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RICE UNIVERSITY
ON THE BOUNDARIES OF SPECIAL
LAGRANGIAN SUBMANIFOLDS

by

Lei Fu

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ABSTRACT

On The Boundaries of Special Lagrangian Submanifolds

by

Lei Fu

An \( n \)-dimensional submanifold \( M \) in \( \mathbb{C}^n = \mathbb{R}^{2n} \) is called Lagrangian if the restriction of \( \omega \) to \( M \) is zero, where \( \omega = \sum_i dz_i \wedge d\bar{z}_i \). It is called special Lagrangian if the restrictions of \( \omega \) and \( \text{Im}dz = \text{Im}(dz_1 \wedge \cdots \wedge dz_n) \) are zero. Special Lagrangian submanifolds are volume minimizing, and conversely, any minimal Lagrangian submanifold can be transformed to a special Lagrangian submanifold by a unitary transformation.

This paper studies the conditions satisfied by the boundaries of special Lagrangian submanifolds. We say an \( n-1 \) dimensional submanifold \( N \) in \( \mathbb{C}^n \) satisfies the moment condition if \( \int_N \phi = 0 \) for any \( n-1 \) form \( \phi \) with \( d\phi \) belonging to the differential ideal generated by \( \omega \) and \( \text{Im}dz \). By Stokes' formula, the boundary of a special Lagrangian submanifold satisfies the moment condition. In fact on \( \mathbb{C}^2 \), the converse is also true, that is, a curve is the boundary of a special Lagrangian submanifold if it satisfies the moment condition. However, we show in this paper that for \( n \geq 3 \), there exist \( n-1 \) dimensional submanifolds which satisfy the moment condition but do not bound any special Lagrangian submanifolds.

Let \( f \) be a real valued function defined on some open subset on \( \mathbb{C}^n \) which contains a special Lagrangian submanifold \( M \). We show that

\[
\Delta_M f = d(J(\nabla f) \| \text{Im}dz)|_M,
\]

where \( \Delta_M \) is the Laplace operator on \( M \) and \( J \) is the standard complex structure. Using this formula and the maximum principle, we prove that if \( d(J(\nabla f) \| \text{Im}dz) \) belongs to the differential ideal generated by \( df, \omega \) and \( \text{Im}dz \), then \( f|_M \) attains its maximum and minimum on the boundary of \( M \). This result is then used to produce some conditions other than the moment condition which are satisfied by the boundaries of special Lagrangian submanifolds.
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Introduction

In [HL1], Harvey and Lawson prove that Redz \(dz = dz_1 \wedge \cdots \wedge dz_n\) is a calibration on \(C^n = R^{2n}\), that is, it is a closed form of comass 1, where \(z = (z_1, \ldots, z_n)\) are the coordinates on \(C^n\). Let \(P\) be an \(n\) dimensional oriented plane in \(R^{2n}\) with an oriented orthonormal basis \(\{e_1, \ldots, e_n\}\). Then \(\zeta_P = e_1 \wedge \cdots \wedge e_n \in \wedge^n R^{2n}\) doesn't depend on the choice of the basis. We say \(P\) is special Lagrangian if Redz(\(\zeta_P\)) = 1.

An \(n\) dimensional oriented submanifold \(M\) of \(C^n\) is called special Lagrangian if each of its oriented tangent planes is special Lagrangian, or equivalently, if its volume form is given by the restriction of the ambient form Redz. Special Lagrangian submanifolds are volume minimizing. One can show that an orientable submanifold \(M\) is special Lagrangian (with respect to a proper orientation) if and only if the restrictions of \(\omega\) and Imdz are zero, where \(\omega\) is the standard Kähler/symplectic form on \(C^n = R^{2n}\). In particular special Lagrangian submanifolds are Lagrangian, (i.e. the restriction of \(\omega\) to \(M\) is zero). A Lagrangian submanifold is minimal if and only if it can be transformed to a special Lagrangian submanifold by a unitary transformation. Consequently the study of minimal Lagrangian submanifolds is reduced to the study of special Lagrangian submanifolds. See [HL1] or the preliminaries of this paper for details.

In this paper we study the conditions satisfied by the boundaries of special Lagrangian submanifolds. There is a natural necessary condition for a submanifold to be the boundary of a special Lagrangian submanifold, which we will call the moment condition. Denote by \(I\) the differential ideal generated by \(\omega\) and Imdz. An \(n - 1\) dimensional submanifold \(N\) of \(C^n\) satisfies the moment condition if for any smooth \(n - 1\) form \(\phi\) with \(d\phi \in I\), we have \(\int_N \phi = 0\). By Stokes' formula, the boundary of a special Lagrangian submanifold satisfies the moment condition.

On \(C^2\) special Lagrangian geometry is equivalent to the complex geometry. (See Remark 0.8 in the preliminaries of this paper.) The results in [HL2] then show that the moment condition is also sufficient for a one dimensional submanifold of \(C^2\) to bound a special Lagrangian submanifold (probably with singularities). However, this is no longer true already for \(C^3\). In Section 1 of this paper, we prove that when \(n \geq 3\), there exist forms \(\phi_1, \ldots, \phi_{n(n+2)}\) on \(C^n\) such that a closed
submanifold $N$ of $\mathbb{C}^n$ satisfies the moment condition if and only if $N$ is isotropic and $\int_N \phi_k = 0$ ($k = 1, \ldots, n(n+2)$). Based on this result, in Section 2 we construct a two dimensional submanifold of $\mathbb{C}^3$ which satisfies the moment condition but does not bound any special Lagrangian submanifold. Actually, we prove that it does not bound any special Lagrangian rectifiable current.

Let $f$ be a function on $\mathbb{C}^n$ and $M$ be a special Lagrangian submanifold of $\mathbb{C}^n$. In Section 3, we find a useful new formula for $\Delta_M f$, where $\Delta_M$ is the Laplace operator on $M$ with respect to the induced metric. Using this formula we prove that if $f$ is a harmonic Hermitian polynomial, that is

$$f = c + \sum_{k=1}^n (b_k z_k + \bar{b}_k \bar{z}_k) + \sum_{k, l=1}^n a_{kl} z_k \bar{z}_l$$

for some real number $c$, complex numbers $b_k$ and $a_{kl}$ with $\bar{a}_{kl} = a_{lk}$ and $\sum_{k=1}^n a_{kk} = 0$, then the restriction of $f$ to every special Lagrangian submanifold is harmonic. The converse is also true when $n \geq 3$. In addition, the formula for $\Delta_M f$ combined with the maximum principle is used to prove that if $d(J(\nabla f) \perp \text{Im}dz)$ belongs to the differential ideal generated by $df$, $\omega$ and $\text{Im}dz$, (where $J$ is the standard complex structure on $\mathbb{C}^n$), then $f|_M$ attains its maximum and minimum on the boundary of $M$. In particular, if $f$ is constant on the boundary of $M$, then $f$ is also constant on the interior of $M$. Using this result, we find some conditions on the boundary of a special Lagrangian submanifold which do not come from the moment condition. There are many functions $f$ such that $d(J(\nabla f) \perp \text{Im}dz)$ belongs to the differential ideal generated by $df$, $\omega$ and $\text{Im}dz$. Some examples of such functions are given in the same section.

In section 4 we prove that a function $f$ is harmonic on every minimal Lagrangian submanifold if and only if $f$ is a harmonic Hermitian polynomial. Hence there is no difference between functions which are harmonic on every minimal Lagrangian submanifold and functions which are harmonic on every special Lagrangian submanifold when $n \geq 3$. But this is not true when $n = 2$.

Recall that if a Lagrangian submanifold $M$ is contained in a hypersurface defined by $f = \text{const}$, then each integral line of the vector field $J(\nabla f)$ which passes through some point on $M$ must be contained in $M$. In particular if we assume
$J(\nabla f)$ is not tangent to $\partial M$, then we can locally recover $M$ from $\partial M$ by taking the union of all the integral lines of $J(\nabla f)$ passing through some point on $\partial M$.

Let $\Gamma$ be an $n - 1$ dimensional submanifold in $\mathbb{C}^n$. Let $\Sigma$ be the union of all the integral lines of $J(\nabla f)$ passing through some point on $\Gamma$. In Section 5 we find necessary and sufficient conditions on $\Gamma$ for $\Sigma$ to be a special Lagrangian submanifold.

In Section 7, we study the boundary value of a harmonic gradient. In the appendix we prove some algebraic results needed in this paper.

We can study special Lagrangian geometry in the category of rectifiable currents. The reader will find the following discussion unnecessary in this paper except in Section 2 and Section 6.

Let $T$ be a rectifiable current of dimension $n$ in $\mathbb{C}^n$. Denote by $\|T\|$ its volume measure, and by $\bar{T}(x)$ its oriented tangent $n$-plane, which is defined for $\|T\|$-a.e. point $x$. We say $T$ is a special Lagrangian rectifiable current if $\partial T$ is rectifiable and for $\|T\|$-a.e. point $x$, we have $\text{Red}_z(\bar{T}(x)) = 1$. We say $T$ is a general special Lagrangian rectifiable current if $\partial T$ is rectifiable and for $\|T\|$-a.e point $x$, we have $\text{Red}_z(\bar{T}(x)) = \pm 1$. Special Lagrangian rectifiable currents are volume minimizing. An $n-1$ dimensional rectifiable current $S$ satisfies the moment condition if $S(\phi) = 0$ for all $n-1$ forms $\phi$ such that $d\phi$ belongs to the differential system $I$ generated by $\omega$ and $\text{Im}dz$.

In [BG], Bryant and Griffiths study the characteristic cohomology of differential systems. They ask whether the moment condition is isomorphic to the conservation law. (And they think the answer is no.) In special Lagrangian geometry, stated in our language, the question is whether the moment condition is enough for a rectifiable current to bound a general special Lagrangian rectifiable current. The submanifold constructed in Section 2 shows that the moment condition is not enough for a rectifiable current to bound a special Lagrangian rectifiable current.

We can also consider a special Lagrangian current (not necessarily rectifiable). We say a current $T$ of dimension $n$ is special Lagrangian if $T(\psi) = 0$ for all forms $\psi$ of degree $n$ which belong to the differential ideal generated by $\omega$ and $\text{Im}dz$. In Section 6, we show that an $n-1$ dimensional current is the boundary of a special
Lagrangian current if and only if it satisfies the moment condition. (Actually we prove this is true for any current defined by a differential system generated by forms with constant coefficients.) So this weak version of the boundary problem is quite different from the geometric one studied in this paper.

0. Preliminaries

In this section, we present some fundamental results on special Lagrangian geometry. They are all due to R. Harvey and H. B. Lawson and can be found in [HL1].

Let \( \omega = \frac{1}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k = \sum_{k=1}^{n} dx_k \wedge dy_k \) be the standard Kähler/symplectic form on \( \mathbb{C}^n \). Recall that a real \( n \) dimensional submanifold in \( \mathbb{C}^n \) is called a Lagrangian submanifold if the restriction of \( \omega \) to it vanishes. This is equivalent to saying that \( J(TM) = (TM)^\perp \), where \( J \) is the standard complex structure on \( \mathbb{C}^n \).

**Lemma 0.1.** If an oriented Lagrangian submanifold \( M \) is minimal, then the volume form of \( M \) is given by the restriction of \( \text{Re}(e^{i\theta} dz_1 \wedge \cdots \wedge dz_n) \) for some constant \( \theta \).

