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Compactification problems in the theory of characteristic currents associated with a singular connection

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COMPACTIFICATION PROBLEMS IN THE
THEORY OF CHARACTERISTIC CURRENTS
ASSOCIATED WITH A SINGULAR CONNECTION

by

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ABSTRACT

COMPACTIFICATION PROBLEMS IN THE
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ASSOCIATED WITH A SINGULAR CONNECTION

JOHN ZWECK

A compactification of the Chern–Weil theory for bundle maps developed by Harvey and Lawson is described. For each smooth section $\nu$ of the compactification $\mathbb{P}(\mathcal{Q} \oplus F) \to X$ of a rank $n$ complex vector bundle $F \to X$ with connection, and for each $\text{Ad}$–invariant polynomial $\phi$ on $\mathfrak{gl}_n$, there are associated current formulae generalizing those of Harvey and Lawson. These are of the form

$$\phi(\Omega_F) + \nu^*(\text{Res}_\infty(\phi)) \text{Div}_\infty(\nu) - \phi(\Omega_0) - \text{Res}_0(\phi) \text{Div}_0(\nu) = dT$$

on $X$,

where $\text{Div}_0(\nu)$ and $\text{Div}_\infty(\nu)$ are integrally flat currents supported on the zero and pole sets of $\nu$, where $\text{Res}_0(\phi)$ and $\text{Res}_\infty(\phi)$ are smooth residue forms which can be calculated in terms of the curvature $\Omega_F$ of $F$, where $T$ is a canonical transgression form with coefficients in $L^1_{\text{loc}}$, and where $\phi(\Omega_0)$ is an $L^1_{\text{loc}}$ form canonically defined in terms of a singular connection naturally associated to $\nu$.

These results hold for $C^\infty$–meromorphic sections $\nu$ which are atomic. The notion of an atomic section of a vector bundle was first introduced and studied by Harvey and Semmes. The formulae obtained include a generalization of the Poincaré–Lelong formula to $C^\infty$–meromorphic sections of a bundle of arbitrary rank. Analogous results hold for real vector bundles and for quaternionic line bundles.
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Chapter I
Complex Vector Bundles

0. Introduction.

This thesis is concerned with a variant of the following general problem. Given a smooth homomorphism between vector bundles relate the degeneracy sets of the homomorphism to the topology of the bundles. The most important example of a degeneracy set of a bundle map is given by the zero set of a section of a vector bundle. The first result in this subject is the Poincaré–Hopf Theorem which states that the number of zeros of a vector field on a compact surface is equal to the Euler characteristic of the surface.

Much work has been done on this problem, especially in the holomorphic case (see for example [B], [BC], [GH], [BGS]). One of the major contributions of my supervisor Reese Harvey and his collaborators Blaine Lawson and Stephen Semmes (see [HL], [HS]) has been to develop a theory which enables one to study this problem for a large class of smooth bundle maps. Their approach is to derive current equations relating the Chern forms of the bundles — whose cohomology classes are the basic topological invariants of the bundle — to certain degeneracy currents supported on the degeneracy sets of the bundle map.

A current of dimension $p$ (or degree $n - p$) on an oriented $n$-dimensional manifold is a continuous linear functional on the smooth differential $p$-forms on the manifold. There are two basic examples of currents. A smooth $p$-form defines a current of degree $p$ by wedging it with an $n - p$ form and integrating the resulting $n$-form over the manifold. On the other hand, an oriented $p$-dimensional submanifold of a manifold defines a $p$-dimensional current by integration.

In this thesis we study $C^\infty$–meromorphic sections of vector bundles. The main
aim is to derive current equations relating the zero and pole sets of the section to the topology of the bundle. These formulae can be regarded as compactifications of the current formulae of Harvey and Lawson. They include a \( C^\infty \)-generalization to arbitrary rank bundles of the Poincaré–Lelong equation of complex analysis. Note that the simplest example of such a formula is the classical argument principle.

Let \( F \to X \) be a rank \( n \) complex vector bundle over an oriented manifold. A connection on \( F \) is a differential operator \( D \) from sections of \( F \) to sections of \( T^*X \otimes F \) which satisfies the Leibniz rule \( D(f \mu) = df \otimes \mu + fD\mu \) for all smooth functions \( f \) on \( X \) and all smooth sections \( \mu \) of \( F \). Any connection \( D \) can be extended to a mapping \( D \) from \( T^*X \otimes F \) to \( \Lambda^2(T^*X) \otimes F \). The curvature of \( D \) is the \( \cdot \)-operator \( D^2 \). Let \( \phi \) be a polynomial on \( \mathfrak{gl}(n, \mathbb{C}) \) which is invariant under the adjoint action. For each choice of connection on \( F \) the Chern–Weil theory constructs a \( d \)-closed differential form \( \phi(\Omega_F) \) on \( X \), called the \( \phi \)-Chern form, which represents the \( \phi \)-Chern class of \( F \) and is defined in terms of the curvature of the connection. Harvey and Lawson generalize this theory by associating to each sufficiently well behaved smooth section \( \mu \) of the vector bundle \( F \) with connection, a canonically defined \( d \)-closed current \( \phi(\Omega_{F,\mu}) \), called the \( \phi \)-Chern current of \( \mu \), which also represents the \( \phi \)-Chern class of \( F \). This current can be expressed as the sum of an "absolutely continuous" part and a singular part which is supported on the zero set of the section \( \mu \). They also construct a canonically defined transgression current \( T = T(\phi, \mu) \) satisfying the current equation

\[
\phi(\Omega_{F,\mu}) - \phi(\Omega_F) = dT \quad \text{on } X.
\]

Suppose that 0 is a regular value of \( \mu \). Then the current \( c_n(\Omega_{F,\mu}) \) associated with the top Chern polynomial \( c_n = (\frac{i}{2\pi})^n \det \) is equal to the current \( [Z] \) of integration
over the zero set $Z$ of $\mu$, and we have the fundamental current equation

$$[Z] - c_n(\Omega_F) = dT \quad \text{on } X.$$ 

In this chapter we study the $\phi$--Chern currents associated to "meromorphic" sections of a complex vector bundle $F$. By a meromorphic section of $F$ we mean a $C^\infty$--section $\nu$ of the bundle $\mathbb{P}(\mathbb{C} \oplus F) \to X$ of projective spaces which compactifies the bundle $F \to X$. The pole set of such a section $\nu$ is the subset $P$ of $X$ defined by $P := \nu^{-1}(\mathbb{P}(\mathbb{C} \oplus F) \sim F)$. In analogy with Harvey and Lawson's work we associate to $\nu$ a canonically defined current $\phi(\Omega_{F,\nu})$ which also represents the $\phi$--Chern class of $F$ and whose singular support is contained in the union of the zero and pole sets of $\nu$. On the complement of the pole set the meromorphic section $\nu$ defines a smooth section $\mu$ of $F$ and the current $\phi(\Omega_{F,\mu})$ is equal to Harvey and Lawson's current $\phi(\Omega_{F,\nu})$. The main aim of this chapter is to study the current $\phi(\Omega_{F,\nu})$ in a neighbourhood of the pole set $P$. In the special case that the zero and pole sets of $\nu$ do not intersect the critical set of the section $\nu$, the $c_n$--Chern current of $\nu$ is of the form

