). \]  \hspace{1cm} (2.1)

This technique is known in engineering as the inverse transform method.

Gaussian random variables are the random variables most commonly used in engineering practice. In this case Eq. 2.1 requires the evaluation of the inverse of the error function $erf = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. For this purpose, statistically independent, zero mean, unit variance Gaussian variables $U_1$ and $U_2$ can be generated more efficiently, from the numerical standpoint, by using the following transformation

\[ U_1 = \sqrt{-2 \log Z_1} \cos(2\pi Z_2), \]

\[ U_2 = \sqrt{-2 \log Z_1} \sin(2\pi Z_2), \]  \hspace{1cm} (2.2)

where $Z_1$ and $Z_2$ are statistically independent.

2.2 Simulation of General Gaussian Vectors

The Covariance Decomposition method for synthesizing a Gaussian random vector $Y$ with the covariance matrix $C$ is quite straightforward

\[ Y = LU, \]  \hspace{1cm} (2.3)

where $U$ is a vector of $N$ independent unit variance Gaussian random variables, and $L$ is a
matrix such that $C = LL'$. The matrix $L$ can be determined by using the eigenvalue
decomposition or the Cholevsky factorization of $C$; whatever is deemed more numerically
expeditious for the given problem. Though appealing in its simplicity, this method
requires significant computational efforts to perform the Cholevsky/eigenvalue decompo-
sition and the matrix-vector multiplications, and it is prone to a considerable round-off
error. Thus, this method can be realistically applied only for synthesizing random vectors
of a relatively modest size.

Reduction of numerical complexity is possible if the vector $Y$ is associated with a
particular random field which has an “additional structure” with respect to the covariance
matrix. Chapter 7 discusses this method of random field synthesis in greater detail and
evaluates the requisite computational resources.

2.3 Simulation of Homogeneous Gaussian Fields

Numerical simulation of a homogeneous Gaussian field is based on a (linear)
transformation of another Gaussian field with a simpler correlation structure: a white noise
or a field with orthogonal increments. Since computational resources are always finite a
field must be discretized beforehand. The synthesis methods discussed in the rest of this
chapter utilize different approaches for random field representations and discretization.
The Spectral method, the Auto-Regressive Moving-Average (ARMA) method, the
Turning Band method (TBM), the Noise Shower method and the scale type methods, such
as the Local Average Subdivision (LAS) method, are considered to be the most versatile
methods for random field synthesis.
2.3.1 Spectral Method

The Spectral method is a powerful and robust procedure for synthesizing random fields. In the homogeneous case it is based on the discretization of the spectral representation of real valued stationary stochastic processes of Eq. 1.25. It can be rewritten as

$$Y(x) = \int_{0}^{\infty} \cos(\omega x) u(\omega) + \sin(\omega x) v(\omega) \, d\omega,$$

(2.4)

where $u(\omega)$ and $v(\omega)$ are real valued, zero mean, independent Gaussian processes with orthogonal increment so that $E[du^2(\omega)] = E[dv^2(\omega)] = 2S^r(\omega) d\omega$. The integral in Eq. 2.4 is well defined by using the concept of the stochastic integral (Wong 1971) as the limit in the mean square sense of the following series

$$Y(x) = \lim_{\Delta \omega \to 0} \sum_{k} \cos(\omega_k x) U_k + \sin(\omega_k x) V_k,$$

(2.5)

where $\omega_k = k\Delta \omega$, $U_k = u(\omega_k + \Delta \omega) - u(\omega_k)$, and $V_k = v(\omega_k + \Delta \omega) - v(\omega_k)$. Further, using the properties of $u(\omega)$ and $v(\omega)$, one can show that the sequences $\{U_k\}$ and $\{V_k\}$ are statistically independent, with mean value equal to zero and variance equal to

$$2 \int_{\omega_k}^{\omega_k + \Delta \omega} S^r(\omega) d\omega = 2S^r(\omega_k) \Delta \omega = \sigma_k^2.$$

(2.6)

The first approximation introduced into Expression 2.5 in order to develop an efficient computer algorithm is removing the limit sign. That is, one can assume that for a sufficiently small value of $\Delta \omega$
\[ Y_{\text{appr}}(x) = \sum_{k=0}^{\infty} \cos(\omega_k x) U_k + \sin(\omega_k x) V_k. \]  \hspace{1cm} (2.7)

Eq. 2.7 induces the following square mean error

\[ \varepsilon = E[(Y(x) - Y_{\text{appr}}(x))^2] = \]

\[ = 2 \sum_{k=0}^{\infty} \int_{\omega_k}^{\omega_k + \Delta\omega} \left( (\cos \omega x - \cos \omega_k x)^2 + (\sin \omega x - \sin \omega_k x)^2 \right) S_{\gamma}(\omega) d\omega. \]  \hspace{1cm} (2.8)

The error defined by Eq. 2.8 is equal to zero if the spectral density function has a quite special form

\[ S_{\gamma}(\omega) = \sum_{k=0}^{\infty} S_{\gamma}(\omega_k) \delta(\omega - \omega_k). \]  \hspace{1cm} (2.9)

Note that this approximation makes stochastic process be periodic with period \( T_0 = 2\pi/\Delta \omega \). Further, Eq. 2.7 still can not be implemented in digital computers because it involves an infinite series. The second approximation providing for a reliable simulation algorithm involves truncating the series in Eq. 2.7. That is,

\[ Y_{\text{appr}}(x) \approx \sum_{k=0}^{N-1} \cos(\omega_k x) U_k + \sin(\omega_k x) V_k. \]  \hspace{1cm} (2.10)

Note that the approximation introduced by Eq. 2.10 does not induce any error if the stochastic process is band-limited: \( S(\omega) = 0 \) for \( \omega > \omega_N \).

Further, changing the variables

\[ U_k = A_k \cos \Phi_k, \ V_k = A_k \sin \Phi_k \]  \hspace{1cm} (2.11)
Eq. 2.10 can be equivalently rewritten as

\[ Y(x) = \sum_{k=0}^{N-1} A_k \cos(\omega_k x + \Phi_k) . \]  

(2.12)

Since \( U_k \) and \( V_k \) are Gaussian random variables, the new random variables \( A_k \) and \( \Phi_k \) are statistically independent and have the Rayleigh distribution with mean \( \sigma_k \sqrt{\pi/2} \) and variance \( \sigma_k^2 (2 - \pi/2) \), and the uniform distribution in the interval \([0,2\pi]\), respectively, where \( \sigma_k \) are given by Eq. 2.6. Eqs. 2.10 and 2.12 require the generation of \( 2N \) random variables for synthesizing a single random field sample. In this regard, Shinozuka (1971) and Shinozuka and Jan (1972) proposed to simplify these equations further by using the following stochastic approximation

\[ Z(x) = \sqrt{2} \sum_{k=0}^{N-1} \sigma_k \cos(\omega_k x + \Phi_k) . \]  

(2.13)

Both formulae 2.12 and 2.13 yield stochastic processes which are zero mean and stationary (Shinozuka and Deodatis 1991, Soong and Grigoriu 1993). However, their distribution and temporal averaging properties are quite different.

Clearly, stochastic processes of Eqs. 2.10 and 2.12 are Gaussian since they are a linear combination of Gaussian random variables. On the other hand, the process defined by Eq. 2.13 is not Gaussian. Thus, a generated realization of this stochastic process is strictly bounded

\[ |Z(x)| \leq \sqrt{2} \sum_{k=0}^{N-1} \sigma_k . \]  

(2.14)

Though this bound can be quite large even for a relatively small value of \( N \), it contradicts
the theoretical concept of Gaussian variates. Nevertheless, this model is extensively used to generate Gaussian samples. This practice is well justified if the number $N$ is sufficiently large. Using the central limit theorem Yang (1973) proved that $Z(x)$ converges in probability to a Gaussian random variable for any $x$ as $N \to \infty$. Also, they estimated numerically the difference between the probability distribution function of the simulated process and the Gaussian distribution. Calculations show (Yang 1973, Shinozuka and Deodatis 1991) that for sufficiently large $N$ the error is small from an engineering standpoint. However, some of the examples given by Soong and Grigoriu (1993) indicate that the process $Z(t)$ is essentially non-Gaussian for small $N$. The reduction of the computational cost due to the lowered number of generated random variates in Eq. 2.13 relative to Eqs. 2.10 and 2.12 can be offset by the need to consider a larger number $N$ of frequencies.

Whereas Eq. 2.13 provides a process which is Gaussian only asymptotically as $N \to \infty$, the stochastic processes of Eqs. 2.10 and 2.12 are strictly Gaussian. On the contrary, Eq. 2.13 yields a stochastic process which is ergodic in the second order sense (Lin 1976), whereas the model of Eqs. 2.10 and 2.12 provides a stochastic process which is ergodic only asymptotically as $N \to \infty$. Indeed, Shinozuka and Deodatis (1991) showed that temporal averaging over the interval $[0, T]$ a sample function $f^{(i)}(x)$ generated by using Eq. 2.13 yields the target zero mean and the auto-correlation function $R_f(\tau)$ for $T = T_0$ or $T \to \infty$. In this regard, it is pointed out that simple algebraic manipulations applied with respect to a sample of the process given by Eq. 2.10 yield

$$R_f^{(i)}(\tau) = \frac{1}{T} \int_0^T f^{(i)}(x + \tau) f^{(i)}(x) dx = \frac{1}{2} \sum_{k=0}^{N-1} ((U_k^{(i)})^2 + (V_k^{(i)})^2) \cos(\omega_k \tau) \quad (2.15)$$

for either $T = T_0$ or $T \to \infty$. Here, the superscript $(i)$ denotes the $i$-th realization of the corresponding random variates. Based on Eq. 2.15, Shinozuka and Deodatis (1991) concluded that processes described by Eq. 2.10 are not ergodic. However, the fact that the
process $Y(t)$ is asymptotically, as $N \to \infty$, ergodic was not considered. Indeed, Eq. 2.15 defines a realization of the random process

$$
\tilde{R}_f(\tau) = \frac{1}{2} \sum_{k=0}^{N-1} ((U_k)^2 + (V_k)^2) \cos(\omega_k \tau).
$$

(2.16)

Note that the mean value of $\tilde{R}_f(\tau)$ is equal to

$$
E[\tilde{R}_f(\tau)] = \sum_{k=0}^{N-1} \sigma_k^2 \cos(\omega_k t) = R_f(\tau).
$$

(2.17)

Similarly, the variance of this stochastic process is found equal to

$$
\text{Var}(\tilde{R}_f(\tau)) = \frac{16\omega_n}{N} \int_0^{\omega_n} S_f(\omega) \cos(\omega \tau)^2 d\omega.
$$

(2.18)

That is, the variance decays proportionally to $1/N$; the stochastic process $\tilde{R}_f(\tau)$ tends toward the target auto-correlation function in the mean square sense. Thus, the processes defined by Eqs. 2.10 and 2.13 are different only for small values of $N$. In this case, one should select whether the Gaussian distribution function or the ergodicity of generated samples is more essential for the particular practical problem.

The FFT algorithm can be used to improve the computational efficiency of the spectral simulation method making the number of the requisite operations for a single field generation of order $N\log N$ (Yang 1972, Vaicaitis et.al. 1973, Wittig and Sinha 1975). This requires the discretization of the stochastic field in the time domain consistently with the discretization of the frequency domain; the samples can not be generated at an arbitrary location in the time domain by adding cosines as in Eqs. 2.10, 2.12 and 2.13. That is, the number of the discretization nodes in the time and frequency domains has to be equal and
be equally spaced. Though the material reviewed in this subsection emphasizes the problem of generating one-dimensional stochastic processes, the multi-dimensional extension is quite straightforward. In this case, the corresponding integrals, sums, and indices in the equations of this section should be interpreted as multi-dimensional ones. Note that the multi-variate version of the Spectral method is available; see for example Kareem and Li (1991).

2.3.2 ARMA Method

The Auto-Regressive Moving-Average (ARMA) method for synthesizing stochastic processes is related to the theory of linear systems. Specifically, the stochastic process is viewed as the response of a linear system subjected to the white noise excitation. Indeed, it is shown in Section 1.4.2 that for the one-dimensional uni-variate case any stationary stochastic processes can be represented as a filtered white noise response of a linear system, Eq. 1.35. A straightforward discretization of Eq. 1.35 yields

\[ y(k) = Y(k\Delta x) = \sum_{i = -\infty}^{\infty} b_i w(k - i) \quad , \quad (2.19) \]

where \( \{ w(k) \} \) are uncorrelated random variables, \( E[w(i)w(k)] = \frac{2\pi}{\Delta x} \delta_{ik} \), and \( \Delta x \) is the sampling time. The discrete white noise \( w(k) \) has the same variance as the continuous process with the spectral density function equal to 1 and band-limited to the frequency \([-\omega_n, \omega_n]\). Further, the sampling time and the cut-off frequency are related through the Nyquist relation

\[ \omega_n = \frac{\pi}{\Delta x} \quad . \quad (2.20) \]
The coefficients of Eq. 2.19 are determined by the selected discretization procedure. That is,

\[ b_k = h(k\Delta x)\Delta x . \]  

(2.21)

Eq. 2.19 constitutes the moving-average (MA) model of the discrete stochastic process \( y_k \). According to Eq. 2.19, the MA process is generated by a convolution of the impulse response function \( h(k) = b_k \) of a discrete time system and the white noise sequence. If the integral in Eq. 1.35 corresponds to a causal system, the corresponding MA model is also causal. That is, \( b_i = 0 \) for \( i < 0 \).

The spectral density function of the MA process can be defined by using the discrete Fourier transformation

\[ S_{MA}(\omega) = |H_{MA}(z)|^2 , \]  

(2.22)

where

\[ H_{MA}(z) = \sum_{k = -\infty}^{\infty} b_k z^{-k} , \]  

(2.23)

\[ z = e^{i\omega \Delta x} . \]  

(2.24)

The MA coefficients determined based on Eq. 2.21 are derived by using a discrete representation of the stochastic integral in Eq. 1.35. This particular choice of the discretization scheme may not approximate the continuous signal in an optimal way. Alternatively, the MA coefficients \( \{ b_i \} \) can be determined by minimizing the following error criterion signifying the energy of the discrepancy between the continuous process and the discrete process:
\[ \varepsilon = E[(y(k) - y_{MA}(k))^2] = \frac{\Delta x}{2\pi} \int_{-\omega_n}^{\omega_n} |H(z) - H_{MA}(z)|^2 d\omega, \quad (2.25) \]

where \( H(z) \) is the spectral density function associated with the discrete sequence \( \{y(k)\} \).

The MA process which minimizes Eq. 2.25 can be readily determined by the equation

\[ b_k = \frac{\Delta x}{2\pi} \int_{-\omega_n}^{\omega_n} H(z)z^{-k} d\omega, \quad k = \ldots, -1, 0, 1, \ldots \quad (2.26) \]

The error in Eq. 2.25 is equal to zero when the MA parameters are determined from Eq. 2.26. Therefore, any discrete signal can be represented exactly as an infinite MA process.

If the spectral density function of the process is integrable, the sequence \( \{b_i\} \), \( i = \ldots, -1, 0, 1, \ldots \), has also a finite sum of squares of all elements

\[ \sum_{i=-\infty}^{\infty} |b_i|^2 < \infty. \quad (2.27) \]

In this case Eq. 2.19 can be further simplified by truncating the infinite sum

\[ y_k = \sum_{i=-q_2}^{q_1} b_i w_{k-i}. \quad (2.28) \]

The associated error defined by Eq. 2.25 can be expressed as

\[ \varepsilon_{MA, min} = \sum_{k < q_2, k > q_1} b_k^2. \quad (2.29) \]

The magnitude of \( \varepsilon_{MA, min} \) can be used as a criterion for the quality of approximation.
Note that Spanos and Mignolet (1990) addressed the problem of MA approximation of multi-variate stationary random processes. In this case, additional computational effort must be devoted to the determination of an appropriate spectral factorization of the target spectral density function.

Auto-Regressive (AR) processes constitute another important class of discrete random processes. An AR process can be represented by the equation

\[ y(k) = - \sum_{i=1}^{p} a_i y(k-i) + b_0 w(k). \]  

(2.30)

Eq. 2.30 can be viewed as the best linear estimate of the present value of the discrete process \( y(k) \) by using \( p \) previous values \( y(k-1), y(k-2), \ldots, y(k-p) \) (Makhoul 1975).

The error of this estimation can be expressed as

\[ e = \frac{\Delta x}{2\pi} E \left[ \left( \sum_{i=1}^{p} a_i y(k-i) \right)^2 \right] = b_0^2. \]  

(2.31)

A straightforward minimization of Eq. 2.31 yields a Toeplitz system of linear equations known as the Yule-Walker equations

\[ \sum_{i=1}^{p} R_y(k-i)a_i = -R_y(k), \quad k = 1, \ldots, p, \]  

(2.32)

where \( R(k-i) = E[y_k y_i] \) is the target auto-correlation function. It can be estimated from a simple record of an ergodic process or can be found as the inverse Fourier transform of the corresponding target spectral density function.
\[ R_y(k) = \int_{-\omega_n}^{\omega_n} S(\omega) \cos(k\omega \Delta x) d\omega. \]  
\hspace{1cm} (2.33)

By substituting Eq. 2.32 into Eq. 2.31, the minimum prediction error, or gain, is found to be equal to

\[ b_0^2 = \frac{\Delta f}{2\pi} \left( R(0) + \sum_{i=1}^{p} R_y(i)a_i \right). \]  
\hspace{1cm} (2.34)

Several quite efficient algorithms have been developed for solving the system of equations 2.32. The matrix of this system is symmetric and positive definite as the covariance matrix of the random vector \( \{ y(1), y(2), \ldots, y(p) \}^T \). Since the process \( y(k) \) is stationary this matrix is a Toeplitz matrix; the elements along any diagonal of this matrix are identical. Levinson (1947) proposed an elegant recursive procedure for solving this kind of equations. This procedure was later substantiated by Robinson (1967). Another efficient method to solve Eq. 2.32 recursively was given by Durbin (1960).

The \( k \)-th step of the Durbin’s recursive algorithm yields the AR parameters of a system of order \( k \). Thus, obtaining the AR model of order \( p \), one actually computes all AR models of order less than \( p \). This fact can be used to evaluate the optimal order of the AR model which provides an adequate approximation with a few parameters. The loss of stability of the computed AR systems can be controlled at every step of the algorithm. Also, the value of the corresponding mean square prediction errors is an intermediate result of the corresponding computations. Specifically, it can be shown that the error \( \varepsilon_i \) defined by Eq. 2.31 is a non-increasing function of the filter length. That is,

\[ 0 \leq \varepsilon_{\omega_0} \leq \varepsilon_1 \leq \varepsilon_{i-1} \leq \ldots \leq \varepsilon_0 = R(0). \]  
\hspace{1cm} (2.35)
Further, from another perspective, the AR process can be viewed as the response of a
discrete linear system defined by Eq. 2.30 to a white noise excitation (Makhoul 1975). The
transfer function of this system can be found by applying Z-transformation to Eq. 2.30.
Specifically,

\[ H_{AR}(z) = \frac{b_0}{A(z)}, \]  

(2.36)

where

\[ A(z) = 1 + \sum_{k=1}^{p} a_k z^{-k}. \]  

(2.37)

One can show that poles of the transfer function of a stable causal system are located
inside the unite circle in the complex Z-domain. The requirement of stability is quite
important for synthesizing stochastic processes. Only stable filters can be implemented
numerically. Remarkably, the Yule-Walker system of Eq. 2.32 ensures that all roots of
\( A(z) \) lie inside the unite circle (Grenander and Szego 1958). Thus, the AR method in
conjunction with the Yule-Walker equation provides a stable simulation procedure.

Further, the error of Eq. 2.31 can be equivalently defined in the frequency domain by
taking the Z-transform of Eq. 2.30

\[ \varepsilon = \frac{\Delta x b_0^2}{2\pi} \int_{-\omega_0}^{\omega_0} S(\omega) \frac{S(\omega)}{S_{AR}(z)} d\omega, \]  

(2.38)

where

\[ S_{AR}(z) = |H_{AR}(z)|^2. \]  

(2.39)
In this regard, the gain factor $b_0$ can be obtained by equating the total energy of the target and approximated spectra

$$R(0) = \int_{-\omega_n}^{\omega_n} S(\omega) d\omega = \int_{-\omega_n}^{\omega_n} S_{AR}(z) d\omega = R_{AR}(0) \quad .$$  \hspace{1cm} (2.40)

Note that the manner of approximation of the target spectrum $S(\omega)$ by the AR spectrum $S_{AR}(z)$ is naturally reflected in the relationship between the corresponding auto-correlation functions. The system of the Yule-Walker equations and Eq. 2.40 yields

$$R(i) = R_{AR}(i) \quad , \quad i = 0, 1, \ldots p \quad .$$  \hspace{1cm} (2.41)

Increasing the range over which the target auto-correlation function coincides with the estimated one, or equivalently increasing the AR order $p$, leads to a better fit between two corresponding spectra. In the limit $p \to \infty$, $R(i) = R_{AR}(i)$ for all values of $i$. Hence, the two spectra become identical. This shows that, theoretically, any spectrum can be approximated by an infinite AR model and any random process can be generated as the white noise response of the corresponding discrete linear system.

One important constraint on the value of the AR prediction error can be obtained by noticing that the minimal prediction error is always equal to $\varepsilon = b_0^2$, Eq. 2.31. Therefore, Eq. 2.38 yields

$$\int_{-\omega_n}^{\omega_n} \frac{S(\omega)}{S_{AR}(z)} d\omega = 2\omega_n \quad .$$  \hspace{1cm} (2.42)

That is, the ratio of the target spectrum $S(\omega)$ to its approximation $S_{AR}(z)$ is close to 1. Eq. 2.42 is valid for any value $p$. In particular, it becomes an identity for $p \to \infty$ or for the
case when the spectrum has a finite number of poles.

Although it is shown that the approximated spectral density function coincides, asymptotically, with the target one, the frequency domain analysis can enhance understanding of the quality of the obtained approximation. In this regard, a useful information can be obtained from evaluating the value of the following integral

$$
\hat{c}_0 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \log(S_{AR}(z)) d\omega = \log b_0^2 - \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \log|A(z)|^2 d\omega.
$$

(2.43)

Poles of a stable AR system are located inside the unite circle. Then, the second part in Eq. 2.43 is equal to zero and

$$
\varepsilon_p = b_0^2 = \hat{c}_0.
$$

(2.44)

Since the error \( \varepsilon_p \) decreases as \( p \) increases (see Eq. 2.35) the minimum occurs as \( p \to \infty \) and is equal to

$$
\varepsilon_{min} = \varepsilon_{c0},
$$

(2.45)

where

$$
c_0 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \log(S(\omega)) d\omega.
$$

(2.46)

This result shows that the minimum prediction error associated with an AR filter of infinite length can be nonzero even when the spectrum \( S_{AR}(z) \) is identical to the power spectral density function \( S(\omega) \). In this regard, note that the value \( \varepsilon_p \) is equal to the portion of the energy that is not representable by the \( p \)-th order filter.
Nevertheless, for a class of processes, called positive semi-definite processes, the minimal error $\varepsilon_{\text{min}}$ is equal to zero. In this case the corresponding power density function is "almost" identical to zero over a range in the frequency domain. Such processes are often used in engineering applications. For example, the Pierson-Moscowitz spectrum is defined in the dimensionless form by the equation (Spanos 1983)

$$S(\omega) = \frac{e^{s/4}}{\omega^5} e^{-\frac{5}{4} \omega^4}.$$  

Clearly, substituting Eq. 2.47 into Eq. 2.46 yields $-\infty$ and the minimum error is equal to zero; the process is perfectly predictable from its past values. The numerical instability of the Yule-Walker equations is the major problem which is associated with the AR approximation of predictable signals. The minimum and maximum eigenvalues, $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$, of the matrix of the Yule-Walker system tend asymptotically, as $p \to \infty$, to the minimum and maximum of the spectral density function, respectively (Grenander and Szego 1958); this results in the extremely large condition number of the matrix, $d = \lambda_{\text{max}}/\lambda_{\text{min}}$, since $\lambda_{\text{min}} = S_{\text{min}} \to 0$. Also, Spanos and Mignolet (1986) observed that for semi-definite processes almost all points of the unit circle $z = e^{i \omega \Delta \tau}$, where $S(\omega) \neq 0$, are poles of the transfer function of the infinite order AR filter. Indeed,

$$\lim_{p \to \infty} \varepsilon_p = \lim_{p \to \infty} \frac{\Delta x b^2}{2\pi} \int_{-\omega_n}^{\omega_n} \frac{S(\omega)}{S_p(z)} d\omega = \lim_{p \to \infty} \frac{\Delta x}{2\pi} \int_{-\omega_n}^{\omega_n} S(\omega)|A_p(z)|^2 d\omega = 0.$$  

(2.48)

Since $S(\omega) \geq 0$, Eq. 2.48 yields

$$\lim_{p \to \infty} |A_p(z)|^2 = 0$$  

(2.49)

where $S(\omega) \neq 0$. Then, the poles corresponding to a high but finite order AR filter can be
arbitrary close to the unit circle. Thus, the AR spectral approximation of semi-definite processes displays sharp peaks. Relying on Eq. 2.41 and on the approximation theory, the spectra $S(\omega)$ and $S_{AR}(z)$ are shown to coincide at $p+1$ frequencies, at least, within the interval $[0, \omega_n]$ since the auto-correlation values $R(k)$ and $R_{AR}(k)$, $k = 0, 1, \ldots p$ are the first $p+1$ coefficients of the Fourier series approximation of the corresponding spectra; the approximation $S_{AR}(z)$ oscillates with a high amplitude around the target spectrum $S(\omega)$.

The preceding comments elucidate the reasons for which it is not possible to determine a reliable AR representation of semi-definite processes in a straightforward manner. Some regularization methods were described by Makhoul (1975). For example, Mignolet and Spanos (1991b) advocated to shift slightly the target spectrum by adding a small positive regularizing constant, $S(\omega) \rightarrow S(\omega) + \delta$.

The Auto-Regression Moving-Average (ARMA) system is the generalization of the AR and MA systems. Unlike the two models described previously, the transfer function of the ARMA system contains simultaneously poles and zeros which enhance the capacity of ARMA models to approximate the target spectra. The $k$-sample of the ARMA$(p,q)$ process can be generated from the $p$ previous ones by using the equation

$$y(k) = -\sum_{i=1}^{p} a_i y(k-i) + \sum_{l=0}^{q} b_l w(k-l). \quad (2.50)$$

Eq. 2.50 defines the ARMA process as the white noise response of a linear casual discrete system with the transfer function

$$H_{ARMA}(z) = \frac{B(z)}{A(z)}, \quad (2.51)$$

where
\[ A(z) = \sum_{k=0}^{p} a_k z^{-k}, \quad \text{and} \quad B(z) = \sum_{l=0}^{q} b_l z^{-l}. \quad (2.52) \]

The ARMA method for random field simulation can be associated with continuous Markov processes of Eq. 1.40. A component of a Markov stationary stochastic process can be generated as the steady-state response of a linear differential equation

\[ D \left( \frac{d}{dx} \right) Y(x) = N \left( \frac{d}{dx} \right) w(x). \quad (2.53) \]

Clearly, the ARMA model for synthesizing \( Y(x) \) digitally can be determined by using the finite difference methods, forward, central or backward, with respect to approximating the derivatives in Eq. 2.53. However, some of the finite difference discretization schemes do not yield stable discrete systems or stability may require quite small sampling time \( \Delta x \).

As an alternative, the ARMA model associated with Eq. 2.53 can be found by using the frequency domain principles of converting analog transfer functions defined as the prototype Laplace transform into discrete transfer functions which are built on the principles of Z-transform. Note that the finite difference schemes can lend themselves to alternative interpretation as special analog-discrete converters. Of the several methods proposed for converting analog systems into the discrete form, two have proven useful for most applications.

The first one is the impulse-invariant method, which equates the impulse response of a discrete filter to the samples of the prototype analog filter. The impulse response of a discrete system is equal to

\[ h_k = h(\Delta x k) \Delta x \quad (2.54) \]

where the constant factor \( \Delta x \) appears from the proper discretization of the continuous
white noise process, see Eq. 2.21. On the other hand, the analog transfer function of Eq. 1.39, can be written as

\[ H(s) = \sum_{i=1}^{l_2} \frac{C_i}{s - s_i}. \] (2.55)

Then, the associated impulse response can be derived as the inverse Laplace transform of Eq. 2.55. Specifically,

\[ h(\Delta x k) = \sum_{i=1}^{l_2} C_i (e^{s_i \Delta x})^k. \] (2.56)

Substituting Eq. 2.56 into Eq. 2.23 and using equality 2.54 yield

\[ H_{ARMA}(z) = \sum_{k=0}^{\infty} \left[ \sum_{i=1}^{l_2} C_i (e^{s_i \Delta x})^k \right] \Delta x z^{-k} = \sum_{i=1}^{l_2} \frac{C_i \Delta x}{1 - e^{-s_i \Delta x} z^{-1}}. \] (2.57)

Thus, this method maps zeros \( s_k \) of \( H(s) \) into zeros \( z_k \) of \( H_{ARMA}(z) \) by using the transformation \( z_k = e^{s_k \Delta x} \).

Though the requirement for the impulse-invariant method is set on the time domain, the study of the relationship between the corresponding analog and discrete frequency responses is also important. Specifically, one can obtain that

\[ H_{ARMA}(e^{i\omega \Delta x}) = \sum_{k=-\infty}^{\infty} H\left(i\left(\omega - \frac{2\pi k}{\Delta x}\right)\right). \] (2.58)

Thus, the frequency response of the ARMA filter is a periodically repeated version of the frequency response of the corresponding analog filter. This method produces an aliasing,
or an overlapping effect. If the analogue filter contains high frequency components and the sampling time is selected relatively large, the discrete and analog transfer functions may not match adequately.

The second method to transform an analog signal into the discrete form is to use a frequency mapping preserving some important characteristics. In this regard, the bilinear transformation extensively used for signal analysis and filter design applications has the property that it yields a stable filter and does not affect the absolute difference between any two digital counterparts of the corresponding analog transfer functions. The bilinear transformation involves the equation

$$H_{ARMA}(z) = H\left(i\omega = \frac{2}{\Delta x z + 1}\right), \quad (2.59)$$

where

$$s = \frac{2 z - 1}{\Delta x z + 1}, \quad \text{and} \quad z = \frac{2/\Delta x + s}{2/\Delta x - s}. \quad (2.60)$$

The bilinear transformation maps the imaginary axis of the s-plane onto the unit circle of the z-plane. The aliasing effect is avoided but the compression of the frequency axis becomes extreme at high frequencies.

In this regard, Spanos (1983) proposed to use the discretization methods described above for generating ocean waves and evaluated numerically the related error as a function of the sampling time. Note that this ARMA modeling approach requires that a Markovian approximation of the target analog spectrum is available. Alternatively, the coefficients of the ARMA model can be found by minimizing the error criteria.
\[
\epsilon_1 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \frac{S(\omega)}{S_{ARMA}(z)} d\omega , \quad (2.61)
\]

\[
\epsilon_2 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} |H(\omega) - H_{ARMA}(z)|^2 d\omega , \quad (2.62)
\]

where

\[
S_{ARMA}(z) = |H_{ARMA}(z)|^2 , \quad S(\omega) = |H(\omega)|^2 . \quad (2.63)
\]

The error criterion of Eq. 2.61 is similar to Eq. 2.38 and has the same frequency domain interpretation, while the error criterion of Eq. 2.62 is equivalent to Eq. 2.25 which defines the "energy" of the difference of two signals. However, minimization of the introduced error criteria is quite cumbersome and requires substantial computations.

To rectify the difficulties associated with the inefficient nonlinear optimization procedures for minimizing the introduced error criteria, the AR parameters of the ARMA system can be identified first by using the fact that the discrete white noise is an uncorrelated sequence of random variables and the ARMA system is causal. That is,

\[
E[y(k)w(k+i)] = R_{yw}(i) = 0 , \quad i > 0 . \quad (2.64)
\]

Multiplying Eq. 2.50 by \( y(k+n) , \ n > q \), taking mathematical expectation, and employing Eq. 2.64 one can derive the modified Yule-Walker equations

\[
\sum_{i=1}^{p} R_y(n-i) a_i = -R_y(n) , \ n = q + 1 , \ldots . \quad (2.65)
\]

The AR parameters can be found by restricting the range of the parameter \( n \) in Eq. 2.65 to \([q + 1, q + p]\). In this case, the system matrix has a structure which is quite similar to the
structure of the matrix of Eq. 2.32. However, it is not symmetric; this fact accentuate the
difficulties of its solution. Specifically, the derived AR parameters do not necessary corre-
spond to a stable filter and are not adequately robust with respect to the error of numerical
estimation of the auto-correlation function. The latter problem was studied by Cadzow
(1980, 1982) who proposed to use an overdetermined system of equations by extending
the range of the index \( n \) in Eq. 2.65 beyond \( q + p \). Because of computational and model
errors the overdetermined modified Yule-Walker equations do not have an exact solution.
The unknown AR parameters are then found by minimization of the sum of square errors
of every equation. Since the value of the auto-correlation function with a larger lag has a
larger numerical error of estimation, equations which correspond to a larger value of the
index \( n \) are assigned smaller weights in the minimization procedure. This provides the, so-
called, weighted least square method for solving overdetermined system of equations.
Cadzow (1982) used the weighting sequence \((N-k)^3\) in this regard.

Similar arguments can be used to determine the MA parameters of the ARMA model.
Specifically, multiplying Eq. 2.50 by \( y(k-n) \) and \( w(k-n) \), respectively, taking
mathematical expectation, and using Eq. 2.64 yield

\[
R_y(-n) + \sum_{i=1}^{p} a_i R_y(i-n) = \sum_{l=max(0,n)}^{q} b_l R_yw(n-l), \quad n = 1, \ldots p, \tag{2.66}
\]

\[
R_yw(-n) + \sum_{i=1}^{min(n,p)} a_i R_yw(i-n) = 2\omega_p b_n, \quad n = 0, \ldots q. \tag{2.67}
\]

The system of equations (2.66) and (2.67) can be solved for the unknown parameters \( b_l \),
\( l = 0, \ldots q \) and \( R_yw(-n), n = 0, \ldots max(p, q) \). However, the solution of this nonlinear
problem is not straightforward or robust. Several alternative methods have been intro-
duced in the literature to determine the MA parameters. Thus, they can be found by the
conditional minimization of Eq. 2.62 when AR parameters are evaluated using Eq. 2.65.
In this regard, Makhoul (1973) proposed "inverse linear prediction (LP) modeling" of the MA parameters by inverting the target spectrum and by applying the q-pole linear prediction analysis. This method explicitly minimizes the error which is similar to Eq. 2.61 with the inverse expression under the sign of the integral. It yields good results if the target spectrum is smooth relative to the model spectrum. An efficient and robust method to estimate MA parameters was proposed by Cadzow (1980) who considered the auto-correlation coefficient of the ARMA model as a discrete signal with the causal and anti-causal parts. That is,

\[
R^+_y(k) = \begin{cases} 
0.5R_y(0), & k = 0 \\
R_y(k), & k > 0 \\
0, & k < 0 
\end{cases}
\]  

(2.68)

and

\[
R_y(k) = R^+_y(k) + R^+_y(-k).
\]  

(2.69)

Then, the spectral density function could be represented as the Fourier transform of the causal and anti-causal auto-correlation sequences

\[
S_{ARMA}(\omega) = \frac{|B(z)|^2}{|A(z)|^2} = \frac{C(z)}{A(z)} + \frac{C(z^{-1})}{A(z^{-1})} = S^+(\omega) + \overline{S^+(\omega)},
\]  

(2.70)

where

\[
C(z) = \sum_{k=0}^{p} c_k z^{-k}.
\]  

(2.71)

It can be easily shown that the parameters \(c_k\) can be evaluated by using the equation
\[ c_k = R_0^+(k) + \sum_{i=1}^{k} a_i R_0^+(k-i). \] (2.72)

Kay (1980) modified this method to ensure the non-negativeness of the spectral approximation. Minolet and Spanos (1990) applied this approach for synthesizing stochastic processes.

