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Does Chaos Matter in the Plasma Sheet?

by

Adam Usadi

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

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April, 1995
Abstract

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Can the average bulk flow of an ensemble of charged particles in Earth’s plasma sheet still be described by adiabatic theory even if the ensemble contains a significant number of particles executing non-adiabatic motion? This is part of a broader spectrum of questions which ask if chaotic microscopic processes can be parametrized as macroscopic ones when ensemble averaged. *Wolf and Pontius [1993]* have shown that at least for a simple 2D, tail-like magnetic field configuration, the average particle drift speed of an appropriately chosen ensemble of particles, including those executing chaotic motion, is given correctly by the simple adiabatic guiding-center drift formula. Here, we extend the proof to 2.5D magnetic fields (3 component, 2 spatial dependences) and include the effects of an electric field. The results of numerical test-particle simulations further show that the dispersion of particles about the mean drift speed tends to decrease due to the presence of chaotic particle scattering. Thus, we have shown that the standard way of representing particle transport in the inner magnetosphere, namely the isotropic pitch angle, bounce averaged drift formalism, is valid for the central plasma sheet despite the presence of non-adiabatic particle motion.
Acknowledgments

The Department of Space Physics and Astronomy has been a wonderful place to spend the last three and a half years. It is a privilege to have been a part of this energetic group. Dr. Richard Wolf has advised many students with his intelligent and compassionate style. I am most grateful for having had the opportunity to study under his guidance. Dr. Mike Heinemann, Dr. Anthony Chan, and Dr. Wendell Horton provided some key insights and suggestions at many points during this work. Dr. Hannes Voigt advised my first year of research and was always a source of interesting discussions. Dr. Jon Weisheit and Dr. Patricia Reiff were instrumental in helping me to come to the department. Dr. Weisheit and Dr. Freeman helped me obtain permission to teach a section of an undergraduate course, a project I have found most gratifying. Dr. Bob Spiro was a source of inspiration for many teaching methods and is the person with whom I most identify both philosophically and morally. Dr. Frank Toffoletto answered many of my computational and cooking questions. Dr. Reginald Dufour, Umbé and Maria made life much simpler for so many of us. I would also like to thank Dr. Huey Huang for participating on my committee and introducing me to negative temperatures in Houston.

I learned much from fellow graduate students who were both a source of inspiration, competition, and occasional vegetation. Hui Li, Michael Murrillo, Colin Law, Mauricio Ruiz-Reyes, C. Ben Boyle, and Vincent Kargatis made me think even when I didn’t want to.

As part of the Graduate Student Researchers Program, NASA’s Goddard Space Flight Center funded my last two years of research. I am grateful to Dr. Michael Hesse for his help in obtaining this fellowship and for many informative scientific discussions. NCSA and the Rice Center for Research in Parallel Computation provided computer time and helpful support.

My parents have always encouraged me and supported my decisions. I hope I will be as generous and loving with my children. My siblings, Karin, Eric, and Lauren and my fiancé Oanh, have been an inspiration and source of strength. Lastly, I would like to mention that even though Karin and Eric received their PhD’s before I did, I was the first to intonate such a plan. Honest!
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Preface

One possible title for this thesis could be one hundred ways to take an average. To avoid confusion, we list the meaning of our different averages. We note that these averages include a gyroperiod averaging which is not explicit in the notation.

<table>
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<tr>
<th>Type of Average</th>
<th>symbol</th>
<th>method</th>
</tr>
</thead>
<tbody>
<tr>
<td>ensemble averaged velocity</td>
<td>$\langle v \rangle_e$</td>
<td>$\frac{\int v f(x,v) dv dx}{\int dv dx}$</td>
</tr>
<tr>
<td>bounce average drift</td>
<td>$\langle v \rangle_b$</td>
<td>$\frac{1}{T_0} \int_0^{T_0} v dt$</td>
</tr>
<tr>
<td>time average drift</td>
<td>$\langle v \rangle_d$</td>
<td>$\frac{1}{T} \int_0^T v dt$</td>
</tr>
<tr>
<td>ensemble averaged drift</td>
<td>$\langle v \rangle_{d,e}$</td>
<td>$\frac{1}{T} \int_0^T \langle v \rangle_e dt$</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Primer on Earth’s Magnetosphere

In a vacuum, Earth’s magnetic field would appear roughly dipolar with its magnetic moment axis tilted by $11^\circ$ from the spin axis. However, the magnetic field is filled with a plasma and is distorted by external and internal currents. The entire region is collectively called the magnetosphere. The dynamic pressure of the solar wind compresses and stretches the magnetosphere so that it resembles a comet’s tail. Nightside of the Earth is a region called the magnetotail. The magnetotail extends out far beyond the moon’s orbit and contains several distinct regions. In one part of the magnetotail the stretched magnetic field forms a thin region of hot, isotropic plasma known as the plasma sheet (see figure 1.1). Average particle energies in the plasma sheet typically range from $300eV - 10keV$. The plasma density is very raredified by laboratory standards, $0.1 - 10/cm^3$. Magnetic field strength typically ranges from $1 - 30nT$ (nanoTeslas). By comparison, the field strength at Earth’s equatorial surface is about $30,000nT$.

Despite its low density and weak field, the plasma sheet is the primary particle population of Earth’s magnetosphere. It is the source of the particles that cause the aurora borealis, and it also drives the primary currents which flow in the interior of the magnetosphere. It is the center of the ‘substorm’, a phenomenon which is the primary natural disturbance in Earth’s magnetosphere and ionosphere. The plasma sheet has to be an important part of any model of large scale magnetospheric dynamics. This is true for purely theoretical models aimed at physical understanding and operational models aimed at representing and forecasting space weather conditions.

The interaction between the solar wind and Earth’s magnetosphere is complex. Both energy and momentum are transferred to the magnetosphere from the solar wind such that large scale internal convection exists. This leads to an average sunward motion in the inner plasma sheet. In addition, the magnetic field embedded in the solar wind gives rise to an electric field in the frame of the ‘stationary’ magnetosphere. This electric field is primarily in the duskward direction (out of page in figure 1.1).
Figure 1.1 Earth's magnetosphere is formed by the interaction of its plasma-filled magnetic field with the solar wind. Short arrows represent flow velocities. Sample solar wind ($B_{IMF}$) and magnetospheric magnetic field lines are also shown. (“Our Magnetosphere”, fused toner on paper, a generous gift from the Colin Law Collection, Houston, TX.)
A magnetospheric modeler hopes to use the simplest set of equations possible to accurately describe this magnetized plasma environment. In many regions, the single fluid picture of magnetohydrodynamics (MHD) is sufficient to accurately describe plasma bulk flow as well the changes in magnetic field. However, the MHD picture is invalid in the inner part of the magnetosphere, including the inner plasma sheet, because the particles do not behave as a single fluid. The next simplest formalism, which has traditionally been used for that region, involves representing the motion of particles by their adiabatic drift velocities (discussed below). Though more complicated than the single-fluid MHD, this formalism is still sufficiently compact computationally to allow self consistent calculations of large-scale electric fields and currents. Unfortunately, test-particle simulations performed by many groups in recent years, due in part to the availability of cheap computing power, have revealed that the adiabatic-drift equations represent a very inadequate picture of the motions of individual ions in the plasma sheet. This result has led many to conclude that adiabatic drift theory is useless for large-scale modeling of plasma sheet dynamics (e.g. Ashour-Abdalla et al. [1994]). Careful examination of this problem is the objective of this thesis.

1.2 Primer on Adiabatic Drift Motion

![Diagram](image)

**Figure 1.2** Gyroperiod averaged guiding center motion.
The motion of a non-relativistic charged particle in an electromagnetic field is exactly described by the Lorentz force equation.

\[
\begin{align*}
\frac{d^2 \mathbf{r}}{dt^2} &= \frac{q}{m} \left( \mathbf{E}[\mathbf{r}(t), t] + \frac{d\mathbf{r}}{dt} \times \mathbf{B}[\mathbf{r}(t), t] \right)
\end{align*}
\]  

(1.1)

The magnetic field is a function of particle position, \( \mathbf{r} \). Hence, if the electromagnetic field varies rapidly, these equations become highly non-linear. When the magnetic field varies slowly w.r.t a gyration period, the high frequency gyration motion of a particle may be averaged out. The remaining particle motion may be described as a gyroperiod averaged motion perpendicular to the magnetic field and a parallel guiding center motion.

\[
\begin{align*}
\frac{d\mathbf{R}_\perp}{dt} &\approx \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mu}{q} \frac{\mathbf{B} \times \nabla B}{B^2} + \frac{v_{\parallel}^2 \mathbf{b} \times \vec{\kappa}_c}{\Omega} \\
\frac{dv_{\parallel}}{dt} &\approx -\frac{\mu}{m} (\mathbf{b} \cdot \nabla) B + \frac{q}{m} E_{\parallel}
\end{align*}
\]  

(1.2) (1.3)

where \( \mathbf{r} \equiv \mathbf{R} + \mathbf{\rho} \),

\[
\begin{align*}
\mathbf{b} &\equiv \frac{\mathbf{B}}{B} \\
v_{\parallel} &\equiv \mathbf{b} \cdot \mathbf{v}
\end{align*}
\]

magnetic field curvature \( \vec{\kappa}_c \equiv (\mathbf{b} \cdot \nabla) \mathbf{b} = \frac{\mathbf{R}_c}{R_c^2} \)

gyro frequency \( \Omega \equiv \frac{qB}{m} \)

1st adiabatic invariant \( \mu = \frac{m v_{\parallel}^2}{2B} \)

gyro radius \( \mathbf{\rho} = \frac{1}{\Omega} \mathbf{b} \times \left( \mathbf{v} - \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right) \)

where \( R_c \) is the radius of curvature. The first adiabatic invariant, \( \mu \), is also called the magnetic moment. \( \mu \) is conserved to high accuracy if the field experienced by the particle changes very little in one cyclotron period.

These equations may be derived from equation 1.1 by Taylor expanding the fields about the guiding center position, \( \mathbf{R} \), and averaging all terms over one gyration cycle. (For example, see Northrop [1963] or Roederer [1970].) The gyroperiod averaged motion due to the first, second, and third terms of equation 1.2, shown in figure 1.3, are called the \( \mathbf{E} \times \mathbf{B} \), gradient, and curvature 'drifts' respectively. The
Figure 1.3  Motion of a positive particle in a magnetic field with a A) gradient, \( v_\perp = \frac{u}{q} \frac{B_\perp \nabla B}{B^2} \), B) curvature \( v_\perp = \frac{v_\perp^2}{B} \hat{b} \times \kappa_c \), and C) an external force perpendicular to the magnetic field, \( v_\perp = \frac{1}{q} \frac{\mathbf{F}_{ext} \times \mathbf{B}}{B^2} \). Notice that when \( \mathbf{F}_{ext} = q\mathbf{E} \) the factors of \( q \) in this expression cancel. We’re left with \( v_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \), which is independent of charge. Thus, an electric field applied to an electrically neutral plasma produces no current, whereas other forces give rise to currents.
validity of these guiding center equations is determined by the ordering parameter 
\[ \epsilon = \frac{\rho \text{(gyroradius)}}{\text{scale lengths of } E, B} \]. If \( \epsilon \geq 1 \), then these equations are invalid and the individual particle motion is non-adiabatic. These equations also neglect acceleration drifts which arise when a particle moves in a time varying electric field or when the \( E \times B \) drift speed changes due to a temporal change in magnetic field. We will show in section 5.3 specific limitations on the electric field strength which arise from this assumption.

In modeling the behavior of a plasma system, one usually wants to know the average drift of a large number of particles. One can analytically model the flux of an ensemble of particles in the magnetotail by imposing some additional assumptions about the nature of individual particle motion there. Particles in Earth’s inner magnetosphere bounce within a magnetic flux tube between the southern and northern hemispheres just as if they were in a large a magnetic bottle (see figure 1.4). We may average particle motion over this additional bounce period. In such a system, there exists an additional adiabatic invariant which is the particle’s parallel momentum integrated over a bounce path,

\[ J = \int P_\parallel ds \]  \hspace{1cm} (1.4)

The second invariant, \( J \), is conserved to high accuracy if the magnetic field, as a function of distance along the bounce path, \( B(s) \), varies only slightly from one bounce to the next.

---

**Figure 1.4** A particle bouncing adiabatically between the northern and southern lobes of a parabolic shaped magnetic field similar to portions of Earth’s magnetotail. The pitch angle, \( \alpha \), is the angle between the particle’s velocity and magnetic field direction. The mirror point, \( B_m = \frac{E_h}{\mu} \), is symmetric in the top and bottom halves of this figure only if \( \mu \) is conserved.
A particle’s pitch angle, \( \alpha \), is defined as the angle between its velocity and the local magnetic field so that \( v_\perp = v_{th} \sin(\alpha) \), where \( v_{th} \) is the particle’s thermal velocity. A particle’s pitch angle is related to \( \mu \) via \( \mu = (1/2)m(v_{th} \sin(\alpha))^2/B \). An adiabatic particle’s pitch angle varies as the magnetic field strength changes in order to keep \( \mu \) constant. When a particle’s motion becomes non-adiabatic, \( \mu \) is no longer conserved. The pitch angle is said to be scattered.

It is possible to combine the gradient and curvature drift terms of equation 1.2. A particularly useful form of this combined drift equation is averaged over the bounce time of a particle’s motion. The bounce averaged gradient-curvature drift equation can be derived by direct integration of equation 1.2 as in Wolf [1995] or Appendix II of Roederer [1970], or from energy principles as applied to an isotropic distribution as in Wolf [1983]. In these derivations, one finds that the gradient-curvature drift may be written in terms of the spatial gradient of the particle’s kinetic energy, \( E_k \). The spatial gradient of \( E_k \) refers to the gradient of particle energies for a given \( \mu \) and \( J \) within a magnetic field model.

\[
\langle v_\perp \rangle_b = \langle v_{GC} \rangle + \langle v_{E \times B} \rangle
\]

\[
= \frac{B \times \nabla [E_k + q\Phi]}{qB^2}
\]

This tells us the drift of a particle, perpendicular to the magnetic field, averaged over a bounce path. For instance, if you keep track of a particle as it crosses the equatorial \((x - y)\) plane, the net drift in this plane due to gradient, curvature, and \( E \times B \) effects is described by this equation. Thus, by averaging over a bounce period, one greatly reduces the amount of information necessary to track a particle’s motion in the magnetotail.

Furthermore, we may describe, with a simple equation, the average motion of an ensemble of particles for some special cases. If the distribution of particles along a field line is isotropic, then we may use thermodynamical arguments \((E_k \propto \mathcal{V}^{(1-\gamma)})\) to transform the gradient-curvature part of the drift to,

\[
\langle v_{GC} \rangle_{b,c} = E_k \mathcal{V}^{2/3} \frac{B \times \nabla \mathcal{V}^{-2/3}}{qB^2}
\]

where the flux tube volume, \( \mathcal{V} \equiv \int \frac{ds}{B(s)} \), and \( E_k \mathcal{V}^{2/3} \) is a unit of magnetic flux (See Wolf [1983]). We emphasize that these traditional derivations of 1.7 assume that all particles move adiabatically, i.e.,

\[
|\rho| << R_c + \left| \frac{B}{\nabla B} \right|
\]
This assumption was thought to be critical for two reasons. It is necessary in order to derive equation 1.2 on which equation 1.6 is based. In addition, in a region where the B-field scale lengths are comparable to a gyroradius, particles drift systematically at a speed comparable to the thermal speed, and the off-diagonal terms in the pressure tensor become large (Ashour-Abdalla et al. [1994]), hence invalidating the derivation of equation 1.7 from equation 1.6.

Relating energy, $\mu$, and $J$

One may write an adiabatic particle's energy as a function of its adiabatic invariants, $\mu$ and $J$, as well as a magnetic field line label. The particle's kinetic energy in terms of relevant quantities is:

$$E_k(\bar{x}, \bar{\nu}) = \frac{1}{2} m v_\perp^2 + \frac{1}{2} m v_{||}^2$$  \hspace{1cm} (1.9)

$$= \mu B + \frac{P_{||}^2}{2m}$$  \hspace{1cm} (1.10)

Therefore

$$P_{||} = \sqrt{2m} \sqrt{E_k - \mu B}$$  \hspace{1cm} (1.11)

$$J(\alpha, \beta, \mu, E_k) = \sqrt{2m} \int \sqrt{E_k - \mu B} ds$$  \hspace{1cm} (1.12)

where $\alpha$ and $\beta$ are field line labels which could be Euler potentials. For a given magnetic field configuration, equation 1.12 allows calculation of $J$ for a given field line (labeled $\alpha$ and $\beta$) and given values of kinetic energy and first invariant, $\mu$. Equation 1.12 can sometimes be inverted numerically to determine the function $E_k(\alpha, \beta, \mu, J)$. This is done explicitly in chapter 4 with a chosen magnetic field model. We emphasize that writing particle energy as a function of $\mu$ and $J$ is meaningless when particle motion is non-adiabatic.