$$c_n(\Omega_{F,\nu}) = [Z] - \text{Res}[P] \quad \text{on } X$$

where $[P]$ denotes the current of integration over the pole set $P$ and Res is a $d$--closed smooth form. Furthermore, just as in Harvey and Lawson's setting, we derive a current equation of the form

$$[Z] - \text{Res}[P] - c_n(\Omega_F) = dT \quad \text{on } X.$$

This is a special case of a current equation which generalizes the Poincaré–Lelong formula of complex analysis. Note that the classical Poincaré–Lelong formula only
holds for meromorphic sections of holomorphic line bundles, whereas the generalizations described in this paper hold for $C^\infty$-meromorphic sections of smooth complex vector bundles of arbitrary rank.

We begin by describing Harvey and Lawson's construction of the Chern currents associated to a bundle map. Let $E$ and $F$ be complex vector bundles of ranks 1 and $n$ defined over a smooth oriented manifold $X$. Endow $E$ and $F$ with connections $D_E$ and $D_F$, and choose hermitian metrics on $E$ and $F$. Let $E \xrightarrow{\alpha} F$ be a smooth bundle map, and let $Z = \{x \in X : \alpha_x \equiv 0\}$ be the zero set of $\alpha$. Note that a section $\mu$ of $F$ defines a bundle map $\alpha$ from the trivial bundle $\mathbb{C}$ to $F$ by $\alpha(1) = \mu$. Harvey and Lawson use the bundle map $\alpha$ to pushforward the connection $D_E$ on $E$ to a smooth connection $\overline{D}$ defined on $F$ over $X \sim Z$ by

$$\overline{D} := \alpha D_E \beta + D_F (1 - \alpha \beta) = D_F - (D_F \alpha - \alpha D_E) \beta,$$

where $F \xrightarrow{\beta} E$ is the bundle map over $X \sim Z$ given by orthogonal projection of $F$ onto the rank 1 subbundle $\text{Im}(\alpha)$ followed by the inverse of the map $E \xrightarrow{\alpha} \text{Im}(\alpha) \subset F$. The connection $\overline{D}$ is called the singular pushforward connection associated with $\alpha$. It provides a way to relate the curvature operators of the bundles $E$ and $F$ to the zero set of the map $E \xrightarrow{\alpha} F$. This relationship is made explicit by computing "Chern currents" for $\overline{D}$.

Let $\phi$ be an Ad-invariant polynomial on $\mathfrak{gl}(n, \mathbb{C})$. The $\phi$-Chern current of $\overline{D}$ at time zero, $\phi(\overline{D}_0)$, on $X$ is defined as follows. First we define a smooth family of smooth connections $\overline{D}_s$, for $0 < s < \infty$, on $F$ over all of $X$ by

$$\overline{D}_s = D_F + (\alpha D_E - D_F \alpha) \alpha^* (\alpha \alpha^* + s^2)^{-1},$$

where $F \xrightarrow{\alpha^*} E$ denotes the hermitian adjoint of $\alpha$. This family has been chosen so that as $s \to 0$, $\overline{D}_s \to \overline{D}$ on $F$ over $X \sim Z$ and as $s \to \infty$, $\overline{D}_s \to D_F$ on $F$ over
Then the Chern current at time zero on $X$ is defined to be the current limit

$$\phi((\overline{D}_0)) := \lim_{s \to 0} \phi(\overline{D}_s) \quad \text{on } X,$$

whenever this limit exists. Actually there are many families $\overline{D}_s$ which have the required limits as $s \to 0$ and $s \to \infty$. We choose to work with the particular family $\overline{D}_s$ defined above because it arises naturally from a geometric construction to be described below.

In order that the Chern current $\phi((\overline{D}_0))$ exist it is necessary to impose some conditions on the behaviour of the map $\alpha$ near its zero set. To that end Harvey and Semmes defined the notion of an atomic bundle map $\alpha$ and defined the zero divisor, $\text{Div}_0(\alpha)$, of such a map. One can think of atomicity as a weak condition which ensures the existence of a zero divisor. The class of atomic sections includes those smooth sections which vanish algebraically on sets of the proper codimension. In particular real analytic sections whose zero sets are not too big are atomic. Harvey and Semmes also proved that the zero divisor is an integrally flat current supported on the zero set of $\alpha$, and that their definition of divisor agrees with the other definitions of divisor found in the literature.

Harvey and Lawson proved that, if the bundle map $\alpha$ is atomic, the Chern current $\phi((\overline{D}_0))$ exists and is of the form

$$\phi((\overline{D}_0)) = \phi(\overline{D}_0) + \text{Res}_\phi(\overline{D}_0) \text{ Div}_0(\alpha) \quad \text{on } X,$$

where $\phi(\overline{D}_0)$ is a $d$-closed $L^1_{\text{loc}}$ form on $X$ which agrees with the smooth form $\phi(\overline{D})$ on $X \sim Z$, and where the residue form, $\text{Res}_\phi(\overline{D}_0)$, is a smooth form on $X$ of classical Chern–Weil type which can be computed as a polynomial in the curvatures of the connections $D_E$ and $D_F$. They also proved that the transgression formula

$$\phi(D_F) - \phi(\overline{D}_s) = dt_{\infty,s},$$
involving smooth differential forms on $X$, converges, in the sense of currents as $s \to 0$, to the current equation

$$\phi(D_F) - \phi(\overline{D}_0) = dT_{\infty,0} \quad \text{on } X,$$

where the transgression current $T_{\infty,0} := \lim_{s \to 0} T_{\infty,s}$ converges in $L^1_{\text{loc}}(X)$.

Let $\text{Hom}(E, F) \to X$ denote the vector bundle of all bundle maps from $E$ to $F$. This bundle can be compactified in the vertical directions by embedding it in the bundle of projective spaces $\mathbb{P}(E \oplus F) \to X$. This embedding is defined by sending a bundle map to its graph. Recall that $\mathbb{P}(E \oplus F) = \text{Hom}(E, F) \cup \mathbb{P}(F)$. Each section $\nu$ of $\mathbb{P}(E \oplus F) \to X$ induces a bundle map $E \to F$ defined over the complement of the pole set $P := \nu^{-1}(\mathbb{P}(F))$ of $\nu$ in $X$. We can apply the work of Harvey and Lawson described above to the induced map $\alpha$ over $X \sim P$ to construct the family $(\overline{D}_s, F)$ over $X \sim P$ and compute the Chern current at time zero on $X \sim P$. The main results of this chapter can be stated as follows. The Chern forms, $\phi(\overline{D}_s)$ for $0 < s < \infty$, extend smoothly across $P$ to all of $X$ (see Theorem 2.28). Provided that the section $\nu$ of $\mathbb{P}(E \oplus F) \to X$ satisfies a suitable atomic hypothesis (Definition 1.8) the Chern forms $\phi(\overline{D}_s)$ converge, in the sense of currents as $s \to \infty$, to the Chern current at time infinity, $\phi(\overline{D}_\infty)$, on $X$ (Corollary 3.4). Furthermore (see Theorem 3.8) the Chern current $\phi(\overline{D}_\infty)$ is of the form

