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The growth of fractures in the Earth

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Rice University, 1994
RICE UNIVERSITY

The Growth of Fractures in the Earth

BY

Kaihong Wei

A Thesis Submitted
in Partial Fulfillment of the
Requirements for the Degree

Doctor of Philosophy

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March, 1994
Abstract

THE GROWTH OF FRACTURES IN THE EARTH

BY

KAIHONG WEI

Fracture growth under compressive loading is studied using the maximum strain energy release rate criterion by means of both the finite element and the boundary element methods. Although this approach is computationally intensive, it is indispensable for this type of problems because other criteria cannot account for the friction effect on the fracture faces. We use a repulsion scheme to handle the frictional contact constraints on the fracture faces: the interpenetration is eliminated by adjusting the normal compressive force (repulsion), and the friction law is satisfied by modifying the friction resistance at each iteration.

Our results explain the fact that a natural fracture under uniaxial compression often grows in its own plane, while an artificial cut grows by means of a kink: the reason lies in the lower friction coefficient on an artificial cut than on a natural fracture.

Fracture growth under simple shear and under transtension occurs by a kink and along a smooth, slightly convex trajectory; the computed path is almost identical to the one obtained in the laboratory. Under transpression, fracture also grows
by a kink and along a smooth trajectory which is of the opposite convexity than in the previous case, when compression is large. Right-stepping fractures under a left-lateral shearing run away from each other when their centers are more than one fracture length distant; when this is not the case, they turn toward each other. Interaction is thus significant only in this last case.

Geologically, our results imply that essentially planar faults may be due to continuing remote compressional stress at about 30° to the fault, while abrupt changes in orientation may indicate that the previous stress has been replaced by a remote shear stress. Finally, a convex fault path may indicate simple shear or transtension, whereas a concave one may indicate transpression.
Acknowledgements

I gratefully acknowledge the tremendous help from my thesis adviser, Dr. De Bremeecker. Under his supervision, I was encouraged, sometimes urged, to pursue the answers to a range of problems until we were both completely satisfied. His criticism often led to debates between us and advanced my understanding of the issues involved. I have also enjoyed the friendship with him, and greatly appreciate his care of my career.

Dr. Avé Lallemand gave me guidance on some structural geology problems. My thesis committee helped me define my thesis research. I am grateful for this help, and sincerely hope that my work will not disappoint them.

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List of Symbols

+, - sub-domains.

$E$ Young’s modulus.

$G$ influence matrix for traction.

$G$ strain energy release rate.

$H$ influence matrix for displacement.

$K_1, K_2, K_3$ stress intensity factors.

$S, \Delta S$ strain energy of a body and its reduction.

$S_a$ artificial strain energy contribution due to multi-domain scheme.

$S_c, S_r$ computed and real strain energy after an extension.

$S_c^-, S_r^-$ computed and real strain energy before an extension.

$T$ boundary traction array.

$\mathbf{T}$ traction on boundary.

$U$ boundary displacement array.

$\mathbf{U}$ displacement constraint on boundary.

$a, b, c$ nodes.

$\mathbf{b}$ body force.

$d$ distance.
\( e_k \)  
\( h \)  
\( i, j, k \)  
\( l_c, l_e \)  
\( \vec{n} \)  
\( p, q \)  
\( r \)  
\( \vec{r} \)  
\( s \)  
\( s_{cr} \)  
\( s_f \)  
\( \vec{t} \)  
\( u, v \)  
\( \vec{u} \)  
\( \tilde{\vec{u}} \)  
\( x, y \)  
\( \vec{x}, \vec{y} \)  
\( \Gamma \)  
\( \Gamma_a \)  
\( \Gamma_c \)  
\( \Gamma_t \)

element associated with node \( k \).
loading array.
indices.
length of a crack and its extension.
normal vector.
nodes.
repulsion array.
repulsion vector.
strain energy density.
critical strain energy density.
strain energy density factor.
tangential vector.
displacement components.
displacement array.
displacement vector.
coordinates of a position vector.
position vectors.
boundary.
cohesive interface.
fracture face.
traction boundary.
<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$\Gamma_u$</td>
<td>displacement boundary.</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>interior of a domain.</td>
</tr>
<tr>
<td>$\alpha_p$</td>
<td>penetration relaxation coefficient.</td>
</tr>
<tr>
<td>$\alpha_s$</td>
<td>sliding shooting coefficient.</td>
</tr>
<tr>
<td>$\beta$</td>
<td>crack orientation.</td>
</tr>
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<td>$\epsilon$</td>
<td>strain tensor.</td>
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<td>$\zeta$</td>
<td>sliding.</td>
</tr>
<tr>
<td>$\eta$</td>
<td>penetration.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>extension orientation.</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>resistance coefficient.</td>
</tr>
<tr>
<td>$\mu$</td>
<td>friction coefficient.</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson’s ratio.</td>
</tr>
<tr>
<td>$\xi$</td>
<td>parameter.</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>stress tensor.</td>
</tr>
<tr>
<td>$\tau$</td>
<td>shear stress.</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Remarks

This thesis is a presentation of research on fracture growth in a brittle elastic material under compression. The basic issue here is the orientation of the stable fracture propagation under such conditions. Theoretically, the maximum strain energy release rate criterion, $G_{\text{max}}$, is the foundation of the research, and, practically, finite element and boundary element methods are used to implement the criterion. Uniaxial compression cases and oblique convergence cases are investigated.

Fracture growth under compression is an important problem in structural geology. The fractures observed on the surface and underground are evidence of past tectonic events. The interpretation of this evidence greatly relies on how well we understand the fracturing process, specifically the relation between the fracture geometry and the corresponding tectonic (mechanical) loading. Fracture propagation is also the source of earthquakes. The study of this process may thus be useful for predicting earthquakes and saving human life.

The study of fracture growth under compressive loading is necessary because
our understanding of this phenomenon is rather limited. From a theoretical point of view, although many criteria for fracture propagation exist, most of them cannot be used for this study due to their assumptions about the conditions on fracture faces or to their incapability of accounting for friction on the fracture faces. From a practical aspect, due to the frictional contact of the fracture faces, the problem is highly nonlinear, and an efficient and reliable method to handle this nonlinearity needs to be devised.

This thesis shows that the $G_{\text{max}}$ criterion gives satisfactory results for a variety of fracture propagation problems, and, suggests it as the fundamental principle for the study of this type of problems. It also proposes a repulsion scheme to handle the constraints on the fracture faces, and gives two numerical implementations of the scheme: a finite element method and a boundary element method. Our investigation of various fracture problems offers an explanation for the different growth of a fracture under uniaxial compression and advances our understanding of fracture growth under transression.
1.2 Background

1.2.1 Fracture mechanics

Fracture propagation is, of course, an issue in fracture mechanics, and has been an interesting problem to engineers for several centuries (see Kanninen and Popelar [1] for a complete review). The earliest formulation of the fracture problem is due to Coulomb [2]. On the basis of his laboratory results of a rock column under uniaxial compression, Coulomb obtained an empirical equation

\[ \tau = c + \mu \sigma. \]  \hspace{1cm} (1.1)

where \( \tau \) is the critical shear stress for a planar fracturing, \( c \) the cohesion of the material, and \( \mu \) the internal friction coefficient. By maximizing \( \tau \) with respect to the fracture orientation \( \theta \), the preferential orientation of the fracture can be found

\[ \theta = \frac{\pi}{4} \pm \frac{\phi}{2}. \]  \hspace{1cm} (1.2)

where \( \phi = \tan^{-1} \mu \). Later, in 1900, Mohr extended this criterion by allowing the dependency of \( \mu \) on \( \sigma \) in order to reconcile the discrepancies found in the laboratory. The physical basis of the Coulomb-Mohr criterion has, however, been questioned [3] since it adds the material cohesion which is meaningful before fracture and the resistance to friction which is meaningful after fracture. Furthermore, experimental data [4, 5] have shown an appreciable departure from the prediction of the criterion in some cases. Finally, since the criterion was obtained empirically from uniaxial
compression experiments for an unbroken rock column, and since an existing crack modifies the pre-existing stress field, the applicability of the Coulomb-Mohr criterion to the growth of an existing fracture is questionable.

At the beginning of this century, fracture mechanics experienced a great advance when Griffith [6] proposed an energy approach to the solution of the fracture stability problem. In this approach, Griffith postulated that the length of a crack at equilibrium must be such that the potential energy is minimum with respect to the length. It turns out that this minimization is equivalent to the statement that the energy supply is equal to the energy consumed in creating fracture faces.

\[ G = \frac{dW}{dA} - \frac{dS}{dA} = \gamma. \]  \hspace{1cm} (1.3)

where \( W \) is the external work done to a body, \( S \) is the strain energy of the body, \( \gamma \) is the surface energy of the material, and \( A \) is the area of fracture extension. When the thermal effect can be neglected, the external work \( W \) would, by the energy conservation law, be equal to the strain energy \( S \) if no crack extension had occurred. Thus, \( G \) measures the strain energy release rate due to a crack extension \( dA \). It is further postulated that for a brittle material, such as glass, a crack will grow when \( G > \gamma \).

Three modes of fracturing, as shown in figure 1.1, are used to facilitate the analysis of fracture problems. On the basis of the Westergaard solution [7] for an open crack (where traction on the crack faces is null of course), Irwin [8] showed
Figure 1.1: Three modes of fracture propagation and the coordinate system associated with the fracture tip.

that the asymptotic solution for the stress field near a crack tip can be written as

\[ \sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta), \quad (1.4) \]

where \( r \) and \( \theta \) are defined in figure 1.1, and \( K \) is the stress intensity factor (SIF) of an appropriate mode. Furthermore, for a two dimensional case, Irwin established the relationship between \( G \) and the \( K \)'s as

\[ G = \frac{1}{E} (K_I^2 + K_{II}^2) + \frac{1}{2\mu} K_{III}^2. \quad (1.5) \]

The derivation of this equation was based on the assumption that the crack grows in its own orientation. The onset of fracture extension is thus characterized by a critical value of the combined stress intensity factors.
To predict the orientation of such a fracture extension, an additional criterion is needed. In 1963, Erdogan and Sih [9] proposed that: 1) an open crack extends when the circumferential tensile stress $(\sigma_{\theta \theta})_{\text{max}}$ reaches a critical, material constant value; 2) an open crack extends at the crack tip and in a radial direction $\theta$; 3) the orientation is determined by the maximizer of $\sigma_{\theta \theta}(\theta)$ around a fracture tip. It is clear that this maximizer is a principal axis and thus $\sigma_{r \theta} = 0$ in the preferential orientation. For a combination of mode I and mode II, $\sigma_{r \theta} = 0$ leads to

$$K_I \sin \theta + K_II (3 \cos \theta - 1) = 0. \quad (1.6)$$

For a pure mode I case ($K_I = 0$) this equation shows that the crack grows in its own orientation ($\theta = 0$). For a pure mode II case ($K_II = 0$) the crack grows by a kink at 70.5°. These conclusions have been confirmed by many analog experiments [9, 10, 11, 12, 13].

In 1974, Sih [14] proposed the use of the maximization of the strain energy density factor, $(s_f)_{\text{max}}$, to determine the orientation of crack propagation. This theory relies on two hypotheses: 1) The crack will extend in the direction of maximum potential energy density; 2) The critical intensity $s_c$ of this potential field governs the onset of crack propagation. The application of this theory to a pure mode I case yields $\theta = 0$, i.e., the crack grows in its own orientation. For a pure mode II case, it yields the following equation

$$\cos \theta = \frac{1 - 2\nu}{3}, \quad (1.7)$$

which gives $\theta = 70.5^\circ$ for $\nu = 0$, and $\theta = 90.0^\circ$ for $\nu = 0.5$. 
As noted by Erdogan and Sih [9], the logical extension of the Griffith theory to the orientation of fracture growth is that the orientation of a fracture extension may be determined by the maximization of $G$ with respect to $\theta$. The $(\sigma_{\theta\theta})_{\text{max}}$ and $(s_j)_{\text{max}}$ criteria are both based on local variables. They are motivated by the difficulty of computing $G_{\text{max}}$ for a general case [9]. In 1974, Hussain et al. [15] derived a closed form of $G$ for a deflected crack under a loading of the mixed mode type with the standard assumption of stress free fracture faces. Their results show that $G$ attains its maximum at $\theta = 0^\circ$ for a pure mode I crack, and at $\theta = 83.0^\circ$ for a pure mode II crack.

Abundant experimental evidence has shown that a crack of mixed mode generally does not propagate in its own orientation. Therefore, the assumption used to derive equation (1.5) is not valid, as pointed out by a number of investigators [14, 15, 16]. An analytic form of $G(\theta)$ is extremely difficult to obtain for a general case, and, therefore, has limited usefulness.

When fracture faces are closed, a substantial stress may exist on these faces. The value of the stress is not known, and, indeed, it is a part of the solution of the fracture propagation problem. This stress invalidates the stress free assumption on the fracture faces, and, thus, the above approaches are inherently incapable of handling closed fracture problems. Furthermore, due to the contact of the fracture faces, frictional sliding takes place. It is clear that this sliding relaxes the stress and strain in the vicinity of the crack and, thus, reduces the total strain energy.
Figure 1.2: An artificial cut under uniaxial compression extends by kinks at its tips.