1.2.1 MHD and Adiabatic Drift

Though MHD is not the subject of this thesis, it is worth noting why the MHD approximation is not valid in certain regions of the magnetosphere and why we must turn to drift theory. Ideal MHD assumes that a plasma is perfectly conducting so that $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$, which implies that in the presence of an electric field, the component of the fluid velocity perpendicular to $\mathbf{B}$ is given by $\mathbf{u} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$. This equation may be interpreted as a condition of infinite conductivity. Magnetic flux and plasma are 'frozen' together. If other particle drifts become large, such as the
gradient or curvature drift, then this relationship must be modified (see, for instance, *Wolf Plasma 1 notes*).

\[
E_\perp + u \times B = \frac{1}{\rho} \left[ \sum_i n_i m_i (v_{GC} + v_{\text{other drifts}_i}) \right] \times B
\]  

(1.13)

In order for the 'frozen-in-flux' condition to hold, the drift velocities on the r.h.s. of this equation must be small compared to the \( E \times B \) drift velocity. In other words, MHD breaks down when these drift velocities get large, as they do in regions of large magnetic field gradients and curvature.

### 1.3 Primer on Chaos/Non-adiabatic Motion

A particle's motion is said to be chaotic if its trajectory is highly sensitive to initial conditions. Two particles which begin their motion close to each other in phase space end up in very different places. The exponential divergence rate of these two particles can be characterized by the Lyapunov exponent, \( \sigma \). Following *Lichtenberg and Lieberman* [1992], let the trajectory of one particle be \( x \) while the nearby one is \( x + \Delta x \).

\[
\sigma(x, \Delta x) = \lim_{t \to \infty} \frac{1}{t} \ln \left( \frac{d(t)}{d(0)} \right)
\]  

(1.14)

where \( d(t) \equiv ||\Delta x|| \). The value of \( \sigma \) is often used to characterize the stochasticity of a near integrable system. However, we're not really interested in chaos itself, but rather in determining ensemble averaged quantities of a system which may contain chaotic solutions. For this purpose, a simpler tool of analysis exists.

#### 1.3.1 The \( \kappa \) Parameter

A \( x-z \) projection of Earth's central plasma sheet magnetic field is roughly parabolic in shape, resembling figure 1.4. Oppositely directed magnetic field lines of the 'lobe' are separated by a neutral sheet at the equatorial crossing region \((z = 0)\). Northward and southward of this region are the 'lobe' portions of the central plasma sheet magnetic field lines. This is not technically correct nomenclature since the 'lobe' specifically refers to the region of the magnetosphere outside of the central plasma sheet with depleted plasma density. We never refer to the true 'lobes' so there shouldn't be any confusion.
Figure 1.5 Three types of particle motion can exist in a parabolic field. Shown here are typical particle trajectories in 3D and projected onto the $X-Z$ and $X-Y$ planes. Whether a particle trajectory is Speiser, chaotic, or integrable depends on its initial conditions as well on the magnetic field.
The magnetotail field shape tends to give rise to three different types of particle motion as shown in figure 1.5. The magnetic field shown here is defined in chapter 5. Different parameters and/or initial conditions lead to different types of orbits. A particle executing a ‘Speiser orbit’ spirals in from the plasma sheet lobe towards the equatorial crossing and is temporarily trapped within a small $\Delta z$ region. The particle is said to ‘meander’ through this region, repeatedly crossing $z = 0$ as the magnetic field turns the particle back towards the neutral sheet. It eventually escapes the neutral sheet, spiraling outward towards the plasma sheet lobes once again (figure 1.5A). A chaotic particle is trapped in the neutral sheet region for an extended period of time and is less regular in its motion (figure 1.5B). An integral orbit exhibits regular motion consistent with the existence of an additional constant of motion (figure 1.5C). Our traditional adiabatic motion is one sort of integrable trajectory.

Were the magnetic field sufficiently straight, the particle’s motion would be completely adiabatic and hence integrable. Indeed, away from the equatorial region, particle motion may be adiabatic. Particle trajectories are categorized by their motion throughout a complete bounce cycle. Local adiabaticity may often be parametrized by examining the ratio of the magnetic field’s radius of curvature, $R_c$, to the particle’s gyroradius, $\rho$. As this ratio goes from $+\infty \to 1$, the particle motion becomes less and less adiabatic. As the ratio decreases even further from $1 \to 0$, particles may continue to exhibit chaotic motion or they may recover a constant of motion and become ‘quasi-adiabatic’. The ratio of maximum gyroradius to minimum radius of curvature serves as a useful parameter to characterize the bounce averaged motion of a particle. For Earth’s magnetotail, this occurs at the equatorial plane of the plasma sheet where the magnetic field is weakest and curvature greatest. $\kappa$ may be changed by changing the energy of the particles or by changing some other field parameters.

\[ \kappa \equiv \sqrt{\frac{R_{\text{min}}}{\rho_{\text{max}}}} \]  

(1.15)

Do not confuse this $\kappa$ with the curvature parameter $|\kappa_c| = |\frac{1}{R_c}|$. $\kappa$ is simpler to use than a Lyapunov exponent, but is less diagnostic since a given particle with a chosen $\kappa$ value may execute any of the above described motions. $\kappa$ does not contain any information about the $x$ gradient of the magnetic field. The type of orbit a particle executes depends on its initial conditions as well magnetic field configuration. This will become clearer when we produce a Poincaré surface of section plot in the next section. It is worth noting that our adiabatic condition (1.8) may be written in terms
of $\kappa$ such that $1 \ll \kappa^2 + B/(\rho |\nabla B|)$. If the scale length of the magnetic field gradient is large, then $\kappa \gg 1$ implies that particles conserve their adiabatic invariants to a high degree of accuracy.

1.3.2 Poincaré Surface of Section

One may use a Poincaré surface of section (SOS) plot to analyze low-dimensional dynamical systems and obtain a geometrical description of the solutions. Particle trajectories are just solutions to Newton's laws. SOS plots can show what class of solutions to a set of differential equations may exist based on initial conditions. It provides much more information than the single $\kappa$ parameter can. We thus briefly describe how one may produce and use a SOS plot. We then apply the technique to particle motion in a magnetic field such as that which we just described.

![Diagram](image)

**Figure 1.6** A general example of how to obtain a Poincaré surface of section plot (after Lichtenberg and Lieberman [1992]). The trajectory of a constant energy particle in 4D phase space is shown with the curved arrow. We may reduce the dimensionality of information to easily visualize types of solutions to O.D.E's.

Consider the motion of a particle in 2D real space. The trajectory of this particle in 4D phase space is depicted by the arrow path of figure 1.6. If energy is conserved, the trajectory lies on a 3D energy surface $H(p_1, p_2, q_1, q_2) = H_0$, so that one of the variables is dependent on the other three. Say $p_2 = p_2(p_1, q_1, q_2)$. We may then consider only the projection of this trajectory onto a 3D volume $(p_1, q_1, q_2)$. We then choose some $q_2$-plane which the trajectory will repeatedly cross. We will thus have a
collection of of \((p_1, q_1)\) points which will, in general, fall in some bounded region of the plane. If there exists an additional constant of the motion, \(I(p_1, q_1, q_2) = I_0\), we may derive a relationship for \(p_1 = p_1(q_1, q_2)\). Successive crossings of the motion with the SOS will lie on a unique curve called a KAM curve after Kolmogorov, Arnold, and Moser. In this way, we may show the existence of invariants graphically. KAM curves do not appear without the existence of such local invariants.

We may apply this theory to the motion of particles in our parabolic field as shown in figure 1.5. As particles cross the \(z = 0\) plane going from \(-z\) to \(+z\) we record \(x\) and \(v_x\). The Poincaré surface of section (SOS) plot of \(v_x\) versus \(x\) for one hundred particles is shown in figure 1.7. Two different \(\kappa\) cases are shown. \(\kappa\) was varied here by varying the ratio of two magnetic field parameters, \(B_a/B_0\). This will be described later. The majority of particles are chaotic and would completely fill the main region were the simulation run long enough. The KAM curves reveal the existence of integral orbits for this magnetic field model. A typical integrable trajectory leading to the creation of the a KAM curve of figure 1.7B is shown in figure 1.5C.

This SOS plot is somewhat different from that shown in Chen [1992]. It was produced by following the motion of particles in an \(x\) dependent parabolic field. Chaotic and Speiser orbits overlap and are indistinguishable in the \(\kappa = 0.207\) case. All particles eventually bounce somewhere in the ‘lobe’ of the central plasma sheet and return to the equatorial plane. No particles are lost. Notice that the integrable region is larger for the \(\kappa = 0.104\) case. In addition, if you look closely at the top figure, you will see a darkened area surrounding the KAM curve region indicating a boundary region, possibly the overlap of regions of chaotic and Speiser particles.

### 1.4 A Statement of The Problem

Many particles in the plasma sheet do not execute adiabatic motion. Nevertheless, we pose the question: Are the adiabatic drift equations still useful? This has become a very important question for magnetospheric physics. In extensive test-particle simulations of magnetotail ions in an observation-based magnetic-field model, Ashour-Abdalla et al. [1992,1993] found that most ions effectively hit a “wall” in the inner plasma sheet where they drift rapidly westward around the Earth, avoiding the inner magnetosphere. This work raises the possibility that chaotic motion might allow ions to undergo much faster cross-tail motion than would be expected from adiabatic-drift theory and that consequently, chaos might be a powerful loss mechanism for plasma
Figure 1.7  This Poincaré SOS plot shows the value of $v_x$ versus $x$ for particles as they cross the $z = 0$ plane going from $-z$ to $+z$ in an $x$ dependent magnetotail type magnetic field. The squares indicate the initial conditions for 100 particles as they were launched from $(x, 0, 0)$ and random pitch angles. A) $\kappa = 0.104$, $\frac{B_n}{B_0} = 0.05$. B) $\kappa = 0.207$, $\frac{B_n}{B_0} = 0.1$. 
sheet particles. If correct, this conclusion would have important consequences for quantitative magnetospheric modeling. It would mean that MHD codes, (e.g., Fedder et al. [1991], Hesse and Birn [1992]), which neglect all the terms on the r.h.s. of equation 1.13, are even more incorrect than previously thought. Even convection-model calculations, (e.g., Spiro and Wolf [1983]), which include gradient/curvature drift and magnetization terms from equation 1.13, would have to be modified to include chaotic-orbit effects if those theoretical techniques are to provide a realistic picture of the plasma sheet.

The conclusion of this thesis is that equation 1.7 is still a valid description for the average drift of an ensemble of particles even if many of those particles are non-adiabatic. This thesis is organized as follows:

1. We present a generalized derivation of the ensemble averaged drift velocity (eq. 1.7) which includes the effects of chaotic particle motion (chapter 2).

2. We derive the explicit form of \( (v)_{d,e} \) for a chosen magnetic field model (chapter 3).

3. We derive the adiabatic particle bounce averaged drift velocities (eq. 1.6) in this magnetic field (chapter 4).

4. We compare the results of a numerical particle tracing program (5) to the predictions of adiabatic theory, show that the drift motion of an ensemble of isotropic particles is well described by bounced averaged adiabatic theory, and show that the dispersion about the mean velocity is somewhat reduced by the presence of chaotic motion (chapter 6). We thus find that the accuracy of equation 1.7 is enhanced by the presence of chaotic motion.
Chapter 2

A Generalized Derivation of Ensemble Averaged Drift

In 1993, Wolf and Pontius presented a new derivation of the bounce averaged gradient curvature drift formula (1.7), hereafter referred to as the WP proof. The validity of the WP proof was limited to particles moving in 2D magnetic fields with no electric field (e.g. \( \mathbf{B}(x, z) = (B_x, 0, B_z) \)). We first show how the WP proof may be easily extended to include a \( B_y \) component. Then, we analytically derive a new form of the bounce averaged drift formula to include the effects of a perpendicular electric field as first shown by Heinemann [1994].

It is important to understand the connection between \( \langle v_y \rangle_{\delta, c} \), the bounce averaged ensemble drift speed at a given time and \( \langle v_y \rangle_{c}(y) \), the ensemble averaged velocity at a chosen \( y \). The WP proof relied on the assumption that these two quantities are identically equal. We show analytically that this is true within the limitations of their proof. Analytic extension of this equality to our generalized case is non-trivial. Numerical simulations, discussed in chapter 6, serve to bridge this important gap in the new proof.

A word of caution: Parts of section 2.1 assume the reader is familiar with the WP proof. Do not despair. Section 2.2 is a generalized version of the original WP proof and relies on many of the same arguments. We only wish to avoid duplication within this thesis. We only wish to avoid duplication within this thesis.

2.1 The WP Proof including \( B_y \)

A 2D magnetic field may be described entirely by one component of the magnetic vector potential. For instance, any magnetic field configuration that has only \( x \) and \( z \) components may be described by \( \mathbf{B} = \nabla \times \mathbf{A} = \nabla A_y(x, z) \times \hat{y} \). (i.e. Euler potentials \( (\alpha, \beta) = (A_y, y) \)). We may add a simple \( B_y \) component to this magnetic field without changing the \( x-z \) projection by using a more general vector potential \( \mathbf{A} = (A_x, A_y, 0) \) so that

\[
\mathbf{B} = -\frac{\partial A_y(x, z)}{\partial z} \hat{x} + \frac{\partial A_x(x, z)}{\partial z} \hat{y} + \frac{\partial A_y(x, z)}{\partial x} \hat{z}
\]  

(2.1)
The Lagrangian for a non-relativistic particle in an electromagnetic field is given by

\[ L = \frac{1}{2}mv^2 - q\Phi + qA \cdot v \]  

(2.2)

\[ = \frac{1}{2}mv^2 - q\Phi + q[A_x(x, z)v_x + A_y(x, z)v_y] \]  

(2.3)

And the canonical momentum is defined by

\[ P_i = mv_i + qA_i \]  

(2.4)

As long as \( A_x, A_y, \) and \( \Phi, \) are not explicit functions of \( y, \) the momentum conjugate to \( y \) is still a constant of the motion since,

\[ \frac{dP_y}{dt} = \frac{\partial L}{\partial y} \]  

(2.5)

The WP derivation of the bounce averaged gradient-curvature drift remains valid with a few subtle modifications. Because there is no \( y \) dependence in the magnetic field, one can continue to describe the projection of the magnetic field model in the \( x-z \) plane by \( A_y \) alone, as can be seen in equation 2.1. Integration of the distribution function over \( x-z \) space may be performed as before with minor clarification. In particular, the spatial differentials may be rotated to lie along and perpendicular to the magnetic field line (see figure 2.1).

\[ dxdz = ds_{xx}dx_{x} = ds_{zz} \frac{dA_y}{|\nabla_{xz}A_y|} = \frac{ds_{xx}}{B_{zx}} dA_y \]  

(2.6)

where we used the fact that

\[ \frac{ds}{B} = \frac{ds_{xx}}{B_{zx}} = \frac{dz}{B_z} \]  

(2.7)

This follows from the definition of a vector field line \( ds \times B = 0, \) which implies that:

\[ (dyB_z - dzB_y)\dot{x} + (dzB_x - dxB_z)\dot{y} + (dxB_y - dyB_x)\dot{z} = 0 \]

or

\[ \frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \]

Therefore

\[ \frac{ds}{B} = \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(B_x)^2 + (B_y)^2 + (B_z)^2}} \]

\[ = \frac{dz}{B_z} \sqrt{\frac{(\frac{dx}{dz})^2 + (\frac{dy}{dz})^2 + 1}{(\frac{B_x}{B_z})^2 + (\frac{B_y}{B_z})^2 + 1}} = \frac{dz}{B_z} \]
Figure 2.1} One can use a transformation to rotate the differentials, $dx\,dz$, to differentials along and perpendicular to the magnetic field lines, $dS_{xz} \, dx_{\perp}$. The field lines labeled $A_{y\pm}$ bound particle motion. Particle guiding centers remain on the central field line, $A_{y0}$.

Similarly,

$$\frac{dS_{xz}}{B_{xz}} = \frac{\sqrt{(dx)^2 + (dz)^2}}{\sqrt{(B_x)^2 + (B_z)^2}} = \frac{dz}{B_z}$$

Thus, the flux tube volume is the same whether it is computed by its 3D definition, $V \equiv \frac{dV}{B}$, its 2D projection, $\oint \frac{dS_{xz}}{B_{xz}}$, or its 1D projection, $\oint \frac{dz}{B_z}$. Consequently, the WP derivation of equation 1.7 can be trivially generalized to the case where $B_y \neq 0$. However, the assumption that $\frac{dB}{dy} = 0$ cannot be easily removed.

### 2.2 Adding an Electric Field

We will now derive a formula for the average velocity of an ensemble of particles at a given $y$. This derivation draws heavily from the work of Heinemann, [1994]. For simplicity, we consider a constant, curl free electric field of the form,

$$\Phi = -E_{y0}y - E_{z0}z$$  \hspace{1cm} (2.8)

$$\mathbf{E} = E_{y0}\mathbf{j} + E_{z0}\mathbf{k}$$  \hspace{1cm} (2.9)

so that, from (2.5),

$$\frac{dP_y}{dt} = -qE_{y0}$$  \hspace{1cm} (2.10)
Our two constants of the motion then become the total particle energy,
\[ \mathcal{E} = \frac{1}{2} m(v_x^2 + v_y^2) - q(E_{y0}y + E_{z0}z) \]  
(2.11)
and the extended $y$ canonical momentum,
\[ \mathcal{P} = mv_y + qA_y - qE_{y0}t \]  
(2.12)

We seek a form for the average velocity in the $y$ direction of an ensemble of particles with a distribution which is a function of only these constants of motion. This form guarantees that the distribution function will be constant along a particle trajectory and thus satisfies the Vlasov equation.

\[ \langle v_y \rangle_e(y) = \frac{\int F(\mathcal{E}, \mathcal{P})v_y d^3v dx dz}{\int F(\mathcal{E}, \mathcal{P})d^3v dx dz} \]  
(2.13)
The distribution function, $F(\mathcal{E}, \mathcal{P})$, is not an explicit function of $y$. However, $\langle v_y \rangle_e$ is $y$ dependent because particle potential energy is a function of $y$. $\langle v_y \rangle_e$ will give us the average speed at a given $y$.