$$\phi(\overline{D}_\infty) = \phi(D_F) + \nu^*(\text{Res}_\phi(\overline{D}_\infty)) \text{ Div}_\infty(\nu) \quad \text{on } X$$

where the pole divisor, $\text{Div}_\infty(\nu)$, of $\nu$ is an integrally flat current supported on the pole set $P$ of $\nu$ and the residue form at infinity, $\text{Res}_\phi(\overline{D}_\infty)$, is a $d$-closed smooth form on $\mathbb{P}(F)$ whose cohomology class can be computed in terms
of the Chern classes of the bundles $E, F$ and the tautological bundle $L$ over $\mathbb{P}(F)$. Further properties of the residue form at infinity will be discussed in Section 4.

In addition the smooth transgression forms $T_{r,s}$, which satisfy the transgression formula

$$\phi(\overrightarrow{D}_r) - \phi(\overrightarrow{D}_s) = dT_{r,s}$$

for $0 < s < r < \infty$ on $X \sim P$, extend smoothly to all of $X$ (Theorem 2.28). As $r \to \infty$ and $s \to 0$ the transgression forms $T_{r,s}$ converge in $L^1_{\text{loc}}(X)$ to the $L^1_{\text{loc}}$ transgression current $T_{\infty,0}$ and the current equation

$$\phi(\overrightarrow{D}_\infty) - \phi(\overrightarrow{D}_0) = dT_{\infty,0} \quad \text{on } X$$

is the limiting form of the smooth transgression formula (Proposition 3.1, Corollary 3.4).

The main ideas used to prove that the smooth Chern and transgression forms, $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$ for $0 < s < r < \infty$, on $X \sim P$ extend smoothly to all of $X$ can be outlined as follows. Let $E, F$ be the pullbacks to $\mathbb{P}(E \oplus F)$ of $E, F$, and let $U \subset E \oplus F$ be the tautological line bundle over $\mathbb{P}(E \oplus F)$. We begin by constructing a certain family of smooth connections $D_{Q,s}$ for $0 < s < \infty$ on the pullback to $X$ via $\nu$ of the quotient bundle $Q = E \oplus F / U$ over $\mathbb{P}(E \oplus F)$. Then we show that the family $(\overrightarrow{D}_s, F)$ is gauge equivalent to the family $(D_{Q,s}, \nu^*(Q))$ over the subset $X \sim P$ of $X$. That is, the family $(\overrightarrow{D}_s, F)$, which is only defined over $X \sim P$, is compactified by the family $(D_{Q,s}, \nu^*(Q))$, defined over all of $X$. Consequently the Chern and transgression forms of the family $(\overrightarrow{D}_s, F)$ over $X \sim P$ extend smoothly to all of $X$ as the Chern and transgression forms of the family $(D_{Q,s}, \nu^*(Q))$ over $X$. In addition we will show that as $s \to \infty$ the family $D_{Q,s}$ converges smoothly over the complement of the pole set $P$ of $\nu$ to a connection $D_{Q,\infty}$ defined on
\nu^*(Q) over \(X \sim P\). Similarly, as the parameter \(s \to 0\) the family \(D_{Q,s}\) converges smoothly over the complement of the zero set \(Z\) of \(\nu\) in \(X\) to a connection \(D_{Q,0}\) defined on \(\nu^*(Q)\) over \(X \sim Z\). This last fact implies that the Chern current at time zero on \(X\) associated with an atomic section \(\nu\) of \(\mathbb{P}(E \oplus F) \to X\) has exactly the same form as it did in the case of a bundle map \(\alpha\).

Note that many of the results concerning the smooth forms \(\phi(\overrightarrow{D}_s)\) induced by \(\nu\) on \(X\) were already well understood by Harvey and Lawson (see [HL: I,8–9]). We conclude this introduction with some additional comments.

The definition of an atomic section \(\nu\) of \(\mathbb{P}(E \oplus F) \to X\) is the one forced upon us in order that the zero and pole divisors of \(\nu\) are well defined flat currents on \(X\). If \([t, u]\) denotes a local homogeneous expression for the section \(\nu\) of \(\mathbb{P}(E \oplus F) \to X\) we simply require that the local \(n\)-vector-valued function \(\frac{t}{u_j}\) and the scalar-valued functions \(\frac{1}{u_j}\) for \(j \in \{1, ..., n\}\) are atomic functions in the sense of [HS].

The main result outlined above is nicely illustrated by seeing what it says in the trivial case. Let \(E = \mathcal{C}\) and \(F = \mathcal{C}^n\) be the trivial rank 1 and \(n\) bundles over \(X = \mathbb{P}(\mathcal{C} \oplus \mathcal{C}^n)\), endowed with the trivial connection \(d\). Let 0 denote the distinguished point \(\mathcal{C} \oplus 0 \in \mathbb{P}(\mathcal{C} \oplus \mathcal{C}^n)\) and let \(\mathbb{P}(\mathcal{C}^n)\) be the hyperplane at infinity in \(\mathbb{P}(\mathcal{C} \oplus \mathcal{C}^n)\). The identity section \(\nu\) of \(\mathbb{P}(\mathcal{C} \oplus \mathcal{C}^n) \to \mathbb{P}(\mathcal{C} \oplus \mathcal{C}^n)\) is atomic with \(\text{Div}_0(\nu) = [0]\) and \(\text{Div}_\infty(\nu) = [\mathbb{P}(\mathcal{C}^n)]\), where \([M]\) denotes the current of integration over a submanifold \(M\) of \(X\). Let \(\phi = c_n\) be the \(n\)th Chern polynomial. Then the Chern currents at time zero and time infinity on \(\mathbb{P}(\mathcal{C} \oplus \mathcal{C}^n)\) are given by

\[
c_n((\overrightarrow{D}_0)) = [0] \quad \text{and} \quad c_n((\overrightarrow{D}_\infty)) = (-c_1(D_L))^{n-1} [\mathbb{P}(\mathcal{C}^n)],
\]

where \(D_L\) is the connection induced on the tautological line bundle \(L \subset \mathcal{C}^n\) by the trivial connection \(d\). Note that the residue at infinity, \((-c_1(D_L))^{n-1}\), is just the
volume form on $\mathbb{P}(\mathbb{C}^n)$ in the Fubini-Study metric. Finally, the smooth volume forms $c_n(D_{Q,s})$ on $\mathbb{P}(\mathbb{C} \oplus \mathbb{C}^n)$ can be regarded as a compactification of a natural family of approximate identities on $\mathbb{C}^n$. This last fact can be generalized by saying that in the "universal case" the smooth forms $c_n(D_{Q,s})$ on $\mathbb{P}(\mathbb{C} \oplus F)$ compactify a smooth family of Thom forms on the bundle $F$.

