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RICE UNIVERSITY

Application of Markov Chains to the Critical Element Model for Determining the Fatigue Life of Composites

by

John David Rowatt

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

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April, 1995
Abstract

Application of Markov Chains to the Critical Element Model for Determining the Fatigue Life of Composites

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A stochastic model for predicting the lifetime of composite laminates subjected to multiaxial fatigue loading is proposed. The model is based on the application of Markov chains to the well known "critical element" model for the fatigue of composite laminates. The model considers the accumulation of fatigue damage as an evolutionary random process characterized by changes in the global compliance of a laminate. These changes are modeled as nonstationary, discrete time, discrete state Markov processes (Markov chains) utilizing stationary Markov chains and polynomial transformations of their indexing parameters. The stationary Markov chains are developed on the assumption of "equivalent damage". Their parameters are determined from experimental data. The Markov chain models yield full cycle dependent probability distributions for the changes in laminate compliance. These changes and their respective distributions are used as input into a mechanical analysis to determine the stresses on the life controlling critical elements of a laminate. The stresses on the critical elements and their derived probability distributions are used in turn to predict the lifetime of a laminate based on Markov chain models of the fatigue behavior of the critical elements. The predictive capability of the proposed model is demonstrated by comparison with experimental results.
Acknowledgments

How blessed is the man who finds wisdom,
And the man who gains understanding.

For its profit is better than the profit of silver,
And its gain more fine that gold. - Proverbs 3:14,15

I wish to thank all of those who have aided in my pursuit of wisdom and understanding. Special thanks to my dissertation advisor Dr. Pol D. Spanos. Thanks are also expressed to Dr. Enrique V. Barrera, Dr. Joel P. Conte, Dr. Scott R. White, Scott Miller, Bobby Eberle, my immediate and extended family, and my wife Cynthia.
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<tr>
<th>Symbol</th>
<th>Description</th>
<th>SI Units$^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>area</td>
<td>$[m^2]$</td>
</tr>
<tr>
<td>$A'$</td>
<td>extensional stiffness tensor</td>
<td>$[N/m]$</td>
</tr>
<tr>
<td>$A''$</td>
<td>extensional compliance tensor</td>
<td>$[m/N]$</td>
</tr>
<tr>
<td>$B$</td>
<td>coupling stiffness tensor</td>
<td>$[N]$</td>
</tr>
<tr>
<td>$B_i$</td>
<td>$i^{th}$ submatrix of a probability transition matrix</td>
<td>[-]</td>
</tr>
<tr>
<td>$B'$</td>
<td>coupling compliance tensor</td>
<td>$[N^{-1}]$</td>
</tr>
<tr>
<td>$b$</td>
<td>dimension of a square probability transition matrix</td>
<td>[-]</td>
</tr>
<tr>
<td>$C$</td>
<td>general stiffness tensor ($2^{nd}$ or $4^{th}$ order)</td>
<td>$[N/m^2]$</td>
</tr>
<tr>
<td>$D$</td>
<td>bending stiffness tensor</td>
<td>$[N \cdot m]$</td>
</tr>
<tr>
<td>$D'$</td>
<td>bending compliance tensor</td>
<td>$[(N \cdot m)^{-1}]$</td>
</tr>
<tr>
<td>$d$</td>
<td>diameter</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$E$</td>
<td>random event</td>
<td>[?]</td>
</tr>
<tr>
<td>$E_{ij}$</td>
<td>Young's moduli ($i = j$) and shear moduli ($i \neq j$)</td>
<td>$[N/m^2]$</td>
</tr>
<tr>
<td>$E_{ij}^e$</td>
<td>effective laminate Young's moduli ($i = j$) and shear moduli ($i \neq j$)</td>
<td>$[N/m^2]$</td>
</tr>
<tr>
<td>$E{\cdot}$</td>
<td>expectation operator</td>
<td>[?]</td>
</tr>
<tr>
<td>$F(x)$</td>
<td>cumulative distribution function for the random variable $X$</td>
<td>[-]</td>
</tr>
<tr>
<td>$F_n(x)$</td>
<td>$n^{th}$ order joint cumulative distribution function of the random variables $X_1, X_2, \ldots, X_n$</td>
<td>[-]</td>
</tr>
</tbody>
</table>

$^*$[-] denotes a dimensionless quantity. [?] denotes a variable quantity.
$F_n(x, t)$  
$n^{th}$ order joint cumulative distribution function of the random process $X(t)$  
[-]

$f$  
frequency of applied loading  
[Hz]

$f(x)$  
Probability density function for the continuous random variable $X$  
[-]

$f_n(x)$  
$n^{th}$ order joint probability mass function of the continuous random variables $X_1, X_2, \ldots, X_n$  
[-]

$f_n(x, t)$  
$n^{th}$ order joint probability mass function of the continuous random process $X(t)$  
[-]

$g$  
time transformation  
[-]

$h$  
thickness  
[m]

$k_{X^p}$  
$p^{th}$ central moment of the random variable $X$  
[?]

$L$  
length  
[m]

$M$  
moment resultant vector  
[N]

$M$  
moment resultant vector  
[Kg/m]

$m_{X^p}$  
$p^{th}$ moment of the random variable $X$  
[?]

$N$  
stress resultant vector  
[N/m]

$n$  
number of applied load cycles  
[-]

$P\{\cdot\}$  
probability operator  
[-]

$P$  
probability transition matrix  
[-]

$p$  
“stay” probabilities associated with a unit-jump probability transition matrix  
[-]

$p(x_i)$  
probability mass function for the discrete random variable $X$  
[-]

$p_n(x)$  
$n^{th}$ order joint probability mass function of the discrete random variables $X_1, X_2, \ldots, X_n$  
[-]

$p_n(x, t)$  
$n^{th}$ order joint probability mass function of the discrete random process $X(t)$  
[-]

$p_0$  
probability mass function (in vector form) of initial damage[-]
\( p_x \) probability mass function (in vector form) of damage after \([\cdot]\) \( x \) duty cycles

\( p_y \) probability mass function (in vector form) of damage after \([\cdot]\) \( y \) duty cycles

\( Q \) reduced stiffness tensor \([N/m^2]\)

\( q \) "jump" probabilities associated with a unit-jump probability transition matrix \([\cdot]\)

\( R \) stress ratio \([\cdot]\)

\( r \) ratio of "stay" to "jump" probabilities in a unit-jump probability transition matrix \([\cdot]\)

\( S \) general compliance tensor \([m^2/N]\)

\( S_1, S_2, S_6 \) longitudinal, transverse, and axial shear strengths \([N/m^2]\)

\( T_\varepsilon \) strain transformation matrix \([\cdot]\)

\( T_\sigma \) stress transformation matrix \([\cdot]\)

\( T \) temperature \([K]\)

\( u \) displacement vector \([m]\)

\( V \) volume \([m^3]\)

\( W \) strain energy density \([N/m^2]\)

\( \{x_1, x_2, x_3\} \) principal material "on-axis" coordinates \([m]\)

\( \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \) nonprincipal "off-axis" coordinates \([m]\)

\( \alpha \) Weibull shape parameter \([\cdot]\)

\( \beta \) Weibull scale parameter \([?]\)

\( \gamma \) Weibull location parameter \([?]\)

\( \Delta \sigma \) mean stress range tensor (1\(^{st}\) or 2\(^{nd}\) order) \([N/m^2]\)

\( \delta \) ineffective length of reinforcing fibers \([m]\)

\( \epsilon \) strain tensor (1\(^{st}\) or 2\(^{nd}\) order) \([\cdot]\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^o$</td>
<td>midplane strain tensor (1st or 2nd order)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>plate curvatures vector</td>
</tr>
<tr>
<td>$\nu_{ij}$</td>
<td>Poisson's ratios</td>
</tr>
<tr>
<td>$\nu_{ij}^0$</td>
<td>effective laminate Poisson's ratios</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>stress tensor (1st or 2nd order)</td>
</tr>
<tr>
<td>$\sigma_{mean}$</td>
<td>mean stress tensor (1st or 2nd order)</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>sample space of a random variable</td>
</tr>
</tbody>
</table>

**Special Notation**

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\cdot)$</td>
<td>“off-axis” property or quantity</td>
</tr>
<tr>
<td>$(\cdot)_a$</td>
<td>applied quantity</td>
</tr>
<tr>
<td>$(\cdot)_c$</td>
<td>compressive property</td>
</tr>
<tr>
<td>$(\cdot)_e$</td>
<td>“critical element” property</td>
</tr>
<tr>
<td>$(\cdot)_m$</td>
<td>matrix property</td>
</tr>
<tr>
<td>$(\cdot)_f$</td>
<td>fiber property</td>
</tr>
<tr>
<td>$(\cdot)_r$</td>
<td>residual property</td>
</tr>
<tr>
<td>$(\cdot)_t$</td>
<td>tensile property</td>
</tr>
</tbody>
</table>

**Acronym**

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td>cumulative damage</td>
</tr>
<tr>
<td>CDF</td>
<td>cumulative distribution function</td>
</tr>
<tr>
<td>CDS</td>
<td>characteristic damage state</td>
</tr>
<tr>
<td>CLPT</td>
<td>classic laminated plate theory</td>
</tr>
<tr>
<td>DC</td>
<td>duty cycle</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>--------------------------------------</td>
</tr>
<tr>
<td>EDF</td>
<td>empirical distribution function</td>
</tr>
<tr>
<td>JCDF</td>
<td>joint cumulative distribution function</td>
</tr>
<tr>
<td>JPDF</td>
<td>joint probability density function</td>
</tr>
<tr>
<td>JPMF</td>
<td>joint probability mass function</td>
</tr>
<tr>
<td>PDF</td>
<td>probability density function</td>
</tr>
<tr>
<td>PGF</td>
<td>probability generating function</td>
</tr>
<tr>
<td>PMF</td>
<td>probability mass function</td>
</tr>
<tr>
<td>PTM</td>
<td>probability transition matrix</td>
</tr>
<tr>
<td>RVE</td>
<td>representative volume element</td>
</tr>
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</table>
Chapter 1

Introduction

Throughout history the evolution of technology has been influenced by materials. This is especially true today where progress in many areas of science and engineering depends on the availability of materials that provide improved performance over conventional engineering materials such as metals and metal alloys. As such, composite materials, with their tailorable material properties, have become an increasingly attractive alternative to conventional engineering materials and have seen great increases in the number, importance, and diversity of their applications.

While attention has been given to composite materials primarily over the last 50 years, their use as a structural material dates back to before recorded history when mixtures of clay, sand, straw, and other naturally occurring materials were used as basic building materials. Their use as civil construction materials continues today with concrete and reinforced concrete, the modern equivalents of these early materials. World War II, however, prompted the development of composite materials for more advanced applications. Fiber reinforced composites were developed in direct response to the demands of the aerospace community for materials that could overcome two major problems associated with metals and their alloys; namely, corrosion and fatigue. By the end of the war, fiberglass reinforced plastics had been successfully used in filament wound rocket motors and a number of other structural applications. Their use expanded rapidly in the 1950’s and 1960’s within the aerospace community but was confined primarily to military applications where the demand for high performance justified their high costs. As advances in manufacturing techniques were made
and subsequent increases in availability and decreases in cost were realized, composite materials began to be used in the commercial aviation industry. Their growth in this area was fueled by the energy crisis of the 1970’s which highlighted the weight savings that their use affords. Success in aerospace applications has led to today’s wide-spread use of composite materials for a number of applications including sports and recreation equipment, marine structures, and automotive structural components to name a few (Reinhart, 1987).

Composite materials are by definition heterogeneous and they are often highly anisotropic. These features are what distinguish composites from homogeneous, isotropic materials such as metals and metal alloys. They are the key to the great success that composite materials have enjoyed. These features allow composite materials to be tailored for specific applications. The flexibility that this affords in the design of composite structures gives composite materials many advantages over other engineering materials. Unidirectional carbon fiber reinforced epoxies, for example, can provide specific tensile strengths (the ratio of tensile strength to density) and specific stiffnesses (the ratio stiffness to density) of approximately 4 to 6 times and 3.5 to 5 times greater than those of steel or aluminum respectively. These properties allow structures fabricated from these materials to provide weight savings of 25 to 50 percent over aluminum structures designed to the same specifications. Furthermore, they are extremely corrosive resistant and they can provide fatigue endurance limits of approximately 60 percent of their ultimate tensile strengths making them much more fatigue resistant than metals and their alloys (Noton, 1987).

As noted by Reifsnider (1991b), composite materials present engineers and scientist with the greatest opportunity in the history of man to create materials with tailored responses to achieve specific performance requirements. To exploit this op-
portunity in an optimal manner requires a thorough understanding of how these materials behave under complex mechanical, thermal, and chemical load environments. This presents a great challenge as heterogeneity and anisotropy, the very properties which give composite materials their many advantages, add inherent complexity to the mechanics that govern their behavior.

An extensive research effort has been put forth over the last few decades in an attempt to develop a concise understanding of the mechanics of composite materials. An appreciable amount of this work has focused on composite laminates which lend themselves naturally to applications involving plates, beams, and shells. From this effort, classical laminated plate theory has emerged. This theory, in conjunction with an appropriate failure criterion, has been widely accepted as a successful predictor of the static behavior of composite laminates. Unfortunately, no one particular method for characterizing the behavior of composite laminates under general dynamic loading has experienced similar success. This is due in part to the complexity of the cumulative damage (CD) process which governs the response of composite laminates under general dynamic or cyclic fatigue loading.

Under mechanical fatigue loading numerous damage mechanisms including matrix cracking, fiber/matrix debonding, delamination, and fiber fracture initiate, propagate, interact, and localize to produce a complex pattern of internal damage within a laminate. The combined effect of these damage mechanisms is the degradation with continued cyclic loading of laminate strength and stiffness on both local and global levels. This degradation in material properties is stochastic in nature and hinders the performance of a laminate over time. Consequently, the performance of structural components constructed from these materials will deteriorate as well. Failure of
such components during their service life must be avoided. However, nothing can be reliably avoided which cannot be reliably predicted.

Successful prediction of the useful life of a composite under fatigue loading requires a thorough understanding and representation of the complex CD process. It is generally accepted, however, that the complexity of the CD process precludes any representation that takes into account all damage mechanisms that are known to occur. As such, progress in this area demands the ability to discern and accurately represent those processes that control the ultimate behavior of a composite. Such a representation should be firmly grounded in mechanics and capable of addressing any random aspects which influence the governing processes. The present work is an attempt at such a representation. It focuses on predicting the lifetime of composite laminates under general multiaxial mechanical fatigue loading. A stochastic model for this purpose is proposed and discussed herein. It is based on the application of Markov chain models for CD processes (Bogdanoff and Kozin, 1985) to the well known “critical element” model (CEM) for the fatigue of composite laminates (Reifsnider, 1986; Reifsnider and Stinchcomb, 1986).

The CEM provides a rational means for addressing the complex CD process under fatigue loading by utilizing a representative volume element (RVE) approach wherein the response of the RVE is assumed to characterize the response of a composite as whole. The RVE is divided into critical and subcritical elements based on their contribution to the response of the RVE. The subcritical elements are those damage mechanisms or material elements whose realization or failure causes a redistribution of stress within a laminate due to changes in the internal geometry of a laminate. The magnitude of the stress redistribution is determined by measuring or predicting changes in the stiffness components of a laminate. The critical elements are those
damage mechanisms or material elements whose realization or failure causes failure of the laminate. The critical elements control the state of a laminate itself and are characterized by phenomenological information on their individual response to fatigue loading.

The random aspects which influence the CD process in composite laminates under fatigue loading are assumed to manifest themselves in the random variation of laminate stiffness (or equivalently compliance) properties. These aspects include random loading, random constituent material properties, and random initiation and accumulation of damage. Changes in laminate compliance are modeled herein as evolutionary stochastic processes. Specifically, these changes are modeled as nonstationary, discrete time, discrete state Markov processes (Markov chains) utilizing stationary Markov chains and polynomial transformations of their indexing parameters. The Markov chain models yield full probability distributions for changes in laminate compliance as a function of the number of applied load cycles. The stationary Markov chains are based on the assumption of "equivalent damage" and their parameters are determined from experimental data. The particular choice of this modeling procedure is motivated by the complexity of the fatigue process in composite laminates. It reflects the general absence of the appropriate physical laws needed to completely describe the CD process and the fact that variations from the mean often occur. The simple structure and comprehensive and coherent nature of the Markov chain models provide a means for assessing the major sources of variability that influence the CD process. Further motivation for their use is provided by the success of Markov chain models for modeling CD processes such as wear and fatigue crack growth in metals.

Markov chain models for changes in laminate compliance are used within the framework of the CEM to determine stress distributions within a laminate in the
presence of damage. Particular attention is given to the enhanced stress states experienced by the critical elements. A probabilistic description of the enhanced stresses on the critical elements is obtained from the evolutionary probability distributions of laminate compliance and classical laminated plate theory. The derived distributions for the random stress state on the critical elements are used in conjunction with a phenomenological description of their fatigue behavior to determine an appropriate class of Markov chain models for determining a probabilistic description of their useful service life. As the critical elements control the life of a laminate, a description of their lifetime provides a lower bound on the lifetime of a laminate. This approach provides a means for incorporating both the random aspects and mechanics of damage accumulation in composite laminates under fatigue loading into a single model. Both features are seen as crucial for the development of a realistically usable life prediction model for certifying composite structures.

Since the proposed life prediction model and many of the life prediction models in the literature are stochastic in nature, a review of those concepts from probability theory pertinent to the understanding of these models is presented in Chapter 2. Chapter 3 provides a review of classical laminated plate theory (CLPT) including its underlying kinematic constraints, formulation of lamina and laminate stress-strain relationships in arbitrary coordinate systems, and its limitations. Chapter 4 discusses the fatigue process in both unidirectional laminae and laminates and the mechanics that govern the process. Attention is focused on identifying generic patterns of damage development, their chronology, their effect on laminate stiffness, and ultimately their effect on the life of a laminate. Methods for characterizing damage for the purpose of predicting changes in laminate stiffness are discussed. A review of life prediction models for composite laminates subjected to fatigue loading is presented in
Chapter 5. It includes an in-depth examination of the CEM. Chapter 6 addresses the use of Markov chain models for modeling CD processes. Special emphasis is placed on their mathematical formulation and the construction of nonstationary Markov chain models from stationary Markov chain models. Chapter 7 combines the concepts of Chapters 2 through 6 in the formulation of the proposed life prediction model. The structure and use of the model are discussed in detail and its applicability is demonstrated by comparison with experimental results.
Chapter 2

Random Variables and Stochastic Processes

2.1 Introductory Remarks

The application of probabilistic methods to the study of fatigue requires an understanding of the fundamentals of probability theory and stochastic processes. This chapter serves as a review of those concepts from probability theory which are pertinent to the present study.

First, the probability of an event is introduced. This leads naturally to the discussion of random variables, their probabilistic description, and finally to the concept of a random process as a generalization of a random variable. The classification of stochastic processes based on statistical regularity and memory is also discussed.

Note that although the present study focuses on the development of a life prediction model based on discrete stochastic processes, both discrete and continuous random variables and random processes are discussed in this chapter. This discussion is motivated by the fact that many of the existing stochastic fatigue models for composites are based on the concepts of continuous random variables and random processes. To analyze existing models and to develop the proposed model, the concepts of discrete and continuous random variables and random processes are presented in parallel.

It is assumed that the reader is familiar with these basic concepts. As such, this review is rather terse and is included for completeness. The interested reader is referred to the many available treatises on the subject of probability theory and stochastic

2.2 Mathematical Probability

To introduce random variables and random processes, consider first that an experiment, which can be repeated arbitrarily often, is to be performed according to a well defined set of rules. Further, consider that the result of the experiment can not be uniquely predicted. A single performance of the experiment is referred to as a trial. The result of the experiment is known as the outcome. The set of all possible outcomes of the experiment is known as the sample space of the experiment and is denoted by \( \Omega \). Each outcome \( \omega \) is an element of \( \Omega \) such that

\[
\omega \in \Omega. \tag{2.1}
\]

An event \( E \) is a collection of outcomes which constitutes a subset of \( \Omega \). That is,

\[
E \subseteq \Omega. \tag{2.2}
\]

If an experiment is repeated a large number of times, the ratio of the number of times that an event \( E \) occurs to the number of times that the experiment is performed gives the relative frequency of \( E \). If the relative frequency of an event remains essentially unchanged each time the experiment is repeated a large number of times, then the experiment is said to exhibit statistical regularity. Closely related to these concepts is the notion of a probability measure \( P\{E\} \) on an event \( E \).

The probability \( P\{E\} \) of an event \( E \) is a number which satisfies the following axioms: nonnegativity, normalization, and additivity. Nonnegativity requires that for every event \( E \) in a sample space \( \Omega \), there is a probability \( P\{E\} \) such that

\[
0 \leq P\{E\} \leq 1. \tag{2.3}
\]
Normalization states that the probability of the certain event is

\[ P\{\Omega\} = 1. \] \hspace{1cm} (2.4)

Additivity gives the probability of the union of two events as

\[ P\{E_1 \cup E_2\} = P\{E_1\} + P\{E_2\} - P\{E_1 \cap E_2\}. \] \hspace{1cm} (2.5)

Of significant importance to the present work is the concept of conditional probability. The conditional probability of some event \( E_2 \) under the condition that some event \( E_1 \) has occurred is the conditional probability of \( E_2 \) given \( E_1 \). This probability is denoted as \( P\{E_2 \mid E_1\} \). The event \( E_1 \) functions as a reduced sample space. The conditional probability of \( E_2 \) given \( E_1 \) is the ratio of the probability of the intersection of events \( E_1 \) and \( E_2 \) to the probability of \( E_1 \). This is stated mathematically in the multiplication rule

\[ P\{E_1 \cap E_2\} = P\{E_1\}P\{E_2 \mid E_1\} = P\{E_2\}P\{E_1 \mid E_2\}. \] \hspace{1cm} (2.6)

The events \( E_1 \) and \( E_2 \) are said to be statistically independent if

\[ P\{E_2 \mid E_1\} = P\{E_2\}. \] \hspace{1cm} (2.7)

If \( E_1 \) and \( E_2 \) are statistically independent then the multiplication rule reduces to

\[ P\{E_1 \cap E_2\} = P\{E_1\}P\{E_2\}. \] \hspace{1cm} (2.8)

2.3 Random Variables

2.3.1 Probability Distributions

The outcome of an experiment does not have to be a number. To associate a number with each outcome \( \omega \) contained in the sample space \( \Omega \) of an experiment a random
variable $X(\omega)$ is defined. More precisely, a random variable is a function defined over the sample space $\Omega$ of an experiment that maps each outcome $\omega$ onto the real line $\mathbb{R}$. It is denoted as

$$X(\omega), \omega \in \Omega.$$  \hfill (2.9)

If the elements of $\Omega$ are mapped into a finite or countably infinite number of values on $\mathbb{R}$, then $X(\omega)$ is a discrete random variable. Conversely, if the mapping produces an infinite number of values on $\mathbb{R}$, then $X(\omega)$ is a continuous random variable. In each case, events that are defined on $\Omega$ are mapped into intervals on $\mathbb{R}$.

The range $R_X$ of a random variable is the subset of values on $\mathbb{R}$ that $X(\omega)$ may assume. The behavior of $X(\omega)$ over $R_X$ is governed by its probability law. The probability law can be characterized in a number of ways; the most common means is through the probability mass function (PMF) for discrete random variables and its counterpart, the probability density function (PDF), for continuous random variables. The PMF and PDF are defined as

$$p(x_i) = P\{X = x_i\}$$  \hfill (2.10)

and

$$f(x) \, dx = P\{x < X \leq x + dx\}.$$  \hfill (2.11)

Closely related to the PMF and PDF is the cumulative distribution function (CDF). The CDF is defined in general as

$$F(x) = P\{X \leq x\}.$$  \hfill (2.12)

More explicitly, for discrete random variables the CDF it has the form

$$F(x) = \sum_{x_i \leq x} p(x_i)$$  \hfill (2.13)
and for continuous random variables it is given as

\[ F(x) = \int_{-\infty}^{x} f(\xi) \, d\xi. \quad (2.14) \]

Clearly, for a discrete random variable the PMF and CDF are related as

\[ p(x_i) = F(x_i) - F(x_{i-1}). \quad (2.15) \]

Similarly, for a continuous random variable the PDF and CDF are related as

\[ f(x) = \frac{dF(x)}{dx}. \quad (2.16) \]

To satisfy the axioms of probability, the CDF for both discrete and continuous random variables must satisfy the equations

\[ F(-\infty) = 0, \quad 0 \leq F(x) \leq 1, \quad F(\infty) = 1. \quad (2.17) \]

It is clear that \( F(x) \) is a nondecreasing function of \( x \). For a discrete random variable \( F(x) \) is piecewise continuous. For a continuous random variable \( F(x) \) is continuous.

2.3.2 Multiple Random Variables

The concept of a random variable and its associated probabilistic characterization can be readily extended to two or more random variables. For a problem which depends on \( n \) random variables, the elements of the sample space for the problem are mapped into the \( n \)-dimensional real space \( \mathbb{R}^n \). Analogously to the case of a single random variable, if the elements of \( \Omega \) map into a finite or countably infinite number of points in \( \mathbb{R}^n \), the corresponding random variables \( X_1, X_2, \ldots, X_n \) are discrete random variables. If the number of points in \( \mathbb{R}^n \) is infinite, then the random variables are continuous.
The most common descriptors for the behavior of a group of random variables $X_1, X_2, \ldots, X_n$ on $\mathbb{R}^n$ are the joint probability mass function (JPMF) for discrete random variables, and the joint probability density function (JPDF) for continuous random variables. Introducing the vector notation $x = \{x_1, x_2, \ldots, x_n\}^T$, the $n^{th}$ order JPMF and the $n^{th}$ order JPDF are defined as

$$p_n(x) = P\{X_1 = x_{1_1} \cap X_2 = x_{2_2} \cap \ldots \cap X_n = x_{n_n}\} \quad (2.18)$$

and

$$f_n(x) \, dx = P\{(x_1 < X_1 \leq x_1 + dx_1) \cap (x_2 < X_2 \leq x_2 + dx_2) \cap \ldots \cap (x_n < X_n \leq x_n + dx_n)\} \quad (2.19)$$

or in terms of conditional PMFs and PDFs as

$$p_n(x) = \prod_{i=1}^n p(x_i|x_{i-1} \ldots x_1) \quad (2.20)$$

and

$$f_n(x) = \prod_{i=1}^n f(x_i|x_{i-1} \ldots x_1), \quad (2.21)$$

where for $i = 1$, $p(x_i|x_{i-1} \ldots x_1) = p(x_1)$ and $f(x_i|x_{i-1} \ldots x_1) = f(x_1)$. Clearly, for a single discrete random variable, $p(x_i) = p_i(x)$ and for a single continuous random variable $f(x) = f_1(x)$.

Alternatively, the random variables $X_1, X_2, \ldots, X_n$ can be defined through the joint cumulative distribution function (JCDF). The JCDF is defined in general terms as

$$F_n(x) = P\{(X_1 \leq x_1) \cap (X_2 \leq x_2) \cap \ldots \cap (X_n \leq x_n)\}. \quad (2.22)$$

More explicitly, for discrete random variables the JCDF has the form,

$$F_n(x) = \sum_{\{x_{i_1} \leq x_1, x_{i_2} \leq x_2, \ldots, x_{i_n} \leq x_n\}} p_n(x) \quad (2.23)$$
and for continuous random variables it is given as
\[ F_n(x) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_n(x) \, dx. \] (2.24)

The JCDF is a nondecreasing function in \( x_1, x_2, \ldots, x_n \) which satisfies the relations
\[ F_n(-\infty) = 0, \quad 0 \leq F_n(x) \leq 1, \quad F_n(\infty) = 1. \] (2.25)

Further, when any one or more of the \( x_j = -\infty \) then \( F_n(x) = 0 \). When all but one of the \( x_j = \infty \) then \( F_n(x) = F(x_j) \) where \( F(x_j) \) is known as the marginal CDF of \( X_j \). In terms of the JPMF for discrete random variables, the marginal PMF of \( X_j \) is defined as
\[ p(x_j) = \sum_{i_n} \cdots \sum_{i_{j+1}} \sum_{i_{j-1}} \sum_{i_1} p_n(x). \] (2.26)

Similarly, for continuous random variables the marginal PDF of \( X_j \) is defined as
\[ f(x_j) = \int_{x_n=-\infty}^{x_\infty} \cdots \int_{x_{j+1}=x_\infty}^{x_\infty} \int_{x_{j-1}=x_\infty}^{x_\infty} \cdots \int_{x_1=-\infty}^{x_\infty} f_n(x) \, dx_1 \cdots dx_{j-1} \, dx_{j+1} \cdots dx_n. \] (2.27)

### 2.3.3 Moments of Random Variables

The PMF of a discrete random variable or the PDF of a continuous random variable (or equivalently the CDF of either) completely characterizes its probabilistic structure. An alternative approach to describing the behavior of a random variable is through its moments. The moments of a random variable are indicators of the dominant features of the behavior of the random variable. To obtain expressions for the moments of a random variable, consider first a general function \( g(X) \) of a random variable. The weighted average of this function, where the weight is the PMF or PDF accordingly, is the expected value value of the function. It is given for discrete and
continuous random variables as

$$E\{g(X)\} = \begin{cases} \sum_{i} g(x_i) \, p(x_i) \\ \int_{-\infty}^{\infty} g(x) \, f(x) \, dx. \end{cases}$$  \hfill (2.28)

The expectation operator $E\{ \cdot \}$ is a linear operator.

An important case arises when $g(X)$ has the form $g(X) = X^p$. In this case equation (2.28) reduces for discrete and continuous random variables to

$$m_{X^p} = E\{X^p\} = \begin{cases} \sum_{i} x_i^p \, p(x_i) \\ \int_{-\infty}^{\infty} x^p \, f(x) \, dx, \end{cases}$$  \hfill (2.29)

where $m_{X^p}$ is known as the $p^{th}$ moment of $X$. For $p = 1$, $m_X$ is known as the mean of $X$. Another important case occurs when $g(X)$ has the form $g(X) = (X - m_X)^p$. For this case equation (2.28) reduces to

$$k_{X^p} = E\{(X - m_X)^p\} = \begin{cases} \sum_{i} (x_i - m_X)^p \, p(x_i) \\ \int_{-\infty}^{\infty} (x - m_X)^p \, f(x) \, dx, \end{cases}$$  \hfill (2.30)

where $k_{X^p}$ is known as the $p^{th}$ central moment of $X$. For $p = 2$, $k_{X^2}$ is known as the variance of $X$. The square root of the variance of a random variable is known as its standard deviation. The moments of a random variable describe the distribution of a random variable about the origin whereas the central moments describe the distribution of a random variable with respect to its mean. As is the case for the distribution functions, these concepts can be generalized to multiple random variables.

A number of alternative methods exist for determining the moments of a single random variable or multiple random variables through transformations. Of particular importance are the probability generating function (PGF) of a discrete random variable, and the characteristic function of a continuous random variable. These
transforms, also known as geometric and Fourier transforms in systems theory, are often more convenient to work with than the expressions for the moments themselves. The PGF of a discrete random variable plays an important role in the present work. It is discussed in detail in section 6.2 in conjunction with Markov chains. For a general discussion of transform techniques and their application to probability theory, the reader is referred to Giffin (1975).

2.4 Random Processes

2.4.1 Probability Description

In the preceding section the notion of a random variable was introduced. A random variable was defined as a map which transforms the outcomes of a random experiment into real numbers according to the laws of probability. More generally, a random variable is a mathematical model of an invariant (static) quantity which takes values according to some probabilistic law. However, in many problems the outcome of a random experiment may depend on one or more parameters such as time or space or both. For such a case, the outcome of the experiment is a random function of its parameters. That is, it is random process. A random process or stochastic process is a mathematical model of a process whose functional dependence on its parameters is governed by some probabilistic law.

Considering dependence on only one parameter, for example time, a random process is a set function of two variables; namely, the time parameter $t$ and the elements $\omega$ of the sample space $\Omega$ of the experiment. It is denoted as

$$X(\omega, t), \ t \in T, \ \omega \in \Omega$$

(2.31)
or simply as $X(t)$ for convenience. The outcome of each trial of the experiment is a realization or sample function of the random process. The collection of all possible realizations is called the ensemble of the random process. For a fixed value of time, a random process is a random variable. Thus, a random process may be viewed either as an ensemble of its realizations or as a parameterized family of random variables.

Analogously to random variables, a random processes can be discrete or continuous, depending on whether it is a family of discrete or continuous random variables. Further, a random process is known as discretely parametered or continuously parametered depending on the nature of its indexing parameter $t$. A random process can thus be a continuously parametered continuous random processes, a continuously parametered discrete random processes, a discretely parametered continuous random processes, or a discretely parametered discrete random processes. A continuously parametered continuous random process is illustrated in Figure 2.1.

![Figure 2.1: Ensemble of realizations of the random process $X(t)$.](image)

Since a random process $X(t)$, $t \in T$ reduces to a random variable for a fixed value of $t$, a random process can be characterized by the joint behavior of a family of random variables $X(t_1), X(t_2), \ldots$ indexed on $T$. More precisely, following the definition of
Soong and Grigoriu (1993), if for every finite set of time instances \( \{t_1, t_2, \ldots, t_n\} \) of \( t \in T \) there corresponds a set of random variables \( X_1 = X(t_1), X_2 = X(t_2), \ldots, X_n = X(t_n) \) with a well defined \( n^{th} \) order JCDF

\[
F_n(x, t) = P\{X(t_1) \leq x_1 \cap X(t_2) \leq x_2 \cap \ldots \cap X(t_n) \leq x_n\}, \quad (2.32)
\]

where \( x = \{x_1, x_2, \ldots, x_n\}^T \) and \( t = \{t_1, t_2, \ldots, t_n\}^T \), then this family of distribution functions defines the random process \( X(t), t \in T \) provided that \( F_n(x, t) \) satisfies the condition of consistency such that for any \( m > n \)

\[
F_n(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n; t_{n+1}, \ldots, t_m) = F_n(x, t), \quad (2.33)
\]

and the condition of symmetry such that it is invariant under an arbitrary permutation of its indices \( 1, 2, \ldots, n \). That is,

\[
F_n(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_n(x_{i_1}, x_{i_2}, \ldots, x_{i_n}; t_{i_1}, t_{i_2}, \ldots, t_{i_n}). \quad (2.34)
\]

The consistency and symmetry conditions are collectively known as the Kolmogorov compatibility conditions. Satisfying the consistency condition ensures that marginal distributions for \( X(t) \) can be generated from higher order distributions.

The probability structure of a random process can be equivalently characterized by a family of JPMFs or JPDFs depending on the whether the random process is discrete or continuous. Similarly to equation (2.32), this requires the existence of a well defined \( n^{th} \) order JPMF or JPDF. These quantities are defined as

\[
p_n(x, t) = P\{X(t_1) = x_{i_1} \cap X(t_2) = x_{i_2} \cap \ldots \cap X(t_n) = x_{i_n}\} \quad (2.35)
\]

and

\[
f_n(x, t) = P\{(x_1 < X(t_1) \leq x_1 + dx_1) \cap (x_2 < X(t_2) \leq x_2 + dx_2) \cap \ldots \cap (x_n < X(t_n) \leq x_n + dx_n)\}.
\quad (2.36)
\]
As in the case of random variables, an alternative approach to characterizing the statistical nature of a random process is through its moments. The expected values of functions of a random process can be obtained by generalization of the expectation operator for random variables. In general, all of the moments of a random process are required for a complete description of its probability structure.

2.4.2 Classification of Random Processes

As previously mentioned, a complete description of the probability structure of a random process requires that all of its joint probability distributions, or equivalently its moments, are known. Such complete information on a random process is generally not known. This leads to an incomplete characterization of a random process often involving only the first and second order joint distributions or moments. As such, many techniques for addressing problems involving random processes focus only on the first and second order information of the processes. Fortunately, in many cases the probability structure of a random process is somewhat simpler in the sense that all of its statistical information is contained in a relatively few distribution functions or moments. Two classifications of random processes with probabilistic structures more amenable to analytic treatment are addressed; namely, classification based on statistical regularity and classification based on memory. Note that these two classifications are not mutually exclusive. Many important random processes contain properties of both.