### 2.2.1 Choosing a Distribution Function

Just as for the WP proof, we must choose a distribution of particles which, when projected into the $x - z$ plane, is a function of only the constants of motion. This means that the particles must be distributed uniformly throughout the phase space accessible to them.

\[ F(\mathcal{E}, \mathcal{P}) \equiv F_0 \delta(\mathcal{E} - \mathcal{E}_0)\delta(\mathcal{P} - \mathcal{P}_0) \]  
(2.14)
This describes a cone of vectors in extended phases space about the extended $y$ canonical momentum. The benefit of using this rather odd distribution function is that it allows us to easily compute $\langle v_y \rangle_e$ for a limited flux tube of monoenergetic particles. It is one slice of a fully isotropic distribution function. To obtain a realistic particle distribution one can integrate our results over the desired energy or $\mathcal{P}$ range.

The requirement that all particles share the same value of $\mathcal{P}_0$ is equivalent to choosing a bound on the flux tube width in which a particle may move. Does this distribution function yield a particle density function which is invariant in the $x - z$ plane as we would like? Let's integrate over velocity space to find out.

\[ N(x) = \int F(x, v) d^3v = \int F(\mathcal{E}, \mathcal{P})2\pi v_y dv_y \frac{1}{2} dv_x^2 \]  
(2.15)
\[ = \pi F_0 \int \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P} - \mathcal{P}_0) dv_y^2 dv_y \]  
(2.16)
Figure 2.2 Our distribution function is one slice of a fully isotropic one. It is a $\delta$ function in energy and canonical $y$ momentum.

To transform the velocity space differentials to energy and momentum differentials, use the Jacobian as follows,

$$d\mathcal{E}d\mathcal{P} = \left| \frac{\partial (\mathcal{E}, \mathcal{P})}{\partial (v_x^2, v_y)} \right| dv_x^2 dv_y$$

$$= \left( \frac{\partial \mathcal{E}}{\partial v_x^2} \frac{\partial \mathcal{P}}{\partial v_y} - \frac{\partial \mathcal{E}}{\partial v_y} \frac{\partial \mathcal{P}}{\partial v_x^2} \right) dv_x^2 dv_y$$

$$= \frac{1}{2} m^2 dv_x^2 dv_y$$

Therefore,

$$N(x) = 2\pi \frac{F_0}{m^2} \int \delta (\mathcal{E} - \mathcal{E}_0) \delta (\mathcal{P} - \mathcal{P}_0) d\mathcal{E}d\mathcal{P}$$

$$N(x) = \begin{cases} 2\pi \frac{F_0}{m^2}, & \text{for } x \text{ such that } \mathcal{P} = \mathcal{P}_0, \mathcal{E} = \mathcal{E}_0 \text{ is possible} \\ 0, & \text{otherwise} \end{cases}$$

The particle density projected into the $x-z$ plane is constant within the chosen flux tube. No particles exist outside of our flux tube bound by $A_{y\pm}$ (see figure 2.1).

2.2.2 Denominator

Let’s take a look at the denominator of the fraction in 2.13. By the definition of a distribution function this integral must be equal to the total number of particles per unit $y$. This is almost identical to the integration we just performed.

$$\mathcal{D} \equiv \int F(\mathcal{E}, \mathcal{P}) d^3v dx dz = \pi \int F(\mathcal{E}, \mathcal{P}) dv_x^2 dv_y dx dz$$
The velocity differentials are once again transformed to energy and momentum space. The spatial differentials may also be transformed according to equation 2.6. Substituting in our distribution function (2.14) the denominator then becomes,

\[ D = 2\pi \frac{F_0}{m^2} \int \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P} - \mathcal{P}_0) d\mathcal{E} d\mathcal{P} \frac{ds}{B} d\mathcal{A}_y = 2\pi \frac{F_0}{m^2} \int \frac{ds}{B} d\mathcal{A}_y \]  

(2.23)

where the integral of $s$ is extended over the whole flux tube and the integral over $A_y$ includes all values consistent with $\mathcal{E} = \mathcal{E}_0$, $\mathcal{P} = \mathcal{P}_0$. The details of the limits of this integration are discussed in section 2.2.4.

### 2.2.3 Numerator

The numerator,

\[ \mathcal{N} \equiv \int F(\mathcal{E}, \mathcal{P}) v_y d^3v dxdz \]  

(2.24)

We substitute the distribution function into this equation and transform the differentials as before to obtain

\[ \mathcal{N} = 2\pi \frac{F_0}{m^2} \int \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P} - \mathcal{P}_0) v_y d\mathcal{E} d\mathcal{P} \frac{ds}{B} d\mathcal{A}_y \]  

(2.25)

This equation may be simplified by substituting in a functional form for $v_y$ obtained by rearranging equation 2.12, i.e. $v_y = (\mathcal{P} - qA_y + qE_y t)/m$. We also integrate over $\mathcal{E}$ to obtain:

\[ \mathcal{N} = 2\pi \frac{F_0}{m^2} \int \delta(\mathcal{P} - \mathcal{P}_0) \frac{1}{m} (\mathcal{P} - qA_y + qE_y t) d\mathcal{P} \frac{ds}{B} d\mathcal{A}_y \]  

(2.26)

\[ = 2\pi \frac{F_0}{m^3} \int (\mathcal{P}_0 - qA_y + qE_y t) \frac{ds}{B} d\mathcal{A}_y \]  

(2.27)

$\mathcal{P}_0$ is the transformed $y$ canonical momentum for all the particles with a given energy within a chosen flux tube. The flux tube is defined by the central field line, $A_{y_0}$, on which all particles’ guiding centers are moving (see figure 2.1). With the cross tail electric field, the entire flux tube moves down the $x$ axis with a velocity of $u = \frac{E_x B}{B^2}$. $\mathcal{P}_0$ is defined by equation 2.12 with $v_{y_0} = 0$ as

\[ \mathcal{P}_0 = qA_{y_0}(t) - E_y t \]  

(2.28)

so that expression 2.27 becomes

\[ \mathcal{N} = 2\pi \frac{F_0}{m^3} \int \{qA_{y_0} - qE_y t - qA_y + qE_y t\} \frac{ds}{B} d\mathcal{A}_y \]  

(2.29)

\[ = 2\pi \frac{F_0}{m^3} \int -q\{A_y - A_{y_0}(t)\} \frac{ds}{B} d\mathcal{A}_y \]  

(2.30)
With $V \equiv \int \frac{dv}{B}$, we obtain the still exact formula,

$$\langle v_y \rangle_e = -\frac{q}{m} \frac{\int [A_y - A_{y0}(t)] V(A_y) dA_y}{\int V(A_y) dA_y}$$  \hspace{1cm} (2.31)

### 2.2.4 Limits of Integration

The limits of integration are defined by the field lines which bound a particle with all of its kinetic energy in the perpendicular direction (see figure 2.1). By definition, this occurs at a particle’s bounce point where $E_K = \frac{1}{2} m v_y^2$ and $|v_{ymax}| = \sqrt{\frac{2E_K}{m}}$. Once again using an inversion of equation 2.12, we derive the integration limits of $A_y$.

$$qA_{y\pm}(t) = P_0 \pm m v_{ymax} + qE_y t$$  \hspace{1cm} (2.32)

which with equation 2.28 becomes,

$$qA_{y\pm}(t) = qA_{y0}(t) - qE_y t \pm m \sqrt{\frac{2E_K}{m}} + qE_y t$$  \hspace{1cm} (2.34)

$$A_{y\pm}(t) = A_{y0}(t) \pm \frac{m}{q} \sqrt{\frac{2E_K}{m}}$$  \hspace{1cm} (2.35)

This makes physical sense as it is intuitively the same as for the case when no electric field exists. The flux tube moves so as to transform out the electric field and the tube widens in $A_y$ space as the kinetic energy of the particles increases.

$$\langle v_y \rangle_e|_{exact} = -\frac{q}{m} \frac{\int_{A_{y-0}(t)}^{A_{y+(t)}} [A_y - A_{y0}(t)] V(A_y) dA_y}{\int_{A_{y-0}(t)}^{A_{y+(t)}} V(A_y) dA_y}$$  \hspace{1cm} (2.36)

This formula is still exact.

### 2.2.5 Taylor Approximation

A simplified, but approximate form of this formula may be obtained if one assumes that a particle’s gyroradius is small compared to the scale length of the flux tube gradients. An explicit Taylor expansion of both the numerator and denominator is performed in Appendix A. After truncating the series after the first term, we obtain

$$\langle v_y \rangle_e|_{max} \approx -\frac{2}{3} \frac{E_K}{q} \left. \frac{1}{V(A_{y0}(t))} \frac{dV(A_y)}{dA_y} \right|_{A_y = A_{y0}(t)}$$  \hspace{1cm} (2.37)
This form is equivalent to the adiabatic approximation of the gradient curvature drift formula, 1.7. We conclude that it is possible to derive equations equivalent to the bounce-averaged adiabatic theory without having to assume that every particle moves adiabatically. The new limit of applicability is now defined,

$$\rho \left| \frac{\nabla \mathcal{V}}{\mathcal{V}} \right| \ll 1$$ (2.38)

This is much less restrictive than the old limit defined by (1.8) and allows for the inclusion of effects due to chaotic particles.

Both our exact (2.36) and approximate (2.37) formulas for the average $y$ velocity of an ensemble of particles include the effects of magnetic field gradients, curvatures, and a constant electric field. Both neglect the effects of acceleration drifts. They have the same form as derived in the WP proof. However, $\langle v_y \rangle$ is not what is generally thought of as the drift velocity. The drift velocity is just the average $y$ displacement divided by time.

We derive the drift velocity from the following formula:

$$\langle v_y \rangle_{d,e} = \frac{1}{T} \int_0^T \langle v_y \rangle_e dt$$ (2.39)

where $\langle v_y \rangle_e$ is given by the exact and approximate formulas we just derived. To assume that the $\langle v_y \rangle_e$ we have just calculated is the same as used in this relationship is not as trivial as you might think. Indeed, the proof presented in the next section may not convince you. However, just as paper and pencil are the poor man’s computer, numerical simulations are the dumb man’s theoretical proof. Our numerical results indicate that the relationship of 2.39 is true.

2.3 The Connection Between $\langle v_y \rangle_e$ and $\langle v_y \rangle_{d,e}$

The average $y$ drift velocity of an individual particle from time 0 to time $T$, is defined by

$$\langle v_y \rangle_d = \frac{1}{T} \int_0^T v_y(x_0, v_0, t) dt$$ (2.40)

where $(x_0, v_0)$ is the particle’s position and velocity at time $t = 0$. We average this single particle’s drift over an ensemble of particles defined by the distribution function, $f(\mathcal{E}, \mathcal{P}_y) = f_0 \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P}_y - \mathcal{P}_{y_0})$.

$$\langle v_y \rangle_{d,e} = \frac{1}{T} \int_0^T dt \int f(y_0) \delta(\mathcal{E}(y_0)) \delta(\mathcal{P}_y - \mathcal{P}_{y_0}) d\mathcal{E} d\mathcal{P}_y$$ (2.41)
where $\delta(y_0)$ indicates that all the particles start at $y = 0$. We change the differential variables to those of a later time since we know from Liouville’s theorem that in a conservative system, the phase space volume of an ensemble is conserved, i.e. $d^3x_0d^3v_0 = d^3xd^3v$.

$$\langle v_y \rangle_{d,e} = \frac{1}{T} \frac{\int_0^T dt \int v_y f(\mathcal{E}, \mathcal{P}) \delta(y_0)d^3xd^3v}{\int f(\mathcal{E}, \mathcal{P})d^3xd^3v}$$ \hspace{1cm} (2.42)

The initial location, $y_0$, may be expressed as an implicit function of its current position and velocity, $y_0 = y_0(x, v, t)$. $f(\mathcal{E}, \mathcal{P}_y)$ has explicit $y$ dependences in the electric field potential. For the WP proof, there is no electric field potential and no other $y$ dependence. Therefore, integrating just over $y$ yields

$$\int f(\mathcal{E}, \mathcal{P}) \delta(y_0) dy = \int f_0 \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P} - \mathcal{P}_0) \delta(y_0[x, v, t]) dy$$ \hspace{1cm} (2.43)

$$= \int f_0 \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P} - \mathcal{P}_0) \left| \frac{\partial y}{\partial y_0} \right|_{x,v,t} dy$$ \hspace{1cm} (2.44)

$$= f_0 \delta(\mathcal{E} - \mathcal{E}_0) \delta(\mathcal{P} - \mathcal{P}_0)$$ \hspace{1cm} (2.45)

where the $\frac{\partial y}{\partial y_0}$ term is equal to unity due to the $y$-independence of the field configuration. This step required that we revert back to the WP proof and assume no $y$ dependence in $\mathcal{E}$. We may integrate the denominator in a similar, even simpler way to obtain,

$$\langle v_y \rangle_{d,e} = \frac{1}{T} \frac{\int_0^T dt \int v_y f(\mathcal{E}, \mathcal{P}) dxdzd^3v}{\int f(\mathcal{E}, \mathcal{P}) dxdzd^3v}$$ \hspace{1cm} (2.46)

This equality indicates that integrating from $0 \rightarrow T$ the ensemble averaged velocity at a given $y$ yields the same result as the ensemble averaged drifted velocity at a chosen time, $T$. For our more general case which includes an electric field, this analytically proof is less obvious. We rely on our numerical comparison to convince you that our calculated $\langle v_y \rangle$ is properly interpreted as the average instantaneous velocity of the ensemble.
Chapter 3

Application Using a Specific Magnetic Field Model

The exact formula of equation 2.36 should hold as long as the $x - z$ projection of the magnetic field may be described by $A_y(x, z)$ alone, the electric field may be described by 2.9, and the particle distribution function is well described by equation 2.14. In addition, the gradient curvature formula of equation 2.37 should hold as long as the Taylor expansion parameter is sufficiently small.

For the purposes of further exploring this problem we must choose a magnetic field model on which to experiment. We require that the magnetic field be $x$ dependent and qualitatively resemble Earth's magnetotail. We further enforce the condition that the magnetic field could satisfy the equation of force balance for an isotropic pressure, $(\mathbf{J} \times \mathbf{B})_{zz} = \nabla_{zz} P_{\text{Pressure}}$. Our particle distribution is intended as an elemental piece of such an isotropic distribution. Since a test-particle tracing program does not self consistently solve for magnetic field perturbations, we choose an equilibrium magnetic field which allows our particle distribution function to remain unchanged over time.

Hilmer and Voigt [1987] suggest a 3 component magnetic field model which may contain an isotropic pressure distribution while remaining in force balance. In their field, $B_y$ is an arbitrary function of $A_y$.

\begin{align*}
A_y(x, z) &= A_0 e^{\alpha x} \cos (C_z z) \quad (3.1) \\
A_x(x, z) &= \int f(A_y) dz \quad (3.2) \\
\mathbf{B} &= C_z \tan (C_z z) A_y \hat{x} + f(A_y) \hat{y} + \alpha A_y \hat{z} \quad (3.3) \\
A_0 &= B_0 \Delta \frac{2}{\pi} \quad (3.4) \\
\alpha &= \frac{\pi}{2} \frac{1}{\Delta} B_0 \quad (3.5) \\
C_z &= \frac{\pi}{2} \Delta \quad (3.6)
\end{align*}

$\Delta$ is half of the plasma sheet thickness. We have the freedom to pick a functional form for $f(A_y)$. With no electric field, the simplest choice is $B_y = f(A_y) = B_0$. We will find, in the next section, that the inclusion of an electric field complicates
matters just slightly. We also note that this $x$ dependent model is not 'bulb' shaped like the model used by Karimabadi et al. [1990] or Burkhart et al. [1992]. Note also that $\alpha = \frac{1}{B} \frac{\partial B}{\partial x}$. $\alpha$ is inversely proportional to the $x$ gradient scale length of $B$.

Using this magnetic field model we may obtain analytical solutions to equations for both the exact (2.36) and approximate (2.37) ensemble averaged velocity. The ensemble averaged drift equations may be obtained by substituting our resulting formula for $\langle v_y \rangle_e$ into equation 2.39.