**Proof.** Let \( \{e_1, \ldots, e_n\} \) be an orthonormal frame for \( M \) near a point \( x \in M \). We extend \( \{e_1, \ldots, e_n\} \) so that it is defined in some neighbourhood of \( x \) in \( \mathbb{C}^n \). Since \( M \) is Lagrangian, \( \{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \) is an orthonormal basis for \( T_y \mathbb{C}^n \) for \( y \in M \) near \( x \). Let \( \{e_1, \ldots, e_n, J\epsilon_1, \ldots, J\epsilon_n\} \) be its dual basis for \( T^*_y \mathbb{C}^n \). Then

\[
 dz_1 \wedge \cdots \wedge dz_n = e^{i\theta}(e_1 + iJ\epsilon_1) \wedge \cdots \wedge (e_n + iJ\epsilon_n),
\]

where \( \theta \) is defined by

\[
 (dz_1 \wedge \cdots \wedge dz_n)(e_1 \wedge \cdots \wedge e_n) = e^{i\theta}.
\]

Denote by \( \nabla \) the standard Levi-Civita connection on \( \mathbb{C}^n \). Then for any tangent
vector $v$, we have
\begin{align*}
v(e^{i\theta}) &= \sum_k (dz_1 \wedge \cdots \wedge dz_n)(e_1 \wedge \cdots \wedge \nabla_v e_k \wedge \cdots \wedge e_n) \\
&= \sum_k (e^{i\theta}(e_1 + iJe_1) \wedge \cdots \wedge (e_n + iJe_n))(e_1 \wedge \cdots \wedge \nabla_v e_k \wedge \cdots \wedge e_n) \\
&= \sum_k e^{i\theta}(e_k + iJe_k)(\nabla_v e_k) \\
&= \sum_k e^{i\theta}((-e_k, \nabla_v e_k) + i(Je_k, \nabla_v e_k)) \\
&= \sum_k i e^{i\theta}(Je_k, \nabla_v e_k),
\end{align*}
where the last equality follows from the fact that $(e_k, \nabla_v e_k) = \frac{1}{2} \nabla_v (e_k, e_k) = 0$. We also have
\begin{align*}
v(e^{i\theta}) &= ie^{i\theta}v(\theta).
\end{align*}

Hence
\begin{align*}
v(\theta) &= \sum_k (Je_k, \nabla_v e_k).
\end{align*}

So
\begin{align*}
v(\theta) &= \sum_k (Je_k, \nabla_v e_k + [v, e_k]) = \sum_k (Je_k, \nabla_v e_k v) \\
&= -\sum_k (e_k, \nabla_v (Jv)) = -(H, Jv),
\end{align*}
where the second equality follows from the fact that $[v, e_k]$ is in $TM$ and $Je_k$ is in $(TM)^\perp$. Since $M$ is minimal, we have $H = 0$. Hence $v(\theta) = 0$. Therefore $\theta$ is constant on $M$. Since $(dz_1 \wedge \cdots \wedge dz_n)(e_1 \wedge \cdots \wedge e_n) = e^{i\theta}$, we have $\text{Re}(e^{-i\theta}dz_1 \wedge \cdots \wedge dz_n)(e_1 \wedge \cdots \wedge e_n) = 1$. Therefore $\text{Re}(e^{-i\theta}dz_1 \wedge \cdots \wedge dz_n)$ is the volume form of $M$.

For convenience, we write $dz = dz_1 \wedge \cdots \wedge dz_n$.

**Lemma 0.2.** For any vectors $v_1, \ldots, v_n \in \mathbb{C}^n$, we have
\begin{align*}
|dz(v_1 \wedge \cdots \wedge v_n)|^2 &= (\text{Re}(e^{i\theta}dz)(v_1 \wedge \cdots \wedge v_n))^2 + (\text{Im}(e^{i\theta}dz)(v_1 \wedge \cdots \wedge v_n))^2 \\
&= |v_1 \wedge \cdots \wedge v_n \wedge Jv_1 \wedge \cdots \wedge Jv_n|.
\end{align*}
Proof. Let \( v_i = (a_{i_1}, \ldots, a_{i_n}) \in \mathbb{C}^n \) and \( a_{ij} = \alpha_{ij} + i\beta_{ij} \) for some real numbers \( \alpha_{ij} \) and \( \beta_{ij} \). Denote \( A = (\alpha_{ij}), B = (\beta_{ij}) \). Then

\[
|dz(v_1 \wedge \cdots \wedge v_n)|^2 = |\det(A + iB)|^2 = \det(A + iB)\det(A - iB).
\]

We have

\[
\det(A + iB)\det(A - iB) = \det\begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix} = \det\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.
\]

Moreover

\[
v_1 \wedge \cdots \wedge v_n \wedge Jv_1 \wedge \cdots Jv_n = \det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} (e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n),
\]

where \( e_i \) is the vector in \( \mathbb{C}^n \) whose \( j \)-th component is \( \delta_{ij} \). Hence

\[
|v_1 \wedge \cdots \wedge v_n \wedge Jv_1 \wedge \cdots Jv_n| = \det\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.
\]

So

\[
|dz(v_1 \wedge \cdots \wedge v_n)|^2 = |v_1 \wedge \cdots \wedge v_n \wedge Jv_1 \wedge \cdots Jv_n|.
\]

Lemma 0.3. (Hadamard) Let \( V \) be a real inner product space and \( v_1, \ldots, v_n \) be vectors in \( V \). Then for any \( 1 \leq k \leq n - 1 \), we have

\[
|v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots v_n| \leq |v_1 \wedge \cdots \wedge v_k||v_{k+1} \wedge \cdots v_n|.
\]

The equality holds if and only if either \( v_1, \ldots, v_n \) are linearly independent and \( v_1, \ldots, v_k \in \text{span}\{v_{k+1}, \ldots, v_n\}^\perp \) or one of the families \( v_1, \ldots, v_k \) and \( v_{k+1}, \ldots, v_n \) is linearly dependent.

Proof. We only consider the case of \( \{v_1, \ldots, v_n\} \) being linearly independent. (The case of one of the families \( \{v_1, \ldots, v_n\} \) and \( \{v_{k+1}, \ldots, v_n\} \) being linearly dependent is straightforward.)

Let \( v_i = v_i' + v_i^\perp \), where \( v_i' \in \text{span}\{v_{i+1}, \ldots, v_n\} \) and \( v_i^\perp \in \text{span}\{v_{i+1}, \ldots, v_n\}^\perp \).

Then

\[
v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots v_n = v_1^\perp \wedge \cdots \wedge v_k^\perp \wedge v_{k+1} \wedge \cdots \wedge v_n.
\]
Obviously we have

\[ |v_1^\perp \wedge \cdots \wedge v_k^\perp \wedge v_{k+1} \wedge \cdots \wedge v_n| = |v_1^\perp \wedge \cdots \wedge v_k^\perp| |v_{k+1} \wedge \cdots \wedge v_n|. \]

Hence

\[ |v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_n| = |v_1^\perp \wedge \cdots \wedge v_k^\perp| |v_{k+1} \wedge \cdots \wedge v_n|. \]

To prove the lemma, it’s enough to show that

\[ |v_1^\perp \wedge \cdots \wedge v_k^\perp| \leq |v_1 \wedge \cdots \wedge v_k| \]

and that the equality holds if and only if \( v_1, \ldots, v_k \in \text{span}\{v_{k+1}, \ldots, v_n\}^\perp \).

For convenience let’s take \( k = 4 \). The proof for the general case is completely the same. Then

\[
\begin{align*}
v_1 \wedge v_2 \wedge v_3 \wedge v_4 &= v_1^\perp \wedge v_2 \wedge v_3 \wedge v_4 + v_1^\perp \wedge v_2^\perp \wedge v_3 \wedge v_4 + v_1^\perp \wedge v_2 \wedge v_3^\perp \wedge v_4 + v_2^\perp \wedge v_3 \wedge v_4^\perp \wedge v_4 \\
&+ v_1^\perp \wedge v_2^\perp \wedge v_3^\perp \wedge v_4^\perp.
\end{align*}
\]

Obviously the terms in the sum are perpendicular to each other. Hence

\[ |v_1^\perp \wedge v_2^\perp \wedge v_3^\perp \wedge v_4^\perp| \leq |v_1 \wedge v_2 \wedge v_3 \wedge v_4|. \]

If the equality holds, then

\[
v_1^\perp \wedge v_2 \wedge v_3 \wedge v_4 = v_1^\perp \wedge v_2^\perp \wedge v_3 \wedge v_4 = v_1^\perp \wedge v_2^\perp \wedge v_3 \wedge v_4 = v_1^\perp \wedge v_2^\perp \wedge v_3^\perp \wedge v_4^\perp = 0.
\]

Since \( v_1, \ldots, v_n \) are linearly independent, we have \( v_i^\perp \neq 0 \) for \( i = 1, \ldots, n-1 \). Moreover \( v_1^\perp, v_2^\perp, v_3^\perp, v_4^\perp \) are perpendicular to each other. So from \( v_1^\perp \wedge v_2^\perp \wedge v_3^\perp \wedge v_4^\perp = 0 \) we get \( v_4^\perp = 0 \). Hence \( v_4 \in \text{span}\{v_5, \ldots, v_n\}^\perp \). Since \( v_1^\perp \wedge v_2^\perp \wedge v_3 \wedge v_4 = 0 \), \( v_1^\perp \) and \( v_2^\perp \) are perpendicular to \( v_3 \) and \( v_4 \), and \( v_1^\perp \wedge v_2^\perp \neq 0 \), we have \( v_3 \wedge v_4 = 0 \). Hence \( v_3^\perp \in \text{span}\{v_4\} \). So \( v_3 \in \text{span}\{v_5, \ldots, v_n\} \). Similarly from \( v_1^\perp \wedge v_2^\perp \wedge v_3 \wedge v_4 = 0 \) we can get \( v_2^\perp \wedge v_3 \wedge v_4 = 0 \). So \( v_2^\perp \in \text{span}\{v_3, v_4\} \). Therefore \( v_2 \in \text{span}\{v_5, \ldots, v_n\} \). Finally from \( v_1^\perp \wedge v_2 \wedge v_3 \wedge v_4 = 0 \) we get \( v_1^\perp \in \text{span}\{v_2, v_3, v_4\} \). So \( v_1 \in \text{span}\{v_5, \ldots, v_n\} \).

Reverse the argument above, we can show that if \( v_1, v_2, v_3, v_4 \in \text{span}\{v_5, \ldots, v_n\} \), then

\[ |v_1^\perp \wedge v_2^\perp \wedge v_3^\perp \wedge v_4^\perp| = |v_1 \wedge v_2 \wedge v_3 \wedge v_4|. \]
Lemma 0.4. Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{C}^n$. Then

$$|\text{Re}(e^{i\theta} dz)(v_1 \wedge \cdots \wedge v_n)| \leq |v_1 \wedge \cdots \wedge v_n|. $$

The equality holds if and only if $\text{Im}(e^{i\theta} dz)(v_1 \wedge \cdots \wedge v_n) = 0$ and $Jv_1, \ldots, Jv_n \in \text{span}\{v_1, \ldots, v_n\}^\perp$.

Proof. By Lemma 0.2 and Lemma 0.3, we have

$$\left(\text{Re}(e^{i\theta} dz)(v_1 \wedge \cdots \wedge v_n)\right)^2 + \left(\text{Im}(e^{i\theta} dz)(v_1 \wedge \cdots \wedge v_n)\right)^2 = |v_1 \wedge \cdots \wedge v_n \wedge Jv_1 \wedge \cdots \wedge Jv_n| \leq |v_1 \wedge \cdots \wedge v_n||Jv_1 \wedge \cdots \wedge Jv_n| = |v_1 \wedge \cdots \wedge v_n|^2$$

and the equality holds if and only if $Jv_1, \ldots, Jv_n \in \text{span}\{v_1, \ldots, v_n\}^\perp$. The lemma follows immediately.

Theorem 0.5. Let $M$ be an $n$-dimensional submanifold in $\mathbb{C}^n$. The following statements are equivalent.

1. $M$ is a homologically volume minimizing Lagrangian submanifold.
2. $M$ is a minimal Lagrangian submanifold.
3. There exists a constant $\theta$ such that the volume form of $M$ is given by the restriction of $\text{Re}(e^{i\theta} dz)$.
4. $M$ is a Lagrangian submanifold and $\text{Im}(e^{i\theta} dz)|_M = 0$ for some constant $\theta$.

Proof. (1)$\implies$(2) is classical. (2)$\implies$(3) follows from Lemma 0.1.