The discussed methods of ARMA approximation yield linear system of equations to find the unknown ARMA parameters. However, they do not have an explicit interpretation in terms of minimization of a physically motivated error criterion in the time or frequency domain. A different framework for ARMA approximation was proposed by Gersch and Foutch (1974), who addressed the structural system identification by using the ARMA modeling method. They performed ARMA approximation in two stages. During the first stage a "long" AR model is identified. A large number of AR parameters ensures that the target and AR spectra are quite close. Then, the AR model can be used instead of the target process. The explicit mathematical representation of the target process as an AR system grants the opportunity to find some quantities which cannot be determined otherwise from the signal records or the target spectrum. In particular, the input-output cross-correlation can be estimated by multiplying Eq. 2.30 by \( w_{k+n} \) and by taking subsequently the mathematical expectation; the auto/cross-correlation of the AR and ARMA models are matched (ACM method)

\[
R_{yw}(k) = \begin{cases} 
0, & k > 0 \\
2 \omega_0 b_0^{AR}, & k = 0 \\
\min(p^{AR}, -k) - \sum_{k=1}^{\min(p^{AR}, -k)} a_i^{AR} R_{yw}(k+i), & k < 0 
\end{cases}
\] (2.73)

Here "AR" in the superscript denotes the quantities associated with the long AR model.
Upon evaluating $R_{yy}(-k), k = 0, 1, \ldots \max(p, q)$ by using Eq. 2.73 and substituting these values into Eqs. 2.66 and 2.67, the ARMA parameters can be found as the solution of a linear system of equations. Gersch and Yonemoto (1977), Samaras et. al. (1983), Spanos and Mignolet (1986) extended this method for synthesizing stochastic processes.

Another two-stage method was formulated by Graupe et.al. (1975). They proposed to determine ARMA parameters by equating the transfer functions of “long” AR and “short” ARMA systems. Specifically,

$$H_{AR}(z) = \frac{b_0^{AR}}{A_{AR}(z)} = \frac{B(z)}{A(z)} = H_{ARMA}(z). \quad (2.74)$$

Cross-multiplying the terms in Eq. 2.74 and equating the coefficients of like powers of $z^{-1}$ yield

$$\sum_{l=0}^{q} b_l a_{n-l}^{AR} = 0, \quad n = p + 1, \ldots p + q, \quad (2.75)$$

$$b_0 = b_0^{AR} \quad (2.76)$$

$$a_n = \begin{cases} \frac{1}{b_0^{AR}} \left( \sum_{l=0}^{n-1} b_l a_{n-l}^{AR} + b_n \right), & n = 1, \ldots q \\ \frac{1}{b_0^{AR}} \sum_{l=0}^{n-1} b_l a_{n-l}^{AR}, & n = q + 1, \ldots p. \end{cases} \quad (2.77)$$

Spanos and Mignolet (1986) used this power order matching (POM) method for synthesizing random see-waves. They gave it a new interpretation of a Padé-type approximation of the transfer function since the difference between the AR and the ARMA transfer functions had the following Laurent expansion
\[
\hat{H}_{AR}(z) - \hat{H}_{ARMA}(z) = \sum_{k = p + q + 1}^{\infty} r_k z^{-k}.
\]

Minolet and Spanos (1987) showed that the ACM and POM methods can be associated with the minimization of the following error criteria

\[
\varepsilon_3 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} |A(z)H_{AR}(z) - B(z)|^2 d\omega.
\]

Note the similarity between the criteria defined by Eq. 2.79 and Eq. 2.62. The first equation can be derived from the second by multiplying the integrand expression by \( |A(z)|^2 \) and assuming that \( H(\omega) = H_{AR}(z) \). Minimization of Eq. 2.79 by using the following constraints

\[
R_{\hat{y}w}(-n) + \sum_{i = 1}^{\min(n, p)} a_i R_{\hat{y}w}(i-n) = 0, \quad n = q + 1, \ldots q + r
\]

yields the ACM method for \( r = 0 \) and the POM for \( r = p \). Eq. 2.80 requires implicitly that the MA coefficients \( b_n, n = q + 1, \ldots q + r \) are identically equal to zero. Thus, a family of two-stage algorithms can be formulated by selecting different values \( r \) in Eq. 2.80. The stability of these two-stage algorithms has been investigated by Spanos and Minolet (1987) and several criteria were found in this regard. In particular, the ACM method always yields a stable solution. Some other error criteria have been also discussed by Minolet and Spanos (1987, 1991a) and Spanos and Minolet (1990) to determine an MA system from the “long” AR model and an ARMA system from the “long” MA model.

Note the significant difference between the criteria defined by Eq. 2.62 and Eq. 2.79. While the criterion of Eq. 2.62 represents the energy of the difference between two
signals, the error defined by Eq. 2.79 lacks a meaningful physical interpretation. Spanos and Zeldin (1995a,b) addressed this problem and developed an iterative algorithm of nonlinear optimization; this iterative algorithm is discussed in Chapter 6 with greater details.

Friendlander and Porat (1984) also appealed to the "energy" criteria to reduce the "long" AR system to the "efficient" ARMA model. They identified energy associated with every mode of the AR system. Clearly, neglecting some modes with energy smaller than some threshold does not significantly affect the spectral approximation. Specifically, this approach can be formulated by expanding the long AR model as follows

\[ H_{AR}(z) = \frac{b_0}{A_{AR}(z)} = \sum_{j=1}^{p_{AR}} \frac{d_j z^{-1}}{1 - z_j z^{-1}}, \tag{2.81} \]

where \( z_j \) are the poles of the AR system. It is assumed that all poles of the AR system are simple. The parameters \( d_j \) can be evaluated by equating the residuals at the poles of \( H_{AR}(z) \). The energy of the \( j \)-th mode can be determined as

\[ E_j = \frac{1}{2 \omega_n} \int_{-\omega_n}^{\omega_n} \left| \frac{d_j z^{-1}}{1 - z_j z^{-1}} \right|^2 d\omega = \frac{|d_j|^2}{1 - |z_j|^2}. \tag{2.82} \]

This method has also been applied in the context of wind and see-wave loading by Krenk and Clausen (1987), Mandal et.al. (1992) and for aerospace engineering applications by Spanos et.al. (1995) and Eberle (1995).

Due to its versatility the ARMA method is extensively used in different engineering applications and industry. Thus, Li and Kareem (1990) applied ARMA method to model wind loading. Further, Grigoriu and Balopoulou (1993) and Gluver and Krenk (1990) studied sampling error associated with the ARMA analog-digital transformation. Mignolet

2.3.3 Scale-Type Methods

Scale type methods for random field simulation have been initially developed for computer graphics and movie industry applications. In this regard, the challenge focuses on simulating artificial textures and landscapes which however look quite natural, on one hand, and are not a "photo-copy" of existing objects, on the other hand. In the late 70s it was realized that realizations of some random fields could be advantageously used in this regard.

Natural landscapes appear to be a quite complicated mixture of several objects of different sizes and locations. However, some fundamental statistical features that make landscapes look naturally can be identified. These characteristics can be described by using the concepts of the statistical self-similarity and self-affinity of fractal curves and surfaces.

A class of curved lines with a new notion of dimension have been constructed by mathematicians. On one hand, a curve of this class has the topological dimension equal to 1. That is, a single point of this line cuts it into two pieces. On the other hand, though the curve is contained within an area of a finite size, its length can not be measured in a conventional way. This curve tends to fill the entire plane with itself. One approach to
define the fractal dimension introduces a ruler of size \( l \) to measure the curve length. The length of a curve relative to the ruler \( l \) can be determined as

\[
L = lN(l),
\]

(2.83)

where \( N(l) \) is the number of steps of the ruler \( l \) in the process of measuring the curve. If the curve has a finite length \( L \), \( N(l) \) behaves asymptotically as \( L/l \). However, for some curves with infinite length, \( N(l) \) has the following asymptotic relation to the ruler length

\[
N(l) \sim 1/l^D,
\]

(2.84)

where \( D \) is the curve fractal dimension.

The curve \( S \) is statistically self-similar if it can be decomposed into \( k \) distinct subsets each of which is scaled down by the ratio \( r \) from the original and is identical in all statistical respects to \( rS \). The fractal dimension in this case can be expressed as

\[
D = \frac{\log k}{\log \frac{1}{r}} = \lim_{l \to 0} \frac{\log N(l)}{\log \frac{1}{l}}.
\]

(2.85)

Similarly, the curve \( S \) is self-affine if it is statistically congruent under “affine” transformation. That is, the curve can be mapped into itself if every coordinate is scaled by a different ratio.

A natural example of a self-similar curve is coastlines. A drive along a coastline is shorter than a walk along the beach. Real coastlines on our planet can, in fact, be characterized by fractal dimensions \( D \) of about 1.15 to 1.25 over a large range of scales (Mandelbrot 1982). Also, the shape of granular materials which has a significant effect on the macro properties of composite engineering materials, such as concrete, soil, and sand, can be modeled by fractal curves with fractal dimension \( D \) in the range from 1.1193 for
smooth elements to 1.1313 for rough profiles (Vallejo and Zhou, 1995).

The property that objects can look statistically similar while at the same time different in details at different scales is the central feature of all methods for synthesizing artificial landscapes. If a realization of a random field is used for landscape simulation, it must be self-affine and have a fraction dimension larger than its topological dimension. The fractional Brownian motion has been selected to model artificial landscapes. This process is defined by Mandelbrot and Ness (1968). Also, the fractional Brownian motion can be approximately associated with \( f \) noise, a process extensively used in engineering to model some natural phenomena (Keshner 1982, Taqqu 1986).

The fractional Brownian motion \( B_H(t) \) is defined by the stochastic integral

\[
B_H(t) = B_H(0) + \frac{1}{\Gamma(H + 0.5)} \left\{ \int_{-\infty}^{0} [(t - s)^{H - 0.5} - (s)^{H - 0.5}] dB(s) \right. \\
+ \int_{0}^{t} [(t - s)^{H - 0.5}] dB(s) \left. \right\}, \tag{2.86}
\]

where \( B(t) \) is the ordinary Brownian motion and \( 0 < H < 1 \). Note that this process reduces to the ordinary Brownian motion for \( H = 0.5 \). The expression for the corresponding auto-correlation function can be readily determined from Eq. 2.86. Specifically,

\[
R_{B_H}(t, s) = V_H(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \tag{2.87}
\]

where

\[
V_H = \frac{-\Gamma(2 - 2H) \cos(\pi H)}{\pi H(2H - 1)}, \tag{2.88}
\]

and \( \Gamma \) is the gamma-function. The fact that the fractional Brownian motion \( B_H(t) \) is a statistically self-affine process is clearly displayed by the equation
\[ E[(B_H(r_{t_2}) - B_H(r_{t_1}))^2] = r^{2H}E[(B_H(t_2) - B_H(t_1))^2] \] .

(2.89)

Further, it can be shown by using Eq. 2.89 that the fractal dimension \( D \) of a realization of \( B_H(t) \) is almost surely equal to

\[ D = N - H, \]

(2.90)

where \( N \) is the topological dimension of the curve + 1.

To develop an efficient procedure for synthesizing the fractional Brownian motion is a challenging problem. A review of related algorithms is summarized by Voss (1985). The fractional Brownian motion is a non-stationary random process. This fact makes difficult the straightforward utilization of the ARMA and spectral methods. Further, it is a process with long, in fact infinite, memory; this requires, for example, a large number of ARMA parameters. Moreover, the simulation algorithm developed for computer graphics applications should have additional margin. Namely, it has to provide for an effective local rendering procedure.

To illustrate this point, consider a view seen from a space-craft approaching a surface of an unknown planet in a science-fiction movie. As the space-craft approaches the surface, its crew observes increasingly smaller objects within an increasingly smaller area of the planet. To animate this view, the algorithm must have the ability to increase efficiently the resolution within a limited local area. The simulation procedure must be able to operate within a non-homogenous discretization mesh. Similarly, the solution of stochastic mechanical problems exhibits much higher rate of variation within geometrical non-homogeneous regions. This requires the introduction of additional nodes for some local areas. In this case, the simulation of mechanics problems also must be performed using non-homogenous meshes.
By realizing the last requirement, Carpenter (1980) and Fournier and Fussell developed a scale-type algorithm simulating the fractional Brownian motion on the increasingly smaller scales. These independent works were consolidated into a single integrated piece and reported in Fournier et.al. (1982). The "random midpoint method", or "recursive subdivision method", is based on the following property of the process \( B_H(t) \) (Mandelbrot and Ness 1968)

\[
E[B_H(v)|B_H(0)=0, B_H(1)=1] = \frac{1}{2}(v^{2H} + 1 - |v - 1|^{2H}). \tag{2.91}
\]

Note that when \( v = 1/2 \) the conditional expectation in Eq. 2.91 is equal to 1/2 independent of \( H \). Then, a field sample can be synthesized on a fine scale by using its realization on the coarser scales. Specifically,

\[
B_H\left(\frac{h}{2}\right) = \frac{1}{2}(B_H(0) + B_H(h)) + \xi, \tag{2.92}
\]

where \( \xi \) is a Gaussian random variable with zero mean and is independent of \( B_H(0) \) and \( B_H(h) \). The variance, \( \Delta \), of \( \xi \) can be selected so that the variance of \( B_H(h/2) \) is equal to its target value. In this regard, Eqs. 2.88 and 2.91 yield

\[
\Delta = E[\xi^2] = E\left[\left(\frac{h}{2}\right)^\frac{H}{4} - B_H(0)^2\right] - \frac{1}{4}E\left[(B_H(h) - B_H(0))^2\right] = \sigma^2 h^{2H}\left(\frac{1}{4}H - \frac{1}{4}\right). \tag{2.93}
\]

The next step of the algorithm is to generate the sample realization on the next finer scale: \( B_H(h/4) \). This procedure continues until a sufficient resolution level is obtained.

Using this scale-type algorithm Fournier et.al. (1982) generated some remarkable landscapes. Haruyama and Barsky (1984) utilized this method for artificial texture
generation. They also experimented with the scale-nonhomogeneous version of this algorithm by prescribing different weights to different scales. Further, Voss (1985) noted that, though the correlation between the new generated points and its “parents” was exact, the cross-correlation of new points was not well approximated. Indeed,

\[
E \left[ \left( B_H \left( \frac{h}{2} \right) - B_H \left( -\frac{h}{2} \right) \right)^2 \right] = 2E \left[ \left( B_H \left( \frac{h}{2} \right) - B_H (0) \right)^2 \right] = \frac{2\sigma^2 h^{2H}}{4^H} .
\]  

(2.94)

This gives the target value \( \sigma^2 h^{2H} \) only for the normal Brownian motion, \( H = 1/2 \). He associated this mathematical inconsistency of the random midpoint method with the creasing of generated images manifesting itself in the appearance of strikes and traces in the generated artificial landscapes. To resolve this problem Voss (1985) proposed a new variation of the scale-type method, “successive random additions”, by recalculating the values of all points in the finer scale based on the coarse scale.

Lewis (1987) developed a generalized stochastic subdivision method which extended the ideas of the random midpoint method and made it applicable to a broader class of stochastic processes. Lewis argued that the creasing effect is the result of estimating the value of the Brownian motion at a point using the values of only its two closest neighbors. Obviously, the subsequent stages of the midpoint simulation algorithm would tend to undermine the target correlation. Then, better results can be obtained by selecting a larger neighborhood for estimating the mean value of a point at the given scale. Specifically, Eq. 2.92 can be replaced by the equation

\[
y((k + 1/2)h) = \sum_{i = 1 - N}^{N} a_i y((k + i)h) + u_k .
\]

(2.95)

The multi-scale simulation procedure which is based on Eq. 2.95 can be applied to any stationary process (Lewis 1987). In this case, the coefficients \( a_i \) can be determined by
using the orthogonality principle

\[
E \left\{ y((k + m)h) \left[ y((k + 1/2)h) - \sum_{i = 1 - N}^N a_i y((k + i)h) \right] \right\} = 0,
\]

(2.96)

where \( m = 1 - N, ..., N \). Similarly, the variance of the Gaussian random variables \( u_k \) can be found by equating the variance of \( y((k + 1/2)h) \) to its target value.

The multiscale method of Eq. 2.95 provides higher accuracy for the generalized subdivision method eliminating, to a some extent, the creasing of the generated samples. Further, Fenton (1990, 1994) and Fenton and Vanmarcke (1990) proposed to incorporate a random field smoothing procedure into the scale recursive simulation. In this case, average values of the random field within increasingly dense meshes are determined first (Vanmarcke 1983). Subsequently, these values are generated by following the Lewis’ approach of Eq. 2.95. At each stage the averages of the field within the given scale are generated. Finally, Rumelin (1991) noted the similarity between the one step scale prediction and the AR method; he identified the scale domain methods with a family of non-homogeneous non-causal AR algorithms.

This dissertation provides a solid theoretical foundation for the scale type methods, addresses the properties of generated fields, and estimates related errors. This foundation is developed in the form of the wavelet expansion of random fields. For this purpose, wavelets and their properties were considered in Chapter 1. The formal study of random fields has two stages. During the first stage, a random field is transformed into a new form having a simpler structure by using the wavelet expansion. Several algorithms are discussed in Chapter 3 to determine the correlation of the wavelet coefficients of general random fields. It is shown that a judicious use of the local and multi-scale structure of Daubechies wavelets yields a family of efficient algorithms which can be implemented in
$O(N)$ operations to find the covariance matrix of the wavelet coefficient of random fields. Further, it is shown that a broad class of random fields is amenable to a simplified representation.

During the second stage, the simplified representation of random fields is used for sample generation. In this regard the wavelet reconstruction algorithm is viewed as a dynamic system in the scale domain. A family of efficient random field simulation algorithms is developed in Chapter 4 by using this underlying multi-scale dynamics of wavelet coefficients. This provides the requisite theoretical generalization of the scale-type simulation methods. The developed algorithms for random field simulation are capable of generating a single random field sample in $O(N)$ numerical operations. The multi-dimensional extension of the proposed algorithms is considered in Chapter 5 by incorporating multi-dimensional wavelets.

2.3.4 Simulation Using Shot Noise, Random Fourier Series, and Noise Shower Processes

In this section simulation of stochastic processes by using summation of random pulses is reviewed. These models have been extensively used for the analytic treatment of dynamic systems to non-Gaussian random excitation (Lin 1976). Also, some physical phenomena, for example traffic loads, naturally lend themselves to this formulation. In the most general form the sum of random pulses can be written as

$$y(x) = \sum_k Q_k(x, \xi_k),$$  \hspace{1cm} (2.97)

where $Q_k(x, \xi_k)$ is the $k$-th random pulse which commences at a random coordinate $\xi_k$. Two models of this class are of particular interest: the random Fourier series, and the shot
noise.

The random Fourier series can be defined as

\[ y(x) = \sum_k p_k \exp(-i\omega_k x) , \tag{2.98} \]

where \( p_k \) and \( \omega_k \) are random variables. Some properties of this stochastic process are studied by Janssen (1979). Specifically, assume that the random variables \( p_k \), \( k = 1, 2, \ldots \), are zero-mean and uncorrelated,

\[ E[p_k] = 0 , \quad E[p_k p_n] = c_n \delta_{k,n} , \tag{2.99} \]

the pairs \((p_k, p_n)\) and \((\omega_k, \omega_n)\) are mutually independent, for every real \( k \) and \( n \), and probability density functions \( f_k \) of the random variables \( \omega_k \) are such that the (infinite) sum

\[ \sum_k c_k f_k \tag{2.100} \]

convergence in the sense of distributions (Gel'fand 1964) to the function \( f_0 \). Then, the random Fourier series is well defined as a generalized stochastic process (Gel'fand 1964, Janssen 1979). Moreover, the auto-correlation function of the process of Eq. 2.98 can be found by squaring the corresponding expression and subsequently taking mathematical expectation. That is,

\[ R_y(x_1, x_2) = E[y(x_1)\overline{y(x_2)}] = E\left[\sum_n \sum_k p_k p_n \exp(-i(\omega_k x_1 - \omega_n x_2))\right] = \sum_k c_k E[\exp(-i\omega_k(x_1 - x_2))] = \sqrt{2\pi} \hat{f}_0(x_1 - x_2) , \tag{2.101} \]

where \( \hat{f}_0 \) is the Fourier transform of the infinite sum of Eq. 2.100. A more rigorous derivation of Eq. 2.101 can be found in Janssen (1979). Thus, any second order stationary
stochastic process can be represented by using the random Fourier series. The spectral density function of this process can be found by performing the Fourier transformation according to Eq. 1.24

\[ S_y(\omega) = f_0(\omega). \]  

(2.102)

Eq. 2.102 provides a clear physical interpretation of the sum of Eq. 2.100; it is the spectrum of the random Fourier series. Then, the stochastic process can be synthesized based on formula 2.98 by generating random variables \( p_k \) and \( \omega_k \) so that the conditions specified by Eqs. 2.99-2.102 are satisfied.

Note that several alternative schemes can be formulated by varying the manner in which the truncated series of Eq. 2.100 approximates \( f_0 \). This approximation is exact if

\[ f_k = \frac{1}{N} \frac{S(\omega)}{\alpha}, \quad c_k = \alpha, \quad k = 1, \ldots, N, \]  

(2.103)

where \( N \) is an arbitrary positive integer number, and \( \alpha = \sigma/N \). This scheme was originally introduced by Mejia and Rodriguez-Iturbe (1974). Note that the spectral method of random field simulation discussed in Section 2.3.1 can also be formulated as the random Fourier series. In this case, the random variables \( \omega_k \) are assigned deterministic values by partitioning the frequency domain with mesh of size \( \Delta \omega \), and \( p_k \) are selected as zero mean random variables with variance \( S(\omega_k)\Delta \omega \). That is, Eq. 2.102 is approximated as

\[ S_y(\omega) = \sum_k S(\omega_k)\Delta \omega \delta(\omega - \omega_k). \]  

(2.104)

Note that Eq. 2.104 is, in fact, equivalent to Eq. 2.9.

Further, the second order white noise process can be also generated by using the random Fourier series. Specifically, this is the case when the white noise intensity
corresponds to the variance of the random variables \( p_k \) and adding the probability distribution functions \( f_k \) yields a constant value; \( f_0 = \text{const.} \).

The shot noise can be derived by taking the formal Fourier transformation of the random Fourier series of Eq. 2.98

\[
y(x) = \sum_k p_k \delta(x - \xi_k),
\]

(2.105)

where the random parameters \( \xi_k \) are also called the impulse arrival time. Under the conditions stated by Eqs. 2.99-2.102, the shot noise is well defined as a generalized stochastic process. The auto-correlation function of the shot noise can be readily found (Janssen 1979)

\[
R_y(x_1, x_2) = E[y(x_1)\overline{y(x_2)}] = f_0\left(\frac{x_1 + x_2}{2}\right)\delta(x_1 - x_2).
\]

(2.106)

This represents the "white-like" noise process with non-stationary power given by function \( f_0 \). Clearly, if \( f_0 \) is a constant, the shot noise becomes the second order stationary white noise.

In this regard, several probabilistic models of random parameters \( \xi_k \) are commonly used in engineering applications. Thus, \( \xi_k \) are often modeled as the Poisson process with stationary increment and parameters \( p_k \) are assumed identically distributed. That is, the probability density function of \( \xi_k \) are (Lin 1976)

\[
f_k(x) = \begin{cases} 
  e^{-\lambda x} \frac{\lambda^k x^{k-1}}{(k-1)!}, & x \geq 0, \\
  0, & \text{otherwise,} \quad k \geq 1,
\end{cases}
\]

(2.107)

where \( \lambda \) is the "impulse arrival rate". Then, the shot noise intensity \( f_0(x) \) can be readily
found as

\[ f_0(x) = \begin{cases} \lambda, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}. \]  

(2.108)

Note that this process tends to a stationary white noise, as \( x \to \infty \), and the non-stationary effect at the origin \( x = 0 \) decreases.

Finally, any second order process can be reliably represented by the noise shower (Janssen 1979) which is the combination of the random Fourier series and the shot noise

\[ y(x) = \sum_k p_k e^{-i \omega_k (x - \xi_k)} \alpha(x - \xi_k), \]  

(2.109)

where the correlation of the random parameters \( \omega_k, \xi_k \) and \( p_k \) must be appropriately selected so that the process \( y(x) \) has a specified time-frequency distribution. A particular form of this model assigning zero to the parameters \( \omega_k \) and assuming the statistical independence of \( p_k \), yields the following auto-correlation function

\[ R_y(x_1, x_2) = \sum_k c_k \int g(x_1 - \xi)g(x_2 - \xi) f_k(\xi) d\xi. \]  

(2.110)

Lin (1976) discussed the case where parameters \( \xi_k \) signify the arrival time of the Poisson distribution. He also showed that if the Poison process is stationary, the noise shower tends to stationary as \( x \to \infty \). Recently, Lin and Cai (1995) used random showers with non-stationary arrival rate for modeling non-stationary processes. Frey (1995) utilized random shower processes to synthesize two-dimensional random fields.
2. 3.5 Turning Band Method

The turning band method for simulation of multi-dimensional random fields was originally developed by Matheron (1973). The main idea of this method is to reduce a multi-dimensional problem to a series of one-dimensional problems. This quite specific objective restricts the class of random fields amenable to simulation by this method. Specifically, only isotropic fields with the auto-correlation function

\[ R(\mathbf{x} - \mathbf{y}) = R(r) , \]  \hspace{1cm} (2.111)

where \( r = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \), fit into this framework.

To address this problem note that the equation

\[ f(x) = f_1(\langle x, \xi \rangle) , \]  \hspace{1cm} (2.112)

where \( f_1(t) \) is a stationary stochastic process, and \( \xi \) is a unit vector in \( R^n \), defines a \( n \)-dimensional random field with the auto-correlation function

\[ R_f(x - y) = E[f(x)f(y)] = E[f_1(\langle x, \xi \rangle)f_1(\langle y, \xi \rangle)] = R_1(\langle x - y, \xi \rangle) . \]  \hspace{1cm} (2.113)

Here, \( R_1(\tau) \) denotes the auto-correlation function of the process \( f_1(t) \). Eq. 2.112 can be further generalized by treating the vector \( \xi \) as a random vector of unit length with a certain distribution. With this assumption, the auto-correlation function of the random field \( f(\mathbf{x}) \) can be found as

\[ R_f(\mathbf{x}) = \int_{\Omega} R_1(\langle \mathbf{x}, \xi \rangle)P(d\xi) , \]  \hspace{1cm} (2.114)

where \( P \) is the probability which is associated with the area \( ds \) of the unit sphere \( \Omega \).

Finally, if the random vector \( \xi \) is uniformly distributed over the unite sphere, the field
$f(x)$ becomes isotropic and Eq. 2.114 can be rewritten as

$$R_f(r) = \frac{2I\left(\frac{1}{2}n\right)}{\sqrt{\pi}I\left(\frac{1}{2}(n-1)\right)} \int_0^1 (1 - v^2)^{\frac{n-3}{2}} R_1(vr) dv . \quad (2.115)$$

Eq. 2.115 has a particular simple form for three-dimensional fields,

$$R_f(r) = \int_0^1 R_1(vr) dv . \quad (2.116)$$

Eq. 2.116 can be readily inverted. This yields a formula for evaluating the auto-correlation function of the one-dimensional stochastic process associated with the three-dimensional isotropic field $f(x)$,

$$R_1(r) = \frac{d}{dr} R_f(r) . \quad (2.117)$$

Two-dimensional random fields are also quite important for engineering applications. In this case Eq. 2.115 yields

$$R_f(r) = \frac{2}{\pi} \int_0^1 (1 - v^2)^{-1/2} R_1(vr) dv . \quad (2.118)$$

Eq. 2.118 is quite complex and can not be directly inverted. However, this expression can be significantly simplified by considering the relationship between the corresponding spectra of two- and one-dimensional fields. Indeed, the spectrum of the stochastic process $f_1(t)$ is defined by Eq. 1.24
\[ R_1(r) = 2 \int_0^\infty \cos(\omega r) S_1(\omega) d\omega . \quad (2.119) \]

Substituting Eq. 2.119 into Eq. 2.118, one can derive

\[ R_f(r) = 2 \int_0^\infty J_0(\omega r) S_1(\omega) d\omega . \quad (2.120) \]

On the other hand, the spectrum of the two-dimensional isotropic random field can be found as

\[ R_f(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\omega) e^{i(\omega_1 x_1 + \omega_2 x_2)} d\omega_1 d\omega_2 = 2\pi \int_{-\infty}^{\infty} J_0(\omega r) S(\omega) d\omega , \quad (2.121) \]

where \( \omega = \sqrt{\omega_1^2 + \omega_2^2} \). Eqs 2.120 and 2.121 yield the quite simple relationship between two spectra

\[ \pi S(\omega) \omega = S_1(\omega) . \quad (2.122) \]

Eqs. 2.117 and 2.122 can be readily used to determine the correlation structure of the one-dimensional stochastic process \( f_1(t) \) corresponding to the two- or three-dimensional fields, respectively. Then, the conditional simulation procedure is used to generate a sample of the multi-dimensional field \( f(x) \). Specifically, the stochastic process \( f_1(t) \) is synthesized first by using the estimated auto-correlation function \( R_1(\tau) \) or the spectrum \( S_1(\omega) \) and, then, a sample of the multi-dimensional field \( f(x) \) is simulated by generating the random vector \( x \) uniformly distributed over the unite sphere. Also, note that the generated sample is not ergodic; this fact may decrease the applicability of this method to certain engineering problems.
Mantoglou and Wilson (1981) studied this method for civil and mechanical engineering applications. To reduce the non-ergodic characteristic of the synthesized samples they used quite inefficient averaging of several samples to produce a single field sample which appear to be homogeneous. Also, Mantoglou and Wilson (1981) combined this method with the stratified sampling method with respect to the random vector $g$.

The error associated with the turning band method was studied by Fenton (1994). He found that the turning band method is numerically efficient but produces some artificial strikes in the synthesized realizations related to their non-ergodic character. These strikes can be eliminated only by averaging of a significant number of the synthesized samples. The experience gained in the practical application of the turning band method (Fenton 1994) suggests that it usually requires more computations, for the same level of error, compared to the Spectral method with FFT or the Local Average Subdivision method. Also, the correlation function is difficult to determine in the two-dimensional case.
Chapter 3. Representation of Random Processes Using Wavelet Bases

In the past decade, wavelets have become an essential tool of engineering analysis and computing systems. A comprehensive perspective of their applications in engineering was presented in Section 1.5.6.

Modeling, analysis, and synthesis steps of signal processing, deterministic or random, involve the transformation of the original signal into a new form. This transformation either yields another signal with simpler structure, reducing the problem complexity, or facilitates the analysis and synthesis stages with simpler mathematical relationships. For example, relying on the spectral representation of stationary stochastic processes one can expedite the computations associated with random signals and systems by using the Fourier basis which produces an uncorrelated set of random variables, the white noise sequence.

Wavelets have some quite useful properties as discussed in Section 1.5; they are useful for representing random fields significantly reducing the requisite computational resources for problems of stochastic mechanics, and other fields. To address this issue, the wavelet transformation of stochastic processes and random fields is considered from a probabilistic standpoint. The capacity of wavelets to reduce numerical operations for stochastic signal processing is studied. In this regard, the main objective of this study is to generalize the wavelet decomposition and reconstruction schemes shown in Figures 1.7 and 1.8, respectively, for the stochastic case. Generally speaking, one must put a stochastic process in the flow of Figures 1.7 and 1.8 which is formalized by Eqs. 1.125-1.127 and extract maximum of useful information regarding the stochastic process.

In this dissertation a comprehensive theoretical foundation for of representing
stochastic processes and systems in wavelet bases is developed. This material will serve subsequently as a "back-bone" for a new wavelet based method for random field synthesis. For this purpose, methods of expanding stochastic processes in wavelet bases are formulated in Section 3.1 and the implicit equations determining the related probability structures are derived. The developed formulae are used as a foundation for a numerical algorithm that evaluates the moments of the wavelet coefficients of stochastic processes by using digital computers; it is introduced in Section 3.2 by incorporating the wavelet transform algorithm for calculating related integrals. This algorithm requires $O(N)$ and $O(N^2)$ operations for stationary and non-stationary processes, respectively. However, a judicious use of the local properties of wavelets and the fact that Daubechies wavelets have several vanishing moments can yield more efficient algorithm which requires $O(\log N)$ and $O(N)$ numerical operations, respectively. Section 3.3 discusses the compressing properties of wavelets related to their ability to make the covariance matrix of associated random variates to be close to a diagonal form. Finally, some concluding remarks summarizing the material of this chapter are given in Section 3.4.

3.1 Random Coefficient Description

The stochastic process $f(x)$ defined by its auto-correlation function $R_f(x_1, x_2)$ can be expanded by using a wavelet basis (Walter 1994, Zeldin and Spanos 1995a,b, Dijkerman and Mazumbar 1994). Specifically,

$$f(x) = \sum_{j,k=\infty}^{\infty} d_{j,k} \psi_{j,k}(x), \quad (3.1)$$

where, for orthonormal wavelets, the coefficients can be expressed as
\[ d_k^j = \int f(x) \psi_{j,k}(x) \, dx . \]  

(3.2)

Thus, a stochastic process is defined by a countable set of random variables \( d_k^j \). The probability structure of the wavelet coefficients is determined by \( f(x) \). Also, if an infinite set of random variables with the prescribed probability distributions is given, Eq. 3.1 specifies uniquely the probability properties of the corresponding stochastic process.