### 3.1 Exact Ensemble Averaged $v_y$

We substitute our magnetic field into equation 2.36 and note that,

$$\mathcal{V}(A_y) = \int_{-\Delta}^{+\Delta} \frac{dz}{B_z} = \frac{2\Delta}{A_y\alpha}$$  \hspace{1cm} (3.7)

#### 3.1.1 Denominator

$$\int_{A_{y-}}^{A_{y+}} \mathcal{V}(A_y) dA_y = \frac{2\Delta}{\alpha} \int_{A_{y-}}^{A_{y+}} \frac{dA_y}{A_y} = \frac{2\Delta}{\alpha} \ln \left( \frac{A_{y+}}{A_{y-}} \right)$$  \hspace{1cm} (3.8)

#### 3.1.2 Numerator

$$\mathcal{N} = \int_{A_{y-}}^{A_{y+}} A_y \mathcal{V}(A_y) dA_y - \int_{A_{y-}}^{A_{y+}} A_{y0} \mathcal{V}(A_y) dA_y$$  \hspace{1cm} (3.9)

$$= \frac{2\Delta}{\alpha} \int_{A_{y-}}^{A_{y+}} dA_y - \frac{2\Delta}{\alpha} A_{y0} \int_{A_{y-}}^{A_{y+}} \frac{dA_y}{A_y}$$  \hspace{1cm} (3.10)

$$= \frac{2\Delta}{\alpha} (A_{y+} - A_{y-}) - \frac{2\Delta}{\alpha} A_{y0} \ln \left( \frac{A_{y+}}{A_{y-}} \right)$$  \hspace{1cm} (3.11)

$$= \frac{4\Delta m}{\alpha q} \sqrt{\frac{2\mathcal{E}_K}{m}} - \frac{2\Delta}{\alpha} A_{y0} \ln \left( \frac{A_{y+}}{A_{y-}} \right)$$  \hspace{1cm} (3.12)

So that the exact average $y$ velocity is given by,

$$\langle v_y \rangle_e|_{exact} = -\frac{q}{m} \left[ A_{y0} - \frac{2 \sqrt{2\mathcal{E}_K}}{m} \frac{m}{\ln \left( \frac{A_{y+}}{A_{y-}} \right) q - A_{y0}} \right]$$  \hspace{1cm} (3.13)

where $\mathcal{E}_K(t) = \mathcal{E}_{K0} + qE_y y(t)$ is the the kinetic energy of the ensemble of particles. This equation is valid with or without a constant cross tail electric field. If $E_y$ is non zero, $\mathcal{E}_K$ and $A_{y0}$ become functions of position and hence time.
A true ensemble of particles will not move monolithically. Rather, different particles will move at different average speeds depending on their initial conditions. In order to derive an explicit equation for the average velocity of the ensemble, we solve for the motion of a single representative particle which has the ensemble averaged $y$ velocity. In other words, we solve this equation analytically or numerically as if it were the motion of a single particle.

### 3.2 Approximate Ensemble Averaged $v_y$

The average adiabatic speed is an approximate form of the above formula. From equation 3.7, we know that

$$\left. \frac{1}{\mathcal{V}(A_{y0}(t))} \frac{d\mathcal{V}(A_y)}{dA_y} \right|_{A_y=A_{y0}(t)} = -\left. \frac{1}{A_y} \right|_{A_y=A_{y0}(t)}$$

Substituting this into equation 2.37 yields

$$\langle v_y \rangle_e \Big|_{p v_y | \ll 1} \approx \frac{2}{3} \frac{E_K}{q A_{y0}}$$

This equation is particularly simple if there are no electric fields. In such a case, $E_K$ is a constant of the motion. Hence, the average velocity of our particle which represents the flux tube ensemble may be calculated trivially using the value of the central field line label, $A_{y0}$.

### 3.3 Ensemble Averaged $v_x$

We can also determine the ensemble averaged $x$ velocity due to the electric field. Here we make a subtle assumption that the flux tube full of particles drifts, on average, with the same speed as the equatorial crossing point of the flux tube. Consider, once again, a representative particle drifting in the equatorial plane along the central field line labeled by $A_{y0}$. At $z = 0$, $\langle v_x \rangle$ becomes:

$$\langle v_x \rangle = \frac{E \times B |_{z=0}}{B^2} = \frac{E_y B_z - E_z B_y}{B_x^2 + B_y^2}$$

$\langle v_x \rangle_e$, is an exact formula. It is fundamentally different than $\langle v_y \rangle_e$ because the particles are bound together within a finite $x$ range due to the conservation of $P_y$. They move together as a fluid towards $+x$. In contrast, the $y$ drift is completely non-fluid like. Particles with different pitch angles drift at different $y$ speeds.
3.4 Flux Tube Motion - \( \frac{dA_y}{dt} \)

We need only an equation which gives the function \( A_y(t) \). At \( z = 0 \) we may determine the convective derivative of \( A_y \).

\[
\frac{dA_y(x, z, t)}{dt} = \frac{\partial A_y}{\partial x} \frac{dx}{dt} \bigg|_{z=0} = \frac{\partial A_y}{\partial x} \frac{\mathbf{E} \times \mathbf{B}_x}{B^2} \bigg|_{z=0} \tag{3.18}
\]

\[
= B_z \frac{E_y B_z - E_z B_y}{B_x^2 + B_y^2} \tag{3.19}
\]

We have just derived equations which we interpret as describing the motion of a representative particle which moves with bounce averaged velocity of our ensemble. If we let \( \frac{dx}{dt} = \langle v_x \rangle \) and \( \frac{dy}{dt} = \langle v_y \rangle \), we obtain a set of coupled differential equations. For the simple case of no electric field, this integration is straightforward and analytical. When the effects of an electric field are added, this set of coupled differential equations, \( \frac{dv_x}{dt}, \frac{dv_y}{dt} \), and \( \frac{dA_y}{dt} \), is easily solved numerically by integrating from \( t = 0 \rightarrow T \). We end up with \( y(T) \) which yields,

\[
\langle v_y \rangle_{d,e} = \frac{y(T)}{T} \tag{3.20}
\]

This theoretical drift speed is compared to particle simulation results in chapter 6.

3.5 Adding \( \mathbf{E} \) when \( \mathbf{B} \) is 3 Component

Adding a cross tail electric field is slightly more complicated with a non-zero \( B_y \) than without because we must ensure that \( E_y = 0 \). In a self consistent particle calculation, as in nature, the parallel electric field will tend to be small since mobile electrons will establish a polarization electric field which will cancel the externally applied electric field. Rather than perform such a self consistent calculation, we mimic its effects by choosing an electric field with no parallel component, i.e. we set \( \mathbf{E} \cdot \mathbf{B} = 0 \). For simplicity, we choose an electric field with no \( x \) component and no \( y \) dependence.

\[
E_y B_y + E_z B_z = 0 \tag{3.21}
\]

\[
E_z = -\frac{B_y}{B_z} E_y \tag{3.22}
\]

We also enforce the curl free condition, \( \nabla \times \mathbf{E} = 0 \), so that

\[
\frac{\partial E_y}{\partial z} = 0 \tag{3.23}
\]
\[
\frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x} \left( \frac{E_y B_y}{B_z} \right) = 0 \quad (3.24)
\]
\[
\frac{\partial E_y}{\partial x} = 0 \quad (3.25)
\]

The ratio \( \frac{E_y B_y}{B_z} \) must be independent of \( x \). The choice of a constant \( E_y = E_{y0} \) is not consistent with a simple, constant \( B_y \) if \( B_z \) is a function of \( x \). We must either choose a more complicated function for \( E_y \) or \( B_y \). The next simplest choice for \( B_y \) is \( B_y = \text{Const} \times A_y \). With this form, we may then set \( E_y = E_{y0} \). Also recall that \( \alpha A_0 = B_n \).

\[
A = \frac{B_{y0}}{C_z} e^{\alpha x} \sin (C_z z) \hat{x} + A_0 e^{\alpha x} \cos (C_z z) \hat{y} \quad (3.26)
\]
\[
B = C_z \tan (C_z z) A_y \hat{x} + \frac{B_{y0}}{A_0} A_y \hat{y} + \alpha A_y \hat{z} \quad (3.27)
\]
\[
E = E_{y0} \hat{y} - \frac{B_{y0}}{B_n} E_{y0} \hat{z} \quad (3.28)
\]

Figure 3.1 A and B show the \( x - z \), \( x - y \), and \( y - z \) projections of two possible magnetic field configurations. Three magnetic field lines are drawn in each picture. Only the field shown in figure 3.1 B was ultimately used. The field shown in figure 3.1 A has a shortcoming. The rotation of field lines along the \( x \) direction, most clearly seen in the bottom frame, makes it impossible to apply an electric field which is both curl free and everywhere perpendicular to the magnetic field.
\[ A_y(x, z) = A_0 e^{\alpha z} \cos(C_z z) \]

\[ A_x(x, z) = B_{yo} z \]
\[ B = C_z \tan(C_z z) A_y \hat{x} + B_{yo} \hat{y} + \alpha A_y \hat{z} \]

\[ A_x(x, z) = \frac{B_{yo}}{C_z} e^{\alpha z} \sin(C_z z) \]
\[ B = C_z \tan(C_z z) A_y \hat{x} + \frac{B_{yo}}{A_0} A_y \hat{y} + \alpha A_y \hat{z} \]

**Figure 3.1** Two different force-balanced, 3D magnetic field models. Three field lines are shown in each figure. Model B can exist self consistently with a constant, curl-free, perpendicular electric field. Model A cannot.
Chapter 4

Particle Dispersion

It is not enough to know the average drift of an isotropic ensemble of particles. A distribution which has a large spread in drift velocities cannot be well described by the average motion. We thus need to compare the dispersion of an ensemble which contains particles executing chaotic and Speiser orbits to the same ensemble were all the particles to move according to the individual adiabatic drift equations (1.2 and 1.3). To make this comparison, we need to derive the drift velocity of an individual particle as a function of its adiabatic invariants rather than just the average drift of an isotropic distribution.

4.1 Individual Particle Bounce Averaged Drift

From adiabatic theory, we know that the bounce averaged perpendicular motion of a charged particle in a magnetic field may be derived from a particle’s kinetic energy, \( E_k \), as in equation 1.6. We also know that for the generalized 2\( \frac{1}{2} \)D B field the kinetic energy of an adiabatic particle at a given \( y \) may be written as a function of \( \mu \) and \( J \) as well as the single field line label, \( A_y \). Therefore,

\[
\langle v_\perp \rangle_b = \frac{B \times \nabla [E_k(A_y, \mu, J) + q\Phi(y)]}{qB^2}
\] (4.1)

where

\[
\nabla E_k(A_y, \mu, J) = \left( \frac{\partial E_k}{\partial A_y} \right)_{\mu, J} \nabla A_y(x, \tilde{z})
\] (4.2)

\[
= \left( \frac{\partial E_k}{\partial A_y} \right)_{\mu, J} [B_x \hat{x} - B_x \hat{z}]
\] (4.3)

Substituting this into the gradient/curvature part of equation 4.1, we obtain

\[
\langle v_{GC \perp} \rangle_b = \frac{1}{qB^2} \left( \frac{\partial E_k}{\partial A_y} \right)_{\mu, J} \left[ -B_y B_x \hat{x} + (B_x^2 + B_z^2) \hat{y} - B_y B_z \right]
\] (4.4)

We want to know the total bounce motion of the particle’s guiding center, not just the component perpendicular to the magnetic field. Because \( B_y \neq 0 \), \( \langle v \rangle \) includes
a parallel component. i.e. $\langle v \rangle_b = \langle v_\perp \rangle_b + \langle v_\parallel \rangle_b$ where $v_\parallel = v_\parallel B / B$. We may solve for $\langle v_\parallel \rangle_b$ by noting that the magnetic flux tube in which our particle is constrained to move is symmetric about the $z = 0$ axis. Hence our particle’s bounce averaged $z$ velocity must be zero.

$$\langle v_\perp z \rangle_b + \langle v_\parallel z \rangle_b = 0$$ (4.5)

Therefore

$$\langle v_\parallel \rangle_b = \frac{B_z}{B} - \frac{B_y B_z}{q B^2} \left( \frac{\partial E_k}{\partial A_y} \right)_{\mu,J} = 0$$ (4.6)

$$\langle v_\parallel \rangle_b = \frac{1}{q} B_y \left( \frac{\partial E_k}{\partial A_y} \right)_{\mu,J}$$ (4.7)

Returning to equation 4.1 and noting that $\mathbf{E} = -\nabla \Phi$, we obtain

$$\langle v \rangle_b = \frac{1}{q} \frac{\partial E_k(A_y, \mu, J)}{\partial A_y} \mathbf{j} + \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$ (4.8)

A particle of a given energy and $P_y$ is confined to move within a flux tube defined by $A_y \pm$ (cf. equation 2.35). On average, particles will drift towards positive $x$ with the $\mathbf{E} \times \mathbf{B}$ drift speed in the equatorial plane (cf. equation 3.17). All particles will gyrate about the same $A_y \phi$ value. The more subtle calculation is that for the $y$ motion. We seek a function for $\left( \frac{\partial E_k}{\partial A_y} \right)_{\mu,J}$

$$dJ = \left( \frac{\partial J}{\partial E_k} \right)_{A_y, \mu} dE_k + \left( \frac{\partial J}{\partial A_y} \right)_{E_k, \mu} dA_y$$ (4.9)

$$\left( \frac{\partial E_k}{\partial A_y} \right)_{J, \mu} = -\frac{\left( \frac{\partial J}{\partial E_k} \right)_{E_k, \mu}}{\left( \frac{\partial J}{\partial E_k} \right)_{A_y, \mu}}$$ (4.10)

We need a functional form for $J$ in terms of $\mu, E_k$ and magnetic field line label, $A_y$. Such a formula, calculated in chapter 1, is rewritten here for convenience.

$$J(\mu, A_y, E_k) = \sqrt{2m} \oint \sqrt{E_k - \mu B} \, ds$$

To solve this integral, we simplify our magnetic field equation.

$$\mathbf{B} = C_z \tan(C_z z) A_y \mathbf{x} + \frac{B_{y0}}{A_0} A_y \mathbf{y} + \alpha A_y \mathbf{z}$$ (4.11)

$$|B| = \alpha A_y f(z)$$ (4.12)

$$ds = dz f(z)$$ (4.13)

where

$$f(z) = \sqrt{\left( \frac{C_z}{\alpha} \right)^2 \tan^2(C_z z) + \left( \frac{B_{y0}}{\alpha A_0} \right)^2 + 1}$$ (4.14)
We now solve for \( J = J(E_k, A_y, \mu) \) by integrating over the entire flux tube.

\[
J(E_k, A_y, \mu) = 2\sqrt{2m} \int_0^{z_{\text{max}}} \sqrt{E_k - \mu A_y f(z)} f(z) dz
\]  \hspace{1cm} (4.15)

Differentiating, we obtain,

\[
\left( \frac{\partial J}{\partial E_k} \right)_{A_y, \mu} = 2\sqrt{2m} \int_0^{z_{\text{max}}} \frac{1}{2} (E_k - \mu A_y f(z))^{-\frac{1}{2}} f(z) dz
\]

\[
\left( \frac{\partial J}{\partial A_y} \right)_{E_k, \mu} = 2\sqrt{2m} \int_0^{z_{\text{max}}} \frac{1}{2} (E_k - \mu A_y f(z))^{-\frac{1}{2}} (-\mu A_y f(z)) f(z) dz
\]  \hspace{1cm} (4.16)

(4.17)

Substituting these into equation 4.10 yields

\[
\langle v_y \rangle_b(\mu, A_y, E_k) = \frac{\mu A_y}{q} \int_0^{z_{\text{max}}} \frac{f(z) [1 - C_X f(z)]^{-\frac{1}{2}}}{f(z) [1 - C_X f(z)]^{-\frac{1}{2}}} dz
\]

\[
\int_0^{z_{\text{max}}} \frac{f(z) [1 - C_X f(z)]^{-\frac{1}{2}}}{f(z) [1 - C_X f(z)]^{-\frac{1}{2}}} dz
\]  \hspace{1cm} (4.18)

where \( C_X \equiv \frac{\mu A_y}{E_k} \). The limit of integration is a function of the pitch angle and energy of the particle. In particular, \( f(z_{\text{max}}) = 1/C_X \), so that

\[
z_{\text{max}} = \frac{1}{C_z} \tan^{-1} \left( \frac{\alpha (1/C_X)^2}{\alpha A_0^2 - 1} \right)
\]

\[
(\alpha A_0)^2 - 1
\]  \hspace{1cm} (4.19)

### 4.1.1 Single Particle Bounce Averaged Motion when \( \mu \to 0 \) and \( \mu \to \mu_{\text{max}} \)

Equation 4.18 is the bounce averaged velocity of a single particle given the field line label, \( A_y, \mu \) and \( J \). What is the value of \( \langle v_y \rangle \) for the extremes of \( \mu = 0 \) and \( \mu = \mu_{\text{max}} = E_k/B(z=0) \)? For \( \mu = 0 \), the particle is field aligned. For \( \mu = \mu_{\text{max}} \), the particle motion is confined to the equatorial plane and is relatively straightforward to calculate. As \( \mu \to \frac{E_k}{B(z=0)}, z_{\text{max}} \to 0 \).

\[
\langle v_y \rangle_b = \lim_{\mu_{\text{max}} \to 0} \frac{\mu A_y}{q} \int_0^{z_{\text{max}}} \frac{f(z) [1 - C_X f(z)]^{-\frac{1}{2}}}{f(z) [1 - C_X f(z)]^{-\frac{1}{2}}} dz
\]

\[
= \lim_{\mu \to \mu_{\text{max}}} \frac{\alpha}{q} \left( \frac{B_y}{\alpha A_0} \right)^2 + 1 = \frac{\alpha E_k}{q \sqrt{B_y^2 + B_z^2}} \left( \frac{B_y}{\alpha A_0} \right)^2 + 1
\]  \hspace{1cm} (4.20)

\[
= \frac{E_k}{q A_y}
\]  \hspace{1cm} (4.21)

Where we recalled that \( B_n = \alpha A_0 \). This is slightly faster than the isotropic ensemble averaged speed (cf. formula 3.15), not an unreasonable value. In the other limit, the
Figure 4.1 An example of how \( \langle v \rangle_b \) varies with \( \mu \). As \( \mu \to 0 \), \( \langle v \rangle_b \to 0 \). As \( \mu \to \mu_{\text{max}} \), \( \langle v \rangle_b \to \frac{E_b}{q A_y} \). The horizontal axis extends just beyond \( \mu_{\text{max}} \). The constant, ensemble averaged drift velocity is drawn and has a value \( \langle v \rangle_e = \frac{2E_b}{3q A_y} \).