Two other special cases are worthy of comment. The first is the case that rank $F = 1$. In this case we have the current equation
\[
dT_{\infty,0} = c_1(D_F)^m - c_1(D_E)^m + \left(\text{Div}_\infty(\nu) - \text{Div}_0(\nu)\right) \frac{c_1(D_F)^m - c_1(D_E)^m}{c_1(D_F) - c_1(D_E)}
\]
for $m \geq 1$ on $X$. In the special case $m = 1$ we have the formula
\[
dT_{\infty,0} = c_1(D_F) - c_1(D_E) + \text{Div}_\infty(\nu) - \text{Div}_0(\nu).
\]

This formula is a $C^\infty$-generalization of the Poincaré–Lelong formula of complex analysis. Note that these formulae are immediate corollaries of [HL: II].

The second special case is that of a section of the compactification $\mathbb{P}(\mathbb{C} \oplus F) \to X$ of a holomorphic vector bundle $F \to X$. It seems more natural to compute Chern currents of a section of the compactified bundle $\mathbb{P}(\mathbb{C} \oplus F)$, rather than those of a meromorphic section of the bundle $F \to X$ (see Section 5). Note that when the base manifold $X$ is a Riemann surface the concepts of a meromorphic section of $F \to X$ and a holomorphic section of $\mathbb{P}(\mathbb{C} \oplus F)$ coincide.

In Chapter II we will describe some analogous results which hold for $C^\infty$-meromorphic sections of a real vector bundle, while in Chapter III we examine the divisors and characteristic currents associated with $C^\infty$-meromorphic maps between quaternionic line bundles. As originally stated Harvey, Lawson and Semmes' definitions and results concerning sections of a real vector bundle $V \to X$ required that orientation assumptions be placed on the bundle $V$ and on the base manifold.
X. In Chapter IV we remove these orientation hypotheses, showing how to extend their definitions and results to arbitrary real vector bundles. Interestingly enough the definitions and results of Chapter IV will be applied ahead of time in Chapter II where we discuss the Euler current at time infinity associated with a section of the compactification of an odd rank oriented real vector bundle. We already need to address orientation issues in Chaper II because the even dimensional real projective spaces are nonorientable.
1. Divisors of atomic sections of a compactified bundle.

Let $E$ and $F$ be hermitian bundles of ranks 1 and $n$ over an oriented manifold $X$. In this section we apply the work of Harvey and Semmes [HS] to define the notion of an atomic section of the compactification $\mathbb{P}(E \oplus F) \to X$ of the bundle $\text{Hom}(E, F) \to X$, define the zero and pole divisors of such a section and give some geometric conditions which ensure that such a section is atomic. Harvey and Semmes make the following definition.

**Definition 1.1.** [HS:1.1]. Let $y = (y_1, \ldots, y_m)$ denote coordinates on $\mathbb{R}^m$ and let $M$ be a smooth manifold. A smooth function $f : M \to \mathbb{R}^m$ ($m > 1$) is called **atomic** if for each form $\frac{du^I}{|y|^p}$ on $\mathbb{R}^m$, with $p = |I| \leq m - 1$, the pullback $f^* \left( \frac{du^I}{|y|^p} \right)$ to $M$ has an $L^1_{\text{loc}}$ extension across the zero set $Z$ of $f$. Also assume that $f$ does not vanish identically in any connected component of its domain $M$.

**Note.** Harvey and Semmes [HS: 1.2] prove that if $f$ is atomic then the zero set $Z$ has Lebesgue measure zero in $M$. Therefore the $L^1_{\text{loc}}$ extension of $f^* \left( \frac{du^I}{|y|^p} \right)$ across $Z$ is unique.

We generalize Definition 1.1 by replacing the trivial bundle $\mathbb{R}^m \to \{0\}$ by a vector bundle $V \to X$ and the zero set $Z$ of $f$ by the inverse image of $X$ under $f$. Here we identify $X$ with the zero section of $V$.

**Definition 1.2.** Let $V \to X$ be a smooth real vector bundle of rank $m$ and let $M$ be a smooth manifold. A smooth mapping $f : M \to V$ is called **atomic with respect to $X$** if for every choice of local fibre coordinate $y$ on $V$ the local function $y \circ f : M \to \mathbb{R}^m$ is atomic.
Remark 1.3. Harvey and Semmes [HS:2.5] define the notion of an atomic section of a bundle \( V \to X \). This definition is simply the one given above in the special case that the mapping \( f \) is a section of the bundle \( V \to X \). Furthermore let \( V \) denote the pullback to \( M \) via \( \pi \circ f : M \to X \) of the bundle \( \pi : V \to X \). Note that a mapping \( f : M \to V \) defines a section \( \tilde{f} \) of the bundle \( V \to M \). Then \( f \) is atomic with respect to \( X \) iff \( \tilde{f} \) is an atomic section of \( V \to M \).

Note. This observation combined with Lemma 2.4 of [HS] ensures that the atomicity with respect to \( X \) of a mapping \( f \) is a well defined notion.

Let
\[
\theta := \sum_{k=1}^{m} (-1)^{k-1} \frac{y_k dy_1 \wedge \cdots \wedge dy_k \cdots \wedge dy_m}{|y|^m}
\]
denote the solid angle kernel on \( \mathbb{R}^m \), and let \( \gamma_m \) denote the volume of the unit sphere in \( \mathbb{R}^m \). The fundamental current equation \( d(\gamma_m^{-1} \theta) = [0] \) on \( \mathbb{R}^m \), where \([0]\) denotes the point mass at the origin, motivates the definition of divisor.

Definition 1.4. Let \( V \) be a smooth oriented real vector bundle of rank \( m \) over a smooth oriented manifold \( X \). Let \( f : M \to V \) be atomic with respect to \( X \) and let \( y \) be any positively oriented local fibre coordinate on \( V \). The divisor of \( f \) with respect to \( X \), denoted \( \text{Div}_X(f) \), is defined by the current equation
\[
\text{Div}_X(f) := d(\gamma_m^{-1} (y \circ f)^{\ast} \theta) \quad \text{on} \ M.
\]

Note. The divisor \( \text{Div}_X(f) \) is a degree \( m \) current supported on the subset \( f^{-1}(X) \) of \( M \).

Remark 1.5. Note that the divisor of \( f \) with respect to \( X \) is simply the divisor of the section \( \tilde{f} \) of \( V \to M \). Therefore we can apply Theorem 2.11 of [HS] to show that the divisor of a mapping which is atomic with respect to \( X \) is well defined.
independent of choice of positively oriented local fibre coordinate on $V$.