Relying on Eq. 3.2 for evaluating the moments of the random wavelet coefficients, the mean value of \( d_k^j \) can be found by taking the mathematical expectation of this equation. Specifically,

\[ E[ d_k^j ] = \int E[f(x)] \psi_{j,k}(x) \, dx . \]  

(3.3)

Similarly, the second order moments of the wavelet coefficients can be evaluated from the equation

\[ r_{k,l}^{j,i} = E[ d_k^j d_k^l ] = E[ \int \int f(x_1) f(x_2) \psi_{j,k}(x_1) \psi_{i,l}(x_2) \, dx_1 \, dx_2 ] = \int \int \int \int R_f(x_1, x_2) \psi_{j,k}(x_1) \psi_{i,l}(x_2) \, dx_1 \, dx_2 \]  

(3.4)

If \( f(x) \) is a stationary stochastic process, \( r_{k,l}^{j,i} \) depends only on the difference of the low indices \( k - 2^{j-i} l \). Similarly, higher order moments of the coefficients \( d_k^j \) can be found. However, if \( f(x) \) is a Gaussian random process, the wavelet coefficients \( d_k^j \), \( j, k = \ldots -1, 0, 1, \ldots \), are also Gaussian random variables. In this case, the probability structure of \( d_k^j \) is completely characterized by their first and second order moments.

Similarly, the scale coefficients are random variables and can be found by relying on the following equation

\[ c_k^j = \int f(x) \phi_{j,k}(x) \, dx . \]  

(3.5)
Eq. 3.5 yields formulae for evaluating the moments of the scale coefficients. Specifically,

$$E[c_k^l] = \int E[f(x)] \phi_{j,k}(x) dx$$ \hspace{1cm} (3.6)

and

$$a_{k,l}^{j,i} = E[c_k^l c_l^j] = \int \int R_f(x_1, x_2) \phi_{j,k}(x_1) \phi_{i,l}(x_2) dx_1 dx_2$$ \hspace{1cm} (3.7)

Finally, the cross-correlation of the wavelet and scale coefficients can be found by multiplying Eq. 3.2 by Eq. 3.5 and taking mathematical expectation. That is,

$$b_{k,l}^{j,i} = E[c_k^l d_l^j] = \int \int R_f(x_1, x_2) \phi_{j,k}(x_1) \psi_{i,l}(x_2) dx_1 dx_2$$ \hspace{1cm} (3.8)

Note that if \(f(x)\) is a stationary random process Eqs. 3.4, 3.7 and 3.8 can also be expressed in the frequency domain by using the spectral density function. Specifically,

$$r_{k,l}^{j,i} = \int \int \int S(\omega) e^{i(x_2 - x_1)\omega} \psi_{j,k}(x_1) \psi_{i,l}(x_2) dx_1 dx_2 = 2 \pi \int S(\omega) \hat{\phi}_{j,k}(\omega) \overline{\hat{\psi}_{i,l}(\omega)} d\omega$$ \hspace{1cm} (3.9)

where

$$\hat{\psi}_{j,k}(\omega) = e^{-i 2^j \omega (k - 1)} 2^{j/2} \hat{\psi}(2^j \omega) \hspace{1cm} (3.10)$$

Similarly,

$$b_{k,l}^{j,i} = 2 \pi \int S(\omega) \hat{\phi}_{j,k}(\omega) \overline{\hat{\psi}_{i,l}(\omega)} d\omega$$ \hspace{1cm} (3.11)

$$a_{k,l}^{j,i} = 2 \pi \int S(\omega) \hat{\phi}_{j,k}(\omega) \overline{\hat{\phi}_{i,l}(\omega)} d\omega$$ \hspace{1cm} (3.12)

where
\[ \hat{\phi}_{j,k}(\omega) = e^{-i2^j \omega (k-1)/2} \hat{\phi}(2^j \omega) . \] (3.13)

The use of Eqs. 3.3, 3.4 and 3.6-3.12 is limited due to the fact that the analytical integration of the corresponding expressions is feasible only for a limited number of stochastic processes and wavelet bases. On the other hand, the direct discretization of these integrals induces computationally costly and inaccurate schemes. Efficient algorithms for numerical calculation of the corresponding integrals are considered in the next section.

3.2 Algorithms for Evaluating the Probability Structure of the Wavelet Coefficients

The efficiency of analyzing and synthesizing stochastic processes by using wavelets is determined, to a great extent, by the computational requisite resources for evaluating moments of the random wavelet coefficients. In this context, the wavelet decomposition algorithm described in Section 1.5.4 is generalized to circumvent the numerical integration involved in Eqs. 3.3, 3.4 and 3.6-3.12.

Assume that for an adequately fine scale, say \( j \), the correlation of the scale coefficients \( a^{j,j}_{k,l} \) is determined based on the analog-discrete converter discussed in Section 1.5.4, see Eqs. 1.130-1.132. Then, the correlation parameters \( r^{j+1,j+1}_{k,l}, b^{j+1,j+1}_{k,l} \) and \( a^{j+1,j+1}_{k,l} \) can be found by using a two-dimensional version of the wavelet decomposition algorithm with respect to \( a^{j,j}_{k,l} \) (see Eqs. 1.125 and 1.126); this yields the following equations which provides the correlation parameters for the next \((j+1)\)-th scale

\[ r^{j+1,j+1}_{k,l} = \sum_{n,m=0}^{2M-1} g_{n+1}^{j+1} g_{m+1}^{j+1} a^{j,j}_{2k+n-1,2l+m-1} , \] (3.14)
\[ b_{k,l}^{j+1,j+1} = \sum_{n,m=0}^{2M-1} g_{n+1}^{j+1} h_{m+1}^{j+1} a_{2k+n-1,2l+m-1}^{j+1,j+1} \]  \hspace{1cm} (3.15) \\
\[ a_{k,l}^{j+1,j+1} = \sum_{n,m=0}^{2M-1} h_{n+1}^{j+1} h_{m+1}^{j+1} a_{2k+n-1,2l+m-1}^{j+1,j+1} \]  \hspace{1cm} (3.16)

Then, Eqs. 3.14-3.16 can be used to evaluate correlation within the next \((j+2)\)-th scale again increasing the index \(j\) by one, etc. The auto-correlation functions of stochastic processes commonly used in engineering decay quite rapidly. Moreover, wavelets can amplify this feature for some stochastic processes. Then, the correlation parameters \(r_{k,l}^{j,j}\), \(b_{k,l}^{j,j}\) and \(a_{k,l}^{j,j}\) can be evaluated only in a neighborhood of the diagonal \(k=l\). A similar procedure has been previously used by Belkin et.al. (1991) to find a representation of mathematical operators in wavelet bases.

The computational procedure formalized by Eqs. 3.14-3.16 can be further simplified for stationary stochastic processes. In this case, \(r_{k,l}^{j+1,j+1}\), \(b_{k,l}^{j+1,j+1}\) and \(a_{k,l}^{j+1,j+1}\) depend on the difference \(k-l\) and the double summation which involved in these expressions can be reduced by a simple change of indices as follows

\[ r_{k-l}^{j+1,j+1} = \sum_{n=1}^{2M-1} \beta_n (a_{2(k-l)+n}^{j,j} + a_{2(k-l)-n}^{j,j}) + \beta_0 a_{2(k-l)}^{j,j} \]  \hspace{1cm} (3.17) \\
\[ b_{k-l}^{j+1,j+1} = \sum_{n=1}^{2M-1} (\gamma_n a_{2(k-l)+n}^{j,j} + \gamma_n a_{2(k-l)-n}^{j,j}) + \gamma_0 a_{2(k-l)}^{j,j} \]  \hspace{1cm} (3.18) \\
\[ a_{k-l}^{j+1,j+1} = \sum_{n=1}^{2M-1} \alpha_n (a_{2(k-l)+n}^{j,j} + a_{2(k-l)-n}^{j,j}) + \alpha_0 a_{2(k-l)}^{j,j} \]  \hspace{1cm} (3.19)

where
\[ \alpha_n = \sum_{k=0}^{2M-1-n} h_{k+1} h_{k+1+n}, \quad (3.20) \]

\[ \beta_n = \sum_{k=0}^{2M-1-n} g_{k+1} g_{k+1+n}, \quad (3.21) \]

\[ \gamma_n^1 = \sum_{k=0}^{2M-1-n} g_{k+1} h_{k+1+n}, \quad (3.22) \]

\[ \gamma_n^2 = \sum_{k=0}^{2M-1-n} h_{k+1} g_{k+1+n}, \quad (3.23) \]

\[ \gamma_0 = \sum_{k=0}^{2M-1} h_{k+1} g_{k+1}. \quad (3.24) \]

Clearly, this computationally costly scheme requires \( O(N^2) \) and \( O(N) \) numerical operations for non-stationary and stationary processes, respectively. The correlation structure of wavelet coefficients can be evaluated more efficiently by using some approximate quadrature formulae developed originally by Belkin et.al. (1991) in the context of determining the structure of a class of mathematical operators in wavelet bases. This method which requires only \( O(N) \) operations is generalized for approximating the correlation of the scale coefficients. Two approaches can be used in this regard. First, Daubechies (1988) has constructed finite orthonormal wavelet bases having \( M \) vanishing moments; see Section 1.5.3. However, it can never happen with the scale function since \( \int \phi(x) dx = 1 \). In 1989 Coifman suggested that it might be worth constructing an orthonormal wavelet basis with some vanishing moments not only for \( \psi \) but also for \( \phi \). That is,

\[ \int x^l \phi(x + \tau) dx = 0, \ l=1,...,M-1. \quad (3.25) \]
Daubechies (1992) showed a method of how this can be accomplished and proposed to call the resulting functions “coiflets”. The corresponding parameters $h_k$ are listed in Table 3.3.1. Coiflets provide a one point quadrature for evaluating $a_{k,l}^{j}$. Indeed, Eq. 3.7 can be rewritten as

$$a_{k,l}^{j} = E[c_k^j c_l^j] = 2^{-j} \int R_f(x_1 + 2^j(k-1), x_2 + 2^j(l-1)) \phi(2^{-j}x_1) \phi(2^{-j}x_2) dx_1 dx_2.$$  

(3.26)

If the auto-correlation function is at least $M$ time differentiable, it can be expanded in the Taylor series as

$$R_f(x_1 + 2^j(k-1), x_2 + 2^j(l-1)) = R_f(2^j(k-1), 2^j(l-1)) +$$

$$x_1 \frac{\partial}{\partial x_1} R_f(2^j(k-1), 2^j(l-1)) + x_2 \frac{\partial}{\partial x_2} R_f(2^j(k-1), 2^j(l-1)) + \ldots$$

$$\frac{x_1^{M-1}}{(M-1)!} \frac{\partial^{(M-1)}}{\partial x_1^{M-1}} R_f(2^j(k-1), 2^j(l-1)) + \ldots$$

$$\frac{x_2^{M-1}}{(M-1)!} \frac{\partial^{(M-1)}}{\partial x_2^{M-1}} R_f(2^j(k-1), 2^j(l-1)) + O(C) \sum_{i=0}^{M} x_1^{-i} x_2^{i}.$$  

(3.27)

where the parameter $C$ is proportional to

$$\frac{1}{M!} \max \left\{ \sup \left[ \frac{\partial^{M}}{\partial x_1^{M} \partial x_2^{M}} R_f(x_1, x_2)(x_1 - 2^j(k-1))^{M-i}(x_2 - 2^j(l-1))^i \right] \right\}.$$  

(3.28)

Substituting Eq. 3.27 into Eq. 3.26 and using Eq. 3.25 one finds

$$a_{k,l}^{j} = 2^j R_f(2^j(k-1), 2^j(l-1)) + 2^{-j} O(C) \sum_{i=0}^{M} x_1^{-i} x_2^{i} \phi_{j,k}(2^{-j}x_1) \phi_{j,l}(2^{-j}x_2) dx_1 dx_2$$

$$= 2^j R_f(2^j(k-1), 2^j(l-1)) + O(2^{j(M+1)}).$$  

(3.29)
Thus, the parameters $a_{k,l}^{j,j}$ can be determined by using Eq. 3.29 sufficiently accurate for adequately fine scales. Then, $r_{k,l}^{j+1,j+1}$ and $b_{k,l}^{j+1,j+1}$ can be found in the neighborhood of the diagonal of the correlation matrix by using the two-dimensional wavelet decomposition algorithm, Eqs. 3.14 and 3.15. The coefficients $a_{k,l}^{j+1,j+1}$ again can be evaluated from Eq. 3.29. When the scale becomes too coarse and the corresponding calculations are not accurate, $r_{k,l}^{j,j}$, $b_{k,l}^{j,j}$ and $a_{k,l}^{j,j}$ are evaluated by relying on Eqs. 3.14-3.24. Obviously, the described procedure requires only $O(N)$ and $O(\log N)$ numerical operations for non-stationary and stationary processes, respectively.

Unfortunately, wavelets with $M$ vanishing moments have support $3M-1$. It is almost 50% longer than the original orthonormal wavelets (Daubechies 1988). It increases proportionally the requisite computations. Alternatively, the one point quadrature given by Eq. 3.29 can be replaced by the following equation

$$a_{k,l}^{j,j} = 2^j \sum_{n, m = 0}^{q-1} \alpha_n \alpha_m R_f(2^j n + 2^j (k - 1), 2^j m + 2^j (l - 1)) + \epsilon , \quad (3.30)$$

where the coefficients $\alpha_n$ are selected so that $\epsilon$ is null if $R_f(x_1, x_2)$ is a polynomial in $x_1$ and $x_2$ of order less than $q$. If $R_f(x_1, x_2)$ is a $q$ times differentiable function the latter requirement yields

$$\epsilon \sim O(2^{j(q+1)}) . \quad (3.31)$$

The coefficients $\alpha_n$ of Eq. 3.30 can be found from the following system of equations

$$\sum_{l = 0}^{q-1} l^n \alpha_l = \int x^n \phi(x) dx . \quad (3.32)$$

An algorithm to evaluate the moments of the scale function in the right hand side of Eq.
3.32 without performing numerical integration is presented in Appendix A.

Further, the auto-correlation functions of a large class of stochastic processes are not smooth. In particular, the auto-correlation functions of non-differentiable in the square mean sense stochastic processes are not differentiable. However, these singularities are localized within the diagonal \( x_1 = x_2 \). Also, the auto-correlation function of some stochastic processes can be differentiable everywhere, but the module of these derivatives is large within a neighborhood of the diagonal. In this case, the constant \( C \) of Eq. 3.28 can be quite large amplifying the errors of the approximation. Then, it is reasonable to evaluate \( a_{k,l}^{i,j} \) in this neighborhood of the diagonal by using the two-dimensional multi-scale relationships of Eqs. 3.14-3.24 and to use the above approximation only outside this region.

Several numerical examples are considered to study the errors of the discussed approximation. The auto-correlation function of a stationary process is taken

\[
R_f(x_2 - x_1) = \frac{1}{1 + ((x_1 - x_2)/\alpha)^2},
\]

(3.33)

where the value \( \alpha = 0.1 \) is taken. The error of the quadrature of Eq. 3.30 is evaluated with respect to the exact formulae given by Eqs. 3.17-3.24. This approximation is applied in the region \( |k - l| > M + 4 \) for the coarsest scale

\[
N = (\log_2(e))/M,
\]

where \( q = M \) and \( \varepsilon = 0.001 \). The Daubechies wavelets with the maximum number of vanishing moments are used. The maximum relative errors of evaluating \( a_{k,l}^{i,j} \) based on Eq. 3.30 for every scale are shown on Figure 3.1(a) for several wavelets with different number of vanishing moments. As theoretically predicted, smaller errors are associated with fine meshes and large values of \( M \). These errors are relatively small and are negligible.
for practical purpose.

Similarly, a stationary stochastic process with the auto-correlation function

$$R_f(x_2 - x_1) = \exp(|x_1 - x_2|/\alpha)$$

(3.34)

is considered, and the value $\alpha = 0.1$ is taken. Note that this stochastic process is not differentiable. Therefore, the corresponding errors of approximation of $a_{k,l}^{h,j}$ shown in Figure 3.1(b) are somewhat, larger but still within admissible limits.

Further, using the numerical values of $a_{k,l}^{h,j}$, parameters $r_{k,l}^{h,j}$ are found for stochastic processes with the auto-correlation function of Eqs. 3.33 and 3.34. The maximum absolute errors of estimating $r_{k,l}^{h,j}$ normalized by the diagonal value $r_{k,k}^{h,j}$ are shown in Figures 3.2(a) and 3.2(b), respectively. The errors of this estimation are acceptable for engineering purposes. For fine scales the wavelet transformation behaves as the semi-differential operation. This fact amplifies small high frequency errors in the estimation of $a_{k,l}^{h,j}$ and affects the corresponding errors in $r_{k,l}^{h,j}$.
Table 3.1: The filter coefficients $h_k/\sqrt{2}$ for compactly supported coiflets with the highest number of vanishing moments compatible with their support width

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3.3 Compressing Properties of Wavelets

To a great extent, the broad application of wavelets to signal processing and computational systems is attributed to their capacity to compress information contained in signals and to reduce the requisite numerical computations. Indeed, consider the wavelet expansion of \( f(x) \in C^M \) in a basis with several vanishing moments. The function \( f(x) \) can be expanded in the following Taylor series

\[
f(x + 2^j (k - 1)) = f(2^j (k - 1)) + f'(2^j (k - 1)) x + \ldots
\]

\[
\frac{1}{(M-1)!} f^{(M-1)}(2^j (k - 1)) x^{M-1} + x^M r(x), \tag{3.35}
\]

where \( r(x) \) is the remainder of the Taylor series

\[
r(x) = \frac{1}{M!} f^{(M)}(\theta x + 2^j (k - 1)), \quad 0 < \theta < 1. \tag{3.36}
\]

Substituting Eq. 3.35 into Eq. 1.123, the first \( M \) terms of this expansion are nullified due to Eq. 1.102. The module of the wavelet coefficients of \( f(x) \) can be bounded as

\[
|d_k^j| = \left| 2^{-j/2} \int x^M r(x + 2^j (k - 1)) \psi(2^{-j} x) dx \right| \leq C 2^{j(M + 1/2)} \int |x^M \psi(x)| dx \tag{3.37}
\]

For small \( j \), the quantity in Eq. 3.37 is negligibly small, unless \( r(x) \sim f^{(M)}(x) \) is very large near \( 2^j (k - 1) \). Then, the wavelet coefficients are significant only in a small neighborhood of singularities of \( f(x) \). For example, in image processing the wavelet coefficients are retained only near the edges and boundaries of different objects. This effect is more substantial if the number \( M \) is large.

Similarly, Meyer (1989), Alpert et.al. (1990), Belkin et.al. (1991,1992), and Belkin (1993) studied the ability of wavelets with some vanishing moments to compress a large
class of operators: pseudo-differential operators, some operators from the Zigmund-Calderon class. They showed that wavelets reduced dense $N \times N$ matrices of these operators to a sparse structure having only $O(N)$ entries by neglecting all elements smaller than a small threshold $\epsilon$. Also, they proved that the global error of this approximation would be still of order $O(\epsilon)$. Using these findings Belkin et.al. (1991) developed a class of algorithms which can yield the product of full non-Toeplitz matrices and vectors in $O(N)$ operations.

In this section the compressing properties of wavelets are summarized and generalized for analyzing stochastic processes. The correlation of the wavelet coefficients of fractal random processes has been previously addressed. This section considers a much broader class of stochastic processes which are commonly used in engineering applications. Clearly, a sparse covariance matrix induces significant reduction in the requisite numerical computations for synthesizing random fields.

### 3.3.1 Compressing Fractal Stochastic Signals

The fractional Brownian motion is a generalization of the usual Brownian motion. This class of processes has been introduced for modeling phenomena that have long memory and/or statistical self-similarity (Mandelbrot and Van Ness 1968). Fractional Brownian motion has been used for image generation and interpretation, texture classification, modeling of burst errors in communication channels, and representation of $1/f$ noise in oscillatory systems and current noise in metal films and semiconductor devices (Keshner 1982, Taqqu, 1986).

A mathematical model for the fractional Brownian motion is introduced in Section 2.3.3 in the context of the review of the scale-type methods for random field synthesis. This non-stationary process is defined (Mandelbrot and Van Ness 1968) as a random
integral over the Brownian motion. Alternatively, this process is specified by the spectrum

\[ S(\omega) = |\omega|^{-2H-1}. \]  \hspace{1cm} (3.38)

The notion of spectrum, or spectral density function, defies the non-stationary nature of the generalized Brownian motion. The interpretation of the spectrum 3.38 in the sense of distributions is not, also, quite satisfactory. Flandrin (1989b, 1992) defined the spectrum for the Fractional Brownian motion by introducing the continuous wavelet transform; this is well suited to the non-stationary nature of this process. He found that for a given scale this process is stationary with a well defined scale-invariant spectrum. This result is not surprising since the wavelet transform, in some sense, is analogous to the differentiation and derivatives of the Fractional Brownian motion are stationary. Further, Ramanathan and Zeitouni (1991) proved the inverse statement which shows that a scale invariant spectrum corresponds only to the generalized Brownian motion. Also, Tewfik and Kim (1992) showed that the wavelet coefficients of the fractional Brownian motion are "almost" white if the number of vanishing moments of the analyzing wavelet is sufficiently large. In particular,

\[ \nu_k^{l,i} \sim O \left( |2^l k - 2^l i|^{2(H-M)} \right). \]  \hspace{1cm} (3.39)

Thus, wavelets provide two useful simplifications in this case. First, wavelets make the fractional Brownian motion stationary; second, they whiten the process.

Wornell (1990) showed that completely uncorrelated wavelet coefficients can be used to model 1/f noises. Indeed, if the variance of the uncorrelated wavelet coefficients is equal to

\[ 2^j \nu \sigma^2, \quad 0 < \nu < 2, \hspace{1cm} (3.40)\]
the corresponding time-averaged spectrum is found to be equal to

\[ \tilde{S}(\omega) = \sigma^2 (2^\gamma - 1) \sum_j 2^{j\gamma} |\psi(2^j \omega)|^2. \] (3.41)

Finally, if the wavelet basis satisfies some quite non-restrictive conditions, Eq. 3.41 provides for constants \( 0 < k_1 \leq k_2 < \infty \) such that

\[ \frac{k_1}{\omega^\gamma} \leq \tilde{S}(\omega) \leq \frac{k_2}{\omega^\gamma}. \] (3.42)

The stochastic processes which satisfy Eq. 3.42 are often called the generalized fractional Brownian motion (Wornell 1990). Note that Wornell and Oppenheim (1992) proposed a method of estimating the characteristics of fractal processes by using the introduced models with uncorrelated wavelet coefficients.

### 3.3.2 Decay of the Variance of the Wavelet Coefficients

As it has been alluded to previously, wavelets behave locally as differential operators. This property significantly simplifies the correlation matrix of the wavelet coefficients for a large class of random fields. Thus, the variance of the wavelet coefficients decreases rapidly with the scale of resolution. Indeed, the variance of the scale coefficients can be estimated by using Eq. 3.30. For a sufficiently fine scale it yields

\[ a_{k,1}^j \sim 2^j R_f(2^j(k - 1), 2^j(l - 1)). \] (3.43)

On the other hand, if the auto-correlation function is, at least, \( Q \) times differentiable and the wavelet function has \( M \) vanishing moments, \( M \geq Q \) the variance of the wavelet coeffi-
cient decreases as

\[ r_{k,i}^j = C 2^j(Q+1). \]  

(3.44)

Eq. 3.44 is developed by expanding the auto-correlation function in the Taylor series of Eq. 3.27, substituting the resulting expression into Eq. 3.4, and using \( M \) vanishing moments for the wavelet function. Therefore, the ratio of these variances behaves as

\[ r_{k,i}^j/a_{k,i}^j = C 2^j. \]  

(3.45)

This ratio can be negligibly small if the scale is adequately fine. Eq. 3.45 indicates that the importance of scales decays exponentially with \( j \). Using this result one can argue that a few scales can provide a reasonably accurate representation of the random field.

If the auto-correlation function is not sufficiently smooth the estimate 3.45 can not be used. However, the frequency localization properties of wavelets can be utilized to evaluate the importance of different scales for stationary stochastic processes. In this case Eq. 3.9 yields

\[ r_{k,i}^{h,j} = 2\pi 2^j S(\omega) |\tilde{\psi}(2^j \omega)|^2 d\omega . \]  

(3.46)

Note that the function \( |\tilde{\psi}(2^j \omega)| \) is concentrated around the frequency \( 2\pi 2^{-j} \) and the parameter \( r_{k,k}^{h,j} \) is approximately proportional to \( S(2^{-j}2\pi) \). Thus, if \( S(\omega) \sim \omega^{-\alpha} \)

\[ r_{k,k}^{h,i}/a_{k,k}^{h,i} \sim 2^{j(\alpha-1)} . \]  

(3.47)

Clearly, Eq. 3.47 coincides with the previous result of Wornell (1990), Eqs. 3.40 and 3.42, for the fractal processes. Note that the power spectral density must decrease with \( \alpha > Q + 1 \) to provide for the auto-correlation function being differentiable \( Q \) times. In this
case, the approximation of Eq. 3.47 coincides with the approximation of Eq. 3.45.

Figure 3.3(a) shows the ratio of the variances of the wavelet and scale coefficients versus the scale and the number of vanishing moments for stochastic process with the auto-correlation function given by Eq. 3.33 with \( \alpha = 0.1 \). These numerical results are in a good agreement with the theoretical estimates of Eq. 3.45. It shows that the wavelet coefficients corresponding to fine scales are relatively insignificant and they can be neglected. This effect is more profound for large \( M \).

Similarly, Figure 3.3(b) shows the decay of the variance of the wavelet coefficients for the non-differentiable in the mean square sense stochastic process with the auto-correlation function given by Eq. 3.34 with \( \alpha = 0.1 \). This auto-correlation function is not differentiable at the diagonal \( x_1 = x_2 \). Therefore, the variance of the wavelet coefficients does not decay as rapidly as in the previous case and is not affected by the number of vanishing moments of the wavelet function.

The above analysis establishes a method to select the number of scales required for an accurate representation of random fields.

3.3.3 Cross-Scale Correlation of the Wavelet Coefficients

In context with the preceding result, it is noted that the covariance matrix of the wavelet coefficients is sufficiently sparse for a large class of random fields. In particular, the correlation of wavelet coefficients from different scales can be evaluated for stationary processes by using Eq. 3.9. Since functions \( \psi(2^i \omega) \) and \( \psi(2^j \omega) \) are concentrated in the different frequency ranges and the energy of their interaction is small, the correlation of the wavelet coefficients from different scales is expected to be also small. Specifically, using Eq. 3.10 this cross-correlation is bounded as
\begin{equation}
\left| r_{k,i}^{j,l} \right| \leq \max(S(\omega))2\pi \int_{-\infty}^{\infty} |\hat{\psi}_{j,k}(\omega)||\hat{\psi}_{k,i}(\omega)| d\omega . \tag{3.48}
\end{equation}

The integral in Eq. 3.48 can be further simplified by substituting Eq. 1.85 and 1.86. Specifically,

\begin{equation}
\left| r_{k,i}^{j,l} \right| \leq \max(S(\omega))2\pi \int_{0}^{\pi} \chi_{|j-i|}(\omega) \left( \sum_{l} |\hat{\phi}(\omega + 2\pi l)|^2 \right) d\omega = \max(S(\omega)) \int_{0}^{\pi} \chi_{|j-i|}(\omega) d\omega , \tag{3.49}
\end{equation}

where the last equality is due to Eq. 1.92, and the function \( \chi_{k}(\omega) \) is defined as

\begin{equation}
\chi_{k}(\omega) = 2^{2+k/2} \left| m_{j}(\omega) \right| \left| m_{0}(\omega) \right| \cdots \left| m_{0}(2^{k-1} \omega) \right| \left| m_{j}(2^{k} \omega) \right| . \tag{3.50}
\end{equation}

This function is plotted in Figures 3.4(a) and 3.4(b) for wavelets with \( M = 1 \) and \( M = 7 \) vanishing moments, respectively. Note, that the cross-scale "decorrelation" properties of wavelets increase as the number of vanishing moments \( M \) becomes larger. The wavelet coefficients of the Shannon wavelets are completely uncorrelated across scales. These figures outline only the most general "decorrelation" properties of wavelets since Eq. 3.49 does not consider the behavior of the power spectrum.

It must be noted that an estimate of the cross-scale correlation of wavelet coefficients was established by Dijkerman and Mazumdar (1994) for the Gauss-Markov process with the auto-correlation function of Eq. 3.34 by using the time domain approach. However, the authors have ambiguously stated that their estimate shows the decay of the cross-scale correlation of the wavelet coefficients in an exponential way. The strength of the cross-correlation of two random variables is determined by the value of the cross-correlation
coefficient. As it has been shown previously, the variance of the wavelet coefficients of this stochastic process also decreases exponentially fast with the scale of resolution. Therefore, more delicate analysis of the energy of interaction of wavelets from different scales is required to properly evaluate the decay of the cross-scale correlation.

3.3.4 In-Scale Correlation of the Wavelet Coefficients for a Class of Stochastic Processes

Relying on the similarity between the auto-correlation function and the kernel of a linear operator, the weakening of the correlation of wavelet coefficients from the same scale can also be shown for some stochastic fields. In particular this fact can be demonstrated for the stochastic processes of the so-called "Meyer-Belkin" class (Meyer 1989, Belkin et.al. 1991) by using the Daubechies wavelet basis with $M$ vanishing moments. The auto-correlation function of these stochastic processes satisfy the equations

$$R(x_1, x_2) \leq \frac{C}{|x_1 - x_2|^\alpha},$$

(3.51)

and

$$\partial^{(\beta)}_{x_1}R(x_1, x_2), \partial^{(\beta)}_{x_2}R(x_1, x_2) \leq \frac{C_\beta}{|x_1 - x_2|^\alpha + \beta}.$$  

(3.52)

By substituting the Taylor series of this auto-correlation function into Eqs. 3.4, 3.7 and 3.8, the following estimates of the magnitude of the wavelet correlation parameters are derived for wavelets with $2M \leq \beta$

$$|a^k_{j, l}| = \left|2^{-j} \int \int R_f(x_1 + 2^j(k - 1), x_2 + 2^j(l - 1))\phi(2^{-j}x_1)\phi(2^{-j}x_2)dx_1 dx_2 \right| \leq$$

$$2^{-j} \left| R_f(\tilde{x}_1 + 2^j(k - 1), \tilde{x}_2 + 2^j(l - 1)) \right| \int \int \phi(2^{-j}x_1)\phi(2^{-j}x_2)dx_1 dx_2 \leq$$
\[ \left| \frac{C}{(\bar{x}_1 - \bar{x}_2 + 2^j (k - l))^{\alpha}} \right|_{\bar{x}_1, \bar{x}_2 \in 2^j [0, 2M - 1]} \leq \frac{C}{(k - l)^\alpha} , \text{ for } |k - l| \geq 2M. \] (3.53)

\[ \left| b_{k, l}^{j, j} \right| \leq 2^{-j} \left\| \frac{\partial^M}{\partial x_2} R_f(x_1 + 2^j (k - l), x_2 + 2^j (l - 1)) \phi(2^{-j} x_1) \psi(2^{-j} x_2) dx_1 dx_2 \right\|_{\bar{x}_1, \bar{x}_2 \in 2^j [0, 2M - 1]} \leq \frac{C}{(k - l)^\alpha + M} , \text{ for } |k - l| \geq 2M. \] (3.54)

Finally,

\[ \left| r_{k, l}^{j, j} \right| \leq 2^{-j} \left\| \frac{\partial^M}{\partial x_1 \partial x_2} R_f(x_1 + 2^j (k - l), x_2 + 2^j (l - 1)) \phi(2^{-j} x_1) \psi(2^{-j} x_2) dx_1 dx_2 \right\|_{\bar{x}_1, \bar{x}_2 \in 2^j [0, 2M - 1]} \leq \frac{C}{(k - l)^\alpha + 2M} , \text{ for } |k - l| \geq 2M. \] (3.55)

The above inequalities show that the correlation of the wavelet and scale coefficients of these stochastic processes behaves as

\[ a_{k, l}^{j, j} \sim \frac{1}{(k - l)^\alpha} , \quad b_{k, l}^{j, j} \sim \frac{1}{(k - l)^{\alpha + M}} , \quad r_{k, l}^{j, j} \sim \frac{1}{(k - l)^{\alpha + 2M}}. \] (3.56)
Note that although the scale coefficients exhibit similar correlation as the stochastic process itself, the correlation of the wavelet coefficients diminishes significantly faster, especially for large values of $M$. Then, the correlation parameters $r_{k,l}^{h,j}$ and $b_{k,l}^{h,j}$ can be neglected even for a relatively small value $(k-l)$. The wavelet transformation acts as a semi-whitening filter and yields an uncorrelated or weakly correlated set of random variables.

A stochastic process with the auto-correlation function given by Eq. 3.33 with $\alpha=0.1$ is considered to assess the whitening effect of the wavelet transformation. Clearly, this process belongs to the Meyer-Belkin class. Figure 3.5 shows the decay of the correlation parameters $a_{k,l}^{h,j}$, $r_{k,l}^{h,j}$, and $b_{k,l}^{h,j}$ as a function of $k-l$ for several Daubechies’s wavelets with the different number of vanishing moments $M$. The results are plotted for the scale $j=-6$ so that the corresponding time step between two neighboring wavelet functions is smaller than the “scale of fluctuation” (Vanmarcke 1983). These results agree with the theoretically developed estimate of Eq. 3.56 quite well. The same results are plotted in Figure 3.6 using the logarithmic coordinate system for a better demonstration of the wavelet “decorrelation”. The dependence of the rate of decay of $r_{k,l}^{h,j}$ and $b_{k,l}^{h,j}$ on the number of vanishing moments $M$ can be clearly observed in Figure 3.6. These wavelet coefficients can be considered “almost white” for practical applications.

Finally, it has been observed that if the auto-correlation function of a stochastic process has some irregularity along the diagonal $x_1 = x_2$, the correlation of the wavelet coefficients for a given scale resemble a diagonal matrix. In this case, wavelets act as differential operators and emphasize the diagonal singularity of the auto-correlation function. The Taylor series can be utilized to evaluate the module of correlation in a fashion similar to the previous case. However, this estimate breaks down near the diagonal. Then, the diagonal values of $r_{k,l}^{h,j}$ are expected to be quite large. The wavelet transform preserves the energy of signals. Thus, the diagonal value of $r_{k,k}^{h,j}$ increases only
at the "expense" of the non-diagonal values. The correlation parameters $a_{k,l}^{i,j}$ and $r_{k,l}^{i,j}$ for the Gauss-Markov process having the auto-correlation function of Eq. 3.34 with $\alpha=0.1$ are plotted for $j=-6$ in Figures 3.7 and 3.8 in the normal and logarithmic coordinate systems, respectively. Wavelets with $M=3$ vanishing moments have been used in this regard. It can be readily seen that the correlation of the wavelet coefficients is quite weak in this case.

3.4 Concluding Remarks

In this chapter it has been shown that stochastic processes can be efficiently represented by a set of random variables by using the wavelet expansion. The probability structure of these random variables is uniquely determined by the probability structure of the corresponding stochastic process, and vice versa.