The ratio of the integrals of equation 4.18 diverges more slowly than \( \mu \) goes to zero. Hence the bounce averaged velocity goes to zero. A plot of \( \langle v_y \rangle_b \) versus \( \mu \), depicted in figure 4.1, was calculated numerically and makes clear that

\[
\langle v_y \rangle_b = \lim_{\mu \to 0} \frac{\mu \alpha \int_{z_{\text{max}}}^{z_{\text{max}} - \Delta} f^2(z) [1 - C_X f(z)]^{-\frac{1}{2}} dz}{q \int_{z_{\text{max}}}^{z_{\text{max}} - \Delta} f(z) [1 - C_X f(z)]^{-\frac{1}{2}} dz} = 0 \quad (4.23)
\]

Thus, the particle bouncing in the equatorial plane drifts most quickly. A field aligned particle never bounces and does not drift. Most particles fall somewhere between these two extremes. In fact, we can analytically determine the number of particles with a certain \( \mu \) value based on our magnetic field model. This calculation is performed in the next section.

We obtain a generalized drift velocity from a bounce averaged one by integrating (numerically) over time.

\[
\langle v_y \rangle_d = \int_0^T dt \langle v_y \rangle_b(\mu, A_y, E_b) \quad (4.24)
\]

\[
\langle v_x \rangle_d = \int_0^T dt \langle v_x \rangle_b(E_y, B) \quad (4.25)
\]
where $A_y$ and $B$ are explicit functions of position and hence implicit functions of time.

### 4.2 Analytical Results for a Particle Distribution

We wish to determine how an isotropic distribution of particles disperses over time as the ensemble drifts according to adiabatic drift theory. We can then compare these results to those of the numerical test particle traces. Equations 4.18, 3.19, 4.24, and 4.25 tell us how a particle with a given $\mu$ and energy gradient-curvature drifts in the $y$-direction. We need also determine an analytical form for the initial particle distribution.

Specifically, we wish to create an analytical histogram of particles as they drift in the $y$-direction. Mathematically, we need to find a function which yields the number of particles per $y$ displacement range, i.e. $\frac{dN}{dy}$. We may then integrate this over the appropriate $y$-range bins and compare to numerical results.

\[
\frac{dN}{dy} = \left( \frac{\frac{dN}{du}}{\frac{d\mu}{du}} \right)
\]  
(4.26)

We start by integrating the distribution function over all of phase space. We then manipulate this phase space integration through a series of Jacobian transformations. We denote the Jacobian as $j$.

The initial distribution function may be written in a few different forms.

\[
f(x, v, t) = f_0 \delta(E_k - E_{k0})\delta(P_c - P_0)
\]  
(4.27)

\[
f(x, v, t = 0) = f_0 \delta(E_k - E_{k0})\delta(A_y - A_{y0})
\]  
(4.28)

The number of particles may be obtained by integration:

\[
N = \int dx \, dz \, d^3v \, f(x, v, t = 0)
\]  
(4.29)

\[
= f_0 \int dx \, dz \, d^3v \, \delta(E_k - E_{k0})\delta(A_y - A_{y0})
\]  
(4.30)

We may further transform the velocity differentials because $j dv || dv_\perp = dE_k d\mu$. In addition, we know that $\mu = \frac{mv^\perp}{2B}$ and $E_k = \frac{1}{2} m \left( v_\perp^2 + v_\parallel^2 \right)$. The Jacobian may be calculated:

\[
j = \begin{vmatrix}
\left( \frac{\partial E_k}{\partial v_\parallel} \right)_{v_\parallel} & \left( \frac{\partial E_k}{\partial v_\perp} \right)_{v_\parallel} \\
\left( \frac{\partial E_k}{\partial v_\parallel} \right)_{v_\perp} & \left( \frac{\partial E_k}{\partial v_\perp} \right)_{v_\perp}
\end{vmatrix}
= \begin{vmatrix}
mv_\parallel & \left( \frac{1}{2} m \right) \\
0 & \left( \frac{m}{2B} \right)
\end{vmatrix}
= \frac{1}{2} m^2 \frac{v_\parallel}{B}
\]
Therefore,
\[ d^3v = \pi dv_1dv_2 = \pi j^{-1}dE_kd\mu = \frac{2\pi B}{m^2v_1^2}dE_kd\mu \]

So equation 4.30 becomes:
\[ N = f_0\frac{2\pi}{m^2} \int dA_y \frac{ds}{B} \frac{dE_k}{v_1} \delta(E_k - E_{k0}) \delta(A_y - A_{y0}) \]  \hfill (4.31)
\[ \frac{dN}{d\mu} = f_0\frac{2\pi}{m^2} \int dA_y \frac{ds}{v_1} dE_k \delta(E_k - E_{k0}) \delta(A_y - A_{y0}) \]  \hfill (4.32)
\[ = f_0\frac{2\pi}{m^2} \int dA_y \int_{E_k} dE_k \delta(E_k - E_{k0}) \int_A \delta(A_y - A_{y0}) \frac{1}{v_1} \]  \hfill (4.33)

Recall that \[ ds = dzf(z) \] where \[ f(z) = \sqrt{\left(\frac{C_z}{\alpha}\right)^2 \tan^2(C_zz) + \frac{B_{z0}}{\alpha A_0} + 1}. \]
\[ \frac{dN}{d\mu} = f_0\frac{2\pi}{m^2} \int_0^{2\max} dzf(z) \int dE_k \delta(E_k - E_{k0}) \int dA_y \delta(A_y - A_{y0}) \frac{1}{v_1} \]  \hfill (4.34)

Since \[ v_1 \] is a function of \[ \mu \] and \[ E_k \], we must substitute the functional form before integrating. We recall that \[ \mu B + \frac{1}{2}mv_1^2 = E_k \] or Therefore \[ v_1 = \sqrt{\frac{2}{m}(E_k - \mu B)} \].
\[ v_1 = \sqrt{\frac{2E_k}{m} \sqrt{1 - C_X f(z)}} \]  \hfill (4.35)

where that \[ C_X = \frac{\mu A_y}{E_k} \] and \[ dC_X = d\mu \frac{A_k}{E_k} = d\mu \frac{B_k}{E_k} \]. We resume simplifying equation 4.34.
\[ \frac{dN}{d\mu} = f_0\frac{4\pi}{m^2} \sqrt{\frac{m}{2E_k}} \int_0^{2\max} \frac{dzf(z)}{\sqrt{1 - C_X f(z)}} \]  \hfill (4.36)

It is worth describing the math used to determine a value for \( f_0 \). Integrating \( \frac{dN}{d\mu} \) over all \( \mu \) should yield the total number of particles.
\[ N_0 = \int_{\mu_0}^{\mu_{\max}} d\mu \frac{dN}{d\mu} = \frac{E_k}{B_z} \int_0^1 dC_X \frac{dN}{d\mu} \]  \hfill (4.37)
\[ = \frac{4\pi}{m^2} \sqrt{\frac{m}{2E_kB_z}} f_0 \int_0^1 dC_X \int_0^{\mu_{\max}} dzf(z) \frac{dzf(z)}{\sqrt{1 - C_X f(z)}} \]  \hfill (4.38)

Focusing on the integrals and substituting \( y \equiv C_X f(z) \) gives
\[ \int_0^1 dC_X \int_0^{\mu_{\max}} \frac{dzf(z)}{\sqrt{1 - C_X f(z)}} = \int_0^\Delta dzf(z) \int_0^{\mu(\Delta)} \frac{dC_X}{\sqrt{1 - C_X f(z)}} \]  \hfill (4.39)
\[ = \int_0^\Delta dz \int_0^1 \frac{dy}{\sqrt{1-y}} \left( \frac{2\sqrt{1-y}}{1} \right)^{1/2}\]  

The total number of particles in terms of \( f_0 \) is

\[ N_0 = \frac{4\pi}{m^2} \sqrt{\frac{m}{2E_k}} \frac{B_z}{2\Delta} f_0 2\Delta \]  

\[ f_0 = \frac{N_0}{8\pi} \sqrt{\frac{2E_k}{m}} \frac{B_z}{E_k \Delta} \]  

Therefore,

\[ \frac{dN}{d\mu} = N_0 \frac{B_z}{E_k \Delta} \int_0^{z_{\text{max}}} \frac{dz f(z)}{\sqrt{1 - C_x f(z)}} \]  

This integral is calculated numerically.

**Now What?**

We have just established a set of analytical equations which we use to track the bounce averaged drift of a particle in our magnetic field given this particle’s \( \mu \) value and initial energy. In other words, we have an explicit form for equation 1.6 given our magnetic field. This is much more detail than the ensemble averaged drift of equation 1.7. A note to the dedicated reader: Your valiant efforts at understanding this derivation should not go unrewarded. You may revel in mathematical rigor. Nevertheless, for a more tangible reward, please let me know you’ve made it here.

Equation 4.43 describes analytically how we should distribute particles within a flux tube, i.e. how many particles per \( \mu \) range there should be. Such particles will uniformly fill the \( x-z \) projection between our bounding field lines, \( A_{y\pm} \) such that the particle density in the \( x-z \) plane is not a function of position (cf. equation 2.21).

We combine these equations to produce analytical results similar to the output produced by our numerical particle tracing program. We will compare the results and derive answers to all of the secrets of nature. We will present some of these answers in chapter 6.
Chapter 5

Test Particle Simulation Model

We wish to compare the adiabatic drift formulas derived in the previous chapters with the actual drift of an ensemble of particles which move according to the exact Lorentz force equation. This chapter describes how we establish appropriate numerical models to which we may compare our adiabatic theory. We also summarize important parameters and their significance when interpreting the numerical results.

5.1 2D Magnetic Field Model

Using the 2D magnetic field model described in chapter 3, we numerically model the distribution of particles given by equation 2.14. This distribution is a function of only the constants of motion. The density of particles in the $x-z$ plane obtained from this distribution function is a constant (cf. equation 2.21). It does not depend on pitch angle or location within the $X-Z$ projection of the flux tube. Adding a perpendicular electric field changes the distribution subtly. We thus describe separately the two situations, with and without an electric field.

No Electric Field

One thousand monoenergetic particles are initially distributed uniformly on the $y = 0$ plane within a magnetic flux tube bounded by the field lines labeled by $A_{y\pm}$ (see figure 5.1). This is accomplished by using a Monte-Carlo type method of filling an area (see Sobel's sequence from Press et al.[1986]). A particle's $x-z$ position uniquely determines it's $A_y$ value. The modeled flux tube is bounded at $x_{min}$ by the equatorial crossing points of $A_{y\pm}$ and extends to infinity in positive $x$. The fraction of flux tube volume contained within a given $x$ range decreases with increasing $x$ due to the $x$ gradient in magnetic field strength. Thus, when using a finite number of particles, there is a practical $x_{max}$ limit beyond which most particles will not venture due to their bounce motion. Only an exactly field aligned particle should be able to travel arbitrarily far down a field line. A large $x_{max}$ was chosen for initializing particle positions. During the actual particle tracing, no artificial boundary was imposed.
\[ A_y(x, z) = A_0 e^{\alpha x} \cos (C_z z) \]
\[ B = C_z \tan (C_z z) A_y \hat{z} + \alpha A_y \hat{z} \]
\[ E = E_{y0} \hat{y} \]

Figure 5.1 A portion of our \( x \) dependent, 2D Magnetic field model with initial particle distribution. Test particle simulations were performed for both zero and non-zero values of \( E_{y0} \).

All particles are assigned the same kinetic energy and \( P_y \). Physically, this implies that the guiding centers of all particles fall along the same central field line defined by \( A_{y0} \). However, the particle itself may fall within any \( A_{y-} \leq A_y \leq A_{y+} \). A particle's \( v_y \) is determined by it's \( A_y \) value since \( v_y = \frac{1}{m} (P_y - qA_y) \). Because the particle's energy is given by \( v_p^2 + v_y^2 = \frac{2}{m} E_k \), we are free to distribute \( v_x \) and \( v_z \) subject to the constraint that \( v_x^2 + v_z^2 = v_y^2 \). On a given field line, we distribute the particle velocities isotropically in the \( v_x - v_z \) plane. Different particle \( \mu \)’s span the available range of values \( (0 \leq \mu \leq \frac{E_k}{B^2}) \).

As discussed in chapter 1, this type of magnetic field allows for three types of particle trajectories depending on magnetic field parameters and particle initial con-
ditions. Because the particle distribution is monoenergetic, for a given set of magnetic field parameters, we may define a $\kappa$ for an entire distribution of particles. The Speiser type motion of a typical $\kappa = 0.2$ particle is shown in figure 1.5A. An integrable trajectory is shown in figure 1.5C. A chaotic orbit for $\kappa = 0.5$ is shown in part B of this figure. The Poincaré surface of section for an ensemble of these types of particles is shown in figure 1.7.

Adding a Cross Tail Electric Field

As for the case with no electric field, particles are initially distributed uniformly within the $y = 0$ plane of the magnetic flux tube. Assigning velocities, however, is a bit more tricky. If we randomize $v_x - v_z$ and then impose a cross tail electric field, the distribution of particle pitch angles will be isotropic in the lab frame, but not in the frame that is $E \times B$ drifting with the flux tube of particles. Each particle will tend to drift with its local $E \times B$ drift speed. We need to add this $E \times B$ drift to the $v_x - v_z$ distribution in order to make the distribution isotropic in this moving frame. The question then arises: what value for $v_{E \times B}$ should we use?

The magnetic field varies substantially within the flux tube. However, on average, the net $E \times B$ drift direction will be towards positive $x$. In fact, the actual drift value should be approximately described by the drift of the equatorial foot of the central field line defined by $A_{y0}$. In other words, the natural choice for $\langle v_x \rangle_{\text{ensemble}}$ should be the $E \times B$ drift with $E$ and $B$ at $(x_{A_{y0}}, z = 0)$. This is described in equation 3.17. Thus, we add this value of $v_x$ to the velocities of all of the particles. We then must ensure that the value of $\mathcal{E} = E_k + q\Phi$ is the same for all of the particles. We can do this in one of two ways. We may renormalize the velocities. We may also move the particles in $y$, lowering the potential energy of a particle with a high kinetic energy and raising the potential of a particle with low kinetic energy. Both methods were used. There was no discernible difference in results.

One note of interest: If the $E \times B$ drift were constant everywhere, then the electric field effects could be trivially transformed away by switching to the deHoffman-Teller frame. However, there is an $x$ dependence in the drift velocity due to the $x$ dependence of $B_z$. 
5.2 $2\frac{1}{2}$D Magnetic Field Model

Just as for the 2D model, particles are initially distributed within the $x-z$ projection of the magnetic flux tube so that their 2D spatial density is uniform. They are then shifted in $Y$ so that they all reside on the same initial flux tube. Specifically, all of the particles have the same value of $\psi$ where,

\[
\psi = y - \int_0^z \frac{B_y}{B_z} dz' = y - \int_0^z \frac{B_0A_y}{\alpha A_0} dz' 
\]

\[
= y - \int_0^z \frac{B_0A_y}{\alpha A_0} dz' = y - \frac{B_0}{\alpha A_0} z 
\]

(5.1)

(5.2)

The average change of $\psi$ is analogous to $\langle v_y \rangle$.

\[
\frac{d\psi}{dt} = \langle v \rangle \cdot \nabla \psi = \langle v_y \rangle 
\]

(5.3)

We also emphasize that no parallel electric field was imposed, i.e. $E \cdot B = 0$. This was accomplished by using an electric field of the form described by equation 3.28.

5.3 Adjusting $\frac{B_n}{B_0}$, $\kappa$, $\frac{B_0}{B_n}$, and $\frac{E_y}{v_{th}B_n}$

For our 2D magnetic field model with no electric field, the physics is determined entirely by a few parameters, $B_0$, $B_n$, $\Delta$, $\alpha$, $E_k$. These variables are not completely independent and may be reduced to a smaller set of dimensionless, adjustable parameters. Let's see how. With the flux tube volume given by equation 3.7 the bounce averaged velocity at $(0,0,0)$ is given by

\[
\langle v_y \rangle = \frac{2}{3} \frac{E_k}{q} \frac{1}{A_0} = \frac{m}{3q} \frac{v_{th}}{v_{th}} \frac{C_z}{B_0} 
\]

\[
= \frac{1}{3} \frac{v_{th}}{qB_n} B_n \frac{C_z}{B_0} = \frac{1}{3} \frac{v_{th}}{\Omega_n} B_n \frac{C_z}{B_0} 
\]

\[
= \frac{1}{3} \frac{v_{th}}{\rho_n} C_z \frac{B_n}{B_0} 
\]

(5.4)

(5.5)

(5.6)

The curvature of our magnetic field model is given by $\kappa_c$ (not to be confused with the adiabaticity parameter, $\kappa \equiv \frac{1}{\sqrt{\kappa_{c_{max}} \rho_n}}$)

\[
\tilde{\kappa}_c(x, z) = C_z \frac{B_n}{B_n} \frac{\sec(C_z z)}{[1 + (\frac{B_0}{B_n})^2 \tan^2(C_z z)]^2} \left[ \hat{x} - \frac{B_0}{B_n} \tan(C_z z) \hat{z} \right] 
\]

\[
\kappa_{c_{max}} = \kappa_c(0, 0) = C_z \frac{B_0}{B_n} 
\]

(5.7)

(5.8)
\[ A_y(x, z) = A_0 e^{\alpha x} \cos(C_z z) \]
\[ B = C_z \tan(C_z z) A_y \hat{x} + \frac{B_{y0}}{A_0} A_y \hat{y} + \alpha A_y \hat{z} \]
\[ E = E_{y0} \hat{y} + E_{z0} \hat{z} \]

\( K=0.50 \, \text{Br/B0}=0.10 \)

**Figure 5.2** A portion of our \( x \) dependent 2\( \frac{1}{2} \)D Magnetic field model with initial particle distribution. The \( x-z \) projection of this field is our 2D model. Test particle simulations were performed for both zero and non-zero values of \( E_{y0} \).

so equation 5.6 becomes,

\[ \langle v_y \rangle = \frac{1}{3} v_{th} \rho_n \kappa_{c_{max}} \left( \frac{B_n}{B_0} \right)^2 \quad (5.9) \]
\[ \frac{\langle v_y \rangle}{v_{th}} = \frac{1}{3} \kappa \left( \frac{B_n}{B_0} \right)^2 \quad (5.10) \]

The average \( y \) speed, normalized to the particles' thermal speed, is determined entirely by the dimensionless parameters, \( \kappa \), and \( \frac{B_n}{B_0} \). \( \frac{B_n}{B_0} \) prescribes how 'tail-like' the field is. And \( \kappa \) characterizes the particles. \( B_y \) does not effect the theoretical average \( y \) velocity because as we showed in section 2.1, the \( \langle v_y \rangle \) formula is valid as long as \( A_y \)
is not a function of $y$. However, there are practical limitations to how large $B_y$ can be made relative to $B_0$ when simulating only 1000 particles. The surface area of the tilted 2d flux tube increases with increasing $\frac{B_y}{B_0}$. If the area becomes too great, our particles will not really fill the flux tube. Statistical error will dominate.