These ideas can be applied to a section of the compactified bundle $\mathbb{P}(E \oplus F) \to X$ as follows. First we recall some of the basic structure of the bundle $\mathbb{P}(E \oplus F) \to X$. The natural embedding

$$\text{Hom}(E, F) \hookrightarrow \mathbb{P}(E \oplus F)$$

$$\alpha_x \hookrightarrow \text{Graph}(\alpha_x),$$

where $\text{Graph}(\alpha_x) = \{ [v, \alpha_x(v)] \in \mathbb{P}(E_x \oplus F_x) / v \in E_x \sim \{0\} \}$, provides a compactification of the bundle $\text{Hom}(E, F) \to X$ in the vertical directions. The total space of the projectivized bundle $\mathbb{P}(F) \to X$ can be regarded as a complex-codimension 1 submanifold of $\mathbb{P}(E \oplus F)$ by the inclusion

$$\mathbb{P}(F) \hookrightarrow \mathbb{P}(E \oplus F)$$

$$[f] \mapsto [0, f].$$

Note that $\mathbb{P}(E \oplus F) = \text{Hom}(E, F) \sqcup \mathbb{P}(F)$. We call $\mathbb{P}(F)$ the projectivized bundle at spatial infinity in $\mathbb{P}(E \oplus F)$. Also recall that $X$ embeds in $\mathbb{P}(E \oplus F)$ as the image of the zero section of $\text{Hom}(E, F) \subset \mathbb{P}(E \oplus F)$ and that the projection

$$\pi : \mathbb{P}(E \oplus F) \sim X \longrightarrow \mathbb{P}(F)$$

$$[e, f] \mapsto [f]$$

has the natural structure of a complex line bundle. Finally, let $Z = \nu^{-1}(X)$ denote the zero set and $P = \nu^{-1}(\mathbb{P}(F))$ the pole set of a section $\nu$ of $\mathbb{P}(E \oplus F) \to X$. Note that $Z \cap P = \emptyset$.

**Definition 1.8.** A smooth section $\nu : X \to \mathbb{P}(E \oplus F)$ is called atomic if the following two conditions hold.

1. The induced section $\nu : X \sim P \to \text{Hom}(E, F)$ is atomic, and
(2) The induced mapping \( \nu : X \sim Z \to \mathbb{P}(E \oplus F) \sim X \) is atomic with respect to \( \mathbb{P}(F) \).

Thanks to Lemma 1.2 of [HS] the zero and pole sets of an atomic section of the bundle \( \mathbb{P}(E \oplus F) \to X \) have Lebesgue measure zero.

**Definition 1.9.** Let \( \nu \) be an atomic section of the bundle \( \mathbb{P}(E \oplus F) \to X \). Then

1. The **zero divisor** of \( \nu \), denoted \( \text{Div}_0(\nu) \), is defined to be the extension from \( X \sim P \) to all of \( X \) of the divisor of the induced section \( \nu : X \sim P \to \text{Hom}(E,F) \).

2. The **pole divisor** of \( \nu \), denoted \( \text{Div}_\infty(\nu) \), is defined to be the extension from \( X \sim Z \) to all of \( X \) of the divisor with respect to \( \mathbb{P}(F) \) of the induced section \( \nu : X \sim Z \to \mathbb{P}(E \oplus F) \sim X \).

Note that the zero (resp. pole) divisor of an atomic section of \( \mathbb{P}(E \oplus F) \to X \) is a codimension (or degree) \( 2n \) (resp. 2) \( d \)-closed flat current on \( X \).

**Remark 1.10.** Let \( e, f \) be local frames for \( E, F \) and let \([t, u]\) be the corresponding local homogeneous coordinate on \( \mathbb{P}(E \oplus F) \). Let \( \mathcal{U}_j = \{ u_j \neq 0 \} \) and let \( t_j := \frac{t}{u_j} \) be a coordinate function on \( \mathcal{U}_j \). Then the atomicity with respect to \( \mathbb{P}(F) \) of the mapping \( \nu : X \sim Z \to \mathbb{P}(E \oplus F) \sim X \) is equivalent to the statement that

\[
\nu^* \left( \frac{dt_j}{t_j} \right) \in L^1_{\text{loc}}(X \sim Z) \quad \text{for all } j \in \{1, \ldots, n\},
\]

and the pole divisor of \( \nu \) is given locally by

\[
\text{Div}_\infty(\nu) = d \left( \nu^* \left( \frac{1}{2\pi i} \frac{dt_j}{t_j} \right) \right) \quad \text{on } \nu^{-1}(\mathcal{U}_j).
\]

Of course we could have defined the notion of atomicity of a section of \( \mathbb{P}(E \oplus F) \to X \) by (1.11) and defined its pole divisor by (1.12). Since \( \mathbb{P}(E \oplus F) \sim X \to \mathbb{P}(F) \) is a line bundle it is particularly easy to check that these notions are well defined.
when defined in this manner.

We now turn to the question: "What sorts of sections of the bundle \( \mathbb{P}(E \oplus F) \to X \) are atomic?" The results which follow are all immediate corollaries of the work of Harvey and Semmes. Choosing local coordinates on the bundle \( \mathbb{P}(E \oplus F) \) and base manifold \( X \) it suffices to study functions \( v = [t, u] : X \to \mathbb{P} (\mathbb{C} \oplus \mathbb{C}^n) \) where \( X \) is an open subset of \( \mathbb{R}^m \). Such a function \( v \) is atomic provided that

1. the coordinate function \( u : X \sim P \to \mathbb{C}^n \) is atomic, and
2. the coordinate functions \( t_j : W_j \to \mathbb{C} \) defined in Remark 1.10 are atomic, where \( W_j = t_j^{-1}(U_j) \) is open in \( X \).

For completeness sake we quote Harvey and Semmes’ theorem which gives weak geometric conditions which ensure that functions such as \( u \) and \( t_j \) are atomic. However first note that if 0 is a regular value of a smooth map \( u : X \to \mathbb{R}^n \) then \( u \) is atomic and the divisor is given by \( \text{Div}(u) = [Z] \). Here \([Z]\) is the current of integration over the appropriately oriented codimension \( n \) closed submanifold \( Z = \text{Zero}(u) \) of \( X \).

**Theorem 1.13. [HS,3.2]** Suppose that \( u \) is a smooth \( \mathbb{R}^n \)-valued function on \( X^{\text{open}} \subset \mathbb{R}^m \) and let \( Z := \{ x \in X : u(x) = 0 \} \). Assume the following:

1. **Algebraic vanishing.** For each compact set \( K \subset X \) there exist constants \( c > 0 \) (small) and \( N \) (large) such that
   \[
   |u(x)| \geq c \text{dist}(x, Z)^N \quad \text{for all } x \in K.
   \]

2. **Strong codimension greater than \( n - 1 \).** The zero set \( Z \) locally has Minkowski codimension strictly greater than \( n - 1 \) in the sense that, for each compact set \( K \subset X \), there exists an \( \epsilon > 0 \) such that the upper Minkowski content of \( Z \cap K \) in dimension \( m - n + 1 - \epsilon \) is finite.
Then $u$ is atomic.