(3)$\iff$(4): Let $\{v_1, \ldots, v_n\}$ be a basis for the tangent space of $M$ at an arbitrary point. Then $\text{Re}(e^{i\theta} dz)$ is the volume form if and only if $\text{Re}(e^{i\theta})(v_1 \wedge \cdots \wedge v_n) = |v_1 \wedge \cdots \wedge v_n|$. By Lemma 0.4, this is equivalent to saying

$$\text{Im}(e^{i\theta} dz)(v_1 \wedge \cdots \wedge v_n) = 0 \text{ and } Jv_1, \ldots, Jv_n \in \text{span}\{v_1, \ldots, v_n\}^\perp,$$

that is $\text{Im}(e^{i\theta} dz)|_M = 0$ and $M$ is Lagrangian.

(3)$\implies$(1): Obviously $\text{Re}(e^{i\theta} dz)$ is a closed form. So there exists a form $\phi$ such that $d\phi = \text{Re}(e^{i\theta} dz)$. Let $M'$ be an $n$-dimensional submanifold with $\partial M = \partial M'$. Then

$$\int_M \text{vol}_M = \int_M \text{Re}(e^{i\theta} dz) = \int_{\partial M} \phi = \int_{\partial M'} \phi = \int_{M'} \text{Re}(e^{i\theta} dz) \leq \int_{M'} \text{vol}_{M'},$$
where the last inequality follows from Lemma 0.4. So $M$ is homologically volume minimizing.

**Definition 0.6.** An oriented $n$-dimensional submanifold $M$ in $\mathbb{C}^n$ is called a special Lagrangian submanifold if it satisfies one of the following equivalent conditions:

1. The volume form of $M$ is given by the restriction of $\text{Re}dz$.
2. $M$ is a Lagrangian submanifold and $\text{Im}dz|_M = 0$.

**Remark 0.7.** On $\mathbb{C}$ special Lagrangian submanifolds are lines parallel to the $x$-axis and their geometry is trivial. Therefore throughout this paper, we only consider $\mathbb{C}^n$ for $n \geq 2$.

**Remark 0.8.** To understand the special Lagrangian geometry on $\mathbb{C}^2$, introduce another copy of $\mathbb{C}^2$, denoted by $\mathbb{C}'^2$, and let $(w_1, w_2)$ be its coordinates. Define a diffeomorphism $F : \mathbb{C}^2 \to \mathbb{C}'^2$ by $F(z_1, z_2) = (w_1, w_2)$, where $w_1 = x_1 + i x_2, w_2 = y_2 + i y_1$, and $x_k, y_k$ are determined by $z_k = x_k + i y_k$ ($k = 1, 2$). It is easy to see that under this diffeomorphism $\text{Re}dz_1 \wedge dz_2$ corresponds to $\frac{i}{2}(dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2)$. That is, the special Lagrangian calibration $\text{Re}dz$ on $\mathbb{C}^2$ corresponds to the Kähler calibration on $\mathbb{C}^2$. So a submanifold of $\mathbb{C}^2$ is special Lagrangian if and only if it corresponds to a complex submanifold of $\mathbb{C}^2$. Therefore the special Lagrangian geometry on $\mathbb{C}^2$ is the same as the complex geometry on $\mathbb{C}^2$.

Let $\tilde{J}$ be the complex structure on $\mathbb{C}^2$ defined by pulling back the complex structure on $\mathbb{C}'^2$, that is

\[ \tilde{J}(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}, \quad \tilde{J}(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_1}, \quad \tilde{J}(\frac{\partial}{\partial y_1}) = \frac{\partial}{\partial y_2}, \quad \tilde{J}(\frac{\partial}{\partial y_2}) = -\frac{\partial}{\partial y_1}. \]

Then we can also say that the special Lagrangian geometry on $\mathbb{C}^2$ is equivalent to complex geometry on $\mathbb{C}^2$ with respect to the complex structure $\tilde{J}$.

**Example 0.9.** Let $u(x_1, x_2)$ and $h(x_1, x_2)$ be two real valued functions satisfying

\[
(1 + u_2^2)u_{11} - 2u_1 u_2 u_{12} + (1 + u_1^2)u_{22} = 0,
\]

\[
(1 + u_2^2)h_{11} - 2u_1 u_2 h_{12} + (1 + u_1^2)h_{22} = 0.
\]

Then the graph of the gradient of $h(x_1, x_2) + x_3 u(x_1, x_2)$ is a special Lagrangian submanifold in $\mathbb{C}^3$. 
Examples 0.10. Let \( v_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{C}^n \), \( i = 1, \ldots, n \) be \( n \) vectors in \( \mathbb{C}^n \). Then \( \text{span}\{v_1, \ldots, v_n\} \) is a Lagrangian submanifold if and only if the matrix \( (a_{ij}) \in \text{U}(n) \). It is a special Lagrangian submanifold if and only if \( (a_{ij}) \in \text{SU}(n) \).

1. The Moment Condition

We work on \( \mathbb{C}^n = \mathbb{R}^{2n} \) for \( n \geq 2 \). (The special Lagrangian geometry on \( \mathbb{C} \) is trivial.) We show that when \( n \geq 3 \) an \( n - 1 \) dimensional closed submanifold \( N \) of \( \mathbb{C}^n = \mathbb{R}^{2n} \) satisfies the moment condition if and only if \( N \) is isotropic and \( \int_M \phi_k = 0 \) (\( k = 1, 2, \ldots, n(n + 2) \)), where \( \phi_k \) are some \( n - 1 \) forms which will be described later.

Define

\[
V = \{ f \mid df = f \text{Im}dz + \omega \wedge \alpha \text{ for some } n - 2 \text{ form } \alpha \text{ and } n - 1 \text{ form } \phi \}.
\]

Let \( \mathcal{I} \) be the differential ideal generated by \( \omega \) and \( \text{Im}dz \), and let \( \mathcal{E} \) be the differential complex of smooth differential forms. Denote

\[
\{ \phi \mid df \in \mathcal{I}^n \}/(d\mathcal{E}^{n-2} + \mathcal{I}^{n-1})
\]

by \( H^{n-1}(\mathcal{E}/\mathcal{I}) \) because it is the \( n - 1 \) dimensional cohomology group of the quotient complex \( \mathcal{E}/\mathcal{I} \). Note that an \( n - 1 \) dimensional oriented submanifold \( N \) satisfies the moment condition if and only if \( \int_N \phi = 0 \) for any differential form \( \phi \) with \( [\phi] \in H^{n-1}(\mathcal{E}/\mathcal{I}) \).

Proposition 1.1. The vector space \( V \) is isomorphic to \( H^{n-1}(\mathcal{E}/\mathcal{I}) \).

Proof. For any differential form \( \phi \) such that \( [\phi] \in H^{n-1}(\mathcal{E}/\mathcal{I}) \), we have \( df = f \text{Im}dz + \omega \wedge \alpha \) for some function \( f \) and \( n - 2 \) form \( \alpha \). Define a map \( T \) by sending \( [\phi] \) to \( f \). Obviously \( T \) is a well-defined linear map from \( H^{n-1}(\mathcal{E}/\mathcal{I}) \) to \( V \). By the definition of \( V \), \( T \) is onto. To show it is one to one, assume \( T([\phi]) = 0 \), i.e. \( df = \omega \wedge \alpha \) for some \( \alpha \), then \( \omega \wedge da = 0 \). Since the map \( L : \gamma \mapsto \omega \wedge \gamma \) is injective on \( \wedge^{n-1}T^*\mathbb{R}^{2n} \), we must have \( da = 0 \). So \( \alpha = db \) for some form \( b \). Then we have
\[ d\phi = d(\omega \wedge \theta). \] So \[ \phi = \omega \wedge \theta + d\rho \] for some form \( \rho \), i.e. \( [\phi] = 0 \) in \( H^{n-1}(\mathcal{E}/\mathcal{I}) \). So \( T \) is one to one.

**Lemma 1.2.** \( df \wedge \text{Im}dz = \omega \wedge (J(\nabla f) \hookrightarrow \text{Im}dz) \), where \( J \) is the standard complex structure.

**Proof.** We have

\[
\omega \wedge \text{Im}dz = \left( \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k \right) \wedge \frac{1}{2i} (dz_1 \wedge \cdots \wedge dz_n - d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n) = 0.
\]

So \( J(\nabla f) \hookrightarrow (\omega \wedge \text{Im}dz) = 0 \). That is

\[
(J(\nabla f) \hookrightarrow \omega) \wedge \text{Im}dz + \omega \wedge (J(\nabla f) \hookrightarrow \text{Im}dz) = 0.
\]

It is easy to see that \( J(\nabla f) \hookrightarrow \omega = -df \), so we get

\[-df \wedge \text{Im}dz + \omega \wedge (J(\nabla f) \hookrightarrow \text{Im}dz) = 0.
\]

This completes the proof.

**Lemma 1.3.** \( f \in V \) if and only if \( d(J(\nabla f) \hookrightarrow \text{Im}dz) = 0 \).

**Proof.** If \( f \in V \), then there are forms \( \phi \) and \( \alpha \) such that

\[ d\phi = f\text{Im}dz + \omega \wedge \alpha. \]

Therefore

\[ df \wedge \text{Im}dz + \omega \wedge d\alpha = 0. \]

So we have

\[ \omega \wedge ((J(\nabla f) \hookrightarrow \text{Im}dz) + d\alpha) = 0 \]

by Lemma 1.2. Since the map \( L : \gamma \mapsto \omega \wedge \gamma \) is injective on \( \wedge^{n-1}T^*\mathbb{R}^{2n} \), we must have

\[ (J(\nabla f) \hookrightarrow \text{Im}dz) + d\alpha = 0. \]

Therefore \( d(J(\nabla f) \hookrightarrow \text{Im}dz) = 0 \).

The reverse of this argument proves the converse.
Lemma 1.4. For $n \geq 3$, we have $d(J(\nabla f) \nmid \text{Im}dz) = 0$ if and only if

$$\frac{\partial^2 f}{\partial z_k \partial z_l} = 0, \quad \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} = 0 \quad (k, l = 1, 2, \ldots, n), \quad \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} = 0,$$

and for $n = 2$, we have $d(J(\nabla f) \nmid \text{Im}dz) = 0$ if and only if

$$\frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2} = 0, \quad \frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial z_1 \partial \bar{z}_2} = 0, \quad \frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2 f}{\partial z_2 \partial \bar{z}_2} = 0.$$

Proof. We have

$$J(\nabla f) \nmid \text{Im}dz$$

$$= \frac{1}{2i} \sum_{k=1}^{n} \left(-\frac{\partial f}{\partial y_k} \frac{\partial}{\partial x_k} + \frac{\partial f}{\partial x_k} \frac{\partial}{\partial y_k}\right) \langle dz_1 \wedge \cdots \wedge dz_n - d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \rangle$$

$$= \frac{1}{2i} \sum_{k=1}^{n} (-1)^{k-1} \left(-\frac{\partial f}{\partial y_k} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_night)$$

$$+ \frac{\partial f}{\partial y_k} d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n$$

$$+ i \frac{\partial f}{\partial x_k} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_n$$

$$+ i \frac{\partial f}{\partial x_k} d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n)$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \left(\frac{\partial f}{\partial z_k} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_n + \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n\right).$$

So
\[
\begin{align*}
&d(J(\nabla f) \wedge \text{Im}dz) \\
&= \sum_{k=1}^{n} (-1)^{k-1} \left( \sum_{l=1}^{n} \frac{\partial^{2} f}{\partial z_{k} \partial \bar{z}_{l}} dz_{l} \wedge dz_{1} \wedge \cdots \wedge \bar{d}z_{k} \wedge \cdots \wedge d_{z_{n}} \\
&\quad + \frac{\partial^{2} f}{\partial z_{k} \partial \bar{z}_{k}} dz_{k} \wedge dz_{1} \wedge \cdots \wedge \bar{d}z_{k} \wedge \cdots \wedge d_{z_{n}} \right) \\
&\quad + \left( \sum_{l=1}^{n} \frac{\partial^{2} f}{\partial \bar{z}_{k} \partial z_{l}} dz_{l} \wedge \bar{d}z_{1} \wedge \cdots \wedge \bar{d}z_{k} \wedge \cdots \wedge \bar{d}z_{n} \right) \\
&\quad + \frac{\partial^{2} f}{\partial \bar{z}_{k} \partial \bar{z}_{k}} \bar{d}z_{k} \wedge \bar{d}z_{1} \wedge \cdots \wedge \bar{d}z_{k} \wedge \cdots \wedge \bar{d}z_{n} \right) \\
&= \sum_{k,l=1}^{n} (-1)^{k-1} \frac{\partial^{2} f}{\partial z_{k} \partial \bar{z}_{l}} dz_{l} \wedge dz_{1} \wedge \cdots \wedge \bar{d}z_{k} \wedge \cdots \wedge d_{z_{n}} \\
&\quad + \sum_{k,l=1}^{n} (-1)^{k-1} \frac{\partial^{2} f}{\partial z_{k} \partial \bar{z}_{l}} dz_{l} \wedge \bar{d}z_{1} \wedge \cdots \wedge \bar{d}z_{k} \wedge \cdots \wedge \bar{d}z_{n} \\
&\quad + \sum_{k=1}^{n} \frac{\partial^{2} f}{\partial \bar{z}_{k} \partial \bar{z}_{k}} d_{z_{1}} \wedge \cdots \wedge d_{z_{n}} + \sum_{k=1}^{n} \frac{\partial^{2} f}{\partial \bar{z}_{k} \partial \bar{z}_{k}} \bar{d}z_{1} \wedge \cdots \wedge \bar{d}z_{n}.
\end{align*}
\]

So the lemma follows. What causes the difference between the case \(n \geq 3\) and the case \(n = 2\) is that in the last expression for \(d(J(\nabla f) \wedge \text{Im}dz)\), when \(n \geq 3\), the differential forms occuring in the sums are linearly independent; when \(n=2\), each form occuring in the first sum also appears in the second sum and vice versa.