Consequently, the orientation of the crack extension may greatly depend on friction.

This feature of the problem is not accounted for by previous formulations.

1.2.2 Some difficulties

Uniaxial compression experiments of rock samples have been extensively studied by many investigators [4, 5, 17]. Laboratory experiments show that shear fractures of rocks under uniaxial compression are essentially planar. Indeed, this observation led to the Coulomb-Mohr criterion and has been reported by Handin and others [5, 17].

The propagation of a pre-existing crack under uniaxial compression has also been extensively examined. In most experiments [18, 19, 20, 21], a pre-existing crack is artificially created at an orientation $\beta$ with respect to the compression axis; the specimen is then loaded in uniaxial compression. A schematical representation of the configuration and of the result of the experiments is given in Figure 1.2. $\theta$ is close to $90^\circ$ at the start, and the crack grows along a curved path toward the compression axis $\sigma_1$. Crack extensions at an angle to the main crack are generally
known as kinks. The statement: "It is not possible for a shear crack in an elastic medium to grow in its own plane. Instead it grows by the generation of Mode I cracks parallel to $\sigma_1$." [22, p.26], appears to be generally accepted.

The differing results from these two types of experiments are puzzling: natural fracturing under uniaxial compression occurs in a plane, while a preexisting crack grows by a kink under the same condition. Note that the experiments on a preexisting crack under compression are inspired by the development of Griffith's theory, and the often made suggestion that a macro-crack is the connection of micro-cracks. However, in Nemat-Nasser and Horii [23] such connection never occurs for a linear array of small artificial cuts [23, Fig. 17, 18, and 19]. This suggests that their analysis is inadequate for explaining the macro-fracturing. Their experiments do, however, raise the question why the growth of these artificial cuts differs so much from the fracturing of an intact sample under the same condition.

A well-known mathematical analysis of the problem of shear crack propagation was given by Nemat-Nasser and Horii [20, 21, 23]. This analysis takes compression and friction on the fracture faces into account. On the basis of their laboratory experiments, they assumed that the crack is extended by two tension cracks, or kinks, on which the stress vanishes (see, [20, eq. A2]; [21, eq. 2.2]; and [23, eq. 12]). The crack orientation was chosen so as to maximize $K_I/|\sigma_1|(\pi c)^{1/2}$ over the interval $[0, \pi]$, [23, Fig.4 - Fig.9], which implicitly assumes that the extension of $\theta \in [0, \pi/2]$ is always open. However, this assumption may not always be correct, e.g., for $\theta = 0^\circ$. 
In fact, our numerical experiments show that for $\theta \leq 14^\circ$ the extension is still closed. Furthermore, the maximization of $K_I/|\sigma_1|(\pi c)^{1/2}$ favors the opening of the kinks and correspondingly reduces the possibility of detecting the propagation of a crack in its own plane (pure mode II extension). As pointed out by Sih [14, p. 8], any fracture criterion based on a single parameter such as $K_I$ alone will not be sufficient to describe general problems.

The mixed loading of mode I and mode II is a combination of tensile and shear loading. Many experimental reports can be found in the literature. However, problems of fracture growth under a combination of compression and shearing are not well studied. This type of loading represents the transpression tectonics which is prevalent in an oblique convergent setting.

### 1.2.3 Numerical methods

When fracture faces are closed due to a compressive loading, frictional sliding may take place. In such a case, neither the displacement nor the traction may be specified on the fracture faces. In fact, the fracture faces are constrained by a kinematic condition — the fracture faces can not interpenetrate — and a physical condition — the tangential traction must obey a friction law.

The constraints on the frictional contact surfaces have been formulated as Signorini's inequality. This inequality states that the normal displacement of a pair of points must be less than or equal to the gap between them. This formulation allows
Figure 1.3: a. Before deformation, the gap between A and a is zero; b. After deformation, the sum of the normal displacements at A and a is not zero. Signorini formulation is too restrictive for a general situation.

A variational statement of the frictional contact problem as reported by Duvaut and Lions [24], Hlavacek et al. [25], and Kikuchi and Oden [26].

As pointed out by Kikuchi and Oden [26], Signorini’s inequality is a linear approximation of the non-interpenetration constraint. For a general situation, this approximation may be too restrictive (see, Fig 1.3). Bathe and Chaudhary [27] proposed a general formulation of this constraint based on a numerical scheme. This formulation states that a contacting node must be on a segment of the contact boundary. Belytschko and Neal [28] proposed another numerical method to check the interpenetration. Based on a finite element scheme, they embed a sphere, called pinball, in each element, and check the interpenetration by determining whether the pinballs have overlapped.

Cornet [29, 30] used a displacement discontinuity scheme [31] to calculate $G$ for various orientations. In this scheme, non-interpenetration is enforced by requiring zero discontinuity of normal displacement for contacting nodes, an idea based on
Signorini's. In 1992, Wei and De Bremaecker [32, 33] devised a finite element scheme to solve the frictional contact problem. The interpenetration is handled as in [27], and the sliding is adjusted iteratively to satisfy the Coulomb friction law. With this scheme, they were able to show that the orientation of fracture initiation can be determined by $G_{\text{max}}$.

1.3 Fundamental issues

Fracturing is a path-dependent and irreversible process. Therefore, it must be handled in an incremental way. At each increment, one must determine where and in what orientation a fracture grows. When fractures grow under a compressive loading, frictional sliding generally takes place. The displacement and the traction on the fracture faces are constrained by a kinematic and a physical condition. However, the exact values of the normal and shear stresses are unknown a priori. The way to solve this frictional sliding problem is still evolving. Furthermore, the varying geometry due to fracture growth is an important issue in choosing and implementing a numerical method, and the sensitivity of the result to the implementation must be carefully checked out.
Chapter 2

Fundamental Assumptions

This chapter presents the fundamental assumptions and concepts of classical elasticity, frictional contact, and fracture criterion. They are the basic elements of the approach to analyze fracture propagation used in this work.

A two-dimensional elastic body is assumed to be defined by its interior $\Omega$ and its boundary $\Gamma$; this boundary consists of three parts: $\Gamma_u$ where a displacement $\bar{U}$ is imposed, $\Gamma_t$ where a traction $\bar{T}$ is imposed, and fracture faces $\Gamma^\pm_c$ where frictional contact may take place. We will call a point $(x, y) \in \Omega$ an interior point and a point $(x, y) \in \Gamma$ a boundary point.
2.1 Classical elasticity

**Displacement.** The displacement \( \bar{u} \) is assumed to have continuous first derivatives in \( \Omega \) and to verify the boundary condition on \( \Gamma_u \)

\[
\bar{u} = \bar{U}.
\]  

(2.1)

**Strain tensor.** It can be shown that the kinetics at a point in a deformable body can be completely determined by its strain tensor. The small displacement hypothesis assumes that the gradients of the displacement are so small that their products are negligible compared with their own values. Thus, for any \( (x, y) \in \Omega \),

\[
\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}),
\]

(2.2)

is sufficient to determine the deformation.

**Stress tensor.** The stress tensor \( \sigma_{ij} \) may be introduced by

\[
T_i = \sigma_{ij} n_j
\]

(2.3)

where \( T_i \) is the traction on an infinitesimal area, or arc, of normal vector \( \bar{n} \).

**Linear elasticity assumption.** For an isotropic material, linear elasticity implies that the stresses and strains are linearly related. For a plane problem,

\[
\sigma_{ij} = \frac{E'}{1 + \nu'} \epsilon_{ij} + \frac{E' \nu'}{(1 + \nu')(1 - 2\nu')} \epsilon_{kk} \delta_{ij},
\]

(2.4)

where \( E' \) and \( \nu' \) are Young’s modulus \( E \) and Poisson’s ratio \( \nu \) respectively for plane stress, and \( E' = \frac{E}{1 - \nu^2} \) and \( \nu' = \frac{\nu}{1 - \nu} \) for plane strain [34, 35].
**Equilibrium equation.** At equilibrium, the resultant force on any volume must be zero. This leads to the equilibrium equation

\[ \sigma_{ij}(\vec{x}) + b_i(\vec{x}) = 0, \ \forall \vec{x} \in \Omega \]  \hspace{1cm} (2.5)

where \( \vec{b} \) is a body force. On \( \Gamma \), the equilibrium implies

\[ \sigma_{ij}(\vec{x}) n_j(\vec{x}) = T_i(\vec{x}), \ \forall \vec{x} \in \Gamma. \]  \hspace{1cm} (2.6)

**Strain energy.** In the absence of thermal process, the work done by an external force \( W \) consists of that done by \( \vec{b} \) on \( \Omega \) and that by \( \vec{T} \) on \( \Gamma \), i.e.,

\[ W = \oint_{\Gamma} T_i U_i \, d\gamma + \int_{\Omega} b_i u_i \, d\omega. \]  \hspace{1cm} (2.7)

Using the divergence theorem and the equilibrium equation, one can show that

\[ \oint_{\Gamma} T_i U_i \, d\gamma + \int_{\Omega} b_i u_i \, d\omega = \int \sigma_{ij} \epsilon_{ij} \, d\omega. \]  \hspace{1cm} (2.8)

Since the strain energy of the body \( S \) is

\[ S = \int_{\Omega} \sigma_{ij} \epsilon_{ij} \, d\omega. \]  \hspace{1cm} (2.9)

we have

\[ W = S. \]  \hspace{1cm} (2.10)

\[ s = \sigma_{ij} \epsilon_{ij}, \]  \hspace{1cm} (2.11)

is called the strain energy density.
2.2 Frictional contact

On $\Gamma_c^\pm$ the displacement $\bar{u}$ must satisfy the non-interpenetration constraint, and the traction $\bar{T}$ at the points in contact must verify a friction law:

$$ T_l \begin{cases} < \pm \mu |T_n| & \text{for sticking} \\ = \pm \mu |T_n| & \text{for sliding} \end{cases} \quad (2.12) $$

where $\mu$ is the friction coefficient of the surface $\Gamma_c^\pm$. The sign of $T_l$ is so chosen that $T_l$ is always opposite to the sliding.

The normal component $T_n$ at a point on a surface must be inward, i.e.,

$$ T_n \leq 0, \quad (2.13) $$

if the point is in contact with the other surface. $T_n$ must be zero if the point is detached from the other surface. In no case is $T_n > 0$ allowed.

Finally, on the contact surface, the traction must be balanced.

$$ \bar{T}_l|_{\Gamma_c^+} = -\bar{T}_l|_{\Gamma_c^-}. \quad (2.14) $$

It should be clear that these contact conditions describe the contact surfaces at equilibrium after loading. Therefore, $T_n$ and $T_l$ should be evaluated with respect to deformed contact surface $\Gamma_c^\pm(\bar{u})$, and not to original $\Gamma_c^\pm$.

2.3 Fracture criterion

In order to study fracture development, an incremental approach is particularly suitable due to the general path-dependence of the phenomenon. One such ap-
proach is to approximate the fracture path by a number of line segments, so called increments or extensions, and to focus on one segment at a time. It is clear that a better approximation can be obtained with a shorter length for each segment. Thus the length is controlled by an accuracy requirement. For an increment, two other problems are to determine the place(s) of the increment onset and the orientation of the increment. We propose to use the following two criteria for this purpose.

**Criterion 2.1** A crack increment start at the point where the strain energy density $s$ first reaches a critical value $s_{cr}$.

**Criterion 2.2** The orientation of a crack increment is such that the strain energy reduction of the body $\Delta S$ is maximum among all virtual orientations for a given increment length.

Although cracks generally extend from their tips, they also can have branches not from their tips, e.g., the en échelon fracturing associated with a basement strike-slip faulting [36, 37]. Extension from a tip occurs often when the crack is under a remote loading. In such cases, crack tips are always points of high stress concentration and, thus, of high strain energy density. When a crack is under local loading, as is the case for en échelon fracturing, where the underlying strike slip imposes a local loading to these en échelon cracks, a crack may not always extend from its tips. Furthermore, for a three-dimensional problem where the crack tip is no longer a point but a curved front in a cracked body, the deformation along this front generally varies from point
to point. Therefore, in a general setting, it is important to have a criterion to determine the point where crack propagation will start. Since the SIF's are defined at a crack tip, they may not suffice for the general case. Although the Griffith theory can be used to assess fracture onset for a given point (a crack tip in Griffith's cases), it is not directly applicable for locating such a point. Thus, the strain energy density is a logical choice though further experimental verification is needed.

Since different orientations of a fracture increment must produce different deformation states in a finite domain around the onset point, it is appropriate to determine the orientation based on a global criterion evaluated after extension. Furthermore, fracture is not only a material discontinuity but also a deformation discontinuity, i.e., a discontinuity of displacement $\vec{u}$. These two discontinuities are interdependent: Without a material discontinuity a deformation discontinuity would be impossible, and without a deformation discontinuity a material discontinuity is only a geometric entity without physical meaning. The deformation discontinuity can be in the form of opening or of sliding. In compressional cases, fracture faces generally remain in contact, and sliding occurs. Such sliding relaxes the shear stress and thus reduces the strain energy of the solid. The amount of sliding depends on the frictional resistance characterized by the sliding friction coefficient $\mu$. Therefore we can anticipate that the preferential orientation for an increment of fracture will depend on $\mu$, as is the case for fracture initiation [32, 33]. Note that this dependence on friction resistance is not accounted for in the SIF, $(\sigma_{yy})_{\text{max}}$, or $(s_f)_{\text{max}}$.
theories since they assume stress free crack faces, i.e., open cracks. Therefore the applicability of these theories to compressional cases is questionable. Criterion 2 is the discretized equivalence of the $G_{\text{max}}$ criterion. Since it essentially compares the energies before and after extension, the friction dependence can be taken into account.