In particular we have the following two corollaries. The first is really a corollary of [HS,3.4] and the second of [HL: II,7.2].

**Corollary 1.14.** Let $\nu : X \to \mathbb{P}(E \oplus F)$ be a real analytic section. Suppose that

(1) each irreducible component of $Z$ has codimension $\geq 2n$, and

(2) each irreducible component of $P$ has codimension $\geq 2$.

Then $\nu$ is atomic.

**Corollary 1.15.** Let $\nu : X \to \mathbb{P}(E \oplus F)$ be a holomorphic section. Suppose that

(1) each irreducible component of $Z$ has codimension $\geq 2n$, and

(2) $P \neq X \sim Z$.

Then $\nu$ is atomic.

**Remark 1.16.** Harvey and Semmes also investigate the structure of the divisor of an atomic function $u$. They show [HS: 1.24] that the divisor of $u$ equals the slice of $X$ by $u$ at 0 (c.f. [F]) and so agrees with all other reasonable definitions of zero divisor found in the literature. Moreover they prove [HS: 4.27–28] that the divisor is a $d$–closed locally integrally flat current. (Recall that currents of the form $R + dS$ with $R$ and $S$ rectifiable are said to be integrally flat.) Consequently if the divisor has locally finite mass it is a locally rectifiable current.
2. The induced singular connections and their smooth approximations.

We begin this section by recalling Harvey and Lawson's definition of the singular pushforward and pullback connections and their smooth approximating families. Then we define the Chern currents at time zero and time infinity and review Harvey and Lawson's results concerning the Chern current at time zero. Next we present the main result of this section, proving that the smooth approximating Chern and transgression forms defined by Harvey and Lawson on the complement of the pole set of a section $\nu$ of $\mathbb{P}(E \oplus F) \to X$ have a natural extension to all of $X$. Finally we derive a formula for the smooth approximating transgression form in a local homogeneous coordinate system on $\mathbb{P}(E \oplus F)$. This formula will be useful in Section 3 where we address the problem of computing the Chern current at time infinity.

Let $E$ and $F$ be rank 1 and rank $n$ hermitian bundles endowed with connections $D_E$ and $D_F$. The Chern forms of a connection $D$ on a complex vector bundle $V$ of rank $m$ are defined as usual (c.f. [BC]). Let $\omega$ be a locally defined connection matrix or gauge for $D$ and let $\Omega := d\omega - \omega \wedge \omega$ be the corresponding curvature matrix. The total Chern form $c$ is defined by

$$c(D) = c(\Omega) := \det(I + \frac{i}{2\pi} \Omega)$$

and the Chern character is defined by

$$\text{ch}(D) = \text{ch}(\Omega) := \text{tr}(e^{\frac{i}{2\pi} \Omega}).$$

More generally, for any polynomial $\phi$ on $\mathfrak{gl}(m, \mathbb{C})$ which is invariant under the adjoint action, the $\phi$-Chern form is defined by

$$\phi(D) := \phi(\Omega).$$
Let $D_s$ $(0 < s < \infty)$ be a family of connections on the bundle $V$ and let $\phi$ be an Ad-invariant polynomial on $\mathfrak{gl}(m, \mathbb{C})$. Set $\phi(A; B) := \left. \frac{d}{dt} \right|_{t=0} \phi(B + tA)$. The \textbf{transgression form} $T_{r,s}$ on $X$ which satisfies the standard transgression formula

\begin{equation}
\phi(D_r) - \phi(D_s) = dT_{r,s} \quad \text{on } X
\end{equation}

is defined (c.f. [BC]) by

\begin{equation}
T_{r,s} := \int_s^r \phi(\frac{\partial \omega}{\partial s}; \Omega_s) ds.
\end{equation}

Following Harvey and Lawson [HL: I,2.10] the connection $D_E$ on $E$ can be pushed forward via a section $\nu$ of $\mathbb{P}(E \oplus F) \rightarrow X$ to a singular connection $\overrightarrow{D}$ defined on $F$ over $X \sim (Z \cup P)$ as follows.

\textbf{Definition 2.3.} Let $E \xrightarrow{\alpha} F$ over $X \sim P$ be the bundle map induced by a smooth section $\nu$ of $\mathbb{P}(E \oplus F) \rightarrow X$ by the inclusion (1.6). The \textbf{singular pushforward connection associated with} $\nu$ is the smooth connection $\overrightarrow{D}$ on $F$ over $X \sim (Z \cup P)$ defined by

$$
\overrightarrow{D} := \alpha D_E \beta + D_F (1 - \alpha \beta) = D_F - (D_F \alpha - \alpha D_E) \beta
$$

where $F \xrightarrow{\beta} E$ is the bundle map over $X \sim (Z \cup P)$ given by orthogonal projection of $F$ onto the subbundle $\text{Im} \alpha$ followed by the inverse of the map $E \xrightarrow{\alpha} \text{Im} \alpha \subset F$.

Similarly the connection $D_F$ on $F$ can be pulled back by $\nu$ to a connection $\overleftarrow{D}$ on $E$ over $X \sim (Z \cup P)$.

\textbf{Definition 2.4.} The \textbf{singular pullback connection associated with} $\nu$ is the smooth connection $\overleftarrow{D}$ on $E$ over $X \sim (Z \cup P)$ defined by

$$
\overleftarrow{D} := \beta D_F \alpha + (1 - \beta \alpha) D_E = D_E + \beta (D_F \alpha - \alpha D_E).
$$
Note. Let $F \overset{\alpha}{\to} E$ denote the adjoint of $\alpha$. Then the bundle map $\beta$ defined above is given by
\[ \beta = (\alpha^* \alpha)^{-1} \alpha^*. \]

Harvey and Lawson considered the case of a smooth bundle map $E \overset{\alpha}{\to} F$ over $X$ (i.e. $P = \emptyset$). One of their main aims was to compute "Chern currents" for the singular connection $\overline{\mathcal{D}}$ over $X$. To do this they constructed a smooth family $\overline{\mathcal{D}}_s$, for $0 < s < \infty$, of smooth connections on $F$ over all of $X$ with the following properties.

As $s \to 0$, $\overline{\mathcal{D}}_s \to \overline{\mathcal{D}}$ as smooth connections on $F$ over $X \sim Z$, and

as $s \to \infty$, $\overline{\mathcal{D}}_s \to \mathcal{D}_F$ as smooth connections on $F$ over $X$.

**Definition 2.5.**

1. Such a family $\overline{\mathcal{D}}_s$ is called a smooth approximating family to the singular connection $\overline{\mathcal{D}}$.

2. If as $s \to 0$ the corresponding Chern forms $\phi(\overline{\mathcal{D}}_s)$ on $X$ converge weakly as currents on $X$, then the limit will be denoted $\phi(\overline{\mathcal{D}}_0)$ and referred to as the $\phi$–Chern current of the singular connection $\overline{\mathcal{D}}$.