An algorithm which is based on the two-dimensional generalization of the wavelet decomposition algorithm has been developed to evaluate numerically the correlation of the wavelet coefficients. This algorithm requires $O(N)$ and $O(N^2)$ numerical operations for stationary and non-stationary stochastic processes, respectively. Alternatively, efficient approximation based on the judicious use of the local and quasi-differential properties of wavelets is introduced. It reduces the number of requisite numerical operations to $O(\log N)$ and $O(N)$, respectively.

The analysis of the covariance matrices associated with the wavelet coefficients has shown that wavelets can be quite useful for representation and synthesis of stochastic processes; the wavelets are capable of compressing random signals which can be characterized by a few correlation parameters. Thus, it has been shown that wavelets effectively restrict the random signal energy to a few scales. Moreover, the wavelet transformation can be viewed as a semi-whitening filter for a large class of stochastic processes. Finally, it has been shown that the "cross-scale" correlation of the wavelet
coefficients is quite weak and the “in-scale” correlation is also “almost” white for the fractional Brownian motion, the Meyer-Belkin processes, and the Markov-Gauss processes.
Figure 3.1. Relative value of the error of approximation of $a_{k,l}$; the black dots indicate the first scale where the approximation is not used.
Figure 3.2. Maximum of the absolute value of the errors of approximation of $r_{k,l}^{i,j}$ normalized by $r_{k,k}^{i,j}$ and evaluated based on the results of Figure 3.1.
Figure 3.3. The ratio $r^{j,i}_{k,k}/a^{j,i}_{k,k}$ as a function of the scale for several values of $M$. 
Figure 3.4. Function $\chi_k(\omega)$ for wavelets with
a) $M=1$ vanishing moment, b) $M=7$ vanishing moments.
Figure 3.5. Parameters $a_{k,l}$, $b_{k,l}$ and $r_{k,l}$ as a function of $k,l$:  
(a) $M=1$ (Haar basis); (b) $M=3$; (c) $M=7$. 
Figure 3.6. Parameters $a_{k,l}$, $b_{k,l}$ and $r_{k,l}$ as a function of $k$-$l$:
(a) $M=1$ (Haar basis); (b) $M=3$; (c) $M=7$. 
Figure 3.7. Parameters $a_{k,l}$, $b_{k,l}$, and $r_{k,l}$ as a function of $k-l$:
(a) $M=1$ (Haar basis); (b) $M=3$; (c) $M=7$. 
Figure 3.8. Parameters $a_{k,l}$, $b_{k,l}$ and $r_{k,l}$ as a function of $k,l$:
(a) $M=1$ (Haar basis); (b) $M=3$; (c) $M=7$. 
Chapter 4. Simulation of Stochastic Processes Using Wavelets

The representation of stochastic processes developed in the previous chapter by using wavelet bases can be viewed as a stochastic analog of the deterministic wavelet decomposition of Eqs. 1.125 and 1.126 schematically shown in Figure 1.7. Similarly, synthesis of stochastic process can be developed as a generalization of the wavelet reconstruction algorithm shown in Figure 1.8. The wavelet reconstruction algorithm, Eq. 1.127, can be interpreted as a dynamic process defined on the scale domain: the process on the next scale, \( \mathbf{c}^{j+1} = \begin{bmatrix} c_1^{j+1} & \ldots & c_{2^j}^{j+1} \end{bmatrix} \), is defined by the scale “initial conditions” \( \mathbf{e}^{j+1} \) and a “noise excitation”, \( \mathbf{d}^{j+1} = \begin{bmatrix} d_1^{j+1} & \ldots & d_{2^j}^{j+1} \end{bmatrix} \), which, in general, is correlated with the initial conditions. Then, an efficient synthesis method can be formulated by utilizing this underlying scale dynamics of the wavelet representation.

In this regard, Section 4.1 develops a simple parametric model of stochastic processes by neglecting the cross-scale correlation of the wavelet coefficients. That is, the random vectors \( \mathbf{c}^j \) and \( \mathbf{d}^j \) are assumed to be statistically independent. The error associated with this simplification is identified and evaluated. The developed parametric model fructifies an efficient method for synthesizing stochastic processes by relying on the wavelet reconstruction algorithm.

However, a more accurate and efficient synthesis method which utilizes the wavelet multi-scale structure is introduced in Section 4.2 by estimating the wavelet coefficients based on the scale coefficients. Then, \( \mathbf{d}^j \) can be accurately and efficiently simulated; this defines the stochastic process on the next scale. The error associated with these approximate models is also discussed. This development yields a class of algorithms capable of synthesizing stochastic processes in \( O(N) \) numerical operations. The numerical
examples prove the usefulness of the proposed synthesis method.

Another interpretation of the proposed synthesis method is introduced in Section 4.3 by using the concepts of dynamic systems on homogeneous trees. Also, the guidelines for ongoing research are outlined. Finally, Section 4.5 summarizes the material of this chapter and presents concluding remarks.

4.1 Simple Scale-Independent Parametric Models of Stochastic Processes

A set of random variables $d^j_k$ defines uniquely a stochastic process $\tilde{f}(x)$ by using the following parametric model

$$\tilde{f}(x) = \sum d^j_k \psi_{j,k}(x).$$

(4.1)

Clearly, if the covariance matrix of the random variables $d^j_k$ satisfies Eq. 3.4, the parametric stochastic process $\tilde{f}(x)$ has the auto-correlation function $R_f(x_1, x_2)$. However, if the correlation structure of the wavelet coefficients is simplified by using some “decorrelation” properties of the wavelet expansion and some correlation parameters are neglected, the corresponding auto-correlation function $\tilde{R}_f(x_1, x_2)$ of $\tilde{f}(x)$ is only an approximation for the target function $R_f(x_1, x_2)$.

For evaluating the errors of this approximation the auto-correlation function $\tilde{R}_f(x_1, x_2)$ can be expressed as

$$\tilde{R}_f(x_1, x_2) = \sum_{k,l,j,i} r^j_i \psi_{j,k}(x_1) \psi_{k,i}(x_2).$$

(4.2)

One of the straightforward simplifications of the parametric model of Eq. 4.1 is neglecting the cross-scale correlation of the wavelet coefficients. That is,
\[ r_{k_i}^{j_i} = r_{k_i}^{j_i} \delta_{j_i} \]  

(4.3)

where \( \delta_{j_i} \) is the Kronecker delta. Substituting Eq. 4.3 into Eq. 4.2 yields

\[ \tilde{R}_f(x_1, x_2) = \sum_j \sum_{k_i} r_{k_i}^{j_i} \psi_{j_i, k}(x_1) \psi_{j_i, l}(x_2) . \]

(4.4)

Note that this assumption can introduce a significant bias into the parametric model. Thus, the process \( \tilde{f}(x) \) is not stationary unless \( M = \infty \), even if the auto-correlation function \( R_f(x_1, x_2) \) depends only on the difference \( x_2 - x_1 \). The concept of the Wigner distribution (Claasen and Mecklenbrauker 1980a,b,c, Martin and Flandrin 1985) can be utilized to study this effect further. Although the Wigner distribution is not positive for the entire time-frequency plane, it has some positive definite properties (Janssen and Claasen 1985) that provide for its approximate treatment as the "energy distribution" over the time-frequency plane. Also, the Wigner distribution reduces to the spectral density function for stationary processes. It is defined as

\[ W(x, \omega) = \int_{-\infty}^{\infty} R\left( x + \frac{\tau}{2}, x - \frac{\tau}{2} \right) e^{-i\omega \tau} d\tau . \]

(4.5)

Substituting Eq. 4.4 into Eq. 4.5 yields

\[ \tilde{W}_f(x, \omega) = \sum_j \sum_{k_i} r_{k_i}^{j_i} W_{\psi_{j_i, k}, \psi_{j_i, l}}(x, \omega) = \sum_j \sum_{k} r_{k_i}^{j_i} \sum_{l} W_{\psi_{0, 0}, \psi_{0, k}} (2^{-j} x - l, 2^{j} \omega) , \]

(4.6)

where

\[ W_{f, g}(x, \omega) = \int_{-\infty}^{\infty} f\left( x + \frac{\tau}{2} \right) g\left( x - \frac{\tau}{2} \right) e^{-i\omega \tau} d\tau \]

(4.7)
is the cross-Wigner distribution of functions \( f(x) \) and \( g(x) \).

The Wigner distribution \( W_\psi(x, \omega) \) of the Haar wavelet \( \psi(x) \) is shown in Figure 4.1. This distribution, as a function of frequency, is concentrated around the frequency \( 2\pi \). However, it is clearly shown that this distribution is not a constant function of time in the segment \([0,1]\). In particular, this function is equal to zero for \( t = 0 \) and \( 1 \). The assumption of the cross-scale independence of the wavelet coefficients implies that this deficiency of the Wigner distribution on one scale is not properly corrected by the finer scales. That is, the variance of the processes \( \tilde{f}(x) \) is under-represented in the neighborhood of the dyadic points \( k2^j \).

Though the process \( \tilde{f}(x) \) is not stationary the concept of averaged spectrum can be used to evaluate the ability of the parametric model of Eq. 4.1 to approximate spectra. The averaged spectrum can be defined as

\[
\tilde{S}(\omega) = \int T \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\omega t} d\tau \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} W(t, \omega) dt . \tag{4.8}
\]

Substituting Eq. 4.6 into Eq. 4.8 and using the fact that \( \sum_{l} W_{\psi_{0,0}, \psi_{0,k}} (2^{-j} t - l, 2^{j} \omega) \) is a \( 2^j \) periodic function of \( t \) one can derive

\[
\tilde{S}(\omega) = \sum_{j} \sum_{k} r_k^j \int_{-\infty}^{\infty} W_{\psi_{0,0}, \psi_{0,k}} (t, 2^j \omega) dt = \sum_{j} P_j(2^j \omega) |\psi(2^j \omega)|^2 , \tag{4.9}
\]

where

\[
P_j(\omega) = \frac{1}{2\pi} \sum_{k} r_k^j e^{i\omega k} \tag{4.10}
\]

is the spectrum of the discrete random sequence \( d_k^j, k = \ldots, -1, 0, 1, 2, \ldots \).
Another, even simpler parametric model of the process \( f(x) \) can be constructed by assuming that the wavelet coefficients are completely uncorrelated across scales as well as within scales. In this case, the average spectrum of the parametric process \( \tilde{f}(x) \) can be found by using Eq. 4.9

\[
\tilde{S}(\omega) = \sum_j r_0^j |\psi(2^j \omega)|^2 .
\]  

(4.11)

Figure 4.2 shows the spectral approximations which are associated with the parametric models with the cross-scale uncorrelated and completely uncorrelated wavelet coefficients. The target spectrum corresponds to the Gauss-Markov process with the auto-correlation function given by the equation

\[
R(x_2 - x_1) = \exp \left( \frac{|x_2 - x_1|}{\alpha} \right)
\]  

(4.12)

with \( \alpha = 0.1 \). The Haar wavelet has been used in this case. Note that both models approximate the target spectrum reasonably well. Observe that the in-scale correlation of the wavelet coefficients do not improve the spectral approximation. This fact can be explained by the whitening effect of the wavelet expansion for the Gauss-Markov processes as it is discussed in Section 3. These results show that the in-scale cross-correlation does not significantly influence the quality of the spectrum approximation; a more significant portion of the error is introduced by disregarding the cross-scale correlation of the wavelet coefficients.

Finally, note that the spectral approximation of Eq. 4.11 with the coefficients \( r_0^j \), \( j = \ldots -1, 0, \ldots \), is not the best approximation. The correlation parameters \( r_0^j \) are found by expanding the auto-correlation function in the wavelet basis, Eq. 4.2, with subsequent neglecting subsequently all other terms. Instead, one can seek the parameters \( \lambda_i \) which
provide the best spectral estimation

\[ \tilde{S}(\omega) = \sum_{j} \lambda^j |\psi(2^j \omega)|^2. \]  

(4.13)

The parameters \( \lambda^j \) can be calculated by minimizing the difference \( S(\omega) - \tilde{S}(\omega) \) for a set of frequencies \( \omega_k, k = 1, \ldots, N \). That is, \( \lambda^j \) are the least square solution of the overdetermined system of equations

\[ \sum_{j=n_2}^{n_1} \lambda^j |\psi(2^j \omega_k)|^2 = S(\omega_k), \quad k = 1, 2, \ldots, N, \quad N > n_1 - n_2. \]  

(4.14)

The results of this modeling procedure are shown in Figure 4.3. Clearly, this procedure can provide significant improvement of spectral approximation by using uncorrelated set of random variables. The parametric model \( \tilde{f}(x) \) in this case can be written as

\[ \tilde{f}(x) = \sum d_i^j \psi_{j, k}(x), \]  

(4.15)

where \( d_i^j \) are statistically independent random variables such that

\[ E[d_i^j d_i^l] = \lambda^j \delta_{jl} \delta_{kl}. \]  

(4.16)

Eqs. 4.15 and 4.16 leads to an efficient algorithm for synthesizing stochastic processes since the summation in Eq. 4.15 can be performed by using the wavelet reconstruction algorithm which requires only \( O(N) \) numerical operations.
4.2 Scale Dependent Parametric Models for Synthesizing Stochastic Processes

Simple scale independent models utilizing the wavelet expansion of stochastic processes have been developed in the preceding section. Due to their simplicity these models provide an efficient procedure for synthesizing stochastic processes. However, it has been shown that the introduced simplifications can lead to significant errors both in the spectral approximation of the stochastic process and in the introduction of quite specific non-stationary features into the process approximation. A simple modification can eliminate these drawbacks which, to a great extent, are the result of neglecting the cross-scale correlation of the wavelet coefficients.

The main objective of this development is to make a better use of the multi-scale structure of the wavelet representation of stochastic processes. As it has been alluded to previously the wavelet reconstruction algorithm can be interpreted as a scale dynamic system. That is, the projection of a stochastic process into the $j$-th scale can be modeled as a dynamic relation between the scale and wavelet coefficients on the $(j+1)$-th scale. In order to synthesize efficiently a stochastic process, the model for the wavelet coefficients must provide both a numerical algorithm with minimal computational cost and an adequate approximation of the corresponding covariance matrix. This modeling problem has been originally addressed by Zeldin and Spanos (1995a,b) and is formulated in the sequel.

4.2.1 Mathematical Formulation

Assume that the stochastic process $f(x)$ is defined by the auto-correlation function $R_f(x_1, x_2)$ within a bounded domain and denote by $f_j(x)$ the projection of $f(x)$ into the
scale \( j \). This projection is specified by the random vector \( \mathcal{Q}^j = \{ c_{1}^{j}, ..., c_{\kappa}^{j} \} \), \( \kappa = 2^{-j} \).

Note that the number of components of \( \mathcal{Q}^j \) is quite small for a sufficiently coarse scale. Then, a realization of \( \mathcal{Q}^j \) can be generated by performing the Cholesky factorization of its covariance matrix. This provides a sample of the projection \( f_j(x) \) which is used to initiate the simulation of \( f(x) \) for finer scales. Thus, the projection of \( f(x) \) into the next \((j-1)\)-th scale can be found by simulating the random vector \( d^j = \{ d_{1}^{j}, ..., d_{\kappa}^{j} \} \); \( \mathcal{Q}^{j-1} \) can be determined by relying on the wavelet reconstruction algorithm, Eq. 2.86. Next, the random vector \( \mathcal{Q}^{j-2} \) can be simulated based on the realization of \( \mathcal{Q}^{j-1} \), etc. This hierarchical procedure generates a sample of the stochastic process \( f(x) \).

For a relatively fine scale, the problem of generating a large random vector with correlated components is quite complicated. Models which neglect completely the cross-correlation of the wavelet coefficients were considered in the previous section. It was shown that these models can induce a significant bias in the approximation of stochastic processes. In this context, it is noted that the random variables \( c_k^j \) and \( d_k^j \) are defined as an average of the stochastic process \( f(x) \) over the same domain weighted by the scale and wavelet functions, respectively. The quasi-differential properties of wavelets ensure that 
\[ c_k^j \sim f(2^{-j}(k - 1)) \quad \text{and} \quad d_k^j \sim f^{(M)}(2^{-j}(k - 1)) \,
\]
Clearly, the derivative \( f^{(M)}(x) \) can be approximated reliably by a finite difference scheme. Thus, one can approximate the coefficient \( d_k^j \) by a linear combination of the components of the vector \( \mathcal{Q}^j \) in the form
\[
d_k^j = \sum_{l} \alpha_{k,l} c_l^j + \beta_k u_k, \tag{4.17}
\]
where \( u_k, k = 1, 2, \ldots \), are uncorrelated zero mean, unit variance random variables which are statistically independent of \( \mathcal{Q}^j \).

Eq. 4.17 also uses the assumption that the components of the vector \( d^j \) are statistically independent, given a realization of \( \mathcal{Q}^j \). Note that this approximation does not imply that
the components of $d^j_l$ are unconditionally statistically independent, but rather that their
dependence is completely reflected by a linear combination of the components of $c^j_l$. Eq. 4.17 provides a simple filter which transforms the white noise process 
$u = \{u_1, u_2, \ldots, u_k\}$ into a sequence of random variables $d^j_l$. This expression can be
efficiently used to simulate the random vector $d^j_l$ based on a realization of $c^j_l$.
Alternatively, Eq. 4.17 can be interpreted as the best linear estimation of the wavelet 
coefficients in the j-th scale by using the scale coefficients in the same scale.

Note that the summation in Eq. 4.17 can be confined to a few adjacent elements, since
the correlation of the wavelet coefficients decreases rapidly with the difference $k-l$

$$d^j_k = \sum_{l=-n}^{n} \alpha^j_{k,l} c^j_{k+l} + \beta^j_{k} u_k.$$  \hspace{1cm} (4.18)

The parameters $\alpha^j_{k,l}$ in Eq. 4.18 can be computed in an optimal way so that the covariance 
matrix of the wavelet and scale coefficients is well approximated. In particular, the 
variance of the wavelet coefficients $d^j_k$ and the cross-correlation of the wavelet and the 
scale coefficients can be equated with the target values. Thus, multiplying Eq. 4.18 by 
c^j_{k+i}, i = -n, \ldots n$ and taking mathematical expectation one finds

$$\sum_{l=-n}^{n} \alpha^j_{k,l} \alpha^{h,j}_{k+l,k+i} = b^{h,j}_{k+i,k}, \quad i = -n, \ldots n.$$  \hspace{1cm} (4.19)

The matrix in Eq. 4.19 is symmetric and positive definite; if the stochastic process is 
stationary, this is a Toeplitz matrix. This system can be readily solved without significant 
computational effort as the number of equations $2n+1$ is small. Further, the parameter $\beta^j_k$
can be found by squaring Eq. 4.18 and taking the mathematical expectation. That is,

$$\beta^j_k = \sqrt{b^{h,j}_{k,k}} - \sum_{l=-n}^{n} \sum_{i=-n}^{n} \alpha^j_{k,l} \alpha^j_{k+l,k+i}.$$  \hspace{1cm} (4.20)
Note that, for the stationary case, Eqs. 4.19 and 4.20 must be solved only once for every scale $j$.

Note that the pyramid structure of the proposed simulation algorithm breaks down in some neighborhood of the boundary. In the present study this problem is addressed by incorporating boundary layers adjoint to the domain of interest. Only certain wavelet parameters are properly handled in these layers to eliminate associated errors. Alternatively, periodic wavelets and functions (Daubechies 1992) can be used.

Finally, note that the scale-type simulation methods which are discussed in Section 2.3 are a rather special case of the proposed approach. Specifically, the Local Average Subdivision method (Fenton and Vanmarcke 1990, Fenton 1990, 1994) can be reproduced by utilizing the Haar basis ($M = 1$) in the preceding development, whereas the Lewis (1987) method can be formulated based on the Shannon wavelets ($M = \infty$).

### 4.2.2 Error Analysis

The approximation of the wavelet coefficients introduced by Eq. 4.17 is quite simple but adequate for most engineering applications. To substantiate it, the corresponding errors can be assessed. Assume that the stochastic process has been simulated within the $j$-th scale exactly. Further, assume that the wavelet coefficients $d^j_k$ are found by using the proposed cross-scale dependent model, Eq. 4.18, and the scale coefficients on the ($j$-1)th scale $c^{j-1}$ are determined by relying on the wavelet reconstruction algorithm. That is,

$$\hat{c}^{j-1}_k = \sum_l (h_{k-2l+2} c^j_l + g_{k-2l+2} d^j_l), \quad (4.21)$$

where the hat denotes the estimated value rather than exact. Multiplying Eq. 4.21 by itself and performing some rather simple algebraic manipulations one can obtain
\[
\hat{a}_{k_1,k_2}^{j-1,j-1} = \langle \hat{c}_{k_1}^{j-1}, \hat{c}_{k_2}^{j-1} \rangle = \sum_{l_1,l_2} a_{l_1,l_2}^{j,j} h_{k_1-2l_1} + 2h_{k_2-2l_2} + 2 + \\
\sum_{l_1,l_2} b_{l_1,l_2}^{j,j} \hat{h}_{k_1-2l_1} + 2g_{k_2-2l_2} + 2 + \sum_{l_1,l_2} b_{l_1,l_2}^{j,j} g_{k_1-2l_1} + 2h_{k_2-2l_2} + 2 + \\
\sum_{l_1,l_2} r_{l_1,l_2}^{j,j} g_{k_1-2l_1} + 2g_{k_2-2l_2} + 2 .
\]

Further, the target correlation of the scale coefficients on the \((j-1)\)-th scale can be found by the equation

\[
a_{k_1,k_2}^{j-1,j-1} = \langle c_{k_1}^{j-1} c_{k_2}^{j-1} \rangle = \sum_{l_1,l_2} a_{l_1,l_2}^{j,j} h_{k_1-2l_1} + 2h_{k_2-2l_2} + 2 + \\
\sum_{l_1,l_2} b_{l_1,l_2}^{j,j} \hat{h}_{k_1-2l_1} + 2g_{k_2-2l_2} + 2 + \sum_{l_1,l_2} b_{l_1,l_2}^{j,j} g_{k_1-2l_1} + 2h_{k_2-2l_2} + 2 + \\
\sum_{l_1,l_2} r_{l_1,l_2}^{j,j} g_{k_1-2l_1} + 2g_{k_2-2l_2} + 2 .
\]

Further, Eqs. 4.19 and 4.20 yield that

\[
b_{l_1,l_2}^{j,j} = b_{l_1,l_2}^{j,j}, \quad l_1, l_2 = -n, \ldots, n ,
\]

\[
\hat{r}_{l_1,l_2}^{j,j} = r_{l_1,l_2}^{j,j} .
\]

However,

\[
\hat{r}_{l_1,k}^{j,j} \neq r_{l_1,k}^{j,j} , \text{ if } k \neq l .
\]

If the summation index in Eq. 4.17 is extended beyond the support of the wavelet function, the difference between the estimated and the corresponding target values of the scale correlation parameters is found by subtracting Eq. 4.22 from Eq. 4.23 and using Eq. 4.24

\[
a_{k_1,k_2}^{j-1,j-1} - \hat{a}_{k_1,k_2}^{j-1,j-1} = \sum_{l_1,l_2} (r_{l_1,l_2}^{j,j} - \hat{r}_{l_1,l_2}^{j,j}) g_{k_1-2l_1} + 2g_{k_2-2l_2} + 2 ,
\]
where the value $r_{l_1,l_2}^{i,j}$ can be estimated by using Eq. 4.18

$$r_{l_1,l_2}^{i,j} = \sum_{k_1,k_2} a_{l_1,l_2}^{i,j} + k_2 l_2 + k_2 \alpha_{l_1,k_1} \alpha_{l_2,k_2} + \beta_{l_1} \beta_{l_2} \delta_{l_1,l_2}.$$  \hspace{1cm} (4.28)

The quantity in the right hand side of Eq. 4.27 is relatively small due to the following factors:

1) For sufficiently fine scales the magnitude of the parameters $r_{l,k}^{i,j}$ in the right hand side of Eq. 4.27 is smaller than the value of parameters $a_{l,k}^{i,j}$ in the left hand side of Eq. 4.27; the parameters $r_{l,k}^{i,j}$ provide "small" correction to $a_{l,k}^{i-1,j-1}$.

2) Eq. 4.18 is the best linear estimate of a random variable $d_k^j$ using the vector $c_k^j$. Eqs. 4.19 and 4.20 ensure that the diagonal error is equal to zero, Eq. 4.25, and the non-diagonal errors are small.

3) The non-diagonal elements $r_{l,k}^{i,j}$ decay rapidly for a large class of random processes for which the wavelet transform can be viewed as a semi-whitening filter; see the discussion in Section 3.3.

4) Eq. 4.27 can be viewed as a two-dimensional high-pass moving average filter which tends to the ideal high pass filter as $M$ becomes large. Then, this high-pass filter eliminates the low-frequency error and allows only the high frequency error. Further, the high frequency range for the scale $j$ becomes low frequency for the scale $j-1$. The error induced by Eq. 4.17 occurs on the separate frequency ranges for each scale and is not accumulated.

Several numerical examples are considered to substantiate the developed theoretical estimation. First, the auto-correlation function

$$R(x_1, x_2) = \frac{1}{1 + ((x_1 - x_2)/\alpha)^2}$$  \hspace{1cm} (4.29)
is considered for $\alpha = 0.1$. The correlation of the wavelet coefficients is estimated by using Eq. 4.28 and is compared with the exact value. This comparison is shown in Figures 4.4 and 4.5 for the Daubechies wavelets with $M=1$ and $M=3$, respectively. The number of adjacent elements in Eq. 4.18 is selected $n=1$. Note that the wavelet correlation parameters $r_{l,k}^{(j)}$ are estimated exactly for the zero lag, as expected. The proposed approximation is quite accurate for fine scales. For coarse scales the sequence of wavelet coefficients is almost white and the corresponding error is also small. The error of estimating the scale correlation parameters is evaluated based on Eq. 4.27 by using the corresponding estimates of the correlation of the wavelet coefficients. These results are shown in Figures 4.11(a) and 4.11(b) for $M=1$ and $M=3$, respectively; the maximum value of the error for every scale is normalized by the variance of the scale coefficients. Note, that this error is small for fine scales and achieves the maximum value, approximately 4%, for the scale ($-2$). The mesh size of this scale is approximately equal to the scale of fluctuation of this stochastic process. The quality of the proposed approximation depends on the value of the parameter $n$ in the summation of Eq. 4.18. Figures 4.11(a) and 4.11(b) show that the approximation is slightly better for larger value $n$. Note that $\alpha_{k,0} = 0$ for $n=0$ in conjunction with the Haar basis, $M=1$. In this case, Eq. 4.18 provides the white noise approximation of the wavelet coefficients discussed in Section 4.1. As a result this yields a significantly larger error especially for fine scales. On the contrary, the approximation error is not reduced significantly for $n \geq 1$ in conjunction with the Daubechies wavelets; fewer adjacent sells can be used in this case.

Next, the auto-correlation function

$$R(x_1, x_2) = \exp\left(-\frac{(x_1-x_2)^2}{\alpha^2}\right)$$

(4.30)

is considered for $\alpha = 0.1$. Similar results of estimation are shown in Figure 4.6 for $M=1$ and $n=3$, and in Figures 4.7 and 4.8 for $M=3$ and $n=3$ and $n=5$, respectively. Again, the
approximation of Eq. 4.18 is reasonably accurate for fine scales while for coarse scales the sequence of the wavelet coefficients is almost white. The error of estimating the scale correlation parameters are evaluated based on Eq. 4.27 by using the corresponding estimate of the wavelet correlation. These results are shown in Figures 4.11(c) and 4.11(d) for \( M=1 \) and \( M=3 \), respectively; the maximum value of the corresponding error for every scale is normalized by the variance of the scale coefficients. Note that this error is small for fine scales and achieves its maximum value, approximately 5%, for the scale (-2). Again, larger values of \( n \) provide more accurate approximation for fine scales; see Figures 4.6, 4.7, 4.8, and 4.11(c,d).

Finally, a stochastic process with the quite irregular correlation function

\[
R(x_1, x_2) = \exp\left(-|x_1 - x_2|/\alpha\right)\cos(\omega|x_1 - x_2|)
\]  

(4.31)

is considered; \( \alpha = 1 \) and \( \omega = 10 \) are taken. Figures 4.9, 4.10, 4.11(e), and 4.11(f) show the results similar to the previous cases. Note that for this auto-correlation function the maximum relative error of estimating the scale coefficients can be as big as 25% for the scale with mesh size comparable to the size of ripples of the auto-correlation function. For this scale the variance of the wavelet coefficients is significantly larger than the variance of the scale coefficients. Then, a relatively small error in estimating the wavelet coefficients is still large in comparison with the variance of the scale coefficients.

4.2.3 Proposed Synthesis Algorithms

Summarizing the preceding discussion, the proposed algorithm for synthesizing random fields specified by the auto-correlation function \( R_f(x) \) can be formulated as follows:

1. Select a wavelet basis; pertinent numerical studies suggest that the Daubechies wavelets
with $M=3$ are quite adequate for forms of $R_f(x)$ encountered in most engineering applications. Note that these wavelets are differentiable functions and the generated field can be readily used in applications necessitating differentiation of the generated field samples.

2. Find the correlation of the wavelet and scale coefficients as discussed in Section 3.

3. Synthesize a sample of the random process for a relatively coarse scale $j$ by simulating a small-dimensional vector $c^j$; the Cholesky factorization of the covariance matrix of $c^j$ is deemed appropriate for this purpose.

4. Generate the vector $d^j$ by using Eq. 4.18.

5. Based on the realizations of $c^j$ and $d^j$, synthesize a sample of the random process on the next ($j-1$) scale by using the wavelet reconstruction algorithm of Eq. 1.127.

6. If the ratio of the variance of the wavelet coefficients to the variance of the scale coefficients is not adequately small, proceed to a refined scale and return to step 4.

Each step of the proposed algorithm requires, at most, $O(N)$ numerical operations. Moreover, the decorrelation and multi-scale properties of wavelets ensure that the requisite number of parameters in Eq. 4.17 is small.

### 4.2.4 Numerical Simulations

A simple computer program is developed to assess the performance of the proposed simulation method. The Daubechies wavelet function with $M=3$ vanishing moments is used in this regard. In this case the generated samples of stochastic processes are differentiable since the selected wavelet function is differentiable.

First, samples of a Gaussian stationary stochastic process with the auto-correlation
function given by Eq. 4.29 are generated for $\alpha = 0.05$, see Figure 4.12(a). The accuracy of the proposed simulation method is studied by using the ensemble average based on $N=200$ realizations. Estimates of the mean value, the variance, and the auto-correlation function of this stochastic process are shown in Figure 4.13. Several estimates of the auto-correlation function are available by taking different points for estimation. The error of estimating the mean value and variance of this stochastic process is compared with the theoretical values in Table 4.1. For this purpose the upper and low bounds of the estimates of the mean and variance, such that only 5% of the corresponding quantities exceed these bounds, are found. The theoretical values of these bounds are found assuming that the corresponding estimates are normally distributed random variables. Table 4.1 shows that the accumulated error is within the theoretical bounds for the given ensemble size. Also, the auto-correlation function is approximated quite accurately as shown in Figure 4.13(c).

Next, samples of a Gaussian stationary stochastic process with the auto-correlation function given by Eq. 4.30 are generated for $\alpha = 0.05$, see Figure 4.12(b). The accuracy of the proposed simulation method is studied by using the ensemble average based on $N=200$ realizations. Estimates of the mean value, the variance, and the auto-correlation function of this stochastic process are shown in Figure 4.14. Again, several estimates of the auto-correlation function are found by taking different points for estimation. A similar comparison of the errors of estimating the mean value and the variance is presented in Table 4.1. Table 4.1 shows that the accumulated error is within the theoretical bounds for the given ensemble size. Also, the auto-correlation function is approximated quite accurately as shown in Figure 4.14(c).

Finally, samples of a Gaussian stationary stochastic process with the auto-correlation function given by Eq. 4.31 are generated for $\alpha = 1$ and $\omega = 33.3$, see Figure 4.12(c). The accuracy of the proposed simulation method is studied by using the ensemble average based on $N=200$ realizations. Estimates of the mean value, the variance, and the auto-
correlation function of this stochastic process are shown in Figure 4.12. Several estimates of the auto-correlation function are available by taking different points for estimation. Although a small systematic bias in estimating the variance and the auto-correlation function can be observed in Figure 4.15, the shape of the auto-correlation function and the variance are evaluated quite accurately as shown in Figure 4.15(c). The error of estimating the mean value and the variance is compared to pertinent theoretical values in Table 4.1. This comparison shows that the accumulated error is within the theoretical bounds for the given ensemble size.

Table 4.1: The error of estimating the mean value and the variance based on the 5% exceeding criteria

<table>
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<tr>
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<th>Mean Value</th>
<th></th>
<th>Variance</th>
<th></th>
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</thead>
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<tr>
<td></td>
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<td>upper</td>
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<td>upper</td>
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<tr>
<td></td>
<td>bound</td>
<td>bound</td>
<td>bound</td>
<td>bound</td>
</tr>
<tr>
<td>Example 1</td>
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<td>0.8442</td>
<td>1.1494</td>
</tr>
<tr>
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<td>0.8317</td>
<td>1.1600</td>
</tr>
<tr>
<td>Example 3</td>
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<td>0.8154</td>
<td>1.1237</td>
</tr>
<tr>
<td>Theory</td>
<td>-0.1163</td>
<td>0.1163</td>
<td>0.8355</td>
<td>1.1645</td>
</tr>
</tbody>
</table>

4.3 The Proposed Synthesis Method as a Realization of a Scale Dynamic System

An interesting interpretation of the proposed simulation method can be obtained by incorporating the theory of dynamic processes on homogeneous trees (Cartier 1974, Arnaud 1980, Dunau 1983, Basseville et.al. 1992a,b,c and Benveniste et.al. 1990). A homogeneous tree $T$ of order $q$ is an infinite acyclic, undirected, connected graph such that every node of $T$ has exactly $(q+1)$ branches to other nodes. With this definition, $q=1$
corresponds to the usual integer structure with the obvious branches from one integer to its two neighbors. Then, the standard ARMA method discussed in Chapter 2 and 6 (see also Samaras et.al 1983, Minolet and Spanos 1987) can be viewed as a realization of a dynamic system defined on a homogeneous tree of order 1. The proposed wavelet based synthesis of stochastic processes naturally lends itself to treatment by the theory of homogeneous trees of order $q=2$. A part of this tree and corresponding dynamic moves are shown in Figure 4.16. Two dimensional random fields can be modeled by using homogenous trees of order $q=4$ (Tanimoto and Pavlidis 1975, Wilson and Spann 1987, Clippingsdale and Wilson 1989, Luettgen et.al. 1994).

The dyadic ($q=2$) tree $T$ has a natural definition of distance between its nodes: $d(u,v)$ is the number of branches along the shortest path between the nodes $u, v \in T$. Then, one can define the notion of an isometry on $T$ which is simply a one-to-one map of $T$ onto itself that preserves distance. For the case of $q=1$, corresponding to the usual discrete time sequence, the group of all possible isometries contains the translations of the integers $(t \rightarrow t + k)$, the reflection $(t \rightarrow -t)$, and concatenations of the two. Dunau (1983) described the group of all possible isometries on trees of general order which is much more complex and contains significantly more elements.