Criterion 2 may be formulated as follows.

$$\Delta S(\theta^*) = \max_{\theta \in [0, 2\pi]} \Delta S(\theta).$$  \hspace{1cm} (2.15)$$

where

$$\Delta S(\theta) = S^- - S(\theta).$$  \hspace{1cm} (2.16)$$

and

$$S^- = \int_{\Omega} \sigma_{ij}^- \varepsilon_{ij}^- \, d\omega.$$  \hspace{1cm} (2.17)$$

$$S(\theta) = \int_{\Omega(\theta)} \sigma_{ij} \varepsilon_{ij} \, d\omega.$$  \hspace{1cm} (2.18)$$

$\sigma_{ij}^-$ and $\varepsilon_{ij}^-$ are, respectively, the stress and strain tensors for the body $\Omega$ before an increment of propagation, while $\sigma_{ij}$ and $\varepsilon_{ij}$ are the same quantities for the deformed body $\Omega(\theta)$ with an increment of propagation at orientation $\theta$. These quantities may be calculated based on the displacement $\bar{u}$ which is the solution of a frictional contact problem. $S^-$ is the strain energy of the body before the extension of the crack under displacement boundary conditions, while $S(\theta)$ is that after the extension under the same conditions.
Equations (2.16) to (2.18) make it clear that the maximization of $\Delta S(\theta)$ is equivalent to the minimization of $S(\theta)$. Furthermore, we have following proposition.

**Proposition 1** For an isotropic linear elastic material, the minimum of $S(\theta)$ is independent of Young's modulus $E$.

*Proof:* For an isotropic linear elastic material, equation (2.4) holds, and it may be rewritten as

$$\sigma_{ij} = E(c_1 \epsilon_{ij} + c_2 \epsilon_{kk} \delta_{ij}), \quad (2.19)$$

where $c_1$ and $c_2$ are two constants related to $\nu$. Substituting it into equation (2.9) for $\sigma_{ij}$, we have

$$S(\theta) = \int_{\Omega(\theta)} E(c_1 \epsilon_{ij} + c_2 \epsilon_{kk} \delta_{ij}) \epsilon_{ij} \, d\omega,$$

$$= E \int_{\Omega(\theta)} (c_1 \epsilon_{ij} + c_2 \epsilon_{kk} \delta_{ij}) \epsilon_{ij} \, d\omega,$$

$$= Ef(\theta). \quad (2.20)$$

It is thus concluded that $E$ does not alter the position of the minimizer. QED.

On the basis of this proposition and of the $G_{\text{max}}$ criterion, it is clear that the orientation of a fracture extension is independent of Young's modulus $E$. 
Chapter 3

Frictional Contact

— Repulsion Scheme

It is extremely difficult, if not impossible, to find a closed form for $S(\theta)$. Therefore, the maximization of the strain energy $S(\theta)$ is done numerically. In practice, this inevitably involves the evaluation of $S(\theta)$ for some $\theta$'s. As shown in equation (2.9), the computation of the strain energy $S(\theta)$ relies on the solutions, $\sigma$ and $\epsilon$, to the deformation problem of a cracked body. This problem is difficult because of the frictional contact of the fracture faces. This chapter gives a numerical scheme for solving the problem.
3.1 Classical formulation

At first, we state the classical formulation for a frictional contact problem in linear elasticity. This formulation is the foundation necessary to establish the repulsion scheme.

To formulate a frictional contact problem, we have the following conditions:

\[ \sigma_{ij}(x,y) + b_i(x,y) = 0 \quad \forall (x,y) \in \Omega, \quad (3.1) \]

\[ \bar{u}(x,y) = \bar{U}(x,y) \quad \forall (x,y) \in \Gamma_u, \quad (3.2) \]

\[ \sigma_{ij}(x,y)n_j(x,y) = T_i(x,y) \quad \forall (x,y) \in \Gamma_i \cup \Gamma_c^\pm. \quad (3.3) \]

For a point \((x,y) \in \Gamma_c^\pm\), we have

\[ \bar{T}(x,y) = 0 \quad \text{if} \ (x,y) \ \text{is detached}, \quad (3.4) \]

\[ T_n(x,y) \leq 0 \quad \text{if} \ (x,y) \ \text{is in contact}, \quad (3.5) \]

\[ T_i(x,y) = \pm \mu |T_n(x,y)| \quad \text{if} \ (x,y) \ \text{is sliding}, \quad (3.6) \]

\[ T_i(x,y) < \pm \mu |T_n(x,y)| \quad \text{if} \ (x,y) \ \text{is sticking}. \quad (3.7) \]

Furthermore, no interpenetration of the contact surfaces is allowed.

3.2 Repulsion

As mentioned before, fracture faces are constrained by the non-interpenetration condition and a friction law. The major difficulty results from the inequality formulation of these constraints. Consequently, the contact geometry, namely the place
and area of the contact, is unknown until the problem is solved; similarly the traction condition, namely the values of the normal and tangential stresses, is also unknown before the solution.

Conceptually, the non-interpenetration is maintained by a traction $T_n$ from the contacted surface to the contacting point. For this reason we call $\vec{T}$ on $\Gamma_c^\pm$ a repulsion and denote it by $\vec{r}$. $r_n$ is in the inner normal direction, and its value depends on the deformation of the body. Although $r_n$ is not known, it is clear that a null $r_n$ generally results in the penetration of the contact surface, and a large inward $r_n$ reduces this interpenetration. In reality, $r_n$ is just sufficient to eliminate interpenetration. Thus, by evaluating the penetration for a given $r_n$ it is possible iteratively to determine the value of $r_n$ which just eliminates penetration.

When friction resistance exists on $\Gamma_c^\pm$, a tangential component of traction, $r_t$, must be taken into account. The direction of $r_t$ at a point on $\Gamma_c^\pm$ is always opposite to the sliding at the point, and the absolute value of $r_t$ is in $[0, |\mu r_n|]$, depending on whether the point is sliding or sticking. Although the exact value of $r_t$ is unknown, it is clear that a zero resistance allows a large sliding, and a large enough resistance can prevent sliding.

The value of $\vec{r}$ is determined iteratively with null initial value of $\vec{r}$ on $\Gamma_c^\pm$. Given this $\vec{r}$ on $\Gamma_c^\pm$ and the loading conditions on $\Gamma_u$ and $\Gamma_t$, a standard boundary value problem exists, and its solution $\vec{u}$ can be obtained by a conventional method. On the basis of the displacement solution $\vec{u}$, the penetration $\eta$ and the sliding $\zeta$ can
be computed. \( r_n \) and \( r_l \) are then modified according to \( \eta \) and \( \zeta \). This process is
repeated until the penetration is completely eliminated \( (\eta \geq 0) \), and the friction law
is satisfied. The overall scheme may presented as follows:

**Overall algorithm.**

Step 1. Let \( k = 0 \).

Step 2. Let \( \vec{r}^k = 0 \).

Step 3. Solve deformation problem for \( \vec{u} \), under boundary conditions \( \vec{u} = \vec{U} \) on \( \Gamma_u \),
\[ \vec{t} = \vec{T} \text{ on } \Gamma_t, \text{ and } \vec{r}^k \text{ on } \Gamma^\pm_c. \]

Step 4. For each point on \( \Gamma^\pm_c \),

Step 5.1 Compute \( \zeta^k \) and \( \eta^k \) on the basis of \( \vec{u}^k \).

Step 5.2 Update \( r_n \) (discussed below).

Step 5.3 Update \( r_l \) (discussed later).

Step 6. If no change in \( \vec{F} \) has been made at any point on \( \Gamma^\pm_c \), exit from algorithm;
otherwise

Step 7. Update \( k = k + 1 \), and

Step 8. Go to step 3.
3.3 Update of repulsion

A penetration ($\eta > 0$) at a point on $\Gamma^\pm$, here called underrelaxation, indicates the inadequacy of the current $r_n$ at the point. Thus a subsequent increase in $r_n$ is needed, and it is reasonable to make this increase proportional to the penetration. However, this increase cannot exactly cancel the interpenetration. Thus, because of the approximate nature of $r_n$ at each iteration, it is possible to have a inward $r_n$ at a point while opening occurs at this same point. This may be called overrelaxation, and it is obviously unrealistic. When this happens, the value of $r_n$ must be reduced till either $\eta = 0$ which indicates a contact, or till $r_n = 0$ and $\eta < 0$ which indicates a real opening, whichever comes first. On the basis of these considerations, the update of $r_n$ in the previous algorithm can be done in the following way.

Step 5.2.1 If $\eta^k = 0$ (contact) and $r_n^k > 0$ (compressive), then $r_n^{k+1} = r_n^k$; otherwise,

Step 5.2.2 If $\eta^k < 0$ (opening) and $r_n^k = 0$ (null traction), then $r_n^{k+1} = r_n^k$; otherwise,

Step 5.2.3

$$r_n^k = r_n^{k-1} + \alpha_p \eta^k; \quad (\alpha_p > 0).$$

(3.8)

where $\alpha_p$ is called the penetration relaxation coefficient.

Step 5.2.4 If $r_n < 0$ (numerically $|r_n| < \varepsilon$), then $r_n = 0$.

It is clear that the tests at steps 5.2.1 and 5.2.2 correctly and completely comply with the non-interpenetration condition, and the exit from the algorithm occurs
only if no interpenetration exists at any point on $\Gamma_c^\pm$. Therefore, convergence of the algorithm indicates the satisfaction of the non-interpenetration condition. Note that $\eta < 0$ for opening. When this happens, step 5.2.3 results in a decrease in $r_n$, correcting the overrelaxation. Step 5.2.4 ensures that the repulsion is always inwardly oriented. Moreover, due to the iterative determination of $r_n^k$, the exact value of $\eta^k$ at a particular iteration is not important, provided that $\eta^k$ correctly indicates penetration ($\eta^k > 0$), contact ($\eta^k = 0$), or opening ($\eta^k < 0$). This provides valuable reliability and flexibility for simplifying the programming.

### 3.4 Update of friction resistance

At iteration $k$, after the determination of $r_n^k$ on the basis of $\eta^k$ at each iteration, the friction law provides a constraint to relate $r^k_t$ to $r_n^k$. However, this constraint is in a form of an inequality (2.12). In order to handle this constraint, we introduce a resistance coefficient $\lambda$ by

$$r_t = \pm \lambda |r_n|, \quad (3.9)$$

while requiring

$$0 \leq \lambda \leq \mu \quad (3.10)$$

When sliding takes place, $r_t = \pm \mu |r_n|$, and hence $\lambda = \mu$; when sticking occurs, $r_t < \pm \mu |r_n|$, and hence $\lambda < \mu$. A value of $\lambda$ which is less than $\mu$ and cannot prevent sliding is called an undershooting. Thus, we may increase $\lambda$ at a point until the
sliding stops ($\zeta = 0$) but only until $\lambda = \mu$. Thus, step 5.3 in the algorithm can be done as follows.

Step 5.3.1 If $|\zeta|^k = 0$ (numerically $|\zeta|^k < \epsilon$), then $\lambda^{k+1} = \lambda^k$; otherwise

Step 5.3.2 If $\lambda^k = \mu$, then $\lambda^{k+1} = \lambda^k$; otherwise

Step 5.3.3

$$\lambda^k = \lambda^{k-1} + \alpha_s \zeta^k, \quad (\alpha_s > 0),$$

(3.11)

where $\alpha_s$ is called the sliding shooting coefficient.

Step 5.3.4 Let $\lambda^{k+1} = \mu$ if $\lambda^{k+1} > \mu$

The tests at steps 5.3.1 and 5.3.2 correctly and completely describe the friction law. Thus the convergence implies the satisfaction of the friction law. When an overshooting occurs, i.e., when the previous $r_t$ is so large that the current sliding is opposite to the previous one, equation (3.11) will correct $r_t$. In order to avoid oscillations, $\alpha_s$ is different for undershooting and for overshooting.

### 3.5 Concluding remarks

From the proposed algorithms, it can be seen that at each iteration a solution $\bar{\zeta}^k$ is obtained by solving the equilibrium equation using $(\bar{\tau})^k$ on $\Gamma_c^\pm$ and the conventional boundary conditions. The iteration stops only when all constraints on the contact surface are satisfied. Thus, when convergence is obtained, the result $\bar{n}$ is the solution to the classical formulation stated in section 3.1.
Although a rigorous proof of convergence has not yet obtained, physical considerations appear to assure convergence. Let us consider the difference $\Delta \bar{u}^k = \bar{u}^k - \bar{u}^{k-1}$. It is easy to show that $\Delta \bar{u}^k$ is the solution to a problem where $\Gamma_u$ is fixed, $\Gamma_\ell$ is stress free, and $\Gamma^\pm_\ell$ is under $\bar{r}^* - \bar{r}^{k-1}$. Physically, these conditions mean that only the portion of the boundary $\Gamma^\pm_\ell$ is under loading $\bar{r}^* - \bar{r}^{k-1}$, while the rest of the boundary is not under any loading. Note that the normal component of $\bar{r}^* - \bar{r}^{k-1}$, where a penetration exists, is inwardly oriented. It is physically clear that $\Delta \bar{u}^k$, the response of the elastic body to the physical conditions, is also inwardly oriented. A reduction of interpenetration is thus obtained. Similarly where the friction law is not yet satisfied, the tangential component of $\bar{r}^* - \bar{r}^{k-1}$ is opposite to the sliding. For the same reason, $\Delta \bar{u}^k$ is also opposite to the sliding. Thus, a closer satisfaction of the friction law is obtained. These considerations are reinforced by the fact that convergence is always obtained in our numerical experiments.