$E_y$ is also an independent parameter. Adding a cross tail electric field adds a $y$ dependence to the energy, $\phi(y) = -qE_y y$, tending to speed up the GC motion. However, it also introduces an $\mathbf{E} \times \mathbf{B}$ drift to the particles. They drift into a region of stronger magnetic field, which tends to slow down the GC drift. We may still use our formalism to describe the ensemble motion as long as the electric field strength does not get too large. How large is too large? As long as the distribution remains isotropic, the exact formula of equation 2.36 should hold true. However, our drift equations are only valid if acceleration drifts are small. The electric field introduces an acceleration drift by moving the particles towards positive $x$ where the magnetic field strength increases. We limit our electric field in order to limit the acceleration drift.

\[
\frac{v_{\text{acceleration drift}}}{v_{\text{GC}}} = \frac{(v \mathbf{E} \cdot \mathbf{B}) v \mathbf{E} \cdot \mathbf{B}}{\alpha v^2_{th}} \ll 1
\]

\[
\frac{(E_x \frac{\Delta}{B_z} \cdot \nabla) E_y}{\alpha v^2_{th}} = \frac{E^2_y}{v^2_{th} E^2_n} \ll 1
\]

5.4 A Different B Field Model

To impress upon you that the results presented in the next chapter are not caused by some fluke characteristic of our magnetic field model, we present similar results for a modified Harris sheet configuration which is not really in force balance. This field was used test particle calculations by several people (e.g. Chen [1992]) and is shown in figure 5.3. Note that $B_0$, $B_n$, and $\Delta$ are all constant as before. This magnetic field lines resemble parabolas, but the field does not explicitly dependent on $x$. Hence, the bounce averaged drift velocity should go to zero. Perfectly reflecting walls are placed at some $x_{max}$ such that $\mu$ is conserved. By the geometry of the field lines, these walls introduce a gradient in the flux tube volume as perceived by the particles which bounce off of it. This artificial gradient in the flux tube volume introduces a net $v_y$ drift described by our drift equations.
\begin{align}
A_y &= -B_0 \Delta \ln \left( \cosh \left( \frac{z}{\Delta} \right) \right) + B_n x \\
B &= B_0 \tanh \left( \frac{z}{\Delta} \right) \hat{x} + B_n \hat{z}
\end{align}
(5.13)
(5.14)

Figure 5.3 Initial distribution for $x$ independent parabolic magnetic field model as used by Chen [1992] and others.

5.5 Numerical Scheme

Particle tracing involves solving the differential equation given in 1.1. We used a highly accurate Bulirsch-Stoer method described in Press et al. [1986]. $E$ and $P_y$ of most particles was conserved to better than one part in $10^6$ over the entire time of the simulation. Additional integrals were solved numerically with Gaussian or Romberg methods also described in Press et al..

The particle simulation as well as numerical and graphical analyses were performed primarily on an HP 750 series 9000. Additional numerical work was performed on Convex, Intel Paragon, Sun, Apollo, Dec Alpha, and Macintosh machines.
Chapter 6

Results

We now present several different types of graphs which reveal the behavior of an ensemble of particles in a magnetotail field model. The average $y$ drift of the ensemble of particles versus time is shown for a variety of parameters. Particle dispersion is revealed with histograms showing the number of particles per $y$ bin at a given time. Both of these quantities are compared with analytical results predicted from adiabatic theory as discussed in previous chapters. Statistical behavior is presented for some of the simulation runs in the form of standard deviation calculations and related quantities. Table 6.1 is a comprehensive listing of the different parameters used corresponding to the different simulations runs presented here. We explore the behavior of ensemble averaged particle drifts for $0.1 \leq \kappaappa \leq 2$. As a rule of thumb, large $\kappaappa$ indicates that more of the particles behave adiabatically. Very small $\kappaappa$ means that many particles execute Chaotic or Speiser-type motion.

Table 6.1 Simulation runs

<table>
<thead>
<tr>
<th>Figures</th>
<th>Particle I.C.</th>
<th>B Model</th>
<th>$\frac{E_n}{B_0}$</th>
<th>$\kappaappa$</th>
<th>$\frac{E_{\phi}}{B_n}$</th>
<th>$\frac{E_{\phi}}{v_{\phi}B_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.6</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>1.99</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.1, 6.2</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>0.207</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.4</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.02</td>
<td>0.200</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.8</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.05</td>
<td>0.500</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.10, 6.11</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.05</td>
<td>0.104</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.12, 6.13</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>1.00</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>6.15, 6.16</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>0.750</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>6.18, 6.19</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>0.500</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>6.21, 6.22</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>0.207</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>6.24, 6.25</td>
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<td>equilibrium</td>
<td>0.1</td>
<td>0.500</td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.26, 6.27</td>
<td>uniform</td>
<td>equilibrium</td>
<td>0.1</td>
<td>0.207</td>
<td>1.0</td>
<td>0.1</td>
</tr>
<tr>
<td>6.28</td>
<td>launched from $z = 0$</td>
<td>equilibrium</td>
<td>0.1</td>
<td>1.00</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6.29</td>
<td>uniform</td>
<td>Harris</td>
<td>0.1</td>
<td>0.260</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
6.1 Interpreting the Graphs

There are many ways to calculate the average speed of a group of particles. We are interested in the average $y$ displacement of the ensemble of particles described by

$$\langle v_y \rangle_{d,o} = \frac{1}{N} \sum_i^N \frac{y_i}{t_i} \approx \frac{1}{N} \sum_i^N \frac{y_i}{T}$$  \hspace{1cm} (6.1)

Simulation averages were calculated by this equation. In addition, two different analytical results were plotted for comparison. Equation 3.13 substituted into 2.39 gives the exact drift, versus time, of a representative particle which moves with the average drift of the ensemble of particles. This equation does not depend on any mathematical approximation and is henceforth labeled “analytical exact average” on the graphs. Any difference between this value and the numerical results is an indication that some factor not yet discussed is at play. Equation 3.15 substituted into 2.39 gives the predicted adiabatic drift speed and is labeled “analytical adiabatic average” on the graphs.

The graphical results presented here make use of the normalized units listed in table 6.2. In particular, note that after normalization the lobe gyroperiod is $2\pi$, the thermal speed is 1, and the lobe gyro-radius is 1. In addition, recall that for the $2\frac{1}{2}$D magnetic field case, a corresponding perpendicular electric field is imposed. Specifying a value for $E_{y0}$ and $B_{y0}$ determines $E_z$.

**Table 6.2** In the plots presented here, the following simulation normalizations are used. The actual values of $B_0$ and $v_{th}$ are not important. The dimensionless quantities such as $\frac{B_0}{B_n}$ determine all of the physics of our system. Time is given in units of $\frac{1}{\Omega_0}$ so that one lobe gyroperiod is $2\pi$.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>lobe gyro-frequency</th>
<th>lobe gyro-radius</th>
<th>lobe gyro-period / (2$\pi$)</th>
<th>thermal velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
<td></td>
<td>$\Omega_0 = \frac{v_{th}}{m_p}$</td>
<td>$\rho_0 = \frac{v_{th}}{\Omega_0}$</td>
<td>$\tau_0 = \frac{1}{\Omega_0}$</td>
</tr>
<tr>
<td>time</td>
<td></td>
<td>$\tau_0$</td>
<td>$v_{th}$</td>
<td>$v_{th} = \rho_0 \Omega_0$</td>
</tr>
</tbody>
</table>

It’s worth walking through an analysis of a typical graph of ensemble averaged drift, e.g. figure 6.1. There are three curves presented on a plot of $\langle v_y \rangle$ versus time. The results indicate that after tens of lobe gyro periods (a few equatorial gyroperiods), our simulation particles begin to drift, on average, according to our exact equations.
Figure 6.1 $\kappa = 0.207$.  $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle = \frac{1}{N} \sum_{i=1}^{N} v_{y,i}$.

$\frac{B_{x0}}{B_0} = 0.0$, $\frac{B_n}{B_0} = 0.1$, $\frac{E_y}{v_{th}B_n} = 0.0$. Notice that the adiabatic approximation differs from the exact value by less than 2%.

$3.13 + 2.39$. Our method of filling the flux tube should yield an average instantaneous $v_y$ which is the same as our exact formula while individual particles' $v_y$ range from $-v_{th} \rightarrow v_{th}$. However, due to the statistical error of using a finite number of particles the simulation takes some time to approach this value.

A Note on Errors

For the data of figure 6.1 $\langle v_y \rangle_{exact}$ is expected to be about $0.08v_{th}$. Since we use only one thousand particles, we expect the statistical error in our average drift to be about $\frac{0.08v_{th}}{\sqrt{1000}} \approx 0.03v_{th}$. Thus the expected percentage error in the initial value of the average drift is pretty big – about 40%. The expected statistical error in the average $v_y$ should decline on a time scale of an equatorial gyroperiod ($2\pi$ in our normalized units). The statistical accuracy of a drift calculation increases with increasing number of particles.

The difference between the analytical exact and adiabatic curves is due to the error of neglecting higher order terms in the Taylor expansion performed in section
2.2.5. This error is less than 2% for this extreme value of \( \kappa \) and is even better for most runs. Note that the exact and adiabatic curves on most plots lie on top of one another.

**Dispersion**

As mentioned in chapter 4, we are not only interested in comparing the mean drift of chaotic particles to that of adiabatic ones. We also wish to determine whether or not chaotic scattering disperses particles about their mean drift faster than predicted by adiabatic theory. As the numerically traced particles bounce around and slowly drift in the \( +y \) direction, we bin them according to their \( y \) position and plot these as histograms. Shown here are the actual particle positions as opposed to their guiding center positions. Binning them according to their guiding center positions would remove some of the noise produced by their gyration in \( y \). However, for \( \kappa < 1 \), the guiding center has no clear meaning. In any event the results, though not shown here, are nearly indistinguishable.

We use equations 4.24, 4.25, and 4.43 to analytically calculate the histograms that would be produced by adiabatically moving particles. We could also have pushed the guiding centers of our actual particles according to equations 1.2 and 1.3 and compared these adiabatic results to actual particle motions.

Consider the histograms depicted in figure 6.2. This is different information about the same data set as used for figure 6.1. The vertical axis indicates the number of particles in each bin. Overlayed on top of this plot is the histogram curve predicted by adiabatic theory using the same bin values. For a given simulation run, three different times (top \( \rightarrow \) bottom = increasing time) are shown. The analytical curve should begin at \( y = 0 \) and extend out beyond what is drawn. The area underneath both plots (were the analytical ones extended outward) should be identically 1000 since the number of particles is conserved. The mean \( y \) value of both curves agrees to within 2% as noted from figure 6.1.

The histogram curves take significantly longer than the ensemble averaged curves to yield results comparable to that predicted by adiabatic theory. This is because the histogram displays individual particle information. Clearly a particle must execute at least one bounce before its bounce averaged drift is revealed. Particles which take longest to bounce are those that begin in the strong plasma sheet 'lobe' field with large pitch angles. They gyrate in tight orbits and gradient drift towards \(-y\).
Figure 6.2 \( \kappa = 0.207 \). Particle dispersion about its mean drift is revealed with a time series of histograms showing the number of particles versus y bin. Numerically traced exact particle motion is compared to adiabatic theory. Chaotic pitch angle scattering narrows the simulation curve and makes its shape more Gaussian. \( \frac{B_0}{B_n} = 0.0, \frac{B_n}{B_0} = 0.1, \frac{E_{\|}}{\nu_n B_n} = 0.0. \)
Until these particles make it to the equatorial plane, the simulated histograms show a smattering of particles at small $y$.

The shape of the analytical curve depends on the magnetic field and the way in which particles were distributed. According to analytical theory, a particle's $y$ drift speed is a function of its adiabatic invariants. These do not change. Thus, a particle which starts out drifting rapidly continues to do so. For our magnetic field, the fastest drifting particle bounces in the equatorial plane and corresponds to adiabatic invariant values of $J = 0$ and $\mu = \mu_{\text{max}}$. Likewise, the slowest particle is entirely field aligned with $\mu = 0$. This particle spends all of eternity going down a lobe field line, never bouncing. Of course, in the real magnetosphere, such a particle would enter a region of high plasma density where it could collide with other particles and drizzle down into the lower ionosphere, never to return to the equator.

To obtain a more quantitative measure of particle dispersion as well as shed some light on particle diffusion, we plot a few statistical quantities. The following formulae are used to calculate the simulated particles' average position, $Y$, standard deviation, $\sigma_y$, and diffusion coefficient, $D_y$ at a particular time.

$$\langle y \rangle = \sum_{i}^{N} y_i$$

(6.2)

$$\sigma_y = \sqrt{\frac{1}{(N-1)} \sum_{i}^{N} (y_i - \langle y \rangle)^2}$$

(6.3)

$$D_y \equiv \frac{\sigma_y^2}{2T}$$

(6.4)

Of particular interest is a comparison of the standard deviation as predicted by adiabatic theory to the simulation values. We may compute an analytical standard deviation by converting the summation to an integral.

$$\sigma_y^2 \equiv \frac{1}{(N-1)} \sum_{i}^{N} (y_i - \langle y \rangle)^2$$

(6.5)

$$= \frac{1}{(N-1)} \int_{0}^{N} (y(i) - \langle y \rangle)^2 \, di$$

(6.6)

$$= \frac{1}{(N-1)} \int_{0}^{\mu_{\text{max}}} (y(\mu, t) - \langle y(t) \rangle)^2 \frac{d\mu}{d\mu} \, dy$$

(6.7)

$$\sigma_y^2(t) = \frac{1}{(N-1)} \int_{\mu_{\text{min}}}^{\mu_{\text{max}}} (y(\mu, t) - \langle y(t) \rangle)^2 \frac{d\mu}{d\mu} \, d\mu$$

(6.8)

We already have a functional form for $\frac{d\mu}{d\mu}$ (cf. equation 4.43). The analytically calculated standard deviation is displayed along with that of the simulated particles.
for several of the simulation runs. One should compare the slopes of these two curves in figure 6.3B. The initial sharp increase of $\sigma_y$ for the simulated particles shows the particles spreading out in $y$ from their initial $y = 0$ positions. Until the mean position of the ensemble drifts an equatorial gyroradius, the standard deviation calculation has a high noise level. For figure 6.3B, the drift velocity of about $0.08v_{th}$ suggests that we must wait at least $125\Omega_0^{-1}$ or about $12\Omega_n^{-1}$ until our calculated $\sigma_y$ is statistically valid.

Even without numerically integrating equation 6.8, we may conclude that $\sigma$ should be linearly proportional to time when no electric field is present. This is because both the individual particle's bounce averaged drift velocity and the ensemble averaged drift velocity are independent of time, hence their positions are linearly proportional to time. The slope of the standard deviation of the chaotic ensemble is always less than that of the adiabatic curve for the parameter ranges tested. The dispersion of particles about the mean drift narrows compared to adiabatic particles. We also point out that the simulation standard deviation is not always linearly proportional to time. Even in these cases, the curvature $\sigma_y$ versus time is always negative.

In figure 6.3 we display some quantitative statistical information about the particle motion. In A) the ensemble average $y$ position reveals a line of constant slope. With no electric field, the average drift velocity (slope) is constant. Error bars indicate the instantaneous minimum and maximum $y$ position of particles at that time and correspond to the 'wings' of the histogram curves. Figure 6.3B shows the standard deviation of the ensemble of particles along with the analytically calculated value according to adiabatic theory. Notice that the slope of the standard deviation of the simulated particles decreases with time. It is not a straight line as is expected from adiabatic theory. On the other hand, it does not increase with time as would be expected if chaotic particle motion gave rise to a fast diffusion process. In figure 6.3C the diffusion coefficient was calculated.
Figure 6.3 $\kappa = 0.207$. A) The average particle position with error bars indicating instantaneous particle $y_{\text{min}}$ and $y_{\text{max}}$, B) the standard deviation according to adiabatic theory and simulated exact particle motion, and C) the diffusion coefficient, $D_y$. The slopes of the two lines in B reveal that non-adiabatic particles experience a slower diffusion rate than adiabatic ones.
Figure 6.4 shows $\langle v_y \rangle_{d,e}$ for a similar ensemble. In this case, the magnetic field parameter $B_n/B_0$ was decreased to 0.02. This reduces the value of $\kappa$ to 0.200. Figure 6.5 shows statistical information about this same simulation run.