Classification Based on Statistical Regularity

In general, the $n^{th}$ joint probability distributions of equation (2.32) that are needed to completely describe the probability structure of a random process are functions
of the time instances $t_1, t_2, \ldots, t_n$. A nonstationary random process is one whose probability distributions depend on its indexing parameter(s). Clearly, most stochastic processes are nonstationary. However, many random processes have the property that their statistical behavior does not vary significantly with respect to their indexing parameter(s). As one might expect, random processes of this type possess a number of mathematically attractive properties and are, in general, much simpler to work with. Specifically, a random process $X(t), t \in T$ is said to be a stationary or strictly stationary random process if its associated probability distributions are invariant under an arbitrary translation of the parametric origin. This implies, in terms of the $n^{th}$ order JCDF, that

$$F_n(x; t_1, t_2, \ldots, t_n) = F_n(x; t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau), \quad (2.37)$$

with $t_i \in T$ and $(t_i + \tau) \in T$ for $i = 1, 2, \ldots, n$. In particular, if $\tau = -t_1$ then the $n^{th}$ order JCDF becomes

$$F_n(x; t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau) = F_n(x; 0, t_2 - t_1, \ldots, t_n - t_1). \quad (2.38)$$

For $n = 1$, the first order JCDF reduces to $F_1(x; 0)$ where $x = \{x_1\}^T$. It exhibits no dependence on the indexing parameter. As such, quantities which depend on the first order probability distributions of $X(t)$, such as the mean of $X(t)$, are constant with respect to the indexing parameter. For $n = 2$, the second order JCDF reduces to $F_2(x; 0, t_2 - t_1)$ where $x = \{x_1, x_2\}^T$. It exhibits dependence only on the difference or lag between any two values of the indexing parameter. Quantities which depend on the second order probability distributions of $X(t)$, such as the auto-correlation of $X(t)$, are thus functions of the parametric lag.

In general, it is quite difficult to determine whether a random process is strictly stationary as equation (2.37) must hold for all values of $n$ and, as previously men-
tioned, higher order joint distributions are often unknown. More realistically, only the first and second order joint distributions may be known. A stochastic process \( X(t), t \in T \) is said to be weakly stationary, second-order stationary, or stationary in the wide-sense if equation (2.37) is satisfied for \( n = 1 \) and \( n = 2 \). Clearly, strictly stationary random processes are also weakly stationary. However, the converse is not always true. An important exception is the Gaussian or normal random process. Its complete probability structure is determined from its mean and covariance functions.

**Classification Based on Memory**

Another important classification of random processes is based on how the present states of a random process are related to its prior states. Obviously, the simplest random process is one without memory. It is called a completely stochastic process and its probability structure is governed entirely by its first order probability distributions. More specifically, a random process \( X(t), t \in T \) is said to have no memory if its probability structure at any fixed time \( t \) is independent of its probability structure at any other value of \( t \). For such a case, the \( n^{th} \) order JCDF is given simply as

\[
F_n(x; t) = \prod_{i=1}^{n} F_1(x_{i}, t_i).
\] (2.39)

Thus, all joint distribution functions can be generated from knowledge of the first order distribution functions \( F_1(x_{i}, t_{i}) \).

Those random processes whose complete probability structure is determined by their second order joint distributions are next in order of complexity. Markov processes are an important class of random processes which possess this property. They form the basis for the present study. Specifically, a random process \( X(t), t \in T \) is said to be a Markov process if for every \( n \) and for \( t_1 < t_2 < \ldots < t_n \) in \( T \)

\[
F_n(x_n, t_n|x_{n-1}, x_{n-2}, \ldots, x_1, t_{n-1}, t_{n-2}, \ldots, t_1) = F_n(x_n, t_n|x_{n-1}, t_{n-1}).
\] (2.40)
In terms of conditional PMFs and PDFs, equation (2.40) is equivalent to

\[ p_n(x_n, t_n|x_{n-1}, x_{n-2}, \ldots, x_1; t_{n-1}, t_{n-2}, \ldots, t_1) = p_n(x_n, t_n|x_{n-1}, t_{n-1}) \]  

(2.41)

for discrete random processes, and to

\[ f_n(x_n, t_n|x_{n-1}, x_{n-2}, \ldots, x_1; t_{n-1}, t_{n-2}, \ldots, t_1) = f_n(x_n, t_n|x_{n-1}, t_{n-1}) \]  

(2.42)

for continuous random processes. Clearly, Markov processes are one-step memory processes. Applying equation (2.41) recursively to the \(n^{th}\) order JPMF of a discrete random process yields the relationship between the \(n^{th}\) order JPMF and the second order JPMFs (in terms of conditional PMFs); namely,

\[ p_n(x_n, t_n|x_{n-1}, x_{n-2}, \ldots, x_1; t_{n-1}, t_{n-2}, \ldots, t_1) = p_1(x_1, t_1) \prod_{i=1}^{n-1} p_{i+1}(x_{i+1}, t_{i+1}|x_i, t_i). \]  

(2.43)

Similarly, for a continuous random process

\[ f_n(x_n, t_n|x_{n-1}, x_{n-2}, \ldots, x_1; t_{n-1}, t_{n-2}, \ldots, t_1) = f_1(x_1, t_1) \prod_{i=1}^{n-1} f_{i+1}(x_{i+1}, t_{i+1}|x_i, t_i). \]  

(2.44)

Thus, given the first order probability distributions and the conditional probabilities the joint distributions which characterize the complete probability structure of the random process can be determined. If the joint probability distributions satisfy the previously mentioned consistency condition, then any lower order marginal distributions for the random process can be generated from the joint distributions by appropriate summation or integration respectively.

Equation (2.40), or equivalently equations (2.41) and (2.42), imply that a Markov process represents a family of trajectories whose conditional probabilities at any given time instant, given all past observations, depend solely on the most recent observation. The conditional probabilities are also known as transition probabilities. To
be consistent, the transition probabilities must satisfy the Chapman-Kolmogorov identity. For discrete random processes the Chapman-Kolmogorov identity is given by

$$p(x_i, x_j; t_i, t_j) = \sum_k p(x_j, t_j | x_k, t_k) P(x_k, t_k | x_i, t_i)$$  \hspace{1cm} (2.45)$$
with $t_j > t_k > t_i \geq t_0$. For continuous random processes it is given as

$$f(x_i, x_j; t_i, t_j) = \int_{-\infty}^{\infty} f(x_j, t_j | x, t) f(x, t | x_i, t_i) \, dx$$  \hspace{1cm} (2.46)$$
with $t_j > t > t_i \geq t_0$. The Chapman-Kolmogorov identity ensures that all possible transitions from state $i$ to state $j$ lead to the same transition probabilities. This concept is illustrated in Figure 2.2.

**Figure 2.2:** Illustration of the Chapman-Kolmogorov identity for a random process $X(t)$.

Clearly, the concept of a random processes with memory can be extended to higher orders. Logically, the next order of complexity would be a random process with two-step memory. The complete probability description of such a process would be governed entirely by its third order joint probability distribution. Extensions to random processes with larger memory are obvious.
Chapter 3

Constitutive Relationships for Composite Materials

3.1 Introductory Remarks

As noted in Chapter 1, the ability to exploit the many advantages that composite materials offer depends on a thorough understanding of the mechanics that govern their behavior under complex mechanical, thermal, and environmental loadings. This chapter reviews the fundamental constitutive relationships that govern the behavior of composite laminates under purely mechanical loading. It is divided into three primary sections. The first section focuses on the stress-strain relationship for a body constructed of a general anisotropic homogeneous material and the simplifications that occurs in this relationship when the material has certain symmetries. The second and third sections center on determination of elastic properties and stress-strain relationships for individual laminae and laminates respectively.

The literature is replete with texts on the mechanics of composite materials. The interested reader is referred to the following sources for a more in-depth treatment of this subject: Christensen (1979), Tsai and Hahn (1980), Halpin (1984), Chawla (1987), and Gibson (1994).

3.2 General Stress-Strain Relationship

A general three-dimensional state of stress at a point in a deformable body is described by nine stress components $\sigma_{ij} (i, j = 1, 2, 3)$. These stresses are illustrated in Figure
3.1 as acting on a differential volume element of a body, represented as a cube, in a coordinate system \( \{x_1, x_2, x_3\} \) whose axes are perpendicular to the faces of the cube. The stress component \( \sigma_{ij} \) represents the force per unit area in the \( x_j \) direction on the face of the cube whose outward normal is in the \( x_i \) direction. Stresses for which \( i = j \) are known as normal stresses. Stresses for which \( i \neq j \) are known as shear stresses. Rotational equilibrium requires that \( \sigma_{ij} = \sigma_{ji} \).

![Figure 3.1: Three-dimensional state of stress at a point.](image)

The displacement of a point in a body with respect to the \( \{x_1, x_2, x_3\} \) coordinate system is given by a displacement vector \( \mathbf{u} = \{u_1, u_2, u_3\}^T \). The \( i^{th} \) component of \( \mathbf{u} \) is projection of \( \mathbf{u} \) onto the \( i^{th} \) coordinate axis \( x_i \).

Corresponding to each stress component \( \sigma_{ij} \), there is a strain component \( \epsilon_{ij} \) describing the deformation at a point in a deformable body. Strains for which \( i = j \) are known as normal strains. Strains for which \( i \neq j \) are known as shear strains. The tensor strains are related to the derivatives of the displacement components through
the strain-displacement relationship

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  

(3.1)

Clearly, \( \varepsilon_{ij} = \varepsilon_{ji} \). For \( i \neq j \) the tensor shear strains \( \varepsilon_{ij} \) are one-half the engineering shear strains \( \gamma_{ij} \) such that

\[ \gamma_{ij} = 2\varepsilon_{ij} \quad \text{for} \ i \neq j. \]  

(3.2)

The engineering shear strain \( \gamma_{ij} \) describes the total distortional change in the angle between lines that were originally parallel to the \( x_i \) and \( x_j \) axes. The tensor shear strain \( \varepsilon_{ij} \) describes the rotation of either of the lines individually.

Stress are related to strains through a stress-strain relationship. The form of the stress strain relationship describes the material behavior of a body. For linearly elastic materials, this relationship is known as Hooke's law. In its most general indicial form it is given as

\[ \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \]  

where repeated indices imply summation. The \( C_{ijkl} \) \((i, j, k, l = 1, 2, 3)\) are the elastic constants or stiffness components of a material. In all, the \( C_{ijkl} \) represent 81 stiffness components. The symmetry of the stress and strain components requires that the stiffness components be symmetric in both the first two indices \( (C_{ijkl} = C_{jikl}) \) and the last two indices \( (C_{ijkl} = C_{jikl}) \). This reduces the number of stiffness components to 36. Introducing a contracted notation, as summarized in Table 3.1, such that \( C_{mn} = C_{ijkl} \), \( \sigma_m = \sigma_{ij} \), and \( \varepsilon_n = \varepsilon_{kl} \) where \( \varepsilon_n (n = 4, 5, 6) \) now represent the engineering shear strains, the generalized Hooke’s law can be rewritten in the following indicial form

\[ \sigma_m = C_{mn}\varepsilon_n \quad \text{for} \ m, n = 1, 2, \ldots, 6 \]  

(3.4)

or in matrix form as

\[ \mathbf{\sigma} = \mathbf{C}\mathbf{\varepsilon}. \]  

(3.5)
Table 3.1: Contraction from double to single indices

\[
\begin{array}{cccccc}
ij \text{ or } kl & 11 & 22 & 33 & 23 & 13 & 12 \\
m \text{ or } n & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Alternatively, the generalized Hooke's law can be rewritten in inverse form as

\[
\epsilon_m = S_{mn} \sigma_n \quad \text{for} \quad m, n = 1, 2, \ldots, 6,
\]  (3.6)

or in matrix form as

\[
\epsilon = S \sigma
\]  (3.7)

where \( S \) is known as the compliance tensor. The compliance and stiffness tensors are related by the equation

\[
S = C^{-1}.
\]  (3.8)

It is appropriate at this point, to discuss the problem of heterogeneity in composite materials. The general stress-strain relationships presented thus far are valid at any point in a body. This poses no problem for a body constructed of a homogeneous material as the elastic constants of a homogeneous material do not vary from point to point. However, for a heterogeneous material such as a composite, the elastic constants vary from point to point as the elastic constants of the fiber and matrix phases are usually quite different. For the macro-mechanical analysis of composites it is convenient to work with volume averaged stresses and strains which are related by the effective elastic properties of an equivalent homogeneous material. Assuming that the scale of the inhomogeneity in a composite is significantly smaller than than the characteristic length over which averaging is to take place, the volume averaged stresses and strains are given by the equations

\[
\tilde{\sigma}_i = \frac{1}{V} \int_V \sigma_i \, dv
\]  (3.9)
and

$$\dot{\varepsilon}_i = \frac{1}{V} \int_V \varepsilon_i \, dv.$$  \hspace{1cm} (3.10)

In terms of the averages stresses and strains, the generalized Hooke's law in terms of stiffness becomes

$$\ddot{\sigma}_i = C_{ij} \dot{\varepsilon}_j$$  \hspace{1cm} (3.11)

or in terms of compliance

$$\ddot{\varepsilon}_i = S_{ij} \dot{\sigma}_j,$$  \hspace{1cm} (3.12)

where the $C_{ij}$ and $S_{ij}$ are now the effective elastic constants of the equivalent homogeneous material in the volume $V$. Methods for determining the effective elastic constants of a lamina are discussed in section 3.3.1. In the remainder of this work all lamina properties are assumed to be effective properties and all stress and strains are assumed to be volume averages. The $\dot{\sigma}_i$ and $\dot{\varepsilon}_i$ notation will be dropped hereafter.

In general, the stiffness tensor of equation (3.4) is a fully populated $6 \times 6$ matrix with 36 independent components. This number can be further reduced by considering the existence of a strain energy density function $W(\varepsilon_{ij})$ (Gould, 1983). The strain energy density function, whose existence is a necessary condition for a linear stress-strain relationship, is a scalar quadratic function of the strain components from which the stress components can be derived as

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i} = C_{ij} \varepsilon_j,$$  \hspace{1cm} (3.13)

where

$$W(\varepsilon_{ij}) = \frac{1}{2} C_{ij} \varepsilon_i \varepsilon_j.$$  \hspace{1cm} (3.14)

Clearly, the stiffness components $C_{ij}$ can be obtained from $W$ as

$$C_{ij} = \frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j}.$$  \hspace{1cm} (3.15)
Since the order of differentiation is irrelevant \( C_{ij} = C_{ji} \). Thus, the stiffness tensor is symmetric. As such, only 21 of the 36 components of \( \mathbf{C} \) are independent and \( \mathbf{C} \) has the form

\[
\mathbf{C} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{33} & C_{34} & C_{35} & C_{36} \\
C_{44} & C_{45} & C_{46} \\
C_{55} & C_{56} \\
sym. & C_{66}
\end{bmatrix}.
\] (3.16)

Further reductions in the number of independent elastic constants are possible if a material exhibits certain symmetries in its response to applied loads. For unidirectional reinforced composite materials the most important type of symmetry is special orthotropic symmetry. A specially orthotropic material has symmetry with respect to all three of its principal material coordinates \( \{x_1, x_2, x_3\} \). The principal material coordinates are associated with the direction of reinforcement. Invariance of the \( C_{ij} \) under coordinate transformations corresponding to reflection of coordinate axes in two orthogonal planes can be used to show that a specially orthotropic material has a stiffness matrix of the form

\[
\mathbf{C} = \begin{bmatrix}
C_{11} & C_{12} & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 \\
C_{33} & 0 & 0 & 0 \\
C_{44} & 0 & 0 \\
C_{55} & 0 \\
sym. & C_{66}
\end{bmatrix}.
\] (3.17)

Clearly, a specially orthotropic material 9 independent stiffness components.
Additional simplifications in the stiffness matrix can be made if the material exhibits directional independence with respect to its stiffness properties. With regard to unidirectionally reinforced composites, the stiffness properties in any direction perpendicular to the reinforcement direction are nearly the same. This results from the near uniform distribution of fibers within such a material. A material which exhibits such symmetry is termed a transversely isotropic material. For example, if the stiffness properties along the \( x_2 \) and \( x_3 \) axes are equal, then \( C_{22} = C_{33}, C_{12} = C_{13}, \) and \( C_{55} = C_{66}. \) It can also be shown that \( C_{44} \) is dependent on \( C_{22} \) and \( C_{23}. \) These symmetries reduce the stiffness matrix for a transversely isotropic material to

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
\frac{1}{2}(C_{22} - C_{23}) & 0 & 0 & 0 & 0 & 0 \\
sym. & C_{66} & 0 & 0 & 0 & C_{66}
\end{bmatrix},
\]

which has 5 independent components.

A final reduction in the form of the stiffness matrix occurs if the stiffness properties of a material exhibit total directional independence. Materials with these symmetries, in which every coordinate axis is an axis of symmetry, are called isotropic materials. For isotropic materials the number of independent stiffness components is two and
the stiffness matrix has the form

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{11} & C_{12} & 0 & 0 & 0 & 0 \\
C_{11} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(C_{11} - C_{12}) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(C_{11} - C_{12}) & 0 & 0 & 0 & 0 & 0 \\
\text{sym.} & \frac{1}{2}(C_{11} - C_{12}) & \frac{1}{2}(C_{11} - C_{12}) & \frac{1}{2}(C_{11} - C_{12}) & \frac{1}{2}(C_{11} - C_{12}) & \frac{1}{2}(C_{11} - C_{12})
\end{bmatrix}.
\]  

(3.19)

Metals typically possess this type of symmetry.

### 3.3 Lamina Analysis

#### 3.3.1 Lamina Elastic Properties

The basic structural unit from which composite laminates are constructed is the composite lamina. A lamina is a heterogeneous structure generally composed of fiber and matrix phases. The fibers may be continuous, woven, discontinuous, or any combination thereof. Typical fiber materials include glass, carbon, graphite, and boron. Commonly used matrix materials are polymers, such as epoxies, or light metals, such as aluminum or titanium alloys. Attention will be focused in the remainder of this work on laminae consisting of continuous unidirectional fiber arrays bound by a polymer matrix.

Polymer matrix laminae are generally in the form of a pre-preg tape. That is, a unidirectional array of fibers pre-impregnated with a polymer matrix material. The thickness of an individual lamina is generally on the order of 10 to 100 times the fiber diameter. Within a lamina, fibers are arranged in either square, hexagonal, or random arrays. The type of arrangement affects the material properties of a lamina. The primary purpose of the reinforcing fibers is to carry the applied loads acting
on a lamina. The function of the matrix material is to bind the fibers together and to transfer load between them. As noted above, a lamina is heterogeneous at the material constituent level with properties that change from point to point. Further, unidirectionally reinforced composite laminae are often highly anisotropic with elastic properties which exhibit directional dependence. Elastic properties in the direction of the reinforcing fibers are significantly larger than those transverse to the direction of reinforcement. This difference may be several orders of magnitude.

One of the fundamental problems associated with composite materials relates to how one can predict the effective elastic properties of a lamina given the elastic properties of its constituents. In the "composites community", models which address this problem are termed "micromechanics" models. The major types of models which have been proposed can be classified as netting, mechanistic, self consistent, variational, exact, statistical, and semiempirical. In any case, experimental data on the elastic properties of the constituent materials are required as input into micromechanical models and similar data on a lamina are required for their verification. The scope of the present work prohibits an in-depth review of these methods and as such only a brief description of each is included herein. The reader is referred to the work of Chamis and Sendeckyj (1968) for a review and critique of each method.

An important aspect of micromechanics models for predicting the elastic properties of a lamina is the assumptions on which the models are based. Each of the mentioned micromechanics models are based in part or entirely on the assumptions that a lamina is macroscopically homogeneous, linearly elastic, and generally orthotropic, the reinforcement fibers are linearly elastic and homogeneous, the matrix material is linearly elastic and homogeneous, fibers and matrix are free of voids, there is perfect
bonding between the fibers and the matrix with no interphase region, a lamina is initially stress free, and the fibers are regularly spaced and aligned.

Netting analysis models assume disjointed constituent response. That is, the fiber and matrix phases are not bonded to one another. The fibers are assumed to account for all of the longitudinal elastic properties of a ply, while the matrix accounts for transverse and shear properties. Mechanics of materials models utilize the elastic properties of both the fiber and matrix phases in conjunction with displacement and force equilibrium conditions to determine the average stress/strain state on a ply. Substitution of these values into definitions of the elastic properties yields the effective elastic properties of a lamina. The well-known “rule of mixtures” is an example of this approach.

In the mechanics of materials models, a simple representative volume element (RVE) is chosen for determining the elastic properties of a ply which accounts only for the relative volume fractions of the constituent materials. The geometry of the constituent materials is ignored. This gives rise to predictions of elastic properties, particularly the transverse and shear properties, which do not agree well with experimentally measured values. To overcome this discrepancy models which account for multiphase geometry have been introduced. The most basic of these models are the self consistent field models. In these models the multiphase geometry of a composite is represented by a single fiber embedded in an outer matrix cylinder. The outer cylinder is embedded in an unbounded homogeneous medium whose material properties are taken to be the effective properties of a lamina. Predictions of elastic properties are made based on strain fields produced in the RVE by application of a uniform load on the RVE at infinity. The results that are obtained, however, are independent of fiber packing geometry. Exact methods, within the context of classical
elasticity theory, take account of the fiber packing geometry. Fibers are assumed to be arranged in a periodic array within a lamina. Common fiber arrangements are square and hexagonal. Predictions for the elastic properties of a lamina are made by solving the resulting elasticity problem via series developments, a complex variable technique, or a numerical solution such as finite differences or finite elements.

Variational models are used to determine bounds on elastic properties of a ply. They employ the energy theorems of classical elasticity theory. Specifically, the minimum complementary energy is used to obtain a lower bound, while minimum potential energy is used to obtain an upper bound. If the upper and lower bounds coincide, the elastic properties are determined exactly.

Statistical models relax the assumption that fibers are arranged in a periodic array within a lamina. In these models a lamina is considered as a homogeneous material whose elastic properties vary from point to point in a random manner. That is, the elastic properties constitute random fields. However, the use of these models requires that the complete probability structure of the fields be known. This makes the use of these models overly difficult. Approximations for these fields using only second order moments have been proposed. However, the approximate results can vary significantly from measured values of the elastic constants.

Semiempirical models have been proposed as an improvement to the simple mechanics of materials models. They account for several variables which are neglected in the simple models but which are known to influence the elastic properties of a lamina. They have relatively simple forms when compared to exact models based on elasticity theory. They contain curve fitting parameters whose values are adjusted such that the elastic properties predicted by the semiempirical models match quite satisfactorily those obtained experimentally or through elasticity theory. These models tend to
retain the physical complexity of the problem while relaxing the mathematical complexity. The most well-known of these type models are the Halpin-Tsai equations.

3.3.2 Lamina Stress-Strain Relationship

As noted in the previous section, micromechanics models or experimental measurements are used to determine the effective elastic properties of a lamina. Once determined, the heterogeneous nature of a lamina is ignored. It is treated as a macroscopically homogeneous, specially orthotropic material. With respect to the results of section 3.2 and the principal material coordinate system \( \{x_1, x_2, x_3\} \) in Figure 3.2, where the direction of reinforcement is parallel to the \( x_1 \) coordinate axis, the stress-

![Figure 3.2: On-axis \( \{x_1, x_2, x_3\} \) and off-axis \( \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \) coordinate systems.](image)
strain relationship for a specially orthotropic material is given by the equation

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 \\
C_{33} & 0 & 0 & 0 \\
C_{44} & 0 & 0 & 0 \\
C_{55} & 0 & 0 & 0 \\
sym. & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix}.
\]  

(3.20)

Alternatively, the stress-strain relationship can be written in terms of compliance as

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix}
= \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{22} & S_{23} & 0 & 0 & 0 \\
S_{33} & 0 & 0 & 0 \\
S_{44} & 0 & 0 & 0 \\
S_{55} & 0 & 0 & 0 \\
sym. & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix},
\]  

(3.21)

or in terms of engineering constants as

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix}
= \begin{bmatrix}
1/E_{11} & -\nu_{21}/E_{22} & -\nu_{31}/E_{33} & 0 & 0 & 0 \\
-\nu_{12}/E_{11} & 1/E_{22} & -\nu_{32}/E_{33} & 0 & 0 & 0 \\
0 & 1/E_{33} & 0 & 0 & 0 \\
0 & 1/E_{23} & 0 & 0 & 0 \\
0 & 1/E_{31} & 0 & 0 & 0 \\
0 & 1/E_{12} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}.
\]  

(3.22)

The engineering constants are the Young’s moduli $E_{ii}(i = j)$, the Poisson’s ratios $\nu_{ij} = -\epsilon_j/\epsilon_i(i \neq j)$, and the shear moduli, $E_{ij}(i \neq j)$. Again, for an specially orthotropic
material only 9 stiffness, compliance, or engineering constants are independent with the later following from \( \nu_{ij}/E_{ii} = \nu_{ji}/E_{jj} \).

The length and width of a lamina are typically much greater than its thickness (\( \approx 1 \text{ mm} \)). As such, a lamina is often assumed to be in a state of plane stress. If the \( x_3 \) axis is perpendicular to the plane of a lamina, then a state of plane stress implies that all stress components derived from forces acting parallel to the \( x_3 \) axis are negligible. Thus, \( \sigma_3 = \sigma_4 = \sigma_5 = 0 \). For a state of plane stress the stress-strain relationship for a specially orthotropic material becomes

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{22} & 0 & 0 \\
sym. & Q_{66} & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix} \tag{3.23}
\]

or

\[
\sigma = Q\epsilon \tag{3.24}
\]

where the \( Q_{ij} \) for \( (i,j = 1,2,6) \) are known as the reduced stiffness components. Alternatively, the in-plane stress-strain relationship can be written in terms of compliance as

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{22} & 0 & 0 \\
sym. & S_{66} & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix}, \tag{3.25}
\]

or

\[
\epsilon = S\sigma. \tag{3.26}
\]

Equivalently, in terms of engineering constants the stress-strain relationship is given as

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix} =
\begin{bmatrix}
1/E_{11} & -\nu_{21}/E_{22} & 0 \\
1/E_{22} & 0 & 0 \\
sym. & 1/E_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix}. \tag{3.27}
\]
From equations (3.23), (3.25), and (3.27) the relationships between the reduced stiffness components, the compliance components, and the engineering constants are expressed as

\[
Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}}
\]

\[
Q_{12} = Q_{21} = \frac{S_{11}S_{22} - S_{12}^2}{S_{11}} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}}
\]

\[
Q_{22} = \frac{S_{11}S_{22} - S_{12}^2}{S_{11}} = \frac{1}{1 - \nu_{12}\nu_{21}}
\]

\[
Q_{66} = \frac{1}{S_{66}} = E_{66}.
\]

(3.28)

For the purpose of analyzing composite laminates it is often necessary to have the stress-strain relationship for a lamina in a nonprincipal (or “off-axis”) coordinate system, such as \(\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}\) in Figure 3.2, as opposed to a principal material (or “on-axis”) coordinate system. This is accomplished by means of a coordinate transformation.

With respect to Figure 3.3(a), consider a lamina whose in-plane material symmetry axes \(\{x_1, x_2\}\) are inclined by an angle \(\theta\) (measured positive counterclockwise) to a set of nonprincipal axes \(\{\bar{x}_1, \bar{x}_2\}\). The relationships for the transformation of stress can derived from equilibrium considerations for a differential wedge element \(dA\) in the two coordinate systems as shown in figures 3.3(b) and 3.3(c). If the differential wedge element in Figure 3.3(b), obtained from slicing a unit differential area parallel to the \(x_2\) axis, is assumed to have a unit thickness and a unit length hypotenuse such that the sides of the wedge adjacent and opposite to the angle \(\theta\) have lengths of \(c = \cos(\theta)\) and \(s = \sin(\theta)\) respectively, summation of the forces acting on the sides of the element in the \(\bar{x}_1\) and \(\bar{x}_2\) directions yields

\[
\sum F_{\bar{x}_1} = c\bar{\sigma}_1 + s\bar{\sigma}_6 - c\sigma_1 + s\sigma_6
\]

\[
\sum F_{\bar{x}_2} = s\bar{\sigma}_2 + c\bar{\sigma}_6 - s\sigma_1 - c\sigma_6.
\]

(3.29)

(3.30)
If the off-axis stresses $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_6$ are assumed known, solving for $\sigma_1$ and $\sigma_6$ gives

$$\sigma_1 = c^2 \bar{\sigma}_1 + s^2 \bar{\sigma}_2 + 2cs \bar{\sigma}_6$$ \hspace{1cm} (3.31)$$
$$\sigma_6 = -cs \bar{\sigma}_1 + cs \bar{\sigma}_2 + (c^2 - s^2) \bar{\sigma}_6.$$ \hspace{1cm} (3.32)

Applying the same analysis to the wedge element of Figure 3.3(c), obtained from slicing a unit differential area parallel to the $x_1$ axis, yields

$$\sigma_2 = s^2 \bar{\sigma}_1 + c^2 \bar{\sigma}_2 - 2cs \bar{\sigma}_6.$$ \hspace{1cm} (3.33)

Rewriting the equations (3.31), (3.32), and (3.33) in matrix form yields the transformation from off-axis stress to on-axis stress; namely,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix},$$ \hspace{1cm} (3.34)

or

$$\sigma = T_\sigma \bar{\sigma}.$$ \hspace{1cm} (3.35)
A similar analysis for strains yields the transformation from off-axis strain to on-axis strain; namely,
\[
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_6
\end{pmatrix} =
\begin{bmatrix}
c^2 & s^2 & cs \\
2s^2 & c^2 & -cs \\
-2cs & 2cs & c^2 - s^2
\end{bmatrix}
\begin{pmatrix}
\bar{\varepsilon}_1 \\
\bar{\varepsilon}_2 \\
\bar{\varepsilon}_6
\end{pmatrix}
\] (3.36)

or
\[
\varepsilon = T_\varepsilon \bar{\varepsilon}. \quad (3.37)
\]

It should be noted that the use of two different transformation matrices \( T_\sigma \) and \( T_\varepsilon \) is necessary to account for the fact that the shear strains \( \varepsilon_6 \) and \( \bar{\varepsilon}_6 \) are engineering shear strains. Substituting equations (3.35) and (3.37) into equation (3.24) yields the off-axis or generally orthotropic stress-strain relationship for a lamina. Specifically,
\[
\bar{\sigma} = T_\sigma^{-1} Q T_\varepsilon \bar{\varepsilon} = \bar{Q} \bar{\varepsilon}, \quad (3.38)
\]

where
\[
\bar{Q} = T_\sigma^{-1} Q T_\varepsilon =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
sym. & \bar{Q}_{66}
\end{bmatrix}
\] (3.39)

is the off-axis stiffness tensor.

### 3.4 Laminate Analysis

#### 3.4.1 Laminate Constitutive Relationship

Composite laminates are constructed by bonding together two or more individual unidirectional laminae with generally differing reinforcement orientations. A laminate constructed as such is designated by three quantities including the fiber material, the matrix material, and a description of the stacking sequence of the individual laminae.
The stacking sequence of a general \( n \)-ply laminate is given as \([\theta_1, \theta_2, \ldots, \theta_n]\), where \(\theta_i\) denotes the orientation of the \(i^{th}\) ply with respect to the longitudinal axis of the laminate. Positive values of \(\theta\) represent counterclockwise rotations. Here, \(i = 1\) denotes the bottom ply and \(i = n\) the top ply.

When a laminate has symmetry with respect to its midplane two other notations are commonly used to designate the stacking sequence. If the number of plies in a laminate is even, then the stacking sequence is given by \([\theta_1, \theta_2, \ldots, \theta_{n/2}]_S\), where the subscript \(S\) denotes midplane symmetry. Similarly, if the number of plies in a laminate is odd the stacking sequence is given by \([\theta_1, \theta_2, \ldots, \bar{\theta}_{(n+1)/2}]_S\), where the bar above the last value of \(\theta\) represents midplane symmetry with respect to that ply. That is, the midplane of the laminate is located in the center of the ply whose orientation is \(\bar{\theta}\). These concepts are illustrated in Figure 3.4.

**Figure 3.4:** Stacking sequences and ply numbering schemes for a \([0_2^\circ, 90_2^\circ, \pm45_3^\circ]_S\) laminate and a \([0_2^\circ, \pm45^\circ, 90^\circ]_S\) laminate.

Given the stacking sequence of a laminate and the elastic properties of the individual lamina, the stress-strain relationship for the laminate can be determined via classical laminated plate theory (CLPT). A number of basic assumptions are incor-
porated in CLPT. The thickness of a laminate is assumed to be much smaller than its lateral dimensions. A laminate is assumed to be in a state of plane stress. A perfect bond between the individual lamina is assumed. Finally, the in-plane displacements are assumed to be a linear functions of the thickness coordinate $\bar{x}_3$ with respect to a reference off-axis coordinate system $\{\bar{x}_1, \bar{x}_3, \bar{x}_3\}$. The later assumption is known as the Kirchhoff assumption.