Further, a boundary point of a homogeneous tree can be defined as an equivalent class of sequences of connected nodes where two sequences are equivalent if they differ by a finite number of nodes. For $q=1$ there are two boundary points corresponding to sequences toward $+\infty$ and $-\infty$. For $q>1$ the set of boundary points is uncountable. Then, one point can be chosen and denoted $-\infty$. Ones $-\infty$ is identified, see Figure 4.16, there is a unique path (equivalent class) from any point $t \in T$ toward $-\infty$. The paths of any two nodes to $-\infty$ must meet at some node denoted by $s \wedge t$. Causal processes on dyadic trees can be defined by using the notion of preceding. Namely, $s \leq t$ ($s$ is at least as close to $-\infty$ as $t$) if
\[ d(s, s \land t) - d(t, s \land t) \leq 0, \quad (4.32) \]

\( s \triangleright t \) (equivalence relation on the tree) if

\[ d(s, s \land t) - d(t, s \land t) = 0. \quad (4.33) \]

Dynamic processes are defined by introducing corresponding "time-like" directions of shifts toward or away from \(-\infty\). These shifts are a counterpart of the shift operators \( z \) and \( z^{-1} \) for time sequences. A bigger variety of operators can be defined for dyadic trees as they are illustrated in Figure 4.16:

- **0**: identity operator (no move)
- **\( \alpha \)**: left forward shift (move one step away from \(-\infty\) toward the left)
- **\( \beta \)**: right forward shift (move one step away from \(-\infty\) toward the right)
- **\( \overline{\gamma} \)**: backward shift (move one step toward \(-\infty\))
- **\( \delta \)**: interchange operator (move to the nearest equivalent point)
- \( \gamma = \frac{1}{2}(\alpha + \beta) \): averaged (smoothed) forward shift.

A stochastic process \( x(j,k) \) can be assigned to the nodes of the dyadic tree. Here, \( j \) denotes the level of the tree (scale) and \( k \) denotes translation along the level of the tree (\( k \) increases from left to the right along the level). Then, the node \( t \) of the tree can be represented by its two coordinates \( t = (j, k) \). The operators defined above act in the following manner

\[ \alpha t = (j-1, 2k) \quad (4.34) \]
\[ \beta t = (j-1, 2k + 1) \quad (4.35) \]
\[ \overline{\gamma} t = \left( j + 1, \left\lfloor \frac{n}{2} \right\rfloor \right), \quad (4.36) \]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \).
A general process $y(t)$ can be defined on the dyadic homogeneous tree by using the following state-space realization (Basseville et.al 1992a)

$$
x(t) = A(t)x(\tau t) + B(t)w(t)
$$

$$
y(t) = C(t)x(t) + v(t)
$$

where $x(t)$ is the state of the process which accommodates the internal history, and $A$, $B$, and $C$ are matrices which in general depend on $t$. The properties of the dynamic processes defined by Eqs. 4.37 and 4.38 have been studied by Basseville (1992 a,b,c). The Kalman filtering theory on dyadic trees has been developed by Chou et.al. (1989,1991), Chou and Willsky (1990), and Basseville et.al. (1992a). The theory of isotropic Auto-Regressive processes on dyadic trees is addressed by Benveniste et.al. (1990) and Basseville et.al. (1992a,b,c). Unfortunately, time stationary sequences do not exactly correspond to any isotropy on the dyadic tree. Note that the point $v$ in Figure 4.13 relates differently to $u$ and $t$. A simplified variation of this model in conjunction with the two-dimensional Haar basis has been successfully used by Clippingdale and Wilson (1989) and by Wilson and Spann (1987) for image processing. In particular, Clippingdale and Wilson (1989) studied the filtering of noise images and developed a robust method for image reconstruction. Similar framework was used by Luettgen et.al. (1994) in conjunction with a broader class of wavelet functions.

The theory of dyadic trees and the corresponding dynamic relations are quite similar to the dynamic relationships of the wavelet decomposition and reconstruction algorithms. Then, these two separately developed theories can be unified. Indeed, let the scale coefficients of a stochastic process be associated with nodes of the corresponding level $j$ of the dyadic tree. Then, the scale coefficients for the next finer scale can be assigned to the nodes of the next level ($j-1$). Substituting Eq. 4.18 into the wavelet reconstruction algorithm, Eq. 1.127, one can derive
\[ c_k^j = \sum_l \left[ h_{k-2l+2} c_l^j - 1 + g_{k-2l+2} \left( \sum_{s=-n}^n \alpha_{l,s} c_{l+s}^j - 1 + \beta_{l}^j u_l^j \right) \right] = \sum_l \tilde{\alpha}_{k,l}^j c_l^j - 1 + \sum_l \tilde{\beta}_{k,l}^j u_l^j, \]  

(4.39)

where

\[ \tilde{\alpha}_{k,l}^j = h_{k-2l+2} + g_{k-2l+2} \sum_{s'=-l}^{l'} \alpha_{l,s'-l}^j, \quad \tilde{\beta}_{k,l}^j = \beta_{l}^j g_{k-2l+2}. \]  

(4.40)

This equation can be rewritten in the form of Eqs. 4.37 and 4.38 if \( \tilde{\alpha}_{k,l}^j \) and \( \tilde{\beta}_{k,l}^j \) are thought of as elements of matrices \( A \) and \( B \), respectively. Therefore, the proposed synthesis of random processes can be viewed as a realization of a dynamic system defined on the homogeneous tree of order 2.

Also, Dijkerman and Mazumdar (1994) considered several similar models which are based on the dyadic tree structure for synthesizing Gauss-Markov stochastic processes. They associated the wavelet coefficients with the state variables of the dynamic system 4.37 and 4.38 and utilized the “decorrelation properties” of wavelets to justify these models. However, it has been pointed out previously in Chapter 3 that their theoretical estimates do not provide a clear justification of the “decorrelation properties” of wavelets.

Also, the authors could not identify the parameters of random fields which affect the optimal selection of the scale-dynamic model.

### 4.4 Some Extensions

The developed method for synthesizing stochastic processes is amenable to extension. The following comments are pertinent.
1. It has been shown that the proposed synthesis method can be viewed as a realization of a scale dynamic system defined on the dyadic tree by Eqs. 4.37 and 4.38. Then, additional studies of this scale dynamic system can highlight the properties of the wavelet based synthesis procedure. In particular, the correlation and spectral properties of stochastic processes can be associated with matrices $A(t)$, $B(t)$, and $C(t)$ of Eq. 4.37.

2. The problem of identification of dynamic systems with the time-scale structure of dyadic trees can be addressed. In this case, a realization (record) of a stochastic process can be viewed as the response of a scale dynamics system to the white noise excitation. The parameters of this model can be identified and used for subsequent simulations. The theory of the Kalman filtering on dyadic trees (Chou et.al. 1989, 1991) can be used in this regard.

3. Note that wavelets perform a dyadic (octave-type) partitioning of the frequency domain. That is, the wavelet coefficients carry the spectral information of the whole octave of the signal frequency content. This fact and the proposed simplification of the wavelet correlation matrix can cause a significant bias for the approximation of some stochastic processes. Alternatively, wavelet packages (Daubechies 1992, Gopinath and Burrus 1992) can be used to provide an arbitrary partitioning of the time-frequency plane.

4. Wavelets provide strong cross-scale decorrelation of stochastic processes; see Chapter 3. However, the weakening of the in-scale correlation is shown only for several classes of stochastic processes: fractal processes, the Markov-Gauss processes, the Meyer-Belkin processes. Also, the proposed scale-dependent parametric models for random field synthesis provide a close approximation of the cross-scale correlation. The in-scale correlation is captured only for fine scales; see the discussion in Section 4.2.2. Then, the standard in-scale AR model can improve the approximation properties of the proposed multi-scale simulation method. That is, the model defined by Eq. 4.18 can be replaced by the equation
\[ d^l_k = \sum_{l=-n}^{n} \alpha^l_{k,l} \epsilon^l_{k+1} - \sum_{s=1}^{a} \gamma^l_{k,s} d^l_{k-s} + \beta^l_k u_k. \]  \hspace{1cm} (4.41)

5. The proposed method for synthesizing stationary stochastic processes can be readily extended to the non-stationary case by utilizing the time-frequency localization properties of wavelets. A numerical algorithm can be developed which performs simulation of a non-stationary stochastic processes in \( O(N) \) operations and provides an approximation of the time-frequency distributions of non-stationary processes compatible with the uncertainty principle (Chui 1992a, Janssen 1979).

6. Chapter 5 considers a generalization of the proposed simulation method for multidimensional random fields.

4.5 Concluding Remarks

Scale independent parametric models associated with the wavelet expansion of stochastic processes have been studied and their approximation properties have been investigated in this chapter. It has been shown that these models induce some non-stationary features and a bias in the spectral approximation. However, the corresponding error is found to be relatively small for a class of stochastic processes. The introduced models naturally yield an efficient scheme for synthesizing stochastic processes in \( O(N) \) numerical operations.

The wavelet based spectral approximation can be further improved by introducing alternative scale-dependent models which approximate, quite accurately, the auto-correlation functions of stochastic processes. A class of efficient algorithms for synthesizing stochastic processes in \( O(N) \) operations has been developed by using the local and multi-scale structure of wavelets.
The error introduced by the proposed scale-dependent models has been closely investigated. The developed theoretical and numerical estimates have shown that this error is not significant for the considered stochastic processes commonly used in engineering applications.

Several stochastic processes have been synthesized by using the developed simulation method. This numerical study shows that the mean value, the variance, and the auto-correlation function of the simulated processes approximate quite accurately the corresponding target quantities.

Finally, an interpretation of the proposed simulation method has been discussed by using the theory of stochastic processes on homogeneous trees. It has been shown that the developed models of stochastic processes are realizations of scale dynamic systems.
Figure 4.1. The Wigner distribution of the Haar wavelet.
Figure 4.2. The power spectrum and corresponding wavelet approximations.

1) target spectrum
2) approximation using uncorrelated wavelet coefficients
3) approximation using cross-scale uncorrelated wavelet coefficients.
Figure 4.3. The power spectrum and corresponding wavelet approximations.
1) target spectrum
2) optimal approximation.
Figure 4.4. In-scale correlation of the wavelet coefficients; the Haar basis, $M=1$ vanishing moment, and $n=1$ + exact values; o estimated values.
Figure 4.5. In-scale correlation of the wavelet coefficients; the Daubechies wavelet with $M=3$ vanishing moments and $n=1$ + exact values; o estimated values.
Figure 4.6. In-scale correlation of the wavelet coefficients; the Haar basis, $M=1$ vanishing moment, and $n=3$ + exact values; o estimated values.
Figure 4.7. In-scale correlation of the wavelet coefficients;
the Daubechies wavelet with $M=3$ vanishing moments and $n=3$
+ exact values; o estimated values.
Figure 4.8. In-scale correlation of the wavelet coefficients;
the Daubechies wavelet with $M=3$ vanishing moments and $n=5$
+ exact values; o estimated values.
Figure 4.9. In-scale correlation of the wavelet coefficients;
the Daubechies wavelet with $M=2$ vanishing moments and $n=1$
+ exact values; o estimated values.
Figure 4.10. In-scale correlation of the wavelet coefficients; the Daubechies wavelet with $M=7$ vanishing moments and $n=1$ + exact values; o estimated values.
Figure 4.11. The maximum error of estimating the scale correlation normalized by $\sigma_{k,k}^{j-1,j-1}$.
Figure 4.12. Samples of stochastic processes.
Figure 4.13. Estimates based on the ensemble average of size $N=200$
(a) the mean value, (b) the variance, (c) the auto-correlation function.
Figure 4.14. Estimates based on the ensemble average of size $N=200$

(a) the mean value, (b) the variance, (c) the auto-correlation function.
Figure 4.15. Estimates based on the ensemble average of size $N=200$

(a) the mean value, (b) the variance, (c) the auto-correlation function.
Figure 4.16. Homogeneous tree of order 2 and corresponding dynamic moves.
Chapter 5. Multi-Dimensional Random Field Simulation Using Wavelets

A class of algorithms for synthesizing stochastic processes was developed in the previous chapter. These algorithms were based on the representation of stochastic processes by using a wavelet basis and on a subsequent iterative simulation of the wavelet coefficients on the increasingly finer scales. The proposed synthesis method can be generalized to the multi-dimensional case by utilizing relevant multi-dimensional wavelet bases. In this regard, multi-dimensional wavelets are discussed and the associated multi-dimensional wavelet reconstruction and decomposition algorithms are formulated. The wavelet representation of multi-dimensional random fields is developed using the framework of Chapter 3. Similarly to the one-dimensional case, the wavelet expansion of the multi-dimensional random fields can be viewed as a semi-whitening procedure. A method for synthesizing multi-dimensional random fields is developed by using the local and multi-scale structure of wavelets. Since wavelets simplify the correlation of a class of random fields, the corresponding error is expected to be small. The developed algorithms can be executed in $O(N)$ operations. Finally, the usefulness of the proposed simulation method is examined by synthesizing several random fields.

This chapter focuses on the two-dimensional case; an analogous generalization for higher-dimensional fields is outlined.

5.1 Multi-dimensional Wavelets

Several methods to construct multi-dimensional wavelets are known in the literature (Daubechies 1992, Jaffard and Laurencot 1992). In this regard, only the two-dimensional
case is considered in this chapter for simplicity. A two dimensional orthogonal wavelet basis for $L^2(R^2)$ can be built from a one-dimensional orthonormal wavelet basis for $L^2(R)$ by taking the tensor product of functions $\psi_{j,k}(x) = 2^{-j/2}\phi(2^{-j}x-k+1)$ generated by two one-dimensional bases

$$\psi_{j_1,k_1;j_2,k_2}(x_1,x_2) = \psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2).$$

(5.1)

Indeed, the resulting set of functions $\{\psi_{j_1,k_1;j_2,k_2}(x_1,x_2);j_1,j_2,k_1,k_2 \in Z\}$ is an orthonormal basis for $L^2(R^2)$. In this case, the variables $x_1$ and $x_2$ are dilated separately. Though this construction is quite simple and is frequently used in some applications, it has some drawbacks critical to random field analysis and synthesis. These wavelet bases contain functions representing different scales in different directions. Thus, the corresponding expansion of random fields may have a complicated structure not well suited for random field synthesis applications. Moreover, the unrelated localization in $x_1$ and $x_2$ directions makes them play a privileged role, which can be less natural for isotropic random fields. Also, the multi-scale relationship for the scale and wavelet coefficients is not available in this case.

Another construction which fits better for random field simulation uses the tensor product of two one-dimensional multiresolution analyses rather than of the corresponding wavelet bases. To be more precise, let $V_j$, $j \in Z$ be a subspace of $L^2(R^2)$ defined as

$$V_0 = V_0 \otimes V_0 = \text{Span}\{F(x,y)=f(x)g(y);f,g \in V_0\},$$

(5.2)

$$F \in V_j \iff (2^j \cdot , 2^j \cdot ) \in V_0.$$  

(5.3)

Clearly, the multiresolution analysis in two dimensions is defined as

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots,$$

(5.4)
\[ \text{clos}_{L^2} \bigcup_{j \in \mathbb{Z}} V_j = L^2, \]  \hfill (5.5) \\
\[ \bigcap_{j \in \mathbb{Z}} V_j = \{0\}. \]  \hfill (5.6)

Since the set \( \{\phi(\bullet - n), n \in \mathbb{Z}\} \) constitutes an orthonormal basis for \( V_0 \), the product functions

\[ \Phi_{0;n_1,n_2}(x,y) = \phi(x - n_1)\phi(x - n_2), n_1, n_2 \in \mathbb{Z}, \]  \hfill (5.7)

comprise an orthonormal basis for \( V_0 \). As in the one-dimensional case, \( W_j \) is defined as the orthogonal compliment of \( V_j \) to \( V_{j-1} \)

\[ V_{j-1} = V_{j-1} \otimes V_{j-1} = (V_j \oplus W_j) \otimes (V_j \oplus W_j) = \]  \hfill (5.8) \\
\[ = (V_j \otimes V_j) \oplus [(V_j \otimes W_j) \oplus (W_j \otimes V_j) \oplus (W_j \otimes W_j)] = V_j \oplus W_j. \]

It follows that \( W_j \) has three pieces. Therefore, three wavelet functions are

\[
\begin{align*}
\Psi^{(1)}(x,y) &= \phi(x)\psi(y) \\
\Psi^{(2)}(x,y) &= \psi(x)\phi(y) \\
\Psi^{(3)}(x,y) &= \psi(x)\psi(y)
\end{align*}
\]  \hfill (5.9)

Note that the set of functions

\[ \left\{ \Psi_{j_0; n_1, n_2}^\lambda : j_0, n_1, n_2 \in \mathbb{Z}, \lambda = 1, 2, 3 \right\} \]  \hfill (5.10)

is an orthonormal basis for \( L^2(R^2) \). If the original one-dimensional scale and wavelet functions have compact support, so have \( \Phi \) and \( \Psi^{(\lambda)} \).

Moreover, the interpretation of the wavelet decomposition algorithm in terms of the subband filtering carries over to two dimensions. This filtering can be done on "rows" and
"columns" of the corresponding two-dimensional arrays. In this manner, the two-dimensional counterpart of Eqs. 1.125 and 1.126 is written as

\[
c^j_{(k_1, k_2)} = \sum_{l_2=0}^{2M-1} g_{l_2} + 1 \left( \sum_{l_1=0}^{2M-1} h_{l_1} + 1 \cdot c^j_{(2k_1 + 1, l_1 + 1, 2k_2 + l_2 - 1)} \right),
\]

(5.11)

\[
d^{1;j}_{(k_1, k_2)} = \sum_{l_2=0}^{2M-1} g_{l_2} + 1 \left( \sum_{l_1=0}^{2M-1} h_{l_1} + 1 \cdot c^j_{(2k_1 + 1, l_1 + 1, 2k_2 + l_2 - 1)} \right),
\]

(5.12)

\[
d^{2;j}_{(k_1, k_2)} = \sum_{l_2=0}^{2M-1} g_{l_2} + 1 \left( \sum_{l_1=0}^{2M-1} g_{l_1} + 1 \cdot c^j_{(2k_1 + 1, l_1 + 1, 2k_2 + l_2 - 1)} \right),
\]

(5.13)

and

\[
d^{3;j}_{(k_1, k_2)} = \sum_{l_2=0}^{2M-1} g_{l_2} + 1 \left( \sum_{l_1=0}^{2M-1} g_{l_1} + 1 \cdot c^j_{(2k_1 + 1, l_1 + 1, 2k_2 + l_2 - 1)} \right).
\]

(5.14)

Figure 1.7 is, in two dimensions, a schematic representation of Figure 5.1. Note that if the original vector \( c^j \) consists of an \( K \times K \) array, then (apart from the border effect) every array \( d^{3;j+1} \) consists of \( K/2 \times K/2 \) elements. The initial two-dimensional signal represented by the scale coefficients \( c^j \) is decomposed into four signals of one quarter the size. Once the projection of the signal in the next coarser scale is available \( c^{j+1} \), the entire procedure can be repeated.

Similarly, the two-dimensional wavelet reconstruction algorithm is given by the equation

\[
c^j_{(k_1, k_2)} = \sum_{l_1} \left( \sum_{l_2} \left( c^j_{(l_1, l_2)} h_{k_2 - 2l_2 + 2} + d^{1;j}_{(l_1, l_2)} g_{k_2 - 2l_2 + 2} \right) h_{k_1 - 2l_1 + 2} + \sum_{l_2} \left( d^{2;j}_{(l_1, l_2)} h_{k_2 - 2l_2 + 2} + d^{3;j}_{(l_1, l_2)} g_{k_2 - 2l_2 + 2} \right) g_{k_1 - 2l_1 + 2} \right).
\]

(5.15)
The computational process of this equation is schematically shown in Figure 5.2. At every stage of this algorithm four fields are recombined into one field of the size four times larger and the process repeats itself again.

Note that the wavelet decomposition and reconstruction algorithms require only $O(N)$ numerical operations, where $N$ is the size of the discrete two-dimensional signal. It provides a unique and efficient tool for multi-dimensional signal analysis and synthesis.

The above methods of generating multi-dimensional wavelets are based on a tensor product structure. Alternatively, multi-dimensional wavelets can be generated by the multiresolution analysis where $V_0$ is not a tensor product of two one-dimensional spaces (Meyer 1990). Again, this construction yields three wavelets function $\Psi^{(1)}$, $\Psi^{(2)}$, and $\Psi^{(3)}$. This method provides, for example, wavelets adapted for certain transformation of functional spaces. However, this approach has more theoretical subtleties and does not yield wavelets with a finite support.

The above construction of two-dimensional wavelet bases can be readily generalized to the n-dimensional case yielding $2^n - 1$ wavelets.

### 5.2 Representation of Multi-Dimensional Random Fields Using Wavelet Bases

Upon selecting a wavelet basis for $L^2(R^2)$, any separable two-dimensional random field $f(x_1, x_2)$ can be represented as a linear combination of the corresponding basis functions. Similarly to the one-dimensional case one can write

$$f(x_1, x_2) = \sum_{j = -\infty}^{\infty} \sum_{k_1, k_2 = -\infty}^{\infty} \sum_{\lambda = 1}^{3} d_{k_1, k_2}^{\lambda;j} \Psi_{j;\lambda i(k_1, k_2)}^\lambda (x_1, x_2), \quad (5.16)$$
where for the Daubechies orthonormal wavelets

\[ d_{(k_1, k_2)}^{x,j} = \int \int f(x_1, x_2) \psi_{j_1(k_1, k_2)}(x_1, x_2) dx_1 dx_2. \]  

(5.17)

Further, the projection of a field into the j-th scale of the multiresolution analysis is determined by the scale coefficients that can be found as

\[ c_{(k_1, k_2)}^j = \int \int f(x_1, x_2) \phi_{j_1(k_1, k_2)}(x_1, x_2) dx_1 dx_2. \]  

(5.18)

The correlation of the wavelet and scale coefficients can be found in the manner similar to the one-dimensional case, see Chapter 3. In this context, the following correlation parameters are defined

\[ a_{k_1, l}^{i,j} = E[c_{k_1}^i c_{l}^j] = \int \int R_f(x, y) \phi_{j_1(k_1, k_2)}(x) \phi_{l}(y) dx dy, \]  

(5.19)

\[ b_{k_1, l}^{i,j} = E[c_{k_1}^i d_{l}^j] = \int \int R_f(x, y) \phi_{j_1(k_1, k_2)}(x) \psi_{l}^j(y) dx dy, \]  

(5.20)

\[ r_{k_1, l}^{i,j} = E[d_{k_1}^i d_{l}^j] = \int \int R_f(x, y) \psi_{j_1(k_1, k_2)}^i(x) \psi_{l}^j(y) dx dy, \]  

(5.21)

where \( k = (k_1, k_2), x = (x_1, x_2) \). The integrals in Eqs. 5.19-5.21 can be evaluated without numerical integration. Indeed, the wavelet decomposition algorithm, Eqs. 5.11-5.14, yields the two-dimensional generalization of Eqs. 3.14-3.16

\[ a_{(k_1, k_2);(l_1, l_2)}^{j,j} = \]  

(5.22)

\[ \sum_{n_1, n_2, m_1, m_2 = 0}^{2M-1} h_{n_1} h_{n_2} h_{m_1} h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)}, \]

\[ b_{(k_1, k_2);(l_1, l_2)}^{1;i,j} = \]  

(5.23)
\[\sum_{n_1, n_2, m_1, m_2 = 0}^{2M - 1} h_{n_1} + h_{n_2} + h_{m_1} + h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)} \]

\[b_{(k_1, k_2);(l_1, l_2)}^{2; j} = \] (5.24)

\[\sum_{n_1, n_2, m_1, m_2 = 0}^{2M - 1} h_{n_1} + h_{n_2} + h_{m_1} + h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)} \]

\[b_{(k_1, k_2);(l_1, l_2)}^{3; j} = \] (5.25)

\[\sum_{n_1, n_2, m_1, m_2 = 0}^{2M - 1} h_{n_1} + h_{n_2} + h_{m_1} + h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)} \]

\[r_{(k_1, k_2);(l_1, l_2)}^{1; 1; j} = \] (5.26)

\[\sum_{n_1, n_2, m_1, m_2 = 0}^{2M - 1} h_{n_1} + h_{n_2} + h_{m_1} + h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)} \]

\[r_{(k_1, k_2);(l_1, l_2)}^{2; 2; j} = \] (5.27)

\[\sum_{n_1, n_2, m_1, m_2 = 0}^{2M - 1} h_{n_1} + h_{n_2} + h_{m_1} + h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)} \]

\[r_{(k_1, k_2);(l_1, l_2)}^{3; 3; j} = \] (5.28)

\[\sum_{n_1, n_2, m_1, m_2 = 0}^{2M - 1} h_{n_1} + h_{n_2} + h_{m_1} + h_{m_2} + a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)} \]

\[r_{(k_1, k_2);(l_1, l_2)}^{1; 2; j} = \] (5.29)
\[ \sum_{n_1, n_2, m_1, m_2 = 0}^{2M-1} h_{n_1 + 1} g_{n_2 + 1} g_{m_1 + 1} g_{m_2 + 1} a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)}, \]

\[ r_{(k_1, k_2);(l_1, l_2)} = \]

\[ \sum_{n_1, n_2, m_1, m_2 = 0}^{2M-1} h_{n_1 + 1} g_{n_2 + 1} g_{m_1 + 1} g_{m_2 + 1} a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)}, \]

\[ r_{(k_1, k_2);(l_1, l_2)} = \]

\[ \sum_{n_1, n_2, m_1, m_2 = 0}^{2M-1} g_{n_1 + 1} h_{n_2 + 1} g_{m_1 + 1} g_{m_2 + 1} a_{(2k_1 + n_1 - 1, 2k_2 + n_2 - 1);(2l_1 + m_1 - 1, 2l_2 + m_2 - 1)}, \]

Eqs. 5.22-5.31 can be further simplified for homogeneous random fields consistently with the developments of Eqs. 3.3.17-3.3.24.

The procedure described by Eqs. 5.22-5.31 requires \( O(N^2) \) operations to evaluate the correlation parameters \( a_{k_1,l_1}^j, b_{k_1,l_1}^{ij}, \) and \( \lambda_{1,1}^{ij}; \lambda_{2,2}^{ij} \). Alternatively, numerically efficient algorithms are developed either by using special wavelets having some vanishing moments for the scale function or by utilizing special quadrature rules. Since multi-dimensional wavelets can be generated by a tensor product of corresponding multiresolution analyses the formulation of these algorithms is a straightforward generalization of Eqs. 3.29 and 3.30. First, \( a_{k_1,l_1}^j \) are found based on the selected quadrature rule. Then, Eqs. 5.22-5.31 are evaluated in the neighborhood of the diagonal \( x = (x_1, x_2) = (y_1, y_2) = y \).

Finally, the local and quasi-differential properties of wavelets can be used to justify the wavelet ability to represent compactly the covariance matrices of multi-dimensional random fields. Similarly to the one-dimensional case, the wavelet representation of random fields is confined to a few scales. Further, the cross-scale correlation of the wavelet
coefficients is weak due to the frequency localization of the wavelet function. Finally, the wavelet transformation reduces the length of correlation for a large class of random fields. The analysis of Section 3.3 can be directly adopted for the case of multi-dimensional random fields.

5.3 Wavelet Based Synthesis of Multi-Dimensional Random Fields

Wavelet based synthesis of multi-dimensional random fields can be developed by utilizing the principles of the one-dimensional case. For every scale, the wavelet coefficients are modeled as the best linear estimate based on the scale coefficients. That is, the wavelet coefficients are assumed statistically independent for the generated realization of the scale coefficients. At the same time, the exact cross-correlation of the wavelet and scale coefficients is retained. From another perspective, the proposed method can be viewed as a scale dynamic process defined on the homogeneous tree of order 4. For a given scale, the scale coefficients are used as the initial conditions while the wavelet coefficients can be considered as the "noise input" of this dynamic process. This dynamic process is governed by Eq. 5.15 and is schematically shown in Figure 5.2. It determines the realization of the random field for the next finer scale. A method schematically outlined above is formulated in detail in the following.

The linear estimate of the random vector $d_k^j = (d_{1k}^j, d_{2k}^j, d_{3k}^j)$ can be expressed as

$$d_k^j = A_k \varepsilon_k^j + B_k u_k,$$  \hspace{1cm} (5.32)

where $A_k$ is a $3 \times p$ matrix, $B_k$ is a $3 \times 3$ matrix and $u_k$ is a Gaussian, zero mean random vector having 3 components uncorrelated with $\varepsilon_k^j = (c_1^j, c_2^j, \ldots c_p^j)$, such that
\[ E[u_k u_l] = \delta_{k,l} I_{3 \times 3} \]  
\[ E[u_k^t u_l^t] = O_{p \times 3}, \]  
where \( I_{3 \times 3} \) is the unit \( 3 \times 3 \) matrix, and \( O_{p \times m} \) denotes the zero matrix of size \( p \times m \).

Again, similarly to the one-dimensional case, the cross-correlation of the wavelet and scale coefficients is assumed to decrease rapidly. That is, \( d_k^j \) is determined by a few elements of \( \epsilon^j \) with indices from a small neighborhood of the index \( k = (k_1, k_2) \). The number \( n \) denotes the size of this neighborhood. Thus, apart from the boundary effect, \( A_k \) reduces to a \( 3 \times (2n-1)^2 \) matrix.

Taking the mathematical expectation in Eq. 5.32 shows that the wavelet coefficients have zero mean, as it is expected. The matrix \( A_k \) is determined so that the cross-correlation of the wavelet coefficients and the scale coefficients is exact. That is, multiplying Eq. 5.31 by \( (\epsilon^j)^t \) and taking mathematical expectation one finds

\[ A_k = (E[\epsilon^j(\epsilon^j)^t])^{-1} E[d_k^j(\epsilon^j)^t] \]  

Further, \( B_k \) is determined so that the covariance matrix of the vector \( d_k^j \) is equal to the target value. Multiplying Eq. 5.32 by itself and taking mathematical expectation yields

\[ B_k B_k^t = E[d_k^j(d_k^j)^t] - A_k E[\epsilon^j(\epsilon^j)^t] A_k^t \]  

The matrix \( B_k \) can be arbitrary as long as it satisfies Eq. 5.36. In particular, it can be found by using the Cholevsky factorization of the left-hand side of Eq. 5.36. The index \( n \) is selected to be small and, therefore, Eqs. 5.35 and 5.36 require just a few operations to perform. Note that for homogeneous random fields, these equations must be solved only once for every scale.
The error of the developed algorithm can be analyzed in a manner similar to the one-dimensional case. Thus, Eqs. 5.35 and 5.36 ensure that the cross-correlation between the scale coefficients and the wavelet coefficients, and the variance of the wavelet coefficients are exact, while the small correlation of the wavelet coefficients is approximated to simplify the simulation of the random fields. Note that the wavelet transform behaves locally as the numerical differentiation introducing the "finite difference" relation between the scale and the wavelet coefficients for fine scales. On the other hand, the wavelet coefficients for coarse scales are almost uncorrelated for a large class of random fields.

In the next section several examples of synthesizing two-dimensional random fields address the usefulness of the proposed algorithms. Note that simulation of three- and higher-dimensional fields using the proposed framework is quite straightforward. In this case, one should simultaneously manipulate $2^n - 1$ wavelet coefficients associated with the multi-dimensional index $k = (k_1, k_2, \ldots, k_n)$. Accordingly, all equations of this section remain valid, but $d_k^I$ and $u_k$ become $(2^n - 1)$-dimensional vectors.

### 5.4 Numerical Examples

Several two-dimensional random fields are synthesized to elucidate the usefulness of the developed method. Errors are evaluated by using ensemble averaging. The two-dimensional generalization of the Daubechies wavelet function with $M=3$ vanishing moments are used in this regard.

First, samples of a Gaussian homogeneous random field with the auto-correlation function

$$ R_f(d) = \exp(-d/\alpha) , $$

(5.37)
where

\[ d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \tag{5.38} \]

are synthesized for \( \alpha = 0.1 \), see Figure 5.3(a). Large high frequency components which are attributed to the small regularity of this auto-correlation function can be clearly observed. Note that this auto-correlation function describes a non-differentiable in the mean square sense random field. The accuracy of the proposed synthesis method is assessed by averaging two hundred field realizations. The error of estimating the mean and the variance is compared in Table 5.1 with the theoretical bounds. For this purpose, the upper and low bounds of the estimates of the mean and variance, such that only 5% of the corresponding quantities exceed these bounds, are found. Table 5.1 shows that the proposed synthesis method induces a small error in estimating the variance. Nevertheless, this error is satisfactory for engineering applications. Moreover, the shape of the auto-correlation function is approximated quite accurately as shown in Figure 5.4(a). Several estimates of the auto-correlation function are available by taking different points and directions for calculations.

Next, samples of a Gaussian homogeneous random field with the auto-correlation function

\[ R_f(d) = \exp\left(-\left(d/\alpha\right)^2\right), \tag{5.39} \]

are synthesized for \( \alpha = 0.1 \), see Figure 5.3(b). The realization of this field is quite regular since the high frequency components of this field are not substantial. Two hundred field realizations are assembled to estimate the first and second order characteristics of the simulated random field. The error of estimating the mean and the variance is compared in Table 5.1 with the theoretical bounds. The proposed synthesis method induces a small error
in estimating the variance. This error is satisfactory for engineering applications. The shape of the auto-correlation function is approximated quite accurately as shown in Figure 5.4(b). Several estimates of the auto-correlation function are available by taking different points and directions for calculations.

Table 5.1: The error of estimating the mean value and the variance based on the 5% exceeding criteria

<table>
<thead>
<tr>
<th></th>
<th>Mean Value</th>
<th></th>
<th></th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lower bound</td>
<td>upper bound</td>
<td>lower bound</td>
<td>upper bound</td>
</tr>
<tr>
<td>Example 1</td>
<td>-0.1354</td>
<td>0.1165</td>
<td>0.8724</td>
<td>1.2443</td>
</tr>
<tr>
<td>Example 2</td>
<td>-0.0888</td>
<td>0.1046</td>
<td>0.8126</td>
<td>1.2126</td>
</tr>
<tr>
<td>Theory</td>
<td>-0.1163</td>
<td>0.1163</td>
<td>0.8355</td>
<td>1.1645</td>
</tr>
</tbody>
</table>

5.5 Efficiency of the Proposed Algorithms.

The efficiency of the proposed method is satisfactory due to the reduction of the requisite data storage capacity and to the small number of numerical calculations which are necessary for synthesizing a single random field sample. To perform the synthesis for a given scale, one has to store the scale and wavelet coefficients from the coarser scale. Thus, it is required to store 2N elements. Also it is required to store the coefficients of equations which approximate the wavelet coefficients using the scale coefficients. For the homogeneous case it is further required to store $O(logN)$ numbers, while for the general random fields $O(N)$ additional storage locations are needed.