![Graph showing $\langle v_y \rangle_{d,e}$ versus time for $\kappa = 0.200$. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{y_i}{t_i}$. \[ \frac{B_{y0}}{B_0} = 0.0, \quad \frac{B_{y1}}{B_0} = 0.02, \quad \frac{E_{x0}}{v_{th}B_n} = 0.0. \]
Figure 6.5 $\kappa = .200$. A) The average particle position with error bars indicating instantaneous particle $y_{\text{min}}$ and $y_{\text{max}}$, B) the standard deviation according to adiabatic theory and simulated exact particle motion, and C) the diffusion coefficient, $D_y$. 
Figure 6.6 shows \( \langle v_y \rangle_{d,e} \) for a case when the magnetic field parameter \( B_n/B_0 = 0.1 \) and \( \kappa = 1.99 \). We expect this simulation run to exhibit characteristics similar to an adiabatic ensemble. Particle dispersion of the adiabatically predicted ensemble and simulated particles is about the same. This is evident from figure 6.7B, which shows that the slopes of the standard deviation curves of the adiabatic and simulated particles is about the same.

![Graph showing \( \langle v_y \rangle_{d,e} \) versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where \( \langle v_y \rangle = \frac{1}{N} \sum_i v_{yi} \).

\[ \frac{B_{n0}}{B_0} = 0.0, \quad \frac{B_n}{B_0} = 0.1, \quad \frac{E_{n0}}{v_{iBn}} = 0.0. \]

Figure 6.6 \( \kappa = 1.99 \). \( \langle v_y \rangle_{d,e} \) versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where \( \langle v_y \rangle = \frac{1}{N} \sum_i v_{yi} \).

For a given \( \frac{B_n}{B_0} \) ratio, increasing particle energy decreases \( \kappa \) and increases the gradient-curvature drift speed. No surprises here. As particle energy is increased, the "adiabatic" approximation of the particle drift becomes progressively worse as can be predicted completely analytically. However, for the cases shown here, the difference between the exact and adiabatic drift formula never exceeds 2% and is usually much lower. In other words, the basic regime of validity for the ensemble averaged drift equation far exceeds that predicted by single particle adiabatic approximations.
Figure 6.7 $\kappa = 1.99$. A) The average particle position with error bars indicating instantaneous particle $y_{\text{min}}$ and $y_{\text{max}}$, B) the standard deviation according to adiabatic theory and simulated exact particle motion, and C) the diffusion coefficient, $D_y$. The slope of the two lines in B are nearly the same.
Figure 6.8 shows \( \langle v_y \rangle_{d,e} \) for a case when the magnetic field parameter \( B_n/B_0 = 0.05 \) and \( \kappa = 0.500 \). This short-time simulation run yields the predicted ensemble averaged drift speed. However, particle dispersion is dominated by the initial conditions. Figure 6.9A reveals that not even all of the particles have drifted a single equatorial gyroradius.

**Figure 6.8** \( \kappa = 0.500 \). \( \langle v_y \rangle_{d,e} \) versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where \( \langle v_y \rangle = \frac{1}{N} \sum_{i}^{N} v_{yi} \). \( \frac{B_{\parallel 0}}{B_0} = 0.0, \frac{B_n}{B_0} = 0.05, \frac{E_{\parallel 0}}{v_{th} B_0} = 0.0 \).
Figure 6.9 $\kappa = 0.500$. A) The average particle position with error bars indicating instantaneous particle $y_{\text{min}}$ and $y_{\text{max}}$, B) the standard deviation according to adiabatic theory and simulated exact particle motion, and C) the diffusion coefficient, $D_y$. In this short run case, $\sigma_y$ is dominated by the initial spreading out of particles.
Figure 6.10 shows $\langle v_y \rangle_{d,e}$ for a case when the magnetic field parameter $B_n/B_0 = 0.05$ and $\kappa = 0.104$, the most extreme value we tested. The Poincaré SOS plot for these parameters was shown in chapter 1. The adiabatic drift speed differs slightly from the exact drift speed due to a large gradient in the flux tube volume. Histograms (figure 6.11) reveal that the distribution of particles in this drift velocity space is less Gaussian than might be expected from such a low $\kappa$ case.

![Graph showing drift velocity vs time](image)

**Figure 6.10** $\kappa = 0.104$. $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle = \frac{1}{N} \sum_i^{N} \frac{v_i}{t_i}$.

$\frac{B_{y0}}{B_0} = 0.0$, $\frac{B_n}{B_0} = 0.05$, $\frac{E_{y0}}{v_{th}B_n} = 0.0$. 


Figure 6.11 $\kappa = 0.104$. Histogram plots revealing particle dispersion. Notice that the simulated particle distribution resembles the adiabatic distribution to a greater degree than other figures. This indicates less randomization of pitch angle due to chaotic scattering. $B_{x0}/B_0 = 0.0, \quad B_{z0}/B_0 = 0.05, \quad E_{y0}/v_{th}B_0 = 0.0$. 
6.1.1 Simulations with $E_y \neq 0$

The effect of a perpendicular electric field is to $E \times B$ drift the particles towards $+x$, a region of increasing magnetic field strength. This tends to decrease $\langle v_y \rangle_{d,e}$. The electric field also energizes the particles as they drift towards $+y$, a region of lower electric field potential. This tends to increase drift speed. These two effects do not exactly cancel each other for the cases shown here. In fact, the net effect is that $\langle v_y \rangle_{d,e}$ decreases with time. This is clearly evident in all runs which include an electric field. In addition, the results are not altered significantly by the presence of $B_y(x, z)$.

Just as for previous cases, we display three types of information. $\langle v_y \rangle_{d,e}$ versus time, a time series of histogram plots, and statistical information. Figures 6.12–6.14 show the case when $B_n/B_0 = 0.1$, $E_{y0}/(v_{th}B_n) = 0.1$, $\kappa = 1.0$. As expected, the ensemble averaged drift velocity slows in time and is well predicted by our adiabatic formula. Figure 6.14B shows that the standard deviation of this ensemble becomes a straight line after the initial noise is washed out.

![Graph](image.png)

**Figure 6.12** $\kappa = 1.0$. $\langle v_y \rangle_{d,e}$ versus time for a case with a nonzero electric field. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle = \frac{1}{N} \sum_i^N \frac{y_i}{t_i}$, $B_{y0}/B_0 = 0.0$, $B_{y0}/B_0 = 0.1$, $E_{y0}/v_{th}B_n = 0.1$. 
Figure 6.13 \( \kappa = 1.0 \). Histogram plots revealing particle dispersion. \( \frac{B_{x0}}{B_0} = 0.0, \frac{B_y}{B_0} = 0.1, \frac{E_{y0}}{v_{th}B_n} = 0.1 \).
Figure 6.14 \( \kappa = 1.00 \). A) The average particle position with error bars indicating instantaneous particle \( y_{\text{min}} \) and \( y_{\text{max}} \), B) the standard deviation, and C) the diffusion coefficient, \( D_y \).
Figures 6.15–6.17 show a case when $B_n/B_0 = 0.1$, $E_{y0}/(v_{th}B_n) = 0.1$, $\kappa = 0.750$. As expected, the ensemble averaged drift velocity slows in time and is well predicted by our adiabatic formula. The histograms of figure 6.16 are narrowed more than in figure 6.13. The reasons for this trend is discussed in section 6.2. Figure 6.17B shows that the standard deviation of this ensemble becomes a straight line after the initial noise is washed out.

![Simulation average drift](image)

**Figure 6.15** $\kappa = 0.750$. $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle_{d,e} = \frac{1}{N} \sum_i^N v_{yi}$, $\frac{B_{y0}}{B_0} = 0.0$, $\frac{B_a}{B_0} = 0.1$, $\frac{E_{y0}}{v_{th}B_n} = 0.1$. 
Figure 6.16 $\kappa = 0.750$. Histogram plots revealing particle dispersion. $B_{0}/B_0 = 0.0$, $B_{0}/B_0 = 0.1$, $E_{0}/v_{th}B_0 = 0.1$. 
Figure 6.17  $\kappa = 0.750$. A) The average particle position with error bars indicating instantaneous particle $y_{\min}$ and $y_{\max}$; B) the standard deviation, and C) the diffusion coefficient, $D_y$. 
Figures 6.18–6.20 show a case when $B_n/B_0 = 0.1$, $E_{y0}/(v_{th}B_n) = 0.1$, $\kappa = 0.500$. As expected, the ensemble averaged drift velocity slows in time and is well predicted by our adiabatic formula. However, a slight divergence from theory is observed. This is discussed in section 6.2. The histograms of figure 6.19 show the narrowing due to pitch angle scattering. Figure 6.20B is approximately linear.

![Graph showing average drift](image)

**Figure 6.18** $\kappa = 0.5$. $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle_{d,e} = \frac{1}{N} \sum_{i=1}^{N} y_i$, $\frac{B_{y0}}{B_0} = 0.0$, $\frac{B_n}{B_0} = 0.1$, $\frac{E_{y0}}{v_{th}B_n} = 0.1$. 
Figure 6.19 \( \kappa = 0.5 \). Histogram plots revealing particle dispersion. \( \frac{B_n}{B_0} = 0.0, \frac{B_n}{B_0} = 0.1, \frac{E_{\text{rot}}}{\nu_{\text{th}}B_n} = 0.1 \).
Figure 6.20 $\kappa = 0.500$. A) The average particle position with error bars indicating instantaneous particle $y_{\text{min}}$ and $y_{\text{max}}$, B) the standard deviation, and C) the diffusion coefficient, $D_y$. 
Figures 6.21–6.23 show a case when $B_n/B_0 = 0.1$, $E_y v_0/(v_{th} B_n) = 0.1$, $\kappa = 0.207$. As expected, the ensemble averaged drift velocity slows in time and is well predicted by our adiabatic formula. The histograms of figure 6.22 show a high degree of narrowing due to pitch angle scattering. Figure 6.23B shows that the standard deviation of this ensemble also becomes a straight line after the initial noise is washed out.

We note that a comparison of figures 6.14C, 6.17C, 6.20C, and 6.23C reveals that one cannot derive a simple relationship between $D_y$ and $\kappa$.

![Plot of drift velocity vs time](image)

**Figure 6.21** $\kappa = 0.207$. $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program. $\frac{E_y v_0}{B_0} = 0.0$, $\frac{B_n}{B_0} = 0.1$, $\frac{E_y v_0}{v_{th} B_n} = 0.1$. 
Figure 6.22 $\kappa = 0.207$. Histogram plots revealing particle dispersion. $\frac{B_{02}}{B_0} = 0.0$, $\frac{B_{03}}{B_0} = 0.1$, $\frac{B_{y0}}{B_0} = 0.1$, $v_{th}B_n = 0.1$. 
Figure 6.23 $\kappa = 0.207$. A) The average particle position with error bars indicating instantaneous particle $y_{\text{min}}$ and $y_{\text{max}}$, B) the standard deviation, and C) the diffusion coefficient, $D_y$. 
6.1.2 Simulations with $B_y \neq 0$

The addition of a $B_y$ component does not affect our results. Figures 6.24 and 6.25 should convince you of this fact. Recall, however, that we are now comparing $\langle v_y \rangle_{d,e}$ to our function, $\psi$ as discussed in chapter 5.

**Figure 6.24** $\kappa = 0.5$. $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle_{d,e} = \frac{1}{N} \sum_i^N \frac{y_i}{t_i}$. $\frac{B_{eo}}{B_0} = 1$, $\frac{B_{eo}}{B_0} = 0.1$, $\frac{E_{eo}}{v_{th}B_0} = 0.0$. 
Figure 6.25 $\kappa = 0.5$. Histogram plots revealing particle dispersion. $B_y/B_0 = 1$, $B_n/B_0 = 0.1$, $E_{y0}/B_n = 0.0$.
6.1.3 Simulations with $B_y \neq 0$, $E \neq 0$, and $E \cdot B = 0$

Even in the presence of a perpendicular electric field, the addition of a $B_y$ component does not affect our results. Figure 6.26 is identical to a portion of figure 6.21. The histogram curves of figure 6.27 are similarly unchanged from those produced without $B_y$.

![Graph showing simulation results](Image)

**Figure 6.26** $\kappa = 0.5$. $\langle v_y \rangle_{d,e}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program where $\langle v_y \rangle_{d,e} = \frac{1}{N} \sum_i^{N} \frac{\nu_i}{t_i}$. $\frac{B_y}{B_0} = 1$, $\frac{B_n}{B_0} = 0.1$, $\frac{E \cdot B}{v_n B_n} = 0.1$. 
Figure 6.27 \( \kappa = 0.207 \). Histogram plots revealing particle dispersion. \( \frac{B_0}{B_e} = 1, \frac{B_n}{B_0} = 0.1, \frac{E_{\parallel 0}}{B} = 0.1, \frac{\nu_{hi} B_n}{B} = 0.1 \).
6.1.4 A Different Distribution and Magnetic Field Model

To explore the limits of validity of these results, simulation runs were made for different initial distributions and B fields. In figure 6.28 the average $y$ drift velocity is shown for a run where particles were all initially launched from the equatorial plane, $y = 0, z = 0$. This distribution function does not initially satisfy our requirement that it be function only of constants of motion. However, due to the ergodic nature of the particle orbits for the $\kappa = 1$ case, particles eventually fill up the entire phase space accessible to them. The trend is for the particles to approach the adiabatic value.

![Graph showing average drift velocity over time](image)

**Figure 6.28** $\kappa = 1.00$. $(v_y)_{d,0}$ versus time. The dashed lines indicate the analytic curves predicted by exact and adiabatic theory. The line/points indicate the results of a numerical particle tracing program. $\frac{B_{\text{ho}}}{B_0} = 0.0, \frac{B_{\text{h}}}{B_0} = 0.1, \frac{E_{\text{ho}}}{v_{\text{th}}B_{\text{n}}} = 0.0$. For this run, the initial particle distribution was changed so that they are all launched from the equatorial plane $z = 0, y = 0$.

In using a parabolic field model with no explicit $x$ dependence, we introduce an $x$ gradient in the magnetic flux tube volume by imposing a reflecting wall at some fixed $x_{\text{max}}$. For geometrical reasons, a field lines with their equatorial feet at different $x$ values map to different $z$ locations along this bounding wall. Hence, they have different field line lengths and flux tube volume values.
Figure 6.29 $\kappa = 0.260$. $\langle v_y \rangle_d$ versus time for a Harris type magnetic field model. An $x$ dependence in the flux tube volume was introduced by putting hard walls at some chosen $x$ value. indicate the results of a numerical particle tracing program. $B_{x0} = 0.0$, $B_{y0} = 0.1$, $E_{\phi 0}/v_{th y0} B_{z0} = 0.0$.

6.2 Discussion

Note that particles drift in $y$ most rapidly close to $z = 0$. It is here that the magnetic field direction is primarily towards $+z$ and curvature is strongest. This is true for all $\kappa$ values.

6.2.1 Why the Distribution Narrows

Both the high end and low end of the $y$ distributions are cut off as compared to what adiabatic theory predicts (cf. histogram figures). The standard deviation calculations reinforce this conclusion and indicate that the effect of including particles executing chaotic motion in our ensemble was to reduce the dispersion as compared to adiabatic theory and to make the histograms more Gaussian in shape. This trend may come as a surprise, but can be easily explained. As mentioned previously, the adiabatic curve consists of particles drifting at various rates which are functions of their adiabatic constants of motion. The equatorially bouncing particles drift fastest. A field aligned particle drifts most slowly. The effect of chaotic pitch angle scattering (discussed below) is to prevent non-adiabatic particles from retaining their adiabatic constants
of motion. The fastest particle does not maintain its speed. Neither does the slowest. If chaotic scattering is truly ergodic, then, on average, the number of particles within a given range of $\mu$ or equatorial pitch angles is conserved. However, no single particle can maintain an extremely fast or extremely slow drift speed for a long time. Thus, the ‘wings’ of the distribution are cut off. The stronger this randomizing effect becomes, the more Gaussian the histogram becomes.

I should emphasize that the histograms we are plotting indicate dispersion in the $y$ drift velocity. They are not equivalent to the distribution function of particles in phase space. Thus, it is no contradiction to say that chaotic particle scattering of pitch angles can narrow the distribution of our histograms. The distribution of particles in $v_x - v_z$ space remains isotropic throughout the simulation. In other words, along a field line, the particle distribution function depends on $v_x$ and $v_z$ only through the sum of their squares. This dependence does not change with time. Chaos does not decrease entropy.

For a given ratio of $\frac{\mathbb{E}_0}{\mathbb{B}_0}$, decreasing $\kappa$ from $1.00 \to 0.207$ (by increasing energy) increases this scattering effect with or without an electric field (compare figures 6.13, 6.19, and 6.22). In this parameter range, lowering the value of $\kappa$ decreases the dispersion in drift velocities relative to the average drift velocity. According to our explanation, this implies that pitch angle scattering is more effective for the lower $\kappa$ cases and is not necessarily maximized at $\kappa = 1$.