From equation (3.1), the strains and displacements at a point in a deformable body are related as

\[
\begin{align*}
\dot{\varepsilon}_{11} &= \frac{\partial \bar{u}_1}{\partial \bar{x}_1} \\
\dot{\varepsilon}_{12} &= \dot{\varepsilon}_{13} = \frac{\partial \bar{u}_1}{\partial \bar{x}_2} + \frac{\partial \bar{u}_2}{\partial \bar{x}_1} \\
\dot{\varepsilon}_{22} &= \frac{\partial \bar{u}_2}{\partial \bar{x}_2} \\
\dot{\varepsilon}_{33} &= \frac{\partial \bar{u}_3}{\partial \bar{x}_3} \\
\dot{\varepsilon}_{23} &= \frac{\partial \bar{u}_2}{\partial \bar{x}_3} + \frac{\partial \bar{u}_3}{\partial \bar{x}_2},
\end{align*}
\tag{3.40}
\]

where $\dot{\varepsilon}_{ij}$, $i \neq j$ are the engineering shear strains. From the Kirchhoff assumption the displacement fields are given as

\[
\begin{align*}
\bar{u}_1 &= \bar{u}_1^0(\bar{x}_1, \bar{x}_2) + \bar{x}_3 g_1(\bar{x}_1, \bar{x}_2) \\
\bar{u}_2 &= \bar{u}_2^0(\bar{x}_1, \bar{x}_2) + \bar{x}_3 g_2(\bar{x}_1, \bar{x}_2),
\end{align*}
\tag{3.41}
\]

where $\bar{u}_1^0$ and $\bar{u}_2^0$ are midplane displacements. Further, from the assumption of plane stress and the Kirchhoff assumption

\[
\begin{align*}
\dot{\varepsilon}_{13} &= g_1(\bar{x}_1, \bar{x}_2) + \frac{\partial \bar{u}_3}{\partial \bar{x}_1} = 0 \\
\dot{\varepsilon}_{23} &= g_2(\bar{x}_2, \bar{x}_2) + \frac{\partial \bar{u}_3}{\partial \bar{x}_2} = 0
\end{align*}
\tag{3.42}
\]

from which it can be derived that

\[
\begin{align*}
g_1(\bar{x}_1, \bar{x}_2) &= -\frac{\partial \bar{u}_3}{\partial \bar{x}_1} \\
g_2(\bar{x}_2, \bar{x}_2) &= -\frac{\partial \bar{u}_3}{\partial \bar{x}_2}
\end{align*}
\tag{3.43}
\]
In addition, the assumption of plane stress implies \( \bar{\varepsilon}_{33} = 0 \) and thus the out-of-plane displacement \( \bar{u}_3 \) has the form

\[
\bar{u}_3 = \bar{u}_3(\bar{x}_1, \bar{x}_2).
\]  
(3.44)

Considering equations (3.43) in conjunction with equations (3.41) and equations (3.40) yields for the in-plane strains the expressions

\[
\begin{align*}
\bar{\varepsilon}_{11} &= \frac{\partial \bar{u}_1}{\partial \bar{x}_1} = \frac{\partial \bar{u}_1}{\partial \bar{x}_1} - \bar{x}_3 \frac{\partial^2 \bar{u}_3}{\partial \bar{x}_2^2} = \bar{\varepsilon}^0_{11} + \bar{x}_3 \bar{\kappa}_{11} \\
\bar{\varepsilon}_{22} &= \frac{\partial \bar{u}_2}{\partial \bar{x}_2} = \frac{\partial \bar{u}_2}{\partial \bar{x}_2} - \bar{x}_3 \frac{\partial^2 \bar{u}_3}{\partial \bar{x}_1 \partial \bar{x}_2} = \bar{\varepsilon}^0_{22} + \bar{x}_3 \bar{\kappa}_{22} \\
\bar{\varepsilon}_{12} &= \frac{\partial \bar{u}_1}{\partial \bar{x}_2} + \frac{\partial \bar{u}_2}{\partial \bar{x}_1} = \frac{\partial \bar{u}_1}{\partial \bar{x}_2} + \frac{\partial \bar{u}_2}{\partial \bar{x}_1} - 2\bar{x}_3 \frac{\partial^2 \bar{u}_3}{\partial \bar{x}_1 \partial \bar{x}_2} = \bar{\varepsilon}^0_{12} + \bar{x}_3 \bar{\kappa}_{12},
\end{align*}
\]  
(3.45)

where the \( \bar{\kappa}_{ij} \) are plate curvatures. Utilizing single subscript notation and rewriting equations (3.45) in matrix form yields the following form for the in-plane strains in an off-axis coordinate system \( \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \)

\[
\begin{pmatrix}
\bar{\varepsilon}_1 \\
\bar{\varepsilon}_2 \\
\bar{\varepsilon}_6
\end{pmatrix} = \begin{pmatrix}
\bar{\varepsilon}^0_1 \\
\bar{\varepsilon}^0_2 \\
\bar{\varepsilon}^0_6
\end{pmatrix} + \bar{x}_3 \begin{pmatrix}
\bar{\kappa}_1 \\
\bar{\kappa}_2 \\
\bar{\kappa}_6
\end{pmatrix}
\]  
(3.46)

or

\[
\bar{\varepsilon} = \bar{\varepsilon}^0 + \bar{x}_3 \bar{\kappa}.
\]  
(3.47)

For a laminate composed of \( n \) plies, the stress-strain relationship for the \( k^{th} \) ply in an off-axis coordinate system \( \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \) is given as

\[
\bar{\sigma}_k = \bar{Q}_k \bar{\varepsilon}_k.
\]  
(3.48)

Upon substituting equation (3.47) from CLPT into equation (3.48), the stress-strain relationship becomes

\[
\bar{\sigma}_k = \bar{Q}_k \bar{\varepsilon}^0 + \bar{x}_3 \bar{Q}_k \bar{\kappa}.
\]  
(3.49)
Clearly the dependence of \( \bar{\sigma}_k \) on \( \bar{x}_3 \) allows the individual ply stresses within a laminate to vary from ply to ply. As such, it is customary to define laminate stress and moment resultants which provide a statically equivalent system of forces and moments acting on the midplane of a laminate. The stress resultants are defined as

\[
\bar{N}_i = \int_{-h/2}^{h/2} \bar{\sigma}_i \, d\bar{x}_3 \quad \text{for} \quad i = 1, 2, 6. \tag{3.50}
\]

The moment resultants are given by the equation

\[
\bar{M}_i = \int_{-h/2}^{h/2} \bar{\sigma}_i \bar{x}_3 \, d\bar{x}_3 \quad \text{for} \quad i = 1, 2, 6. \tag{3.51}
\]

The stress and moment resultants are illustrated in Figure 3.5. Substituting equation (3.49) into equations (3.50) and (3.51) yields in matrix form

\[
\bar{\mathbf{N}} = \sum_{k=1}^{n} \int_{\bar{x}_{3k-1}}^{\bar{x}_{3k}} \bar{\sigma}_k \, d\bar{x}_3
\]

\[
= \sum_{k=1}^{n} \left( \int_{\bar{x}_{3k-1}}^{\bar{x}_{3k}} \bar{Q}_k \bar{\epsilon'} \, d\bar{x}_3 + \int_{\bar{x}_{3k-1}}^{\bar{x}_{3k}} \bar{Q}_k \bar{K} \bar{x}_3 \, d\bar{x}_3 \right), \tag{3.52}
\]

and

\[
\bar{\mathbf{M}} = \sum_{k=1}^{n} \int_{\bar{x}_{3k-1}}^{\bar{x}_{3k}} \bar{\sigma}_k \bar{x}_3 \, d\bar{x}_3
\]

\[
= \sum_{k=1}^{n} \left( \int_{\bar{x}_{3k-1}}^{\bar{x}_{3k}} \bar{Q}_k \bar{\epsilon'} \bar{x}_3 \, d\bar{x}_3 + \int_{\bar{x}_{3k-1}}^{\bar{x}_{3k}} \bar{Q}_k \bar{K} \bar{x}_3^2 \, d\bar{x}_3 \right), \tag{3.53}
\]

where \( \bar{x}_{3b} \) and \( \bar{x}_{3t} \) are respectively the bottom and top surfaces of the laminate.

Introducing the notation

\[
\bar{A}_{ij} = \sum_{k=1}^{n} \bar{Q}_{ij} \left( \bar{x}_{3k} - \bar{x}_{3k-1} \right)
\]

\[
\bar{B}_{ij} = \frac{1}{2} \sum_{k=1}^{n} \bar{Q}_{ij} \left( \bar{x}_{3k}^2 - \bar{x}_{3k-1}^2 \right) \tag{3.54}
\]

\[
\bar{D}_{ij} = \frac{1}{3} \sum_{k=1}^{n} \bar{Q}_{ij} \left( \bar{x}_{3k}^3 - \bar{x}_{3k-1}^3 \right)
\]

for \( (i, j = 1, 2, 6) \), equations (3.52) and (3.53) can be combined and rewritten in a matrix form to yield the constitutive relationship in terms of stiffness for an arbitrary
Figure 3.5: Stress and moment resultants for a laminate in an off-axis \( \{ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \} \) coordinate system.

Laminate. Specifically,

\[
\begin{bmatrix}
\tilde{N} \\
\vdots \\
\tilde{M}
\end{bmatrix} =
\begin{bmatrix}
\tilde{A} & \tilde{B} \\
\vdots & \vdots & \vdots \\
\tilde{B} & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
\tilde{\epsilon}^o \\
\vdots \\
\tilde{\kappa}
\end{bmatrix},
\]

(3.55)

where \( \tilde{A}, \tilde{B}, \) and \( \tilde{D} \) are respectively the extensional, coupling, and bending stiffness tensors. Alternatively, the constitutive relationship can be expressed in terms of compliances, see Appendix A, as

\[
\begin{bmatrix}
\tilde{\epsilon}^o \\
\vdots \\
\tilde{\kappa}
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}' & \tilde{B}' \\
\vdots & \vdots & \vdots \\
\tilde{B}' & \tilde{D}'
\end{bmatrix}
\begin{bmatrix}
\tilde{N} \\
\vdots \\
\tilde{M}
\end{bmatrix},
\]

(3.56)

where \( \tilde{A}', \tilde{B}', \) and \( \tilde{D}' \) are the extensional, coupling, and bending compliance tensors.

Clearly, the constitutive relationship for a composite laminate is much more complex than the constitutive relationship for an isotropic material. The stress resultants are related not only to the extensional and shear strains but also to the bending curvatures. Fortunately, the constitutive equations become uncoupled when a laminate
exhibits symmetry with respect to its thickness. This follows from equation (3.54), since the $\bar{B}_{ij}$ are even functions of the individual ply thicknesses.

### 3.4.2 Laminate Engineering Constants

The effective engineering constants of a laminate may be obtained from its extensional compliance tensor $\bar{A}'$. The constitutive relationship for a symmetric laminate in an off-axis coordinate system $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$, where $\bar{B}' = 0$, is given in indicial form by the equation

$$\bar{N}_i = \bar{A}_{ij} \bar{\varepsilon}^o_j \quad \text{for} \quad i, j = 1, 2, 6$$  \hspace{1cm} (3.57)

or in inverted form, see Appendix A, as

$$\bar{\varepsilon}^o_i = \bar{A}'_{ij} \bar{N}_j = \bar{A}'_{ij} \bar{N}_j \quad \text{for} \quad i, j = 1, 2, 6.$$  \hspace{1cm} (3.58)

Recalling that the engineering moduli are defined in terms of strains resulting from simple applied stresses, it follows that the effective longitudinal and transverse Young’s moduli, $E_{11}^o$ and $E_{22}^o$, and the effective in-plane shear modulus, $E_{66}^o$, for a laminate are obtained as

$$E_{ii}^o = \frac{\bar{\sigma}_i}{\bar{\varepsilon}_i^o} = \frac{\bar{N}_i}{\bar{A}_{ii}^o \bar{N}_i} = \frac{1}{h \bar{A}_{ii}^o} \quad \text{for} \quad i = 1, 2, 6.$$  \hspace{1cm} (3.59)

Similarly, the effective in-plane Poisson’s ratios are defined as

$$\nu_{ij}^o = -\frac{\bar{\varepsilon}_{ij}^o}{\bar{\varepsilon}_i^o} = \frac{\bar{A}_{ij}^o \bar{N}_i}{\bar{A}_{ii}^o \bar{N}_i} = \frac{\bar{A}_{ij}^o}{\bar{A}_{ii}^o} \quad \text{for} \quad i, j = 1, 2$$  \hspace{1cm} (3.60)

and the effective in-plane shear coupling coefficients are given by the equations

$$\eta_{i,6}^o = \frac{\bar{\sigma}_6}{\bar{\varepsilon}_i^o} = \frac{\bar{A}_{i6}^o \bar{N}_i}{\bar{A}_{ii}^o \bar{N}_i} = \frac{\bar{A}_{i6}^o}{\bar{A}_{ii}^o} \quad \text{for} \quad i = 1, 2, 6.$$  \hspace{1cm} (3.61)

and

$$\eta_{6,i}^o = \frac{\bar{\sigma}_i}{\bar{\varepsilon}_6^o} = \frac{\bar{A}_{6i}^o \bar{N}_6}{\bar{A}_{66}^o \bar{N}_6} = \frac{\bar{A}_{6i}^o}{\bar{A}_{66}^o} \quad \text{for} \quad i = 1, 2.$$  \hspace{1cm} (3.62)
A similar analysis can be used to obtain the effective in-plane bending moduli in terms of the flexural compliance of a laminate.
Chapter 4

Strength, Failure, and Fatigue of Composite Materials

4.1 Introductory Remarks

As noted by Reifsnider (1991a), the applications for which composite materials offer a clear advantage over conventional engineering materials are precisely those applications in which the degradation of the strength, stiffness, and life of composites by fatigue processes is most likely and most severe. Crucial then to the success of composite materials for such applications is the capacity to predict their behavior under fatigue loading. This requires a thorough understanding of the fatigue process in composite materials. To this end, this chapter provides a review of the current understanding of the physical nature of the fatigue of composite materials, and in particular composite laminae and laminates, in an effort to provide continuity between the fatigue process and the life prediction models to be discussed in subsequent chapters. It begins with a discussion of the strength of composite laminae and the fundamental damage mechanisms which govern their failure under both static and fatigue loading. This leads naturally to a discussion of the fatigue behavior of composite laminae and composite laminates. In view of the complexity of the fatigue process in composite laminates, the chapter ends with a discussion of the important subject of damage characterization.
4.2 Strength and Failure of Composite Laminae

In chapter 3 several "micromechanical" methods for determining the elastic properties of a lamina, given the elastic properties of its constituents were discussed. Experience has shown that reliable predictions for axial stiffness properties can be obtained by simple strength of materials models. Reasonable predictions for transverse and shear properties are more difficult to obtain and require more sophisticated analyses. Unfortunately, methods for predicting the strength of a lamina given the strengths of its constituents and other material properties are not as well developed. This follows from the fact that elastic properties, especially axial properties, are relatively insensitive to the microstructure of a lamina while strength is very dependent on the microstructure. Furthermore, advanced fibers such as carbon and boron are known to exhibit a significant amount of variability in local strength which contributes to the local microstructure sensitivity making prediction of lamina strength a difficult task. To account for the variability in fiber strength a statistical analysis is often used.

A common assumption is that the axial tensile strength of the reinforcing fibers $S_{1r1}$ follows a two parameter Weibull distribution whose cumulative distribution function (CDF) is given by the equation

$$F(S_{1r1}) = 1 - e^{-\alpha L \beta^\beta} \quad \sigma_1 \geq 0, \alpha \geq 0, \beta \geq 0 \quad (4.1)$$

where $\sigma_1$ is the applied stress parallel to the reinforcement direction, $L$ is the length of the fibers, and $\alpha$ and $\beta$ are respectively the distribution scale and shape parameters.

In addition to microstructure sensitivity, the strength of a lamina depends on the orientation of the applied load as the strength of a lamina, much like it stiffness, is highly directionally dependent. For a specially orthotropic lamina, five individual strengths are needed to characterize the overall in-plane strength of a lamina.
These include the axial tensile and compressive strengths, the transverse tensile and compressive strengths, and the axial shear strength. The axial strengths are fiber dominated and can in general be up to an order of magnitude greater than transverse and shear strengths. The transverse and shear strengths are dominated by the matrix phase of a lamina. The individual strengths, again like the stiffness properties of a lamina, are determined by application of simple loading configurations. What follows is a brief discussion of the individual strengths and the failure modes which control their values. The reader is referred to Chawla (1987), Rosen and Hashin (1987), and Gibson(1994) for a thorough treatment of this subject.

4.2.1 Axial Tensile Strength

The axial tensile strength is the most well understood of the five strengths needed to characterize the strength of a composite lamina. It is dominated by the tensile strengths of the reinforcing fibers and their variability. The variability in fiber tensile strength is assumed to result from a distribution of imperfections along the length of a fiber. The two most important consequences of variability in fiber tensile strength are that not all fibers in a lamina will be stressed to their maximum value simultaneously and fibers which break during loading cause a perturbation in the stress field near the break. This perturbation induces high shear stresses along the fiber-matrix interface and causes a load concentration in neighboring fibers. The interfacial shear stresses act to reload a fiber to its original load. The shear stresses are a maximum near a fiber break and decay rapidly along the length of the fiber. The load concentration (tensile) in adjacent fibers is necessary to account for the tensile stress that is shed by a fiber upon breaking. The perturbed stress field near a fiber break is shown schematically in Figure 4.1. As a result of a fiber break and the ensuing redistribution of stress, several
possible failure modes may initiate and propagate before the final failure event. The type of failure mode which occurs is dependent on the strengths of the fiber, matrix, and fiber-matrix interface.

**Weakest Link Failure**

When a fiber breaks and sheds its load on adjacent fibers it causes a load concentration in those fibers. If this load concentration is of significant magnitude such that it causes failure of the adjacent fibers then cracks may initiate and propagate within the lamina. Continued growth of these cracks will lead to the ultimate failure of the lamina and the lamina will be no stronger than the strength of its weakest fiber. That is, weakest link failure will occur at the stress level which causes the weakest fiber to break. Zweben (1968) has determined the expected value of this stress for a group of fibers based on the assumption that the strength of the fibers follows the two parameter Weibull distribution of equation (4.1). The expected value of the axial

**Figure 4.1:** Perturbed stress field resulting from fiber fracture.
tensile strength under this mode of failure is given by the equation

\[ E\{S_{1r}\} = \left( \frac{\beta - 1}{NL\alpha \beta} \right)^{\frac{1}{\beta}}, \]  

(4.2)

where \( N \) is the number of fibers in the group. It should be noted that the occurrence of a first fiber break is a necessary but not sufficient condition for lamina failure. Experience has shown that the fracture of a single fiber rarely leads to complete lamina failure as fiber reinforced composites are very redundant materials. For practical purposes this type of failure is not of significant importance.

**Cumulative Weakening Failure**

If catastrophic failure of a lamina does not occur when the first fiber failure occurs, it is possible to increase the applied load on a lamina. With increasing load, fibers will begin to break randomly throughout a lamina due to imperfections along their lengths. As noted earlier, when a fiber breaks large interfacial shear stresses develop near the break and act to transfer load back to the broken fiber. That is, the matrix material localizes the damage of a fiber break. However, the transfer of load back to a fiber and to its original magnitude is gradual and occurs over a distance known as the ineffective length \( \delta \) of a fiber. The magnitude of \( \delta \) is dependent on many parameters including the elastic properties of both the fiber and matrix, interfacial strength, debonding, matrix fracture, and matrix yielding. A common estimate for the ineffective length is

\[ \delta = \frac{S_{1r} - d_f}{4S_{6m}}, \]  

(4.3)

where \( S_{1r} \) is the axial tensile strength of the fibers, \( S_{6m} \) is the shear yield strength of the matrix, and \( d_f \) is the diameter of the fibers. Rosen (1964) proposed that the fracture of a composite is due to local imperfections and suggested that a composite
lamina could be represented as a chain of bundles of fibers where each link in the chain was of length $\delta$. This concept is illustrated in Figure 4.2. As such, any fiber which fails within a bundle would be unable to be reloaded to its original level and the load carried by the broken fiber would be distributed among the other fibers in the bundle according to some specified load sharing rule. Due to the enhanced load on the unbroken fibers within a bundle, the amount of further loading that the bundle could sustain without failure would decrease, and hence a cumulative weakening of the bundle, and consequently the lamina, would occur.

The validity of the chain of bundles approach is provided by the dependence of fiber strength on fiber length. However, the chain of bundles model neglects the possibility of overlapping damage which limits its success. More importantly, it neglects the coupling effect between fiber breakage and variability in fiber strength. That is, as more fibers within a bundle break, the increased probability that further fiber breaks will initiate and propagate is ignored.
Fiber Break Propagation Failure

To account for the fact that variable fiber strength and variable fiber stress resulting from fiber breaks within a bundle will tend to lead to growth in both the number and size of damage regions within a lamina, the fiber break propagation failure mode has been proposed by Zweben (1968). Zweben conjectured that the first fracture of an overstressed fiber in a two dimensional material could be used as an indicator of the tendency for additional fiber breaks to initiate and propagate and hence the first fiber fracture could be used as a failure criterion. This idea was extended to three dimensional materials by Zweben and Rosen (1970) and included the effects of load concentration on the cumulative weakening failure mode. This criterion has provided good correlation with experimental data for laminae with small fiber volume fractions but it has not worked well for laminae with higher fiber volume fractions. Furthermore, the resulting expression for composite strength resulting from the work of Zweben and Rosen is a sequence which requires a large number of terms for convergence making the model difficult to use. Harlow and Phoenix (1978a, 1978b) have studied this failure mode for two dimensional materials and have utilized conditional probabilities based on assumptions for load sharing between fibers within a bundle to obtain more tractable expressions for lamina strength.

Cumulative Group Mode Failure

The fourth and final fiber dominated axial failure mode is cumulative group mode failure. Models for this failure mode take into account the fact that as fiber failures accumulate within a bundle the local shear stress in the fiber-matrix interface increases. Such increases may lead to shear failure of the matrix material or to fiber-matrix debonding, both of which serve to arrest propagation of fiber breakage.
such, with increasing stress levels from those at which fiber breaks are initiated to the final failure event, a lamina will have dispersed groups of fiber failures. Each group will have an ineffective length that will increase with the size of the groups of broken fibers before matrix failure and with increasing stress after matrix failure. This type of failure mode has been studied by Rosen and Zweben (1972). It is complicated by groups of broken fibers which vary in both the number of broken fibers and the ineffective length of the broken fibers. The cumulative group mode failure can be seen as a generalization of cumulative weakening failure. In practice, it is the most common failure mode. It is illustrated in Figure 4.3.

Figure 4.3: Cumulative group mode failure under tensile loading.
4.2.2 Axial Compressive Strength

Microbuckling Failure

The axial compressive strength of a lamina, much like that of an elastic column, is governed by instability considerations. Consequently, the analysis of the axial compressive strength of composite lamina has been treated in much the same way as the buckling of elastic columns. Through photoelasticity experiments it has shown that two different types of compressive failure due to buckling of reinforcement fibers are possible; namely, in-phase and out-of-phase buckling. The failure modes are shown in figures 4.4a and 4.4b. In-phase buckling involves shear deformation of the matrix material. An energy method analysis yields the following expression for the compressive strength in the in-phase buckling mode

\[ S_{1c} = \frac{E_{12m}}{(1 - V_f)} = \frac{E_{11m}}{2(1 - \nu_{12m})(1 - V_f)}, \quad (4.4) \]

where the subscript \( m \) denotes a matrix property and where \( V_f \) is the fiber volume fraction. Out-of-phase buckling involves transverse tensile and compressive strains. The compressive strength for this failure mode is given, again by energy balance, as

\[ S_{1c} = 2V_f \sqrt{\frac{V_f E_{11m} E_{11f}}{3(1 - V_f)}}. \quad (4.5) \]

The type of compressive failure mode which occurs for a given lamina is governed by the fiber volume fraction as indicated by equations (4.4) and (4.5). For low fiber volume fractions, up to approximately \( V_f = 0.2 \), the out-of-phase mode will occur. At higher fiber volume fractions, neighboring fibers exhibit more influence on one another and the in-phase or coupled failure mode will occur. A variation of the in-phase mode that has also been observed is "shear crippling" due to kink band formation. This failure mode arises from the high shear stresses that develop within a fiber when buckling occurs.
Figure 4.4: Compressive failure modes for a composite lamina.

Transverse Tensile Rupture

A second major type of axial compressive failure mode is transverse tensile rupture. This failure mode is shown in Figure 4.4c. It is a matrix failure mode in which the transverse tensile strain arising from application of an axial compressive load is of sufficient magnitude to cause the fracture of the weaker matrix phase. An elementary
estimate for the compressive strength of a lamina which fails by transverse tensile rupture can be obtained in terms of the effective elastic constants of a lamina as

$$S_{1_C} = \frac{E_{11}c_2}{\nu_{12}}. \quad (4.6)$$

**Shear Failure Without Buckling**

The final axial compressive failure mode is shear failure without buckling. In this failure mode shear failure at an angle of $45^\circ$ to the load axis of both the fibers and the matrix occurs without buckling of the fibers. This type of failure is shown in Figure 4.4d. An elementary estimate for the compressive strength of a lamina which fails by shear failure can be obtained from a simple rule of mixtures analysis noting that $\sigma_{12_{\text{max}}} = S_{1_C}/2$. This estimate gives

$$S_{1_C} = 2 \left( S_{1C_f} V_f + S_{1C_m} (1 - V_f) \right), \quad (4.7)$$

where $S_{1C_f}$ and $S_{1C_m}$ are respectively the axial compressive strengths of the fibers and matrix.

**4.2.3 Transverse Tensile, Compressive and Axial Shear Strengths**

The transverse tensile and compressive strengths, as well as the axial shear strength, of a lamina are difficult quantities to determine as failure under transverse tension, transverse compression, or axial shear can occur without failure of the reinforcing fibers. As such, these strengths are governed by the properties of the matrix material and/or the fiber-matrix interfacial strength. Their prediction is difficult as the state of stress in the matrix material is highly nonuniform due to the presence of the reinforcing fibers and is dependent on the local microstructure. As such, micromechanical models for these strengths are typically used. Values for these strengths are generally determined experimentally.
Individually, the transverse tensile strength of a lamina is governed by the tensile strength of the matrix material or the interfacial fiber-matrix strength depending on whichever is lower. Consequently, the transverse tensile strength of a lamina may be no greater, or possibly even lower, than the tensile strength of the matrix itself. The two possible failure modes which govern the transverse tensile strength are shown in Figure 4.5. The transverse compressive strength is governed by shear stresses

![Diagram](image)

**Figure 4.5:** Transverse tensile failure modes.

which develop in a plane containing the reinforcement fibers and those in a plane normal to the fibers. In planes parallel to the fibers, the fibers themselves offer no significant reinforcement to the matrix material alone. However, in planes normal to the reinforcing fiber, the fibers can have a significant effect on the compressive strength of a lamina particularly at higher fiber volume fractions. The axial shear strength of a lamina is more difficult to characterize. Plasticity theorems in which fibers are idealized as elastic-brittle materials and the matrix as an elastic-perfectly plastic material which obeys the von Mises failure criterion have been utilized to determine bounds on the axial shear strength of a lamina. Analysis has shown that
the presence of fibers does provide a reinforcing effect in terms of the axial shear strength on the order of 25 percent.

4.2.4 Strength and Failure Under Combined Loading

Methods for determining the individual strengths of a lamina based on micromechanical considerations, while providing a great deal of insight into factors that influence strength, have suffered from lack of correlation with experimental data. As such, values for the individual lamina strengths, for the purpose of design, must be determined experimentally. Such experimental data alone are not sufficient to characterize the strength of a lamina as the individual strengths are determined from application of simple load configurations. In reality, lamina are often subjected to combined multiaxial loading, especially when they are incorporated into a laminate. For such cases, the loading is at least two dimensional and sometimes three dimensional. For practical purposes it is necessary to determine the strength of a lamina under combined loading. However, given the complexity of determining the strength of a lamina from micromechanics under simple loading, it is unlikely that this approach will provide useful results for complex loading configurations. Furthermore, experimental verification is also impractical because of the significant number of tests that would be required to fully characterize the strength of a lamina under all possible load configurations. To circumvent these problems, a number analytical failure criteria based on the individual strengths of a lamina have been proposed.

In the formulation of a lamina failure criterion two basic assumptions are generally employed. The first is identical to the one employed in determination of the stress-strain relationship for a lamina; namely, a lamina is assumed to be a homogeneous, specially orthotropic material in a state of plane stress. Second, it is assumed that
the failure criterion can be represented in the normalized form

\[ g(\sigma) = 1, \]  

(4.8)

which describes a surface in three dimensional space with coordinates \( \{\sigma_1, \sigma_2, \sigma_6\} \). As all failure stress states are finite, the surface must be closed. Stress states inside the surface are realizable and do not cause failure. Those stress states which lie on the surface cause failure. Those which lie outside the surface are not realizable.

The most basic failure criterion is the maximum stress criterion. It assumes that failure will occur when any one of the individual stress components reaches its ultimate value regardless of the values of the other stress components. That is, failure occurs if one of the following equalities are met

\[ \sigma_1 \leq S_{1T} \quad \text{or} \quad S_{1C} \]
\[ \sigma_2 \leq S_{2T} \quad \text{or} \quad S_{2C} \]  
\[ \sigma_6 \leq S_6, \]  

(4.9)

where \( S_{1T}, S_{1C}, S_{2T}, S_{2C}, \) and \( S_6 \) are respectively the five individual strength components; namely, the axial tensile and compressive strengths, the transverse tensile and compressive strengths, and the axial shear strength. The maximum strength criterion is unrealistic, however, as it disregards all effects of combined stresses which are known to influence the strength of a lamina. Furthermore, it overestimates the strength of a lamina by doing so.

A more realistic and common failure criterion is the quadratic failure criterion proposed by Tsai and Wu (1971). It takes the form of a quadratic polynomial, including linear terms, in the applied stresses \( \sigma_1, \sigma_2, \) and \( \sigma_6 \). The Tsai-Wu failure
criterion can be expressed as
\[ g_{11}\sigma_1^2 + g_{22}\sigma_2^2 + g_{66}\sigma_6^2 + 2g_{12}\sigma_1\sigma_2 + 2g_{16}\sigma_1\sigma_6 + 2g_{26}\sigma_2\sigma_6 + g_1\sigma_1 + g_2\sigma_2 + g_6\sigma_6 = 1. \] (4.10)

Since the axial shear strength of a lamina is invariant to the sign of the axial shear stress, it follows that all powers of the shear stress in equation (4.10) must be even. Omitting odd powers of the axial shear stress \( \sigma_6 \), equation (4.10) becomes
\[ g_{11}\sigma_1^2 + g_{22}\sigma_2^2 + g_{66}\sigma_6^2 + 2g_{12}\sigma_1\sigma_2 + g_1\sigma_1 + g_2\sigma_2 = 1, \] (4.11)

where \( g_{11}, g_{22}, g_{66}, g_1, \) and \( g_2 \) are strength parameters related to the individual lamina strengths as
\[
\begin{align*}
g_{11} &= \frac{1}{s_{1t} s_{1c}} \\
g_{22} &= \frac{1}{s_{2t} s_{2c}} \\
g_{66} &= \frac{1}{s_6} \\
g_1 &= \frac{1}{s_{1t}} - \frac{1}{s_{1c}} \\
g_2 &= \frac{1}{s_{1t}} - \frac{1}{s_{1c}}.
\end{align*}
\] (4.12)

The stress parameter \( g_{12} \) can not be related directly to the individual lamina strengths. It must be determined from biaxial testing. For simplicity it is often taken as either zero or as
\[ g_{12} = -\frac{1}{2}\sqrt{g_{11}g_{22}}, \] (4.13)
as proposed by Tsai and Hahn (1980) by relating equation (4.11) to the von Mises failure criterion for isotropic materials.

The Tsai-Wu failure criterion of equation (4.11) is based on the stress at a point in a lamina. It suffers from the fact that it does not differentiate between different modes of failure. Thus, it can not be applied in a rational manner to a lamina.
whose failure is dominated by either the fiber or matrix phases. Several modifications to the Tsai-Wu failure criterion have been proposed which take into account fiber and matrix dominated failures by dropping certain terms from the criterion which do not contribute to a particular mode of failure. Swanson (1991) provides a review of these modifications and their effectiveness when compared to experimental data. Hart-Smith (1992, 1993) points out some of the shortcomings of the Tsai-Wu failure criterion and its extensions and provides further insight into the choice of an appropriate failure criterion for practical applications based on different failure modes.

4.3 Interlaminar Stresses and Edge Effects

Before extending the analysis of the strength and failure of composite laminae to composite laminates, it is prudent to discuss the interlaminar stresses and edge effects which may occur in a laminate. Recall from section 3.4 that the constitutive relationship for a laminate was based on the assumptions that each lamina within a laminate is in a state of plane stress, and that the stresses in each lamina are constant but they vary from lamina to lamina. These assumptions are valid for an infinitely wide laminate. However, for a finite width laminate the in-plane shear stress within each ply must vanish at the free edges of the laminate to satisfy equilibrium constraints. Additional stresses are required for this purpose and these stresses are the out-of-plane interlaminar stresses $\sigma_{33}, \sigma_{23}, \text{and} \sigma_{13}$ or in contracted notation $\sigma_3, \sigma_4, \text{and} \sigma_5$. Pipes and Pagano (1970) studied this problem and determined that interlaminar stresses are confined to distances from the free edges of a laminate comparable to its thickness. On the interior of a laminate the assumption of plane stress is valid. Further, it was found that the stacking sequence of a laminate effects the magnitude as well as the sign of the interlaminar stresses. If the interlaminar stresses are tensile then
a laminate is likely to delaminate. That is, the individual laminae may debond and separate. Determination of interlaminar stresses generally involves numerical solution of boundary value problems from the theory of elasticity for anisotropic materials. As will be discussed later, interlaminar stresses may also develop in the interior of a laminate in the presence of damage.

4.4 Strength and Failure of Composite Laminates

The analysis of the strength and failure of composite laminates is based on the analysis and failure of the individual laminae which make up the laminate. As one might expect, the failure of certain plies within a laminate is preferential over others depending on their orientation and the applied load on the laminate. The typical approach to predicting the strength of a laminate is to determine, for a given loading in off-axis coordinates, the individual ply stresses in each lamina in principal material coordinates according to the transformations presented in chapter 3. The stresses in each ply are then input into an appropriate failure criterion to determine if a given ply has failed. The ply in which the failure criterion is satisfied for the lowest load will determine the initial or “first-ply” failure of the laminate. In many cases, however, a laminate has a considerable amount of strength remaining after the initial failure. It remains then to determine when subsequent ply failures or total laminate failure will occur. This is a difficult problem and one which is still a topic of intense research. However, over the past 20 years or so the so-called “ply-discount” method has emerged as a means for estimating these quantities.

In the ply discount method, after first-ply failure and depending on the type of failure mode which caused the initial failure, the appropriate moduli for a given ply are discounted by a given amount or set to zero. Specifically, if the initial failure was
a fiber dominated failure then $E_{11}$ for the failed ply is discounted. If ply failure was caused by failure of the matrix then $E_{22}$ and $E_{66}$ for the failed ply are discounted. After this procedure, the laminate stiffness matrices are recomputed and a stress analysis for the remaining undamaged plies is carried out as before. A failure criterion is then applied to each of the undamaged plies to determine the second-ply failure at which point the appropriate moduli are again reduced. This procedure is repeated until last-ply failure is predicted. The load at which last-ply failure occurs is considered to be the strength of the laminate. A typical stress-strain curve predicted by the ply discount method for a uniaxially loaded laminate is shown in Figure 4.6. From

![Stress-Strain Curve](image)

**Figure 4.6:** Typical laminate stress-strain as predicted by the ply discount method.

this figure it is seen that the ply-discount method produces a piecewise continuous stress-strain curve. Experience has shown that the ply-discount method is accurate to within 10 to 15 percent when compared to experimental results (Reifsnider and Jamison, 1982).
When interlaminar stresses contribute to or dominate the failure of a laminate predicting the strength of a laminate is much more complex. This complexity arises not only from the fact that ply failure criteria generally consider only in-plane stresses but also from the fact that singularities in the interlaminar stresses may exist. In general, numerical analyses such as finite element analysis are necessary for determining the strength of a laminate under such conditions although some analytical approaches based on fracture mechanics have been proposed. Ochoa and Reddy (1992) discuss the role of finite element analysis for delamination as well as other damage modes in composite laminates.

4.5 Fatigue of Composite Laminae

The failure mechanisms, or in more general terms, the damage mechanisms which occur in composite laminae under fatigue loading are the same as those that occur under static loading. That is, fiber fracture, matrix cracking, and fiber-matrix debonding. The difference between statically and dynamically loading a lamina is in how damage accrues. All other things being equal, under static loading damage accumulation is a function on the magnitude of the applied load with larger loads giving rise to a greater accumulation of damage. Under fatigue loading, damage accumulation is a function not only of the applied load but of the number of fatigue cycles as well. In either case, the type of damage which occurs is governed by the orientation of a lamina with respect to the primary load axis. The following sections discuss the fatigue of composite lamina under uniaxial tensile loading. The reader is referred to Talreja (1987) for a more complete discussion of this topic.
4.5.1 Fatigue Under Loading Parallel to Fibers

Three primary damage modes are known to occur within a lamina under fatigue loading parallel to the reinforcing fibers. These damage modes are illustrated in Figure 4.7 and include fiber breakage, matrix cracking, and fiber-matrix debonding.

![Figure 4.7: On-axis fatigue failure modes for a composite lamina.](image)

due to interfacial shear failure. As in the case of static loading, fiber breakage will occur if the maximum applied load (cyclic in this case) exceeds the strength of the weakest fiber in a lamina. When a fiber breaks, shear stresses develop in the fiber-matrix interface as the matrix transfers load back to the fiber. If the shear stress is of sufficient magnitude the interface will fail. The length of the debond is dependent on the strength of the interface and is generally quite small and of the order of a few fiber diameters. The discontinuity caused by fiber-matrix debonding induces a stress concentration for the axial tensile stress. If the augmented tensile stress exceeds the tensile strength of the weaker matrix material, then small transverse matrix cracking will occur. This damage mode is illustrated in Figure 4.7a and is considered as a
non-progressive damage mode. It is non-progressive in the sense that no region of
damage initiates and propagates throughout the lifetime of a lamina.