**Lemma 1.5.** The solutions to the system of differential equations in Lemma 1.4 are

1. When \(n \geq 3\), \(f\) is a harmonic Hermitian polynomial. That is

\[
f = c + \sum_{k=1}^{n} (b_{k}z_{k} + \bar{b}_{k}\bar{z}_{k}) + \sum_{k,l=1}^{n} a_{kl}z_{k}\bar{z}_{l}
\]

for some real number \(c\), complex numbers \(b_{k}\) and \(a_{kl}\) with \(a_{kl} = \bar{a}_{lk}\) and \(\sum_{k=1}^{n} a_{kk} = 0\).

In particular the solution space is of dimension \(n(n + 2)\).

2. When \(n = 2\), \(f\) is pluri-harmonic with respect to the complex structure \(\bar{J}\) defined in the introduction. In particular the solution space is infinite dimensional.

**Proof.** (1) Since \(\frac{\partial^{2} f}{\partial z_{k} \partial z_{l}} = 0\) and \(\frac{\partial^{2} f}{\partial z_{k} \partial \bar{z}_{i}} = 0\), all the third order derivatives of \(f\) are
0. So $f$ is a polynomial of degree less than or equal to 2. Moreover $\sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} = 0$ and $f$ is real valued, so $f$ is of the stated form.

(2) Use the same notation as in the introduction. It is easy to see that under the diffeomorphism $F$, $\text{Im}dz_1 \wedge dz_2$ corresponds to $\text{Re}dw_1 \wedge dw_2$ and $\omega$ corresponds to $\text{Im}dw_1 \wedge dw_2$. By Lemma 1.3 and 1.4, $f$ is a solution of the system if and only if $f \in V$, i.e. if and only if there exist function $g$ and form $\phi$ such that

$$d\phi = f \text{Im}dz_1 \wedge dz_2 + g \omega.$$ 

On $C^2$, this equality can be rewrite as

$$d\phi = f \text{Re}dw_1 \wedge dw_2 + g \text{Im}dw_1 \wedge dw_2 = (\frac{f - jg}{2})dw_1 \wedge dw_2 + (\frac{f + jg}{2})d\bar{w}_1 \wedge d\bar{w}_2.$$ 

Then the form on the right must be closed, and we have in particular $\frac{\partial}{\partial \bar{w}_k}(\frac{f - jg}{2}) = 0$ ($k = 1, 2$). So $\frac{f - jg}{2}$ is holomorphic. Therefore $f$ is pluri-harmonic.

Combine Proposition 1.1, Lemma 1.3, Lemma 1.4 and Lemma 1.5, we get

**Theorem 1.6.** $H^{n-1}(E/I)$ is of dimension $n(n + 2)$ when $n \geq 3$ and infinite dimensional when $n = 2$.

For $n \geq 3$, choose once and for all some $n - 1$ forms $\phi_k$ ($k = 1, 2, \ldots, n(n + 2)$) such that $[\phi_k]$ form a basis of $H^{n-1}(E/I)$.

**Corollary 1.7.** Assume $n \geq 3$ and $\phi$ is an $n - 1$ form. Then $d\phi \in \mathcal{I}^n$ if and only if

$$\phi = \sum_{k=1}^{n(n+2)} r_k \phi_k + \omega \wedge \theta + d\rho$$

for some real numbers $r_k$, $n - 3$ form $\theta$ and $n - 2$ form $\rho$.

**Proof.** Follows from the choice of $[\phi_k]$ and the definition of $H^{n-1}(E/I)$.

On $C^2$, consider the differential system of the differential forms with complex coefficients which have no type $(1, 1)$ component. The two dimensional integral submanifolds of this differential system are complex curves in $C^2$. We say that a one dimensional submanifold $N$ of $C^2$ satisfies the complex moment condition if for
every 1-form $\phi$ such that $d\phi$ has no type $(1,1)$ component, we have $\int_N \phi = 0$. The boundary of a complex curve in $\mathbb{C}^2$ satisfies this moment condition.

As we have seen in the proof of Lemma 1.5 (2), under the diffeomorphism $F$, $\text{Im}dz$ corresponds to $\text{Re}dw$ and $\omega$ corresponds to $\text{Im}dw$. So $F$ will transform the moment condition in special Lagrangian geometry into the complex moment condition defined before. Therefore it is enough to study the complex moment condition.

**Proposition 1.8.** Suppose that $\phi$ is a 1 form on $\mathbb{C}^n$, and that $d\phi$ has no type $(1,1)$ component. Then there exist a holomorphic 1 form $\phi_1$, an antiholomorphic 1 form $\phi_2$ and a function $g$ such that $\phi = \phi_1 + \phi_2 + dg$.

**Proof.** Write $\phi = \psi_1 + \psi_2$ with $\psi_1$ being type $(1,0)$ and $\psi_2$ being type $(0,1)$. Since $d\phi$ has no type $(1,1)$ component, we have $\bar{\partial}\psi_1 + \partial\psi_2 = 0$. So $\bar{\partial}\partial\psi_1 + \partial\partial\psi_2 = 0$, i.e. $\partial\psi_1$ is holomorphic. Since we also have $\partial\partial\psi_1 = 0$, there exists a holomorphic 1 form $\phi_1$ such that $\bar{\partial}\phi_1 = \partial\psi_1$. Similarly there exists an antiholomorphic form $\phi_2$ such that $\bar{\partial}\phi_2 = \partial\psi_2$. Then we have $d\phi = d\psi_1 + d\psi_2 = d\phi_1 + d\phi_2$. So $d\phi = d(\phi_1 + \phi_2)$ and there exists a function $g$ such that $\phi = \phi_1 + \phi_2 + dg$.

**Theorem 1.9.** (1) $n \geq 3$, $N$ is a closed $n - 1$ dimensional submanifold of $\mathbb{C}^n$. $N$ satisfies the moment condition if and only if $N$ is isotropic and $\int_N \phi_k = 0$ ($k = 1, 2, \ldots, n(n + 2)$).

(2) $n = 2$, $N$ is a closed 1 dimensional submanifold of $\mathbb{C}^2$. $N$ satisfies the complex moment condition if and only if $N$ is $\int_M \phi = 0$ for every holomorphic form $\phi$.

**Proof.** Use Theorem 1.7 and Proposition 1.8 and the fact that $N$ is isotropic if and only if $\int_N \omega \wedge \theta = 0$ for any $n - 3$ form $\theta$.

**2. An Example**

In this section, we construct a two dimensional submanifold in $\mathbb{C}^3$ which satisfies the moment condition but doesn’t bound any special Lagrangian rectifiable current.
Let $B^2$ be the closed unit ball in $\mathbb{R}^2$, $B^3$ the closed unit ball in $\mathbb{R}^3$ and $S^2$ the unit sphere in $\mathbb{R}^3$. Let $u_1$ and $u_2$ be two $C^\infty$ functions on $B^2$. Denote the graph of the map

$$S^2 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0)$$

by $\Gamma$ and the graph of the map

$$B^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0)$$

by $\Sigma$.

**Lemma 2.1.** $\Gamma$ bounds a special Lagrangian rectifiable current if and only if $u_1$ and $u_2$ satisfy the Cauchy-Riemann equations,

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \quad \text{and} \quad \frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2}.$$

**Proof.** The tangent space of $\Gamma$ at $(x_1, x_2, x_3, u_1(x_1, x_2), u_2(x_1, x_2), 0)$ is spanned by

$$\frac{\partial}{\partial x_1} = \left(1, 0, -\frac{x_1}{x_3}, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, 0\right) \quad \text{and} \quad \frac{\partial}{\partial x_2} = \left(0, 1, -\frac{x_2}{x_3}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_2}, 0\right),$$

where $(x_1, x_2, x_3) \in S^2$, and without loss of generality, we assume $x_3 \neq 0$.

If $\Gamma$ bounds a special Lagrangian rectifiable current $\Sigma'$, then since $\Sigma'$ is volume minimizing, we have

$$\text{supp} \Sigma' \subset \text{convex hull of } \Gamma \subset \mathbb{R}^5 \times \{0\}.$$
since \( J\bar{\nu} \perp \text{Im}dz \) is closed and the restriction of \( J\bar{\nu} \perp \text{Im}dz \) to \( \Sigma' \) vanishes. Therefore \( \int_{\Gamma} g(J\bar{\nu} \perp \text{Im}dz) = 0 \) for any function \( g \). So the restriction of \( J\bar{\nu} \perp \text{Im}dz \) to \( \Gamma \) vanishes. Similarly, the restriction of \( \omega \) to \( \Gamma \) also vanishes.

By the above claim, we have \( \omega\left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right) = 0 \), from which we get \( \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \). We also have \( (J\bar{\nu} \perp \text{Im}dz)(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}) = 0 \), from which we get \( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \). So \( u_1 \) and \( u_2 \) satisfy the Cauchy-Riemann equations.

Conversely, if \( u_1 \) and \( u_2 \) satisfy the Cauchy-Riemann equations, it is easy to verify that \( \Sigma \) is a special Lagrangian submanifold with boundary \( \Gamma \).

Let \( v \in C^\infty(B^2) \). If we take \( u_1 = \frac{\partial v}{\partial x_1} \) and \( u_2 = \frac{\partial v}{\partial x_2} \) and define \( \Sigma \) and \( \Gamma \) as before. Then \( \Sigma \) is a Lagrangian submanifold with boundary \( \Gamma \).

**Lemma 2.2.** \( \Gamma \) satisfies the moment condition if and only if \( \int_{\Sigma} f \text{Im}dz = 0 \) for any \( f \in V \), where \( V \) is the space defined in the beginning of the last section.

**Proof.** If \( d\phi = f \text{Im}dz + \omega \wedge \alpha \) for some function \( f \) and 1-form \( \alpha \), then by Stokes formula and the fact that \( \Sigma \) is Lagrangian, we have

\[
\int_{\Gamma} \phi = \int_{\Sigma} d\phi = \int_{\Sigma} (f \text{Im}dz + \omega \wedge \alpha) = \int_{\Sigma} f \text{Im}dz.
\]

So \( \Gamma \) satisfies the moment condition if and only if \( \int_{\Sigma} f \text{Im}dz = 0 \) for any \( f \in V \).