### 6.2.2 Why is $\langle v_y \, \text{drift} \rangle$ too high when $\mathbb{E} \neq 0$?

If the initial velocity distribution of particles is not corrected for the presence of the electric field (as described in section 5.1), the particles tend to accumulate within the equatorial region of the flux tube. This is because in the $\mathbb{E} \times \mathbb{B}$ drifting frame, the particle distribution is not isotropic, but rather skewed towards $-x$ and hence towards the equatorial crossing region of the flux tube. Since the $y$ drift velocity is significantly higher in this region of high curvature than it is in the lobe region, the average drift of the ensemble would increase.

Perhaps, if we wait long enough, the distribution would naturally evolve towards isotropy in the $\mathbb{E} \times \mathbb{B}$ drifting frame as the particles were pitch angle scattered as was the case when a non-uniform distribution of particles was allowed to evolve without the electric field (figure 6.28). Why do we have to worry so much about setting up the proper initial distribution of particles? Shouldn’t chaotic scattering smooth out any
weird kinks we may inadvertently put into the distribution? In fact, even when we do take great care in establishing our initial distribution, there is a small disagreement between our analytical average drift velocity curves and the simulation results. For example, see figure 6.21. The disagreement is only a few percent. Nevertheless, it highlights a limitation of this sort of analysis.

It may be that our electric field strength was a bit high. We previously stated that we must limit our electric field strength to prevent inertial drifts from becoming too large (cf. equation 5.12). In the results presented here, \( \left( \frac{E_{\phi}}{v_{B_n}} \right)^2 = 0.01 \). This doesn’t seem too large.

Another possible explanation may be in our interpretation of the analytical results. Recall that we are comparing the average drift of the simulated particles with an analytical formula which technically describes the motion of a single energy representative particle. Our simulation particles spread out in \( y \) and hence pick up a small dispersion in kinetic energy even though total energy is conserved. The drift velocity formulas depend on kinetic energy. If this dispersion of kinetic energy becomes too great, as it does when \( E \) gets too large, our comparison of simulation results to theory becomes invalid. This sort of error may be minimized if the \( x \) gradient scale length of \( B_z \) is long compared to the length of the foot of the flux tube (i.e. \( \alpha \rho_n \ll 1 \)). This might also explain why we must take care in making sure that the initial particle velocity distribution includes the \( v_{E \times B} \) effect. Without initializing the distribution properly, the dispersion in particle kinetic energy grows too large for proper comparison to a single energy, representative particle.

6.3 What’s Old? What’s New?

While this work is a direct extension of the WP proof, there is quite a bit of history in analyzing particle dynamics in the magnetotail. Chen [1992] has an extensive list of references which we will not attempt to repeat here. Rather, we will draw your attention to what we see as a particular train of thought regarding the applicability of adiabatic theory to non-adiabatic particles. We also discuss previous work which supports our claim that chaotic particle scattering does not lead to enhanced particle loss or cross tail acceleration.
6.3.1 1D B fields

There are a number of papers on particle dynamics in 1D magnetic field models, such as the Harris type described by equation 5.14. Sonnerup [1971] studied the extension of adiabatic theory to particle motion in regions of magnetic field reversals. He characterized such motion by the third adiabatic invariant, $I = \oint p_z dz$. In 1975, Stern and Palmadesso showed that an adiabatic particle moving within a parabolic magnetic field with no $x$ gradient has a zero net drift because the curvature drift in the equatorial plane is exactly canceled by the gradient drift in the lobe. Cowley [1978] showed that this result is quite general and includes non-adiabatic particles and an electric field, $E_z$, which cannot be transformed away. The result fell out of the simplicity of the magnetic field model. Plugging any $x$-independent magnetic field model into the bounce averaged drift equation yields a net $y$ drift of zero.

We have already mentioned that a particle's pitch angle, energy, and $\mu$ are geometrically related to each other. When a particle crosses the equatorial plane non-adiabatically, its pitch angle is said to be scattered or changed. Though this process is non-adiabatic, it may not be random. Considerable effort has been devoted to the characterization of this chaotic scattering process. For instance, Chen [1992] calculates a finite time version of the Lyapunov exponent shown in equations 1.14 and 6.9. He then averages this modified Lyapunov exponent over an ensemble of monoenergetic particles with different pitch angles to derive the average exponential divergence rate (AEDR).

\[
\sigma^* \equiv \frac{1}{L\Delta \tau} \sum_j^L \left[ \ln \left( \frac{d_j}{d_0} \right) \right] \quad (6.9)
\]

\[
\text{AEDR} \equiv \frac{1}{M} \sum_m \sigma_m^* \quad (6.10)
\]

where $L$ is the total number of times steps from the first to last $z = 0$ crossing. $\sigma_m^*$ is the divergence rate for the $m^{th}$ particle trajectory; there are $M$ trajectories in the ensemble. Chen claims that AEDR characterizes the degree of divergence during the process of chaotic particle scattering. His results indicate that AEDR increases with decreasing $\kappa$ and does not peak at $\kappa = 1$. This agrees with our results which show that, for a given $\frac{B_\parallel}{B_0}$ ratio, the $y$ particle distribution narrows more as $\kappa$ decreases.

A seemingly contrary result is presented by Burkhart et al. [1995], who support the traditional notion that particle pitch angle scattering is maximized for $\kappa = 1$ when $\kappa\frac{B_\parallel}{B_n} > 1$. Their conclusion is based on a calculation of the average change in a
particle's pitch angle per equatorial crossing. It does not reveal the average change of an ensemble of isotropically distributed particles, which is our interest.

![Particle trajectory near equatorial plane showing a net $\Delta y$ motion.](image)

**Figure 6.30** Particle trajectory near equatorial plane showing a net $\Delta y$ motion.

Recall that the motion of a Speiser type particle as it crosses the equatorial plane may involve some net $\Delta y$ as shown in figure 6.30. For a 1D field, a relatively simple relationship between incoming pitch angle, outgoing pitch angle, particle velocity, and $\Delta y$ was derived by Cowley [1978].

$$\Delta y \propto \sqrt{E_k \left| \cos \beta_2 \right| + \left| \cos \beta_1 \right|}$$

(6.11)

where $\beta_{1,2}$ are the incoming and outgoing pitch angles. Burkhart and Chen [1992] examine the dependence among $\beta_1$, $\beta_2$, and particle energy. Holland and Chen [1992] have generalized Cowley’s result to show that the ensemble averaged incoming and outgoing pitch angles $(\langle \beta_1 \rangle$ and $\langle \beta_2 \rangle$ respectively) of a distribution of particles is well conserved except when the incoming distribution is highly collimated. This result is philosophically quite similar to ours. Namely, one can recover an approximate invariant when averaging over a fairly isotropic ensemble of particles. We should also note that a major point of the paper by Holland and Chen is a refutation of the idea that one can derive a ‘chaotic conductivity’ as discussed in Horton and Tajima [1991b]. A particle’s kinetic energy gain is linearly proportional to $\Delta y$ where $\Delta y$ is independent of electric field amplitude as can be seen in equation 6.11. If this is the
case, then one cannot assume that \( J_y \propto E_{y0} \) nor that \( \Delta E_k \propto E_{y0}^2 \). All of the particles’ energization is described by equation 6.11. Chaotic effects are irrelevant.

_Chen and Holland_ also point out the bulk of the contribution to the cross tail current in the equatorial plane is due to chaotic particles. Particles with orbits that are integral do not contribute to this current density. Their conclusion is based entirely on the fact that their magnetic field model is \( x \) independent. In \( x \) dependent fields, even integral orbits contribute to the current density. Just look at figure 1.5C for an obvious example.

We would be remiss if we did not mention the important work of _Büchner and Zelenyi_ [1989,1990] and _Hurricane et al_. [1994]. In the spirit of _Sonnerup, Büchner and Zelenyi_ have developed a ‘quasi-adiabatic’ theory which allows them to characterize particle motion in \( x \) independent models of the magnetotail. _Hurricane et al._ have derived a linear Vlasov kinetic theory to treat the effect of waves on stochastic ions. Their computations of the properties of the zero-order solutions are generally very consistent with this thesis.

6.3.2 \( x \) Dependent Magnetic Field Models

_Karimabadi et al._ [1990] have discussed some of the differences between basic particle dynamics in magnetic field models with and without an \( x \) dependence. They note that in the presence of a cross tail electric field, \( E_y \), particles can accelerate to much higher energies in \( x \) dependent magnetic fields than in Harris-type models for the simple reason that particles spend more time near the \( z = 0 \) region. They also show that equation 6.11 is still valid in \( x \) dependent models if the \( x \) gradient scale length, \( L_x \), is much larger than the equatorial gyroradius. For our equilibrium magnetic field model this means,

\[
\frac{1}{B_z} \frac{\partial B_z}{\partial x} \rho_n = \alpha \rho_n = \left( \frac{B_n}{\kappa B_0} \right)^2 \ll 1
\]  

(6.12)

Our treatment, which concentrates on average drift rates of all particles in a flux tube includes cases when equation 6.12 is satisfied and some when it is not.

_Burkhart and Chen_ [1992] show that this regime where \( L_x \approx \rho_n \), is a region of enhanced integrability. In other words, the integrable area of the Poincaré sos plot should increase relative to the stochastic area (compare A abd B of figure 1.7). _Chen_ [1992] points out that the actual inner magnetotail is well described by an \( x \) independent magnetic field model in the sense that \( \rho_n \ll L_x \). However, the \( x \)
dependence in the magnetic flux tube volume makes all the difference when calculating bounce averaged drifts.

6.3.3 Other Limitations and Future Work

The choice of distribution function is not as limiting as it may seem. We have solved the drift equation for a monoenergetic distribution of particles within the same $X - Z$ projection of a chosen flux tube. In order to apply these results to a realistic distribution of particles, one can integrate equation 1.7 over all non-relativistic kinetic energies and flux tubes. Hence, once could mimic any desired distribution of particles provided one assume isotropy in $v_x - v_z$.

Since we have not really pushed the limits of the parameter regimes to test for validity, it is possible that there is some combination of parameters for which our results crumble. We have already noted a limit on the electric field strength as well as a limit on the $x$ magnetic field gradient. The numerical analysis presented here may be extended to test whether or not a $y$ dependence in the magnetic field might have significant consequences for ensemble averaged drifts. In particular, one could run test-particle simulations for fully 3D magnetic fields and compare the results to those of adiabatically pushed test-particles. Might we hazard a guess that adiabatic theory might even prevail in a truly 3D magnetic field?

More significant is the very real possibility that non-random scattering processes might significantly alter the ensemble averaged drift quantities. The results presented here require that a distribution of particles be isotropic in pitch angle. This is one definition of the plasma sheet. A simple interpretation of chaotic pitch angle scattering suggests that most distributions might attempt to isotropize. And yet, there is ample evidence for the existence of non-isotropic distribution functions near the tail current sheet (see Frank et al. [1994]). Thus, the tendency in nature for pitch angle distribution functions to isotropize is not always overwhelming. A simple example might include a non-linear $y$ dependence of the electric field potential. Should the electric field vary on a time scale comparable to the gyroperiod motion of particles, non-isotropic pitch angle scattering could skew the distribution of particles in phase space and invalidate the analytical approach began by Wolf and Pontius.
6.4 Conclusions

We have compared the predictions of adiabatic, analytical theory to numerical simulations of exact particle motion. The bounce-averaged gradient-curvature drift of an isotropic ensemble of chaotic particles in a 2D, $x$-dependent magnetic field model is well described by adiabatic theory provided that the gradient in magnetic field flux tube volume is not large compared to the equatorial gyroradius. This restriction is much weaker than the traditional one of $\kappa \gg 1$. A comparison of the dispersion about the mean drift between adiabatic and chaotic particles reveals that chaotic scattering of particle pitch angles tends to narrow the spread of particles. The distribution of our test particles in $y$ is always more narrow than the equivalent adiabatic particles. This would suggest that the usefulness of the ensembled averaged drift equations is improved by the presence of chaotic particle motion. We also note that the agreement between theory and numerical simulations justifies our interpretation of the meaning of the ensemble averaged drift equations.

For a given $\frac{B_A}{B_0}$ ratio, the effect of this particle pitch angle scattering increases with decreasing $\kappa$ to at least $\kappa = 0.1$. Changing the $\frac{B_A}{B_0}$ ratio changes the fraction of particles which execute integrable orbits, and hence changes the dispersive effect. However, in the parameter regime we analyzed, chaotic pitch angle scattering always contributes to a narrowing of the distribution of particles in $\langle v_y \text{ drift} \rangle$.

We have shown that the standard way of representing particle transport in the inner magnetosphere, namely the isotropic pitch angle, bounce averaged drift formalism, is valid for the central plasma sheet despite the presence of non-adiabatic particle motion. It legitimizes the use of bounce-averaged drift equations to enhance the effectiveness of ideal MHD models in treating the inner magnetosphere. It also eliminates the possibility that chaotic motion is the mechanism that nature uses to resolve the pressure-balance inconsistency, a long standing theoretical paradox involving thermodynamical arguments about the plasma content of the inner magnetotail (Erickson and Wolf, [1980]).
Chapter 7

References


8. Heinemann, M, Personal Communication, 1994


24. Wolf, R.A. and D. H. Pontius, Particle Drift in The Earth’s Plasma Sheet, 
Appendix A

Taylor Expansion

We show below the explicit Taylor expansion of equation 2.36 copied here for ease.

\[
\langle v_{y \text{exact}} \rangle_e = -\frac{q}{m} \int_{A_{y-(t)}}^{A_{y+(t)}} \frac{[A_y - A_{y0}(t)]V(A_y) dA_y}{\int_{A_{y-(t)}}^{A_{y+(t)}} V(A_y) dA_y}
\]

For simplicity of notation, let

\[
V(A_y) = \sum_{n=0}^{\infty} \frac{1}{n!} V^{(n)}(A_{y0}(t))(A_y - A_{y0}(t))^n \quad (A.1)
\]

\[
V^{(n)}(A_{y0}(t)) \equiv \left. \frac{d^n V(A_y)}{dA_y^n} \right|_{A_y = A_{y0}(t)} \quad (A.2)
\]

The denominator then becomes

\[
\mathcal{D} \equiv \int_{A_{y-(t)}}^{A_{y+(t)}} V(a_y) dA_y \quad (A.3)
\]

\[
= \int_{A_{y-(t)}}^{A_{y+(t)}} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} V^{(n)}(A_{y0}(t))(A_y - A_{y0}(t))^n \right] dA_y \quad (A.4)
\]

\[
= \left[ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} V^{(n)}(A_{y0}(t))(A_y - A_{y0}(t))^{n+1} \right]_{A_{y-(t)}}^{A_{y+(t)}} \quad (A.5)
\]

\[
= \left[ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} V^{(n)}(A_{y0}(t)) \left\{ \left( + \frac{m}{q} \sqrt{\frac{2\xi_K}{m}} \right)^{n+1} - \left( - \frac{m}{q} \sqrt{\frac{2\xi_K}{m}} \right)^{n+1} \right\} \right] \quad (A.6)
\]

\[
= 2 \sum_{n=0,2A \text{ even}}^{\infty} \frac{1}{(n+1)!} V^{(n)}(A_{y0}(t)) \left( \frac{m}{q} \right)^{n+1} \left( \frac{2\xi_K}{m} \right)^{n+1} \quad (A.7)
\]

\[
= \frac{2m}{q} \sqrt{\frac{2\xi_K}{m}} V(A_{y0}(t)) + 2 \left( \frac{m}{q} \right)^2 \frac{\xi_K}{m} \frac{dV(A_y)}{dA_y} \bigg|_{A_y = A_{y0}(t)} + \ldots \quad (A.8)
\]

Similarly, the numerator becomes

\[
\mathcal{N} \equiv \int_{A_{y-(t)}}^{A_{y+(t)}} (A_y - A_{y0}(t)) V(a_y) dA_y \quad (A.9)
\]
\[ \int_{A_y^+(t)}^{A_y^-(t)} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} V^{(n)}(A_{y_0}(t))(A_y - A_{y_0}(t))^{n+1} \right] dA_y \]  
(A.10)

\[ = \left[ \sum_{n=1, \text{odd}}^{\infty} \frac{1}{n!(n+2)} V^{(n)}(A_{y_0}(t))(A_y - A_{y_0}(t))^{n+2} \right]_{A_y^+(t)}^{A_y^-(t)} \]  
(A.11)

\[ = 2 \sum_{n=1, \text{odd}}^{\infty} \frac{1}{n!(n+2)} V^{(n)}(A_{y_0}(t)) \left( \frac{m}{q} \right)^{n+2} \left( \frac{2E_K}{m} \right)^{n+2} \]  
(A.12)

\[ = \left. \frac{2}{3} \left( \frac{m}{q} \right)^3 \left( \frac{2E_K}{m} \right)^{3/2} \frac{dV(A_y)}{dA_y} \right|_{A_y=A_{y_0}(t)} \]

\[ + \left. \frac{2}{8} \left( \frac{m}{q} \right)^4 \left( \frac{2E_K}{m} \right)^2 \frac{d^2V(A_y)}{dA_y^2} \right|_{A_y=A_{y_0}(t)} + \ldots \]  
(A.13)

If we include only the first terms by assuming:

\[ \rho \frac{|\nabla V(A_y)|}{V(A_y)} \bigg|_{A_y=A_{y_0}(t)} \ll 1 \]  
(A.14)

\[ |\nabla V| = \frac{dV}{dA_y} |\nabla A_y| = \frac{dV}{dA_y} |B| \]  
(A.15)

where

\[ \rho = \frac{v_{\text{thermal}}}{\Omega_{\text{cyclotron}}} , \quad v_{\text{th}} = \sqrt{\frac{2E_K}{m}} , \quad \Omega_c = \frac{qB}{m} \]

then equation 2.37 is obtained.