If a lamina is subjected to fatigue loading, the constrained matrix material is
subjected to strain controlled fatigue. If the magnitude of the cyclic strain exceeds
the strain limit for the matrix material the second damage mode, transverse matrix
cracking in the absence of fiber breakage, can occur. This concept is illustrated
in Figure 4.7b. The initiation and growth of transverse matrix cracks under fatigue
loading of a lamina perpendicular to the fibers is similar to the initiation and growth of
fatigue cracks in metals. That is, there is a process of nucleation and of intermittent
growth until a critical length, or in this case a fiber, is reached. When a short
transverse matrix crack encounters a fiber in its path of growth one of two actions
may occur. If the cyclic strain is small, the crack may be arrested at the fiber-matrix
interface. Alternatively, if the stress concentration at the tip of the matrix crack
is sufficiently large, as in the case of large cyclic strains, then fracture of the fiber
may occur. If the later occurs, the matrix crack will grow and can propagate as a
macro-crack in an opening mode. If the size of the macro-crack is large enough, the
crack may be diverted from an opening mode to a sliding mode when it encounters
a fiber as the stresses at the crack tip may be sufficient to cause failure of the fiber-
matrix interface as illustrated in Figure 4.7c. Clearly these two damage modes are
progressive in nature.

Note that these three damage modes may occur simultaneously. However, experi-
ence has shown that one of the damage mechanisms usually dominates the other two
depending on the range of the applied maximum strain. Based on this observation,
Talreja (1987) has noted that a fatigue-life diagram plotting the applied maximum
strain against fatigue life will have three distinct regions corresponding to the differ-
ent damage modes as shown in Figure 4.7. The choice of maximum applied strain as

![Diagram]

**Figure 4.8:** Schematic representation of a fatigue-life diagram.

an independent variable is motivated by the fact that for a given applied load both the fibers and the matrix will experience the same strain. The stresses in the fibers and matrix will be different due to their different stiffnesses. The relative stiffnesses of the fibers and the matrix control the size of each region of the fatigue life diagram. Since the fracture strain for a lamina is generally equal to the fracture strain of the reinforcing fibers, high stiffness fibers will cause the fatigue life to be dominated by the non-progressive damage mode since a very limited range of strains can be applied to a lamina with high stiffness fibers without causing complete fracture. If the fracture strain of the fibers falls below the fatigue limit strain of the matrix, no progressive damage will occur. In contrast, low stiffness fibers will allow a lamina to experience a much greater range of strains and thus the progressive failure modes will have a much more pronounced effect on the fatigue life of a lamina.
4.5.2 Fatigue Under Loading Inclined to Fibers

As one might expect, when the angle of the applied load is inclined to the direction of the reinforcing fibers in a lamina the fatigue behavior of a lamina will be influenced to a larger extent by the weaker matrix phase of the lamina. For inclined loading, a lamina will experience two displacement components: an opening mode and a sliding mode. If the opening component of the applied load exceeds the transverse strength of a lamina, as governed by the weaker of the matrix strength and the fiber-matrix interfacial strength, matrix cracking or fiber-matrix debonding can be expected to occur. This is illustrated in Figure 4.9. The nature of the applied load leads to

![Diagram](image)

Figure 4.9: Off-axis fatigue failure modes for a composite lamina.

mixed-mode crack growth parallel to the reinforcing fibers. The limiting value of the crack tip displacement under which no growth of a matrix or interfacial crack will occur will be governed by the angle of inclination of the applied load (the off-axis angle). For a given applied strain, the opening mode crack tip displacement will increase with increasing off-axis orientation. From fracture mechanics it is known that opening mode crack growth requires less energy than the sliding mode and as
such the limiting applied strain below which no crack growth will occur will decrease with increasing off-axis orientation. The limiting case ($\theta = 90^\circ$) were the applied load is perpendicular to the fibers is thus the weakest orientation. Matrix and/or fiber-matrix interfacial cracks in this case and other large off-axis orientations may grow instantaneously across the width of a lamina. Further, the cracks are known to span the thickness of a lamina on formation.

4.6 Fatigue of Composite Laminates

The same damage mechanisms that occur in composite laminae subjected to fatigue loading naturally occur in composite laminates subjected to fatigue loading with the added damage mechanism of delamination. Delamination occurs in the matrix rich interface between plies and is a matrix controlled damage mode. The relative roles of fiber breakage, matrix cracking, fiber-matrix debonding, and delamination depend on the configuration of a laminate. That is, the damage that develops within a ply or between plies is dependent on the constraint imposed by adjacent plies.

The accumulation of fatigue damage in composite laminates is characterized by the degradation of laminate strength and stiffness from their respective static values. The magnitudes of these reductions increase with the number of load cycles. Further, these reductions are direct consequences of changes in the internal geometry of a laminate. The introduction of a crack, resulting from either fiber failure or matrix failure, reduces the load carrying cross section of a laminate and makes a laminate more compliant. The action of the individual damage modes on the strength and stiffness of a laminate has been the subject of a great deal of research. Although the effect of the individual damage mechanisms varies from laminate to laminate, a general pattern concerning their chronology is evident. The typical chronology of these
damage modes, as noted by Reifsnider and Talug (1980), Reifsnider et al. (1983a, 1983b), Jamison et al. (1984), Jamison (1986), and Stinchcomb and Reifsnider (1988) is illustrated schematically in Figure 4.10. From this figure the underlying pattern

![Diagram of damage chronology](image)

*Note: Fiber breakage occurs in each stage*

**Figure 4.10:** Chronology of damage in a composite laminate subjected to fatigue loading.

of damage development can be seen as a causative sequence of matrix related effects, each in conjunction with fiber breakage, leading ultimately to laminate failure. Reifsnider (1991c) has given a comprehensive review of the fatigue process in composite laminates.

From Figure 4.10 the initial stage of damage development consists of primary matrix cracking in the off-axis plies of a laminate parallel to the reinforcing fibers. The number of cracks in a given ply increases with the number of applied load cycles at a rate determined by the magnitude of the applied load. The crack density in a
ply will increase until a saturation spacing known as the characteristic damage state (CDS) is obtained (Reifsnider, 1977; Masters and Reifsnider, 1982).

As previously noted, primary matrix cracks extend through the thickness of a ply on formation. This places the primary matrix crack tips at the interface between adjacent plies. Here, the primary matrix crack encounter the reinforcing fibers in the adjacent plies. The stress concentration induced by the primary matrix cracks causes preferential fiber breakage near the crack tips. Destructive testing of composite laminates subjected to fatigue loading has shown that fiber breakage occurs in bands near the tips of the primary matrix cracks with spacing between the bands approximately equal to the saturation spacing of the primary matrix cracks. The region of influence of the primary matrix crack tips on fiber breakage in adjacent plies is on the order of a few fiber diameters. Approximately two thirds of the total number of fibers that fracture before final failure do so in the first one third of the fatigue life of a composite due to primary matrix cracking.

In addition to fiber breakage, primary matrix cracking also initiates secondary matrix cracking. The secondary matrix cracks extend short distances away from the interface between a ply sustaining primary matrix cracking and an adjacent ply. The secondary matrix cracks are generally perpendicular to the primary matrix cracks. They are caused by the opening tensile stresses at the tips of the primary matrix cracks.

Continued cyclic loading leads to coupling of primary and secondary matrix cracking in adjacent plies which in turn leads to interfacial debonding between plies. Experimental results have shown that intersections of primary and secondary matrix cracks in adjacent plies lead to high out-of-plane interlaminar stresses which cause interior delaminations to initiate and propagate with additional cyclic loading. It
should be noted that delamination relaxes the constraint between adjacent plies, and among other things tends to separate one ply from another, thus causing a decrease in the number of fiber fractures until the end of the life cycle is approached.

As the fatigue life of a laminate is approached, a rapid growth and localization of all damage mechanisms occurs which culminates in the final fracture of the laminate as a whole. The final fracture of a laminate is generally dominated by the large-scale tensile fracture of the reinforcing fibers in the plies most aligned with the primary load axis. Often associated with the final fracture event is the phenomenon of longitudinal splitting in which individual volumes of fibers in the load bearing plies of a laminate are isolated from the rest of a laminate due to longitudinal cracking and coalescence of interior delaminations. These isolated fiber volumes support the load on the laminate in parallel, but with a much reduced cross section, leading to overstressing of the fibers within the volumes. Successive failure of these fiber volumes leads to the final fracture event.

Talreja (1987) notes that the individual damage mechanisms occur simultaneously, but for each damage mechanisms there exists a range of the fatigue life of a laminate over which each of the individual damage mechanisms dominate. Furthermore, if the applied strain on a laminate approaches the average failure strain of the reinforcing fibers in the load bearing plies then damage development will be non-progressive and will be statistical in nature. In such cases, generalizations of the aforementioned models for the axial strength of a composite lamina may be suitable for analysis of fatigue failure. Modifications of these models to account for the constraint imposed on the load bearing plies by adjacent plies would of course be necessary.
4.7 Fatigue Damage Characterization

Given the complexity of the cumulative damage (CD) process in a composite laminate subjected to fatigue loading it is natural to ask the question "How can damage be characterized for the sake of predicting the mechanical behavior of a laminate?". Damage characterization is a fundamental problem in the study of the fatigue of composite materials. It remains a great challenge and is a topic of intense research yet today. From the study of damage characterization the discipline of damage mechanics has emerged. Damage mechanics focuses on the development of a physical understanding of the CD processes and the incorporation of this understanding into a mathematical framework, such as a constitutive relationship, which provides a connection between the underlying CD process and the variables used to characterize the process based on the physics and mechanics which govern the process.

Talreja (1991) provides an in-depth discussion of the damage characterization problem and its possible solutions. Specifically, he notes that the various approaches to damage characterization can be broadly classified as micromechanics approaches and internal variable characterizations. Input into either approach consists of data obtained from either destructive or nondestructive testing. The output of either approach consists of mechanical properties of a laminate including its stiffness and/or strength and/or life. Of these, only stiffness is a property which can be measured nondestructively throughout the life of a laminate. Furthermore, Reifsnider and Stinchcomb (1986) note that stiffness changes in a laminate provide a direct measure of the damage in a laminate as stiffness changes are proportional to damage development. As such, many micromechanics and internal variable approaches to characterizing fatigue damage predict stiffness changes in a laminate as a function of the number of applied load cycles.
The micromechanics approach utilizes a representative volume element (RVE) whose response is assumed to describe the laminate behavior as a whole. To meet this assumption, must contain all pertinent material details including microstructural and macro-structural details. Furthermore, it must contain details of the damage entities including their size, shape, and orientation. An appropriate boundary value problem is then formulated for the RVE whose solution yields the response of the RVE and hence the response of the laminate. The micromechanics approach, while straightforward and unambiguous, can become very complex even for simple damage modes. However, it is robust in the sense that more complex damage modes or combinations of damage modes can be analyzed with relatively good accuracy. This approach has been taken by a number of researchers including Hashin (1991) and Nemat-Nasser and Hori (1993). Variations on this approach including variational methods (Hashin, 1985, 1987; Highsmith and Reifsnider, 1986) and fracture mechanics (O'Brien, 1982, 1985) have also been used. These methods, each with their inherent advantages and disadvantages, have focused on stiffness reduction due to intralaminar and interlaminar (delamination) matrix cracking alone. This reflects the limited understanding of the mechanics of the complete damage accumulation process. Reifsnider (1991c) notes that at the present it is not possible to give a fundamental development of the damage mechanics based on first principles which describes in detail the strength, stiffness, and life of a laminate as a general function of the loading.

The internal variable approach is based on continuum mechanics. It characterizes damage indirectly by associating damage with a set of internal state variables. The relationship between the internal state variables and the damage itself is a phenomenological relationship whose parameters are determined experimentally. The internal state variables can be scalars, vectors, or higher order tensors with the choice
being dictated by the details of the damage mode or modes being described. As this approach is based on continuum mechanics it is well suited for the first state of fatigue damage development where damage, in the form of primary matrix cracks, is dispersed throughout a laminate. However, as damage accumulation progresses and localization occurs, the assumption of a continuous medium becomes invalid and difficulties with this approach may arise. This approach has been used by Talreja (1986, 1987, 1991a, 1991b), Allen (1988), and Allen, Highsmith, and Lo (1991) to model stiffness changes due to matrix cracking and delamination.
Chapter 5

Life Prediction Methods

5.1 Introductory Remarks

In the previous chapter it was noted that damage accumulates throughout the life of a composite laminate under fatigue loading and manifests itself in the degradation of laminate strength and stiffness. Furthermore, the magnitudes of these changes increase with the number of applied load cycles. In view of the detrimental effects of the CD process, two fundamental problems associated with the fatigue of composite laminates are the prediction of the useful lifetime of a laminate and the prediction of the safe operating load that a laminate can sustain after a given number of applied load cycles. This chapter will review major approaches to resolving these two issues with emphasis on the former. Specific models will be discussed where warranted. The reader is referred to Wang (1987) and Sendeckyj (1991) for a comprehensive review and discussion of life prediction methods.

5.2 General Formulation of Life Prediction Methods

The development of rational life prediction methods for any material subjected to general fatigue loading is a complex task. As noted by Sendeckyj (1991) it involves experimental observation of the cumulative damage (CD) process which controls the life of the material, formulation of a damage metric which characterizes the CD process, formulation of a CD model including a damage summation rule in terms of the damage metric, experimental characterization of model parameters, and experimental
verification of the predicted lifetimes. Furthermore, successful life prediction methods for composite laminates, with their variability in material properties and their infinite number of possible configurations, should be capable of taking into account all pertinent material, testing, and environmental variables, correlating the data for a large class of materials, predicting laminate lifetimes from lamina fatigue data, and accounting for scatter in fatigue data.

The general approach to formulating life prediction methods based on the first itemized list has been successfully utilized for predicting the fatigue life of metallic structures where the damage metric is the length of a dominant fatigue crack. It has resulted in the formulation of simple crack growth equations such as the well-known Paris-Erdogan equation (Paris and Erdogan, 1963). However, in composite materials, where damage accumulation is characterized by the action and interaction of several distinct damage mechanisms, no single dominant damage mechanism exists. Consequently, a number of different life prediction methods for composite materials, and specifically composite laminates, based on a number of different damage metrics have been proposed. These life prediction methods can be classified as constant amplitude methods and cumulative damage methods with the later being capable of predicting fatigue lifetimes under spectrum and/or random fatigue loading. With regard to the statistical considerations of the second list, many of the proposed life prediction methods for composite laminates are probabilistic in nature.

5.3 Constant Amplitude Life Prediction Methods

5.3.1 Empirical Methods

The most basic fatigue life prediction methods for composite laminates are empirical fatigue theories which describe the stress-life (S-N) behavior of a laminate under
uniaxial, constant amplitude cyclic loading. These theories include the classical power law fatigue failure criterion

\[ \frac{\tilde{\sigma}_{ia}}{\tilde{S}_i} = N^{k_i} \quad \text{for} \quad i = 1, 2, 6 \]  \hspace{1cm} (5.1)

which gives a straight line S-N curve on a log-log plot of stress versus life and its counterpart on a semi-log plot of stress versus life

\[ \tilde{\sigma}_{ia} = \tilde{S}_i - k_i \log_{10} N \quad \text{for} \quad i = 1, 2, 6. \]  \hspace{1cm} (5.2)

Here, \( k_i \) is a material constant, \( \tilde{S}_i \) is the strength of a laminate in the \( i^{th} \) off-axis coordinate direction, \( \tilde{\sigma}_{ia} \) is the maximum or minimum applied cyclic stress in the \( i^{th} \) off-axis coordinate direction depending on the sign of \( \tilde{S}_i \). That is, if the fatigue load is tensile in nature, then \( \tilde{S}_i = \tilde{S}_{tp} \) and \( \tilde{\sigma}_{ia} \) is the maximum applied cyclic stress. Likewise, if the fatigue load is compressive then \( \tilde{S}_i = \tilde{S}_{tc} \) and \( \tilde{\sigma}_{ia} \) is the minimum applied cyclic stress. Several uniaxial, constant amplitude empirical fatigue theories based on modifications of equations (5.1) and (5.2) which incorporate the stress range

\[ \Delta \tilde{\sigma}_i = \tilde{\sigma}_{imax} - \tilde{\sigma}_{imin} \]  \hspace{1cm} (5.3)

and the mean stress

\[ \tilde{\sigma}_{im}\text{ean} = \frac{\tilde{\sigma}_{imax} + \tilde{\sigma}_{imin}}{2} \]  \hspace{1cm} (5.4)

into their structure have also been proposed.

As uniaxial, constant amplitude fatigue loading rarely occurs in the service life of a composite structure, a number of extensions, such as that proposed by Rotem (1979), to extend uniaxial, constant amplitude fatigue theories to include multiaxial fatigue loading have been proposed. These extensions involve the generalization of the point-stress static failure criteria discussed in section 4.4 to fatigue loading. In the multiaxial, constant amplitude empirical fatigue theories the static strengths of
the individual lamina are replaced by their individual residual strengths after cyclic loading, thus introducing the effect of cyclic loading. The general normalized form of the multiaxial, constant amplitude empirical fatigue theories is given by the equation

\[ \max_i g_i(\boldsymbol{\sigma}, n) = 1, \]  

(5.5)

where \( i \) ranges over the number of plies in a laminate and \( \boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_6\}^T \) are the stresses in principal material coordinates at a point in an individual lamina. The form of the fatigue failure criterion itself is not dependent on the number of applied load cycles and typically has the same form as one of the static failure criteria discussed earlier. Specifically, it usually has the form of the maximum stress, Tsai-Hill, or the Tsai-Wu failure criteria or one of their many variations. The functional dependence of the fatigue failure criterion on the number of applied load cycles is governed by the cycle dependence of the individual strength components of each lamina. Each individual lamina strength component \( S_j \in \{S_{1r}, S_{1c}, S_{2r}, S_{2c}, S_6\} \) in principal material coordinates has the form

\[ S_j = S_j(n), \]  

(5.6)

or more generally

\[ S_j = S_j(\sigma_a, T, M, R, \omega, n), \]  

(5.7)

where \( \sigma_a, T, M, R, \) and \( \omega \) are respectively the maximum or minimum applied cyclic stresses, temperature, moisture content, stress ratio \( (R = \sigma_{\text{min}}/\sigma_{\text{max}}) \), and frequency of the applied load. The relationship between the individual lamina strengths and the number of applied load cycles is typically given by a uniaxial, constant amplitude fatigue theory such as equation (5.1), equation (5.2), or one of their many modifications.
Several specific multiaxial, constant amplitude life prediction methods have been proposed in the literature (Hahn, 1979; Sims and Brogdon, (1977); and Rotem and Hashin, 1976). The most common forms for the fatigue failure criterion \( g(\sigma) \) and residual strength function \( S_f(n) \), as noted by Sendeckyj (1991), are shown in Table 5.1. In principle, multiaxial, constant amplitude empirical fatigue theories are capable of predicting the fatigue life of a laminate subjected to multiaxial, constant amplitude fatigue loading using constant amplitude fatigue data for the individual laminae, classical laminated plate theory (CLPT), a damage summation rule, and a ply discount method modified for fatigue loading. The absence of a verified damage summation rule and ply-discount method have, however, prevented the successful application of this life prediction method. Furthermore, this method suffers from the fact that it does not take individual failure modes, stress redistribution, or interfacial (interlaminar) stresses into account.

<table>
<thead>
<tr>
<th>Failure Criterion, ( g(\sigma) )</th>
<th>Residual Strength Function, ( S_f(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max{\frac{\sigma_1}{S_1}, \frac{\sigma_2}{S_2}, \frac{\sigma_3}{S_3}} )</td>
<td>( a_j - b_j \log_{10} n )</td>
</tr>
<tr>
<td>( \frac{\sigma_1^2}{S_1^2} - \frac{\sigma_2\sigma_2}{S_2^2} + \frac{\sigma_3^2}{S_3^2} = 1 )</td>
<td>( a_j + \frac{\sigma_2}{n^2} - \frac{\sigma_1}{A^2} ) with ( A = \frac{1-R}{1+R} )</td>
</tr>
<tr>
<td>( \max{\frac{\sigma_1}{S_1}, \left(\frac{\sigma_2^2}{S_2^2} + \frac{\sigma_3^2}{S_3^2}\right)} = 1 )</td>
<td>( a_j - b_j \log_{10} n )</td>
</tr>
</tbody>
</table>

**Table 5.1:** Multiaxial, constant amplitude empirical fatigue theories where \( a_k, b_k, c_k, x, \) and \( y \) are material constants.

### 5.3.2 Methods Based on Residual Strength Degradation

Given the fact that the CD process in composite laminates subjected to fatigue loading causes a decrease in the strength of a laminate, the magnitude of which increases
with the number of applied load cycles, many researchers have proposed life prediction methods based on residual strength degradation. These methods are currently the most widely used and accepted methods for predicting the lifetime of a laminate. They generally focus on the longitudinal (axial) strength of a laminate. This follows from the convention in the "composites community" that the longitudinal axis of a laminate, in off-axis coordinates, coincides with the primary load axis. Thus, the longitudinal strength of a laminate controls the strength of a laminate as a whole. With this convention, the majority of these methods are based on three basic assumptions. Namely, the longitudinal static strength of a laminate follows a two parameter Weibull distribution, the residual longitudinal strength of a laminate, \( \bar{S}_{1r} \), after \( n \) cycles is related to the initial longitudinal static strength, \( \bar{S}_1 \), by an empirical deterministic equation, and fatigue failure occurs when the residual longitudinal strength is reduced to the level of the maximum applied cyclic stress (in the absolute sense).

The deterministic equation relating the residual strength after fatigue loading and the initial static strength is generally given as a rate equation where the rate of residual strength reduction is a power function of the applied longitudinal cyclic stress. In its most basic form this rate of residual strength degradation is expressed as

\[
\frac{d \bar{S}_{1r}(n)}{dn} = -\frac{g_1}{g_2 \bar{S}_{1r}^{g_2-1}} \bar{\sigma}_{1a}^{g_2},
\]  

(5.8)

where \( \bar{\sigma}_{1a} \) is the maximum or minimum applied cyclic stress in the longitudinal off-axis coordinate direction depending on the sign of \( \bar{S}_{1r} \), and \( g_1 \) and \( g_2 \) are dimensionless functions which depend on the mechanical and environmental loads on a laminate. Equation (5.8) can be integrated with respect to the number of load cycles, \( n \), to determine the relationship between the residual strength after \( n \) cycles and the initial
static strength. Integration yields

\[ \tilde{S}_1(n) = \tilde{S}_1 + g_1 \tilde{S}_1(n - 1). \]  

(5.9)

If failure of a laminate occurs on or during the application of the first load cycle, then \( \tilde{S}_1 = \tilde{S}_1 \). Letting \( g_3 = 1/g_2 \) and rearranging terms leads to the following expression for the initial longitudinal static strength

\[ \tilde{S}_1(n) = \tilde{\sigma}_1 [\left( \frac{\tilde{S}_1}{\tilde{\sigma}_1} \right)^{1/g_3} + g_1(n - 1)]^{g_3}. \]  

(5.10)

If fatigue failure occurs when \( \tilde{S}_1 = \tilde{\sigma}_1 \) and if \( N \) represents the number of load cycles to failure, then equation (5.10) yields the following expression for the S-N behavior of a laminate

\[ \tilde{S}_1 = \tilde{\sigma}_1 [1 + g_1(N - 1)]^{g_3}. \]  

(5.11)

On a log-log plot of equation (5.11) the dimensionless functions \( g_1 \) and \( g_3 \) can be interpreted as a measure of the flat portion of the S-N curve for low \( \tilde{\sigma}_1 \) and the slope of the S-N curve for high \( \tilde{\sigma}_1 \) respectively.

An implicit assumption in the formulation of life prediction methods based on residual strength degradation is that parameters which enter the expression for the residual strength of a laminate are material constants. As such, all scatter in the fatigue data for residual strength must be attributed to scatter in the initial distribution of static strength. With the assumption that the initial static strength of a laminate follows a two parameter Weibull distribution, the initial static strength has a cumulative distribution function (CDF) of the form

\[ F(\tilde{s}_1) = P\{\tilde{S}_1 \leq \tilde{s}_1\} = 1 - \exp\{- \left( \frac{\tilde{s}_1}{\beta} \right)^\alpha \}, \]  

(5.12)

where \( \alpha \) and \( \beta \) are respectively the shape and scale parameters of the distribution. In addition, it is generally assumed that the strength of a laminate satisfies the
"equal rank" assumption proposed by Hahn and Kim (1975, 1976). The equal rank assumption implies that the order (rank) of a group of laminates, in terms of initial static strength, is preserved upon measurement of the residual strength after a given number of load cycles. Substituting equation (5.12) into equation (5.11) and ignoring the small but finite probabilities that some laminates may fail during the first load cycle yields the CDF for the lifetime of a laminate

$$F(n) = P\{N \leq n\} = 1 - e^{-\left(\frac{n-\hat{\gamma}}{\hat{\beta}}\right)^{\hat{\alpha}}}$$

(5.13)

which has the form of a three parameter Weibull distribution with shape ($\hat{\alpha}$), scale ($\hat{\beta}$), and location ($\hat{\gamma}$) parameters of

$$\hat{\alpha} = \alpha g_3$$

(5.14)

$$\hat{\beta} = \frac{1}{g_1} \left(\frac{\beta}{\bar{\sigma}_{1a}}\right)^{1/g_2}$$

(5.15)

$$\hat{\gamma} = \frac{g_1 - 1}{g_1}.$$  

(5.16)

A number of specific constant amplitude fatigue theories based on residual strength degradation have been proposed in the literature (Chou and Croman, 1978, 1979; Sendeckyj, 1981; Whitney, 1982, 1983; Yang 1977, 1978; Yang and Jones, 1978, 1980, 1981; Yang and Liu, 1977; Yang, Miller, and Sun, 1980; and Yang and Sun, 1980). The most common forms for the dimensionless functions $g_1$ and $g_3$, as noted by Sendeckyj (1991), are summarized in Table 5.2. The model parameters in Table 5.2 can be determined by parameter estimation procedures including the method of moments and maximum likelihood estimates among others. Sendeckyj (1981) has proposed a procedure for determining these model parameters and determining whether or not the individual fatigue models fit a given set of experimental fatigue data.

It should be noted that the shape parameter of equation (5.14) may be negative due to disregarding the probabilities that some specimens may fail during the first
<table>
<thead>
<tr>
<th>$g_1$</th>
<th>$g_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$S_0$</td>
</tr>
<tr>
<td>$C$</td>
<td>$S_0$</td>
</tr>
<tr>
<td>$C(1 - R)^G$</td>
<td>$S_0$</td>
</tr>
<tr>
<td>$C(1 - R)^G$</td>
<td>$S_0(1 - R)^G$</td>
</tr>
<tr>
<td>$C(1 - R)^G$</td>
<td>$S_0 + D(1 - R)$</td>
</tr>
<tr>
<td>$C(1 - R)^G$</td>
<td>$S_0(1 - R)^D$</td>
</tr>
</tbody>
</table>

Table 5.2: Uniaxial, constant amplitude fatigue theories based on residual strength degradation with parameters $S_0$, $D$, $C$, and $G$.

load cycle. This can be avoided by using conditional probabilities in the derivation of the CDF of laminate lifetime, however, such a derivation does not lead to a common form for the CDF. Furthermore, the assumption that the parameters which define the residual strength degradation rate equation are material constants may not be correct. Since the residual strength of a laminate cannot be measured nondestructively during testing, it is not feasible to assess the variability of these parameters. One possible solution to this problem is to consider these parameters as random variables.

5.3.3 Methods Based on Stiffness Change - The Critical Element Model

While providing a more realistic method for predicting the life of laminate, life prediction methods based on residual strength degradation suffer from two major problems. The first and foremost is the fact that residual strength cannot be measured during the lifetime of a laminate without destroying the laminate itself. In addition, the residual strength of a laminate may not change significantly until failure of a lam-
inate is imminent. As such, changes in the residual strength of a laminate are not very good indicators of damage accumulation within a laminate.

In an effort to overcome these difficulties, many researchers have proposed utilizing changes in the stiffness of a laminate as a damage metric. A number of studies including those by O'Brien (1980), Reifsnider, et al. (1979), and Highsmith and Reifsnider (1982) have shown that stiffness changes are directly related to the accumulation of damage in a laminate. Furthermore, stiffness changes provide an excellent measure of internal stress redistribution within a laminate since the damage mechanisms which produce stiffness changes produce stress redistribution in direct proportion. In addition, changes in the stiffness of a laminate occur throughout the life of a laminate and are of greater magnitude than changes in the residual strength. Most importantly however, is the fact that stiffness changes can be measured nondestructively throughout the life of a laminate. Measurements of laminate stiffness may be made by continuous monitoring of the stiffness itself or through models, such as those of Talreja (1986, 1987, 1991a, 1991b) and O’Brien (1982, 1985) among others, which relate quantities obtainable through nondestructive testing, such as matrix crack density and delamination size, to changes in laminate stiffness.

The most well known of the life prediction methods based on changes in the stiffness of a laminate is the critical element model (Reifsnider, 1986, 1991c, 1991d, 1992; Reifsnider and Carman, 1992; Reifsnider and Stinchcomb, 1986). The critical element model is a generalization of life prediction methods based on residual strength degradation which provides a rational means for addressing the complex damage accumulation process by utilizing a representative volume approach. Based on the anticipated failure mode of a laminate, the response of a representative volume element (RVE) is assumed to characterize the response of the laminate as a whole.
The RVE is divided into critical and subcritical elements based on their contribution to the response of the RVE.

The subcritical elements are those material elements whose "failure" causes a redistribution of stress in a laminate due to the changes in the local geometry and/or constraints of a laminate. The magnitude of the stress redistribution is determined by measuring or predicting changes in the stiffness properties of a laminate as changes in the stiffness properties are proportional to stress redistribution. The critical elements are those material elements which control the failure or fracture of the laminate. The critical elements describe the state of the laminate itself and are characterized by phenomenological information on their response such as S-N diagrams. To this extent the critical element model is a hybrid model in which changes in the internal stress state in a laminate, induced by subcritical element damage, are used to characterize the state of the material in terms of the residual strength of the critical elements. For typical high modulus, fiber reinforced laminates, where the plies most directly aligned with the primary load axis (on-axis plies) are designed to carry the majority of the applied loads, the subcritical elements are the off-axis plies. They are subcritical in the sense that their "failure", in terms of matrix cracking and delamination, does not contribute directly to the failure of a laminate. The critical elements are the load bearing plies of a laminate whose failure, in terms of fracture or buckling, causes global failure of a laminate.

For uniaxial and constant amplitude loading the critical element model uses the ratio of the number of applied load cycles to the fatigue life of a laminate, $n/N$, as an independent generalized time variable upon which one can calculate damage accumulation. This is a departure from previous life prediction methods which generally use the number of applied load cycles as a "time" variable. The motivation for choosing
a generalized time variable will become apparent in section 5.4.3. With this choice, the critical element model, in its normalized form, is given as

\[
\frac{\bar{S}_{1 re}(n)}{\bar{S}_{1c}} = 1 - \left(1 - g_c(\bar{\sigma}_{1 ac})\right) \left(\frac{n}{N}\right)^k,
\]

where \(\bar{S}_{1 re}, \bar{S}_{1c}, \) and \(\bar{\sigma}_{1 ac}\) are the longitudinal residual strength of the critical element, the longitudinal ultimate static strength of the critical elements, and the maximum or minimum longitudinal applied stress on the critical element in the longitudinal off-axis coordinate direction. The parameter \(k\) is a material constant introduced to account for the nonlinearity in the residual strength reduction curve. The function \(g_c(\bar{\sigma}_{1 ac})\) is a normalized failure criterion which represents the tendency of the internal stress state in the critical elements to cause failure of the critical elements. For the simple case of uniaxial and constant amplitude loading on the critical elements, \(g_c(\bar{\sigma}_{1 ac})\) can be taken as the ratio of the applied load \(\bar{\sigma}_{1 ac}\) on the critical elements to the ultimate static strength of the critical elements. In this case equation (5.17) becomes

\[
\frac{\bar{S}_{1 re}(n)}{\bar{S}_{1c}} = 1 - \left(1 - \frac{\bar{\sigma}_{1 ac}}{\bar{S}_{1c}}\right) \left(\frac{n}{N}\right)^k.
\]

Clearly, when the life fraction \(n/N = 1\), indicating fatigue failure has occurred, the longitudinal residual strength of the critical elements is equal to the appropriate maximum or minimum applied cyclic stress on the critical elements.

In reality the applied cyclic stress on the critical elements is not of constant amplitude even when the globally applied cyclic stress is of constant amplitude. This follows from the accumulation of subcritical element damage which manifests itself in changes in the stiffness of a laminate. The magnitude of these changes increases with the number of applied load cycles. Their effect is to augment the applied cyclic stresses on the critical elements. Since \(\bar{\sigma}_{1 ac} = \bar{\sigma}_{1 ac}(n)\) is no longer a constant function
of the number of applied load cycles, equation (5.18) must be rewritten in integral form as
\[ \frac{\bar{S}_{1e}(n)}{\bar{S}_{1e}} = 1 - \int_{0}^{\eta/H} \left( 1 - \frac{\bar{\sigma}_{1e}(n)}{\bar{S}_{1e}} \right)^{k} \left( \frac{n}{N} \right)^{k-1} d\frac{n}{N(n)}, \] (5.19)
where \( \eta/H \) is a specific value of \( n/N \). It should be noted that the lifetime of the critical elements, \( N = N(n) \), is a function of the number of applied load cycles as well since the lifetime of the critical elements, obtained from a phenomenological characterization of the fatigue behavior of the critical elements, is a function of the applied cyclic stress on the critical elements.

As noted, the magnitude of the applied cyclic stress on the critical elements is determined by changes in the stiffness of a laminate due to subcritical element damage. Such changes are measured or predicted on the basis of information obtained from nondestructive evaluation of a laminate during its service life. The complexity of such measurements or predictions will vary from situation to situation with the configuration of the laminate and the type and magnitude of the applied loading. For the case of uniaxial loading the longitudinal applied cyclic stress on the critical elements can be written as
\[ \bar{\sigma}_{1ac} = \bar{\sigma}_{1ac}(\bar{\sigma}_{1a}, \bar{E}_{1e}, \bar{E}_{1}^{o}), \] (5.20)
where \( \bar{\sigma}_{1a} \), \( \bar{E}_{1e} \), and \( \bar{E}_{1}^{o} \) are the globally applied cyclic stress, and the longitudinal Young's moduli of the critical elements and the laminate respectively in the longitudinal off-axis coordinate direction. If it is assumed that \( \bar{E}_{1e} \) does not degrade with cyclic loading and that \( \bar{\sigma}_{1a} \) is of constant amplitude, a basic rule of mixtures approach to micromechanics yields the applied cyclic stress on the critical elements as
\[ \bar{\sigma}_{1ac} = \frac{\bar{E}_{1e}}{\bar{E}_{1}^{o}(n)} \bar{\sigma}_{1a}. \] (5.21)
Clearly, if $\bar{E}_0(n)$ is a decreasing function of $n$, the applied cyclic stress on the critical elements will be enhanced leading to failure of the critical elements due to overstressing. Equations (5.19) and (5.21), while very elementary, illustrate the fundamental concepts associated with the critical element model.