Now we are ready to describe the example. Recall that \( \Gamma \) is defined to be the graph of the map

\[
S^2 \to R^3, \quad (x_1, x_2, x_3) \mapsto \left(\frac{\partial v}{\partial x_1}(x_1, x_2), \frac{\partial v}{\partial x_2}(x_1, x_2), 0\right).
\]

Take \( v = 7x_1^5 - 10x_1^3 \). Since \( v \) is not harmonic, \( \frac{\partial v}{\partial x_1} \) and \( \frac{\partial v}{\partial x_2} \) don’t satisfy the Cauchy-Riemann equations. By Lemma 2.1, \( \Gamma \) doesn’t bound any special Lagrangian rectifiable current. On the other hand, by Lemma 1.3, 1.4, 1.5 (1), 2.2 and straightforward computation, we can show that \( \Gamma \) satisfies the moment condition.

### 3. The Laplace Operator on a Special Lagrangian Submanifold

Let \( U \) be an open subset of \( \mathbb{C}^n \), \( f \) be a function defined on \( U \) and \( M \) be a special Lagrangian submanifold contained in \( U \).
Lemma 3.1. We have

\[ *df = (J(\nabla f) \llcorner \text{Im}dz)|_M, \]
\[ \Delta_M f = *(d(J(\nabla f) \llcorner \text{Im}dz))|_M, \]

where \( \Delta_M \) and \( * \) are the Laplace operator and the Hodge star operator on \( M \) respectively.

Proof. It’s enough to show the first formula.

Let \( x \in M \) and \( \{e_1, \ldots, e_n\} \) be an orthonormal basis for \( T_xM \). Since \( M \) is special Lagrangian, \( \{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \) is an orthonormal basis for \( T_x\mathbb{C}^n \).

Let \( \{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \) be its dual basis for \( T^*_x\mathbb{C}^n \). Then

\[ dz_1 \wedge \cdots \wedge dz_n = (e_1 + iJe_1) \wedge \cdots \wedge (e_n + iJe_n). \]

On the other hand, we have

\[ \nabla f = \sum_k ((df, e_k)e_k + (df, Je_k)Je_k). \]

Hence

\[ J(\nabla f) = \sum_k ((df, e_k)Je_k - (df, Je_k)e_k). \]

Therefore

\[ J(\nabla f) \llcorner \text{Im}dz \]
\[ = \text{Im}(J(\nabla f) \llcorner dz) \]
\[ = \text{Im} \sum_k ((df, e_k)Je_k - (df, Je_k)e_k) \llcorner ((e_1 + iJe_1) \wedge \cdots \wedge (e_n + iJe_n)) \]
\[ = \text{Im} \sum_k (-1)^{k-1}(i(df, e_k)(e_1 + iJe_1) \wedge \cdots \wedge (e_k + iJe_k) \wedge \cdots \wedge (e_n + iJe_n) \]
\[ - (df, Je_k)(e_1 + iJe_1) \wedge \cdots \wedge (e_k + iJe_k) \wedge \cdots \wedge (e_n + iJe_n) \]

Noting that \( Je_k|_M = 0 \), we get

\[ (J(\nabla f) \llcorner \text{Im}dz)|_M \]
\[ = \text{Im} \sum_k (-1)^{k-1}(i(df, e_k)e_1 \wedge \cdots \wedge \widehat{e}_k \wedge \cdots \wedge e_n \]
\[ - (df, Je_k)e_1 \wedge \cdots \wedge \widehat{e}_k \wedge \cdots \wedge e_n) \]
\[ = \sum_k (-1)^{k-1}(df, e_k)e_1 \wedge \cdots \wedge \widehat{e}_k \wedge \cdots \wedge e_n. \]
Hence
\[ \varepsilon_k \wedge (J(\nabla f) \lrcorner \text{Im}dz)|_M = (df, \varepsilon_k)\varepsilon_1 \wedge \cdots \wedge \varepsilon_n. \]

But at the point \( x \) we have \( \varepsilon_1 \wedge \cdots \wedge \varepsilon_n = \text{vol}_M \). So by the definition of Hodge star operator, we have
\[ *(df) = (J(\nabla f) \lrcorner \text{Im}dz)|_M. \]

**Theorem 3.2.** A function \( f \) is harmonic on every special Lagrangian submanifold contained in \( U \) if and only if
\[ d(J(\nabla f) \lrcorner \text{Im}dz) = 0. \]

Therefore when \( n \geq 3 \), \( f \) is a harmonic Hermitian polynomial, and when \( n = 2 \), \( f \) is pluri-harmonic with respect to the complex structure \( \tilde{J} \) defined in the introduction.

**Proof.** By Lemma 3.1, \( f \) is harmonic on every special Lagrangian submanifold if and only if \( d(J(\nabla f) \lrcorner \text{Im}dz)|_M = 0 \) for every special Lagrangian submanifold \( M \). The last statement is equivalent to say that \( d(J(\nabla f) \lrcorner \text{Im}dz) \) belongs to the differential ideal generated by \( \omega \) and \( \text{Im}dz \). One can show that this is equivalent to \( d(J(\nabla f) \lrcorner \text{Im}dz) = 0 \) (see Proposition 3 in Appendix A.) The second part of the theorem follows from Lemma 1.5.

**Theorem 3.3.** Assume that \( d(J(\nabla f) \lrcorner \text{Im}dz) \) belongs to the differential ideal generated by \( df \), \( \omega \) and \( \text{Im}dz \), and that \( M \) is a special Lagrangian submanifold contained in the domain of \( f \). Then \( f|_M \) attains its maximum and minimum on \( \partial M \). In particular, if \( f|_{\partial M} = \text{const} \), then \( f|_M = \text{const} \).

**Proof.** By assumption, there exist an \( n-1 \) form \( \phi \), an \( n-2 \) form \( \alpha \) and a function \( g \) such that
\[ d(J(\nabla f) \lrcorner \text{Im}dz) = df \wedge \phi + \omega \wedge \alpha + g\text{Im}dz. \]

Since \( M \) is special Lagrangian, on \( M \) we have
\[ d(J(\nabla f) \lrcorner \text{Im}dz)|_M = (df \wedge \phi)|_M. \]

By Lemma 3.1, we get
\[ \Delta_M f = *(df \wedge \phi)|_M = *(df|_M \wedge \phi|_M). \]
Our assertion then follows from the maximum principle.

The following is an anologue of Theorem 3.3 in the complex case.

**Theorem 3.4.** Let $f$ be a function defined on an open subset of $\mathbb{C}^2$ which satisfies $\partial f \wedge \bar{\partial} f \wedge \partial \bar{\partial} f = 0$. Assume that $M$ is a complex submanifold of $\mathbb{C}^2$ of complex dimension 1 on which one of $\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial \bar{z}_1}, \frac{\partial f}{\partial z_2}$ or $\frac{\partial f}{\partial \bar{z}_2}$ is nowhere zero. Then $f|_M$ attains its maximum and minimum on $\partial M$. In particular, if $f|_{\partial M} = \text{const}$, then $f|_M = \text{const}$.

**Proof.** One can show that $\partial \bar{\partial} f$ belongs to the differential ideal generated by $df$, $Re dz$ and $Im dz$ if and only if $\partial f \wedge \bar{\partial} f \wedge \partial \bar{\partial} f = 0$ (See Proposition 4 in Appendix A.) Use this fact and the formula $\Delta_M f = 2i(\Delta f)$ and the same argument as the proof of Theorem 3.3, we can prove this theorem.

**Theorem 3.5.** Suppose $f$ is a function such that $d(J(\nabla f) \cdot \text{Im} dz)$ belongs to the differential ideal generated by $df$, $\omega$ and $\text{Im} dz$. $N$ is an $n-1$ dimensional submanifold such that $f$ is constant on $N$. If $N$ bounds a special Lagrangian submanifold $M$ contained in the domain of $f$, then $(J(\nabla f) \cdot \text{Im} dz)|_N = 0$. Moreover, if $J(\nabla f)$ is not tangent to $N$, then $M$ can be constructed from $N$ by taking the union of all the integral lines of the vector field $J(\nabla f)$ passing through some point on $N$.

**Proof.** By Theorem 3.3, $f$ is constant on $M$. Therefore $\nabla f(p) \in (T_p M)^{\perp}$ for every $p \in M$. Since $M$ is Lagrangian, $TM = J((TM)^{\perp})$. So we have $J(\nabla f)(p) \in T_p M$. Since $M$ is special Lagrangian, we have $\text{Im} dz|_M = 0$. So $(J(\nabla f) \cdot \text{Im} dz)|_M = 0$. In particular $(J(\nabla f) \cdot \text{Im} dz)|_N = 0$ since $N$ is a submanifold of $M$. Moreover since $J(\nabla f)(p) \in T_p M$ for every $p \in M$, the integral lines of $J(\nabla f)$ passing through some point on $M$ must lie on $M$. So we can reconstruct $M$ from its boundary $N$ by taking the union of all the integral lines of $J(\nabla f)$ passing through some point on $N$.

**Remark 3.6.** Under the same assumptions as in Theorem 3.5, we see that in order for $N$ to be the boundary of a special Lagrangian submanifold, it is necessary that $(J(\nabla f) \cdot \text{Im} dz)|_N = 0$. This condition doesn't follow from the moment condition.

**Remark 3.7.** There are many functions $f$ such that $d(J(\nabla f) \cdot \text{Im} dz)$ belongs to the differential ideal generated by $df$, $\omega$ and $\text{Im} dz$. Hermitian harmonic polynomials
are such examples since they satisfy $d(J(\nabla f) \cdot \Im dz) = 0$ by Lemma 1.5. If \( u(x_1, x_2) \) satisfies the minimal surface equation

\[
(1 + u_2^2)u_{11} - 2u_1u_2u_{12} + (1 + u_1^2)u_{22} = 0,
\]

then the function \( f = y_3 - u(x_1, x_2) \) defined on \( \mathbb{C}^3 \) is also such an example by Proposition 5 in Appendix A. Therefore by Theorem 3.6, if the boundary of a special Lagrangian manifold in \( \mathbb{C}^3 \) is contained in the minimal hypersurface \( y_3 = u(x_1, x_2) \), then the interior is also contained in the same minimal hypersurface. (In [HL1], a special Lagrangian submanifolds in \( \mathbb{C}^3 \) is said to have degenerate projection if it is contained in a minimal hypersurface.)

4. Harmonic Functions on Minimal Lagrangian Submanifolds

In this section we study real valued functions whose restrictions to every minimal Lagrangian submanifold are harmonic with respect to the induced metric.

**Theorem 4.1.** \( f \) is harmonic on every minimal Lagrangian submanifold of \( \mathbb{C}^n \) if and only if

\[
f = c + \sum_k (b_k z_k + \overline{b}_k \overline{z}_k) + \sum_{k,l} a_{kl} z_k \overline{z}_l
\]

for some real number \( c \), complex numbers \( b_k \) and \( a_{kl} \) with \( \overline{a}_{kl} = a_{lk} \) and \( \sum_k a_{kk} = 0 \).

So we see that when \( n \neq 1,2 \) there is no difference between functions which are harmonic on every minimal Lagrangian submanifold and functions which are harmonic on every special Lagrangian submanifold. See Theorem 3.2.

**Theorem 4.2.** If

\[
f = c + \sum_k (b_k z_k + \overline{b}_k \overline{z}_k) + \sum_{k,l} a_{kl} z_k \overline{z}_l
\]

for some real number \( c \), complex numbers \( b_k \) and \( a_{kl} \) with \( \overline{a}_{kl} = a_{lk} \) and \( \sum_k a_{kk} \geq 0 \), then \( f \) is subharmonic on every minimal Lagrangian submanifold.
For each real number $c$, complex vector $\tilde{b} = (b_k) \in \mathbb{C}^n$ and Hermitian $n \times n$ matrix $A = (a_{kl})$ with $\text{tr}A \geq 0$, we define a subset $H(c, \tilde{b}, A)$ of $\mathbb{C}^n$ by

$$H(c, \tilde{b}, A) = \{z \mid c + \sum_{k=1}^{n} (b_k z_k + \bar{b}_k \bar{z}_k) + \sum_{k,l=1}^{n} a_{kl} z_k \bar{z}_l \leq 0\}.$$ 

For any subset $S$ of $\mathbb{C}^n$, define $C(S)$ to be the intersection of all those $H(c, \tilde{b}, A)$ containing $S$. By Theorem 4.2 and the maximum principle, we have

**Corollary 4.3.** If $M$ is a minimal Lagrangian submanifold, then $M \subset C(\partial M)$. 

**Proof of Theorem 4.1.** Sufficiency: Let $M$ be a minimal Lagrangian submanifold. Then the restrictions of $z_k$ and $\bar{z}_k$ ($k = 1, 2, \ldots, n$) to $M$ are harmonic. Moreover, constant functions are harmonic on $M$. So it suffices to show that the restriction of $\sum_{kl} a_{kl} z_k \bar{z}_l$ to $M$ is harmonic.