The number of calculations for a single simulation is determined by the selected implementation scheme of the wavelet reconstruction algorithm and by the transformation of the scale coefficients into the wavelet coefficients. Some efficient procedures for the
wavelet reconstruction algorithm are discussed in Herley and Vetterli (1993) which can prove advantageous especially for large values of $M$. Also, the developed scale type dynamic algorithm is a realization of an ARMA scale dynamic system which requires only a few parameters for approximating the correlation properties of random fields due to the decorrelation properties of wavelets. Thus, the developed method can be implemented much faster than the time domain ARMA system. In any case, the proposed simulation method requires at most $O(M)$ operations for synthesizing a random field sample. Table 1 compares the number of numerical operations needed for the generation of a single sample versus the required resolution level, that is, the field size, and the number $M$. This method compares quite favorably with alternative methods of random field simulation and merits further investigation.

Table 5.2: The number of floating operations to implement a single random field simulation using the proposed algorithms.

<table>
<thead>
<tr>
<th>Type of random fields</th>
<th>Size of random fields</th>
<th>Number of flops ($\times 10^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$M=1$</td>
</tr>
<tr>
<td>1-dimensional</td>
<td>64</td>
<td>3.26</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>5.58</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>10.08</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>18.93</td>
</tr>
<tr>
<td></td>
<td>1024</td>
<td>36.48</td>
</tr>
<tr>
<td></td>
<td>2048</td>
<td>71.45</td>
</tr>
<tr>
<td>2-dimensional</td>
<td>32 × 32</td>
<td>75.9</td>
</tr>
<tr>
<td></td>
<td>64 × 64</td>
<td>221.9</td>
</tr>
<tr>
<td></td>
<td>128 × 128</td>
<td>741.4</td>
</tr>
<tr>
<td></td>
<td>256 × 256</td>
<td>2,695.6</td>
</tr>
<tr>
<td></td>
<td>512 × 512</td>
<td>10,272.3</td>
</tr>
</tbody>
</table>
5.6 Concluding Remarks

This chapter has discussed the application of wavelets for analysis and synthesis of multi-dimensional random fields. It has been shown that multi-dimensional wavelet bases which are built by using the tensor product of multiresolution analyses are adequate for these applications; they have compatible multi-scale structure with respect to the axis of the coordinate system and some vanishing moments in, at least, one direction.

A multi-dimensional generalization of the wavelet decomposition and reconstruction algorithms associated with the introduced multi-dimensional wavelet bases has been used for analysis and representation of multi-dimensional random fields. An efficient approximation procedure for evaluating the covariance matrix of the wavelet coefficients in $O(N)$ numerical operations has been discussed.

The wavelet representation of multi-dimensional random fields has been efficiently used to formulate a new class of algorithms for synthesizing random fields by generalizing the corresponding one-dimensional concepts discussed in Chapter 4. The developed algorithms utilize efficiently the local and multi-scale structure of wavelet basis functions and use the capacity of wavelets to simplify correlation properties of random fields.

To examine the usefulness of the proposed simulation method several two-dimensional random fields have been synthesized. This numerical study has elucidated the ability of the wavelet based algorithms to approximate the target second order moments quite accurately. The theoretical study and the numerical simulations have shown that the proposed synthesis algorithms can be executed in $O(N)$ numerical operation; this feature demonstrates the efficiency and usefulness of this new simulation method.
Figure 5.1. Schematic representation of the two-dimensional wavelet decomposition algorithm; ↓ stands for downsampling by a factor of 2.

Figure 5.2. Schematic representation of the two-dimensional wavelet reconstruction algorithm; ↑ stands for upsampling by a factor of 2.
Figure 5.3. Realizations of two-dimensional random fields.
Figure 5.4. The estimates of the auto-correlation functions of two-dimensional random fields based on the ensemble average of size $N=200$. 
Chapter 6. An Optimal ARMA Approximation for Random Field Simulation

The auto-regressive moving-average (ARMA) method has been established as an efficient technique for synthesizing stationary stochastic processes with specified spectral characteristics. It requires constructing an ARMA filter which adequately approximates the target spectrum. Several methods for ARMA modeling were discussed in Chapter 2. A two-stage ARMA approximation was shown to be one of the most versatile techniques in this regard (Gersch and Yonemoto 1977, Samaras et al. 1983, Spanos and Minolet 1986); a "long" auto-regressive (AR) filter was constructed first and, then, the AR filter was approximated by a "short" ARMA filter.

This chapter presents an efficient method for the two-stage ARMA approximation. An error criterion which quantifies the "energy" of the discrepancy between the AR and ARMA approximations is utilized. An iterative algorithm is developed to solve the associated error minimization problem. To elucidate the robustness of the proposed approximation method, spectra that are commonly used in offshore and earthquake engineering are considered.

6.1 AR Approximation

In this section the properties of AR processes are reviewed first, since the proposed ARMA simulation method requires that the target process is initially approximated by an AR model. Certain concepts of the AR modeling procedure were discussed in Chapter 2; in this chapter they are formulated for the more general multi-variate case.

The value at time $k$ of an $n$-variate AR process $y$ of order $m$ can be computed as
\[
y(k) = - \sum_{i=1}^{m} a_i^{AR} y(k-i) + b_0^{AR} w(k),
\]
where \( w \) is a bandlimited \([-\omega_b, \omega_b]\) white noise process
\[
E[w(k)(w(l))^T] = 2\omega_n I_n \delta_{kl},
\]
where \((\ )^T\) denotes the operator of matrix transposition, \( I_n \) is the unity matrix and \( \delta_{kl} \) is the Kronecker delta.

Alternatively, the process defined by Eq. 6.1 can be deemed as the white noise response of a discrete linear system with the following transfer function
\[
H_{AR}(z) = A_{AR}^{-1}(z)b_0^{AR},
\]
where
\[
A_{AR}(z) = \sum_{k=0}^{m} a_k^{AR} z^{-k},
\]
\( z = e^{j\omega T} \) and \( a_0 = I_n \). The spectral matrix of the AR process can be found from the equation
\[
S_{AR}(\omega) = H_{AR}^\ast(z)H_{AR}(z),
\]
where \((\ )^\ast\) denotes complex conjugation.

The AR parameters can be evaluated by multiplying Eq. 6.1 by \( y(k-l) \) and taking mathematical expectation. As the future input excitation is statistically independent from the present system response, the following set of Yule-Walker equations can be derived.
\[ R_\xi(k) + \sum_{i=1}^{m} a_i^{AR} R_\xi(i-k) = 0, \quad k = 1, \ldots, m, \quad (6.6) \]

where the auto-correlation function \( R_\xi(k) \) is equated to the auto-correlation function of the target process at the first \( m + 1 \) time lags. If the target spectrum is specified, the target auto-correlation function can be determined by relying on the following equation

\[ R_\xi(k) = \int_{-\omega_n}^{\omega_n} S_\xi(\omega) \cos(\omega T) d\omega. \quad (6.7) \]

The gain factor \( b_0^{AR} \) of the AR model can be estimated by using the following equation

\[ b_0^{AR} = \frac{1}{2\omega_n} \left[ R_\xi(0) + \sum_{i=1}^{m} a_i^{AR} R_\xi(i) \right]. \quad (6.8) \]

Note that the solution of Eq. 6.8 is not unique. In this study the matrix \( b_0^{AR} \) is determined by performing the Cholesky factorization of the expression in the right hand side of Eq. 6.8. Upon evaluating the AR parameters, some important characteristics of the AR process can be determined. Thus, the input-output cross-correlation given by \( R_{\xi w}(l) = E[\xi(k)w(k+l)] \) can be evaluated recursively by using the equations

\[ R_{\xi w}(l) = 0, \quad l > 0 \quad (6.9) \]

\[ R_{\xi w}(l) = 2 \omega_n b_0^{AR}, \quad l = 0 \quad (6.10) \]

\[ R_{\xi w}(l) = -\sum_{i=1}^{\min(m,-l)} a_i^{AR} R_{\xi w}(l+i), \quad l < 0 \quad (6.11) \]

Also, the auto-correlation function of the AR process satisfies the equation...
\[ R_x(-k) + \sum_{i=1}^{m} a_i^{AR} R_x(i-k) = b_0^{AR} R_{\epsilon\epsilon}(-k) , \quad k = 0, \pm 1, \ldots \quad (6.12) \]

Note that Yule-Walker equations can be derived from Eq. 6.12 by restricting \( k \) to the values \( 1, 2, \ldots m \).

### 6.2. New ARMA Approximation

#### 6.2.1. Definition of the ARMA system

The \( k \)-th sample of an \( n \)-variate ARMA\((p,q)\) process \( \chi \) can be generated using the equation

\[ \chi(k) = - \sum_{i=1}^{p} a_i \chi(k-i) + \sum_{l=0}^{q} b_l \epsilon(l) . \quad (6.13) \]

The ARMA process can also be interpreted as the white noise response of a discrete linear system with the following transfer function

\[ H_{ARMA}(z) = A^{-1}(z)B(z) , \quad (6.14) \]

where

\[ A(z) = \sum_{i=0}^{p} a_i z^{-i} , \quad a_0 = I_n \quad (6.15) \]

and

\[ B(z) = \sum_{l=0}^{q} b_l z^{-l} . \quad (6.16) \]
6.2.2 Review of methods for evaluation of ARMA parameters

Several alternative procedures to obtain the unknown ARMA coefficients are utilized in practice. The commonly used auto-cross-correlation matching (ACM) method has been developed by Gersch and Yonemoto (1977) and consequently has been applied to engineering mechanics problems by Samaras et al. (1983) and Spanos and Minolet (1986). Multiplying Eq. 6.13 by $y^f(k - j)$ and $w^f(k - j)$ and taking mathematical expectation yield the system of linear equations with unknown ARMA parameters

$$
\sum_{i=0}^{p} a_i R_{y^2}(i - k) - \sum_{l=0}^{q} b_i R_{w^2}(l - k) = 0, \quad k = 1, \ldots, p, \quad (6.17)
$$

$$
- \sum_{i=0}^{p} a_i R_{yw}(i - k) + 2\omega_n b_k = 0, \quad k = 0, \ldots, q. \quad (6.18)
$$

However, these equations contain additional unknown input-output cross-correlation functions. The ACM method assumes that the auto-correlation function of the ARMA process is equal to the target auto-correlation function at the first $p + 1$ time lags, and the input-output cross-correlation of the ARMA process is found by using the predetermined AR model based on Eqs. 6.9-6.11. Minolet and Spanos (1987) and Spanos and Minolet (1987) have shown that this method is equivalent to the minimization of the error criterion

$$
\varepsilon_1 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} |A(z)H_{AR}(z) - B(z)|^2 d\omega \quad (6.19)
$$

with the constrain $a_0 = I_n$. The symbol $|U|$ denotes the Euclidean norm of an arbitrary matrix $U$. That is,
\[ |U|^2 = \text{tr}(U^* U^t) = \text{tr}((U^t)^* U) \]  

(6.20)

where \( \text{tr}(\ ) \) stands for the trace operator.

Alternatively, the variance of the difference between the responses to the white noise of the ARMA and AR systems can be minimized

\[
\varepsilon_2 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| H_{AR}(z) - A^{-1}(z)B(z) \right|^2 d\omega = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| H_{AR}(z) - H_{ARMA}(z) \right|^2 d\omega .
\]

(6.21)

This error lends itself to immediate physical interpretation. From a deterministic perspective, the error criterion 6.21 represents the "energy" of the difference between the impulse responses of the AR and ARMA systems. That is, this error criterion can be expressed in the time domain as

\[
\varepsilon_2 = \sum_{k=0}^{\infty} \left| h_{AR}(k) - h_{ARMA}(k) \right|^2 ,
\]

(6.22)

where \( h_{AR} \) and \( h_{ARMA} \) are the impulse responses of the AR and ARMA systems, respectively. Clearly, the error criterion defined by equations 6.21 and 6.22 has a straightforward engineering sense and is more appealing for engineering approximations. Note that Eq. 6.19 can be equivalently rewritten as

\[
\varepsilon_1 = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| A(z)(H_{AR}(z) - H_{ARMA}(z)) \right|^2 d\omega .
\]

(6.23)

That is, the error criterion \( \varepsilon_1 \) is the integral of the difference between the transfer functions of the AR and ARMA systems weighted by the matrix \( A(e^{j\omega T}) \) which tends to decrease
the influence of $H_{AR}(z) - H_{ARMA}(z)$ on the frequencies where the magnitude of the transfer function $|H_{ARMA}(z)| = |A^{-1}(z)B(z)|$ is large. In particular, the ACM method provides less accurate approximation for systems with poles close to the unite circle.

Note that an “energy” based reduction of long AR systems to short ARMA models has been previously pursued by Friedlander and Porat (1984) who have identified the modes which are associated with the largest amount of energy. This procedure has also been examined for ocean engineering problems by Mandal et.al (1992), and for aerospace engineering applications by Spanos et.al. (1995).

Clearly, the error criterion $e_2$ is not a quadratic function of the unknown ARMA parameters and, in general, requires application of non-linear optimization schemes. In this regard, Porat and Friedlander (1984) and Spanos et.al. (1992) have utilized a version of the steepest descent optimization algorithm by establishing analytical expressions for the corresponding gradients. Cheng and Chang (1991) have reduced this optimization problem to the linear programming for the model reduction application.

In this paper an iterative algorithm is adopted for the minimization of the criterion $e_2$. Similar iterations were used by Sanathanan and Koerner (1963) and Steilitz and Mcbride (1965) for system identification applications. At every iteration the ARMA parameters which are determined from the previous iteration are used to improve spectral estimation.

6.2.3 New method formulation: uni-variate case

In the uni-variate case, one can introduce the sequence

$$
e_i = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| \frac{A_i(z)}{A_{i-1}(z)} \right|^2 |H_{AR}(z) - \frac{B_i(z)}{A_i(z)}|^2 \, d\omega =$$
\[ \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| A_i(z)Q_1^{i-1}(z) - B_i(z)Q_2^{i-1}(z) \right|^2 d\omega, \quad (6.24) \]

where

\[ Q_1^{i-1}(z) = \frac{H_{AR}(z)}{A_{i-1}(z)}, \quad Q_2^{i-1}(z) = \frac{1}{A_{i-1}(z)}. \quad (6.25) \]

Note that if \( A_{i-1}(z) = 1 \), Eq. 6.24 reduces to Eq. 6.19. However, when \( A_i(z) = A_{i-1}(z) \), Eq. 6.24 represents the energy error criterion 6.21. Thus, the following condition for convergence of the algorithm is specified

\[ \left| \varepsilon_2^i - \varepsilon_2^{i-1} \right| / \varepsilon_2^i < \delta, \quad (6.26) \]

where \( \varepsilon_2^i = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| H_{AR}(z) - B_i(z)/A_i(z) \right|^2 d\omega \), and \( \delta \) is a number reflecting acceptable error.

After performing some algebraic manipulations, Eq. 6.24 can be rewritten in the equivalent form

\[ \varepsilon_i = \frac{1}{2\omega_n} \left( \sum_{k=0}^{p} \sum_{l=0}^{q} a_k R_{11}(k-l)a_l - \sum_{k=0}^{p} \sum_{l=0}^{q} b_k R_{12}(l-k)a_l - \sum_{k=0}^{p} \sum_{l=0}^{q} a_k R_{12}(k-l)b_l - \sum_{k=0}^{p} \sum_{l=0}^{q} b_k R_{22}(k-l)b_l \right), \quad (6.27) \]

where the superscript index indicating the number of iterations is omitted for notational convenience,
\[ R_{rl}(k) = \int_{-\omega_n}^{\omega_n} Q_r^*(z) Q_l(z) e^{jk\omega} d\omega, \quad r, l = 1, 2. \quad (6.28) \]

Clearly, \( R_{11} \) and \( R_{22} \) can be thought of as the auto-correlation functions of AR processes \( y_1 \) and \( y_2 \) with the transfer functions \( Q_1(z) \) and \( Q_2(z) \) given by Eq. 6.25. Similarly, \( R_{12} \) can be viewed as the cross-correlation of \( y_1 \) and \( y_2 \). These auto- and cross-correlation functions can be found by constructing the state space realization of these systems (Willems 1991) and solving the corresponding Lyapunov equations as it is shown in Appendix B.

The error criterion \( \epsilon_i \) expressed by Eq. 6.27 is a positive definite quadratic form involving the unknown ARMA parameters. For its minimization it is sufficient to set the derivatives of \( \epsilon_i \) with respect to the ARMA parameters \( a_k^i, k = 1, \ldots, p, \) and \( b_l^i, l = 0, \ldots, q \) equal to zero. This leads to a small dimensional system of linear equations involving a symmetric positive definite matrix

\[
\sum_{i=0}^{p} a_i R_{11}(i-k) - \sum_{l=0}^{q} b_l R_{21}(l-k) = 0, \quad k = 1, \ldots, p, \quad (6.29)
\]

\[
- \sum_{i=0}^{p} a_i R_{12}(i-k) + \sum_{l=0}^{q} b_l R_{22}(l-k) = 0, \quad k = 0, \ldots, q, \quad (6.30)
\]

where \( a_0 = 1. \)

Note that if the iterative algorithm is initiated with \( A^0(z) = 1 \), the proposed method yields the ACM solution as the first iteration. Further iterations are performed until the condition specified by Eq. 6.26 is satisfied.
6.2.4 New method formulation: multi-variate case

The proposed method can be generalized for the multi-variate case by introducing the following sequence

$$
\varepsilon_i = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| A_i(z)(A_{i-1}^{-1}(z)H_{AR}(z)) - A_{i-1}^{-1}(z)B_i(z) \right|^2 d\omega .
$$

(6.31)

Again, if $A_{i-1}(z) = I_n$, Eq. 6.31 reduces to Eq. 6.19 which provides the ACM solution. However, when $A_i(z) = A_{i-1}(z)$ Eq. 6.31 represents the error criterion of Eq. 6.21. Then, the convergence of this iterative procedure is determined by Eq. 6.26 where

$$
\varepsilon_2^i = \frac{1}{2\omega_n} \int_{-\omega_n}^{\omega_n} \left| H_{AR}(z) - A_i^{-1}(z)B_i(z) \right|^2 d\omega .
$$

Note that since the general matrices do not commute, a straightforward minimization of Eq. 6.31 leads to algebraic expressions which are not amenable to obvious physical interpretation. To circumvent this problem the procedure of minimizing the error criterion 6.31 is pursued in two steps.

During the first stage $\varepsilon_2^i$ is only minimized with respect to the AR parameters $a_k^i$, $k = 1, \ldots, p$, for the specified MA parameters $b_l$, $l = 0, \ldots, q$. Performing similar manipulations as in the uni-variate case, $\varepsilon_i$ can be rewritten in an equivalent form

$$
\varepsilon_i = \frac{1}{2\omega_n} \text{tr} \left( \sum_{k=0}^{p} \sum_{l=0}^{p} a_k R_{11}(k-l) a_l^i - \sum_{l=0}^{p} R_{21}(-l) a_l^i - \sum_{l=0}^{p} a_l R_{12}(l) + R_{22}(0) \right) ,
$$

(6.32)

where

$$
R_{rl}(k) = \int_{-\omega_n}^{\omega_n} Q_r^*(z) Q_l^i(z) e^{jk\omega T} d\omega , \quad r, l = 1, 2 ,
$$

(6.33)
is the auto/cross - correlation of the processes with the following transfer functions

\[ Q_1^{-1}(z) = A_{i-1}(z)H_{AR}(z) = (A_{AR}(z)A_{i-1}(z))^{-1}b_0^{AR} , \]  

(6.34)

and

\[ Q_2^{-1}(z) = A_{i-1}(z)B_{i-1}(z) . \]  

(6.35)

An efficient procedure to evaluate the auto- and cross-correlation functions defined by Eq. 6.33 is given in Appendix B.

Again, the minimum of \( \varepsilon_i \) can be obtained by setting its derivatives with respect to \( a_k \), \( k = 1, \ldots, p \), equal to zero. It yields the following system of linear equations.

\[ \sum_{i=0}^{p} a_i R_{11}(i-k) - R_{21}(-k) = 0 \quad , \quad k = 1, \ldots, p , \]  

(6.36)

where \( a_0 = I_n \). The system of Eq. 6.36 can be readily solved providing a new estimate for the AR parameters.

Similarly, during the second stage the optimal error criterion \( \varepsilon_i = \varepsilon_2 \) is only minimized with respect to the MA parameters \( b_i^l \), \( l = 0, \ldots, p \), for the given AR part \( a_k \), \( k = 0, \ldots, q \). In this case, the second equality in the definition of the norm of matrices 6.20 better suits for numerical calculations. Then, the associated error criterion can be expressed as

\[ \varepsilon_2 = \frac{1}{2\omega_n} tr \left( R_{11}(0) - \sum_{i=0}^{q} b_i^l R_{21}(i) - \sum_{i=0}^{q} R_{12}(-i) b_i + \sum_{k=0}^{q} \sum_{i=0}^{q} b_i^l R_{22}(k-i) b_k \right) , \]  

(6.37)

where the cross and auto-correlation functions which appear in the equation are defined by
Eq. 6.33 with

\[ Q_1(z) = H_{AR}^i(z) = (b_0^{AR})^i (A_{AR}^i(z))^{-1} = ((b_0^{AR})^i A_{AR}^i(z)(b_0^{AR})^{-i})^{-1} (b_0^{AR})^i, \]

and

\[ Q_2^{-1}(z) = A_i^{-1}(z). \]

Setting the derivatives of \( \varepsilon_i \) with respect to \( b_k^i, k = 0, \ldots, p \), equal to zero yields the system of linear equations

\[ \sum_{i=0}^{q} b_k^i R_{22}(i- k) - R_{12}(-k) = 0, \quad k = 0, \ldots, q. \]

The system of Eqs. 6.40 can be readily solved to determine a new estimate for the MA parameters. Then, the convergence criterion 6.26 can be evaluated and a new iteration cycle can be performed, if necessary.

### 6.3 Numerical Examples

The iterative procedure developed in the previous section has been applied to spectra which are used in offshore and earthquake engineering.

The Kanai-Tajimi spectrum (Madsen et.al. 1986)

\[ S(\omega) = S_0 \frac{\omega_0^4 + 4 \zeta_g^2 \omega_0^2 \omega^2}{(\omega_0^2 - \omega^2)^2 + 4 \zeta_g^2 \omega_0^2 \omega^2} \]

is selected for the first application; the values of \( \zeta_g, \omega_0, \) and \( S_0 \) are \( 8\pi, 0.8, \) and \( 1, \)
respectively. The results of ARMA(2,2) modeling are shown in Figure 6.1(a). The approximation errors of the ACM method and of the proposed method are shown in Figure 6.1(b). Clearly, the proposed method can yield improved spectral matching.

The Pierson-Moskowitz spectrum (Spanos 1983)

\[ S(\omega) = \frac{e^{5/4}}{\omega^{5/2}} \exp\left(\frac{5}{4\omega^4}\right). \]  

(6.42)

is often used in ocean engineering. ARMA(4,4) approximations derived by the proposed method and the ACM method, along with corresponding approximation errors, are given in Figure 6.2(a,b). It is seen again that the proposed iterative method leads to improved spectral representation.

Finally, a two-variate random process which describes the earthquake ground motion of two ground sites is approximated by using the proposed ARMA modeling method. This spatial correlation model has been proposed by Harichandran and Vanmarcke (1986) by analyzing the data recorded during the Taiwan earthquake of December 17, 1982. The spectral form is

\[ S(\omega) = \begin{bmatrix} S_{11}(\omega) & \rho_{12}(\omega) \sqrt{S_{11}(\omega)S_{22}(\omega)} \\ \rho_{12}(\omega) \sqrt{S_{11}(\omega)S_{22}(\omega)} & S_{22}(\omega) \end{bmatrix}, \]  

(6.43)

where \( S_{11}(\omega) = S_{22}(\omega) \) is assumed to be the Kanai-Tajimi spectrum given by equation 6.43; the same numerical values for the coefficients are used. The cross-correlation function is (Harichandran and Vanmarcke 1986)

\[ \rho_{12}(\omega) = 0.736 \exp\left(\frac{5.063r}{\theta(\omega)}\right) + 0.294 \exp\left(\frac{0.744r}{\theta(\omega)}\right), \]  

(6.44)
where

$$\theta(\omega) = 5.210 \sqrt{1 + \left(\frac{\omega}{2.18\pi}\right)^{2.78}},$$  \hspace{1cm} (6.45)$$

and $r$ denotes the distance between two sites. The constant $r$ is selected equal to 0.3. Note that this multi-variate spectrum is not a rational function of frequency. ARMA(2,2) approximations which are determined by using the proposed method and the ACM method are given in Figure 6.3. Clearly, the energy error criterion provides a closer approximation of the target spectrum, especially for the range of frequencies associated with the spectral peak.

Note that for the considered numerical examples two iterations have been found sufficient to achieve convergence in the sense of Eq. 6.26.

### 6.4 Concluding Remarks

A useful method of ARMA spectral approximation in conjunction with the problem of simulating random processes has been presented in this chapter by adopting a frequency domain error criterion which lends itself readily to physical interpretation. The method is iterative and requires solving linear equations only; this can be readily performed by modern digital computers. Pertinent numerical examples have shown the usefulness of the method for practical applications; it yields quite low-order, and thus computationally efficient, ARMA simulation algorithms even for spectra which, as the Pierson Moskowitz spectrum of ocean waves, are mathematically intractable for AR approximation.
Figure 6.1. (a) Target Kanai-Tajimi spectrum and its approximations; (b) error of approximations.
Figure 6.2. (a) Target Pierson-Moskowitz spectrum and its approximations; (b) error of approximations.
Figure 6.3. (a) $S_{11}(\omega)$, the target two-variate spectrum and its approximations, (b) $S_{12}(\omega)$, the target two-variate spectrum and its approximations, (c) error of approximations of $S_{11}(\omega)$, (d) error of approximations of $S_{12}(\omega)$. 
Chapter 7. Random Field Simulation using Covariance Method with Eigenvalue Decomposition

The covariance method for random field simulation is discussed in this chapter in conjunction with the eigenvalue decomposition of covariance matrices. In this regard, the basic principles of this simulation method are introduced in Section 7.1. The eigenvalues of the covariance matrices are positive and their magnitudes decay sufficiently fast for a large class of random fields commonly used in engineering applications. The fast decay of the magnitude of eigenvalues facilitates the random field simulation since only several eigenvalues can be retained for analysis and synthesis. Thus, the eigenvalue decomposition requires the determination of only certain eigenvalues and of the corresponding eigenvectors on the extremum of the covariance matrix spectrum. Further, the Lanczos based methods described in Section 7.2 are well suited for addressing the corresponding eigenvalue problem by performing only a few vector-matrix manipulations and storing only a few additional vectors. Without retaining the bandwidth structure of the covariance matrix the eigenvalue decomposition can be obtained by using $O(N^2)$ numerical operations. This decomposition is done once for all subsequent simulations. Then, the random field can be synthesized in $O(kN)$ numerical operations, where $N$ is the size of the discrete random field, and $k$ is the number of retained eigenvalues.

Note that the main factor determining the numerical efficiency and usefulness of the covariance method is the rate of decay of the covariance matrix eigenvalues. This decay is analyzed in Section 7.3 for several random fields. In this regard, an asymptotic relationship between the square mean error and the number of the retained eigenvalues is developed. This theoretical formula is verified numerically and is used to study the properties of random fields. Several numerical examples are considered in Section 7.4.
7.1 Introduction

The covariance method for simulating Gaussian random fields is the most straightforward and well understood technique. Note that only discrete random fields can be simulated using digital computers. Then, a continuous random field must be discretized and approximated, beforehand, by a random vector $f$, $i=1,...,N$ with the covariance matrix $R = \{R_{ij}\}$. This random vector can be synthesized by using the linear transformation

$$f = L\xi,$$

where $\xi$ is a vector of $N$ independent Gaussian unit variance random variables, and $L$ is a matrix such that

$$R = LL^T.$$

The matrix $L$ can be selected arbitrarily as long as it satisfies Eq. 7.2. In this regard, the Cholesky factorization of $R$ is often used in the area of stochastic mechanics yielding a triangular matrix $L$ in Eq. 7.2 (Scheuer and Stoller 1962). Several modifications of this method utilize additional qualitative information regarding the specific properties of the stochastic mechanics problem to simplify computations (Elishakoff 1979, Yamazaki, et.al. 1989). Though appealing in its simplicity, this method requires significant computations and is subjected to a considerable round-off error. To generate a single sample of the field it is required to multiply a full triangular matrix by a vector. That is, one needs to perform $O(N^2)$ numerical operations. Also, the Cholesky decomposition itself further requires $O(N^2)$ additional operations. In this original form this method can be realistically applied only for synthesizing relatively small random fields.

Alternatively, the eigenvalue decomposition of the covariance matrix $R$ can be used (Yamazaki and Shinozuka 1990, Zeldin and Spanos 1992, Spanos and Zeldin 1993, 1994).
That is,

\[ L = V \Sigma^{1/2}, \]  

(7.3)

where \( V \) is the matrix of eigenvectors and \( \Sigma = \text{diag}(\sigma) \) is the diagonal matrix of eigenvalues of \( R \)

\[ RV = V \Sigma. \]  

(7.4)

In this case only a few components of the vector \( \sigma \) must be retained for an adequate approximation.

Since \( f_\ast \) is the discrete approximation of the continuous, “smooth”, random field \( f(x) \), its covariance matrix has some additional “structure”. Then, this “structure” can be used to assess the usefulness of the eigenvalue decomposition. In this regard, Eqs. 7.1, 7.3 and 7.4 can be viewed as the Loève-Karhunen expansion of the random vector \( f_\ast \). Therefore, the elements of the matrix \( \Sigma \) can be viewed as the approximation of the corresponding eigenvalues of the Loève-Karhunen expansion of the continuous random field \( f(x) \). The Loève-Karhunen expansion of \( f(x) \) can be written as (Traina et.al. 1986, Ghanem and Spanos 1991,1993)

\[ f(x) = \sum_{k=1}^{\infty} \xi_k \sqrt{\sigma_k} v_k(x), \]  

(7.5)

where

\[ \int_{D_T} R_f(x, y)v_k^\top(y)dy = \sigma_k v_k^\top(x), \]  

(7.6)

\[ \int_{D_T} v_l(x)v_k^\top(x)dx = \delta_{lk}, \]  

(7.7)
\[ \xi_k = \frac{1}{\sqrt{\sigma_k \tau}} \int f(x)\psi^\tau_k(x)dx , \quad (7.8) \]

and \( x = (x_1, x_2, \ldots, x_n) \), \( n \) denotes the dimension of the field. The parameter \( \tau \) in Eqs. 7.6-7.8 signifies the size of the field. That is,

\[ \int_{D_\tau} dx = \tau^n \quad (7.9) \]

and the superscript in the notation of \( \psi^\tau_k(x) \) indicates that this function is associated with the domain \( D_\tau \).

Ghanem and Spanos (1991) showed that only a few elements of the Loeve-Karhunen expansion in Eq. 7.5 can be retained for the adequate field representation. The mean square error resulting from truncating the Loeve-Karhunen expansion is minimal in this case. Indeed, the difference between the target and the approximated fields is equal to

\[ \varepsilon_M(x) = \sum_{k = M + 1}^{\infty} \xi_{\sigma_k} \sqrt{\sigma_k} \psi^\tau_k(x) . \quad (7.10) \]

Substituting Eq. 7.8 into Eq. 7.10 and using the orthogonality condition of Eq. 7.7 the mean square error can be expressed as

\[ \varepsilon_M^2 = \int_{D_\tau} E[\varepsilon_M^2(x)]dx = \sum_{k = M + 1}^{\infty} \int_{D_\tau} \int_{D_\tau} R_f(x, y)\psi^\tau_k(x)\psi^\tau_k(y)dxdy . \quad (7.11) \]

Then, minimization of the functional in the right hand side of Eq. 7.11 over all square integrable functions subjected to the orthogonality constraint of Eq. 7.7 yields a system of equations identical to Eq. 7.6. That is, the truncated Loeve-Karhunen expansion is the best linear estimate of the random field by using \( M \) random variables. Further, substituting Eq.
7.6 back into Eq. 7.11 one finds

$$
\varepsilon^2_M = \sum_{k = M + 1}^{\infty} \sigma_k.
$$

That is, the minimal mean square error is equal to the sum of the neglected eigenvalues. Note that the error criterion of Eq. 7.12 is slightly different from the criterion proposed by Traina et.al. (1986) who used the summation of squared eigenvalues.

It is pointed out that Eq. 7.12 shows that if the neglected eigenvalues are small, the truncated Loeve-Karhunen expansion is a good approximation, in the mean square sense, of the random field. This representation of random fields can be efficiently utilized for problems of stochastic mechanics by reducing the reliability analysis to the transformation of several random variables. In particular, Traina et.al. (1986) incorporated this technique for evaluating the response of a linear system to a non-stationary input excitation, whereas Ghanem and Spanos (1993) developed the approximation for nonlinear stochastic dynamics problems. Further, Spanos and Ghanem (1989) and Ghanem and Spanos (1991) showed the usefulness of approximating stochastic finite elements by several random variables. Similar ideas guided Zeldin and Spanos (1992), and Spanos and Zeldin (1993, 1994) who addressed the variance reduction method in this regard; see also Chapter 8.

Similarly, substituting Eq. 7.3 into Eq. 7.1 in the discrete case the truncated Loeve-Karhunen expansion of the random vector \( \mathbf{f} \) can be found as

$$
\mathbf{f}_M = \sum_{n = 1}^{M} \xi_n \sqrt{\sigma_n} \psi_n,
$$

where \( \psi_n \) are eigenfunctions of the covariance matrix. In this regard, the difference between the discrete field and its \( M \)-th order approximation is equal to
\[ \varepsilon_M = f - f_M = \sum_{n = M + 1}^{N} \xi_n \sqrt{\sigma_n} \varepsilon_n. \] (7.14)

The mean square error can be defined in this case as the mathematical expectation of the Euclidean norm of the vector \( \varepsilon_M \). That is,

\[ \varepsilon_M^2 = E[\varepsilon_M' \varepsilon_M] = \sum_{k = M + 1}^{N} \psi_k' R \psi_k = \sum_{k = M + 1}^{N} \sigma_k. \] (7.15)

Minimization of the error in Eq. 7.15 over all orthonormal vectors coincides with the eigenvalue problem of Eq. 7.4. Again, the truncated Loeve-Karhunen series is the optimal representation of the random vector by a linear combination of several random variables.