The extension of the critical element model to the case of multiaxial fatigue loading is straightforward. Realizing that the state of stress in the critical elements is generally two-dimensional and occasionally three-dimensional, equation (5.19) is recast by replacing the applied stress ratio $\sigma_{1c}(n)/\bar{S}_{1e}$ by a general multi-axial failure criterion, $g_c(\sigma_a, n)$, for the critical elements. The form of this failure criterion is completely arbitrary, however; the Tsai-Hill, Tsai-Wu, or one of their many modifications are often used. Equation (5.19) then becomes

$$\frac{\bar{S}_{1e}(n)}{\bar{S}_{1c}} = 1 - \int_{0}^{n/N} (1 - g_c(\sigma_a, n)) k \left( \frac{n}{N} \right)^{k-1} d\frac{n}{N(n)}. \quad (5.22)$$

5.3.4 Methods Based on Damage Mechanisms

A weakness of the life prediction methods based on residual strength degradation and stiffness changes is that neither class of methods accounts for the actual damage mechanisms which occur in the CD process. In an effort to overcome this weakness Wang and co-workers (Wang, Chou, and Lei, 1984; Wang, 1987) have proposed a fatigue theory based on some of the actual damage mechanisms which occur in the CD process. The damage mechanisms that are considered are intralaminar and interlaminar matrix cracking. The proposed theory of Wang and co-workers functions on the basis of three primary assumptions. First, the laminae which make up a laminate contain intrinsic defects that can be modeled as small intralaminar cracks parallel to the reinforcing fibers or small interlaminar cracks between laminae. Second, the intrinsic cracks have random sizes and are distributed randomly within the laminae.
And third, the growth of the intrinsic cracks can be predicted using linear elastic fracture mechanics. Using finite element analysis to accurately determine the crack driving force associated with each intrinsic crack and Monte Carlo simulation, the probability distributions of crack size and density as a function of the number of applied load cycles have been determined. The results agree well with experimentally observed values of these quantities. This method suffers, however, from the fact that it is limited to damage mechanisms involving matrix cracking. As fiber fracture is a general prerequisite for the failure of a laminate under fatigue loading, this method can not be used to predict the lifetime of a laminate in its present form.

Talreja (1987, 1991c) has proposed a more general approach to fatigue life prediction based on damage mechanisms. This approach centers on the fact that CD process in composite laminates under fatigue loading is an extremely complex process that can not be modeled accurately or easily. Rather than focusing on the individual damage mechanisms themselves, Talreja’s approach focuses on two essential features of the CD process. First, the CD process evolves in two dominant stages. The first stage involves dispersed, non-interactive damage (primary matrix cracking) which is generally confined to individual plies. The second stage consists of localized, interactive damage resulting from the interaction of the individual damage mechanisms. Second, the transition between the first and second stages occurs upon attainment of the characteristic damage state (CDS).

Given these two features, this approach functions by modeling the CD process in two stages. The first stage is modeled as a general weakening phase using a residual strength degradation rate equation similar to equation (5.8) which relates the residual strength after n cycles to the initial static strength and the residual strength at the CDS. As before, the residual strength is considered to be a random
property of a laminate. The expression for the CDF of residual strength, depending on its chosen form, can be utilized to determine the CDF of the number of applied load cycles necessary to obtain the CDS either analytically or numerically. The second stage is much more difficult to model due to the extreme complexity associated with localization of damage. As such, it is assumed that all damage modes which are active in the localization of damage can be considered as a single equivalent crack which releases the same amount of energy as the sum of the individual damage modes themselves. Based on this assumption, an empirical relationship relating the residual strength of a laminate to the characteristic size of the equivalent crack is chosen. Further, it is assumed that the growth of the equivalent crack can be characterized by a rate equation such as the Paris-Erdogan equation or a similar model. Given the CDF, or equivalently the PDF, of the residual strength at the CDS and the relationships between residual strength and the equivalent crack size and its growth rate, an expression for the CDF of the residual strength after attainment of the CDS can be found. Again, this relationship can be utilized to find the CDF for the number of applied load cycles necessary to cause fatigue failure either by analytical or numerical means.

5.4 Cumulative Damage Life Prediction Methods

In the previous section, it was assumed that the applied load on a laminate was of constant amplitude. In reality such loading rarely occurs and therefore it is of great practical importance to develop life prediction models capable of predicting the useful lifetime of a laminate under spectrum fatigue loading. A number of such methods have been proposed. In general, these methods are based on one of two possible approaches. The first approach consists of empirically formulating a damage summa-
tion rule without regard to any observed characteristics of the damage accumulation process. The Palmgren-Miner rule (Miner, 1945) is a classic example of this approach wherein damage is assumed to be proportional to the life fractions at different cyclic stress levels and where fatigue failure is assumed to occur when damage reaches some critical value. The second approach, which is a generalization of the methods in the previous section, consists of formulating a damage summation rule based on a damage measure that in some way reflects the CD process in composite laminates under spectrum fatigue loading. This section focuses on fundamental aspects of these two approaches and discusses specific models where applicable. The reader is referred to the work of Sobczyk and Spencer (1992) for a broad perspective of the various approaches to CD models for life prediction. Although this reference focuses on applications to metals, it provides a thorough discussion of issues concerning CD modeling that are relevant to all engineering materials.

5.4.1 Empirical Methods

Hashin and Rotem (1978) have proposed a CD life prediction model based on the concept of equivalent damage curves, as defined in terms of the residual lifetime of a material, that is applicable to any engineering material. Actual damage mechanisms and their effects are ignored. Materials are treated as a "black box" with information on the applied loading as input and lifetimes under the applied loading as output. The model consists simply of defining the necessary experimental information on the applied loading and formulating equivalent damage curves based on this information and a general postulate for the lifetime of a laminate under general spectrum loading. This postulate is the equivalent loading postulate. Stated briefly, it says that cyclic
loads which are equivalent for one particular stress level are equivalent for all stress levels.

To clarify this approach consider the equivalent damage curve in Figure 5.1. For an initial loading of \( n_1 \) cycles at an applied load level \( \bar{\sigma}_{a_1} \) a given amount of damage is assumed to accumulate and it is characterized by the point \((n_1, \bar{\sigma}_{a_1}/\bar{S})\) in the S-N plane. An equivalent amount of damage due to \( n_2 \) cycles of an applied load \( \bar{\sigma}_{a_2} \) is described by the point \((n_2, \bar{\sigma}_{a_2}/\bar{S})\) on the same curve. The horizontal distance between the point \((n_2, \bar{\sigma}_{a_2}/\bar{S})\) and the S-N curve itself defines the residual lifetime \( N_r \) of a material. That is, \( N_r \) defines the number of applied load cycles at an applied load level \( \bar{\sigma}_{a_2} \) that can be sustained before failure given that \( n_1 \) cycles at an applied load level \( \bar{\sigma}_{a_1} \) occurred beforehand. The extension to multi-stage spectrum loading is obvious.

Since it is impossible to establish a complete family of curves for a given material, it is assumed that the damage curves, as well as the S-N curve itself, pass through the static strength point \((0, 1)\) and the endurance limit of a material. The endurance limit is the applied load below which infinite fatigue loading of a material can occur. This is illustrated in Figure 5.1. Unfortunately, experimental evidence suggests that these two assumptions are not true. This model has seen limited success based in part on this observation and the empirical nature of the method itself.

5.4.2 Methods Based on Residual Strength Degradation

Constant amplitude fatigue life prediction methods based on residual strength degradation are readily extendible to the case of spectrum loading. The early work of Broutman and Sahu (1972) provided a foundation for such methods. They proposed a linear CD rule based on residual strength degradation, in terms of the longitudinal
strength of a laminate in off-axis coordinates, of the form

$$\frac{d\tilde{S}_{1r}(n)}{dn} = -\frac{\tilde{S}_1 - \tilde{\sigma}_{1a}}{N}$$

(5.23)

where $N$ is the number of load cycles to failure under an applied stress $\tilde{\sigma}_{1a}$ (obtained experimentally) and $\tilde{S}_{1r}$, $\tilde{S}_1$, and $\tilde{\sigma}_{1a}$ are as defined before. Upon integration, and for a constant amplitude fatigue load, the following expression for the residual strength as a function of the number of applied load cycles, $n$, is obtained

$$\tilde{S}_{1r}(n) = \tilde{S}_1 - (\tilde{S}_1 - \tilde{\sigma}_{1a}) \frac{n}{N},$$

(5.24)

where $n/N$ is the life fraction at the applied stress level. For a two-stage spectrum fatigue load consisting of $n_1$ load cycles at applied load level $\tilde{\sigma}_{1a_1}$ and $n_2$ load cycles at applied load level $\tilde{\sigma}_{1a_2}$, integration of equation (5.23) yields

$$\tilde{S}_{1r} = \tilde{S}_1 - (\tilde{S}_1 - \tilde{\sigma}_{1a_1}) \frac{n_1}{N_1} - (\tilde{S}_1 - \tilde{\sigma}_{1a_2}) \frac{n_2}{N_2}.$$  

(5.25)

Generalization to an arbitrary load spectrum follows in an obvious manner.
The method of Broutman and Sahu can be recast as a CD life prediction method based on equivalent damage curves similar to those proposed by Hashin and Rotem as discussed above with the advantage that the damage curves are based on a damage metric. Specifically, for a two-stage spectrum fatigue, load where failure is assumed to occur during the second stage of loading such that $S_{1r} = \bar{\sigma}_{1a2}$ at failure, equation (5.25) can be rewritten as

$$\frac{\bar{S}_1 - \bar{\sigma}_{1a1}}{\bar{S}_1 - \bar{\sigma}_{1a2}} \frac{n_1}{N_1} + \frac{n_2}{N_2} = 1. \quad (5.26)$$

It is desired to find the equivalent number of applied load cycles, $n_e$, under a constant amplitude fatigue load of magnitude $\bar{\sigma}_{1a2}$ that will produce the same degradation in residual strength as $n_1$ cycles under an applied load of $\bar{\sigma}_{1a1}$. An expression for $n_e$ can be obtained by multiplying equation (5.26) by $N_2$ which yields

$$n_e + n_2 = N_2, \quad (5.27)$$

where

$$n_e = \frac{\bar{S}_1 - \bar{\sigma}_{1a1}}{\bar{S}_1 - \bar{\sigma}_{1a2}} \frac{n_1 N_2}{N_1}. \quad (5.28)$$

Thus, $n_e + n_2$ load cycles of magnitude $\bar{\sigma}_{1a2}$ will produce an equivalent degradation in residual strength as $n_1$ load cycles at $\bar{\sigma}_{1a1}$, followed by $n_2$ load cycles at $\bar{\sigma}_{1a2}$.

The concept of equivalent damage is a very useful means for computing the response of a laminate to complex load spectra. In essence, it allows a complex load spectrum to be replaced by a constant amplitude fatigue load. This concept will play an important role in the proposed life prediction model for the fatigue of composite laminates.

Sendeckyj (1991) has provided a more general life prediction method based on residual strength degradation. It follows from equation (5.8) upon taking $c_1 = 1$ and
integrating over the number of load cycles in each block of the load spectrum. For example, the residual strength $\bar{S}_{1r_1}$ after $n_1$ cycles at a cyclic stress of $\bar{\sigma}_{1a_1}$ is found to be

$$\bar{S}^{1/c_3}_{1r_1} (n) = \bar{S}^{1/c_3}_1 - \bar{\sigma}^{1/c_3}_{1a_1} (n_1 - 1).$$

(5.29)

Similarly, after an additional $n_2$ cycles at a cyclic stress of $\bar{\sigma}_{1a_2}$ the residual strength is found to be

$$\bar{S}^{1/c_3}_{1r_2} (n) = \bar{S}^{1/c_3}_1 - \bar{\sigma}^{1/c_3}_{1a_1} (n_1 - 1) - \bar{\sigma}^{1/c_3}_{1a_2} (n_2 - 1)$$

$$= \bar{S}^{1/c_3}_{1r_1} (n) - \bar{\sigma}^{1/c_3}_{1a_2} n_2.$$  

(5.30)

This procedure can be generalized to find the residual strength of a laminate subjected to a general load spectrum. The number of cycles to failure for a given spectrum can be calculated by as before be equating the residual strength at a particular stage in the spectrum with the applied load associated with that stage.

As one might expect, this life prediction method can be expressed in terms of equivalent damage as well. For example, for a two-stage spectrum load the number of load cycles $n_2$ for an applied load of $\bar{\sigma}_{1a_2}$ necessary to produce an equivalent reduction in the residual strength of a laminate as $n_1$ load cycles at an applied load of $\bar{\sigma}_{1a_1}$, assuming fatigue failure occurs during the second stage of loading, can be found setting $\bar{\sigma}_{1a_2} = \bar{S}_{1r}$ in equation (5.30) and rewriting equation (5.30) as

$$n_e + n_2 = N_2$$

(5.31)

where

$$N_2 = \left( \frac{\bar{S}_1}{\bar{\sigma}_{1a_2}} \right)^{1/c_3}$$

(5.32)

and

$$n_e = \left( \frac{\bar{\sigma}_{1a_1}}{\bar{\sigma}_{1a_2}} \right)^{1/c_3} (n_1 - 1) + 1.$$  

(5.33)
Again, \( n_1 + n_2 \) load cycles of magnitude \( \sigma_{1u2} \) will produce an equivalent degradation in residual strength as \( n_1 \) load cycles at \( \sigma_{1u1} \) followed by \( n_2 \) load cycles at \( \sigma_{1u2} \) will produce.

### 5.4.3 Methods Based on Stiffness Change - The Critical Element Model

The critical element model of Reifsnider and co-workers (Reifsnider, 1986, 1991c, 1991d, 1992; Reifsnider and Carman, 1992; Reifsnider and Stinchcomb, 1986) in the form given by equation (5.19) as

\[
\frac{\tilde{S}_{1re}(n)}{S_{1c}} = 1 - \int_0^{n/H} (1 - g_c(\sigma, n)) \left( \frac{n}{N} \right)^{k-1} d \frac{n}{N(n)}
\]

is suitable for predicting the lifetime of a laminate under spectrum loading. This follows from the fact that the global applied load on a laminate, whether constant amplitude or time varying, does not enter into equation (5.34) explicitly. Equation (5.34) requires as input the applied stresses on the critical elements, as determined from the globally applied loads, and does not assume any particular form for these stresses. It is to be expected, regardless of the type of global loading, that the stresses on the critical elements will be time varying. The choice of a generalized time variable, taken as the life fraction \( n/N \), is important as this variable is continuous even when the applied load spectrum is continuously varying in time.

### 5.4.4 Methods Based on Damage Mechanisms

The life prediction method based on damage mechanisms as proposed by Wang and co-workers (Wang, Chou, and Lei, 1984; Wang, 1987) is based on general crack growth equations and linear elastic fracture mechanics. Further, the method proposed by Talreja (1987, 1991c) is based on the same principle in conjunction with a residual strength degradation rate equation. Thus, it should be expected that these methods
can be extended to the case of spectrum loading. This is indeed the case. The crack
growth equations in both methods are simple rate equations, similar to the residual
strength degradation equation (5.8), where the rate of crack growth is generally taken
as a power function of the instantaneous crack length. As such, crack growth rate
equations can be integrated in an analogous manner to equation (5.8) to obtain
the crack growth as a function of applied load spectrum. The generalization of the
residual strength degradation equation is as discussed above. Sendeckyj (1991) notes
that by combining life prediction methods based on stiffness reduction (the critical
element model) and life prediction methods based on damage mechanisms, it should
be possible to formulate a general life prediction method which accurately describes
the CD process in composite laminates under fatigue loading.
Chapter 6

Markov Chain Models of Cumulative Damage Processes

6.1 Introductory Remarks

The previous chapter reviewed some of the major approaches to predicting the safe operating lifetime of a composite laminate subjected to fatigue loading. It was noted that many of these models, in particular those based on residual strength degradation, were random in nature, reflecting the fact the cumulative damage (CD) process which controls the fatigue life of a laminate is a stochastic process. The random nature of the CD process results from any one or a combination of random material properties, random defects and imperfections, and random loads and environments.

There are two fundamental means by which to introduce randomness into a life prediction model. The first is to formulate a deterministic model and then replace some or all of its parameters by random quantities. This is the approach taken in the life prediction models based on residual strength degradation where the initial static strength of a laminate is assumed to be a random variable with some given probability distribution (usually Weibull). The probability distribution for the residual strength after any number of applied load cycles is then determined from the distribution of initial static strength and the deterministic residual strength degradation equation. An alternative approach is to consider the accumulation of damage, as measured by changes in some physical observable, as an evolutionary stochastic process. In this approach damage is assumed to be an inherently random quantity whose distribution
evolves with the number of applied load cycles. The later approach is adopted in the present work.

The proposed life prediction model for composite laminates subjected to fatigue loading, discussed in detail in the ensuing chapter, is constructed from a well-known class of general probabilistic models of CD based on a specific class of discrete evolutionary random processes known as Markov chains. This chapter will focus on the basic structure of Markov chains and the CD models derived from them. The discussion of the CD models will center on their underlying assumptions, their basic stationary form, parameter estimation, the transformation of stationary models of CD into nonstationary models of CD, and how spectrum loading can be incorporated into their structure.

6.2 Markov Chains

The theory of Markov chains is well developed and has been applied to many random phenomena. The literature on Markov chain theory is rich in both content and number and it provides a sound foundation on which to build a CD model. This section focuses on aspects of Markov chains pertinent to the present study. A thorough introduction to Markov chain theory can be found in the works of Isaacson and Madsen (1976) and Ross (1983) among others.

Markov chains are a class of evolutionary random processes characterized by certain restrictions on the random process itself. The first of these restrictions is that the process is considered to be a discrete time process indexed on the set on nonnegative integers. The second restriction is that the process has a finite or countably infinite state space. Finally, the process must satisfy the Markov property as discussed in section 2.4.2.
To illustrate the basic ideas associated with Markov chains, consider an experiment consisting of a sequence of \( n \) trials whose outcomes, or states, are the random events \( E_1, E_2, \ldots, E_b \) which are mutually exclusive and collectively exhaustive. Let the outcome of each trial be modeled by a random variable \( D \). Clearly, the events \( E_i \) for \( i = 1, 2, \ldots, b \) constitute a partition of the sample space \( \Omega \) of \( D \), and thus only one event occurs for any given trial. If multiple trials are performed, then \( D \) is a function of the number of trials, \( n \), and thus constitutes a random process \( D(n) \).

To initiate the experiment some knowledge of the probability of being in each state \( E_i \) initially is required. Let these probabilities be given by \( P\{E_{i_0}\} = p_{i_0} \). Now suppose interest is centered on finding the probability of being in some specific state after the \( n^{th} \) trial. If the outcome of the sequence of \( n \) trials is denoted by \( E_{i_0} E_{i_1} \ldots E_{i_n} \), the probability of \( E_{i_n} \) can be found from the equation

\[
P\{E_{i_n}\} = \sum_{i_0} \sum_{i_1} \ldots \sum_{i_{n-1}} P\{E_{i_0} E_{i_1} \ldots E_{i_n}\}.
\]  

(6.1)

The right hand side of equation (6.1) can be readily evaluated if the events \( E_i \) are statistically independent or if the \( E_i \) satisfy the Markov property. That is, if the sequence of trials is Markovian, the probability of each event in the sequence is dependent only on the probability of the event itself and the probability of the last known event in the sequence. Mathematically stated, the Markov property implies that

\[
P\{E_{i_0} E_{i_1} \ldots E_{i_n}\} = P\{E_{i_0}\} P\{E_{i_1} \mid E_{i_0}\} P\{E_{i_2} \mid E_{i_1}\} \ldots P\{E_{i_n} \mid E_{i_{n-1}}\}.
\]  

(6.2)

If the conditional probabilities \( P\{E_{i_j} \mid E_{i_{j-1}}\} \), also known as transition probabilities, are denoted as

\[
P\{E_{i_j} \mid E_{i_{j-1}}\} = p_{i_{j-1}i_j},
\]  

(6.3)
and are assumed to be constant, or stationary (i.e., independent of the trial number), the probability of a specific sequence of \(n\) trials is given by the equation

\[
P \{ E_{i_0} E_{i_1} \ldots E_{i_n} \} = p_{i_0} p_{i_0 i_1} \ldots p_{i_{n-1} i_n}.
\] (6.4)

Substituting equation (6.4) into equation (6.1) gives the probability for a specific event after \(n\) trials as

\[
P \{ E_{i_n} \} = \sum_{i_0} \sum_{i_1} \ldots \sum_{i_{n-1}} p_{i_0} p_{i_0 i_1} \ldots p_{i_{n-1} i_n}.
\] (6.5)

Clearly, if there are \(b\) possible outcomes in the experiment there are \(b^2\) possible transition probabilities.

At this point it is convenient to introduce matrix-vector notation. Recall from section 2.4.1 that for a fixed time instance (trial number) a discrete random process reduces to a discrete random variable with an associated PMF. If the sample space of the random variable maps into a finite number of values on the real line then the PMF can be conveniently written in vector form. The collection of initial probabilities associated with each possible state \(E_i\) for \(i = 1, 2, \ldots, b\) can thus be written as a \((1 \times b)\) row vector \(p_0\) as follows

\[
p_0 = \{ P \{ E_{1_0} \}, P \{ E_{2_0} \}, \ldots, P \{ E_{b_0} \} \} = \{ p_{1_0}, p_{2_0}, \ldots, p_{b_0} \}.
\] (6.6)

Likewise, if the probabilities associated with being in each state \(E_i\) after \(n\) trials are given by \(P \{ E_{i_n} \} = p_{i_n}\), then the PMF for the outcome of the experiment after \(n\) trials is given by the \((1 \times b)\) row vector \(p_n\)

\[
p_n = \{ P \{ E_{1_n} \}, P \{ E_{2_n} \}, \ldots, P \{ E_{b_n} \} \} = \{ p_{1_n}, p_{2_n}, \ldots, p_{b_n} \}.
\] (6.7)
Similarly, the transition probabilities can be conveniently written in matrix form as

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1b} \\
p_{21} & p_{22} & \cdots & p_{2b} \\
\vdots & \vdots & \ddots & \vdots \\
p_{b1} & p_{b2} & \cdots & p_{bb}
\end{bmatrix},
\]

(6.8)

where \( P \) is a \( (b \times b) \) matrix known as the probability transition matrix (PTM).

Utilizing the matrix-vector notation, the probabilities associated with each of the possible outcomes of the experiment after \( n \) trials, as given by equation (6.5), can be written in terms of matrix-vector multiplication as

\[
p_n = p_0 \prod_{i=1}^{n} P_i.
\]

(6.9)

Equation (6.9) is a useful representation for a general Markov chain. However, since the individual transition probabilities are assumed to be independent of the trial number, and thus \( P_i = P \), equation (6.9) reduces to

\[
p_n = p_0 P^n.
\]

(6.10)

Equation (6.10) is the most basic representation for a stationary Markov chain.

The structure of the PTM is of direct interest in the formulation of a CD model. The elements \( p_{ij} \) of the PTM represent the probability of the random process \( D(n) \) having a realization \( E_j \) after a trial, given that it had a realization \( E_i \) before the trial. Thus, a fully populated PTM describes a truly random Markov chain in the sense that any state may be obtained from any other state after a given trial.

Consider now the case where the PTM is nonzero only on the main diagonal and the superdiagonal. Let the nonzero entries be denoted by \( p_i \) and \( q_i \) respectively. The
PTM then has the form

\[ P = \begin{bmatrix}
p_1 & q_1 & 0 & \ldots & 0 \\
0 & p_2 & q_2 & 0 & \ldots \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & p_{b-1} & q_{b-1} \\
0 & \ldots & \ldots & 0 & 1
\end{bmatrix}. \quad (6.11) \]

This form of the PTM, which proves particularly useful in what follows, implies that after a given trial the \( D(n) \) can only occupy the same state or the next higher state. A Markov chain with such a PTM can be visualized by the flow diagram as shown in Figure 6.1. In this figure, the states 1 through \((b-1)\) are referred to as transient states since \( D(n) \) can both enter and exit these states. In contrast, state \( b \) is known as an absorbing state since once it is entered in cannot be vacated. Such a PTM is referred to as a “unit-jump” PTM.

### 6.3 Application to Cumulative Damage Processes

#### 6.3.1 Basic Assumptions

The Markov chain model discussed in this section is an extension of Markov chain theory to the physical process of damage accumulation. This section reviews some of

![Figure 6.1: Flow diagram representation of a “unit-jump” Markov chain](image)
the more fundamental concepts associated with this model. A much more thorough
description of these aspects, as well as a discussion of the model's finer points, can
be found in the works of Bogdanoff (1978a, 1978b), Bogdanoff and Krieger (1978),
Bogdanoff and Kozin (1980), and in particular Bogdanoff and Kozin (1985).

Cumulative damage refers to the irreversible accumulation of damage in a compo-
nent under time varying loads. This damage may be manifested by features such as
loss of material (wear), propagation of a flaw, or loss of strength or stiffness. In each
case some physical property deteriorates with the loading until some critical level of
damage accumulation occurs at which point the component is retired or fails. Damage
of this form stems from material behavior at the atomic, molecular, or micro level.
A great deal of effort has been devoted to developing CD models based on material
behavior at these levels. While some progress has been made towards this goal for
simple materials under tightly controlled laboratory conditions, it is still generally
not possible to develop CD models which characterize the macroscopic behavior of
a material based on fundamental physical laws at these levels. As such, CD models
must be phenomenological in nature (at least in part). That is, CD models must be
based on the current understanding of the CD process on the macroscopic level. In
addition, realistic CD models must be probabilistic in nature if realizations (sample
functions) of the CD process, obtained from monitoring a physical property which
deteriorates with time or the number of load cycles for the case of fatigue loading, ex-
hibit random behavior which varies sufficiently from the mean. Markov chain models
of CD reflect these issues and are both phenomenological and probabilistic.

The physical situation which the models attempts to describe is as follows. A
component is subjected to some sort of repetitive or cyclic loading which is possibly
random in nature. The time over which the load acts is discretized and is measured
in units of duty cycles (DCs). A DC represents some repetitive period of operation in which damage can accumulate and could be a specific number of load cycles, revolutions, hours of operation, etc. Damage is assumed to monotonically increase with the number of DCs and takes on discrete values (states). The initial state of damage and the state of damage at failure may be random as well. The level of damage is considered only at the end of each DC and is assumed to depend only on the severity of the DC itself and the level of damage prior to the DC. The later assumption is the Markov assumption and such a CD process can be modeled as an embedded discrete time, finite state Markov chain.

Markov chain models of CD are formulated from a sample function point of view. They attempt to describe the probabilistic distribution of damage in a component as a function of time, or load cycles, by assuming that the CD process is an evolutionary stochastic process. The phenomenological nature of the models requires their parameters to be determined from experimental data. The strength of the Markov chain models lies in their comprehensive and coherent structure. The models are simple, unified, and they accurately predict behavior for conditions not covered by data. In addition, these models are capable of addressing the inherent sources of variability common in CD processes; namely, variability in initial damage, variability in severity and order of loading, variability in the state of damage at failure, and variability in inspection and replacement standards. The later two items, while they can easily be incorporated into the models and are important in their own right, will not be discussed herein. The reader is referred to Bogdanoff and Kozin (1985) for further discussion on these topics.
6.3.2 Stationary Models

To examine the basic structure of the stationary Markov chain models, consider the case where the damage in a component is discretized into a finite number of damage states indexed on the nonnegative set of integers \( i = 1, 2, \ldots, b \), with the magnitude of the damage increasing as the index of the damage state. Furthermore, assume that during any given DC the level of damage in the component can remain unchanged or increase by one unit to the next higher damage level culminating in failure when damage state \( b \) is attained. Under such conditions, a unit-jump Markov chain model is obtained. For such a model, the probability mass function (PMF) of damage is completely determined by the number and severity of the DCs and the initial distribution of damage in the component before the CD process began.

At this point assume that the DCs are repetitive and of constant severity. For the unit-jump model, this implies that the \((b \times b)\) PTM, which defines the severity of the DCs, has the form

\[
P = \begin{bmatrix}
p_1 & q_1 & 0 & \ldots & \ldots & 0 \\
0 & p_2 & q_2 & 0 & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & & \vdots \\
0 & \ldots & \ldots & 0 & p_{b-1} & q_{b-1} \\
0 & \ldots & \ldots & 0 & 1 & \end{bmatrix},
\]

with \( p_i \geq 0, q_i \geq 0, \) and \( p_i + q_i = 1 \). The banded and upper triangular form of the PTM reflects the assumption that the damage level in a component may "stay" at the current level or "jump" to the next higher level during a DC. Further, the damage level is monotonically increasing. The "stay" and "jump" probabilities are given by \( p_i \) and \( q_i \) respectively. The magnitudes of these probabilities are a measure of the
severity of the DC with larger "jump" probabilities indicating a greater probability for the damage level to increase during a DC.

Let the initial distribution of damage be represented by a \((1 \times b)\) row vector

\[
p_0 = \{p_{10}, p_{20}, \ldots, p_{b0}\},
\]

(6.13)

with \(p_{i0} \geq 0\) and \(\sum_{i=1}^{b} p_{i0} = 1\), where \(p_{i0}\) denotes the probability that damage was in state \(i\) before the CD process began. Furthermore, express the distribution of damage after \(x\) DCs in a similar form

\[
p_x = \{p_{1x}, p_{2x}, \ldots, p_{bx}\},
\]

(6.14)

with \(p_{ix} \geq 0\) and \(\sum_{i=1}^{b} p_{ix} = 1\), where \(p_{ix}\) is the probability that damage is in state \(i\) after \(x\) DCs. Clearly \(p_0\) and \(p_x\) constitute PMFs. It follows then from the results of the preceding section on Markov chains that \(p_x\) is related to \(p_0\) and \(P\) by the equation

\[
p_x = p_0 P^x = p_{x-1} P,
\]

(6.15)

where \(P^0 = I\) is the \((b \times b)\) identity matrix.

Given the marginal PMF of damage, \(p_x\), from equation (6.15), statistical information on the state of damage after \(x\) DCs, the probability of failure of a component after \(x\) DCs, and other quantities of interest may be readily found. Let \(D_x = D(x)\) be a random variable representing the level of damage in a component after \(x\) DCs. If \(p_x\) represents the PMF of \(D_x\), then the probability of damage of being in state \(i\) after \(x\) DCs is clearly given by the equation

\[
P(D_x = i) = p_{ix} \quad \text{for} \quad i = 1, 2, \ldots, b.
\]

(6.16)
It follows from the results of chapter 2 that the CDF, mean, and variance of damage after \( x \) DCs can be expressed as

\[
F_{D_x}(j; x) = P\{D_x \leq j\} = \sum_{i=1}^{j} p_{ix}
\]

\[
m_{D_x} = E\{D_x\} = \sum_{i=1}^{b} ip_{ix}
\]

\[
k_{D_x} = E\{(D_X - m_{D_x})\} = \sum_{i=1}^{b}(i - m_{D_x})p_{ix}.
\]

Information on the time to absorption or failure in state \( b \), denoted by the random variable \( X_b \), can be found in a similar fashion. Since attainment of state \( b \) denotes failure, then clearly the probability of failure of a component after \( x \) DCs is given by the equation

\[
P_F(x) = p_{bx}.
\]  

(6.18)

The CDF of the time to failure is also equivalent to \( p_{bx} \) since the probability that failure has occurred in \( x \) DCs or less is equal to the probability that state \( b \) has been obtained in \( x \) DCs. Stated mathematically,

\[
F_{X_b}(x) = P\{X_b \leq x\} = p_{bx}.
\]  

(6.19)

Given the CDF for the time to failure \( F_{X_b}(x) \), the PMF for the time to failure \( p_{X_b}(x) \) can be obtained directly from

\[
p_{X_b}(x) = F_{X_b}(x) - F_{X_b}(x - 1),
\]  

(6.20)

where \( p_{X_b}(0) = 0 \). Expressions for the mean and variance of the time to failure, based on the PMF of the time to failure, follow immediately. Analytical results which relate the moments of the time to failure to the parameters of the Markov chain model can be found by employing probability generating functions (PGFs). These results are derived in Appendix B and are used to determine the parameters of the Markov chain model.
6.3.3 Parameter Estimation

Since the Markov chain models of CD are phenomenological in nature, their parameters must be found from experimental data. For a stationary CD process, these parameters are the “stay” and “jump” probabilities, \( p_i \) and \( q_i \), and the size, \( b \), of the PTM. The initial distribution of damage can be found in a variety of ways. Some of the more common of these methods are fitting a histogram to experimental results, adopting some assumptions on the underlying process, or utilizing subjective information such as expert opinion. The question then arises as to how numerical data on the CD process can be related to the model parameters. One possible approach is to utilize the method of moments.

With a simple PTM, such as that given by equation (6.12), it is possible to obtain concise analytical results relating the model parameters to the moments of the times to transfer from one state to another. The derivation of these results is given in Appendix B. Fortunately, such expressions are in accordance with the kinds of data commonly associated with CD processes such as the time to failure or the time to reach a specific damage state.

For example, consider a constant amplitude fatigue test on a number of identical specimens. Assume that all of the specimens are damage free prior to the testing and that they are cyclically loaded until failure occurs. Further, assume that the number of cycles to failure is recorded for each specimen along with the mean and variance of the time to failure for the group. If the damage free state is taken as state 1 and the failure state as state \( b \), it follows then from equation (B.26) of Appendix B that the relationships between the model parameters and the mean and variance of the time (in DCs) to transition from state 1 to state \( b \), denoted by the random variable \( X_{1,b} \),
are given by the equation

\[ m_{X_{1,b}} = \sum_{i=1}^{b-1} (1 + r_i) \]
\[ k_{X_{2,b}} = \sum_{i=1}^{b-1} r_i (1 + r_i), \]

where \( r_i = \frac{p_i}{q_i} \) is as noted earlier. Equation (6.21) yields two equations in \( b \) unknowns. One possibility in solving this system of equations is to choose \( r_i = r \) as a constant leaving only the parameters \( r \) and \( b \) to be determined. This leads to a rather simple model. Simple models such as this may not be sufficient to model a complex CD process but they serve to illustrate some important points about Markov chain models. The first point, as it is known from probability theory, is that the availability of the first few moments for the time to failure is not sufficient to model a CD process uniquely as several different processes may have the same or very similar distributions on the time to failure. This feature does not pose a serious problem if interest is directed on ultimate failure alone. However, one of the primary goals of any model is to predict behavior not covered by experimental data and therefore a more unique model is essential. Additional information on the CD process is therefore required. This information must be obtained from nondestructive testing techniques and it must be decided how measured inspection quantities can be related to model damage states.

Heretofore, the damage in a component has been indexed solely by "state" without regard as to what the "state" physically represented, with the exception of state \( b \) for failure. If additional data, that is second order moment data, are known on the time to reach subcritical damage levels, such as a particular crack length, then the results of Appendix B can be modified to associate a physical property with the individual damage states. Furthermore, the additional information serves to define the CD process in a more unique manner.
The modification of the results of Appendix B can be carried out as such. Consider the case where the mean and variance of the time to reach \( l \) distinct damage levels are known. The unit-jump PTM can then be modeled as having \( l \) submatrices such that

\[
P = \begin{bmatrix}
B_1 \\
& B_2 \\
& & \ddots \\
& & & B_l
\end{bmatrix}
\] (6.22)

Assuming that the duty cycles are sufficiently small so that damage can increase by only one state during a DC, the individual submatrices \( B_k \) within \( P \) can be written as

\[
B_k = \begin{bmatrix}
p_k & q_k \\
p_k & q_k \\
& \ddots & \ddots \\
p_k & q_k
\end{bmatrix}
\] (6.23)

Recall that the restriction on the \( p_{ij} \) from equation (6.12) requires that \( p_i + q_i = 1 \). The parameters to be determined for each submatrix are then the ratio of the “stay” to “jump” probabilities, \( r_k \), and the size, \( b_k - b_{k-1} (k = 1, 2, \ldots, l) \), of each submatrix. The \( r_k \) of each submatrix are assumed to be constant. Thus, the Markov chain model will transition through \( b_k \) states before reaching the model state corresponding to the \( k^{th} \) known damage level. That is, model state \( b_k \) corresponds to the \( k^{th} \) known level of damage. The intermediate model states can be related to unique level of damage by equating the mean time in DCs necessary for the model to reach a given state to the mean time necessary to reach a given level of observed damage or as interpolated from the observed levels of damage. The expressions relating the model parameters to the mean and variance for the time in DCs to reach a given level of observed damage
then become

\[
\begin{align*}
    m_{X_k} &= \begin{cases} 
    (b_1 - 1)(1 + r_1) & \text{for } k = 1 \\
    m_{X_{k-1}} + (b_k - b_{k-1})(1 + r_k) & \text{for } k > 1 
    \end{cases} \\
    k_{X_k}^2 &= \begin{cases} 
    (b_1 - 1)r_1(1 + r_1) = r_1 m_{X_1} & \text{for } k = 1 \\
    \sigma_{X_{k-1}}^2 + r_k(b_k - b_{k-1})(1 + r_k) & \text{for } k > 1.
    \end{cases}
\end{align*}
\]

(6.24)

Thus, for each submatrix the parameter estimation process reduces to solving two equations in two unknowns.