It should be remarked that the repulsion scheme is independent of the numerical method.
Chapter 4

Numerical Implementations

$\Delta S(\theta)$ is a function of one variable for the cases considered in this work. Many procedures to search for its maximizer can be found in Dennis and Schnabel [38]. In this work, we present the graph of the function over a range of $\theta$ in order to reveal the structure of the function. Thus, the implementation of the repulsion scheme is the major concern here.

In this chapter we report finite element and boundary element implementations of the repulsion scheme. The basic elements of the FEM and the BEM have been well established, e.g., see Becker et al. [39], Owen [40], and Brebbia and Dominguez [34]. We focus on a way to use these methods to implement the repulsion scheme.
4.1 Finite element method

A two dimensional domain can be divided into a number of elements of simple shape. For simplicity, we assume that the body forces are absent and use linear triangle elements to illustrate the implementation. For an element, the displacement \( \bar{u}(x, y) \) is interpolated from its values at the vertices of the element by the associated base functions. The vertices are called nodes and are shared by adjacent elements to provide continuity of the displacement \( \bar{u} \). In general, we may assume that the total number of nodes is \( n \), the first \( p \) are on \( \Gamma_i \), and the last \( q \) on \( \Gamma_e \). Without loss of generality, we assume that \( \bar{u}(x, y) = 0 \ \forall (x, y) \in \Gamma_u \). With these approximations to the physical space and the function space, the virtual work principle leads to the following equation about the nodal values of \( \bar{u} \):

\[
Ku = h, \quad (4.1)
\]

where \( K \) is the stiffness matrix,

\[
u^T = (\bar{u}_1, \cdots, \bar{u}_n) \quad (4.2)
\]

is the array of the nodal values of the solution \( \bar{u}(x, y) \), and

\[
h^T = (\bar{h}_1, \cdots, \bar{h}_n) \quad (4.3)
\]

is called a loading array.

Under the assumptions mentioned earlier, only the first \( p \) and the last \( q \) members of \( h \) are nonzero. For a distributed traction \( \bar{T} \) on \( \Gamma_t \) and a distributed repulsion \( \bar{r} \)
on $\Gamma_{c}^{\pm}$, these members are given by

$$
\vec{h}_{k} = \int_{e_{k}^{-}} \phi_{k}(\vec{x}) \vec{T}(\vec{x}) \, d\gamma(\vec{x}) + \\
\int_{e_{k}^{+}} \phi_{k}(\vec{x}) \vec{T}(\vec{x}) \, d\gamma(\vec{x}) \quad k = 1, \cdots, p, \tag{4.4}
$$

$$
\vec{h}_{k} = \int_{e_{k}^{-}} \phi_{k}(\vec{x}) \vec{r}(\vec{x}) \, d\gamma(\vec{x}) + \\
\int_{e_{k}^{+}} \phi_{k}(\vec{x}) \vec{r}(\vec{x}) \, d\gamma(\vec{x}) \quad k = 1, \cdots, q, \tag{4.5}
$$

where $e_{k}^{+}$ and $e_{k}^{-}$ are two elements adjoined at node $k$, and $\phi_{k}(\vec{x})$ is the base function associated with this node. For a point force loading $\vec{T}_{k}$ and a point repulsion $\vec{r}_{k}$, they are simply

$$
\vec{h}_{k} = \vec{T}_{k} \quad k = 1, \cdots, p, \tag{4.6}
$$

$$
\vec{h}_{k} = \vec{r}_{k} \quad k = 1, \cdots, q. \tag{4.7}
$$

### 4.1.1 Frictional contact

We first establish a proposition.

**Proposition 2** With linear elements, the satisfaction of the frictional contact conditions, namely the non-interpenetration and the friction law, can be assessed at their nodes.

*Proof:* First we examine the friction law. Suppose that $a$ and $b$ are two end nodes defining a linear element. It is clear that the direction of the normal is constant on the element. This allows us to proceed with the proof in scalar form. To verify the
proposition, we will show that \( r_t = \pm \lambda|r_n| \) at nodes \( a \) and \( b \) implies \( r_t = \pm \lambda|r_n| \) for every point on the element. Indeed, with a linear element

\[
\begin{align*}
    r_n &= \xi(r_n)_a + (1 - \xi)(r_n)_b \\
    r_t &= \xi(r_t)_a + (1 - \xi)(r_t)_b
\end{align*}
\] (4.8) (4.9)

for \( 0 \leq \xi \leq 1 \). By assumption, \( r_t = \pm \lambda|r_n| \) at nodes \( a \) and \( b \). Therefore, we have

\[
r_t = \pm(\xi\lambda_a|(r_n)_a| + (1 - \xi)\lambda_b|(r_n)_b|). \quad (4.10)
\]

Note that \( (r_n)_{a,b} \leq 0 \) (equation 2.13). Thus,

\[
\begin{align*}
    r_t &= \mp(\xi\lambda_a(r_n)_a + (1 - \xi)\lambda_b(r_n)_b) \\
    &\leq \mp \max(\lambda_a, \lambda_b)(\xi(r_n)_a + (1 - \xi)(r_n)_b) \\
    &= \mp \max(\lambda_a, \lambda_b)r_n \\
    &= \pm \max(\lambda_a, \lambda_b)|r_n|. \quad (4.11)
\end{align*}
\]

Similarly we can prove that

\[
r_t > \pm \min(\lambda_a, \lambda_b)|r_n|. \quad (4.12)
\]

These two inequalities show that

\[
\begin{align*}
    r_t &= \pm \lambda|r_n|, \\
    0 \leq \min(\lambda_a, \lambda_b) &\leq \lambda \leq \max(\lambda_a, \lambda_b) \leq \mu. \quad (4.13) (4.14)
\end{align*}
\]

Therefore, the satisfaction of the friction law at the end nodes implies the the satisfaction of the friction law everywhere on the element.
It is trivial to see that polygon $A$ overlies polygon $B$ if and only if at least one vertex of polygon $A$ is in polygon $B$. Thus, the satisfaction of the geometry constraint can be determined by the nodes. QED.

This proposition allows us to implement the repulsion scheme based on the situation at each contact node. It is clear that frictional contact is a phenomenon that occurs between surfaces. Numerically, this means the interaction between a pair of contacting elements. Since a node is shared by two adjacent elements, it is necessary to consider the repulsion from both elements.

The boundary and the fracture faces consist of piecewise linear segments. The boundary is oriented in such a way that the interior of the body always lie to its left-hand-side. To facilitate the computations of $\eta$ and $\zeta$, the nodes on the two sides of the fracture, $\Gamma^+_c$ and $\Gamma^-_c$, are defined at the same points.

### 4.1.2 Discontinuity of $\vec{r}$

A non-smooth change of fracture trajectory may exist and result in a discontinuity of the traction or repulsion $\vec{r}$, as can be seen in figure 4.1. As we can see, an opening is created when sliding occurs (Fig., 4.1-b). In this case, the repulsion $\vec{r}$ is non-zero on segment $\overrightarrow{ab}$ but is null on $\overrightarrow{bc}$. For the case of figure 4.1-c, the fracture is locked, and its faces are under normal compression. Due to the ambiguity of the normal vector at node $b$, a discontinuity of the traction at $b$ exists.

Special techniques must be used to handle this discontinuity. In our implemen-
4.1.3 Computation of $\vec{r}$

Suppose that $a$, $b$, and $c$ are the end nodes of two adjacent boundary elements, and that the orientation of the elements is from $c$ to $b$ to $a$, as shown in figure 4.2. At node $b$ there are two tangential vectors, $\vec{t}^+$ and $\vec{t}^-$, with respect to element + and element - respectively. $\vec{t}^\pm$ are computed in the following way.

$$\vec{t}^+ = \frac{\vec{x}_a - \vec{x}_b}{\|\vec{x}_a - \vec{x}_b\|},$$  \hspace{1cm} (4.15)

$$n^+_x = -t^+_y \quad n^+_y = t^+_x.$$  \hspace{1cm} (4.16)

where $\vec{n}^+$ is the outer normal vector at node $b$ with respect to element +.

The repulsion at node $b$, $\vec{r}$, is treated as a point force in our scheme, and it is given by

$$\vec{r} = \vec{r}^+ + \vec{r}^-,$$  \hspace{1cm} (4.17)

$$\vec{r}^+ = -\gamma^+_n (\vec{n}^+ + \lambda^+ \vec{t}^+),$$  \hspace{1cm} (4.18)
Figure 4.2: a) Collocation of nodes on $\Gamma_c^\pm$. b) Normal and tangential vectors of the elements adjoined at $b$. c) Penetration and sliding of $b$ with respect to $\overline{AB}$.

$$\vec{r}^- = -r_n^- (\vec{n}^- + \lambda^- \vec{t}^-).$$  \hspace{1cm} (4.19)

It is important to note that $\vec{r}$ must be the vector sum of the repulsion from the two adjacent elements, otherwise the penetration into an element may not be eliminated by the repulsion scheme. $r_n^\pm$ and $\lambda^\pm$ are computed using equations (3.8) and (3.11) on the basis of $\eta^\pm$ and $\zeta^\pm$.

**Computation of $\eta$ and $\zeta$**

With $\vec{r}_t^\eta$, a solution $\vec{u}_t^\eta$ is obtained. Thus the geometry of $(\Gamma_c^\pm)_t$ can be determined. The penetration of node $b$, $\eta_b$, with respect to a segment, say, $\overline{AB}$, is defined as the distance of the node from the segment (length of $\overline{BB}$ in Fig. 4.2). The computation of this distance is straightforward. We assign a sign to this distance: positive for penetration and negative for opening. Let $b'$ be the projection of node $b$ on element $\overline{AB}$. Sliding $\zeta_b$ is conveniently defined as the distance of $b'$ from $B$, as shown in figure 4.2-c. We also assign a sign to the distance: positive for sliding in the direction of the boundary and negative otherwise.
4.1.4 Update of \( h \)

We treat the repulsion \( \vec{f} \) as a point force. This allows the update of the loading array \( h \) to be done by

\[
(h^\gamma)^T = (\vec{T}_1, \ldots, \vec{T}_p, 0, \ldots, 0, \vec{r}_1^\gamma, \ldots, \vec{r}_q^\gamma).
\]

(4.20)

It is important to see that the formulation of the constraints on the contact faces determines the accuracy of the solution when convergence of the iteration is reached, and the way to comply with the constraints determines the efficiency and the reliability of the iteration. Proposition 2 ensures the accuracy of the solution, and in practice, convergence of the iteration is always achieved. The simplification consisting of treating the repulsion \( \vec{f} \) as a point force is thus acceptable.

4.2 Boundary element method

4.2.1 Direct boundary element method

The boundary element method is a well established numerical technique in dealing with a variety of elasticity problems. The theoretical basis for this method is the following displacement boundary integral equation:

\[
c_{ij}(\vec{y}) U_j(\vec{y}) + \oint_{\Gamma} S_{ij}(\vec{y}, \vec{x}) U_j(\vec{x}) \, d\gamma(\vec{x}) = \oint_{\Gamma} D_{ij}(\vec{y}, \vec{x}) T_j(\vec{x}) \, d\gamma(\vec{x}),
\]

(4.21)

where \( D_{ij}(\vec{y}, \vec{x}) \) and \( S_{ij}(\vec{y}, \vec{x}) \) are Kelvin fundamental fields with a source at point \( \vec{y} \) on \( \Gamma \), \( U_j(\vec{x}) \) and \( T_j(\vec{x}) \) are, respectively, the components of the displacement and
traction at a point $\bar{x}$ on the boundary $\Gamma$, and $c_{ij}$ is a number determined from the geometry of the boundary at $\bar{y}$ [34, 35]. The equation depends on the source point $\bar{y}$; a change of $\bar{y}$ results in a different integral equation.