Note that Eq. 7.15 provides a criterion for the number of random variables which must be generated for random field synthesis in conjunction with the covariance method. Several numerically efficient algorithms can be used to find \( M \) of the largest eigenvalues and the corresponding eigenvectors for a symmetric positively definite matrix. The Lanczos method has two advantages in this regard. First, it requires to perform only matrix-vector products involving matrix \( R \) together with a few vectors to store in the memory. Second, the largest eigenvalues tend to merge well quite rapidly. Also, every step of the Lanczos method yields an estimate for the next largest eigenvalue. Then, the computational process can be maintain dynamically without specifying initially the number \( M \) of the eigenvalues to be computed for the stochastic field. This eigenvalue solver is reviewed in the next section.
7.2 Lanczos Eigenvalue Methods

There are many methods to solve the linear eigenvalue problem. But very few are capable of handling large matrices. Among them the Lanczos method (Lanczos 1950) proves to be a versatile technique for finding several eigenvalues and the corresponding eigenvectors. The \( k \) steps of the Lanczos procedure yields an orthogonal \( N \times k \) matrix \( V_k \) and a symmetric tridiagonal matrix \( T_k \) so that

\[
RV_k = V_k T_k + f_k e_k^T,
\]

where \( e_k \) is the \( k \)-th coordinate basis vector, and \( f_k \) is a vector of length \( N \) such that \( V_k^T f_k = 0 \). The eigenvalues of \( T_j \) approximate the largest eigenvalues of \( R \) and the error of this approximation is given by the equation

\[
\|Ax - x\mu\| = \|f_k\|\|e_k^T y\|,
\]

where \( \mu \) and \( y \) are the eigenvalue and the corresponding eigenvector of the matrix \( T_k \), respectively, and \( x = V_k y \). Eigenvalues and eigenvectors of the symmetric tridiagonal matrix \( T_k \) are determined by using the symmetric \( QR \) method (Cullum and Willoughby 1985).

Unfortunately, the original Lanczos process has some numerical difficulties resulting from the loss of orthogonality among the Lanczos vectors, the appearance of spurious eigenvalues in the spectrum of \( T_k \), the undetermined storage requirement, and the round-off errors (Cullum and Willoughby 1985). These drawbacks have been a subject of considerable research efforts. Sorensen (1990) proposed a new implicitly restarted Lanczos procedure which limited the storage requirements and allowed for the enforcement of full numerical orthogonality of the Lanczos basis vector. This algorithm can be described as follows.
1) Initialize \( V(:, 1) = v_1, T = v_1^T R v_1, f = R v_1 - v_1 T \).

2) Perform the standard Lanczos procedure \( k \) times as it follows

   a) \( \beta = \| f \| \);
   b) \( T = \begin{bmatrix} T & \beta e_i^T e_i \\ \beta e_i e_i^T & \alpha \end{bmatrix} \), \( v_i = \frac{1}{\beta} f \), \( V = \begin{bmatrix} V & v_i \end{bmatrix} \);
   c) \( w = R v_i \);
   d) \( \alpha = v_i^T w, T = \begin{bmatrix} T & \beta e_i^T e_i \\ \beta e_i e_i^T & \alpha \end{bmatrix} \);
   e) \( f = w - \alpha v - \beta V e_{k-1} \);
   d) while \( \| s \| > \varepsilon \| f \| \), \( s = V^T f \), \( f = f - V s \).

3) For \( m = 1, 2, \ldots \) until convergence

   a) if \( \| f \| < tol \), then stop;
   b) perform the standard Lanczos procedure \( p \) times as it is described above;
   c) compute eigenvalues of \( T \) and select \( p \) unwanted eigenvalues (the smallest ones);
   d) \( Q = I_k + p \)
   e) for \( j = 1, 2, \ldots, p \)

\[ T = Q_j^T T Q_j, \quad Q = Q Q_j, \text{ where } T - \gamma_j I = Q_j R_j, \quad \gamma_j \text{ is one of the unwanted eigenvalues, } Q_j \text{ is an orthogonal matrix, and } R_j \text{ is an upper-triangular matrix}; \]

   f) \( v = [VQ] e_{k+1}, \quad V = [VQ] \begin{bmatrix} I_k \\ 0 \end{bmatrix} \);
   g) \( \beta_k = e_{k+1}^T e_k, \quad \sigma_k = e_{k+p}^T Q e_k, \text{ and } f = v \beta_k + f \sigma_k \).

Gupta et.al. (1992) used the block Lanczos algorithm to solve large eigenvalue problems describing the dynamic behavior of a two-dimensional solid medium. Sorensen (1990) successfully applied the iteratively restarted Lanczos method for eigenvalue
analysis of composite membranes with up to 10,000 degrees of freedom. Using this method Smith et.al. (1993) developed an iterative procedure for solving large non-linear eigenvalue problems.

7.3 Eigenvalues of the Covariance Matrix

The efficiency of representing random fields, continuous or discrete, by using the truncated Loeve-Karhunen series depends, to a great extent, on the decay of the corresponding eigenvalues. This decay is determined by infinitely many parameters describing the shape of the auto-correlation function. However, several factors which are particularly important in this regard can be educed for a large class of random fields encountered in engineering applications.

7.3.1 Spacial/Time Domain Analysis

Consider a general positive definite function \( R_f(x) \) which is normalized so that

\[
R_f(0) = 1, \tag{7.18}
\]

\[
\int_{\mathbb{R}^n} R_f(x) \, dx = 1. \tag{7.19}
\]

Eq. 7.19 confines somehow the class of fields under consideration excluding fields with the spectral density function equal to zero at the origin. However, a minor modification can eliminate this restriction (Vanmarcke 1983). Since \( R_f(x) \) is a positive definite function, a parameter dependent family of homogeneous random fields can be defined by the following family of auto-correlation functions
\[ R^\theta_f(x) = R_f(x/\theta), \quad 0 < \theta < \infty. \] (7.20)

The parameter \( \theta \) was discussed by Vanmarcke (1979) who associated it with the *scale of fluctuation* of stochastic processes. Vanmarcke and Grigoriu (1983) used this parameter as the criterion for random field discretization in conjunction with the local average method for stochastic finite elements. Vanmarcke (1983, 1994) extended this approach to the multi-dimensional case. The scale, or in the multi-dimensional case the measure, of fluctuation is defined as

\[ \Theta = \int_{R^n} R^\theta_f(x) dx = \theta^n. \] (7.21)

The parameter \( \theta \) explicitly determines the decay of the auto-correlation function at infinity (Vanmarcke and Grigoriu 1983). This provides another useful interpretation for the parameter \( \theta \).

Note that the Loeve-Karhunen expansion of a random field from this family of fields can be found by solving the following equation

\[ \int_{D_1} \frac{x^n}{\theta^n} R_f^\theta(x-y) \nu_k(y) dy = \frac{\sigma_k}{\theta^n} \nu_k(x), \] (7.22)

where \( D_1 \) is a normalized domain of size 1, and \( \nu_k(x) \) is the eigenfunction of the field defined on \( D_1 \) with the scale of fluctuation \( \theta/\tau \). With this normalization

\[ \nu_k^\tau(x) = \frac{1}{\tau^{n/2}} \nu_k\left(\frac{x}{\tau}\right), \] (7.23)

and the Loeve-Karhunen expansion of Eq. 7.5 can be rewritten as
\[ f(x) = \sum_{k=1}^{\infty} \xi_k \frac{\sigma_k}{\tau^n} \nu_k(x/\tau), \quad x \in D_x. \] (7.24)

Eq. 7.22 shows that for the given auto-correlation function the decay of the eigenvalues in the Loeve-Karhunen expansion is determined by the ratio of the scale of fluctuation and the size of the field. Indeed, two extreme cases can be considered in this regard.

First, if \( \tau/\theta \to 0 \),

\[ R\left(\frac{\tau x}{\theta}\right) \to R(0) = 1. \] (7.25)

Substituting Eq. 7.25 back into Eq. 7.22 one can derive

\[ \frac{\sigma_k}{\tau^n} \nu_k(x) \to 1. \] (7.26)

Eqs. 7.9 and 7.26 and the orthonormal properties of eigenfunctions yield

\[ \sigma_k \to \tau^n \delta_{1,k}, \quad \nu_1(x) \to 1. \] (7.27)

That is, eigenvalues of the Loeve-Karhunen expansion tend to decrease infinitely fast. Substituting Eq. 7.27 into Eq. 7.22 into Eq. 7.22 shows that the random field is reduced to a single random variable, \( f(x) = \xi_1 \).

Second, as \( \tau/\theta \to \infty \),

\[ \frac{\tau^n}{\theta^n} R\left(\frac{\tau x}{\theta}\right) \to \delta(x). \] (7.28)

Substituting Eq. 7.28 into Eq. 7.22 yields
\[ \sigma_1 = \sigma_2 = \ldots = \theta^n, \quad (7.29) \]

and the eigenfunctions can be taken from an arbitrary orthonormal system of functions. Eq. 7.29 shows that in this extreme, when the scale of fluctuation is significantly smaller than the field size, the random field behaves as the finite power white noise and all modes are equally important for the field representation. The Loeve-Karhunen expansion can not be truncated beyond just a few elements and must retain a significant number of eigenfunctions for adequate representation. However, substituting Eq. 7.29 into Eq. 7.24 shows that all eigenvalues are also proportional to \((\theta/\tau)^n\) and the field can be adequately modeled as deterministic.

In connection with the preceding analysis it is noted that engineering materials exhibit often non-homogeneous properties with grain or multi-crystalline structures. However, the random effects diminish for large-scale objects with size significantly larger than the size of the corresponding material non-homogeneities. In this case a deterministic model is quite accurate for engineering analysis and design. On the other hand, systems comparable in size with material grains or crystals must be designed by using stochastic models. For example, Mirfendereski and Der Kiureghian (1992) developed a method for stochastic modeling of multi-crystalline micro-fabricated structural elements for micro-electronics applications. In the latter case, the size of material grains is comparable with the size of structural elements.

The preceding analysis reflects the dependence of eigenvalues on \(\theta/\tau\). One can conjecture that, as \(\theta/\tau\) changes from infinity to zero, the decay of eigenvalues changes from infinitely fast to infinitely slow, respectively. Previously, a similar analysis appeared in the papers by Yamazaki and Shinozuka (1990), Ghanem and Spanos (1991) for the continuous and discrete cases, respectively. The scale of correlation of random fields was the main criterion in this regard.
Several examples are considered to assess numerically the decay of eigenvalues as a function of the scale of fluctuation. The relative error $\delta$ of the approximation is defined as

$$\delta = \sum_{k=M+1}^{N} \sigma_k / \sum_{k=1}^{N} \sigma_k$$  \hfill (7.30)

which corresponds to the mean square error discussed in Section 6.1.

Discrete stochastic processes with the covariance matrices

$$R_{i,j} = \frac{1}{1 + (\pi|i-j|\Delta/\theta)^2}, \quad i, j = 0, 1, \ldots 100 \ , \Delta = 0.01,$$  \hfill (7.31)

$$R_{i,j} = \exp\left(-\pi\left((i-j)\Delta/\theta\right)^2\right), \quad i, j = 0, 1, \ldots 100 \ , \Delta = 0.01,$$  \hfill (7.32)

$$R_{i,j} = \exp\left(-2(|i-j|\Delta/\theta)\cos(2|i-j|\Delta/\theta) + \sin(2|i-j|\Delta/\theta)\right), \quad i, j = 0, 1, \ldots 200 \ , \Delta = 0.005,$$  \hfill (7.33)

$$R_{i,j} = \exp(-2(|i-j|\Delta/\theta)), \quad i, j = 0, 1, \ldots 500 \ , \Delta = 0.002$$  \hfill (7.34)

are analyzed, first. The parameter $\theta$ represents the scale of fluctuation of the associated continuous stochastic processes. The number of eigenvalues which must be retained so that the error of Eq. 7.30 is equal to $\delta$ is calculated. These results are plotted in Figures 7.1, 7.2, 7.3 and 7.4, respectively, as a function of the scale of fluctuation $\theta$.

Similar results are plotted in Figures 7.5 and 7.6 for the discrete two-dimensional isotropic fields associated with the continuous auto-correlation functions

$$R(x - y) = \exp(-\sqrt{2\pi}d/\theta)$$  \hfill (7.35)

and

$$R(x - y) = \exp(-\pi(d/\theta)^2) \ ,$$  \hfill (7.36)
respectively, where

\[ d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}. \]  

(7.37)

These figures are quite similar to the preceding one-dimensional examples.

Several conclusions can be drawn by examining Figures 7.1-7.6. First, the ratio \( \theta / \tau \) is important for the efficiency of the eigenvalue decomposition method. Random fields associated with Eqs. 7.31-7.33 and 7.36 can be well represented by a few, with respect to the field size, random variables; see Figures 7.1-7.3, 7.6. The number of random variables to achieve adequate accuracy reduces very rapidly as \( \theta / \tau \) becomes large. Of course, the ratio \( \theta / \tau \) is not the only parameter which determines this number. It can only be used as an indicator of the potential advantages of the eigenvalue decomposition.

Further, observe that for quite different random fields used for this numerical study all plotted curves resemble parallel straight lines for different values of the relative error \( \delta \). This suggests a general relationship between the parameters \( M, \theta \) and \( \delta \) that governs this behavior. Also, random fields specified by Eqs. 7.34 and 7.35 can not be represented by a few parameters as efficiently as the other fields; see Figures 7.4 and 7.5. Based on this observation one can conclude that another important factor, other than the scale of fluctuation, makes fields of Eqs. 7.34 and 7.35 be somewhat different, less efficient in terms of the eigenvalue decomposition, from the rest. The only difference which can be found by comparing these fields is the smoothness of the auto-correlation functions. Thus, Eqs. 7.34 and 7.35 specify the auto-correlation functions which have non-differentiable irregularities in the diagonal \( x = y \) whereas other fields have quite regular auto-correlation functions. The smoothness of the auto-correlation function corresponds to the decay of the corresponding spectral density function. Then, the frequency domain analysis is expected to reveal the missing link in what governs the efficiency of the discussed simulation
method.

7.3.2 Frequency Domain Analysis

A frequency domain analysis of the Loeve-Karhunen expansion is attempted in this section to determine the parameter which affect the decay of the covariance matrix eigenvalues. In this regard, assume that $\tau = 1$ and the random field is isotropic. Then, Eqs. 7.18 and 7.19 yield that the spectral density function satisfy the equations

$$S_f(0) = \frac{1}{(2\pi)^n}, \quad (7.38)$$

$$\int_{R^n} S_f(\omega) d\omega = 1, \quad (7.39)$$

where

$$\omega_d = \sqrt{\omega_1^2 + \ldots + \omega_n^2}. \quad (7.40)$$

Further, the family of auto-correlation functions of Eq. 7.18 can be associated with the following family of spectral density functions

$$S_f^\theta(\omega) = \theta^n S_f(\theta \omega). \quad (7.41)$$

Note that the scale of correlation determines, also, the decay of the spectral density function at infinity and lends itself to an equivalent interpretation in the frequency domain.

Further, any power-density function can be adequately approximated by a rational function of $\omega^2$. That is,
where \( N_{l_1}(\cdot) \) and \( D_{l_2}(\cdot) \) are polynomials of degree \( l_1 \) and \( l_2 \), respectively. In the one-dimensional case, Eq. 7.42 can be viewed as the Markovian approximation of the target process. In this regard it is pointed out that for processes with rational spectral density function the integral eigenvalue problem of Eq. 7.22 has an equivalent differential formulation (Ghanem and Spanos, 1991). Specifically,

\[
\sigma D_{l_2} \left( \theta^2 \left( \frac{d^2}{dx_1^2} + \cdots + \frac{d^2}{dx_n^2} \right) \right) \psi(x) = \theta^n N_{l_1} \left( \theta^2 \left( \frac{d^2}{dx_1^2} + \cdots + \frac{d^2}{dx_n^2} \right) \right) \psi(x), \quad (7.43)
\]

and appended boundary conditions. It is well known that the eigenvalues of Eq. 7.43 decrease asymptotically as

\[
\sigma_k = \frac{\theta^n}{(\theta^2(k_1^2 + \cdots + k_n^2))^p} = \frac{1}{\theta^{2p-n}(k_1^2 + \cdots + k_n^2)^p}, \quad (7.44)
\]

where \( p = l_2 - l_1 \) and \( k \) is the multi-dimensional index \( (k_1, k_2, \ldots k_n) \). Also, for the finite variance processes \( 2p > n \). Eq. 7.44 shows that the decay of eigenvalues, in fact, depends on the parameter \( p \) which signifies the rate of decay of the spectral density function at infinity. Note that the scale of fluctuation describes merely a constant of this decay.

The relative error \( \delta \) defined by Eq. 7.30 can be found as a function of \( \theta \) and \( M \). Indeed, the sum of all eigenvalues is determined based on the fact that the eigenfunctions form an orthonormal basis.
\[ R_f(x-y) = \sum_{k=1}^{\infty} \sigma_k v_k(x)v_k(y). \] (7.45)

Then,

\[ 1 = R(0) = \sum_{k=1}^{\infty} \sigma_k v_k^2(x). \] (7.46)

Integrating Eq. 7.46 over the domain of the random field and using the orthonormal properties of the eigenfunctions one can derive

\[ \sum_{k=1}^{\infty} \sigma_k = 1. \] (7.47)

On the other hand, using Eq. 7.44 one can evaluate the truncated sum of eigenvalues. Specifically,

\[ \sum_{k=M+1}^{\infty} \sigma_k \sim \sum_{k_1, \ldots, k_n} \frac{1}{\theta^{2p-n} (k_1^2 + \ldots + k_n^2)^p} \sim \frac{1}{\theta^{2p-n} l^{2p-n}}, \] (7.48)

where

\[ l^n = M. \] (7.49)

Substituting Eqs. 7.47-7.49 into Eq. 7.30 yields the asymptotic relationship

\[ \delta \sim \frac{1}{\theta^{2p-n} M^{(2p-n)/n}} = \left( \frac{1}{\theta^n/M^n} \right)^{2p-n}. \] (7.50)

Again, Eq. 7.50 shows that, in fact, the parameter \( p \) that manifests the rate of decay of the power spectral density function plays a quite important role in determining the error of
truncating the Loeve-Karhunen expansion.

Further, for a given value of the relative error $\delta$, Eq. 7.50 defines the number of eigenvalues necessary for adequate approximation as a function of the scale of fluctuation. That is,

$$M \sim \frac{1}{\delta^{n/(2p-n)}\sigma^n}. \quad (7.51)$$

Eq. 7.51 shows that in the one-dimensional case the number of eigenvalues adequate for the approximate representation of random fields is inversely proportional to the scale of fluctuation. The family of curves defined by Eq. 7.51 in the log-log coordinate system should look like a family of parallel lines with the slope 1:1. Numerical results plotted in Figures 7.1-7.4 support this assertion. In the two dimensional case, the corresponding curves should form a family of lines in the log-log coordinate system with the slope 2:1. Again, numerical results shown in Figures 7.5 and 7.6 agree quite well though the agreement of Figure 7.6 is less satisfactory.

Note that the mesh for a stochastic field discretization scheme is commonly selected by using the scale of fluctuation as the main criterion. Moreover, Vanmarcke and Grigoriu (1983) used the concept of the scale of fluctuation exclusively for describing the variation of random fields within some finite elements and developed approximate formulae for the correlation of this finite elements. Similarly, the field discretization can be analyzed by using the eigenvalue decomposition method. Indeed, the mesh must be adequately fine to approximate the high frequency components of the random field. In the one dimensional case the $k$-th eigenfunction of the Loeve-Karhunen expansion is asymptotically equal to

$$v_k(x) \sim \sin \frac{Cx}{(\sigma_k \theta^{2p-1})^{1/2p}}, \quad (7.52)$$
where $C$ is a constant. Then, the mesh size must be proportional to

$$ h \sim (\sigma_k \theta^p - 1)^{1/p} \quad (7.53) $$

to represent the spatial variability of the $k$-th eigenfunction. Substituting Eq. 7.44 into Eq. 7.53 one can find

$$ h \sim 1/k \quad (7.54) $$

Substituting the number of the eigenfunction adequate for the field approximation, Eq. 7.51, into Eq. 7.54 yields the asymptotic relationship between the element size of the discretization and the scale of fluctuation

$$ h \sim 2^{p - \sqrt[3]{\delta}} \theta. \quad (7.55) $$

As one can expect, the mesh of discretization must be proportional to the scale of fluctuation.

Now engineering applications where the eigenvalue decomposition of covariance matrices is useful and efficient, can be identified based on the scale of fluctuation and the smoothness of the auto-correlation functions of associated random fields.

Stochastic processes with the scale of fluctuation small in comparison to the decay of the impulse response function of engineering dynamic systems are common for random vibration problems. Also, these processes are quite irregular. Often, white noise is a good approximation for the input excitation. Then, a significant number of eigenvalues is requisite to approximate the excitation properly. However, eigenvalues associated with the high frequency components can be filtered out by the dynamic system and have small or no effect on the response quantities. Note that methods based on the eigenvalue decomposition coincide with the spectral analysis of stochastic signals and systems for the white noise
excitation.

The randomness in the material properties or geometry variability is commonly associated with quite large values of $\theta/\tau$ and $p$. Hence, for this application, eigenvalue analysis is expected to significantly reduce the requisite numerical computations. However, the filtering side-effects is much smaller for these applications especially with respect to the stress quantities.

Finally, methods used for random field discretization (Brenner 1991) in conjunction with the stochastic finite element analysis can be studied by utilizing the eigenvalue decomposition. Quite few papers have addressed the convergence of stochastic mechanics problems as a function of the mesh size. Thus, Der Kiureghian and Ke (1988) determined numerically the required mesh size of stochastic finite elements for the midpoint and local average methods of discretization. One-dimensional random fields, describing the spatial variability of material properties and loading by using the exponential auto-correlation functions, were analyzed for the beam problem. It was shown that the element size should be one-eighth to one-quarter of the scale of fluctuation to provide for the adequate convergence. However, Figure 6.4 shows that in this case the random field within a finite element of that size can be adequately represented by a single random variable.

Upon discretization of the random field the eigenvalue decomposition of associated random variables are often used to reduce the numerical complexity of the stochastic finite element problem; see for example Liu et.al. (1986, 1990). However, this procedure yields merely an approximation of the global discretization of the Loeve-Karhunen method. Note that all discretization methods asymptotically provide the same number of requisite random variables which is proportional to $1/\theta^n$. In this respect, the Loeve-Karhunen expansion proposed by Spanos and Ghanem (1989) has the additional advantage of being optimal in the square mean sense.
7.4 Numerical Examples

Several two-dimensional random fields are synthesized by using the covariance method. The error induced by neglecting some random variables in conjunction with the Loeve-Karhunen expansion is evaluated by using ensemble averaging.

1) First, a Gaussian homogeneous random field with the auto-correlation function given by Eq. 7.35 is simulated in the case \( \theta = 0.5 \), and its realization is shown in Figure 7.7(a) where \( M=200 \) random variables are retained. Large high frequency components are attributed to the fact that the random field is not differentiable in the mean square sense. Two hundred field realizations are assembled to estimate the first and second order moments of the simulated random field. The error of this estimation is given in Table 7.1 for several values of \( M \). For this purpose, the upper and low bounds of the estimates of the mean and the variance, such that only 5% of the corresponding quantities exceed these bounds, are found. Table 7.1 shows that this field can be adequately approximated only by retaining a large number of random variables. The same can be observed in Figure 7.7 where one can see the significant influence of the high frequency components. Nevertheless, the shape of the auto-correlation function is represented quite accurately as it is shown in Figure 7.8(a). Several estimates of the auto-correlation function in several directions are determined by keeping \( M=200 \) random variables.

2) Next, a Gaussian homogeneous random field with the auto-correlation function given by Eq. 7.36 is simulated for the case \( \theta = 0.2 \) and its realization is shown in Figure 7.7(b) where \( M=20 \) random variables are retained. This realization is quite smooth since the spectral density function decays infinitely fast. Two hundred field realizations are assembled to estimate the first and second order moments of the simulated random field. The error of estimating the mean and the variance is given in Table 7.1 for several values of \( M \). This shows that this field can be accurately approximated by a few random variables.
The shape of the auto-correlation function is approximated quite accurately as it is shown in Figure 7.8(b) for $M=20$. Several estimates of the auto-correlation function in several directions are determined.

The comparison of two fields plotted in Figure 7.7 emphasizes that not only the scale of fluctuation but, also, the decay of the spectral density function are important factors which determines the behavior of random fields. Interestingly, the scale of fluctuation of the field shown in Figure 7.7(b) is 2.5 times smaller than the random field of Figure 7.7(a).

### Table 7.1: The error of estimating the mean value and the variance based on the 5% exceeding criteria

<table>
<thead>
<tr>
<th></th>
<th>$M$</th>
<th>$\delta$</th>
<th>Mean Value</th>
<th></th>
<th></th>
<th>Variance</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>lower bound</td>
<td>upper bound</td>
<td>lower bound</td>
<td>upper bound</td>
<td></td>
</tr>
<tr>
<td>Example 1</td>
<td>100</td>
<td>0.1379</td>
<td>-0.0699</td>
<td>0.1054</td>
<td>0.7537</td>
<td>1.0013</td>
<td></td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>0.1019</td>
<td>-0.0595</td>
<td>0.1865</td>
<td>0.7756</td>
<td>1.0474</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.0771</td>
<td>-0.1444</td>
<td>0.0766</td>
<td>0.7805</td>
<td>1.0601</td>
<td></td>
</tr>
<tr>
<td>Example 2</td>
<td>10</td>
<td>0.2425</td>
<td>-0.1593</td>
<td>0.0821</td>
<td>0.5291</td>
<td>0.9601</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.1235</td>
<td>-0.0673</td>
<td>0.0972</td>
<td>0.6869</td>
<td>1.0484</td>
<td></td>
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<tr>
<td></td>
<td>20</td>
<td>0.0628</td>
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<td>0.1059</td>
<td>0.8086</td>
<td>1.0531</td>
<td></td>
</tr>
<tr>
<td>Theory</td>
<td></td>
<td></td>
<td>-0.1163</td>
<td>0.1163</td>
<td>0.8355</td>
<td>1.1645</td>
<td></td>
</tr>
</tbody>
</table>
7.5 Concluding Remarks

In this chapter the covariance method for random field simulation has been examined. It has been shown that the eigenvalue decomposition of the covariance matrix can be viewed as the Loeve-Karhunen expansion of a random vector. The eigenvalues and eigenvectors of the covariance matrix approximate the corresponding eigenvalues and eigenfunctions of the Loeve-Karhunen expansion of the associated continuous field. The Loeve-Karhunen expansion has been shown to be optimal in the mean square sense. Then, the random field can be synthesized by generating a small number of uncorrelated random variables if the eigenvalues of the covariance matrix decay sufficiently fast.

Next, computational procedures capable of solving large eigenvalue problems have been discussed. It has been shown that if only several eigenvalues of the covariance matrix are needed, the Lanczos method for solving large eigenvalue problems can be efficiently used to reduce the computational and memory requirements.

Further, it has been shown that the corresponding eigenvalues decay faster for large values of the ratio of the scale of fluctuation to the size of the random field \( \theta/\tau \). Another factor which is, maybe, quite important in this context, is the decay of the spectral density function at infinity. An asymptotic relationship between the scale of fluctuation \( \theta \), the rate of decay of a power density function \( p \), the number of retained eigenvalues and the mean square error has been developed. A class of random fields which is well suited for eigenvalue analysis has been identified.

Then, it is shown that all random field discretization methods are asymptotically equivalent. However, the Loeve-Karhunen method has the advantage of yielding a set of uncorrelated random variables. These random variables approximate the random field in the most efficient manner by minimizing, in the mean square sense, the error of the approximation.
Finally, several random fields have been synthesized by using the eigenvalue decomposition. In this context it has been found that the rate of decay of the spectral density function at infinity is quite important for analysis and simulation of random fields.
Figure 7.1. The number of retained eigenvalues to achieve the required approximation error $\delta$ as a function of the scale of fluctuation $\theta$. 
Figure 7.2. The number of retained eigenvalues to achieve the required approximation error $\delta$ as a function of the scale of fluctuation $\theta$. 
Figure 7.3. The number of retained eigenvalues to achieve the required approximation error $\delta$ as a function of the scale of fluctuation $\theta$. 
Figure 7.4. The number of retained eigenvalues to achieve the required approximation error $\delta$ as a function of the scale of fluctuation $\theta$. 
Figure 7.5. The number of retained eigenvalues to achieve the required approximation error $\delta$ as a function of the scale of fluctuation $\theta$. 
Figure 7.6. The number of retained eigenvalues to achieve the required approximation error $\delta$ as a function of the scale of fluctuation $\theta$. 
Figure 7.7. Realizations of two-dimensional random fields.
Figure 7.8. The estimates of the auto-correlation functions of two-dimensional random fields based on the ensemble average of size \( N=200 \).
Chapter 8. Galerkin Sampling Method for Stochastic Mechanics Problems

The Monte Carlo simulation method is commonly used in the field of stochastic mechanics, primarily because of its versatility. Often it is the only option available to solve complex problems. However, indiscriminate use of this method can not be advocated due to its considerable computational cost, even if it is implemented by using modern computers. In fact, several variance reduction techniques have been developed in this regard. Important sampling, stratified sampling, conditional sampling and other methods are examples of this development. In this regard, Section 8.1 discusses the Monte Carlo method and some variance reduction techniques.

Previous chapters focused on synthesizing random fields in conjunction with the Monte Carlo method. In particular, the advantages of representing random fields by using wavelets or eigenfunctions were discussed. Numerical algorithms for synthesizing random fields with low computational cost were formulated and analyzed. Note that representation of a random field, continuous or discrete, using a family of basis functions reduces it to an (uncountable in the continuous case) set of random variables. The correlation structure of these random variables is simpler, in some sense, than the correlation structure of the original field. Thus, the wavelet coefficients are “almost” uncorrelated for a large family of random fields and reflect the multiscale structure of the corresponding wavelet basis, whereas, the eigenvalue decomposition yields a set of uncorrelated random variables.

Rather than using the straightforward simulation, the Monte Carlo method of solving stochastic mechanics problems can incorporate some quantitative information about the problem. In particular, the random variables can be retained for analytical or numerical analysis that provides reduction of the requisite computations. It can be especially efficient
in conjunction with the eigenvalue decomposition of random fields since only several random variables are adequate for the accurate approximation.

The purpose of this chapter is to present a numerical method for solving efficiently stochastic mechanics problems by representing the solution by a small number of parameters. It can be interpreted as a Galerkin approximation of the solution in the sample space involving the mathematical concept of simple random variables (Wong, 1971). This new method is formulated and discussed in Section 8.1. The choice of the basis functions of the projective space makes it a generalized sampling method. Further, the developed solution exhibits certain desirable features due to the orthogonality of the Galerkin projection. The different interpretations of this method which can be deduced using the properties of the Galerkin projection and simple random variables are given in Section 8.2. The usefulness of the proposed method is examined by analyzing the behavior of a beam with random rigidity using a finite difference approximation (Vanmarcke and Grigoriu, 1983) and the Loeve-Karhunen expansion (Traina et.al. 1986); the determination of the natural frequencies and of the seismic response of the beam is discussed in Section 8.3.

8.1 Formulation

8.1.1 Preliminary Remarks

Consider the problem of calculating the integral

\[ I = \int_{\Omega} u(\xi) p_\xi(\xi) d\xi, \quad (8.1) \]

where \( \xi = (\xi_1, \xi_2, ... \xi_M) \) is a M-dimensional random vector, \( p_\xi(\xi) \) is the probability density function of \( \xi \), and \( u(\xi) \) is a random variable. The following algorithm can be
used to determine the value of $I$ numerically by relying on random simulation concepts (Mckay et. al. 1979, Rubinstein 1981). First, divide the region $\Omega$ into $m$ disjoint subregions or strata $\Omega_i$, $i=1,2,...,m$, that is, $\Omega = \bigcup_{i=1}^{m} \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$ where $\emptyset$ is an empty set. Then, the integral of Eq. 8.1 can be approximated by the expression

$$I = \bar{I} = \sum_{i=1}^{m} \frac{P_i}{N_i} \sum_{k_i=1}^{N_i} u(\xi_{k_i})$$

(8.2)

where

$$P_i = \int_{\Omega_i} p_{\xi}(\xi) d\xi$$

(8.3)

is a probability mass of the $i$-th stratum, and $N_i$ denotes the number of samplings taken within the $i$-th stratum. The symbol $N = \sum N_i$ denotes the total number of samplings. Clearly, $\bar{I}$ is a random variable. It is a consistent estimate of $I$ and its variance is given by the expression

$$\text{var}(\bar{I}) = \sum_{i=1}^{m} \frac{P_i^2 \sigma_i^2}{N_i}.$$  

(8.4)

In this equation $\sigma_i$ is the local standard deviation of simulating in the $i$-th stratum $\Omega_i$; it can be expressed as

$$\sigma_i^2 = \frac{1}{P_i \Omega_i} \int_{\Omega_i} u^2(\xi) p_{\xi}(\xi) d\xi - \frac{1}{P_i^2 \Omega_i} \left( \int_{\Omega_i} u(\xi) p_{\xi}(\xi) d\xi \right)^2.$$  

(8.5)

Eq. 8.4 indicates that the variance of the error of this simulated value of $I$, that is $\bar{I}$, depends on the relative proportion of the number of samples assigned to each subregion. Rubinstein (1981) showed that if the sample size $N_i$ was proportional to the stratum probability mass $P_i$, the variance of this stratified sampling method was less than the
variance of the sample-mean Monte Carlo method for any function \( u(\xi) \). For practical purposes all probability masses \( P_i \) can be assumed equal, and \( N_i = 1 \).

From a stochastic mechanics prospective the function \( u(\xi) \) may be seen as a solution of a differential equation

\[
L(\xi)u = f(\xi), \tag{8.6}
\]

where \( \xi = (\xi_1, \xi_2, \ldots, \xi_M) \) is a random vector, \( L(\xi) \) is a mathematical operator describing the performance of the system, and \( f(\xi) \) denotes the load. Clearly, the evaluation of the statistical moments of \( u(\xi) \) can be accomplished by repeating the process of solving Eq. 8.6 for different sampled values of the vector \( \xi \). This can be done in a straightforward manner similarly to a straightforward simulation of the value of \( I \) without any stratification. Alternatively, the solution \( u(\xi) \) of Eq. 8.6 can be represented in an expeditious form using the Galerkin projection with a special choice of the projective space; this can be seen as analogous to the judicious stratification of the sample space for estimating \( I \) by \( \tilde{I} \). In particular, \( u(\xi) \) can be expressed as a linear combination of the indicator functions of the strata. These functions form an orthogonal basis for the space of simple random variables which are constant on every stratum (Wong, 1971). This space can be made arbitrarily dense in the set of all random variables by an adequately fine stratification. Then, Eq. 8.6 can be solved by making the error of the approximation orthogonal to the space of simple random variables. That is, by extending the Galerkin method to approximation of random variables. Note that the orthogonality property of the Galerkin projection yields certain desirable properties. Because of these properties Eq. 8.6 can be solved by this method for a relatively small number of times and still yield the same simulation error which is induced by the standard, and hence, computationally costly Monte Carlo method. Certainly, the
proposed method requires the evaluation of the local moments of the vector \( \xi \). However, the computations of these moments may be readily performed. Further, analytical expressions may be available for certain distributions of \( \xi \).