Harvey and Lawson constructed smooth approximating families to a given singular connection $\overline{\mathcal{D}}$ with the aid of "approximate ones" (c.f. [HL: I,4.1]). Different choices of approximate one yield different approximating families. Because of its geometrical significance (see Theorems 2.23 and 2.38 below) we will restrict our attention to the algebraic approximation mode $\chi : [0, \infty] \to [0, 1]$ which is defined by $\chi(t) = \frac{t}{1+t}$.

Returning to the case of a section $\nu$ of $\mathbb{P}(E \oplus F) \to X$ we have the following definitions.
**Definition 2.6.** The singular pushforward connection $\overline{D}$ associated with $\nu$ is smoothed over $X \sim P$ by the approximating family of smooth connections $\overline{D}_s \ (0 < s \leq \infty)$ on $F$ over $X \sim P$ defined by

$$\overline{D}_s = D_F + (\alpha D_E - D_F \alpha) \alpha^*(\alpha \alpha^* + s^2)^{-1} = (s^2 D_F + \alpha D_E \alpha^*) (\alpha \alpha^* + s^2)^{-1}.$$ 

Note that

(2.7)  

as $s \to 0$, $\overline{D}_s \to \overline{D}_0 = \overline{D}$ as smooth connections on $F$ over $X \sim (Z \cup P)$, and

(2.8)  

as $s \to \infty$, $\overline{D}_s \to \overline{D}_F$ as smooth connections on $F$ over $X \sim P$.

Similarly in the pullback case we define

(2.9)  

$$\overline{D}_s := D_E + (\alpha^* \alpha + s^2)^{-1} \alpha^* (D_F \alpha - \alpha D_E) = (s^2 D_E + \alpha^* D_F \alpha)^{-1}.$$ 

Note that $\overline{D}_\infty = D_E$.

To simplify the discussion we will restrict our attention to the pushforward case. The discussion in the pullback case proceeds along similar lines.

We will show that for $0 < s < \infty$ the Chern forms $\phi(\overline{D}_s)$ and the corresponding transgression forms $T_{r,s}$ defined initially over $X \sim P$ extend across the pole set $P$ to smooth forms on all of $X$. Suppose now that $\nu$ is an atomic section of $\mathbb{P}(E \oplus F) \to X$. The main aim of this chapter is to show that the Chern current at time zero,

(2.10)  

$$\phi(\overline{D}_0) := \lim_{s \to 0} \phi(\overline{D}_s),$$

and the Chern current at time infinity,

(2.11)  

$$\phi(\overline{D}_\infty) := \lim_{r \to \infty} \phi(\overline{D}_r),$$
exist on all of $X$, and to compute them. In particular we will show that the Chern current at time infinity is of the form

$$\phi(\mathcal{D}_\infty) = \phi(D_F) + \nu^*(\text{Res}_\phi(D_\infty)) \, \text{Div}_\infty(\nu) \quad \text{on } X$$

where the residue form at infinity $\text{Res}_\phi(D_\infty)$ is a $d$-closed smooth form on $\mathbb{P}(F)$.

Harvey and Lawson have solved this problem in the case that the pole set $P$ is empty, i.e. in the case of an atomic bundle map $E \xrightarrow{\alpha} F$ over $X$.

**Theorem 2.12.** [HL: III] Let $E \xrightarrow{\alpha} F$ be an atomic bundle map. Then

1. The transgression forms $T_{\infty,s}$ converge as $L_{\text{loc}}^1$-currents on $X$ to the transgression current $T_{\infty,0} := \lim_{s \to 0} T_{\infty,s}$.

2. Therefore the Chern current $\phi(D_0) := \lim_{s \to 0} \phi(D_s)$ exists on $X$ and is given by

$$\phi(D_0) = \phi(D_F) + dt_{\infty,0}.$$ 

Here $\phi(D_0)$ is a $d$-closed flat current on $X$ which represents the $\phi$-characteristic class of $F$.

3. Let $\phi(D_0)$ denote the smooth form $\phi(D)$ on $X \sim Z$. Then $\phi(D_0)$ extends by zero across $Z$ to a $d$-closed $L_{\text{loc}}^1$ form on all of $X$, also denoted by $\phi(D_0)$. Note that $\phi(D_0)$ is called the $L_{\text{loc}}^1$ part of the Chern current $\phi(D_0)$.

4. The Chern current $\phi(D_0)$ is given by

$$\phi(D_0) = \phi(D_0) + \text{Res}_\phi(D_0) \, \text{Div}_0(\alpha)$$

where the residue form $\text{Res}_\phi(D_0)$ is a $d$-closed smooth form on $X$. Note that the residue form is independent of the choice of bundle map $\alpha$ and that the divisor is independent of the choice of invariant polynomial $\phi$. 
(5) The current equation

\[ \phi(D_F) - \phi(D_0) - \text{Res}_\phi(D_0) \text{Div}_0(\alpha) = dT_{\infty,0} \quad \text{on } X \]

is the limiting form as \( s \to 0 \) of the smooth equation

\[ \phi(D_F) - \phi(D_s) = dT_{\infty,s} \quad \text{on } X \]

(6) In particular if \( \phi = c_n \) is the top Chern polynomial and \( E = \mathbb{C} \) is the trivial bundle we have

\[ c_n(D_F) - \text{Div}_0(\alpha) = dT_{\infty,0} \quad \text{on } X. \]

When \( \text{dim } X = \text{rank } F = n \) we can pair this equation with the constant function 1 on \( X \) and recover the Poincaré-Hopf equation of [BC],

\[ \int_X c_n(D_F) = \# \text{ zeros of } \alpha. \]

Note. Analogous results hold in the pullback case (see [HL: III,8]) and in the case of a section of a real vector bundle [HL: IV,1-2].

In order to review Harvey and Lawson's definition of the residue form \( \text{Res}_\phi(D_0) \) it is necessary to discuss the universal case.

Remark 2.13. The universal case. Let \( (E, D_E) \) and \( (F, D_F) \) be complex vector bundles with connections and hermitian metrics as above. Consider the induced bundle

\[ \pi : \text{Hom}(E, F) \longrightarrow X. \]

Let \( E = \pi^*E \) and \( F = \pi^*F \) be the pullbacks of \( E \) and \( F \) to \( \text{Hom}(E, F) \) via \( \pi \), and let \( D_E \) and \( D_F \) be the pullbacks to \( E \) and \( F \) of the connections \( D_E \) and \( D_F \). Over
\( \text{Hom}(E, F) \) there is a tautological or universal homomorphism \( E \xrightarrow{\alpha} F \) which at a point \( \alpha \in \text{Hom}(E_x, F_x) \) above \( x \in X \) is simply defined to be \( \alpha \). The zero set of \( \alpha \) is the zero section \( X \) of the bundle \( \pi : \text{Hom}(E, F) \to X \). Note that a smooth bundle map \( E \xrightarrow{\alpha} F \) can be regarded as a section of \( \text{Hom}(E, F) \) and that
\[
\alpha^*E = E \quad \text{and} \quad \alpha^*F = F
\]
as bundles with connections, and that \( \alpha^*(\alpha) = \alpha \). So every homomorphism is a pullback of the universal one \( \alpha \).