Denote the Laplace operator and the gradient operator on $M$ by $\Delta_M$ and $\nabla_M$ respectively. We have

$$\Delta_M(\sum_{kl} a_{kl} z_k \bar{z}_l) = \sum_{kl} a_{kl}(z_k \Delta_M \bar{z}_l + \bar{z}_l \Delta_M z_k + 2 \nabla_M z_k \cdot \nabla_M \bar{z}_l).$$

Again, since $M$ is minimal, we have $\Delta_M z_k = \Delta_M \bar{z}_l = 0$. So

$$\Delta_M(\sum_{kl} a_{kl} z_k \bar{z}_l) = 2 \sum_{kl} a_{kl} \nabla_M z_k \cdot \nabla_M \bar{z}_l.$$ 

Let $\pi$ be the orthogonal projection from $\mathbb{C}^n$ to the tangent space of $M$. Then $\nabla_M f = \pi(\nabla f)$ for any smooth function defined on $\mathbb{C}^n$, where $\nabla f$ is the gradient operator on $\mathbb{C}^{2n}$. Moreover, since $M$ is Lagrangian, we have $J(TM) = (TM)$. Hence for any two vectors $u$ and $v$ in $\mathbb{C}^n$, we have

$$\pi u \cdot \pi v + \pi(Ju) \cdot \pi(Jv) = u \cdot v.$$
Therefore

\[ \Delta_M \left( \sum_{kl} a_{kl} z_k \bar{z}_l \right) = 2 \sum_{kl} a_{kl} \nabla_M z_k \cdot \nabla_M \bar{z}_l = 2 \sum_{kl} a_{kl} \pi(\nabla z_k) \cdot \pi(\nabla \bar{z}_l) \]

\[ = 2 \sum_{kl} a_{kl} \left( \pi \left( \frac{\partial}{\partial x_k} + i \pi \left( \frac{\partial}{\partial y_k} \right) \right) \right) \cdot \left( \pi \left( \frac{\partial}{\partial x_l} - i \pi \left( \frac{\partial}{\partial y_l} \right) \right) \right) \]

\[ = 2 \sum_{kl} a_{kl} \left( \pi \left( \frac{\partial}{\partial x_k} \right) \pi \left( \frac{\partial}{\partial x_l} \right) + \pi \left( J \left( \frac{\partial}{\partial x_k} \right) \right) \pi \left( J \left( \frac{\partial}{\partial x_l} \right) \right) \right) \]

\[ + i \pi \left( \frac{\partial}{\partial y_k} \right) \pi \left( \frac{\partial}{\partial y_l} \right) - \pi \left( \frac{\partial}{\partial y_k} \right) \pi \left( \frac{\partial}{\partial y_l} \right) \]

\[ = 2 \sum_{kl} a_{kl} \left( \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial y_l} \right) \]

\[ + i \frac{\partial}{\partial x_k} \cdot \frac{\partial}{\partial y_l} - \frac{\partial}{\partial y_k} \cdot \frac{\partial}{\partial y_l} \]

\[ = 2 \sum_{kl} a_{kl} = 0. \]

So \( \sum_{kl} a_{kl} z_k \bar{z}_l \) is harmonic.

Necessity: Since special Lagrangian submanifolds are also minimal Lagrangian submanifolds, we can use Theorem 3.2 to give a proof. Here is another proof. It is divided into six steps.

**Step 1.** \( \Delta f = 0 \), where \( \Delta \) is the Laplace operator in \( \mathbb{C}^n = \mathbb{R}^{2n} \).

The coordinate planes \( \text{Span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\} \) and \( \text{Span}\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\} \) are Lagrangian. Write down the harmonic equations for \( f \) on them and take the sum, we get \( \Delta f = 0 \).

**Step 2.** For every \( A \in U(n) \), \( \text{tr}(A^t AH) + \text{tr}(\bar{A}^t \bar{A} \bar{H}) = 0 \), where \( H = \left( \frac{\partial^2 f}{\partial x_k \partial x_l} \right) \).

In fact, for every \( A \in U(n) \), the image \( P \) of the map

\[ (t_1, \ldots, t_n) \mapsto \left( \sum_k a_{k1} t_k, \ldots, \sum_k a_{kn} t_k \right), \quad t_i \in \mathbb{R} \]

is a Lagrangian plane. Denote the Laplace operator on \( P \) by \( \Delta_P \). Since \( A \in U(n) \), the vectors \( \tilde{v}_1 = (a_{11}, \ldots, a_{1n}), \ldots, \tilde{v}_n = (a_{n1}, \ldots, a_{nn}) \) form an orthonormal basis
for $P$. Use the formula
\[ \frac{\partial f}{\partial v_i} = \sum_k \left( \frac{\partial f}{\partial z_k} a_{ik} + \frac{\partial f}{\partial \bar{z}_k} \bar{a}_{ik} \right) \]
and the fact that $A \in U(n)$, we have
\[ \Delta_P f = \sum_i \frac{\partial^2 f}{\partial v_i^2} \]
\[ = \sum_{ikl} \left( \frac{\partial^2 f}{\partial z_k \partial z_l} a_{ik} a_{il} + \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} a_{ik} \bar{a}_{il} + \frac{\partial^2 f}{\partial \bar{z}_k \partial z_l} \bar{a}_{ik} a_{il} + \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} \bar{a}_{ik} \bar{a}_{il} \right) \]
\[ = \text{tr}(A^t AH) + \text{tr}(\bar{A}^t \bar{A}H) + 2 \sum_k \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k}. \]
That is,
\[ \Delta_P f = \text{tr}(A^t AH) + \text{tr}(\bar{A}^t \bar{A}H) + 2 \sum_k \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k}. \]
Since $\Delta_P f = 0$ by assumption and $\sum_k \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} = 0$ by step 1, we have
\[ \text{tr}(A^t AH) + \text{tr}(\bar{A}^t \bar{A}H) = 0. \]

**Step 3.** For any $A \in U(n)$, $\text{tr}(A^t AH) = 0$

Take $B = e^{it} I$ for, then $B \in U(n)$. By step 2, for any $A \in U(n)$, we have
\[ 0 = \text{tr}((AB)^t (AB)H) + \text{tr}(\overline{(AB)^t (AB)} \bar{H}) = e^{2it} \text{tr}(A^t AH) + e^{-2it} \text{tr}(\bar{A}^t \bar{A}H). \]
That is
\[ e^{2it} \text{tr}(A^t AH) + e^{-2it} \text{tr}(\bar{A}^t \bar{A}H) = 0. \]
Combine this equality with the one in step 2,
\[ \text{tr}(A^t AH) + \text{tr}(\bar{A}^t \bar{A}H) = 0, \]
we get $\text{tr}(A^t AH) = 0$ if we choose $\theta$ such that $\det \begin{pmatrix} e^{2i\theta} & e^{-2i\theta} \\ 1 & 1 \end{pmatrix} \neq 0$.

**Step 4.** For $k = 1, \ldots, n$, we have $\frac{\partial^2 f}{\partial z_k^2} = 0$.

Take $A = \begin{pmatrix} e^{i\theta_1/2} & \cdots & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & e^{i\theta_n/2} \end{pmatrix} \in U(n)$, where $\theta_1, \ldots, \theta_n$ are independent variables. We have
\[ \text{tr}(A^t AH) = e^{i\theta_1} \frac{\partial^2 f}{\partial z_1^2} + \cdots + e^{i\theta_n} \frac{\partial^2 f}{\partial z_n^2}. \]
So by step 3, we have
\[ e^{i\theta_k} \frac{\partial^2 f}{\partial z_k^2} + \cdots + e^{i\theta_n} \frac{\partial^2 f}{\partial z_n^2} = 0. \]

Therefore \( \frac{\partial^2 f}{\partial z_k^2} = 0 \) \((k = 1, \ldots, n)\).

**Step 5.** For \( k, l = 1, 2, \ldots, n \), we have \( \frac{\partial^2 f}{\partial z_k \partial z_l} = 0 \) and \( \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} = 0 \).

Take \( A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{i}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathbb{U}(n) \). Since \( \frac{\partial^2 f}{\partial z_k^2} = 0 \) \((k = 1, \ldots, n)\) by step 4, it is easy to see that \( \text{tr}(A^t A H) = 2i \frac{\partial^2 f}{\partial z_1 \partial z_2} \). So \( 2i \frac{\partial^2 f}{\partial z_1 \partial z_2} = 0 \) by step 3. Therefore \( \frac{\partial^2 f}{\partial z_k \partial z_l} = 0 \). Similarly we have \( \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} = 0 \) when \( k \neq l \). Combine these with step 4, we get \( \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} = 0 \) \((k, l = 1, \ldots, n)\). Moreover, \( f \) is real valued, so we also have \( \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} = 0 \).

**Step 6.** There exist real number \( c \), complex numbers \( b_k \) and \( a_{kl} \) \((k, l = 1, \ldots, n)\) with \( \bar{a}_{kl} = a_{lk} \) and \( \sum_k a_{kk} = 0 \) such that \( f = c + \sum_k (b_k z_k + \bar{b}_k \bar{z}_k) + \sum_{kl} a_{kl} z_k \bar{z}_l \).

By step 5, all the third order derivatives of \( f \) are zero. So \( f \) is a polynomial of degree less than or equal to two. By step 5 again and step 1, \( f \) is of the stated form.

**Proof of Theorem 4.2.** Note that by the proof of the sufficiency of Theorem 4.1, we have \( \Delta f = \sum_k a_{kk} \). The theorem follows.

5. **Constructing a Special Lagrangian Submanifold from Its Boundary**

Let \( S \) be a codimension 1 smooth hypersurface in \( \mathbb{C}^n = \mathbb{R}^{2n} \). Locally it is defined by \( f = 0 \) for some smooth function \( f \). The Hamiltonian vector field associated to \( f \) is \( J(\nabla f) \), where \( J \) is the standard complex structure.

Let \( M \) be a Lagrangian submanifold contained in \( S \). Then for every point \( p \in M \), we have \( T_p M = J(T_p M)^{\perp} \). Since \( (\nabla f)(p) \in (T_p S)^{\perp} \subset (T_p M)^{\perp} \), we have \( J(\nabla f)(p) \in T_p M \). So the restriction of \( J(\nabla f) \) to \( M \) defines a vector field on \( M \). In particular, we have,
Proposition 5.1. If an integral line of the Hamiltonian vector field $J(\nabla f)$ passes through some point on a Lagrangian submanifold $M$, then the whole integral line lies on $M$.

In particular, let $N$ be an $n - 1$ dimensional submanifold of $M$. (For example, take $N = \partial M$.) Assume $J(\nabla f)$ is not tangent to $N$. Then we can recover $M$ from $N$ by taking the union of those integral lines of $J(\nabla f)$ passing through some point on $N$.

Let $\Gamma$ be an $n - 1$ dimensional submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$. Denote by $\Sigma$ the union of those integral lines of $J(\nabla f)$ passing through some point on $\Gamma$. In this section, we are interested in conditions which guarantee that $\Sigma$ is special Lagrangian.

Denote the Hamiltonian flow associated to $J(\nabla f)$ by $\phi_t$. It is standard that $\phi_t^* \omega = \omega$ and $\phi_t^*(df) = df$.

Proposition 5.2. Assume that $J(\nabla f)$ is not tangent to $\Gamma$. Then $\Sigma$ is Lagrangian if and only if $df|_\Gamma = 0$ (i.e. $f$ is constant on $\Gamma$) and $\omega|_\Gamma = 0$. If these hold, then $f$ is also constant on $\Sigma$.