The foregoing results illustrate how a simple stationary Markov chain model of CD with a unit-jump PTM can be constructed given only the mean and variance of the time to reach failure or a given number of damage levels prior to and including failure. Again, if second order moment data on the time to reach subcritical levels of damage prior to failure are known, the model states can be directly related to an actual physical observable. Further, the more data that are available, the more unique the Markov chain model will be. These are important concepts in the ensuing chapter.

Note that a more complex CD model could easily be constructed by simply modifying the structure of the PTM. Experience with this class of CD models has shown, however, that not much in the way of increased predictive accuracy is gained by doing so. Further, when the PTM has a more complex structure it is extremely difficult, if not impossible, to derive the analytical expressions relating the model parameters to the experimental data. As such, the unit-jump model seems to be a good choice for most CD processes. For more complex CD processes the results of the following section can be used to determine more sophisticated nonstationary Markov chain models of CD.
6.3.4 Nonstationary Models

In this section the extension of the stationary Markov chain models of CD to nonstationary Markov chain models of CD is examined. Nonstationary models must be used when DC severity and/or environment and/or material properties change with time. This discussion plays an important role in the forthcoming chapter where the model is used in connection with composite laminates whose strength and stiffness both degrade with cyclic loading. There are several methods by which to introduce a nonstationary structure into the Markov chain models. However, one method has proven particularly useful and is discussed herein; it is the time transformation-condensation method.

Time Transformations

A measure of the predictive capability of Markov chain models is how well the CDFs for model states corresponding to observed levels of damage match the empirical distribution functions (EDFs) for those levels of damage (if known). Clearly, if the second order moment data from a nonstationary CD process are used to determine the parameters of a stationary Markov chain model, discrepancies between the CDFs of the model states corresponding to observed levels of damage and the EDFs of those damage levels should be expected. The magnitude of such discrepancies depends on both the CD process itself and the sophistication of the stationary Markov chain model. The purpose of the time transformation method is to "correct" the CDFs of a stationary Markov chain model such that they are accurate descriptions of the EDFs of the corresponding damage levels by transforming the indexing parameter of the stationary model. More specifically, the time transformation-condensation method
allows a stationary Markov chain model with its simple structure to be used to model a more complex nonstationary CD process.

Mathematically, a time transformation maps the CDF $F_{X_k}(x)$ of the number of DCs necessary to reach model state $b_k$, as predicted by a stationary Markov chain model, into the CDF $F_{Y_k}(y)$ of the number of DCs necessary to reach state $b_k$ as predicted by a nonstationary model. This is done by requiring that $F_{Y_k}(y)$ is an accurate description of the EDF $F_{Y_k}(y)$ of the number of DCs to reach the damage level corresponding to model state $b_k$. The map between $F_{X_k}(x)$ and $F_{Y_k}(y)$ is defined in terms of the indexing parameters $x$ and $y$ and is generally taken as a polynomial of the form

$$y = g_k(x) = c_{1k}x + c_{2k}x^2 + \cdots + c_{\beta_k}x^\beta,$$

(6.25)

where $g_k(0) = 0$, $g_k(x)$ has a continuous first derivative, and where $dg_k(x)/dx > 0$ for $x > 0$. Thus, $g_k^{-1}(y)$ exists and is unique for all $x > 0$. The unknown coefficients $c_{\alpha_k}$ for $\alpha = 1, 2, \ldots, \beta$ are determined by requiring that $F_{Y_k}(x)$ and $F_{Y_k}(y)$ coincide for $\beta$ arbitrary ordinate values. That is, it is desired to find coefficients such that when $F_{X_k}(x)$ is mapped into $F_{Y_k}(y)$ that $F_{Y_k}(y)$ will match exactly $F_{Y_k}(y)$ at the chosen ordinate values. In terms of equation (6.25), this requires that the abscissa values $x_\alpha$ and $y_\alpha$ for which $F_{X_k}(x_\alpha)$ and $F_{Y_k}(y_\alpha)$ are equal to the chosen ordinate values be known. This yields a set of $\beta$ ordered pairs $\{(x_1, y_1), (x_2, y_2), \ldots, (x_\beta, y_\beta)\}$. Placing the respective pairs into equation (6.25) yields a set of $j$ linear equations in the unknown coefficients $c_{1k}, c_{2k}, \ldots, c_{\beta_k}$ whose solution defines the time transformation. Once the transformation is determined, $F_{X_k}(x)$ is mapped into $F_{Y_k}(y)$ utilizing equation (6.25). If $F_{Y_k}(y)$ sufficiently describes $F_{Y_k}(y)$ the procedure is complete. Otherwise, a new transformation based on a new stationary Markov chain model is chosen.
The choice of the \( j \) ordinate values for which \( F_{Y_k}(y) \) and \( F_{\hat{Y}_k}(y) \) are to be equivalent has a significant effect on the quality of the fit between \( F_{Y_k}(y) \) and \( F_{\hat{Y}_k}(y) \) at other ordinate values. Specifically, if a good overall fit between \( F_{Y_k}(y) \) and \( F_{\hat{Y}_k}(y) \) is desired, then \( \beta \) well spaced ordinate values on the interval \([0, 1]\) should be chosen. If, however, interest is focused on finding \( F_{Y_k}(y) \) such that it very closely fits \( F_{\hat{Y}_k}(y) \) in the tails of the distribution, the ordinate values upon which the transformation is determined should be closely spaced and grouped within the tail of interest. For example, if a fourth order time transformation is to be utilized and a good overall fit is desired, four ordinate values of say \( 0.1, 0.4, 0.6, \) and \( 0.9 \) could be chosen such that a reasonable fit between \( F_{Y_k}(y) \) and \( F_{\hat{Y}_k}(y) \) will occur over the entire distribution. In contrast, if a good fit between \( F_{Y_k}(y) \) and \( F_{\hat{Y}_k}(y) \) is desired in the upper tail of the distribution, four ordinate values of say \( 0.9, 0.925, 0.95, \) and \( 0.975 \) could be utilized to determine the time transformation. The flexibility that this aspect of the time transformation method affords is very useful since it allows the accuracy of the nonstationary Markov chain models to be increased over areas of the distributions where interest is centered for a particular study.

The time transformation method has somewhat of a trial and error nature, but if a suitable transformation is found much can be deduced about the nonstationary CD process. Specifically, the time transformation acts to scale time, in DCs, between the stationary and nonstationary models. If the derivative of a transformation, \( dg_k(x)/dx \), is greater than unity, then DC severity in the nonstationary model is less than that in the stationary model as a given value of \( x \) will result in a value of \( y > x \) indicating that more DCs \((y)\) for the nonstationary model are necessary to cause the same amount of damage as a given number of DCs \((x)\) for the stationary model. If \( dg_k(x)/dx \) is an increasing function of \( x \), then DC severity in the nonstationary model is steadily
decreasing. In contrast, if \( dg_k(x)/dx \) is less than unity and greater than zero, then DC severity in the nonstationary model is greater than that in the stationary model. Here, if \( dg_k(x)/dx \) is a decreasing function of \( x \) then DC severity in the nonstationary model is steadily increasing. If \( dg_k(x)/dx \) is a variable function of \( x \) which both increases and decreases then DC severity in the nonstationary model varies in comparison with DC severity in the stationary model. It follows that high powers of \( x \) in equation (6.25) with positive coefficients cause \( F_{Y_k}(y) \) to approach the asymptote of unity slowly, which indicates a slow accumulation of damage. Conversely, high powers of \( x \) with negative coefficients cause \( F_{Y_k}(y) \) to approach the asymptote of unity quickly indicating rapid damage accumulation.

**Condensation Method**

In the previous section it was noted that the objective of the time transformation-condensation method is to utilize a simple stationary Markov chain model defined by the PTM \( P \) to model a nonstationary CD process. The time transformation procedure consists of transforming each of the CDFs for the number of DCs \( x \) necessary to reach a model state corresponding to an observed level of damage such that the transformed CDFs were accurate descriptions of the EDFs of the number of DCs \( y \) required to reach the observed damage levels. This procedure only provides part of the solution as to how stationary Markov chain models can be used to model complex nonstationary CD processes. It remains to be shown as to how the PMFs of damage, as predicted by a stationary Markov chain model after a given number of DCs \( x \), can be determined such that they are an accurate description of the true PMFs of damage associated with the nonstationary CD process. This requires the use of the condensation method.
Recall that time, in terms of DCs, is measured on the set of nonnegative integers with respect to both stationary and nonstationary Markov chain models. This requires \( x \) and \( y \) to be integers. However, placing integer values of \( x \) into equation (6.25) does not necessarily produce integer values of \( y \) as needed. The condensation method functions to preserve discrete indexing parameters by determining appropriate PTMs \( \hat{P}_{iy} \) for a nonstationary Markov chain model from the PTM \( P \) of a stationary Markov chain model and a time transformation \( y = g(x) \).

To illustrate how the condensation method is employed, consider a stationary Markov chain model for a nonstationary CD process defined by the equation

\[
p_x = p_0 P^x.
\]  
(6.26)

Now suppose that interest is centered on finding an accurate nonstationary model for the damage level corresponding to the \( k^{th} \) model state \( b_k \). Since the time transformation \( y = g_k(x) \) which maps the CDF \( F_{X_k}(x) \) of a stationary model into the CDF \( F_{Y_k}(x) \) is a one-to-one mapping between \( x \) and \( y \), it follows that the exact PMF of damage after \( y \) DCs should be given by the equation

\[
p_y = p_0 P^{g_k^{-1}(y)}.
\]  
(6.27)

However, since \( g_k^{-1}(y) \) may not be an integer value, \( P^{g_k^{-1}(y)} \) may not be a valid PTM. That is, the elements of \( P^{g_k^{-1}(y)} \) may be negative and/or the row sums of \( P^{g_k^{-1}(y)} \) may not be unity. As such, equation (6.27) can not be used to model the nonstationary CD process. To circumvent this problem, the condensation method finds for each integer value of \( y \) the nearest integer value of \( x \) just less than or equal to \( g_k^{-1}(y) \), then determines appropriate PTMs \( \hat{P}_{iy} \) for the nonstationary Markov chain model given by

\[
p_y = p_0 \prod_{i_y=1}^{y} \hat{P}_{iy}
\]  
(6.28)
based on the PTM $P$ of the stationary Markov chain model and the integer value of $x$ so found. More specifically, let $x_1, x_2, \ldots$ be determined by solving

$$g_k(x_{i_y}) - i_y = 0 \quad \text{for} \quad i_y = 1, 2, \ldots \tag{6.29}$$

Given the roots $x_{i_y}$ of equation (6.29), the following intervals are formed $(0, x_1], (x_1, x_2], \ldots$ The PTMs of the nonstationary Markov chain model $\hat{P}_{i_y}$ are determined according to the number of integer values within each of these intervals through the relationship

$$\hat{P}_{i_y} = \begin{cases} 
I & \text{if } (x_{i_y-1}, x_{i_y}] \text{ contains no integer} \\
P & \text{if } (x_{i_y-1}, x_{i_y}] \text{ contains one integer} \\
P^2 & \text{if } (x_{i_y-1}, x_{i_y}] \text{ contains two integers,} \\
& \vdots
\end{cases} \tag{6.30}$$

where $I$ is the $(b \times b)$ identity matrix. Given the PTMs, $\hat{P}_{i_y}$, and the initial distribution of damage, $p_0$, the nonstationary CD process is modeled according to equation (6.28).

Note that for any positive integer $y$, $\prod_{i_y=1}^y \hat{P}_{i_y}$ is the integer power of $P$ which is the largest integer equal to or just less than $g_k^{-1}(y)$. This implies that the condensation method, with its restriction to produce integer values of $y$, does not produce the exact powers of $P$ as required by $g^{-1}(y)$. However, this difference never exceeds $P$.

Furthermore, it is shown by Bogdanoff and Kozin (1985) that the error between $p_y$ and the exact distribution of damage is related to the "jump" probabilities of the PTM $P$ which are generally quite small. Consequently, the error is often negligible.

### 6.3.5 Spectrum Loading

Spectrum loading is often used to model the applied load histories encountered by a component during its service life. It is useful for a variety of practical situations
including those were the applied loads on a component vary neither in a periodic, transient, or completely random manner. The ability of a life prediction model to incorporate spectrum loading within its structure is of great importance since the applied loads on a component are rarely constant. This ability is readily incorporated into Markov chain models of CD.

With regard to spectrum loading, life prediction models try to determine the fatigue life of a component under the load spectrum based on the constant amplitude fatigue behavior of a component under each component of the load spectrum alone. Several issues surrounding this approach with regard to Markov chain models of CD must be addressed. Specifically, what kinds of life behavior can be predicted, how can the load spectrum be characterized and samples of the spectrum generated for test purposes, how can the order of the applied loads be taken into account, and how can load interaction be detected and characterized.

The answer to the first question is that under spectrum loading the data which may be obtained from Markov chain models of CD are precisely those data which can be obtained from stationary Markov chain models of CD based only on the time to failure for a given component. That is, the distribution of damage after a given number of DCs may be obtained as well as the distribution for the time to failure or absorption in the final model state. The reasons which limit the models to predicting these data alone will become apparent momentarily.

If a life prediction model aims to predict the fatigue life of a component based on the constant amplitude fatigue behavior of the component under each component of the load spectrum alone, several aspects of the load spectrum must be known. Specifically, the magnitudes of each of components of the load spectrum along with their relative frequencies in the load spectrum must be known. In addition, data
on the time to failure in DCs under each of the load spectrum components must be known. For constructing Markov chain models of CD, these data must include the mean and variance of these times. Given this information, it must be decided as to how realizations of the load spectrum can be generated. If the load spectrum is deterministic this requires no further information. However, if the load spectrum exhibits random variations then several possibilities for specifying the load spectrum are possible. For example, if a component is subjected to a load spectrum containing \( j \) possible load components with magnitudes \( \sigma_{ia} \) and relative frequencies \( f_i \) for \( i = 1, 2, \ldots, j \), Bogdanoff and Kozin (1985) note that three possibilities for generating sample functions of the load spectrum exist. First, select successive loads \( \sigma_{ia} \) randomly based on their relative frequencies \( f_i \). Second, form \( j \) blocks of applied loads each consisting of an applied load \( \sigma_{ia} \) of a different magnitude alone and each containing the same number of applied load cycles; successive blocks of the applied loads are then chosen according to their relative frequencies \( f_i \). And thirdly, form \( j \) blocks of applied loads each consisting of an applied load \( \sigma_{ia} \) of a different magnitude alone and each containing a different number of applied load cycles proportional to their relative frequencies \( f_i \). Successive blocks of applied loads are then chosen independently with equal probabilities. The first method induces a completely random load history. The second and third methods create load histories with distinctive patterns dependent on the which method is being used. The choice of which method to use depends on the load histories that a component experiences during its service life.

Of further interest with regard to generating samples of the load spectrum is how such samples are applied to a group of components. If the group is gang tested, that is if all components are subjected to the same sample of the load spectrum, only
one sample function for the random CD process being considered will be obtained. In contrast, if each component is subjected to a different sample of the load spectrum (i.e., random testing) a number of sample functions of the CD process will be generated. The later method is of more practical use for simulation purposes as statistical data concerning the time to failure may be obtained from the collection of sample functions.

To incorporate spectrum loading into the Markov chain models, the parameters of a unit-jump PTM for a stationary Markov chain model with constant state-to-state transition probabilities must be determined for each of applied loads in the spectrum. That is, the size, $b_i$, and the ratio of the stay to jump probabilities, $r_i$, for each of the applied loads, $\sigma_{ia}$ for $i = 1, 2, \ldots, j$, are determined according to equation (6.21) from the mean and variance on the time to "failure" under each of the loads alone. In anticipation of what follows, the PTMs must be of the same size as they will be multiplied together. If the size of all of the PTMs are set equal to the size of the largest PTM, thus removing one parameter of the two possible parameters from use, two possibilities exist for determining the remaining free parameter. The first of these possibilities is to determine the $r_i$ for each of the remaining PTMs by utilizing the first of equation (6.21) with $b_i$ fixed. However, utilizing this method only allows data on the mean time to "failure" to be incorporated into the model. The other alternative, and the preferable one, is to divide the remaining PTMs into two blocks as in equation (6.22), of fixed but arbitrary size, and use data on both the mean and variance of the time to "failure" in conjunction with equation (6.24) to determine the ratio of the stay to jump probabilities for each block within the PTM. The choice of the size of the individual blocks is arbitrary with the restriction that it produces ratios of stay to jump probabilities for the individual blocks that are real and positive. If each of the PTMs so determined is denoted as $P_i$ for $i = 1, 2, \ldots, j$, a nonstationary Markov
chain model of the CD process can be obtained as

\[ p_y = p_0 \prod_{k=1}^{y} P_{i_k}, \quad (6.31) \]

where the choice of the \( P_i \) for each value of \( k \) depends on how samples of the load spectrum are generated as discussed above. The structure of equation (6.31) answers the third question posed above. Specifically, the order of applied load cycles is taken into account via the fact that matrix multiplication is not commutative, and thus the order of the DCs has an effect on the distribution of damage, \( p_y \), after a given number of DCs.

Several useful results concerning the times to failure predicted by equation (6.31) warrant further attention. Successive application of different samples of the load spectrum to individual components via random testing will produce an EDF for the time to failure of the CD process which will approximate the CDF of the mean time to failure of the CD process. Bogdanoff and Kozin (1985) give a rigorous proof of this result. The CDF of the mean time to failure of the CD process may also be approximated by taking the expectation of equation (6.31) assuming that the initial distribution of damage is known. This yields

\[ E\{p_y\} = p_0 \prod_{k=1}^{y} E\{P_{i_k}\}, \quad (6.32) \]

where

\[ E\{P_{i_k}\} = \sum_{i=1}^{j} f_i P_i. \quad (6.33) \]

Comparison of the CDFs for the time to failure as predicted by equations (6.31) and (6.32), see Bogdanoff and Kozin (1985), illustrates the fact that mean behavior under spectrum loading can be replaced by constant amplitude loading. That is, there exists a constant amplitude load of some unknown magnitude which when applied to a component will induce a level of damage equivalent to that induced by spectrum
loading. This is the well-known "equivalent damage" assumption often employed in life prediction models.

The remaining question to be answered concerning spectrum loading is how load interaction is to be detected and characterized. Load interaction refers to the fact that in some materials the CD process is either accelerated or retarded by changes in the amplitude of the applied loads. This question is easily answered. If load interaction is present in a CD process, it can be detected by noting where the CDF of the time to failure, as predicted by a Markov chain model with spectrum loading, lies in comparison to the EDF for the time to failure as determined experimentally or by observation. Specifically, if the predicted CDF lies to the left of the EDF, retardation of the CD process is present since the Markov chain model is predicting lifetimes shorter than the empirical data suggest. Likewise, if the predicted CDF lies to the right of the EDF, then the CD process is accelerated since the Markov chain model is predicting lifetimes longer than the empirical data suggest. To account for such load interaction effects, the time transformation-condensation method is employed as described in section 6.3.3.
Chapter 7

Markov Chain Models for Life Prediction of Composites

7.1 Introductory Remarks

The three major descriptors of the mechanical behavior of a composite laminate are its strength, stiffness, and fatigue life. Chapter 4 of this work is dedicated to a discussion of these properties and the action of the various damage mechanisms, both individually and collectively, which control their magnitudes. Chapter 5 gives a review of the various approaches to predicting the fatigue life of composite laminates including discussions of their strengths, weaknesses, and limitations. In chapter 6, a general class of probabilistic models for cumulative damage (CD) processes based on the theory of Markov chains is discussed. This chapter brings together elements from each of these chapters, as well as chapter 3, to establish a new model for determining the fatigue life of composite laminates based on the application of the Markov chain models of CD to the critical element model for the fatigue of composites. It focuses on the general philosophy of the model, its mathematical formulation, its application, its strengths, and its weaknesses and limitations. The chapter concludes with a presentation of experimental results aimed at verifying the model and a numerical example of its application.
7.2 Objectives of the Proposed Model

It is prudent at this point to detail the motivation and objectives of the proposed model. Attention will be centered on composite laminates with symmetric stacking sequences constructed of continuous fiber, polymer matrix, unidirectionally reinforced plies. The analysis of such laminates, for structural applications, is generally based on the assumption of material continuity and homogeneity. That is, the individual plies within a laminate are assumed to be homogeneous, specially orthotropic materials whose mechanical properties are determined experimentally or from micromechanics. With respect to chapter 4, however, it is clear that the fatigue process in composite laminates is a complex CD process which takes place at or below the scale of the constituent materials and which manifests itself in the initiation and accumulation of several distinct damage mechanisms. These damage mechanisms, as noted previously, include both primary and secondary intralaminar matrix cracking, interlaminar matrix cracking (delamination), and fiber fracture. They act individually and collectively to induce changes in either the constituent material properties or the micro-geometry associated with the constituent materials, thus destroying the assumption of material continuity. The loss of continuity in turn effects the assumption of material homogeneity since the introduction of the individual damage mechanisms often isolates the constituent materials from one another as in the case of fiber/matrix debonding.

The mechanics of damage development is as complex as CD process itself. A great deal of effort has been devoted over the last two decades to characterize the mechanics which govern the initiation and accumulation of fatigue damage. Reifsnider (1991c) gives a comprehensive account of the progress towards this goal which emphasizes the effects of the initiation, growth, and coupling of the individual damage mechanisms and the stress redistributions they induce on the strength, stiffness, and fatigue life of
a laminate. Of these descriptors only stiffness is a parameter which can be measured nondestructively. As such, much of the current understanding of the mechanics of damage accumulation comes from the mechanics of stiffness reduction. A variety of methods have been used to model changes in laminate stiffness properties including shear-lag analysis (Reifsnider, 1977; Masters and Reifsnider, 1982), micro-mechanics (Hashin, 1991; Nemat-Nasser and Hori, 1993), finite element analysis (Ochoa and Reddy, 1992), variational methods (Hashin, 1985, 1987; Highsmith and Reifsnider, 1986), continuum mechanics (Talreja, 1986, 1987, 1991a, 1991b; Allen et al., 1988, 1991), and fracture mechanics (O'Brien, 1982, 1985). These methods, each with inherent advantages and disadvantages, have focused on stiffness reduction due to intralaminar and interlaminar (delamination) matrix cracking alone, reflecting the limited understanding of the mechanics of the complete damage accumulation process. Clearly, further research in this area is necessary.

Given the effects of the CD process in composite laminates under fatigue loading and the scale on which they occur, it is clear that a life prediction model for composite laminates should ideally be based on a mechanical analysis of the CD process at the micro-level. Based on such an analysis, a successful life prediction model should be able to quantify the effects of the CD process under fatigue loading in terms of the macro-level mechanical properties of strength, stiffness, and fatigue life. Reifsnider (1991c) notes, however, that given the complexity of the damage accumulation process, at the present it is not possible to give a fundamental development of the mechanisms which control damage accumulation at the micro-level based on first principles which describes in detail the strength, stiffness, and life of a laminate as a general function of the loading. As such, life prediction methods based on the current understanding of the macro-level effects of the CD process in composite laminates
under fatigue loading must be used. This is reflected in the fact that at present the
most widely used and well accepted life prediction models for composite laminates are
based on changes in the macro-level strength and stiffness properties of a laminate.

In view of the above considerations, the first objective of the proposed model is
that it be formulated in terms of readily measurable macro-level mechanical property
which in some manner reflects the level of damage accumulation within a laminate.
Laminate stiffness was chosen as a damage metric for this purpose. The choice of
laminate stiffness, or equivalently laminate compliance, was motivated by the fact
that a number of studies including those by O'Brien (1980), Reifsnider, et al. (1979),
and Highsmith and Reifsnider (1982) have shown that stiffness changes are directly
related to the accumulation of damage in a laminate. Furthermore, stiffness changes
provide an excellent measure of internal stress redistribution within a laminate since
the damage mechanisms which produce stiffness changes produce stress redistribu-
tion in direct proportion. In addition, changes in the stiffness of a laminate occur
throughout the life of a laminate and are of greater magnitude than changes in the
residual strength. Most importantly, however, is the fact that stiffness changes can be
measured nondestructively throughout the life of a laminate. Measurements of lami-
nate stiffness may be made by continuous monitoring of the stiffness itself or through
models, such as those of Talreja (1986, 1987, 1991a, 1991b) and O'Brien (1982, 1985)
among others, which relate quantities obtainable through nondestructive testing, such
as matrix crack density and delamination size, to changes in laminate stiffness.

Given the infinite number of possible configurations for a laminate in terms of
both constituent materials and stacking sequences, it is not feasible to characterize
the fatigue behavior of composite laminates of different configurations via traditional
methods such as S-N diagrams. For a life prediction model to be practical in this re-
garded it must be formulated in such a manner that existing characterizations, whether mechanical or phenomenological, of the fatigue behavior of the individual plies themselves can be used to characterize the fatigue behavior of a laminate. The second objective of the proposed model is to incorporate such capability into its formulation. This is especially important for those plies whose fatigue behavior governs the fatigue behavior of the laminate as a whole. The critical element model as proposed by Reifsnider and co-workers (Reifsnider, 1986, 1991c, 1991d, 1992; Reifsnider and Carman, 1992; Reifsnider and Stinchcomb, 1986) provides the framework for meeting this objective.

The capacity of the proposed model to address multiaxial fatigue loading is its third objective. As noted in chapter 5, a number of the life prediction methods that have been proposed in the literature are based on the uniaxial fatigue loading. However, such loading rarely occurs in the service life of a composite structure. As such, a realistic life prediction model for composite laminates must be capable of incorporating multiaxial loading into its structure. This is of particular importance for composite materials due to the anisotropic nature of their mechanical properties which can vary by as much as an order of magnitude in differing directions. Due to the fact that composite laminates are generally much smaller through their thickness in comparison to their lateral dimensions, the multiaxial capability should be at least two dimensional.

The fourth objective of the proposed model, like the third, is related to the capability of a life prediction model to handle complex loading configurations. Specifically, it is desired to formulate a life prediction model which can incorporate spectrum loading (uniaxial or multiaxial) into its structure. This objective, again like the third,
aims at making the proposed model as practical as possible as constant amplitude loading, like uniaxial loading, rarely occurs during the life of a laminate.

Another problem of practical importance which must be addressed by a comprehensive life prediction model is that of initial damage within a component. It is well known that regardless of how stringent the manufacturing standards for a material may be, initial flaws may be present within a material. These flaws may be in the form of voids or inclusions formed during the manufacture of a material or they may be caused by machining of the material during the manufacture of composite components. Regardless of their form or origin, their presence may effect the fatigue life of a structure; thus, their effect must be accounted for if possible. The incorporation of the effects of initial damage into the proposed life prediction model constitutes the fifth objective of the model. Closely related to presence of flaws within a material is the fact that for critical structural applications, such as those in the aviation industry, inspection for such flaws and replacement of materials or structural components containing such flaws is commonplace. Clearly, such procedures will act to extend the fatigue life of a composite structure and the effect of such procedures should be incorporated into a life prediction model. This is the sixth objective of the proposed model.

The final objective is that the proposed model be formulated in such a manner that in can address the effects of random variation in the above quantities. That is, the proposed model should be stochastic in nature and capable of treating random variation in the damage metric (stiffness change), the applied loading, the distribution of initial damage within a material or structure, and the uncertainty associated with the detection of flaws during inspection.
The broad, but practical, objectives of the proposed model make its formulation very challenging. The choice of a class of models upon which to build it is a difficult choice. However, the comprehensive and coherent structure of the Markov chain models of CD, as present in chapter 5, make them a logical choice for this purpose. Their structure allows for many of the objectives to be met in a relatively straightforward and easy manner, especially the final objective. Further, when applied to the critical element model they provide a means for addressing each of the stated objectives. Of particular importance is the fact that coupling the critical element model with the Markov chain models allows for the development of a life prediction model which can incorporate the mechanics of damage development into its structure. This feature affords a more complete and accurate characterization of the CD process in composite laminates under fatigue loading based on basic principles. While other methods may be more suitable for addressing specific objectives of the model, the author knows of no other specific method or model which has such a comprehensive scope. The following sections describe in detail how each of the above objectives, with the exception of inspection and replacement, are incorporated into the proposed model. The reader is referred to Bogdanoff and Kozin (1985) for a discussion of how inspection and replacement procedures can be incorporated into the Markov chain models of CD.

7.3 The Critical Element Approach

As noted in section 5.3.3, the critical element model provides a rational means for addressing the complex CD process in composite laminates subjected to fatigue loading. The model is a generalization of the life prediction models based on residual strength degradation which utilizes changes in laminate stiffness as a damage metric. The
defining feature of the critical element model is the manner in which it divides a laminate into critical and subcritical elements based on their contribution to the life of a laminate. The subcritical elements are those elements whose failure causes a redistribution of stress within a laminate but does not cause failure of a laminate on a global level. The critical elements are those elements whose failure does lead directly to the global failure of a laminate. The subcritical elements participate in the CD process by shedding the applied loads that they carry onto the critical elements as damage accumulates within them. That is, for a given remote stress the increased strain within the subcritical elements, resulting from the formation of cracks, fiber/matrix debonds, delaminations, and fiber fractures, causes a reduction in the stiffness, or equivalently an increase in the compliance, of the subcritical elements. With a reduction in stiffness comes a reduction in the load carrying capacity of the subcritical elements. As a result of this reduction, the critical elements are forced to bear an augmented portion of the applied loads. The enhanced stress on the critical elements in turn reduces the fatigue life of a laminate as the load carrying plies will become over stressed and fail as damage accumulates.

As noted in section 5.3.2, many of the life prediction methods based on residual strength degradation focus on changes in the longitudinal strength of a laminate. This follows from the common convention in composites analyses that laminates are designed such that they exhibit their greatest strength along the primary load axis. This axis is generally assigned to be the longitudinal axis, in global coordinates, of a laminate. As such, changes in the longitudinal strength of a laminate control the changes in the strength of a laminate as a whole. Plies within a laminate whose fiber orientations more or less coincide with the longitudinal axis of a laminate \((\theta \approx 0^\circ)\) are known as “on-axis” plies. Those with other orientations are known as “off-axis”
plies. Using this convention, the critical elements in a laminate are typically the on-axis load bearing plies. The subcritical elements are the off-axis plies.

The fatigue behavior of the critical elements is generally characterized by a phenomenological relationship such as an S-N curve or P-S-N curve (probability-stress-life curve). The applied stresses on the critical elements which define their fatigue lives are determined from the mechanics of damage accumulation which relate changes in the stresses on the critical elements to changes in the stiffness or compliance of a laminate due to subcritical element failure. Further, as the critical elements are usually the on-axis plies within a laminate, the phenomenological constitutive information which characterizes their fatigue behavior can be used for any number of laminate configurations which contain on-axis plies with the same orientation. This is of great practical importance as it circumvents the need for extensive testing programs.

The critical element approach, as applied herein, consists of determining the critical and subcritical elements within a laminate, gathering appropriate phenomenological information of the fatigue behavior of the critical elements, and finding an appropriate means for determining the enhanced stresses on the critical elements due to subcritical element failure based on mechanics. For the laminates considered here, the critical elements will be the on-axis plies within a laminate and the subcritical elements will be the off-axis plies. The fatigue behavior of the critical elements is assumed to be described by a family of P-S-N curves. The stresses on the critical elements are determined from increases in the compliance components of a laminate through classical laminated plate theory (CLPT) as discussed in chapter 3. The increases in each of the laminate compliance components are considered as evolutionary random processes which can be modeled as nonstationary Markov chains. Utilizing this approach allows the first and second objectives of the proposed model to be met.
7.4 Stochastic Modeling of Laminate Compliance

Recall from section 3.4 that the in-plane constitutive relationship for a symmetric laminate, in terms of compliance, under static loading is described in single subscript notation by the equation

\[
\begin{bmatrix}
\bar{e}_1^o \\
\bar{e}_2^o \\
\bar{e}_6^o
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_{11}' & \bar{A}_{12}' & \bar{A}_{16}' \\
\bar{A}_{22}' & \bar{A}_{26}' & \\
sym. & \bar{A}_{66}'
\end{bmatrix}
\begin{bmatrix}
\bar{N}_1 \\
\bar{N}_2 \\
\bar{N}_6
\end{bmatrix}
\]

or

\[
\bar{e}^o = \bar{A}' \bar{N},
\]

where \( \bar{e}_i^o \) are the laminate mid-plane strains, \( \bar{N}_j \) are the stress resultants, and \( \bar{A}_{ij}' \) are the laminate compliance components in the off-axis \( \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \) coordinate system as shown if Figure 3.2. Note that the subscript 6 refers to a shear component in the \( \{\bar{x}_1, \bar{x}_2\} \) plane. The stress resultants are related to the applied stresses on a laminate, \( \bar{\sigma}_{ia} \), through the relationship

\[
\bar{\sigma}_{ia} = \frac{1}{h} \bar{N}_i,
\]

where \( h \) is the total thickness of a laminate. Furthermore, the laminate compliance and stiffness matrices for a symmetric laminate are related by the equation

\[
\bar{A}' = \bar{A}^{-1},
\]

where the \( \bar{A}_{ij} \) denote arithmetic averages of the individual ply stiffnesses, \( \bar{Q}_{ij} \), in the global coordinate system. The \( \bar{A}_{ij} \) are defined as

\[
\bar{A}_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij} d\bar{x}_3.
\]

The stiffness components, \( \bar{Q}_{ij} \), for a ply with a given orientation \( \theta \) are obtained from the ply stiffness components, \( Q_{kl} \) for \( k, l = 1, 2, 6 \), in the material symmetry
coordinate system \( \{x_1, x_2, x_3\} \) through the transformation

\[
\bar{Q} = T^{-1}_\sigma Q T_c,
\]

(7.6)

where \( T_\sigma \) and \( T_c \) are defined in equations (3.34) through (3.37). Equations (7.4) through (7.6) provide a means for obtaining laminate stiffness from laminate compliance or vice versa.

For fatigue loading, the applied loads, and consequently the stress resultants, may vary with the number of load cycles \( n \). Furthermore, the laminate compliance components vary as a function of \( n \) due to the accumulation of fatigue damage within a laminate. Introducing this cyclic dependence into equation (7.1) yields the cycle dependent constitutive relationship for a symmetric laminate in terms of compliance as

\[
\begin{bmatrix}
\bar{\varepsilon}_1^e(n) \\
\bar{\varepsilon}_2^e(n) \\
\bar{\varepsilon}_6^e(n)
\end{bmatrix}
= \begin{bmatrix}
\bar{A}_{11}(n) & \bar{A}_{12}(n) & \bar{A}_{16}(n) \\
\bar{A}_{22}(n) & \bar{A}_{26}(n) & \\
sym. & & \bar{A}_{66}(n)
\end{bmatrix}
\begin{bmatrix}
\bar{N}_1(n) \\
\bar{N}_2(n) \\
\bar{N}_6(n)
\end{bmatrix}
\]

(7.7)

or

\[
\bar{\varepsilon}^e(n) = \bar{A}'(n) \bar{N}(n).
\]

(7.8)

Clearly, for a known load history the six in-plane compliance components, \( \bar{A}_{ij} \) for \( i, j = 1, 2, 6 \), must be known or predicted to determine the three in-plane mid-plane strain components \( \bar{\varepsilon}_i^e \) acting on a laminate. Changes (increases) in each of these compliance components due to the accumulation of fatigue damage in a laminate are to be used as a measure of CD process. The changes in the individual compliance components are assumed to constitute an evolutionary random process and are to be modeled using the Markov chain models of CD as discussed in section 6.3.

The assumptions employed in constructing Markov chain models for each compliance component are as follows. The time over which the fatigue load acts is discretized
and is measured in units of duty cycles (DCs). Herein, a DC refers to a given number of load cycles. Fatigue damage, as measured by changes in laminate compliance, is assumed to increase monotonically with the number of DCs and takes on discrete values (states). The initial state of damage may be random and the level of damage is considered only at the end of each DC. Furthermore, the level of damage after a DC is assumed to depend only on the severity of the DC itself and the level of damage prior to the DC; thus the increases in each compliance component can be modeled as an embedded discrete time, finite state Markov chain.