To find $\bar{U}$ and $\bar{T}$ on $\Gamma$ numerically, the boundary $\Gamma$ is divided into $m$ segments or boundary elements. On an element $\gamma$, $n_\gamma$ nodes at $\bar{x}_k^\gamma$, $k = 1, \cdots, n_\gamma$, may be defined so that $\bar{U}$ and $\bar{T}$ as well as a point $\bar{x}^\gamma$ inside the element can be found by their nodal values, $\bar{U}(\bar{x}_k^\gamma)$, $\bar{T}(\bar{x}_k^\gamma)$, and $\bar{x}_k^\gamma$ by using some base functions $\phi_k(\xi)$. $n_\gamma$ is usually less than 4. Thus,

\begin{align}
\bar{x}^\gamma(\xi) &= \sum_{k=1}^{n_\gamma} \bar{x}_k^\gamma \phi_k(\xi), \\
\bar{U}(\bar{x}^\gamma(\xi)) &= \sum_{k=1}^{n_\gamma} \bar{U}(\bar{x}_k^\gamma) \phi_k(\xi), \\
\bar{T}(\bar{x}^\gamma(\xi)) &= \sum_{k=1}^{n_\gamma} \bar{T}(\bar{x}_k^\gamma) \phi_k(\xi).
\end{align}

where $\xi$ is a parameter defined on an interval $[a, b]$ over which $\bar{x}(\xi)$ describes all the points of element $\gamma$. With these approximations, the boundary equations, obtained by locating source $\bar{y}$ at each node on $\Gamma$, leads to (see [34, 35] for the derivation)

$$
H U = G T,
$$

where $H$ and $G$ are two influence matrices, and $U$ and $T$ are respectively the arrays of the nodal values of $\bar{U}$ and $\bar{T}$ on $\Gamma$.

When any two of $U_x$, $U_y$, $T_x$, and $T_y$ at a node on $\Gamma$ are known, these equations are sufficient for solving the other two unknowns at the node. This approach is often called the direct boundary element method. For example, when the displacement
Figure 4.3: The domain is partitioned along the fracture and a convenient cohesive interface \( \Gamma^{\pm}_a \).

\( U_P \) at nodes \( P = \{p\} \) and the traction \( T_Q \) at nodes \( Q = \{q\} \) are given, \( T_P \) and \( U_Q \) can be found by solving

\[
A X = Y, \tag{4.26}
\]

where

\[
A = \begin{pmatrix} -G_P & H_Q \end{pmatrix}, \tag{4.27}
\]

\[
X^T = \begin{pmatrix} T_P & U_Q \end{pmatrix}, \tag{4.28}
\]

\[
Y = \begin{pmatrix} -H_P & G_Q \end{pmatrix} \begin{pmatrix} U_P \\ T_Q \end{pmatrix}, \tag{4.29}
\]

where \( G_P \) and \( H_P \), respectively, consist of the columns of \( G \) and \( H \) associated with \( P \); similarly, \( G_Q \) and \( H_Q \) are associated with \( Q \).

### 4.2.2 Sub-domain scheme

Due to the coplanar surfaces of a fracture, the BEM suffers from degeneration [41].

This is because a pair of source points on each side of the fracture are at an identical
location, and the corresponding integral equations are thus linearly dependent. In order to avoid this difficulty, a multidomain scheme has been devised [34, 41, 42].

The basic idea of a multidomain scheme is that the fractured body under consideration may be partitioned into two sub-domains through the fracture faces $\Gamma^\pm_\varepsilon$ and some convenient cohesive interface $\Gamma^\pm_\alpha$ (see Fig. 4.3). For the nodes $A$ on $\Gamma^\pm_\alpha$, the continuity of the $\vec{u}$ and of the $\sigma$ must be maintained.

\[ U^+_A = U^-_A, \quad \text{ (4.30)} \]
\[ T^+_A = -T^-_A. \quad \text{ (4.31)} \]

For sub-domain $+$, let $B$ be the complement node set to $A$; then the boundary element equation (4.25) can be written as

\[
\begin{pmatrix}
H^+_B & H^+_A \\
H^-_B & H^-_A
\end{pmatrix}
\begin{pmatrix}
U^+_B \\
U^+_A
\end{pmatrix} =
\begin{pmatrix}
G^+_B & G^+_A \\
G^-_B & G^-_A
\end{pmatrix}
\begin{pmatrix}
T^+_B \\
T^+_A
\end{pmatrix}. \quad \text{ (4.32)}
\]

Similarly, for sub-domain $-$, let $D$ be the complement node set to $A$, then we have

\[
\begin{pmatrix}
H^+_D & H^-_A \\
H^-_D & H^-_A
\end{pmatrix}
\begin{pmatrix}
U^-_D \\
U^-_A
\end{pmatrix} =
\begin{pmatrix}
G^-_D & G^-_A \\
G^-_D & G^-_A
\end{pmatrix}
\begin{pmatrix}
T^-_D \\
T^-_A
\end{pmatrix}. \quad \text{ (4.33)}
\]

Matrices $H^\pm_{A,B,D}$, and $G^\pm_{A,B,D}$ consist respectively of the columns of $H^\pm$ and $G^\pm$ associated with the corresponding node sets.

Combined with the continuity conditions, the above two equations may be written in a matrix form:

\[
\begin{pmatrix}
H^+_B & H^+_A & 0 \\
0 & H^-_A & H^-_D
\end{pmatrix}
\begin{pmatrix}
U^+_B \\
U^+_A \\
U^-_D
\end{pmatrix} =
\begin{pmatrix}
G^+_B & -G^+_A & 0 \\
0 & G^-_A & G^-_D
\end{pmatrix}
\begin{pmatrix}
T^+_B \\
T^+_A \\
T^-_D
\end{pmatrix}. \quad \text{ (4.34)}
\]
We further assume that for domains $\pm$, the traction $T_{P\pm}^\pm$ is known at $P^\pm$ nodes on $\Gamma^\pm$, $U_{Q\pm}^\pm$ is known at $Q^\pm$ nodes on $\Gamma^\pm$, and the repulsion $r^\pm$ is known at $C$ nodes on fracture faces $\pm$. Correspondingly, we may partition the matrices $H^\pm$ and $G^\pm$ by column as

$$H = (H_{P^\pm}^\pm, H_A^\pm, H_C^\pm, H_{Q\pm}^\pm),$$

$$G = (G_{P^\pm}^\pm, G_A^\pm, G_C^\pm, G_{Q\pm}^\pm).$$

Thus, the equation to be solved is

$$AX = Y + R,$$

where

$$A = \begin{pmatrix} H_{P^\pm}^\pm & G_{Q^\pm}^\pm & H_A^\pm & -G_A^\pm & 0 & 0 \\ 0 & 0 & H_A^- & G_A^- & G_{Q^-}^- & H_{P^-}^- \end{pmatrix},$$

$$XT = (U_{P^\pm}^\pm & T_{Q^\pm}^\pm & U_A^- & T_A^- & T_{Q^-}^- & U_{P^-}^-),$$

$$Y = (G_{P^\pm}^\pm & H_{Q^\pm}^\pm & H_{Q^-}^- & G_{P^-}^-),$$

$$R = (G_C^\pm & G_C^-)(r^\pm).$$

With the repulsion scheme, only $r$ changes during the iteration. Consequently, $A$ needs to be decomposed once, which greatly speeds up the computations.
4.2.3 Straight discontinuous elements

At a corner of a boundary, a discontinuity in traction generally exists. One way to model this discontinuity is to use discontinuous element. In fracture problems, nonsmooth geometry such as kinks and tips exists. These geometric irregularities are places not only of traction discontinuity but also of stress concentration. In such cases, discontinuous elements often generate more accurate results than ordinary ones [34].

Discontinuity is achieved by displacing the end nodes of an element to the interior of the element so that the adjacent elements do not share an end node. In our implementation, a quadratic discontinuous element [43] is used everywhere on \( \Gamma \). The nodes of this element are located \( \xi = -\frac{2}{3}, 0, +\frac{2}{3} \), and the base functions are:

\[
\begin{align*}
\phi_1(\xi) &= \xi \left( \frac{9\xi}{8} - \frac{3}{4} \right) \\
\phi_2(\xi) &= (1 - \frac{3\xi}{2})(1 + \frac{3\xi}{2}) \\
\phi_3(\xi) &= \xi \left( \frac{9\xi}{8} + \frac{3}{4} \right)
\end{align*}
\]

(4.42) (4.43) (4.44)

\(-1 \leq \xi \leq +1\)  

(4.45)

The discontinuous element also assures the smoothness of the boundary at each node. Consequently in equation (4.21) \( c_{ij} = \frac{1}{2} \delta_{ij} \) for all nodes [34, 41, 35].
4.2.4 Computation of $\eta$ and $\zeta$

Because of the quadratic base functions, a deformed element is approximated by a parabola. For elements + and - in figure 4.4, their shape are given by

$$
\bar{p}^{\pm}(\xi) = (\bar{x}_{k \pm -1} + \bar{u}_{k \pm -1}) \phi_1(\xi) + \\
(\bar{x}_{k \pm} + \bar{u}_{k \pm}) \phi_2(\xi) + \\
(\bar{x}_{k \pm +1} + \bar{u}_{k \pm +1}) \phi_3(\xi)
$$

(4.46)

Thus, the distance of a point $k^+$ from an element defined by $k^- + 1$, $k^-$, and $k^- - 1$ is the solution to

$$
\eta_{k^+} = d(\xi') = \min_{-1 \leq \xi \leq 1} d(\bar{p}_{k^+}, \bar{p}^- (\xi))
$$

(4.47)

and the sliding $\zeta_{k^+}$ is simply the difference between $\xi'$ and the $\xi$ corresponding to node $k^-$. In the case of figure 4.4, $\xi = 0$ for node $k^+ - 1$.

$r_n$ and $\lambda$ are computed by equations (3.8) and (3.11). The normal and tangential vectors, $\vec{n}$ and $\vec{t}$, at node $k^+$ determine the orientation of the repulsion. They are computed on the basis of the geometry of element $+$.

$$
\vec{t} = \frac{d\bar{p}^+(\xi)}{d\xi} \\
n_x = t_y \\
n_y = -t_x
$$

(4.48) (4.49) (4.50)
4.2.5 Computation of $\vec{r}$

$\vec{r}$ at a contact node is given by

$$\vec{r} = -r_n (\vec{n} + \lambda \vec{r})$$  \hspace{1cm} (4.51)

Since we use discontinuous elements, nodes belong only to a single element; consequently only one repulsive force need to be considered at a node, even for elements at corners of the boundary. Although contact conditions are evaluated at the nodes of an element, the repulsion is imposed on that element in distributed form by using the base functions.
Chapter 5

Case Studies

5.1 Uniaxial compression

5.1.1 Setup and Procedure

Experiments with the finite element method were carried out on a model problem of uniaxial compression (see Figure 5.1). In this figure both $\beta$ and $\theta$ are counted positive counter-clockwise, and are positive as shown; $\beta$ is counted from the direction of the imposed compressive normal stress, while $\theta$ is zero if the crack propagates in its own plane. The major issue to be investigated here is why a natural fracture often develops in a plane, while an artificial cut generally extends by kinks. One increment of propagation suffices for this purpose.

In the present case we assume that the the medium is homogeneous, isotropic, and elastic-brittle, that the fracture is planar, and that the Amonton-Coulomb frie-
Figure 5.1: Configuration of the numerical experiments under uniaxial compression; $\theta$ and $\beta$ are positive counterclockwise.

tion law obtains. We also assume plane stress and the absence of body forces.

The block of rock has a Young’s modulus $E = 2.3 \times 10^{10}$ Pa, a Poisson’s ratio $\nu = 0.33$, a width of 16 m, and a length of 8 m. The stresses and displacements may be scaled by Young’s modulus and the length of the sides of the block, respectively. A compression in terms of displacement is imposed on the east and west sides of the block, and the north and south sides are left free (see Fig. 5.1). A pre-existing crack 4 m in length is assumed to exist at the center of the block with a given orientation $\beta$. In most experiments a virtual extension of length equal to one sixth of the pre-existing crack was introduced at the two crack tips at varying orientations $\theta$ with respect to the pre-existing crack. A different extension length was used in a few test cases (see below).

A quasi-static solution was obtained at each orientation. The finite element method was used, and linear triangles were employed (see Fig. 5.2). For open cracks the $r^{-1/2}$ stress singularity at the crack tip is generally modelled by using quarter
486 nodes and 908 elements

Figure 5.2: Triangulation of the body.

point triangles around the crack tip [44]. As we previously noted, the existence and/or the nature of the singularity at the crack tip is not known for closed cracks. For this reason we decided not to use special elements near the crack tips (see also [32, 33]).

As previously stated, we must compute the strain energy $S$ for varying orientations of the virtual crack extension. Although this poses no theoretical difficulty, it is not numerically straightforward because $S$ is computed by numerical integration; consequently its computed value depends on the triangularization, i.e., on the number and the geometry of the finite elements. This was confirmed by preliminary experiments in which we used a general mesh algorithm [45] to determine the mesh: mesh changes caused numerical errors of the same order as or larger than those due to the change of the orientation itself. Near the crack tips, therefore, we use the
special fixed mesh shown in figure 5.3. Each virtual extension corresponds to one of the edges radiating from the crack tip; these edges are spaced by $10^\circ$. It is obvious that the aspect ratio of the elements ahead of the crack tip is far from optimal. The important point, however, is that the mesh does not change as the orientation of the extension changes; the change in the computed strain energy thus depends only on the orientation of the extension, and the position of its minimizer is thus meaningful.