Proof. ($\Rightarrow$) By the definition of $\Sigma$, for every point $p \in \Sigma$, we have $J(\nabla f) \in T_p \Sigma$. Take any $v \in T_p \Sigma$, since $\Sigma$ is Lagrangian, we have $\omega(J(\nabla f), v) = 0$, that is, $(J(\nabla f) \perp \omega)(v) = 0$. It is easy to see that $J(\nabla f) \perp \omega = -df$. So we get $df(v) = 0$. Hence $df|_\Sigma = 0$, i.e. $f$ is constant on $\Sigma$.

Since $\Gamma$ is a submanifold of $\Sigma$ and $\Sigma$ is Lagrangian, we have $df|_\Gamma = 0$ and $\omega|_\Gamma = 0$.

($\Leftarrow$) Take a basis $v_1, \ldots, v_{n-1}$ of $T_p \Gamma$. Then $J(\nabla f), \phi_{t*}(v_1), \ldots, \phi_{t*}(v_{n-1})$ is a basis of $T_{\phi_t(p)} \Sigma$. We have

$$\omega(J(\nabla f), \phi_{t*}(v_k)) = \phi_t^*(J(\nabla f) \perp \omega)(v_k) = -\phi_t^*(df)(v_k) = -df(v_k) = 0,$$

and

$$\omega(\phi_{t*}(v_k), \phi_{t*}(v_l)) = \phi_t^*(\omega)(v_k, v_l) = \omega(v_k, v_l) = 0.$$

Therefore $\omega|_{T_{\phi_t(p)} \Sigma} = 0$. Since every point of $\Sigma$ is of the form $\phi_t(p)$ for some $t$ and $p \in \Gamma$, we have $\omega|_{\Sigma} = 0$. So $\Sigma$ is Lagrangian.
Theorem 5.3. Assume that \( J(\nabla f) \) is not tangent to \( \Gamma \). Then \( \Sigma \) is special Lagrangian if and only if \( df|_\Gamma = 0, \omega|_\Gamma = 0 \) and \( \phi_t^*(J(\nabla f) \perp \text{Im} dz)|_\Gamma = 0 \).

Proof. Notation as above. Suppose \( \Sigma \) is special Lagrangian, we then have
\[
\text{Im} dz(J(\nabla f), \phi_{t_1}(v_1), \ldots, \phi_{t_n}(v_{n-1})) = 0.
\]
It is easy to see that
\[
\text{Im} dz(J(\nabla f), \phi_{t_1}(v_1), \ldots, \phi_{t_n}(v_{n-1}) = \phi_t^*(J(\nabla f) \perp \text{Im} dz)(v_1, \ldots, v_{n-1}).
\]
So we have
\[
\phi_t^*(J(\nabla f) \perp \text{Im} dz)(v_1, \ldots, v_{n-1}) = 0.
\]
Hence \( \phi_t^*(J(\nabla f) \perp \text{Im} dz)|_\Gamma = 0 \). Combine with Proposition 5.2, we get the necessity. The reverse of the argument above proves the converse.

If we put more restrictions on \( f \), then the conditions on \( \Gamma \) can be simplified.

Theorem 5.4. Assume that \( J(\nabla f) \perp d(J(\nabla f) \perp \text{Im} dz) \) belongs to the differential ideal generated by \( df \) and \( \omega \). Then \( \Sigma \) is special Lagrangian if and only if \( df|_\Gamma = 0, \omega|_\Gamma = 0 \) and \( (J(\nabla f) \perp \text{Im} dz)|_\Gamma = 0 \).

Proof. Denote by \( L_{J(\nabla f)} \) the Lie derivative associated to \( J(\nabla f) \). We have
\[
L_{J(\nabla f)}(J(\nabla f) \perp \text{Im} dz)
= d(J(\nabla f) \perp (J(\nabla f) \perp \text{Im} dz)) + J(\nabla f) \perp d(J(\nabla f) \perp \text{Im} dz)
= J(\nabla f) \perp d(J(\nabla f) \perp \text{Im} dz).
\]

One can show that \( L_{J(\nabla f)}(J(\nabla f) \perp \text{Im} dz) \) belongs to the differential ideal generated by \( df \) and \( \omega \) if and only if \( \phi_t^*(J(\nabla f) \perp \text{Im} dz) - J(\nabla f) \perp \text{Im} dz \) belongs to the same differential ideal using the fact that this differential ideal is invariant under the Hamiltonian flow \( \{\phi_t\} \) and the fact
\[
\frac{d}{dt} \phi_t^*(J(\nabla f) \perp \text{Im} dz) = \phi_t^*(L_{J(\nabla f)}(J(\nabla f) \perp \text{Im} dz)).
\]
Therefore if \( J(\nabla f) \perp d(\nabla f) \perp \text{Im} dz \) belongs to the differential ideal generated by \( df \) and \( \omega \), then \( \phi_t^*(J(\nabla f) \perp \text{Im} dz) \) belongs to the differential generated by \( df, \omega \) and \( J(\nabla f) \perp \text{Im} dz \). Our assertion then follows from Theorem 5.3.
6. A Weak Version of the Boundary Problem

Let $U$ be a convex open subset on $\mathbb{R}^m$. Denote $\mathcal{E}^n(U)$ the Fréchet space of smooth differential $n$-forms on $U$. Let $\psi_1, \ldots, \psi_k$ be some $n$-forms with constant coefficients. Define

$$\mathcal{I} = \{ \sum_{i=1}^k f_i \psi_i | f_i \in C^\infty(U) \}.$$ 

Let $T$ be a current of dimension $n$ with compact support. We say $T$ is an $\mathcal{I}$-current if $T(\psi) = 0$ for all $\psi \in \mathcal{I}$.

Let $S$ be a current of dimension $n-1$ with compact support. We say $S$ satisfies the $\mathcal{I}$-moment condition if $S(\phi) = 0$ for all $n-1$-form $\phi$ with $d\phi \in \mathcal{I}$. Obviously, the boundary of an $\mathcal{I}$-current satisfies the $\mathcal{I}$-moment condition.

6.1. Theorem. If $S$ satisfies the $\mathcal{I}$-moment condition, then there is an $\mathcal{I}$-current $T$ such that $\partial T = S$.

Proof. Since $S$ satisfies the moment condition, $\partial S = 0$. So by de Rham's Theorem, there exists a current $T_0$ such that $\partial T_0 = S$. Define

$$R : \mathcal{I} + d\mathcal{E}^{n-1} \to \mathbb{R}$$

by

$$R(\psi + d\phi) = T_0(\psi).$$

Since $S$ satisfies the moment condition, it is easy to see that $R$ is well-defined. If we can show that $R$ is continuous, then by Hahn-Banach's Theorem, we can extend $R$ to $\mathcal{E}^n$. Take $T = T_0 - R$, then $T$ is an $\mathcal{I}$-current and $\partial T = S$.

So we only need to show $R$ is continuous. Define

$$R' : \mathcal{I} \times d\mathcal{E}^{n-1} \to \mathbb{R}$$

by

$$R'(\psi, d\phi) = T_0(\psi)$$

and

$$Q : \mathcal{I} \times d\mathcal{E}^{n-1} \to \mathcal{I} + d\mathcal{E}^{n-1}$$
by

\[ Q(\psi, d\phi) = \psi + d\phi. \]

Obviously, \( R' \) and \( Q \) are continuous and

\[ R' = RQ. \]

If we can show that \( Q \) is an open map, then \( R \) is also continuous.

So it suffices to show \( Q \) is an open map. Note that \( \mathcal{I} \) is closed, so is \( d\mathcal{E}^{n-1} \) since it is the kernel of \( d \). If we can prove that \( \mathcal{I} + d\mathcal{E}^{n-1} \) is also closed, then \( \mathcal{I} \times d\mathcal{E}^{n-1} \) and \( \mathcal{I} + d\mathcal{E}^{n-1} \) are Fréchet space. So \( Q \) is an open map by the open mapping theorem since it is onto and continuous.

So we only need to prove

**Lemma 6.2.** \( \mathcal{I} + d\mathcal{E}^{n-1} \) is closed.

**Proof.** It is easy to see that \( \mathcal{I} + d\mathcal{E}^{n-1} = d^{-1}(d(\mathcal{I})) \). So it is enough to show \( d(\mathcal{I}) \) is closed. Since \( \mathcal{I} \) is generated by the constant coefficients forms \( \psi_1, \ldots, \psi_k \), this follows from the Ehrenpreis' Theorems. (See [E], especially Theorem 6.1 and Theorem 5.20. Also Theorem 4.2 which shows that PLAU implies LAU.)

### 7. The Boundary Value of a Harmonic Gradient

Let \( f \) be a real valued function defined in some open subset of \( \mathbb{R}^n \). Then the graph of \( \nabla f \) is Lagrangian. It is special Lagrangian if and only if

\[ \text{Im } \det(I + i\text{Hess}f) = 0. \]

The linearization of this differential equation at the zero solution is the Laplace equation

\[ \Delta f = 0. \]

This suggests that the harmonic gradient might be related to special Lagrangian geometry. At this time this relationship is still unclear.
The graph of a harmonic gradient is an integral submanifold of the differential system generated by \( \omega \) and \( \psi = \sum_i (-1)^{i-1} dy_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx}_i \wedge \cdots \wedge dx_n \). Let's call this differential system the harmonic gradient system. In this section we work on the boundary problem of the integral submanifold of this system. We say an \( n-1 \) dimensional submanifold \( N \) in \( \mathbb{R}^{2n} \) satisfies the moment condition if for any \( \phi \) such that \( d\phi \) belongs to the differential system mentioned above, we have \( \int_N \phi = 0 \). By Stokes formula, the boundary of an integral submanifold of the harmonic gradient system satisfies the moment condition.

In [BDS], the following is proved:

**Lemma 7.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( F = (f_1, \ldots, f_n) \) be a vector valued smooth function defined on \( \partial \Omega \). Then \( F \) can be extended to a harmonic gradient if and only if

\[
\int_{\partial \Omega} \left( \sum_i \frac{\partial u}{\partial x_i} e_i \right) \left( \sum_j (-1)^{j-1} e_j dx_1 \wedge \cdots \wedge \widehat{dx}_j \wedge \cdots \wedge dx_n \right) \left( \sum_k f_k e_k \right) = 0
\]

for any harmonic function \( u \), where \( \{e_1, \ldots, e_n\} \) is the standard basis for \( \mathbb{R}^n \) and the product of the elements in this basis is taken according to Clifford's rule, that is, \( e_i^2 = -1 \) and \( e_i e_j = -e_j e_i \).

We use this Lemma to prove the following:

**Theorem 7.2.** Under the same assumption as the Lemma. Then \( F \) can be extended to a harmonic gradient in \( \Omega \) if and only if the graph of \( F \) satisfies the moment condition.

**Remark 7.3.** This theorem can be thought as a geometric version of Lemma 7.1. It is the first step toward characterizing the boundaries of the integral submanifolds of the harmonic gradient system. It is still an open question whether the moment condition characterizes the boundaries of the integral submanifolds of the harmonic gradient differential system.