Let the changes in each compliance component be discretized into a finite number of states indexed on the non-negative set of integers \( k = 1, 2, \ldots, b^{(ij)} \) with the magnitude of the changes increasing as the index of the state. State \( b^{(ij)} \) represents an absorbing or “failure” state for the \( ij^{th} \) compliance component \( \bar{A}_{ij}^*(n) \) for \( i, j = 1, 2, 6 \). From Markov chain theory, the probability mass function (PMF) of damage \( p^{(ij)}_x \) after a given number of DCs \( x \) is completely determined from the PMF of initial damage \( p^{(ij)}_0 \) and the probability transition matrices (PTMs) \( P^{(ij)}_{ix} \). The accumulation of fatigue damage in the \( ij^{th} \) compliance component is modeled as a Markov chain of the form

\[
p^{(ij)}_x = p^{(ij)}_0 \prod_{i_x=1}^{x} P^{(ij)}_{ix},
\]

where \( p^{(ij)}_0 \) and \( p^{(ij)}_x \) are \( 1 \times b^{(ij)} \) row vectors and where \( P^{(ij)}_{ix} \) is a \( b^{(k)} \times b^{(k)} \) matrix. The elements \( p^{(ij)}_{kl} \) of \( P^{(ij)}_{ix} \) represent the probability of the \( ij^{th} \) compliance component being in state \( l \) after the \( i_x^{th} \) DC, given that it was in state \( k \) before the \( i_x^{th} \) DC. Recall that for \( p^{(ij)}_0 \) and \( p^{(ij)}_x \) to be valid PMFs their elements must be positive and sum to unity. Further, for \( P^{(ij)}_{ix} \) to be a valid PTM its elements must be positive and its rows must sum to unity. The assumption that changes in the \( ij^{th} \) compliance component are monotonically increasing restricts the form of \( P^{(ij)}_{ix} \) to upper triangular.
Equation (7.9) is valid for both stationary and nonstationary cumulative damage processes. For stationary processes, where the elements of the probability transition matrices are constant for each DC such that \( P_{i^*_x}^{(ij)} = P^{(ij)} \), equation (7.9) reduces to

\[
    p_x^{(ij)} = p_0^{(ij)} \left[ P^{(ij)} \right]^x.
\]  

(7.10)

For nonstationary processes, such as those where DC severity and/or environmental and/or material properties change, equation (7.9) requires that a probability transition matrix be specified for each DC. Clearly, since the stiffness properties of a laminate change with cyclic loading and the loading itself may vary with time, as in the case of spectrum loading, nonstationary models for changes in the individual compliance components are needed.

Sections 6.3.4 and 6.3.5 were devoted to determining nonstationary Markov chain models. In section 6.3.4 nonstationary Markov chain models were constructed from stationary Markov chain models through the use of the time transformation-condensation method. Construction of a nonstationary Markov chain model in this manner is appropriate when the environment and/or material properties of a material change under constant amplitude loading. When spectrum loading occurs, the methods of section 6.3.5 are more appropriate. However, the methods of section 6.3.5 prevent data in terms of the mean and variance of the number of DCs necessary to reach observed levels of damage prior to failure from being incorporated into the nonstationary model. This in turn prevents the states of nonstationary models determined from these methods from corresponding to values of a physical observable. This restriction comes about from the requirement that the PTMs which define the life of a material or component under each of the loads in the load spectrum must be of the same size since they are multiplied together. This requirement restricts the number of free parameters that can be used to define the PTMs. As much is to be gained in
terms of physical insight into a CD process from associating the states of the Markov chain models for the CD process with actual levels of damage, this restriction poses a significant obstacle to the use of the Markov chain models. There are two possibilities for circumventing this problem.

To examine the first possibility, consider that the mean number of DCs necessary to reach a given number of damage levels in a component are known when the component is acted on by each load in the load spectrum alone. For this case, PTMs of a fixed size for each load component could be constructed with each subdivided into a given number of submatrices, equal to the number observed damage levels, of the same size. The mean number of DCs necessary to reach each level of observed damage under each of the applied loads could then be utilized in conjunction with the first of equations (6.24) to determine the ratio of the stay to jump probabilities associated with each submatrix. The restriction on the size of the individual submatrices limits the data which may be used to define the PTMs to only the mean values of the number of DCs necessary to reach each level of damage. This method is further hampered by the assumption that the mean number of DCs necessary to reach the same levels of damage under each component of the load spectrum is known. Such data may not, and probably will not, be available.

A more reasonable alternative is to make use of the observation of section 6.3.5 that mean behavior under spectrum loading can be replaced by constant amplitude loading. That is, the assumption of "equivalent damage" may be employed. Specifically, it is assumed that a constant amplitude load of unknown magnitude exists such that when it is applied to a component it will induce an equivalent level of damage in the same number of DCs as the load spectrum. Employing this assumption allows a stationary Markov chain model for changes in the individual laminate compliance
components to be determined and used in conjunction with the time transformation-condensation method to determine an appropriate nonstationary model for the compliance changes. The most important aspect of this approach is that it allows both the mean and variance of the number of DCs necessary to reach observed levels of compliance increases prior to and including failure to be utilized in constructing Markov chain models whose states can be associated with actual values of laminate compliance. Thus, at the small expense of employing the equivalent damage assumption, a class of models can be determined which incorporates as much data on the CD process as possible into its structure. This is the approach utilized in the remainder of this work for modeling changes in laminate compliance.

Employing the assumption of equivalent damage, a stationary Markov chain model of the form of equation (7.10) with an initial PMF of compliance change \( p_0^{(ij)} \) and a PTM \( P^{(ij)} \) is constructed for each of the individual laminate compliance components \( \tilde{A}'_{ij} \). Specifically, if the mean and variance of the number of DCs necessary to reach \( l \) distinct levels of compliance increase are known, a unit-jump PTM \( P^{(ij)} \) of a stationary Markov chain model for changes in the \( ij^{th} \) compliance component is determined by subdividing \( P^{(ij)} \) into \( l \) submatrices as in equation (6.22). The individual submatrices are of the form of equation (6.23) and are assumed to have constant stay and jump probabilities. The parameters to be determined for each submatrix are thus the ratio of the stay to jump probabilities \( r_k^{(ij)} \) and its size \( b_k^{(ij)} - b_{k-1}^{(ij)} \) with \( b_0^{(ij)} = 0 \) for \( k = 1, 2, \ldots, l \). These parameters are determined accordingly from equation (6.24).

The stationary Markov chain model of equation (7.10) for each compliance component is utilized to determine the CDFs of the number of DCs, \( x \), necessary to reach each model state \( t_k^{(ij)} \) for \( k = 1, 2, \ldots, l \) corresponding to an observed level of compliance increase. It should be expected that the stationary model will yield values of
the mean and variance of the number of DCs to reach these states, as computed from their CDFs or equivalently their PMFs, which agree well with those observed experimentally. This follows from the fact that parameters of the stationary Markov chain models, as determined by equation (6.24), are such that the mean and variance of the arrival times in states $b_k^{(i)}$ match the experimentally observed values as closely as possible. The inaccuracy in the stationary models becomes evident when the CDFs of the arrival times in states $b_k^{(i)}$ are compared with the EDFs of the compliance changes to which they correspond. If the CDFs predicted by the stationary model do not provide an accurate description of the EDFs, the time transformation-condensation method must be employed.

Several remarks about the choice of an appropriate time transformation are in order at this point. First, given a stationary model for changes in the $ij^{th}$ compliance component $\Delta t_{ij}$ based on the mean and variance of the number of DCs necessary to reach $l$ observed increases in the value of $\Delta t_{ij}$ and EDFs for the arrival times of the increases themselves, at least $l$ different time transformations can be constructed according to procedure discussed in section 6.3.4. The coefficients which define the time transformations are dependent on the CDF and EDF of the particular compliance change being considered. This raises the question of which transformation should be used, if only one of these transformations is to be used with the condensation method to determine a nonstationary Markov chain model. The answer to this question depends on where interest is centered in the CD process and the nature of a Markov process itself. Recall that the fundamental assumption associated with Markov chain models of CD is that the damage after a DC has occurred is dependent only on damage prior to the DC and the severity of the DC itself. That is, a Markov chain model of CD has one-step memory. However, a Markov chain model of CD will
always retain some memory of prior damage development as it has accumulated. In addition, the CDF for model state $b_k^{(i)}$ obtained from a nonstationary Markov chain model will agree most accurately with the EDF of the damage level corresponding to model state $b_k^{(i)}$ if this EDF and the CDF of model state $b_k^{(i)}$ obtained from a stationary model were used to determine the time transformation used in the nonstationary model. In other words, based on the memory and accuracy of the Markov chain models, if interest is centered on damage levels prior to and including the damage level corresponding to model state $b_k^{(i)}$, a time transformation based on the CDF of model state $b_k^{(i)}$ as obtained from a stationary model and the EDF for the damage level corresponding to model state $b_k^{(i)}$ should be used in conjunction with the condensation method. The accuracy of the results predicted by the nonstationary model will increase as model state $b_k^{(i)}$ is approached. That is, predicted results for damage levels nearer the one on which the time transformation was based will be more accurate that those farther removed in either direction from the one on which the time transformation was based.

An alternative to this approach can be formulated in such a manner that a nonstationary model for each compliance component can be determined which will yield CDFs for model states corresponding to observed levels of compliance increase that provide a much more accurate description of all of the EDFs for the observed compliance increases. The only condition placed on this approach is that the EDFs for the observed compliance increases do not overlap. The reason for this restriction will become apparent momentarily. First, consider that a stationary Markov chain model for each compliance component has been constructed as discussed above and that CDFs for each model state $b_k^{(i)}$ for $k = 1, 2, \ldots, l$ corresponding to $l$ observed levels of compliance increases have been computed using the stationary model. Further,
with reference to equation (6.25), assume a $\beta$th order polynomial is to be used for each of the $l$ transformations associated with each compliance component and that the $\beta$ ordered pairs of abscissa values $(x_{\alpha k}, y_{\alpha k})$ for $\alpha = 1, 2, \ldots, \beta$ corresponding to the $\beta$ chosen ordinate values for which $F_{Y_k}(x)$ and $F_{Y_k}(y)$ are to be equivalent have been determined. These steps are the same as those in the previous method for determining the time transformations. The alternative method for determining the time transformations deviates at this point from the previous method as follows.

It is desired to construct a time transformation based on the CDF of model state $b_1^{(ij)}$ as predicted by the stationary model and the EDF of the first observed damage level as this time transformation, when used with the condensation method, will provide the most accurate results, in terms of predicted probability distributions, for model states up to and including state $b_1^{(ij)}$. Assume that this transformation is found and denoted as $g_1^{(ij)}(x)$. Let this transformation be used with the condensation method for values of $y$ up to and including the value of $y$ for which the CDF $F_{Y_1}^{(ij)}(y)$ of model state $b_1^{(ij)}$ is essentially unity. Denote this value of $y$ as $y_{c_1}$. Thus, changes in $\tilde{A}_{ij}$ are modeled as a nonstationary Markov chain for a given number of DCs, $y$, up to and including $y_{c_1}$ as

$$p_{ij}^{y} = p_{ij}^{y} \prod_{y=1}^{y_{c_1}} p_{ij}^{y}, \quad (7.11)$$

using a stationary Markov chain model with PTM $P^{(ij)}$ and a time transformation $g_1^{(ij)}(x)$ in conjunction with the condensation method. Now denote the value of $x$ for which the CDF $F_{X_1}^{(ij)}(x)$ of model state $b_1^{(ij)}$ is essentially unity as $x_{c_1}$. It is desired to find a new transformation valid for values of $x > x_{c_1}$ based on the CDF of model state $b_2^{(ij)}$ as predicted by the stationary model and the EDF of the second observed damage level as this time transformation, when used with the condensation method, will provide the most accurate results, in terms of predicted probability distributions,
for model states $b^{(ij)}_1 + 1$ through $b^{(ij)}_2$. Recall, however, from section 6.3.4 that a general time transformation $y = g(x)$ is formulated such that it is valid for all $x > 0$. As interest is centered here on values of $x > x_{c1}$, the formulation of the time transformation must be modified. This is most easily accomplished by simply subtracting $x_{c1}$ from the $\beta$ ordered abscissa pairs $(x_{\alpha_k}, y_{\alpha_k})$ for $\alpha = 1, 2, \ldots, \beta$ to find a new set of $\beta$ ordered abscissa pairs $(x_{\alpha_k} - x_{c1}, y_{\alpha_k} - x_{c1})$ on which to determine a new time transformation $g^{(ij)}_2(x)$ valid for $x > x_{c1}$. Let this transformation be used with the condensation method for values of $y > y_{c1}$ up to and including the value of $y$ for which the CDF $F_2^{(ij)}(y)$ of model state $b_2^{(ij)}$ is essentially unity. Denote this value of $y$ as $y_{c2}$. Thus, changes in $\tilde{A}'_{ij}$ are modeled as a nonstationary Markov chain for a given number of DCs, $y > y_{c1}$, up to and including $y_{c2}$ as given by

$$p_{y}^{(ij)} = p_{y_{c1}}^{(ij)} \prod_{y=y_{c1}+1}^{y_{c2}} p_{y}^{(ij)},$$

(7.12)

using a stationary Markov chain model with PTM $P^{(ij)}$ and a time transformation $g^{(ij)}_2(x)$ in conjunction with the condensation method. This approach is repeated as necessary for the remaining observed levels of compliance increase.

This method functions by using different time transformations for different values of $y$ by simply translating the time “origin” of the nonstationary Markov chain model accordingly. The assumption that the EDFs for changes in the individual compliance components do not overlap is made to avoid ambiguity with regard to which time transformation to utilize for a given value of $y$. The use of this method allows for a nonstationary Markov chain model for changes in each compliance component to be constructed such that each model state corresponds to a given level of compliance increase. The predictive capability of the nonstationary models using this method is demonstrated by the fact the CDFs, for model states corresponding to observed levels of compliance increase, provide an exceptional fit the EDFs of the observed compli-
ance increases. This lends confidence to the results predicted by the nonstationary models for the intermediate model states. It should be noted that changing from one time transformation to another in the course of modeling compliance changes has the physical interpretation that the fatigue process which causes such changes is changing in severity at this point. Mathematically, changing from one time transformation to another can be interpreted as compensating for the fact that if only one time transformation is used to model the fatigue process the predicted results will be erroneous. Examining the CDFs for the intermediate model states reveals that when the change in time transformations is made vertical discontinuities of various magnitudes may appear in the CDFs for these states. However, in view of the fact that no experimental data are known on the arrival times to these states and the fact that even with these discontinuities the predicted results using this method are much more plausible than those using the previous method, it appears that this approach provides a superior means for modeling changes in the individual laminate compliance components. These results are illustrated in the numerical example which follows later in this chapter.

7.5 Stochastic Stress-Strain Analysis

Recall from equation (7.8) that the in-plane mid-plane strains experienced by a laminate in the global \( \{ \bar{x}_1, \bar{x}_2, \bar{x}_3 \} \) coordinate system due to the action of time varying applied loads are given by the equation

\[ \bar{e}^o(n) = \bar{A}'(n) \bar{N}(n). \]  

(7.13)

From equation (3.37) the laminate in-plane mid-plane strains are related to the ply strains in material symmetry coordinates \( \{ x_1, x_2, x_3 \} \) for a ply with an orientation
angle of $\theta$, measured counterclockwise positive with respect to the $\bar{x}_1$ axis, as

$$\epsilon^\theta(n) = T_\epsilon \bar{\epsilon}^\theta(n), \quad (7.14)$$

where $T_\epsilon$ is the in-plane strain transformation matrix as defined in equation (3.36). From equation (3.24) the ply stresses in material symmetry coordinates are related to the ply strains through the constitutive relationship for a ply as

$$\sigma^\theta(n) = Q(n) \epsilon^\theta(n). \quad (7.15)$$

Combining equations (7.13) through (7.15), the ply stresses in material symmetry coordinates can be related to laminate in-plane compliance components and the applied stress resultants in a global laminate coordinate system as

$$\sigma^\theta(n) = Q(n) T_\epsilon \bar{A}'(n) \bar{N}(n). \quad (7.16)$$

Note that the ply stiffness matrix $Q$ is in general a function of the number of applied load cycles, $n$, due to the accumulation of fatigue damage within a ply which, as noted earlier, causes a degradation of ply stiffness. Thus, given the applied stresses acting on a laminate and the in-plane laminate compliance components in a global laminate coordinate system, the ply stresses in material symmetry coordinates for any given ply in a laminate can be determined.

A few words about the role of the applied cyclic stresses in the formulation of the Markov chain models are in order at this point. Clearly, knowledge of the three applied cyclic stresses, $\bar{\sigma}_{ua}(n)$, acting on a laminate, whether they are constant amplitude or spectral in nature, is necessary for determining the individual ply stresses within a laminate on a DC by DC basis. However, the actual values of the applied cyclic stresses do not enter into the formulation of the Markov chain models. The only manner in which the applied cyclic stresses participate in the formulation of the
Markov chain models is in how they affect the mean and variance of the number of DCs necessary to reach given levels of damage in a component and/or failure of the component. This reflects the fact that Markov chain models of CD are phenomenological.

Another point of interest concerning the applied loading is the relationship between the number of applied load cycles and DCs. Specifically, for general in-plane loading where the three applied cyclic stresses, \( \bar{\sigma}_{1a} (n) \), may have different frequencies, one must adopt a definition of DCs. This is related to the kind of data that are available for constructing the Markov chain models. Recalling that a DC is simply a period of repetitive operation, if multiaxial fatigue data are available for constructing the Markov chain models, then a DC should be defined as repetitive application of each of the applied loads. For example, one DC corresponds to \( x \) cycles of \( \bar{\sigma}_{1a} \), \( y \) cycles of \( \bar{\sigma}_{2a} \), and \( z \) cycles of \( \bar{\sigma}_{6a} \). If, however, only uniaxial fatigue data are available for determining the Markov chain models, a DC should correspond to a given number of applied load cycles of the given uniaxial load. In any case, the ratio of the number of applied load cycles to a DC is arbitrary as a DC is simply a counting measure for the Markov chains. Larger ratios make the Markov chains more computationally efficient as fewer matrix multiplications are necessary. Smaller ratios, however, allow more details of the CD process under consideration to be captured by the Markov chain models. The value of these ratios should be chosen in a manner that is consistent with the complexity of the CD process of interest.

As stated previously, changes (increases) in each of the six independent laminate compliance components, \( A'_{ij}(n) \) for \( i, j = 1, 2, 6 \), due to a known sequence of fatigue loads are to be modeled as nonstationary Markov chains using a stationary Markov chain model and appropriate time transformations in conjunction with the condensa-
tion method. To determine the parameters of the stationary Markov chain model for each compliance component, the mean and variance of the number of DCs necessary to reach observed increases in each of the compliance components must be known. If the applied fatigue loads on a laminate are of constant amplitude, this procedure is straightforward and each state in each of the models can be related to a given level of compliance increase. Recall, however, that if spectrum loading is applied, then the assumption of equivalent damage is employed to preserve the ability to related each model state with a given level of compliance increase.

Working with the ply stresses of equation (7.16) and in terms of DCs, each compliance component, $A'_{ij}(y)$ for $i,j = 1,2,6$, is a random variable for a given number of DCs $y$ whose marginal PMF $p_{y}^{ij}$ can be determined from an appropriate Markov chain model utilizing the procedure outlined above. The ply stiffness matrix $Q$ in equation (7.16) is in general a function of $y$ due to the accumulation of fatigue damage in the individual plies. However, for the critical elements in a laminate, that is those plies most aligned with the primary load axis such that $\theta \approx 0^o$ it will be assumed that the elements of $Q$ are constant. This assumption is made on the grounds that although the stiffness components of a laminate degrade as a function of the number of load cycles under fatigue loading, those of the load bearing plies remain essentially constant due to the absence of substantial damage in the load bearing plies themselves. Employing this assumption and knowledge of the marginal PMFs of the $A'_{ij}(y)$ for a given $y$ leads to the direct determination of the marginal PMF $p_{y}^{(i)}$ for each of the ply stresses, $\sigma_i^{(i)}(y)$ for $i = 1,2,6$, in the material symmetry coordinate system $\{x_1,x_2,x_3\}$. For notational convenience and with reference to Table 3.1, let the double subscript notation, $A'_{ij}$ for $i,j = 1,2,6$, for each individual compliance component be replaced by the single subscript notation, $A'_{m}$ for $m = 1,2,\ldots,6$. With this no-
tation, the values that \( \sigma_i^\theta(y) \) may assume for the \( y^{th} \) DC are found by placing each unique combination of compliance components, \( \bar{A}_m \), into equation (7.16) given the applied cyclic stress resultants, \( \bar{N}_j(y) \), associated with the \( y^{th} \) DC. Thus, if the \( m^{th} \) compliance component has \( b^{(m)} \) possible states then \( \sigma_i^\theta(y) \) will have at most \( \prod_{m=1}^{6} b^{(m)} \) possible states. The probability associated with each possible value of \( \sigma_i^\theta(y) \) is the probability of the intersection of each of the compliance components which make up each of their possible combinations. That is,

\[
P\{ \sigma_i^\theta(y) = \sigma_{i_{i_{m}}}^\theta(y) \} = P\{ \cap_{m=1}^{6} \bar{A}_{m_{m_{m}}(y)} \},
\]

(7.17)

where \( i_{i_{m}} \) and \( i_{m} \) are indices over the possible states of \( \sigma_i^\theta \) and each of the \( \bar{A}_{m} \) respectively. Since the individual compliance components are statistically independent, the probability of a specific combination of compliance components is the product of the probabilities of each compliance component being in its respective state associated with the particular combination. Thus, equation (7.17) reduces to

\[
\mathbb{P}_{y}^{(\sigma_{i_{i_{m}}}^\theta(y))} = P\{ \sigma_i^\theta(y) = \sigma_{i_{i_{m}}}^\theta(y) \} = \prod_{k=1}^{6} P\{ \bar{A}_{m_{m_{m}}(y)} \}.
\]

(7.18)

Thus, if suitable Markov chain models for the changes in the individual laminate compliance components can be constructed in which the individual model states correspond to given levels of compliance increase for a known loading condition, then for any given number of DCs the marginal PMFs of the ply stresses on the critical elements along with the values that they can obtain can be determined. That is, the mechanics of stress redistribution due to subcritical element failure can be used with the Markov chain models for compliance increases to determine the augmented stresses which act on the critical load bearing plies of a laminate and their corresponding PMFs.
7.6 Life Prediction

The results of the previous two sections indicate how the ply stresses acting on the critical elements, along with their marginal PMFs, can be determined from Markov chain models of changes in laminate compliance and CLPT for a known load history. Regardless of whether constant amplitude or spectrum loading is applied to a laminate, the changes that occur in laminate compliance under the action of the applied loads induces a spectrum of in-plane ply stresses on the critical elements. The number of stresses in the spectrum is constant and equal to the number of possible combinations of compliance components, but the values of the stresses themselves will vary from DC to DC depending on the values of the applied stress resultants. The nature of the PMFs of these stresses is such that for increasingly larger values of time, as measured in DCs, the probability that larger ply stresses will occur on the critical elements increases. This, of course, has the effect of increasing the probability of failure of the critical elements, and consequently a laminate containing the critical elements, as cyclic loading continues. Following the critical element model, it is desired to use this information to characterize the fatigue life of a laminate in terms of a phenomenological characterization of the fatigue behavior of the critical elements under the action of the ply stresses which act upon them such as a S-N or P-S-N diagram. Attention will be focused on determining an appropriate Markov chain model for the fatigue behavior of the critical elements under uniaxial fatigue loading in the direction of fiber reinforcement. This follows from the fact that uniaxial fatigue data, in the form of S-N or P-S-N curves, are typically all that is available to characterize the fatigue behavior of the critical elements. Further, the fatigue behavior of the critical elements under longitudinal fatigue loading is of special interest as the longitudinal axis of a laminate coincides with the primary load axis. The lack
of further information regarding the fatigue behavior of the critical elements requires that the Markov chain models of their fatigue behavior be as simple as possible.

Bogdanoff and Kozin (1985) have shown how to construct simple stationary Markov chain models of the CD process in a component leading to its failure given P-S-N curves relating the probability \( P \) of a component failing after the application of \( N \) cycles of a constant amplitude fatigue load \( S \). This procedure simply converts data given on P-S-N curves, such as the mean time to failure and the survival time of a given percentage of components (typically 95%) under a given applied load to the mean and variance of the time to failure as needed to determine the parameters of a stationary Markov chain model. Specifically, if the mean time to failure, \( m_N \), and the 95% survival time, \( n_{95} \), under a given load are known, all that is needed to construct a stationary Markov chain model is to convert the 95% survival time to the variance of the time to failure \( k_{N}^2 \). If raw S-N data for a component is the only data available, the procedure proposed by Yen (1969) can be used to determine the mean and 95% survival curves.

To demonstrate the use of this procedure, consider that a stationary Markov chain model with a unit-jump PTM, \( \mathbf{P} \), with a constant ratio of stay to jump probabilities and an initial distribution of damage, \( p_0 = \{1, 0, \ldots, 0\} \), based on a mean time to failure of \( m_X = m_N = 1000 \) DCs is constructed for various values of the variance of the time to failure \( k_X^2 = k_N^2 \). Here 1 DC = 1 load cycle. Further, assume that the 95% survival time, \( x_{95} = n_{95} \), (i.e., the number of DCs necessary to obtain \( p_{95} = 0.05 \)) is computed for each value of the variance. If these data are tabulated in normalized form (i.e., \( m_X / \sqrt{k_X^2} \) and \( m_X / x_{95} \)) as in Table 7.1, the relationship between \( m_X / \sqrt{k_X^2} \) and \( m_X / x_{95} \) can be obtained graphically as in Figure 7.1. Figure 7.1 can be used
to determine \( k_X^2 \) from \( x_{.95} \) given \( m_X \) as necessary for construction of a Markov chain model for the CD process.

<table>
<thead>
<tr>
<th>( \frac{m_X}{\sqrt{\sigma_X^2}} )</th>
<th>( m_X )</th>
</tr>
</thead>
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<tr>
<td>7.023</td>
<td>1.277</td>
</tr>
</tbody>
</table>

**Table 7.1**: Ratios of mean to 95% survival time as a function of the ratios of mean to standard deviation of time to failure.

For the purpose of characterizing the lifetime of the critical elements under spectrum loading, a Markov chain model for the lifetime of the critical elements based on a phenomenological characterization of their longitudinal fatigue behavior will be constructed. It is assumed that the longitudinal fatigue behavior of the critical elements is given in the form of a P-S-N diagram containing mean and 95% survival times for the critical elements under the individual action of each of the possible longitudinal ply stresses. This can be accomplished by applying the procedure described above for each of the possible values of the longitudinal stress on the critical elements. That is, a simple unit-jump, stationary Markov chain model with PTM \( P_{i\sigma} \) of size \( b_{i\sigma} \) and with a constant ratio of stay to jump probabilities \( r_{i\sigma} \) is to be constructed for each of the possible values of longitudinal stress, \( \sigma_{1\sigma} \), on the critical elements.
Recalling from section 6.3.5 that when the effects of spectrum loading on a component are to be studied, the PTMs which define the DC severity under each of the loads in the spectrum must be of the same size. The size of each PTM is arbitrary but it is generally taken to be the largest of the $b_{i\sigma}$. Following the procedure of section 6.3.5, the PTMs for the remaining loads are determined by subdividing each of the PTMs into two blocks of fixed but arbitrary size then using equations (6.24) to determine the ratio of the stay to jump probabilities for each of the two blocks. With the PTMs, $P_{i\sigma}$, determined and with an initial PMF, $P_0$, the Markov chain model for the fatigue behavior of the critical elements is given by the equation

$$p_{c_y} = p_0 \prod_{i=1}^y P_{i\sigma c}.$$ 

(7.19)

One possibility for determining the choice of which PTM to use for each DC is to pick a PTM at random based on the evolutionary probabilities of the longitudinal ply
stresses on the critical elements associated with each PTM. Choosing PTMs in this manner will give rise to a random load history on the critical elements and a sample function of the fatigue behavior of the critical elements. Successive application of this procedure will produce an EDF for the time to failure of the critical elements which will approximate the CDF of the mean time to failure of the critical elements. An alternative means of obtaining an approximation of the mean CDF of the time to failure of the critical elements, analogous to that discussed in section 6.3.5, is to determine first the mean PTM associated with each DC based on the probabilities associated with the longitudinal ply stresses acting on the critical elements for a given DC; one can use these PTMs in equation (7.19) to model the fatigue behavior of the critical elements. That is, for a given DC \( y \), each of the longitudinal stresses, \( \sigma_{y} \), acting on the critical elements has an associated probability which is the \( i_{\sigma}^{th} \) component of \( p_{y}^{(\sigma \theta)} \), where \( \theta \) is the orientation of the critical element plies (\( \theta \approx 0^\circ \)). If this probability is denoted as \( p_{y_{i\sigma}} \), the mean PTM for the \( y^{th} \) DC is

\[
E\{p_{i\sigma}\} = \sum_{i_{\sigma}} p_{y_{i\sigma}} P_{i\sigma}.
\]  

(7.20)

Using either method, the evolutionary probability of failure of the critical elements can be determined. It follows that since state \( b_{i\sigma} \) of the Markov chain model for the fatigue behavior of the critical elements corresponds to failure of the critical elements, the evolutionary probability of failure of the critical elements, \( P_{F_{y}} \), after \( y \) DCs is the last component of \( p_{c_{y}} \). As failure of the critical elements controls the failure of a laminate itself, the probability of failure of the critical elements provides a measure of the reliability of a laminate. Further, this measure is conservative since the critical elements must fail before the laminate itself fails, and thus the probability of failure of the critical elements will be greater than or equal to the probability of failure of a laminate. The probability of failure of the critical elements is thus an upper bound
on the probability of failure of a laminate. However, since the critical elements are
defined as those elements whose failure causes failure of a laminate, it is reasonable
to expect that the probability of failure of the critical elements will be very near to
or coincide with the probability of failure of the laminate.

The last point to be mentioned with regard to the Markov chain model of the
fatigue behavior of the critical elements is that there is in general no EDF for the
time to failure of the critical elements under spectrum loading with which to compare
the mean CDF of the time to failure of the critical elements as predicted by the
Markov chain model. As such, there is no means by which to tell if load interaction
or multiaxial loading on the critical elements have a significant effect on the lifetime of
the critical elements. If load interaction and/or multiaxial loading acts to accelerate
the degradation of the critical elements, the proposed model will predict results that
are too "liberal". On the other hand, if load interaction and/or multiaxial loading
acts to retard the degradation of the critical elements, the proposed model will predict
results that are too "conservative". If data, in the form of an EDF of the time to
failure of the critical elements under spectrum loading or an EDF of the time to failure
of a laminate are available and the mean CDF of the time to failure as predicted
by the Markov chain model does not agree with this data, the time transformation-
condensation method as discussed in section 6.3.5 can be employed to determine a
"corrected" Markov chain model for the fatigue behavior of the critical elements. It
is recognized that this is a limitation of the proposed model. However, the benefits
that the proposed model affords seem to outweigh this limitation and lend validity to
its use for predicting the lifetime of a laminate.
7.7 Application of the Proposed Model

The previous sections of this chapter have provided physical motivation and mathematical formulation for the proposed life prediction model. The purpose of this section is to provide the framework for its practical application. The flow diagram of Figure 7.3 is intended to aid in this discussion by providing a visual picture of those aspects which are pivotal to the application of the model.

The first step in the application of the model to a structure or component fabricated from a composite laminate is to identify its material constituency, its geometry, and applied loads to which it is subjected. The first two of these tasks are relatively straightforward. Specification of the material system includes the type of laminate (e.g., graphite/epoxy), its stacking sequence, and its pertinent material properties including its initial stiffness components, engineering constants, or compliance components. Specification of the geometry should include such information as the general shape and dimensions of the structure or component and whether holes, notches, cut-outs, thickness variations, etc. are present such that their influence can be accounted for during stress-strain analysis if necessary.

Specification of the applied loading acting on a composite structure or component is a potentially difficult but very important task in the application of the proposed model. Recall that the applied loads themselves do not participate in determining the parameters of the Markov chain models for either the changes in the compliance components of a laminate or the fatigue life of the critical elements. The only manner in which applied loads effect the model parameters is in how they effect the mean and variance of the time, as measured in DCs, necessary to reach given levels of compliance increase and/or failure of laminate and the time to failure of the critical elements. However, the applied loads themselves are used to determine the ply-level
stresses on the critical elements for each DC. Three in-plane stress resultants, \( \bar{N}_i \) for \( i = 1, 2, 6 \), are needed for this purpose and these quantities are related to the applied stresses \( \bar{\sigma}_{ia} \) through equation (7.3). As such, each of the applied stresses on a composite structure or component must be determined for the application of the proposed model. These stresses can be deterministic or random. All that is required for use with the proposed model is that the time history or a collection of time histories of these stresses, depending on whether the applied stresses are deterministic or random, be known. If these stresses are deterministic then this task is straightforward. If, however, these stresses are random in nature realizations or sample time histories of the applied stresses must be used or determined for use with the proposed model. For example, sample time histories can be obtained directly from accelerometer data if available. Alternatively, sample time histories can be obtained from a power spectral density (PSD) of the applied stresses through the use of an autoregressive (AR) model or an autoregressive moving average (ARMA) model by filtering white noise of an appropriate variance. In either case, the time histories are discrete and their components are of known magnitude.

The next step in the application of the proposed model is the determination of the damage modes and patterns and the anticipated failure mode of the structure or component based on a description of the applied stresses. For an accurate determination of these quantities the description of the applied stresses must include information on their magnitudes, whether they are tensile or compressive, and whether they are uniaxial or multiaxial. The type and pattern of damage that occurs in a structure or component determines the subcritical elements of the material system from which it is constructed. Likewise, the type of failure that occurs determines the critical elements. For example, in a structure or component where tensile stresses domi-
nate, damage initiates and accumulates in the form of primary and secondary matrix cracks, fiber/matrix debonds, delaminations, and fiber fractures which concentrate in the plies of a laminate not aligned with the primary load axis. These damage modes act to increase the compliance of a laminate on both local and global levels and further act to increase the ply-level stresses on those plies most aligned with the primary load axis. Since the later are precisely the plies which carry the majority of the applied loads, their failure (by tensile fracture in this case) would control the ultimate failure of the structure or component. As such, the subcritical elements of the material system would be the off-axis plies of the laminate which makes up the structure or component and the critical elements would be the on-axis plies.

With the critical and subcritical elements determined, the application of the proposed model proceeds through the determination of an appropriate nonstationary Markov chain model for changes (increases) in each of the independent laminate compliance components, $\bar{A}_m$ for $m = 1, 2, \ldots, 6$, due to the accumulation of damage in the subcritical elements according to the procedure of section 7.4. Specifically, for a structure or component subjected to either deterministic or random spectrum loading the assumption of equivalent damage is employed which allows for the construction of a suitable stationary Markov chain model in which each model state can be directly related to a given level of compliance increase. The data necessary for estimating the parameters of the stationary model are the mean and variance of the number of DCs necessary to reach given values compliance increase under the deterministic load history or samples of a random load history. These data may be obtained in two ways. First, these data may be obtained through nondestructive testing of a group of structures or components subjected to actual service loads so as to determine quantities such as the size, shape, and density of interlaminar and/or
intralaminar matrix cracks which can be related through appropriate damage models to changes in the compliance components, or equivalently the stiffness components, of a structure or component. Alternatively, such data can be obtained in the laboratory by testing specimens of the same material make-up as the structure or component under simulated service loads determined as discussed above.