Although mathematical certainty is rarely if ever attainable by the use of numerical methods, we further examined the validity of the spline interpolation, especially in the region of a rapid change (between $-10^\circ$ and $10^\circ$). For this purpose we slightly altered the special mesh near the crack tip by moving the node nearest to the desired position, and re-computing $S(\theta)$. For $\theta = -4^\circ$, for example, node 0 was moved, and
Figure 5.4: a) Movement of node 0 to create an extension of $-4^\circ$; b) Movement of node 1 to create an extension of $-6^\circ$. Dashed line represents the extension.

For $\theta = -6^\circ$, node 1 was moved (see Fig. 5.4). Because of the very small change thus introduced in the mesh, the corresponding numerical error may be presumed to be small. Figure 5.5 shows the spline curve determined by the data at a $10^\circ$ interval and the additional data points; as one can see, the basic features of the curve are real. The preferential orientation of the extension is the minimizer. Finally, it should be noted that our main concern in the present work is not to determine the orientation of fracture propagation with a precision of the order of one degree, but to determine whether shear fractures propagate essentially in their own plane, or by means of kinks or wings at roughly $70^\circ$ to the crack.

Since the iteration is devised to comply with the constraints on the fracture faces, we test convergence by substituting the iterates in the relevant constraints, and determining how well the constraints are satisfied. Such a test is very rigorous.
Figure 5.5: Plot of strain energy (in Joules) vs. $\theta$ to verify the accuracy of the spline interpolation. The line is the interpolation, the asterisks (*) show data points.

Since the iterations are designed to null the interpenetration $\eta$, the error on $\eta$ must be small. On the other hand, if the crack opens, the faces must be stress free, and this condition must be tested. Thus:

- For an open crack we require $|\sigma^*| < 10^{-11}|\sigma_a|$ where $\sigma_a$ is the stress on the east and west faces due to the imposed displacement;

- For a closed crack we require $|\eta| < 100\mu m$ which is $\frac{1}{4000}$ of the crack length. In addition we require that $\mu^* = \mu$ for all sliding nodes, and that the amount of sliding be less than $0.1 \mu m$ for all sticking nodes.

For $\beta = 45^\circ$, tables 5.1 and 5.2 give the amount of penetration $\eta$ and sliding on the fracture faces for $\theta = 0^\circ$ and $\theta = -50^\circ$ at convergence. The position of the nodes is schematically shown in figure 5.6. As one can see, $|\eta|$ is smaller than
the convergence criterion in both tables. Table 5.1 corresponds to a crack which stays closed and propagates in its own plane; as expected, sliding decreases from the center of the crack (nodes 474 and 480) to both tips (nodes 483 and 484). Table 5.2 corresponds to a crack which propagates by means of an open kink: as expected there is an opening (negative penetration) at the kink points (nodes 485 and 486) where, as one can see, the repulsive force is null.

5.1.2 Preliminary discussion

In order to understand the direction of extension of a crack, it is useful to examine the stress field in the crack vicinity before extension. This was done for a typical case, where $\beta = 45^\circ$. The compression results in a right lateral shearing on the crack. Figures 5.7 and 5.8 show the hoop stress $\sigma_{\theta\theta}$ and the shear stress $\sigma_{r\theta}$ around the crack tip, and figure 5.9 shows the strain energy density distribution in the vicinity of the fracture. $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are computed at the nodes on the semi-circle around the tip using the special mesh for the vicinity of crack tips (see Fig. 5.3). These figures show that: 1) A small tensile hoop stress exists near $\theta = -45^\circ$; this
Table 5.1: Results for $\theta = 0^\circ$ and $\beta = 45^\circ$.

<table>
<thead>
<tr>
<th>Node No.</th>
<th>Normal repulsion $MN$</th>
<th>Penetration $\mu m$</th>
<th>Sliding $mm$</th>
</tr>
</thead>
<tbody>
<tr>
<td>477</td>
<td>14.2630</td>
<td>45</td>
<td>-3.175</td>
</tr>
<tr>
<td>476</td>
<td>16.3174</td>
<td>-60</td>
<td>-4.061</td>
</tr>
<tr>
<td>475</td>
<td>15.1542</td>
<td>98</td>
<td>-4.740</td>
</tr>
<tr>
<td>474</td>
<td>15.7077</td>
<td>30</td>
<td>-4.867</td>
</tr>
<tr>
<td>473</td>
<td>15.1449</td>
<td>99</td>
<td>-4.734</td>
</tr>
<tr>
<td>472</td>
<td>16.2458</td>
<td>-59</td>
<td>-4.054</td>
</tr>
<tr>
<td>486</td>
<td>13.9800</td>
<td>42</td>
<td>-3.170</td>
</tr>
<tr>
<td>484</td>
<td>0.0</td>
<td>-00</td>
<td>0.000</td>
</tr>
<tr>
<td>471</td>
<td>14.2336</td>
<td>46</td>
<td>-3.170</td>
</tr>
<tr>
<td>478</td>
<td>16.3011</td>
<td>-60</td>
<td>-4.054</td>
</tr>
<tr>
<td>479</td>
<td>15.1737</td>
<td>99</td>
<td>-4.734</td>
</tr>
<tr>
<td>480</td>
<td>15.7075</td>
<td>30</td>
<td>-4.867</td>
</tr>
<tr>
<td>481</td>
<td>15.1279</td>
<td>98</td>
<td>-4.740</td>
</tr>
<tr>
<td>482</td>
<td>16.2738</td>
<td>-59</td>
<td>-4.061</td>
</tr>
<tr>
<td>485</td>
<td>14.0060</td>
<td>41</td>
<td>-3.175</td>
</tr>
<tr>
<td>483</td>
<td>0.0</td>
<td>-00</td>
<td>0.000</td>
</tr>
</tbody>
</table>

is the direction of the compressive normal stress. 2) Near $\theta = 0^\circ$, the shear stress $\sigma_{r\theta}$ is very large and in the same sense as that of the sliding on the crack, while near $\theta = \pm 90^\circ$ it is very large but in the opposite sense to that of the sliding; 3) A strip of high strain energy density starts at the tips and is oriented essentially in the direction of the crack. 4) The middle part of the fracture faces is less stressed than the part near the tips.

5.1.3 Results without residual strain energy

The first group of experiments was conducted to study the extension of a pre-existing artificial cut. In such a case no strain energy remains from the creation of
Table 5.2: Results for $\theta = -50^\circ$ and $\beta = 45^\circ$.

<table>
<thead>
<tr>
<th>Node No.</th>
<th>Normal repulsion $MN$</th>
<th>Penetration $\mu m$</th>
<th>Sliding $mm$</th>
</tr>
</thead>
<tbody>
<tr>
<td>477</td>
<td>5.1158</td>
<td>25</td>
<td>-3.043</td>
</tr>
<tr>
<td>476</td>
<td>14.9618</td>
<td>47</td>
<td>-4.051</td>
</tr>
<tr>
<td>475</td>
<td>15.8116</td>
<td>-43</td>
<td>-4.559</td>
</tr>
<tr>
<td>474</td>
<td>14.7455</td>
<td>98</td>
<td>-4.849</td>
</tr>
<tr>
<td>473</td>
<td>15.7971</td>
<td>-42</td>
<td>-4.553</td>
</tr>
<tr>
<td>472</td>
<td>14.8561</td>
<td>49</td>
<td>-4.044</td>
</tr>
<tr>
<td>486</td>
<td>0.0</td>
<td>-2312</td>
<td>-1.971</td>
</tr>
<tr>
<td>484</td>
<td>0.0</td>
<td>00</td>
<td>0.000</td>
</tr>
<tr>
<td>471</td>
<td>5.0817</td>
<td>26</td>
<td>-3.038</td>
</tr>
<tr>
<td>478</td>
<td>14.9437</td>
<td>47</td>
<td>-4.044</td>
</tr>
<tr>
<td>479</td>
<td>15.8400</td>
<td>-41</td>
<td>-4.553</td>
</tr>
<tr>
<td>480</td>
<td>14.7450</td>
<td>98</td>
<td>-4.849</td>
</tr>
<tr>
<td>481</td>
<td>15.7710</td>
<td>-43</td>
<td>-4.559</td>
</tr>
<tr>
<td>482</td>
<td>14.8744</td>
<td>49</td>
<td>-4.051</td>
</tr>
<tr>
<td>485</td>
<td>0.0</td>
<td>-2317</td>
<td>-1.973</td>
</tr>
<tr>
<td>483</td>
<td>0.0</td>
<td>-0.0</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The fracture; by contrast, on a natural fracture, part of the strain energy remains from the creation (or the propagation) of the fracture.

*Constant $\mu$ on the pre-existing crack and its extension* - The simplest assumption is that $\mu$ has the same constant value on both the pre-existing crack and on its natural extension. As we will see, such an assumption is probably not realistic, but it provides a convenient reference point. Different orientations of the pre-existing crack were used in order to examine the dependence of $\theta^*$, the direction of the real crack extension, on $\beta$. (It will be recalled that, for a given increment length, $\theta^*$ is the maximizer of the strain energy release, or the minimizer of the remaining strain energy.) Similarly for a given $\beta$, a number of $\mu$'s were used in order to examine
Figure 5.7: Circumferential tensile stress $\sigma_{\theta\theta}$ (in Pa) vs. $\theta$ around the crack tip before extension.

the dependence of $\theta^*$ on $\mu$. For $\beta = 30^\circ, 35^\circ$, and $45^\circ$, figures 5.10, 5.11, and 5.12, respectively, show the remaining strain energy $S(\theta)$ vs. $\theta$, the virtual orientation of the extension, for various $\mu$. These curves lead to four different observations:

1. The largest minimum always occurs near $\theta = 0^\circ$, which implies that a pre-existing shear crack propagates in its own plane.

2. As $\mu$ increases, the release of strain energy decreases.

3. As $\mu$ increases, a secondary minimum becomes more pronounced and migrates toward $-80^\circ$, indicating that $\mu$ may play a large role in the deflection of the extension.

4. When the orientation $\beta$ of the pre-existing crack changes from $30^\circ$ to $45^\circ$, the secondary minimum exists for almost all $\mu$. 
Figure 5.8: Shear stress $\sigma_{r\theta}$ (in Pa) vs. $\theta$ around the crack tip before extension.

These observations can be explained as follows:

1. The direction of propagation is largely determined by the fact that the shear stress induced by the motion on the pre-existing crack (see figures 5.8 and 5.9) can be greatly reduced by sliding on an extension of the same orientation; naturally, such a stress reduction leads to a large reduction of the strain energy. This is the case even when the orientation of the crack, $\beta$, is not the preferential one for crack initiation, because the pre-existing crack always introduces a strong shear stress zone ahead of its tips in its own orientation.

2. As $\mu$ increases, sliding becomes more difficult. Consequently the extension of a pre-existing crack in its own plane may not result in appreciable sliding, and will thus not greatly reduce the existing shear stress. Thus the strain energy reduction due to the opening of a kink extension becomes significant, as shown
Figure 5.9: Strain energy density (J/m$^2$) distribution in the vicinity of the preexisting crack at $\beta = 45^\circ$ before extension.
Figure 5.10: Remaining strain energy $S(\theta)$ (in Joules) vs. virtual orientation $\theta$ for a preexisting crack at 30° and various $\mu$'s when the residual strain energy is not taken into account. Circles (o) represent data points.

by observation 3.

3. Let $\beta^*$ be the orientation of crack initiation corresponding to the largest reduction in strain energy, $G_{Max}$; for such a crack $\theta \approx 0$ will obviously also give the largest energy release. For the $\mu$'s used in these experiments, the values of $\beta^*$ range from 28° to 32° (see [32, Fig. 3a]). In figures 13 an 14 the values of $\beta$ are respectively 35° and 45°. As the difference between $\beta$ and $\beta^*$ increases, sliding with $\theta \approx 0$ becomes more difficult, i.e., the reduction in strain energy decreases, while that due to a deflected extension is much less affected. The secondary minimum thus becomes much more pronounced, just as is observed in 4.
Figure 5.11: see Fig. 5.10, but crack at 35°.

Different $\mu$ on the pre-existing crack and its extension - As we have previously noted, it is generally thought that a shear crack grows by means of open kinks oriented close to the direction of the compressional axis, and numerous observations made with pre-existing cracks bolster this opinion. We will show that these observations in no way contradict the results of our previous computations; indeed, they are to be expected if the friction coefficient $\mu$ on the virtual crack extension is appreciably larger than on the crack itself.

Let us consider a crack with friction coefficient $\mu_c$ and its virtual extension with friction coefficient $\mu_e$, and assume that $\mu_e$ is appreciably larger than $\mu_c$. Such a situation is to be expected: natural faults have fractal surfaces [46], their friction coefficient is thus large. This is of course also the case on a natural crack extension. By contrast the faces of artificially made cracks are relatively smooth, and their
Figure 5.12: see Fig. 5.11, but crack at 45°.

friction coefficient is thus fairly low. We first consider the case in which the extension is in the plane of the crack. Because of the high $\mu_e$ sliding on such an extension is relatively difficult, i.e., it does not release much strain energy. Moreover, because the crack and its extension are in a straight line, sliding is inhibited on the crack itself as well.

The situation is quite different if the virtual extension is greatly deflected towards the compressional axis: on the crack itself sliding now occurs due to the low value of $\mu_e$, and is not inhibited by the high value of $\mu_e$; on the extension, opening takes place, just as the classical observations show, and this opening releases some of the strain energy near the crack tips. We need hardly point out that both our experiments and this reasoning assume the absence of confining pressure.
Figure 5.13: $S(\theta)$ (in Joules) vs. $\theta$ when $\mu_e = \mu_c = 0.7$ and $\mu_e = 1.8$ and $\mu_c = 0.7$. Extension lengths are $1/6$ of the crack length.