The preceding conceptual description of the proposed method will be mathematically described in the next section.

### 8.1.2 Mathematical Description

Consider a stochastic mechanics problem with random parameters governed by Eq. 8.6. The solution \( u = u(\xi) \) satisfies this equation for every realization \( \xi^* = (\xi^*_1, \xi^*_2, \ldots \xi^*_M) \) of the random vector \( \xi \). In this context, the sample space \( \Omega \) can be divided into \( N \) subdomains or strata \( \{ \Omega_i, i = 1, N \} \) with prescribed probability mass, analogously to the stratified sampling method. In particular, these strata can have the shape of \( M \)-dimensional disjoint rectangles as shown in Figure 8.1, for \( M=2 \). Next, the set of functions or spline basis \( \{ \varphi_i, i = 1, \ldots N \} \) is introduced such that

\[
\varphi_i(\xi) = \begin{cases} 
1, & \text{if } \xi \in \Omega_i \\
0, & \text{otherwise}
\end{cases} \tag{8.7}
\]

Clearly, \( \varphi_i(\xi) \) and \( \varphi_j(\xi) \) have disjoint supports. That is,

\[
\varphi_i(\xi) \varphi_j(\xi) = 0 \quad \text{if} \quad i \neq j. \tag{8.8}
\]

Therefore,

\[
\int_{\Omega} q(\xi) \varphi_i(\xi) \varphi_j(\xi) p_\xi(\xi) d\xi = 0 \quad \text{if} \quad i \neq j, \tag{8.9}
\]

where \( q(\xi) \) is an arbitrary random variable. From this perspective, the set
\{ \varphi_{i} \mid i = 1, \ldots, N \} \) is an orthogonal basis for the class of simple random variables which are constant for every stratum \( \Omega_j \). Any random variable can be approximated adequately by the use of these basis functions provided that the partition of \( \Omega \) is fine. The solution of Eq. 8.6 is approximated by a linear combination of functions \( \{ \varphi_{i} \mid i = 1, \ldots, N \} \). That is,

\[
\begin{align*}
u(\xi) &= \tilde{u}(\xi) = \sum_{i=1}^{N} c_i \varphi_i(\xi),
\end{align*}
\]  

where the coefficients \( c_i \) are to be determined.

The induced solution error can be expressed as

\[
\varepsilon = L(\xi)\tilde{u} - L(\xi)u = L(\xi)\tilde{u} - f(\xi).
\]  

Then, the approximation given by Eq. 8.10 can be construed by making the error \( \varepsilon \) defined by Eq. 8.11 orthogonal to the space spanned by \( \{ \varphi_{i} \mid i = 1, \ldots, N \} \). That is, using the notation \( < > \) for the operator of mathematical expectation, one can write

\[
\begin{align*}
\langle L(\xi) \sum_{i=1}^{N} c_i \varphi_i(\xi) \varphi_j(\xi) \rangle &= \langle f(\xi) \varphi_j(\xi) \rangle, \quad j = 1, \ldots, N.
\end{align*}
\]  

Eqs. 8.9 and 8.12 yield

\[
\begin{align*}
\langle L(\xi) \rangle_j c_j &= \langle f(\xi) \rangle_j, \quad j = 1, \ldots, N.
\end{align*}
\]

where \( \langle \rangle_j \) denotes averaging over the stratum \( \Omega_j \). This sequence of deterministic equations can be solved to find the coefficients \( c_j \).

Often, in solving stochastic mechanics problems, the operator \( L(\xi) \) can be expressed as
\[ L(\xi) = \sum_{i=1}^{M} \xi_i L_i , \] (8.14)

where \( L_i \) is a deterministic differential operator. Specifically, if the randomness in \( L(\xi) \) is induced by a field representing uncertain properties of the medium, the Loeve-Karhunen expansion can be especially effective for this purpose. Specifically, a relatively small number of random variables can represent adequately randomness; see discussion in Chapter 7. Substituting Eq. 8.14 into Eq. 8.13 gives

\[ \sum_{i=1}^{M} \langle \xi_i \rangle_j L_i c_j = \langle f(\xi) \rangle_j , \] (8.15)

where

\[ \langle \xi_i \rangle_j = \frac{1}{P_i} \int_{\Omega_j} \xi_i P_\xi(\xi) d\xi \] (8.16)

is the local, over the \( j \)-th stratum, mean of the random variable \( \xi_i \).

Upon deriving \( c_i \), the statistical properties of the solution can be estimated by relying on Eq. 8.10. Specifically,

\[ \langle u(\xi) \rangle = \sum_{i=1}^{N} c_i P_i \quad ; \quad \langle u^2(\xi) \rangle = \sum_{i=1}^{N} c_i^2 P_i . \] (8.17)

Similarly, the distribution function of the solution can be found using the equation

\[ P_u(v) = Pr(u < v) = \sum_{i=1}^{N} \chi_v(c_i) P_i , \] (8.18)

where

\[ \chi_v(x) = \begin{cases} 1 & \text{if } x \leq v \\ 0 & \text{if } x > v \end{cases} . \] (8.19)
8.2 Method Interpretation

1. The proposed method may be viewed as a Galerkin-type procedure for random media. The idea of using projection procedures in conjunction with random variables has been previously considered for stochastic mechanics problems. In this regard, the stochastic field has been discretized by the use of the Loeve-Karhunen expansion (Spanos and Ghanem 1989; Ghanem and Spanos 1991, 1993; Jensen and Iwan 1992), or by the midpoint method (Grigoriu 1991). In this manner the problem is first characterized by a finite set of random variables. Then, the solution can be derived by a Galerkin projection into finite dimensional spaces spanned by orthogonal chaos polynomials (Ghanem and Spanos 1990, 1991,1993), or just linear functions (Grigoriu 1991). However, these bases yield a large order system of coupled equations which must be solved.

In this context, Eq. 8.7 can be construed as introducing an advantageous, over the existing methods, basis for the solution representation derived by Eq. 8.10; see also Zeldin and Spanos (1992) and Spanos and Zeldin (1993,1994). The concept of using spline type approximations has been discussed extensively in the area of computational mechanics in connection with the finite element method. From this perspective the system of functions \( \{ \varphi_i \}_{i=1}^N \) represents the simplest spline of piecewise-constant functions. Thus, the proposed method involves approximation of the random variables in the finite dimensional subspace of splines defined by some partition of the sample space. Additional advantages of this representation relate to Eqs. 8.8 and 8.9, since each term \( c_i \) in the expansion of Eq. 8.10 can be found independently and, therefore, can be readily determined.

2. From another perspective, the use of piece-wise constant functions makes this method a generalized sampling procedure. Indeed, examining Eq. 8.13 it can be deduced that this equation is equivalent to
\[
\int_{\Omega_i} [L(\xi) c_i - f(\xi)] p_{\xi}(\xi) d\xi = 0 .
\] (8.20)

If the Lebesgue integral involved in Eq. 8.20 can be interpreted in the Riemann sense, and all pertinent quantities are adequately smooth, the mean-value theorem states that there exists some \(\xi^*_i \in \Omega_i\) such that

\[
L(\xi^*_i) c_i = f(\xi^*_i) , \quad i = I, \ldots, N
\] (8.21)

This equation shows that \(c_i\) represents just a solution of Eq. 8.6 for the realization \(\xi^*_i\) of the random vector \(\xi\). Then, the sequence of Eq. 8.21 for \(i = I, \ldots, N\) can be interpreted as a sequence of "indirect samplings". Moreover, as \(\xi^*_i \in \Omega_i\), the proposed method can be viewed as a method analogous to stratified sampling (McKay et al 1979, Rubinstein 1981). But unlike the stratified sampling method, the points inside every stratum are not sampled randomly; they are computed to make the error defined by Eq. 8.11 orthogonal to the chosen basis and minimal for a given stratification. In this sense, this method is equivalent to finding inside each stratum a point doing for this problem what Gaussian points do for the problem of numerical quadrature (Isaacson and Keller 1966). Namely, the choice of these points minimizes the error given by Eq. 8.11 and would represent the exact solution for splines of a certain degree. In case \(L(\xi)\) is given by Eq. 8.14 and the load factor \(f(\xi)\) in Eq. 8.6 is independent of the random vector \(\xi\), these points of "indirect simulation" represent the local means of \(\xi\) over regions \(\Omega_i\) as it can be seen from Eq. 8.15.

3. Regression analysis can also be used (Pfeiffer 1990) to study the properties of the proposed method. Any random variable \(\Theta(\xi)\) which is a function of \(\xi\) on the sample space can be estimated by \(\hat{\Theta}\) using the set of random variables \(\{\varphi_i, i = 1, \ldots, N\}\) defined by the Eq. 8.7. This estimate provides a minimal variance for the difference \(\Theta - \hat{\Theta}\). That is,
\[ \langle (\Theta - \hat{\Theta})^2 \rangle \] (8.22)

is minimized. As \( \{\varphi_i, i = 1, \ldots N\} \) are indicator functions of disjoint sets, the estimate \( \hat{\Theta} \) can be found using linear regression analysis and the solution can be expressed as

\[ \hat{\Theta} = \sum_{i=0}^{\infty} \theta_i \Phi_i(\xi), \] (8.23)

where

\[ \theta_i = \langle \Theta|\Omega_i \rangle = \langle \Theta \Phi_i \rangle, \] (8.24)

and \( \langle \cdot | \cdot \rangle \) denotes conditional expectation.

The estimate \( \hat{\varepsilon} \) of the error \( \varepsilon \) given by Eq. 8.11 can be derived using Eqs. 8.23 and 8.24. Thus, if the coefficients of the expansion of Eq. 8.10 are taken from a system of Eqs. 8.12, then \( \hat{\varepsilon} = 0 \). That is, the proposed method ensures that the error defined by the Eq. 8.10 has a zero mean square estimate from the indicator functions of the chosen stratification.

4. A related perspective can be established using some algebra concepts (Pfeiffer 1990). In this regard the vector \( \xi \) defines a sigma-algebra \( G \) in the sample space, and the stratification shown in Figure 8.1 defines a coarse pure atomic \( \sigma \)-algebra \( G_1 \subset G \) with indicator functions \( \{\varphi_i, i = 1, \ldots N\} \). Then, the preceding regression analysis applied to the error \( \varepsilon \) leads to

\[ \langle \varepsilon \rangle = 0 \quad , \quad \hat{\varepsilon} = \langle \varepsilon | G_1 \rangle = 0. \] (8.25)

In other words, this method yields the minimal error \( \varepsilon \) defined by Eq. 8.11 with respect to the coarse \( \sigma \)-algebra of a given stratification.

5. Note that if the operator \( L(\xi) \) is linear and can be inverted, one can write
\[ \ddot{u} - u = L^{-1}(\xi)e(\xi) . \] (8.26)

Moreover, if \( L^{-1}(\xi) \) is bounded uniformly for all \( \xi \), that is,

\[ K = \sup_{\xi \in \Omega} \| L^{-1}(\xi) \| < \infty, \] (8.27)

where \( \| \cdot \| \) is some specific norm, then

\[ \| \ddot{u} - u \| \leq \| L^{-1}(\xi) \| \| e(\xi) \| \leq K \| e(\xi) \| . \] (8.28)

Thus, the error \( e \) defined by Eq. 8.11 bounds the error of the approximation.

### 8.3 Numerical Examples

#### 8.3.1 Preliminary Remarks

The proposed method is applied for the analysis of the dynamic behavior of a beam of unit length. The beam problem can be described by the equation

\[ (EI(x)u''(x, t))'' = f(x, t), \] (8.29)

where \( u \) is the beam deflection, and \( q \) denotes the distributed force acting on the beam which in general is taken as a stochastic process. The symbol \( EI(x) \) denotes the beam bending rigidity which is assumed to be a normal homogeneous stochastic process with mean equal to 1 and autocorrelation function

\[ R_{EI}(x_1, x_2) = \sigma^2 \exp \left( \frac{(x_2 - x_1)^2}{c} \right), \] (8.30)
where \( \sigma \) and \( c \) are constants. Thus, randomness is manifested in this problem through the operator and the load.

In implementing the proposed method, first the finite difference scheme (Vanmarcke and Grigoriu 1983) is applied to approximate the continuous problem in the nodal points. Thus, the beam operator of Eq. 8.29 can be written as

\[
Lu(x_i) = \frac{1}{h^4}(EI_{i-1}u_{i-2} - 2(EI_{i-1} + EI_i)u_{i-1} + (EI_{i-1} + 4EI_i + EI_{i+1})u_i - 2(EI_i + EI_{i+1})u_{i+1} + EI_{i+1}u_{i+2}),
\]

(8.31)

where \( u_i \) and \( EI_i \) are random variables denoting the displacement and the rigidity in the nodal points, respectively. Appending to Eq. 8.31 the corresponding boundary condition, a system of linear stochastic algebraic equations can be derived in the form

\[
Au = f,
\]

(8.32)

where \( A \) is a matrix of system rigidity with random variables as its entries, and \( u \) and \( f \) are random vectors representing the beam displacement and the force in the nodal points, respectively.

As it has been alluded to previously, the Loeve-Karhunen expansion can be used effectively for reducing the number of requisite random variables needed to represent a random process. In this regard it is noted that the entries of the matrix \( A \) are taken from the vector \( \mathbf{R} = (EI_1, EI_2, \ldots EI_n)' \), where \( (\cdot)' \) denotes transposition. Clearly, the components of \( \mathbf{R} \) are correlated. Therefore, a transformation of it into a vector with uncorrelated elements is adopted. To proceed in this manner the following eigenvalue problem must be solved

\[
Cov(\mathbf{R})\Phi = \Phi\Sigma,
\]

(8.33)
where $\text{Cov}(\overline{R})$ is the covariance matrix of the vector $\overline{R}$, and $\Phi$ and $\Sigma$ are the matrices of its eigenvectors and the eigenvalues, respectively. The diagonal matrix $\Sigma$ is also the covariance matrix of a new random vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)'$ which is related to $R$ by the equation

$$R = \langle R \rangle + \Phi \xi.$$  \hfill (8.34)

Some elements of $\xi$ having the smallest variance can be omitted in Eq. 8.34 leaving a small set of $M$ uncorrelated random variables having the largest contribution to the random variability; see related discussion in Chapter 6.

Substituting Eq. 8.34 into Eq. 8.32 and taking into account only first $M$ elements of the vector $\xi$ one derives

$$(A_0 + \xi_1 A_1 + \ldots + \xi_M A_M)u = f.$$  \hfill (8.35)

Next, specific numerical examples of application of the proposed method are presented; the numerical values $\sigma = 0.3$ and $c = 0.5$ are used.

### 8.3.2 Beam Eigenvalue Problem

The eigenvalue problem of a clamped-clamped beam is considered first. Then, the force in Eq. 8.29 takes the form $f(x) = \lambda u(x)$. This kind of problem is quite difficult either for an analytical or for a numerical treatment. A description of pertinent analytical methods was presented by Boyce (1966). In the papers of Goodwin and Boyce (1964) and Hasselman and Hart (1979) some numerical examples of solution of stochastic eigenvalue problems can be found.

In implementing the proposed method the random domain is divided into a set of
rectangles \( \left\{ \xi_j^i | \xi_j^i \leq \xi_i \leq \xi_j^i, \quad i=1,\ldots,M; \quad j=1,\ldots,N \right\} \) of prescribed equal probability mass. Then, the basis \( \{ \varphi_i, i = 1, \ldots,N \} \) is constructed to conform with Eq. 8.7. Next, \( u \) and \( \lambda \) are expressed in the form

\[
u(\xi) = \sum_{i=1}^{N} \psi_i \varphi_i(\xi) \quad (8.36)
\]

and

\[
\lambda(\xi) = \sum_{i=1}^{N} c_i \varphi_i(\xi), \quad (8.37)
\]

where the scalars \( c_i \) and the vectors \( \psi_i \) are to be determined. Substituting Eqs. 8.36 and 8.37 into Eq. 8.35, multiplying it by \( \varphi_j(\xi) \), and taking mathematical expectation yields

\[
\left( A_0 + \langle \xi_1 \rangle_i A_1 + \cdots + \langle \xi_M \rangle_i A_M \right) \psi_i = c_i \psi_i, \quad i=1,\ldots,N, \quad (8.38)
\]

Then, statistical analysis can be performed in conjunction with Eq. 8.17 to estimate the moments of the eigenvalues and eigenvectors.

The theoretical values of the first two eigenvalues for the deterministic case, when the beam rigidity is set equal to the mean rigidity of the problem under consideration, are \( \lambda_1^{det} = 22.37 \) and \( \lambda_2^{det} = 61.67 \). The corresponding deterministic finite difference approximation with 41 node points gives \( \lambda_1^{findif} = 22.32 \) and \( \lambda_2^{findif} = 61.33 \). It was found that despite the considerable variability in the rigidity of the beam, the variability in the eigenvectors is negligible. However, the variability in the eigenvalues is essential and it is of the order of 10%; see Figure 8.2. Further, the influence of different numbers \( M \) of used random variables \( \xi_i \) has been studied; the contribution of terms beyond \( \xi_2 \) in Eq. 8.35 is negligible. The convergence of this method for different order of partition of axes \( \xi_1 \) and \( \xi_2 \), that is for different values of number \( N \) or "indirect samplings", can be assessed by examining Figure 8.2(a) for the standard deviation of \( \lambda_1 \), and Figure 8.2(b) for the
standard deviation of \( \lambda_2 \). It is seen that the proposed method yields quite good approximations even when the value of number \( N \) is quite small. However, the Montecarlo method, that is when the parameters are sampled arbitrarily, yields reliable results only if the number of the used simulations is large.

### 8.3.4 Beam Response to Deterministic Load

The second problem involves a continuous system with random parameters exposed to deterministic excitation. Specifically, the response of a cantilever beam to earthquake-type base excitation is considered. In this case the force term in Eq. 8.29 can be expressed as

\[
 f(x, t) = -a_g(t) - \ddot{u}(x, t) - \alpha \dot{u}(x, t),
\]

(8.39)

where \( \alpha \) is a coefficient of damping, \( u \) represents the displacement of the beam relative to the base, and \( a_g(t) \) is taken as the time history of the ground acceleration produced by the North-South component of El Centro earthquake recorded on station No 117 and reported in *Strong Motion...* (1976). Figure 7.3 shows this earthquake excitation. Further, in Eq. 8.39 it is assumed that the beam has unit mass per length. The discretization of the beam by the finite difference method, Eq. 8.31, built upon 20 nodes in the spatial domain, is used in conjunction with the Loeve-Karhunen expansion of the bending rigidity. Then, the solution is expressed in the form of Eq. 8.36, where in this case \( v_i = \varphi_i(t) \) are deterministic vector-functions. Substituting this expression into the resulting equation, multiplying it by \( \varphi_j(t) \), and averaging, an uncoupled system of deterministic ordinary differential equations is derived. Each equation of this system is solved numerically using the central difference scheme. Finally, the mean value and the standard deviation of the free end displacement are determined by relying on Eq. 8.17.
The time history of the free end displacement of the cantilever beam having the mean characteristic for the stiffness and damping is plotted in Figure 8.4. Further results of the calculations are shown in Figure 8.5(a,b) for several values of the number \( N \) of indirect samplings. In Figure 8.5, \( N_1, N_2, \) and \( N_\alpha \) denote the number of strata in the domains of \( \xi_1, \xi_2, \) and \( \alpha, \) respectively. The case with \( \alpha = 0.4 \) is considered; it corresponds to damping of approximately 6% of critical for the first mode of the system with deterministic rigidity equal to the mean of the corresponding stochastic problem. Also the case where \( \alpha \) is a random variable statistically independent from \( \xi_2 \) and uniformly distributed between 0.1 and 0.7 is examined. The computations show that three strata in the domain of \( \alpha \) can adequately represent the dependence of the solution on the damping variability. Also, the calculations reveal that only the first two components of the vector \( \xi_2 \) influence significantly the beam response. It can be seen that the response of the beam to the deterministic excitation is strongly affected by the rigidity variability.

### 8.3.4 Beam Response to Stochastic Load

The third problem is described again by Eqs. 8.29 where the force term is expressed as

\[
f(x, t) = -a_g(t) - \ddot{u}(x, t),
\]

with \( a_g(t) \) taken as a stationary random process. Upon discretizing the beam consistently with Eq. 8.31 and using 20 nodes along its length, a damping term is added. It is assumed that the dissipative term induces equal critical damping ratios for all modes of vibration. The solution is expressed in the form of Eq. 8.36. In this case \( y_i = \psi_i(t) \) is not a deterministic vector-function, but a stochastic vector-process. That is, \( \psi_i(t) \) is the response of a deterministic system to random excitation. A number of techniques exist for the solution of this problem. In particular, a spectral approach is applied, provided that \( a_g(t) \)
is a second order stochastic process. In the latter case the second order characteristics of the solution can be determined from the formulae

\[
E[y] = \sum_{i=1}^{N} E[y_i(t)] p_i , \quad \text{and} \quad R_y(t_1, t_2) = \sum_{i=1}^{N} R_{y_i}(t_1, t_2) p_i . \tag{8.41}
\]

Then, this approach can be viewed as a semi-analytical method analogous to the directional sampling (Ditlevsen et al. 1988). At first the vector \( \xi \) is simulated, and then the solution for the given simulation is calculated using known analytical techniques with subsequent application of some averaging as in Eq. 8.41.

Again two cases for \( \alpha \) are considered. First \( \alpha \) is a deterministic coefficient, and second \( \alpha \) is uniformly distributed random variable; it induces for each mode of the discrete model of the beam damping 6\%, and 2\% to 10\% of critical, respectively. The power spectral density of the displacement of the free end is calculated using the proposed method; \( a_g(t) \) is a white noise process with unit two sided spectral density. The data for different numbers of indirect samplings are plotted in Figures 8.6 and 8.7 together with the corresponding solution for the response of a deterministic system with rigidity equal to the mean rigidity of the stochastic system. Figure 8.6 shows these results in a logarithmic scale for the frequency range of the first two eigenvalues of the deterministic system, while Figure 8.7 shows the same results in the standard coordinates for the frequency range of the first eigenvalue of the deterministic system. Figures 8.6 and 8.7 reveal that the randomness of the system has a significant effect on the system response variability and reduces the peaks of the response power spectral density. The calculations indicate that only the first two components of the vector \( \xi \) influence the first two moments of the solution significantly. Further, the effect of the damping variability can be captured using only two strata in its domain.
8.4 Concluding Remarks

A numerical method for solving stochastic mechanics problems has been presented in this chapter. Specifically, it has been proposed to use the Galerkin projection into the space of simple random variables. This space can be spanned by piece-wise constant spline functions with a chosen partition of the sample space. Further, it has been shown that the proposed method can be construed as a generalized sampling. It is related to the stratified sampling method and it is optimal in the sense of Eq. 8.25. That is, the approximation of the problem from the space of simple random variables (Wong 1971) produces an error with mean and conditional expectation, given the sigma-algebra induced by this partition, equal to zero. It has also been shown that this error has a zero estimate from the set of indicator functions of the given stratification. Some stochastic mechanics problems have been studied utilizing the proposed method in conjunction with the Loeve-Karhunen expansion which is a versatile tool for the approximation of a stochastic field by a finite set of random variables. These examples have demonstrated that the proposed method can be applied to treat a broad class of stochastic mechanics problems.
Figure 8.1. Stratified sample space.
Figure 8.2. Standard deviation of eigenvalues of a clamped-clamped beam with random rigidity versus the number of samples for the Monte Carlo method and the new method
(a) first eigenvalue, (b) second eigenvalue.
Figure 8.3. Earthquake-type excitation.

Figure 8.4. Time history of the free end displacement of a deterministic cantilever beam under earthquake base excitation.
Figure 8.5. Base excited response of the top of a cantilever beam with random rigidity (a) mean value, (b) standard deviation; curve 1: $N=63$ ($N_1=9$, $N_2=7$); curve 2: $N_1=7$; curve 3: $N=45$ with random damping and ($N_1=5$, $N_2=3$, $N_0=3$.)
Figure 8.6. Power spectral density of the displacement of the free end of a cantilever beam within frequency [0,30];

- curve 1: \(N=63\) \((N_1=9, N_2=7)\); curve 2: \(N_1=7\);
- curve 3: \(N=45\) with random damping \((N_1=5, N_2=3, N_\alpha=3)\)
- curve 4: deterministic system with mean characteristics.
Figure 8.7. Power spectral density of the displacement of the free end of a cantilever beam within frequency [0,10];
curve 1: \(N=63\) \((N_f=9, N_2=7)\); curve 2: \(N_f=7\);
curve 3: \(N=45\) with random damping and \((N_f=5, N_2=3, N_\alpha=3)\)
curve 4: deterministic system with mean characteristics
Chapter 9. Summary

Several important issues related to the development of accurate and computationally efficient models of engineering uncertainties have been considered in the dissertation. In this regard, the research efforts have focused on the development of methods for representation and synthesis of random fields that are particularly well suited for engineering analysis and design. The developed models address the trade-off between the level of the requisite mathematical rigor for the description of engineering uncertainties and the efficiency of computational algorithms that can be used for analyzing the corresponding engineering structures. Specifically, the development of efficient numerical algorithms that can be used to synthesize multi-dimensional random fields in conjunction with the Monte Carlo method of reliability analysis of engineering structures has been discussed comprehensively.

Upon establishing the research objectives, the models of homogeneous random fields have been reviewed in Chapter 2. The advantages and drawbacks of the considered algorithms for synthesizing random fields have been also discussed in conjunction with the associated numerical simplifications and the corresponding computational requirements.

A new efficient representation of multi-dimensional random fields which can be used for the analysis and synthesis applications has been discussed in Chapters 3, 4, and 5 by introducing the concepts of the theory of wavelet bases. It has been shown that the correlation of random fields can be represented by a few parameters associated with the covariance matrices of the random wavelet coefficients. These correlation parameters can be readily evaluated by using the developed computational algorithms which utilize an approximate relationship between the wavelet coefficients associated with different scales.
Further, the introduced expansion of random fields in term of a wavelet basis has been used as a motivation for the development of a new class of scale-type methods for synthesizing random fields. In this regard, Chapter 4 has discussed two approaches for simulating stationary stochastic processes. Specifically, the wavelet coefficients associated with different scales have been assumed to be correlated and uncorrelated, respectively. The numerical efficiency and computational requirements of the developed synthesis algorithms have been determined. Also, it has been shown that the developed scale-type procedures can be treated by using the theory of stochastic processes on homogeneous trees. In Chapter 5 the developed models and numerical procedures have been further generalized for the case of multi-dimensional random fields. Numerical simulation of one and two-dimensional random fields commonly used in practical applications has examined the usefulness of the proposed synthesis methods. This numerical study has elucidated the capacity of the wavelet based algorithms to approximate the target second order moments quite accurately and to provide improved numerical efficiency for the computer software.

A useful method of ARMA spectral approximation has been presented in Chapter 6 by adopting a frequency domain error criterion which lends itself readily to physical interpretation. The method is iterative and requires solving linear equations only; this can be readily performed by modern digital computers. Pertinent numerical examples have demonstrated the usefulness of the method for practical applications; it yields quite low-order, and thus computationally efficient, ARMA simulation algorithms even for spectra which, as the Pierson Moskowitz spectrum of ocean waves, are mathematically intractable for AR approximation.

The covariance method for random field simulation has been studied in Chapter 7. It has been shown that this method is closely related to the Loeve-Karhunen representation. Thus, the eigenvalue decomposition of the covariance matrix can yield an efficient simulation procedure for some random fields if the Loeve-Karhunen expansion requires
just a few elements for an adequate approximation. The parameters which determine the
decay of the eigenvalues of the Loeve-Karhunen expansion have been identified. Since this
method may require evaluation of several eigenvalues and eigenvectors for large matrices,
computational procedures capable of solving large eigenvalue problems have been
discussed. Several random fields have been synthesized by using the eigenvalue
decomposition of the covariance matrix elucidating the usefulness of this method.

Finally, a numerical method for solving stochastic mechanics problems has been
presented in Chapter 8. Specifically, it has been shown that the Galerkin projection is
applicable for the stochastic mechanics problems. The proposed method can be viewed as
a generalized, for computational efficiency, sampling procedure. Some stochastic
mechanics problems have been studied utilizing the proposed method in conjunction with
the Loeve-Karhunen expansion. These examples have demonstrated that the proposed
method can be applied for treating a broad class of stochastic mechanics problems.

It is the author's believe that mathematical modeling of engineering uncertainties and
the development of reliability design procedures will progress in the future by resolving
the trade-off between the efficiency of the related computational methods and the
mathematical complexity and accuracy of the utilized probabilistic models.
Appendix A. Determination of Moments of the Scale Function

In this Appendix an efficient method is presented to find the integrals $\int x^m \phi(x) dx$ which appear in Eq. 3.27.

Using the Fourier transform one can derive

$$\chi_m = \int x^m \phi(x) dx = \int x^m \phi(x) e^{-ix\xi} dx \bigg|_{\xi = 0} = \frac{\sqrt{2\pi}}{(-i)^m} \hat{\phi}^{(m)}(0) . \quad (A.1)$$

Also, Eq. 1.89 yields

$$\hat{\phi}(0) = \frac{1}{\sqrt{2\pi}} , \text{ and } \chi_0 = m_0(0) = 1 . \quad (A.2)$$

Further, differentiating Eq. 1.85 $m$ times yields

$$\hat{\phi}^{(m)}(0) = \left( m_0 \left( \frac{M}{2} \right) \hat{\phi} \left( \frac{M}{2} \right) \right)^{(m)} \bigg|_{\xi = 0} = \sum_{k = 0}^{m} \frac{1}{2^m} \binom{m}{k} m_0^{(k)}(0) \hat{\phi}^{(m-k)}(0) = \frac{1}{2^m} \sum_{k = 1}^{m} \binom{m}{k} m_0^{(k)}(0) \hat{\phi}^{(m-k)}(0) , \quad (A.3)$$

where the last equality holds because of Eq. A.2. The value $m_0^{(k)}(0)$ can be determined from Eq. 1.87 as

$$m_0^{(k)}(0) = (-i)^k \kappa_k , \quad (A.4)$$

where
\[ \kappa_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2M-1} l^k h_{l+1} \]  

(A.5)

are readily available. Finally, substituting Eqs. A.3 and A.4 into Eq. A.1 yields

\[ \chi_m = \frac{1}{2^m - 1} \sum_{k=1}^{m} \binom{k}{m} \kappa_k \chi_{m-k} \]  

(A.6)

Thus, Eq. A.6 defines a recursive algorithm for evaluating the moments of the scale function.

**Appendix B. Determination of the Cross-Correlation Function of ARMA Processes**

Consider two ARMA processes, \( y_1(k) \) and \( y_2(k) \), that are specified by the transfer functions

\[ Q_1(z) = A_1^{-1}(z)B_1(z) \quad \text{and} \quad Q_2(z) = A_2^{-1}(z)B_2(z) , \]  

(B.1)

respectively, where

\[ A_r(z) = \sum_{k=0}^{m} a_k^r z^{-k} , \quad B_r(z) = \sum_{k=0}^{m} b_k^r z^{-k} , \quad r = 1, 2 , \]  

(B.2)

and \( a_0^r = I_n \). The cross-correlation function of these random processes can be evaluated by solving a system of linear equations. This system of equations can be assembled by multiplying the input-output relationship of equation 6.13 by the pertinent input and output quantities with subsequent averaging. However, the following procedure is more effi-
cient for numerical calculations and it has been used in the present study.

The transfer function of the ARMA system can be written using different order of multiplication of its AR and MA parts. That is,

\[ Q_r(z) = A_r^{-1}(z)B_r(z) = \tilde{B}_r(z)\tilde{A}_r^{-1}(z) \quad , \quad r = 1, 2, \text{ (B.3)} \]

where the parameters \( \tilde{a}_k^r, \ldots p_r, \) and \( \tilde{b}_k^r, 0, \ldots q_r \) can be found by solving the following equation

\[ A_r(z)\tilde{B}_r(z) = B_r(z)\tilde{A}_r(z) \quad , \quad r = 1, 2. \text{ (B.4)} \]

Then, the cross-correlation \( R_{12}(k) \) of the ARMA processes \( \chi_1(k) \) and \( \chi_2(k) \) can be determined by using the following equation

\[
R_{12}(k) = \int_{-\omega_n}^{\omega_n} Q_1^*(z)Q_2(z)z^k \, dz = \sum_{m = 0}^{q_1} \sum_{l = 0}^{q_2} \tilde{b}_l^1 \int_{-\omega_n}^{\omega_n} \tilde{A}_1^{-1}(z)\tilde{A}_2^{-1}(z)z^{-l} \, dz \, dz = \sum_{m = 0}^{q_1} \sum_{l = 0}^{q_2} \tilde{b}_m^1 \tilde{R}_{12}(k - l + m) \tilde{b}_l^2 , \text{ (B.5)}
\]

where \( \tilde{R}_{12}(k) \) is the cross-correlation function of two AR processes \( \tilde{\chi}_1(k) \) and \( \tilde{\chi}_2(k) \) with the following transfer functions

\[ Q_1(z) = \tilde{A}_1^{-1}(z) \quad \text{and} \quad Q_2(z) = \tilde{A}_2^{-1}(z) , \text{ (B.6)} \]

respectively. This cross-correlation can be found by using the equivalent state-space representation for the AR processes (Willems 1991)

\[ \psi_r(k) = F_r\psi_r(k - 1) + C_r w(k) \quad , \quad r = 1, 2, \text{ (B.7)} \]

where
\[ \nu_r(k) = \begin{bmatrix} \tilde{y}_r(k - p_r + 1) \\ \tilde{y}_r(k - p_r + 2) \\ \vdots \\ \tilde{y}_r(k) \end{bmatrix}, \quad F_r = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & I_n & \cdots & 0_n \\ 0_n & 0_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\tilde{a}_r^r & -\tilde{a}_r^{r-1} & -\tilde{a}_r^{r-2} & \cdots & -\tilde{a}_1^r \end{bmatrix}, \quad C_r = \begin{bmatrix} 0_n \\ 0_n \\ \vdots \\ I_n \end{bmatrix}. \] (B.8)

Multiplying Eq. B.7 by its transposition and taking mathematical expectation one can derive the following Lyapunov equation

\[ \tilde{R}_{12} = F_1 \tilde{R}_{12} F_2^T + 2 \omega_b C_1 C_2^T, \] (B.9)

where

\[ \tilde{R}_{12} = \begin{bmatrix} \tilde{R}_{12}(p_1 - p_2) & \cdots & \tilde{R}_{12}(p_1 - 1) \\ \vdots & \ddots & \vdots \\ \tilde{R}_{12}(1 - p_2) & \cdots & \tilde{R}_{12}(0) \end{bmatrix} \] (B.10)

is the Toeplitz matrix which contains the elements of the cross-correlation function of the AR systems.

Thus, calculation of the auto-correlation function of two ARMA systems can be performed based on equations B.5 and B.9.
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