Given a stationary model for changes in each independent laminate compliance component, the CDF of the number of DCs to reach each model state is determined and the mean arrival time for each model state is computed from this CDF. This information is used to establish the relationship between each model state and a given level of compliance increase by determining the level of compliance increase whose mean number of DCs to attainment is equivalent to the mean number of DCs necessary to reach a given model state. Appropriate time transformations are then found based on the CDFs of those model states corresponding to observed levels of compliance increase and the EDFs for those increases. These time transformations, along with the stationary Markov chain model, are used in conjunction with the condensation method to determine the appropriate PTM associated with each DC in the nonstationary Markov chain model. These PTMs, along with an initial PMF of compliance change, define the nonstationary Markov chain model for each compliance component. The nonstationary Markov chain model for changes in each compliance component yields a discrete set of possible changes in each compliance component and the evolutionary PMF of the set. It should be noted again that the use of nonstationary Markov chain models, and in particular the time transformation method, as described herein allows for greatly improved predictions of the number of DCs necessary to reach a given level of compliance change to be obtained. The nature of the time transformations is such that coefficients of the transformations can be
determined in such a manner that an improved fit between the CDF of a model state corresponding to a given level of compliance increase and the EDF for the increase can be realized in any portion of the CDF/EDF. This is especially important with regard to the tails of the distributions for such changes as interest is often centered in these areas when performing reliability studies.

The possible values for changes in each laminate compliance component along with the evolutionary PMFs of such changes are used in conjunction with CLPT to determine the ply-level stresses acting on the critical elements and their corresponding PMFs. It should be noted that the number of possible values for each stress component, $\sigma_i$ for $i = 1, 2, 6$, in material symmetry coordinates acting on the critical elements will be the same for each DC and equal to the number of possible combinations of the individual compliance components. However, the values of these stresses will vary from DC to DC depending on the value of the applied stress resultants $\tilde{N}_i$ for each DC. Further, it should be noted that if discontinuities such as holes, notches, cut-outs, thickness variations, etc. are present in the structure or component, then CLPT will not yield correct values for the stresses acting on the critical elements as it is based on the assumption of a uniform stress state within a laminate. If such discontinuities do appear, an alternate means of relating changes in laminate compliance to ply-level stresses which can address the affects of discontinuities must be used.

Based on the spectrum of ply-level stresses which act on the critical elements and a phenomenological characterization of their fatigue behavior under these stresses, such as a family of P-S-N curves, the next step in the proposed model is to determine an appropriate Markov chain model for predicting the fatigue life of the critical elements according to the techniques of section 6.3.5. Specifically, for every possible critical
element stress state a unit-jump PTM must be found based on the constant amplitude fatigue behavior of the critical elements under the action of each stress state. Recall that the PTMs so determined must be modified accordingly such that they are of the same size. Given the modified PTMs, the fatigue behavior of the critical elements under the spectrum of stresses associated with each DC is modeled by choosing an appropriate PTM for use with each DC based on the probabilities associated with each of the possible stress states corresponding with each PTM. The evolutionary probability of failure of the critical elements is computed from this model and corresponds to the probability of being in the final model state after each DC. Again, it should be noted that this probability of failure forms an upper bound on the probability of failure of the laminate in the absence of significant load interaction effects. Further, the phenomenological data which are typically available to describe the fatigue behavior of the critical elements are generally obtained from uniaxial testing of the critical elements. As noted earlier, if the structure or component is designed such that the longitudinal axis of the critical elements is most aligned with the primary load axis and if the transverse and shear stresses on the critical elements are negligible in comparison with the longitudinal stresses, then the fact that only uniaxial fatigue data for the critical elements are available is a problem of little significance. If, however, significant load interaction or multiaxial stress states do occur, they must and can be accounted for through the use of the time transformation-condensation method given an EDF for the time to failure of the composite structure or component.

7.8 Example

The purpose of this section is to demonstrate the use of the proposed life prediction model for composite laminates and to examine its predictive capability. Specifically,
it is desired to illustrate the procedures for determining nonstationary models of changes in laminate compliance, stochastic stress-strain analysis, and life prediction of the critical elements. Unfortunately, the requisite data for this purpose are not available in the open literature to the best of the author's knowledge. To circumvent this problem, an experimental program was initiated to obtain such data.

The experimental program was simple in nature and consisted of subjecting a number of composite specimens to fatigue loading and measuring changes in their elastic properties due to the accumulation of damage under the given loading conditions. The material system used for this program was Hercules AS4/3501-6 unidirectionally reinforced graphite/epoxy pre-preg tape with material properties as given in Table 7.2. A six ply, cross-plied laminated panel with a $[0^\circ, 90_2^\circ]$s stacking sequence

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Property Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal Young's Modulus $(E_{11})$</td>
<td>140 GPa (20.3e6 psi)</td>
</tr>
<tr>
<td>Transverse Young's Modulus $(E_{22})$</td>
<td>8.36 GPa (1.21e6 psi)</td>
</tr>
<tr>
<td>In-Plane Shear Modulus $(E_{66})$</td>
<td>4.31 GPa (0.625e6 psi)</td>
</tr>
<tr>
<td>Major In-Plane Poisson's Ratio $(\nu_{12})$</td>
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<td>Ply Thickness</td>
<td>149e-6 m (5.87e-3 inches)</td>
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<tr>
<td>Fiber Volume Fraction (after cure)</td>
<td>0.7</td>
</tr>
</tbody>
</table>

*Table 7.2: Material properties for Hercules AS4/3501-6 graphite/epoxy pre-preg tape*

was fabricated from this material. The material properties of the panel as predicted by CLPT are given in Table 7.3. Seventeen identical specimens were cut from this panel and machined to their final width and length dimensions of 2.54 cm (1 inch) and 30.5 cm (12 inches), respectively. Each specimen was then placed in an Instron servo-hydraulic test stand with serrated grips and subjected to constant amplitude, sinusoidal, tension-tension fatigue loading with a stress ratio of $R = 0.1$ and a max-
<table>
<thead>
<tr>
<th>Material Property</th>
<th>Property Value</th>
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<tbody>
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<tr>
<td>In-Plane Shear Modulus ($G_{66}^o$)</td>
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</tr>
<tr>
<td>Major In-Plane Poisson’s Ratio ($\nu_{12}$)</td>
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</tr>
<tr>
<td>Laminate Thickness</td>
<td>894e-6 m (35.2e-3 inches)</td>
</tr>
</tbody>
</table>

Table 7.3: Laminate properties as predicted by CLPT for a $[0^\circ, 90^\circ]^s$
AS4/3501-6 graphite/epoxy laminate

imum cyclic stress of $0.7\bar{S}_{1T} = 420$ MPa (60.9ksi) for intervals of 20,000 load cycles and a total of 100,000 load cycles. Emory paper was used as an interleav between the grips and the specimens to prevent specimen damage and to aid in the transmission of tensile forces to the specimens (Hart-Smith, 1990). Upon completion of each interval of cyclic loading, each specimen was removed from the servo-hydraulic test stand and placed in an Instron static test stand for measurement of its longitudinal Young’s modulus $E_{11}^o$. The same grips were used on both machines. The value Young’s for each specimen was taken as the initial slope of the stress-strain curve for the specimen under quasi-static tension loading. Strain was measured via a longitudinal extensometer with a 2.54 cm (1 inch) gage length. The initial modulus of each specimen and its residual modulus after loading is shown in Table 7.4.

Upon completion of the experimental program, the data for the reduction in the longitudinal Young’s modulus $E_{11}^o(n)$ of each specimen were converted to data for increases in the longitudinal compliance $\bar{A}_{11}(n)$ of each specimen using equation (3.59) noting that for symmetric laminates $\bar{A}_{11}(n) = \bar{A}_{11}^*(n)$. These data were then normalized by the initial longitudinal compliance and interpolated (linearly) to give sample functions or realizations of the changes in $\bar{A}_{11}$ as a function of the number of load cycles. These sample functions are shown in Figure 7.3 and conform with the as-
Table 7.4: Initial and residual values of longitudinal Young’s modulus $\bar{E}_{II}^0(n)$ after cyclic loading. Note: all values are in units of GPa.

<table>
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<tr>
<th>Specimen</th>
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<td>41.2</td>
</tr>
<tr>
<td>17</td>
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<td>47.5</td>
<td>46.2</td>
<td>45.4</td>
<td>44.8</td>
<td>44.7</td>
</tr>
</tbody>
</table>

The assumption of monotonically increasing damage used in the formulation of the Markov chain models for changes in laminate compliance.

The sample functions for changes in normalized longitudinal laminate compliance were used to determine the EDFs of the number of DCs necessary to reach given percentages of compliance increase. The chosen increases were 2.5%, 5%, 7.5%, 10%, 12.5%, and 15% respectively. One DC was taken to be 50 load cycles. The EDFs so determined are shown in Figure 7.4. The mean and standard deviation of the number of DCs to reach each of these percent increases were computed from their respective EDFs. These values are shown in figures 7.5 and 7.6 respectively and are indicated by circles. The solid line in each figure representing the mean and standard deviation of
the number of DCs to reach a given percentage of increase in normalized longitudinal laminate compliance was obtained by linearly interpolating between each of the data points.

The EDFs for 5% and 15% compliance increases were used to construct a non-stationary Markov chain model for changes in the normalized longitudinal laminate compliance recalling the restriction that damage levels with nonoverlapping EDFs must be used for this purpose. The mean and variance of the number of DCs to reach a 5% increase were 197 DCs and 2,116 DCs respectively. Likewise, the mean and variance of the number of DCs to reach a 15% increase were 1262 DCs and 294,849 DCs. These data were used to determine the parameters of a stationary Markov chain model for changes in the normalized longitudinal laminate compliance according to the procedure of section 7.4. Specifically, a stationary Markov chain model with PTM $P^{(11)}$ was constructed where $P^{(11)}$ was of unit-jump form and divided into two blocks each with a constant ratio of stay to jump probabilities. The parameters to be determined were the size of each block, $(b_1 - 1)$ and $(b_2 - b_1)$, and the ratio of the stay to jump probabilities in each block, $r_1$ and $r_2$. The values of these parameters are given in Table 7.5. The time transformation method was then employed to determine

\[
\begin{array}{ccc}
\hline
i & b_i & r_i \\
\hline
1 & 18 & 10.6 \\
2 & 22 & 265 \\
\hline
\end{array}
\]

**Table 7.5:** Parameters of the stationary Markov chain model for changes in normalized longitudinal laminate compliance.

two suitable transformations of the indexing parameter for the stationary Markov chain model as discussed in section 7.4 given a stationary Markov chain model and the EDFs for the two observed levels of compliance increase. A fourth order poly-
nomial of the form of equation (6.25) was used for both transformations. Ordinate values for the CDFs/EDFs of 0.06, 0.41, 0.82, and 0.94 were used to determine the coefficients of each transformation. These values were chosen such that the CDFs for the model states of the nonstationary Markov chain model corresponding to observed levels of compliance increase would provide a good fit to the EDFs for those increases over their entire domain. The coefficients so determined for each transformation are given in Table 7.6. Given the stationary Markov chain model and the time trans-

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<th>i</th>
<th>c_{i1}</th>
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Table 7.6: Time transformation coefficients.

formations, the condensation method was employed to implement the nonstationary Markov chain model for changes in the normalized longitudinal laminate compliance according to the procedure of section 7.4. The CDFs for model states 18 and 22, corresponding to 5% and 15% increases in normalized longitudinal laminate compliance, as predicted by the stationary and nonstationary Markov chain models are shown in Figure 7.7 along with the EDFs of each of these increases. This figure illustrates the dramatic improvement in the predictive capability of the Markov chain models when a nonstationary Markov chain model is used. Of special importance is the fact that a much better fit is obtained with the nonstationary models in the tails of the EDFs were interest is often centered for reliability studies. It should be noted that this fit could be further enhanced by concentrating the CDF/EDF ordinate values used to determine the time transformation in either of tails depending on where interest is focused.
The mean number of DCs necessary to reach each model state as predicted by the nonstationary Markov chain model was computed and used in conjunction with the experimental data for the mean number of DCs necessary to reach a given percentage increase in normalized longitudinal laminate compliance, as shown in Figure 7.5, to determine the percent increase in compliance associated with each model state as discussed in section 6.3.3. This relationship is shown in Figure 7.8 and is used to determine the ply-level stresses which act on the critical elements. Specifically, each possible value of compliance is used in conjunction with the applied stress state on the laminate for a given DC to determine each of the possible stress states acting on the critical elements according to equation 7.16. The critical elements for this example are the 0° plies. Due to lack of additional information on the changes in the other laminate compliance components due to the accumulation of fatigue damage in the subcritical elements, all other laminate compliance components were assumed to be constant. As such, since there are 22 possible values of normalized longitudinal laminate compliance there are 22 possible values of each stress component acting on the critical elements. Since the axial stress component is the only stress component that is dependent on the longitudinal laminate compliance, it is the only random stress component. The possible values of each stress component acting on the critical elements are shown in Figure 7.9. Clearly, the evolutionary behavior of the changes in laminate compliance are such that as cyclic loading continues the value of the axial stress on the critical elements is increasing in nature.

Given the spectrum of stresses acting on the critical elements, it remains to find an appropriate Markov chain model for the fatigue behavior of the critical elements under this spectrum of loads. Recall from section 6.3.5 that this requires a phenomenological characterization of the constant amplitude fatigue behavior of the critical elements
under each of the critical element stress states alone. It is assumed that this characterization is in the form of a P-S-N diagram which contains the mean and 95% survival times of the critical elements under each possible stress state. Since such data are typically based on uniaxial loading, the fatigue behavior of the critical elements will be characterized in terms of their behavior under each of the possible values of the axial stress, $\sigma_{11}$, which act upon them. It assumed that the mean and 95% survival curves for the critical elements under uniaxial loading are described by a straight line on a semi-log plot of stress versus life through an equation of the form

$$\frac{\sigma_{11}}{S_1} = 1 - c \log_{10} N.$$  \hspace{1cm} (7.21)

The slope of the mean and 95% survival curves were taken as 0.0275 and 0.0285 respectively, and $S_1$ was taken as 1500 MPa. Note that $N$ in this equation represents load cycles. These curves are shown in Figure 7.10. The mean and 95% survival times in load cycles for the critical elements under each of the possible values of axial stress states were computed from equation (7.21) and converted to DCs noting that 1 DC is taken as the equivalent of 50 load cycles. The mean and 95% survival times in DCs were converted to the mean and variance of the time to failure of the critical elements under each of the axial critical element stress utilizing Figure 7.1. These values are shown in terms of the mean value and signal-to-noise ratio ($m_X / \sqrt{\sigma_X^2}$) in Table 7.7.

An appropriate PTM was determined for each value of the axial stress on the critical elements for characterizing the constant amplitude fatigue behavior of the critical elements. Each PTM was chosen to have a fixed size. Further, each PTM was divided into two blocks of fixed size and of unit-jump form. Specifically, $b_1$ and $b_2$ for each block were taken as 2 and 38 respectively. It should be noted that this choice is arbitrary and was one of many that could have been chosen that satisfy the
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<th>$\sigma_{11}$ (GPa)</th>
<th>$m_X$ (DC)</th>
<th>$m_X/\sqrt{k_X^2}$</th>
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<th>$b_2$</th>
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Table 7.7: Ratio of stay to jump probabilities for each block of the PTMs associated with each value of the axial stress acting on the critical elements.

The requirement that the ratio of the stay to jump probabilities for each block must be real and positive. The ratio of the stay to jump probabilities for each block of each of the PTMs was then determined from equation (6.24). The values of these ratios are shown in Table 7.7. The PTMs and the evolutionary probabilities associated with each value of the axial stress on the critical elements were used to determine the mean PTM associated with each DC as given by equation (7.20). The mean PTM for each DC was used in conjunction with equation (7.19) to determine the evolutionary
probability of failure of the critical elements. For this example, \( P_{F_y} = p_{c_y} \). The evolutionary probability of failure is shown in Figure (7.11). From this figure it is clear that the probability of failure of the critical elements, and hence the laminate, rises sharply as damage begins accumulate in the laminate. As noted before, this probability of failure constitutes an upper bound on the probability of failure of the laminate barring any significant load interaction or multiaxial load effects. Additional information such as an EDF of the time to failure of the laminate would be necessary to account for such effects if they occur.
Figure 7.2: Flow diagram of the proposed model for predicting the fatigue life of composite laminates subjected to fatigue loading
**Figure 7.3:** Sample functions for changes in the normalized longitudinal laminate compliance

**Figure 7.4:** EDFs for 2.5%, 5%, 7.5%, 10%, 12.5%, and 15% increases in normalized longitudinal laminate compliance.
Figure 7.5: Mean number of DCs to reach a given level of compliance increase.

Figure 7.6: Standard deviation of the number of DCs to reach a given level of compliance increase.
Figure 7.7: Comparison between EDFs (solid) and the CDFs for model states corresponding to a 5% and 15% increase in normalized longitudinal laminate compliance respectively as predicted by a stationary Markov chain model (dashed) and a nonstationary Markov chain model (dash-dot).
Figure 7.8: Relationship between percent increase in normalized longitudinal compliance and the model states of the nonstationary Markov chain model.

Figure 7.9: Possible values of the axial (solid), transverse (dashed), and shear (dash-dot) stress components acting on the critical elements.
Figure 7.10: Mean and 95% survival curves for the fatigue behavior of the critical elements under constant amplitude axial fatigue loading.

Figure 7.11: Evolutionary probability of failure of the critical elements.
Chapter 8

Concluding Remarks

This work has focused on developing a probabilistic life prediction model for composite laminates subjected to fatigue loading. The proposed model involves Markov chain models of cumulative damage (CD) in conjunction with the critical element concept for the fatigue of composite laminates. Changes in laminate compliance due to the accumulation of fatigue damage in a laminate were considered as evolutionary stochastic processes and were modeled as discrete time, finite state nonstationary Markov chains utilizing stationary Markov chains and the time transformation-condensation method. The nonstationary Markov chain models for each compliance component were constructed such that the individual states of the models could be related to a specific increase in each of the individual laminate compliance components. The evolutionary probability mass functions (PMFs) for changes in each compliance component as determined from these models were used to determine the magnitudes of the ply stresses on the critical load bearing plies in a laminate and their PMFs. This information was used in conjunction with a phenomenological characterization of the fatigue behavior of the critical elements (P-S-N curves) to find a simple Markov chain model for the lifetime of the critical elements. This model was used to determine the evolutionary probability of failure of the critical elements, which serves as an upper bound on the probability of failure of a laminate containing the critical elements.

Chapters 1 through 6 provided the necessary elements for constructing the proposed life prediction model. Chapter 1 provided much of the motivation for studying the fatigue of composite laminates. Chapter 2 served to review those concepts from
probability theory concerning random variables and random processes that were pertinent to the development of the proposed model. Since this model is based on changes in the stiffness properties of a laminate, chapter 3 was devoted to developing the constitutive relationship for a composite laminate based on the stress-strain relationships of its constitutive plies. Chapter 4 provided a brief, but broad, overview of strength, failure, and fatigue of composite laminates with emphasis on the CD process which governs the strength, stiffness, and fatigue life of composite laminates. Deterministic and probabilistic methods for predicting the fatigue life of composite laminates were reviewed in chapter 5. Particular emphasis was placed on the critical element model. Chapter 6 provided background on Markov chain models of CD necessary for the development of the proposed model. Chapter 7 contained the crux of the present work. It brought the previous chapters together in the formulation of the proposed model. To the best of the author’s knowledge, this work is the first attempt to develop a life prediction method for composite laminates based on the highly successful Markov chain models of CD.

The proposed model is quite comprehensive; it is based on changes in a readily measurable damage metric and is capable of characterizing the fatigue behavior of a laminate based on the fatigue behavior of its individual plies by incorporating the mechanics of damage development into its structure. Further, it is capable of addressing, in a probabilistic setting, the effects of general multiaxial spectrum loading, initial damage, and of inspection and replacement procedures. While it is realized that other methods may be more suitable for addressing each of these effects individually, it appears no other method offers the versatility of the proposed model for addressing these effects collectively. This makes the proposed model quite promising for further development.
In the formulation of the proposed model a number extensions to the Markov chain models of CD were presented. Specifically, it was shown that nonstationary Markov chain models of CD can be constructed based on the mean and variance of the time to reach subcritical levels of damage prior to failure. This involved a novel approach to finding appropriate time transformations for converting stationary Markov chain models into nonstationary Markov chain models which optimized the predictive capability of the nonstationary models. This approach centered on finding a time transformation for each of the known levels of damage and shifting the time “origin” of the Markov chain models accordingly for use with each transformation. In addition, it was shown how a mean cumulative distribution function for a Markov chain model of CD under spectrum loading could be constructed when the probabilities associated with each of the loads in the load spectrum evolved with time. These extensions, developed with respect to the proposed model for composite laminates, are applicable to Markov chain models for any CD process and add to the long list of attractive features offered by these models.

The greatest limitation associated with the proposed life prediction model, and with Markov chain models of CD in general, is that there is a clear absence of the kind of data that are needed for determination of model parameters. This is a direct result of the fact that the data needed for parameter estimation are statistical in nature, and obtaining such data experimentally is often quite difficult. Fortunately, the structure of the proposed model minimizes that amount of such data necessary for model construction by incorporating the ideas of the critical element model into its structure. Further, the statistical data necessary to determine Markov chain models for changes in laminate compliance can be gathered through a number of models that have been proposed in the literature for determining changes in laminate stiffness.
properties based on quantities that are readily measurable through nondestructive testing techniques. This greatly increases the applicability of the model to "real life" situations.

In conclusion, the proposed life prediction model provides a simple, yet effective, means of modeling the complex CD process in composite laminates subjected to fatigue loading. It utilizes as input data on stiffness loss, or equivalently compliance increase, which can be correlated to the individual damage mechanisms and phenomenological characterizations of the critical load bearing plies within a laminate. From this model a wealth of information of the CD process can be obtained including the evolutionary probability of failure of a laminate and distributions for the time to failure.
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Appendix A

Inversion of the Laminate Constitutive Relationship

The in-plane constitutive relationship for a laminate in terms of stiffness in an off-axis coordinate system \( \{ \bar{x}_1, \bar{x}_2, \bar{x}_3 \} \) is given by the equation

\[
\begin{bmatrix}
\bar{N} \\
\vdots \\
\bar{M}
\end{bmatrix} =
\begin{bmatrix}
\bar{A} & \bar{B} \\
\vdots & \vdots \\
\bar{B} & \bar{D}
\end{bmatrix}
\begin{bmatrix}
\bar{\varepsilon}^o \\
\bar{\kappa}
\end{bmatrix},
\tag{A.1}
\]

or

\[
\bar{N} = \bar{A}\bar{\varepsilon}^o + \bar{B}\bar{\kappa},
\tag{A.2}
\]

and

\[
\bar{M} = \bar{B}\bar{\varepsilon}^o + \bar{D}\bar{\kappa},
\tag{A.3}
\]

where \( \bar{N} \) and \( \bar{M} \) are the stress and moment resultants, \( \bar{A}, \bar{B}, \) and \( \bar{D} \) are the extensional, coupling, and bending stiffness matrices, and \( \bar{\varepsilon}^o \) and \( \bar{\kappa} \) are the laminate midplane strains and curvatures. This relationship can be inverted to obtain expressions for \( \bar{\varepsilon}^o \) and \( \bar{\kappa} \).

Inverting equation (A.2) yields

\[
\bar{\varepsilon}^o = \bar{A}^{-1}\bar{N} - \bar{A}^{-1}\bar{B}\bar{\kappa}.
\tag{A.4}
\]

Substituting this result into equation (A.3) gives

\[
\bar{M} = \bar{B}\bar{A}^{-1}\bar{N} - (\bar{B}\bar{A}^{-1}\bar{B} - \bar{D})\bar{\kappa}.
\tag{A.5}
\]
Combining equations (A.3) and (A.5) one finds the partially inverted constitutive relationship

\[
\begin{align*}
\begin{bmatrix} \bar{\varepsilon}^o \\ \dots \\ \bar{\kappa} \end{bmatrix} &= \begin{bmatrix} \tilde{\mathbf{A}}^* & \tilde{\mathbf{B}}^* \\ \cdots & \cdots & \cdots \\ \bar{\mathbf{C}}^* & \bar{\mathbf{D}}^* \end{bmatrix} \begin{bmatrix} \bar{\mathbf{N}} \\ \cdots \\ \bar{\kappa} \end{bmatrix}, \\
\end{align*}
\]  
(A.6)

or

\[
\bar{\varepsilon}^o = \tilde{\mathbf{A}}^*\bar{\mathbf{N}} + \tilde{\mathbf{B}}^*\bar{\kappa},
\]  
(A.7)

and

\[
\bar{\mathbf{M}} = \bar{\mathbf{C}}^*\bar{\mathbf{N}} + \bar{\mathbf{D}}^*\bar{\kappa},
\]  
(A.8)

where

\[
\begin{align*}
\tilde{\mathbf{A}}^* &= \tilde{\mathbf{A}}^{-1} \\
\tilde{\mathbf{B}}^* &= -\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{B}} \\
\bar{\mathbf{C}}^* &= \bar{\mathbf{B}}\tilde{\mathbf{A}}^{-1} = -\bar{\mathbf{B}}^{\top} \\
\bar{\mathbf{D}}^* &= \bar{\mathbf{D}} - \bar{\mathbf{B}}\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{B}}.
\end{align*}
\]  
(A.9)

Inverting equation (A.8) yields

\[
\bar{\kappa} = (\bar{\mathbf{D}}^*)^{-1}\bar{\mathbf{M}} - (\bar{\mathbf{D}}^*)^{-1}\bar{\mathbf{C}}^*\bar{\mathbf{N}}.
\]  
(A.10)

Substituting equation (A.10) into equation (A.7) gives

\[
\bar{\varepsilon}^o = (\tilde{\mathbf{A}}^* - \tilde{\mathbf{B}}^*(\bar{\mathbf{D}}^*)^{-1}\bar{\mathbf{C}}^*)\bar{\mathbf{N}} + \tilde{\mathbf{B}}^*(\bar{\mathbf{D}}^*)^{-1}\bar{\mathbf{N}}.
\]  
(A.11)

Equations (A.10) and (A.11) can be combined and rewritten in matrix form to yield the inverted laminate constitutive relationship in terms of compliance as

\[
\begin{bmatrix} \bar{\varepsilon}^o \\ \cdots \\ \bar{\kappa} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}' & \tilde{\mathbf{B}}' \\ \cdots & \cdots & \cdots \\ \tilde{\mathbf{B}}' & \tilde{\mathbf{D}}' \end{bmatrix} \begin{bmatrix} \bar{\mathbf{N}} \\ \cdots \\ \bar{\mathbf{M}} \end{bmatrix}
\]  
(A.12)
where

\[
\begin{align*}
\tilde{A}' &= \tilde{A}^* + \tilde{B}^* (\tilde{D}^*)^{-1} (\tilde{B}^*)^T \\
\tilde{B}' &= \tilde{B}^* (\tilde{D}^*)^{-1} \quad \text{(A.13)} \\
\tilde{D}' &= (\tilde{D}^*)^{-1}.
\end{align*}
\]
Appendix B

Derivation of Moment Formulae

It is possible to obtain analytical results for a stationary Markov chain relating the parameters of the Markov chain to the moments for the transition times from one state to another in the Markov chain. These results are obtained through the use of geometric transforms of the evolutionary probability mass function (PMF) of a Markov chain. This appendix focuses on the derivation of these results as related to the present work. A rigorous treatment of this topic can be found in Bogdanoff and Kozin (1985).

In much of the literature relating to discrete probability theory, a slight variation of the z-transform of discrete systems theory is employed for the analysis of discrete sequences. This variation is known as the geometric transform. The geometric transform of a discrete sequence $D[i]$ for $i = 0, \pm 1, \pm 2, \ldots$ is defined as (Giffin, 1975)

$$G(D[i]) = G_D(z) = \sum_{i=-\infty}^{\infty} D[i]z^i.$$  \hspace{1cm} (B.1)

If the discrete sequence is the PMF of a random variable $D$ with with a range $R_D = 0, 1, \ldots$ where the PMF of $D$ is given by the equation

$$p[d_i] = P\{D = d_i\} = p_i,$$  \hspace{1cm} (B.2)

the geometric transform of $p[d_i]$ is

$$G(p[d_i]) = G_p(z) = \sum_{i=0}^{\infty} p_i z^i.$$  \hspace{1cm} (B.3)

The geometric transform of the PMF of a random variable is also known as the probability generating function (PGF) of the random variable.
Clearly, $G_p(z)$ is the expected value of $z^D$. That is,

$$G_p(z) = E\{z^D\} = \sum_{i=0}^{\infty} p_i z^i. \tag{B.4}$$

Upon differentiating with respect to $z$ one finds

$$\frac{d^n G_p(z)}{dz^n} = G_p^{(n)}(z) = E\{D(D-1) \cdots (D-[n-1])z^{D-n}\}. \tag{B.5}$$

Evaluating $G_p^{(n)}(z)$ at $z = 1$ gives

$$G_p^{(n)}(z)|_{z=1} = E\{D(D-1) \cdots (D-[n-1])\}. \tag{B.6}$$

Since the expectation operator is linear, equation (B.6) allows for the $n^{th}$ moment of $D$ to be evaluated from the PGF of $D$. Specifically, if

$$D^{<n>} = \frac{n!}{(D-n)!} = D(D-1) \cdots (D-[n-1]), \tag{B.7}$$

equation (B.6) reduces to

$$G_p^{(n)}(z)|_{z=1} = E\{D^{<n>}\}. \tag{B.8}$$

For the case where $n = 1$, equation (B.8) reduces to

$$G_p^{(1)}(z)|_{z=1} = E\{D\} = m_D. \tag{B.9}$$

Similarly for $n = 2$,

$$G_p^{(2)}(z)|_{z=1} = E\{D(D-1)\} = E\{D^2 - D\} = m_{D^2} - m_D, \tag{B.10}$$

and thus

$$m_{D^2} = G_p^{(2)}(z)|_{z=1} + G_p^{(1)}(z)|_{z=1}. \tag{B.11}$$

The variance of $D$ can be obtained by noting that

$$k_{D^2} = E\{(D - m_D)^2\} = m_{D^2} - m_D^2 \tag{B.12}$$

$$= G_p^{(2)}(z)|_{z=1} + G_p^{(1)}(z)|_{z=1} - \left(G_p^{(1)}(z)|_{z=1}\right)^2.$$
To apply these concepts to Markov chains, consider the stationary Markov chain with \( b \) possible states given by the equation

\[
p_x = p_0 \mathbf{P}^x,
\]

where \( p_x \) is the PMF, written as a \((1 \times b)\) row vector, whose components \( p_{jx} \) for \( j = 1, 2, \ldots, b \) represent the probability of being in a particular state \( j \) after \( x \) trials. The PMF \( p_0 \) is similarly defined. The probability transition matrix (PTM) \( \mathbf{P} \) is a \((b \times b)\) matrix whose elements \( p_{ij} \) represent the probabilities of being in state \( j \) after a trial given that state \( i \) was occupied before the trial.

Multiplying equation (B.13) by \( z \) and summing over \( x \) yields

\[
\chi(z) = p_0 \Psi(z),
\]

where \( \chi(z) \) and \( \Psi(z) \) are defined by the equations

\[
\chi(z) = \sum_{x=0}^{\infty} p^x z^x
\]

and

\[
\Psi(z) = \sum_{x=0}^{\infty} \mathbf{P}^x z^x.
\]

A component \( \chi_j \) of \( \chi(z) \) is the generating function of the probability of being in state \( j \). Similarly, a component \( \psi_{ij}(z) \) of \( \Psi(z) \) is the generating function of the probability of being in state \( j \), given that state \( i \) was occupied at \( x = 0 \). The relationship between these two probability generating functions is

\[
\chi_j = \sum_{i=1}^{b} p_{0i} \psi_{ij}(z).
\]

Utilizing a Neumann series, \( \Psi(z) \) in equation (B.16) can be written as

\[
\Psi(z) = \sum_{x=0}^{\infty} \mathbf{P}^x z^x = (\mathbf{I} - \mathbf{P} z)^{-1},
\]
where $\mathbf{I}$ is the $(b \times b)$ identity matrix. Thus, $\psi_{ij}(z)$ can be written as

$$
\psi_{ij}(z) = (-1)^{i+j} \frac{|m_{ji}|}{|\mathbf{I} - \mathbf{P}z|},
$$

where $|\mathbf{I} - \mathbf{P}z|$ is the determinant of the matrix $(\mathbf{I} - \mathbf{P}z)$ and where $|m_{ji}|$ represents the determinant of the $(ji)$th minor of $(\mathbf{I} - \mathbf{P}z)$ which is the $((b - 1) \times (b - 1))$ determinant found by removing the $j$th row and $i$th column of $|\mathbf{I} - \mathbf{P}z|$.

If $\mathbf{P}$ is a unit-jump PTM with only nonzero elements $p_i$ and $q_i$ on its main diagonal and super diagonal respectively, where $p_i > 0$, $q_i > 0$, and $p_i + q_i = 1$, $(\mathbf{I} - \mathbf{P}z)$ has the form

$$
\mathbf{I} - \mathbf{P}z =
\begin{bmatrix}
1 - p_1 z & -q_1 z & 0 & \ldots & \ldots & 0 \\
0 & 1 - p_2 z & -q_2 z & 0 & \ldots & \\
0 & 0 & \ddots & \ddots & & \\
\vdots & & \ddots & \ddots & \ddots & \\
0 & & & \ldots & 1 - p_{b-1} z & -q_{b-1} z \\
0 & \ldots & \ldots & \ldots & 0 & 1 - z
\end{bmatrix}.
$$

(B.20)

It is readily seen that

$$
\psi_{ij}(z) = 0 \quad \text{for} \quad i > j
$$

and that the determinant $|\mathbf{I} - \mathbf{P}z|$ is simply the product of the elements of the main diagonal. That is,

$$
|\mathbf{I} - \mathbf{P}z| = (1 - p_1 z)(1 - p_2 z) \cdots (1 - p_{b-1} z)(1 - z).
$$

(B.22)

Interest is focused on determining the moments of the transition time from any given state $a$ to the final state $b$. To determine these moments, the PMF for the transition time from one state $a$ to state $b$ must be known. Evaluating equation (B.19) for $i = a$ and $j = b$ yields

$$
\psi_{ab}(z) = \frac{q_a \cdots q_{b-1} z^{b-a}}{(1 - p_a z) \cdots (1 - p_{b-1} z)(1 - z)} \quad \text{for} \quad a = 1, 2, \ldots, b - 1.
$$

(B.23)
Note however, that although equation (B.23) is the generating function of the probability of being in state $b$ given that state $a$ was occupied at $x = 0$, it does not generate the PMF of the transition time from state $a$ to state $b$. This follows from the fact that the PGF of a random variable, as defined in equation (B.3), must equal unity for $z = 1$.

Equation (B.23) can be used to determine the PMF of the transition time from state $a$ to state $b$ by defining a new generating function $\phi_{ab}(z)$ as

$$\phi_{ab}(z) = (1 - z)\psi_{ab}(z).$$  \hspace{1cm} \text{(B.24)}

Utilizing equations (B.24) and (B.23), $\phi_{ab}(z)$ can be rewritten as

$$\phi_{ab}(z) = \frac{q_a \cdots q_{b-1} z^{b-y}}{(1 - p_a z) \cdots (1 - p_{b-1} z)} \quad \text{for} \quad a = 1, 2, \ldots, b - 1. \hspace{1cm} \text{(B.25)}$$

Clearly, $\phi_{ab}(z)|_{z=1} = 1$ and thus $\phi_{ab}(z)$ is a PGF of the transition time from state $a$ to state $b$. As such, the relationship between the moments of the transition time from state $a$ to state $b$ and the parameters of a Markov chain can be readily determined by substituting equation (B.25) into equation (B.8). Upon simplification, this yields the following results for the mean and variance of the transition time, $X_{ab}$, from state $a$ to state $b$

$$m_{X_{ab}} = \sum_{i=a}^{b-1} (1 + r_i) \hspace{1cm} \text{(B.26)}$$

$$k_{X_{ab}^2} = \sum_{i=a}^{b-1} r_i (1 + r_i),$$

where $r_i = p_i / q_i$. 