**Proof of Theorem 7.2.** The necessity follows from the Stokes formula. Let's show the sufficiency.
First we introduce some notations. For every index $k$ we let

$$
\alpha_k = \sum_i (-1)^i \frac{\partial u}{\partial x_i} y_k dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
+ \sum_{i < k} y_i ((-1)^{k-1} \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n \\
+ (-1)^i \frac{\partial u}{\partial x_k} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n) \\
- \sum_{k < i} y_i ((-1)^{i-1} \frac{\partial u}{\partial x_k} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
+ (-1)^k \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n) 
$$

and for every three indices $i < j < k$ we let

$$
\beta_{ijk} = y_i ((-1)^{k-1} \frac{\partial u}{\partial x_j} dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n \\
+ (-1)^j \frac{\partial u}{\partial x_k} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n) \\
- y_j ((-1)^{k-1} \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n \\
+ (-1)^i \frac{\partial u}{\partial x_k} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n) \\
+ y_k ((-1)^{j-1} \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\
+ (-1)^j \frac{\partial u}{\partial x_j} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n). 
$$

By straightforward calculation and using the fact that $u$ is harmonic one can show that

$$
d\alpha_k = \left( \sum_{i < k} (-1)^{i+k-1} \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \widehat{dx_k} \wedge \cdots \wedge dx_n \\
+ \sum_{k < i} (-1)^{i+k} \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \wedge \omega \\
- \frac{\partial u}{\partial x_k} \psi
$$
and

\[ d\beta_{ijk} = ((-1)^{i+k-1} \frac{\partial u}{\partial x_j} dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_k \wedge \cdots \wedge dx_n \\
+ (-1)^{i+j} \frac{\partial u}{\partial x_k} dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n \\
+ (-1)^{j+k} \frac{\partial u}{\partial x_i} dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_k \wedge \cdots \wedge dx_n) \wedge \omega. \]

Hence \(d\alpha_k\) and \(d\beta_{ijk}\) belong to the differential system generated by \(\omega\) and \(\psi\). Since the graph of \(F\) satisfies the moment condition, we have

\[
\int_{\text{Graph}(F)} \alpha_k = 0
\]

and

\[
\int_{\text{Graph}(F)} \beta_{ijk} = 0.
\]

By straightforward calculation and using the Clifford multiplication rule once can show that

\[
\int_{\partial \Omega} \left( \sum_i \frac{\partial u}{\partial x_i} e_i \right) \left( \sum_j (-1)^{j-1} e_j dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n \right) \left( \sum_k f_k e_k \right)
= \int_{\text{Graph}(F)} \left( \sum_k \alpha_k e_k + \sum_{i<j<k} \beta_{ijk} e_i e_j e_k \right).
\]

So if the graph of \(F\) satisfies the moment condition, we have

\[
\int_{\partial \Omega} \left( \sum_i \frac{\partial u}{\partial x_i} e_i \right) \left( \sum_j (-1)^{j-1} e_j dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n \right) \left( \sum_k f_k e_k \right) = 0.
\]

By Lemma 7.1 \(F\) can be extended to a harmonic gradient on \(\Omega\).
Appendix. Some Algebraic Results

In this appendix we prove some algebraic results needed in this paper.

Lemma 1. Let $V$ be a vector space of dimension $m$. Let $\mathcal{J}$ be an ideal of $\Lambda V$ generated by homogeneous forms $\phi_1, \ldots, \phi_k$. Let $\phi \in \Lambda^n V$. Then $\phi \in \mathcal{J}$ if and only if for any $\psi \in \Lambda^{m-n} V$ with $\phi_1 \wedge \psi = 0, \ldots, \phi_k \wedge \psi = 0$, we have $\phi \wedge \psi = 0$.

Proof. The wedge product defines a nondegenerate bilinear form

$$\Lambda^n V \times \Lambda^{m-n} V \to \Lambda^m V \cong \mathbb{R}.$$ 

Moreover $\dim \Lambda^n V = \dim \Lambda^{m-n} V$. So this bilinear form defines an isomorphism between $\Lambda^{m-n} V$ and the dual space $(\Lambda^n V)^*$ of $\Lambda^n V$. Therefore $\phi \in \mathcal{J}^n$ if and only if for any $\psi \in \Lambda^{m-n} V$ such that $\alpha \wedge \psi = 0$ for all $\alpha \in \mathcal{J}^n$, we have $\phi \wedge \psi = 0$. But $\alpha \wedge \psi = 0$ for all $\alpha \in \mathcal{J}^n$ is equivalent to $\phi_1 \wedge \psi = 0, \ldots, \phi_k \wedge \psi = 0$, so the Lemma follows.

We also need the following standard fact.

Lemma 2. Notation as above. Let $v$ be a nonzero element of $V$ and $\phi \in \Lambda V$. Then $v \wedge \phi = 0$ if and only if $\phi = v \wedge \tau$ for some $\tau \in \Lambda V$.

The following proposition is needed in the proof of Theorem 3.2.

Proposition 3. $d(J(\nabla f) \upharpoonright \text{Im} dz)$ belongs to the differential ideal generated by $\omega$ and $\text{Im} dz$ if and only if $d(J(\nabla f) \upharpoonright \text{Im} dz) = 0$.

Proof. By the proof of Lemma 1.4, we know that

$$d(J(\nabla f) \upharpoonright \text{Im} dz)$$

$$= \sum_{k,l=1}^{n} (-1)^{k-1} \frac{\partial^2 f}{\partial \bar{z}_k \partial z_l} d\bar{z}_l \wedge dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_n$$

$$+ \sum_{k,l=1}^{n} (-1)^{k-1} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} dz_l \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n$$

$$+ \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_k \partial z_k} dz_1 \wedge \cdots \wedge dz_n + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$
Assume $n \geq 3$. Take

$$\psi_1 = dz_k \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_k \wedge \cdots \wedge d\bar{z}_n.$$ 

Then

$$\omega \wedge \psi_1 = 0,$$

$$\text{Im} dz \wedge \psi_1 = 0.$$ 

So by Lemma 1, we have

$$d(J(\nabla f) \wedge \text{Im} dz) \wedge \psi_1 = 0.$$ 

By the above formula for $d(J(\nabla f) \wedge \text{Im} dz)$, we have

$$d(J(\nabla f) \wedge \text{Im} dz) \wedge \psi_1 = (-1)^{k-1} \frac{\partial^2 f}{\partial \bar{z}_k \partial z_k} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$ 

So $\frac{\partial^2 f}{\partial \bar{z}_k \partial z_k} = 0$. Similarly, we have $\frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} = 0$.

For each pair $k < l$, take

$$\psi_2 = (dz_k \wedge d\bar{z}_k - dz_1 \wedge d\bar{z}_1) \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_k \wedge \cdots \wedge d\bar{z}_l \wedge \cdots \wedge d\bar{z}_n.$$ 

Then

$$\omega \wedge \psi_2 = 0,$$

$$\text{Im} dz \wedge \psi_2 = 0.$$ 

So

$$d(J(\nabla f) \wedge \text{Im} dz) \wedge \psi_2 = 0.$$ 

But

$$d(J(\nabla f) \wedge \text{Im} dz) \wedge \psi_2 = 2(-1)^{l-k-1} \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} dz_1 \wedge \cdots \wedge dz_n \wedge \bar{d}z_1 \wedge \cdots \wedge \bar{d}z_n.$$ 

So $\frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} = 0$. Similarly, $\frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} = 0$.

Finally, take

$$\psi_3 = dz_1 \wedge \cdots \wedge dz_n + (-1)^n d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$
We have 
\[ \omega \land \psi_3 = 0, \]
\[ \text{Im}dz \land \psi_3 = 0. \]
So
\[ d(J(\nabla f) \land \text{Im}dz) \land \psi_3 = 0. \]
But
\[ d(J(\nabla f) \land \text{Im}dz) \land \psi_3 = 2(-1)^n \sum_k \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} dz_1 \land \cdots \land dz_n \land d\bar{z}_1 \land \cdots \land d\bar{z}_n. \]
So \[ \sum_k \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} = 0. \]
So by the above formula for \( d(J(\nabla f) \land \text{Im}dz) \), we have \( d(J(\nabla f) \land \text{Im}dz) = 0. \)
The \( n = 2 \) case can be proved by the same method.

The following proposition is needed in the proof of Theorem 3.4.

**Proposition 4.** Assume that \( f \) is a real valued function defined on some open subset of \( \mathbb{C}^2 \) and one of \( \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial \bar{z}_1}, \frac{\partial f}{\partial z_2} \) or \( \frac{\partial f}{\partial \bar{z}_2} \) is nowhere zero. Then \( \partial \bar{\partial} f \) belongs to the differential ideal generated by \( df, \text{Red}z \) and \( \text{Im}dz \) if and only if \( \partial f \land \bar{\partial} f \land \bar{\partial} \partial f = 0. \)

**Proof.** First let's find all the differential forms \( \psi \) such that \( df \land \psi = 0, \text{Red}z \land \psi = 0 \) and \( \text{Im}dz \land \psi = 0. \) These are equivalent to \( df \land \psi = 0, dz_1 \land dz_2 \land \psi = 0 \) and \( d\bar{z}_1 \land d\bar{z}_2 \land \psi = 0. \) Since \( df \land \psi = 0 \) and \( df \neq 0 \), there exists a 1-form \( \tau \) such that \( \psi = df \land \tau \) by Lemma 2. Without loss of generality, let's assume \( \frac{\partial f}{\partial \bar{z}_2} \neq 0. \) Then \( dz_1, dz_2, d\bar{z}_1 \) and \( df \) form a basis and we can assume \( \psi = df \land (a_1 dz_1 + a_2 dz_2 + a_3 d\bar{z}_1) \) for some numbers \( a_1, a_2 \) and \( a_3. \) Since \( dz_1 \land dz_2 \land \psi = 0, \) we have \( a_3 = 0. \) Since \( d\bar{z}_1 \land d\bar{z}_2 \land \psi = 0, \) we have \( a_1 \frac{\partial f}{\partial z_1} - a_2 \frac{\partial f}{\partial z_2} = 0. \) But \( \frac{\partial f}{\partial z_2} \neq 0 \) since \( f \) is real valued and \( \frac{\partial f}{\partial z_2} \neq 0. \) So \( \psi \) is a multiple of \( df \land (\frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2) = -\partial f \land \bar{\partial} f. \) Finally, by Lemma 1, \( \partial \bar{\partial} f \) belongs to the differential ideal generated by \( df, \text{Red}z \) and \( \text{Im}dz \) if and only if \( \partial f \land \bar{\partial} f \land \bar{\partial} \partial f = 0. \)

The next proposition is needed in Remark 3.7.

**Proposition 5.** On \( \mathbb{C}^3, \) let \( f = y_3 - u(x_1, x_2), \) where \( u \) satisfies the minimal surface

\[ (1 + u_2^2)u_{11} - 2u_1 u_2 u_{12} + (1 + u_1^2)u_{22} = 0. \]
Then $d(J(\nabla f) \rightarrow \text{Im}dz)$ belongs to the differential ideal generated by $df$ and $\omega$.

**Proof.** We will use notations like $\phi_1 = \phi_2(\text{mod } df, \omega)$ which means $\phi_1 - \phi_2$ belongs to the differential ideal generated by $df$ and $\omega$.

By straightforward calculation, we can show that

$$
\begin{align*}
&\quad d(J(\nabla f) \rightarrow \text{Im}dz) \\
&= (u_{11}dx_1 + u_{12}dx_2)(dy_2dy_3 - dx_2dx_3) - (u_{12}dx_1 + u_{22}dx_2)(dy_1dy_3 - dx_1dx_3) \\
&= (u_{11}dx_1 + u_{12}dx_2)(dy_2du - dx_2dx_3) \\
&\quad - (u_{12}dx_1 + u_{22}dx_2)(dy_1du - dx_1dx_3)(\text{mod } df) \\
&= (u_{11}u_{12} - u_{22}u_{11})dx_1dx_2dy_2 + (u_{22}u_{11} - u_{11}u_{22})dx_1dx_2dy_1 \\
&\quad - (u_{11} + u_{22})dx_1dx_2dx_3(\text{mod } df).
\end{align*}
$$

We also have

$$
\begin{align*}
&\quad dx_1 \wedge \omega = dx_1dx_2dy_2 + dx_1dx_3dy_3 \\
&\quad = dx_1dx_2dy_2 + dx_1dx_3du(\text{mod } df) \\
&\quad = dx_1dx_2dy_2 - u_2dx_1dx_2dx_3(\text{mod } df).
\end{align*}
$$

Therefore

$$
dx_1dx_2dy_2 = u_2dx_1dx_2dx_3(\text{mod } df, \omega).
$$

Similarly

$$
dx_1dx_2dy_1 = u_1dx_1dx_2dx_3(\text{mod } df, \omega).
$$

Substitute these two equalities into the formula for $d(J(\nabla f) \rightarrow \text{Im}dz)$, we get

$$
d(J(\nabla f) \rightarrow \text{Im}dz) = -(1 + u_2^2)u_{11} - 2u_1u_2u_{12} + (1 + u_1^2)u_{22})dx_1dx_2dx_3(\text{mod } df, \omega).
$$

So if $u$ satisfies the minimal surface equation, then $d(J(\nabla f) \rightarrow \text{Im}dz)$ belongs to the differential ideal generated by $df$ and $\omega$. 


BIBLIOGRAPHY