It is relatively easy to examine the validity of the preceding considerations with our numerical method. The crack orientation $\beta = 45^\circ$ was chosen to be the same as used by Horii and Nemat-Nasser [21, Fig. 14]. In one case we used $\mu_e = \mu_c = 0.7$, in the other case $\rho = \mu_e/\mu_c = 1.8/0.7 \approx 2.6$. The results are shown in figure 5.13. As expected the minimum of $S(\theta)$ near $\theta = -65^\circ$ becomes deeper than the one near $\theta = 0^\circ$ for $\rho = 1.8/0.7$. Furthermore, also as expected, we found that the extension is open for values of $\theta < -14^\circ$. When the extension is open, $\mu_e$ has no influence on $S(\theta)$. Since $\mu_c$ is the same for both cases, the two curves coincide for $\theta < -14^\circ$.

5.1.4 Results with residual strain energy

In laboratory experiments and, presumably, in fault propagation, the residual energy remaining after crack initiation is not dissipated; its influence should thus be taken
Figure 5.14: $S(\theta)$ (in Joules) vs. $\theta$ for an initial fracture at $\beta = 30^\circ$ and various $\mu$'s when the residual strain energy is taken into account.

Into account in a numerical model. On the other hand, because the fracture initiates and propagates in a natural way, the friction coefficient on the extension is the same as that on the initial fracture. We carried out numerical experiments which take these considerations into account; the results are shown in figure 5.14. A comparison of figures 5.14 and 5.10 for which the crack orientation is the same, shows that the minimum at $\theta \simeq 0^\circ$ is much more pronounced when the influence of the residual strain energy is taken into account. This is logical since this energy is concentrated near $\theta = 0^\circ$ (see Fig. 5.9). The absence of this influence is thus another factor that causes the extension of an artificial cut, where no residual energy exists, to be deflected toward the compression axis, i.e., to form kinks.
Figure 5.15: see Fig. 5.13, but extension lengths are 1/12 of the crack length.

5.1.5 Influence of the relative extension length

As previously mentioned we also tested the influence of the relative length of the extension compared to that of the crack. If this length is small enough, its influence should, of course, be negligible. In our case it is not evident that this condition is fulfilled, but computational considerations had motivated our choice. We thus repeated the experiments shown in figure 5.13, but with an extension length only half of the previous one; the results are shown in figure 5.15. When the friction coefficients on the crack and its extension are the same, ($\rho = 0.7/0.7$), the curves in these two figures give the same value of the minimizer within the expected precision, i.e. $2^\circ \pm 1^\circ$; the minimum near $55^\circ$ is still secondary. When the two values of $\mu$ differ substantially, ($\rho = 1.8/0.7$), the minimizers differ by roughly $4^\circ$; more importantly, though, the minimum near $60^\circ$ is now the minimizer of $S$ in both Figures. This
agreement clearly shows that the difference in friction coefficients between a natural and an artificial crack is the main cause of the creation of kinks as a mode of crack propagation.

5.2 Shearing

5.2.1 Generality

The propagation of cracks inside or at the edge of a two-dimensional rectangular elastic body has been investigated with the boundary element method. The relative flexibility of the BEM in dealing with changing geometry allows us to use more than one increment of the fracture growth under simple shearing, transtension, and transpression. The interaction of two cracks under simple shearing is also investigated.

As shown in figure 5.16, the loading in terms of a tangential displacement \( u \) and a normal component \( v \) is imposed on the north and south sides of the body. Depending on the value and the sign of \( v \), cases of simple shearing, transpression, and transtension result.

A crack at least 4 m long was introduced into a square, homogeneous, and isotropic elastic body 10 m by 10 m for all cases. A length of 0.3 m is used for all subsequent increments. On the basis of Proposition 1, Young's modulus \( E \) has no influence on the orientation of a fracture extension. We arbitrarily choose \( E = \)
1000.0 for all experiments. Since we are interested in the propagation of cracks under various loading conditions, Poisson's ratio $\nu$ is kept constant (0.33) in all experiments.

With the direct BEM, the solution is obtained in terms of the displacement $\mathbf{u}$ and the traction $\mathbf{T}$ on the boundary of a deformed body. By equation (2.10), the strain energy of the body is equal to the work done on its boundary, i.e.,

$$ S = \oint_{\Gamma} \mathbf{U} \cdot \mathbf{T} \, d\gamma $$

(5.1)

By definition, $\Gamma$ includes the fracture faces $\Gamma_c^\pm$.

As mentioned before, the discontinuity due to the presence of cracks is introduced by the multi-domain technique with a BEM formulation (e.g., Fig. 5.16-d). Various orientations of a crack extension inevitably result in different connections $aa'$. It is expected that such connections may make a contribution to the variation of computed strain energy. We assume that,

$$ S_c(\theta) = S_r(\theta) + S_a(\theta), $$

(5.2)

where $S_c$ and $S_r$ denote the computed and the real strain energy respectively, and $S_a$ denotes the artificial contribution.

Since the connection is not real, its artificial effect must be checked. Note that a crack is only a geometry entity if no displacement discontinuity exists across it. Thus, at each increment, the strain energy before the increment, $S_c^-$, is computed for cohesive extensions of various orientations $\theta$. The cohesive extensions are geo-
metrically identical to the potential virtual orientations. In such a case,

\[ S_c^-(\theta) = S_r^- + S_u(\theta) \]  \hspace{1cm} (5.3)

where \( S_r^- \) is the real strain energy that is independent of how the domain is partitioned. The variation in \( S_c^- \) with respect to \( \theta \) is thus simply due to the artificial effect due to the connection.

In each group of experiments, a typical case is tested. A comparison of the artificial variation and the total variation is made. If the artificial variation is only a insignificant part of the total variation, the minimizer of the real \( S(\theta) \) can be correctly identified by the minimizer of computed \( S(\theta) \); otherwise, we use the maximizer of

\[ \Delta S_c(\theta) = S_c^- - S_c^- = S_r + S_u - S_r^- - S_u = S_r - S_r^- \]  \hspace{1cm} (5.4)

to identify the preferential orientation.

In general, the boundary of a sub-domain consists of three types: a portion \( \Gamma_b \) of the boundary of the two-dimensional body, a cohesive connection \( \Gamma_a \) with another sub-domain, and a fracture face \( \Gamma_c \). We use a characteristic length to control the discretization of each type of boundary. This characteristic length is the desirable length for an element of the boundary of given type; the actual length of the element, determined by the actual length of a boundary, is the closest one to
it. As the characteristic length decreases, the accuracy of the numerical analysis increases. In practice, we found that the smoothness of the curve $S$ vs. $\theta$ indicates the accuracy. Therefore, we keep refining the discretization until a satisfactory smoothness is achieved. In our cases, characteristic lengths of 1.5 m, 1.0 m, and 0.5 m for conventional boundaries, cohesive interfaces, and crack faces respectively are found to be sufficient to achieve the desirable smoothness.

A quasi-static solution is obtained for each virtual orientation of an increment. The orientation of the increment is determined by the maximizer of $\Delta S(\theta)$ or the minimizer of $S(\theta)$. The trajectory of a fracture growth is obtained by carrying several steps of the increment.

### 5.2.2 Growth of a single edge crack

The configuration and the loading conditions of the first group of experiments are shown in figure 5.16 (a-c). The objective here is to investigate the propagation of a single crack under various loading conditions. The elastic body has dimensions of 10 m × 10 m. A preexisting straight crack of length 4 m, parallel to the north side of the body, is located at the middle of the west edge of the body. A load in terms of tangential ($u$) and normal ($v$) displacements was imposed on the north and south sides, while the east and west edges were left traction free ($u$ is positive as shown, $v$ is positive along the outer normal). Up to 9 increments of crack growth under a given loading were used. The length of each increment is 0.3 m, about
one thirteenth of the length of the preexisting crack. \( u \) was chosen to be 0.1 m for all experiments, and \( v \) was chosen to be -0.5, -0.1, 0, 0.1, 0.5, and 1.0 \( \times u \), corresponding to situations of transpression, simple shearing, and transtension.

The body is partitioned into two sub-domains in order to introduce the crack and its extensions (see Fig.5.16-d). As mentioned above, the variation of \( S \), when sliding on the new extension is prohibited, indicates the artificial effect of the multi-domain scheme. Table 5.3 gives \( S_c^- \) (resulting from the artificial effect only) and
Table 5.3: Comparison of the artificial variation, $\lvert \delta S^- \rvert = \lvert S^-_c(\theta_k) - S^-_c(\theta_{k-1}) \rvert$, of $S$ due to a multi-domain scheme with the total variation, $\lvert \delta S_c \rvert = \lvert S_c(\theta_k) - S_c(\theta_{k-1}) \rvert$, of $S$ due to the change of $\theta$.

<table>
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<tr>
<th>$\theta_k^\circ$</th>
<th>no sliding on new extension</th>
<th>sliding on new extension</th>
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<tr>
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<td>$S^-_c(\theta_k)$</td>
<td>$\lvert \delta S^- \rvert$</td>
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<tr>
<td>90</td>
<td>17.963700</td>
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its comparison with $S_c$ (the computed strain energy after extension). For the sake of generality, the case is chosen to be the sixth incremental growth of a crack in a body under transpression. It can be seen that the artificial variation is at most 2.5% (see $\theta = 55^\circ$ in the table) of the total variation, which shows that the artificial effect is practically absent. The minimizers of $S_r$ and those of $S_c$ are thus essentially identical.

The friction coefficient $\mu$ is chosen to be 0.5 for all experiments so as to make them to be comparable. Figure 5.17 shows the change of $S$ vs. $\theta$ for the first extension of the crack under various loading conditions. The circles represent data points obtained at a $5^\circ$ interval. The data points are connected simply by lines.
Figure 5.17: Strain energy (in Joules) vs. direction of crack extension for various loadings. a. transpression with \( \frac{u}{u} = 0.5 \); b. transpression with \( \frac{u}{u} = 0.1 \); c. simple shearing; d. transtension with \( \frac{u}{u} = 0.1 \); e. transtension with \( \frac{u}{u} = 0.5 \); and f. transtension with \( \frac{u}{u} = 1.0 \).

without any smoothing. Figure 5.18 shows the crack growth for up to 9 increments.

Figures 5.17 and 5.18 show that:

1. Under transpression, when compression is insignificant (e.g. \( \frac{u}{u} = 0.1 \)), the crack growth is very much like that under simple shearing (Fig. 5.17 b and Fig. 5.18 b); when compression is significant (e.g. \( \frac{u}{u} = 0.5 \)), the first extension of the crack is at -30°, and the crack grows along a smooth concave path (Fig. 5.17 a and Fig. 5.18 a).

2. Under simple shear, the crack grows with an open kink extension (Fig. 5.17
Figure 5.18: Trajectory of crack growth for various loadings. a. transpression with $\frac{u}{h} = 0.5$; b. transpression with $\frac{u}{h} = 0.1$; c. simple shearing; d. transtension with $\frac{u}{h} = 0.1$; e. transtension with $\frac{u}{h} = 0.5$; and f. transtension with $\frac{u}{h} = 1.0$.

c and Fig. 5.18 c). The first extension is at -70° as given by the minimizer of $S(\theta)$, and the crack grows along a smooth convex path.

3. Under transtension, when the tension is insignificant (e.g. $\frac{u}{h} = 0.1$), the first crack extension is at -65°, very much like that under simple shearing (Fig. 5.17 d and Fig. 5.18 d). For $\frac{u}{h} = 0.5$ and $\frac{u}{h} = 1.0$, the first extension is at -45° and -15° respectively. The crack grows along a smooth convex path (Fig. 5.17 d, e, and f and Fig. 5.18 d, e, and f).

4. The trajectory of the crack growth is smooth in all cases except for the sharp
angle of the first increment with respect to the preexisting crack.

For the first extension of a crack under simple shearing, the numerical results show that both the extension and the crack faces are opened by a small amount of the order of $10^{-3}$ m on the crack and $10^{-2}$ m on the extension. Thus, no contact occurs, and friction has no influence on fracture propagation. In such a case, the SIF theory is adequate for studying the fracturing problem. In the SIF terminology, the crack propagation under simple shear is of pure mode II. On the basis of the maximum circumferential tensile stress criterion, a crack of pure mode II will extend at $70.5^\circ$ (see, e.g., Erdogan and Sih [9]), a result verified by many laboratory experiments [9, 10, 11, 12, 13]. The remarkable agreement between our result ($70^\circ$) and that obtained by previous investigators confirms the validity of the $G_{\text{max}}$ criterion and its numerical implementation. Furthermore, the computed crack trajectory is almost identical to the one in the laboratory results of Erdogan and Sih [9].

For the transtension cases, our numerical results confirm the expectation that the crack and its extension are open. An exact comparison of our numerical results with the laboratory results of Erdogan and Sih [9] cannot be made because the loading conditions of the laboratory experiments are unclear from their paper; however, the fact that, under simple shear, $\theta$ systematically decreases from $70.5^\circ$ toward $0^\circ$ as the tension increases is in agreement with these results. This is due to the fact that with increasing dominance of tensile loading, the crack propagation is getting closer to propagating in its own plane (mode I propagation).
Little study has been reported on cases of transpression. When the normal component \( v \) of the loading is large enough, the friction resistance is sufficient to prevent sliding on the preexisting crack, and no discontinuity in displacement exists across the crack. When this happens, the crack is only a geometrical entity without any physical significance, and does not propagate. We will restrict ourselves to cases where sliding occurs along the crack. Our numerical results show that the crack faces are in contact in such cases. The SIF approach is thus not applicable to these cases because it cannot account for the friction effect. The \( G_{\text{max}} \) criterion is thus indispensable. Our results show that it generates a result consistent with observations or intuitions.

If we look along the crack toward its tip, figure 5.18 shows that in all cases — simple shear, transpression, and transtension — the crack turns to the right for a right-lateral shearing. Clearly the crack will turn to the left in all cases of left-lateral shearing because of symmetry.

Finally, the smooth trajectory of the crack growth and the abrupt change of growth orientation from the orientation of the preexisting crack imply that in those cases the crack and its extension cannot be due to the same loading. An abrupt change of orientation of a fault may thus indicate a polyphase tectonic loading history if heterogeneity or other local irregularities are absent.
Figure 5.19: Two multi-domain schemes. The dashed lines show the cohesive interface between the sub-domains; the stipled line is the reference line used to define $\beta$.

### 5.2.3 Interaction of two edge cracks

Although the interaction of the tips of two cracks under mode I loading has been studied by a number of authors [47, 48, 49], such interaction under simple shearing and transpression loading is not yet clear. Note that the latter cases are more geologically realistic.

Although the elastic body has the same dimension, 10 m $\times$ 10 m, as before, the configuration is changed. Two cracks of length $l_c$ and of orientation $\beta$ start at the east and west edges of an elastic body, as shown in figure 5.19. The length of the extension $l_e$ is still 0.3 m. The crack length $l_c$ and its orientation $\beta$ vary from case to case in order to control the separation of the cracks. The combination of $l_c$ and $\beta$ may result in the overlapping of the two cracks. When this happens, the multi-domain scheme of figure 5.19-a results in sharp corners for the sub-domains, and this is numerically undesirable. In order to avoid this difficulty, the sub-domain scheme shown in figure 5.19-b is adopted.
The mechanical properties of the body such as Young's modulus $E$, Poisson's ratio $\nu$, and friction coefficient $\mu$ are the same as for the cases of a single edge crack. These choices make the results comparable with those of a single crack, and, thus, make it possible to identify the interaction of the cracks.

**Growth of one crack**

The two cracks are symmetric with respect to the center of the plate. Theoretically, they should thus have identical behaviors. However, in reality perfect symmetry may not always be achieved, and the simultaneous growth of two cracks is not often observed [50]. Therefore, in this group of experiments, we restrict the growth to only one crack.

The results are presented in terms of $\Delta S_c$ vs. $\theta$ because the artificial effect may not be ignored in the present cases. The preferential orientation is therefore identified by the maximizer of $\Delta S_c$. The results in figure 5.20 were obtained with $\beta = 2.0^\circ$, and with $l_c = 4.0$ m, 4.5 m, and 4.8 m resulting in increasing closeness of the two right-stepping cracks, respectively 2.0 m, 1.0 m, 0.4 m. In these cases, the two cracks are not overlapping. A left-lateral simple shearing of $u = 0.1$ m was imposed along the north and south sides. Only one increment of propagation was carried out for it suffices to show the interaction of the cracks.

From figure 5.20 we may observe that:

1. The crack on the left turns at 65° away from the one on the right when $l_c =$
Figure 5.20: Interaction of two non-overlapping cracks when their tips get close. Left: strain energy reduction (in Joules) vs. $\theta$. Right: trajectory of crack growth.

4.0 m (Fig. 5.20-a).

2. The crack on the left turns at $-10^\circ$ and $-15^\circ$ toward the one on the right when $l_c = 4.5$ m and 4.8 m respectively when the distance of the two tips is about two to three times the extension length $l_c$.

3. The maximum of $\Delta S$ around $-15^\circ$ is increasingly significant as the two tips get closer, and the angle with the crack direction becomes increasingly negative.

Since the two cracks makes an angle of $2^\circ$ with the orientation of the imposed
shearing, the cracks are under transtension. As previously noted, a single edge crack under such a loading tends to grow at an angle less than 70° from the crack. The growth of the crack with a large separation from the other crack, shown in figure 5.20-a, is similar to that of a single crack; the small difference between 65° and 70° is due to the small tilting of the cracks. Furthermore, a comparison of figure 5.17-d with figure 5.20-a.1 shows that they are very similar, except that the direction of the axes have been reversed. This reversal is due to the reversal of the sense of shear and to the fact that in one case we are plotting the remaining strain energy and in the other the strain energy reduction. This shows that the cracks in figure 5.20-a.2 do not interact with each other. Figure 5.20-b.1 differs appreciably from figure 5.17-d; the local maximum near -15° is now slightly greater than the maximum near 55°, showing that the two cracks now interact. When the two crack tips are even closer (Fig. 5.20-c), the two cracks will eventually interconnect: this corresponds to the fact that the maximum near -15° become significantly larger than that near 55° (Fig. 5.20-d).

Figure 5.21 shows the results for overlapping cracks. In these cases, \( l_c = 6.0 \) m, and \( \beta = 2^\circ, 5^\circ, \) and \( 20^\circ \), resulting in increasing separation of the cracks. The loading is the same as for the previous case.

As in the case of cracks without overlapping, for a large separation (Fig. 5.21-c) the strain energy curve closely resembles that for a single crack, showing that no crack interaction occurs. For \( \beta = 20^\circ \) (Fig. 5.21-c), the crack extends at 60° because
Figure 5.21: Interaction of two overlapping cracks. Left: strain energy reduction (in Joules) vs. $\theta$. Right: trajectory of crack growth.

of the $20^\circ$ difference in the orientations of the crack and the shearing. As the cracks get closer ($\beta$ gets smaller), a local maximum near $-15^\circ$ develops, and finally becomes the global maximum; the cracks tend to connect with each other.

The growth of two cracks under simple shearing is extremely different from that of two cracks under tension [47, 49] where two collinear cracks grow in their own orientations until interaction takes place. When this happens, the two cracks first avoid each other, and then tend to connect (see, e.g., [49, Fig. 1]). For the case of simple shearing, a single crack will not grow in its own orientation when the
interaction with the other crack is absent; instead, it grows by a kink near 70°. Therefore, it is unlikely that two collinear cracks will be running toward each other before the interaction occurs. When two cracks are close enough, interaction takes place, and the cracks simply tend to connect with each other.

**Growth of two cracks**

The growth of two right-stepping cracks under right-lateral shearing is studied with the numerical model where $l_c = 4.0$ m, $l_v = 0.3$ m, $u = 0.1$ m, and $v = 0.0$ (initial separation of 2.0 m Fig. 5.23). At each increment, the cracks grow simultaneously from each tip. Due to the symmetry, only one $\theta$ suffices to describe their extensions. The preferential orientation, $\theta^*$, is determined by the minimizer of $S_c(\theta)$ because the artificial effect is insignificant. Figure 5.22 shows $S$ vs. $\theta$ for each of the four steps taken, and figure 5.23 shows the fracture trajectory at the corresponding stages of the crack growth. Note that $\theta$ in figure 5.22 is measured from the orientation of the previous increment.

These figures show that the first extensions of the cracks are deflected from their own orientations at -70°, and the orientation of subsequent extensions are +5° with respect to the previous extension. This is quite similar to the results for the growth of a single crack. The interaction of the two cracks is thus negligible.

Geologically, the two preexisting cracks may represent an old right-stepping en échelon fault system. This experiment shows that when this system is under a
right-lateral shearing, a pull-apart basin may develop in the region confined by the trajectory of the extensions. This idea has been much discussed by geologists (see *Sylvester* and the references therein [51]). Note that the axis of the basin trends approximately 65° with respect to the orientation of the en échelon faults.

### 5.2.4 Growth of two embedded cracks

The growth of two right-stepping cracks under a left-lateral simple shear has been investigated with the BEM. It has been found in laboratory experiments [36, 37] that right-stepping en échelon faults develop on the surface when the basement
Figure 5.23: Growth of two right-stepping horizontal cracks under right-lateral shearing.

underneath undergoes a left-lateral strike-slip faulting. It is thus interesting to compare the numerical results with the laboratory observations.

Two cracks 4 m long are embedded in a rectangular body 16 m by 10 m (Fig. 5.24). The cracks trend 20° with respect to the shearing direction, and are at the center of the body with various separations (2 m, 4 m, and 6 m) between their midpoints. A simple left-lateral shearing of 0.1 m is imposed on the north and south sides of the body while the east and west sides are free of loading or constraint. Young's modulus $E$ and Poisson's ratio $\nu$ are the same as previously.
Figure 5.24 shows $\Delta S_e$ vs. $\theta$ and the trajectory of the cracks. The preferential orientation, $\theta^*$, is determined by the maximizer of the computed strain energy reduction, $\Delta S_e$. This figure shows that:

1. When the two cracks are separated by 6 m (Fig. 5.24-c), there is no interaction between them. Each crack extends at about 30° as if the other was absent (see Fig. 5.18-f), and the cracks turn away from each other.

2. When the two cracks are close (0.5 m in Fig. 5.24-a and 1 m in Fig. 5.24-b), interaction takes place; they now turn toward each other.

Due to the 20° angle with respect to the shearing, the cracks are under transtension. The cracks with no interaction thus turn to the left, in the same sense as the shearing. The interaction of the cracks is thus clear when they are close. One can also observe that there are generally two local maximizers: one near +30° and the other near -25°. Depending on the configuration, namely the separation of the cracks, either of them can be the global maximizer. Therefore, under the same loading conditions faults may grow differently depending on their spacing.

In our numerical experiments, we always extend cracks from their tips. This is clearly not the case in the laboratory observations by Wilcox et al. [36] and Tchalenko [37] that during the late stages en échelon faults connect with each other but not from their tips. We believe that the difference in behavior is due to the difference in loading conditions: In our numerical experiments, loading is imposed on the edge of the body and would provide a continuous stress field if the cracks
Figure 5.24: Growth of two right-stepping embedded cracks under left-lateral shearing. a. separation of 2 m; b. separation of 4 m; and c. separation of 6 m. Left: strain energy reduction (in Joules) vs. $\theta$. Right: trajectory of crack growth.

were absent. The embedded cracks result in stress concentrations at their tips. In the laboratory experiments, shearing is imposed directly underneath the en échelon faults, and the stress and displacement are discontinuous even if the faults are absent. In other words, a strong local shear loading exists in a narrow zone directly above the "basement fault". The crack tips are thus less highly stressed when they reach outside that zone, while the highly stressed zone results in further fracturing; the connection of the en échelon faults then follows.
Chapter 6

Conclusions

From this research we draw following conclusions.

1. The $G_{\text{max}}$ criterion is both theoretically sound and experimentally verified. It is indispensable in solving problems of fracture growth under compressive loading.

2. The repulsion scheme gives the solution to frictional contact problems, and is computationally efficient and reliable. Convergence has been obtained in all cases tested.

3. The implementation of the repulsion scheme is correct on the basis of the comparison of the numerical and the laboratory results available.

4. A natural crack under uniaxial compression propagates in its own plane. This is observed in some laboratory experiments, and is explained by the fact that
extension in the orientation of the crack releases the maximum strain energy.

5. An artificial cut under uniaxial compression extends by a kink at 70°. This is also observed in laboratory experiments. The reason is that $\mu$ is high on natural fracture faces but low on the artificial cut.

6. A single preexisting crack under simple shearing or under transtension turns right under right-lateral shearing, and vice versa. Under simple shearing, the orientation of the first extension is 70.5° from the crack. It decreases when the tension increases, and reaches 0° when the crack is under pure tension. The trajectory of the extensions is convex in all cases.

7. A single preexisting crack under transpression turns right under right-lateral shearing, and vice versa. The orientation of the first extension is less than 70.5°. When the compression is significant, the convexity of the extension trajectory is opposite to that in simple shearing cases or in transtension cases.

8. Two right-stepping cracks under a left-lateral shearing run away from each other when their midpoints are approximately 1.5 crack lengths apart. They turn toward each other when their midpoints are less than one crack length apart.

Laboratory and numerical results show that a natural fracture under uniaxial compression is straight shear fracture, while the trajectory of fracture growth under shearing is a smoothly curved path with an abrupt change in orientation with respect
to the preexisting fracture. This implies that the straight strike-slip faults found in nature may not be due to shear tectonics or to shear component of oblique convergence because a fault under shearing is bound to change its orientation abruptly. The abrupt orientation change of a fault may indicate two tectonics history:

1. An old fault, in which a large amount of fluid existed, was under compression. The friction resistance on the faces of the old fault was much lower than that on the faces of the potential extension, or

2. An old fault was reactivated by shearing.

Furthermore, a convex fault trajectory may indicate simple shear or transtension tectonics, while a concave fault trajectory may indicate transpression tectonics.
Bibliography