Note that we can apply Theorem 2.12 to the universal homomorphism \( \alpha \) (which is atomic with \( \text{Div}(\alpha) = [X] \)) to obtain the current equation
\[
\phi(D_F) - \phi(D_0) - \text{Res}_\phi(D_0)[X] = dT \quad \text{on } \text{Hom}(E, F).
\]

Let \( T \) be the transgression current on \( \text{Hom}(E, F) \) associated to the universal homomorphism \( \alpha \), and let \( \rho \) denote the restriction of \( \pi : \text{Hom}(E, F) \to X \) to the \( \epsilon \)-sphere bundle \( \rho : S_\epsilon(\text{Hom}(E, F)) \to X \). Note that \( T \in C^\infty(S_\epsilon(\text{Hom}(E, F))) \).

**Definition 2.14.** The Residue form \( \text{Res}_\phi(D) \) on \( X \) is defined to be minus the fibre integral (or current pushforward) of \( T \), i.e.
\[
\text{Res}_\phi(D_0) = - \int_{\rho^{-1}(x)} T = -\rho_*(T).
\]

**Remark 2.15.** Harvey and Lawson [HL: III,2-3] establish the following properties of the residue form

1. The residue form is independent of the radius \( \epsilon \) of the \( \epsilon \)-sphere bundle.

2. \[
\text{Res}_\phi(D) = \int_{\pi^{-1}(x)} \phi(D_s) \quad \text{for any } 0 < s < \infty.
\]
(3) The residue is a $d$-closed smooth form on $X$.

(4) The residue form is a classical Chern-Weil form in that it can be expressed as a universal $Ad$-invariant polynomial in the curvatures of the connections $D_E$ on $E$ and $D_F$ on $F$. In particular it is completely determined by computing its associated cohomology class.

We now return to the problem of showing that in the case of a section $\nu$ of $P(E \oplus F) \to X$ the smooth Chern and transgression forms of the smooth approximating families $\overline{D}_s$ and $\overline{D}$ over $X \sim P$ extend across the pole set $P$ to smooth forms on all of $X$. We analyse this problem at the level of connections. Actually the results described here do not require the assumption that rank $E = 1$. So for the time being let us assume that $p = \text{rank } E$ and $q = \text{rank } F$ are arbitrary. The bundle $\text{Hom}(E, F) \to X$ is compactified in the vertical directions by embedding it in the Grassmannian bundle $G_p(E \oplus F) \to X$ of $p$-planes in $E \oplus F$ via the graphing map (c.f. (1.6)). Let $P = \nu^{-1}(G_p(E \oplus F) \sim \text{Hom}(E, F))$. Just as in the case $p = 1$ we can use $\nu$ to define singular pushforward and pullback connections $\overline{D}$ and $\overline{D}$ and construct smooth approximating families $\overline{D}_s$ and $\overline{D}_s$.

Now the families $(\overline{D}_s, F)$ and $(\overline{D}_s, E)$ for $0 < s < \infty$ do not extend from $X \sim P$ to all of $X$. However we will be able to construct families $(D_{Q,s}, Q)$ and $(D_{U,s}, U)$ defined over all of $X$ whose Chern forms and transgression forms provide smooth extensions to all of $X$ of the Chern forms and transgression forms of the families $(\overline{D}_s, F)$ and $(\overline{D}_s, E)$ on $X \sim P$. The construction of the families $(D_{Q,s}, Q)$ and $(D_{U,s}, U)$ is most easily described using a compactification of the universal case.

Remark 2.16. The universal compactification. Let $E = \pi^*E$ and $F = \pi^*F$
be the pullbacks of $E$ and $F$ to $\mathbb{P}(E \oplus F)$ via the projection map $\pi : \mathbb{P}(E \oplus F) \to X$, and let $D_E$ and $D_F$ be the pullbacks to $E$ and $F$ of the connections $D_E$ and $D_F$. The induced bundle $G_p(E \oplus F) \to G_p(E \oplus F)$ has a tautological or universal section $\nu$ which sends a $p$-plane $P \in G_p(E \oplus F)$ to itself. For any section $\nu$ of $G_p(E \oplus F) \to X$ we have

$$\nu^*E = E \quad \text{and} \quad \nu^*F = F$$

as bundles with connections, and $\nu^*(\nu) = \nu$. So to prove results of a smooth nature on or over $X$ which involve a section $\nu$ it often suffices to prove them on or over $G_p(E \oplus F)$ for the universal section $\nu$ and then pullback the result to $X$ via $\nu$.

**Definition 2.17.** Let $V, W$ be isomorphic vector bundles. A family of connections $D_{V,s}$ $(0 < s < \infty)$ on $V$ is said to be **equivalent** to a family of connections $D_{W,s}$ $(0 < s < \infty)$ on $W$ if there is a bundle isomorphism $\varphi : V \to W$ (independent of $s$) so that

$$(2.18) \quad D_{W,s} = \varphi \circ D_{V,s} \circ \varphi^{-1}$$

for all $s \in (0, \infty)$.

This concept is important because of the following fact.

**Lemma 2.19.** Equivalent families of connections have equal Chern and transgression forms.

**Proof.** Using the notation of Definition 2.17 let $e$ be a local frame for $V$ and $f := \varphi \circ e$ the corresponding frame for $W$. Define a local gauge $\omega_s$ by $D_{V,s}e = \omega_se$. Then by (2.18) $D_{W,s}f = \omega_sf$ and so the transgression forms for $V$ and $W$ both equal $\int_s^r \phi(\omega_s ; \Omega_s) ds$. □
In the universal case the family \((D_{U,s}, U)\) is constructed over \(G_{p}(E \oplus F)\) as follows. Let \(U\) denote the tautological or universal \(p\)-plane bundle over \(G_{p}(E \oplus F)\). Now consider the flow \(\psi_s : E \oplus F \to E \oplus F\) defined by

\[
\psi_s(e, f) = (se, f) \quad \text{for } 0 < s < \infty.
\]

This induces a flow \(\Psi_s : G_{p}(E \oplus F) \to G_{p}(E \oplus F)\). For each \(s\) we let \(U_s := \psi_s^*(U)\) denote the pullback of the universal bundle \(U\) via \(\Psi_s\).

We can induce a connection \(D_{U_s}\) on \(U_s\) over \(\mathbb{P}(E \oplus F)\) by regarding \(U_s\) as a subbundle of \(E \oplus F\) and defining

\[
D_{U_s} = \text{pr}_s \circ D_{E \oplus F} \quad \text{operating on sections of } U_s,
\]

where \(\text{pr}_s : E \oplus F \to U_s\) denotes orthogonal projection. Then we can define the family of connections \(D_{U,s}\) on \(U\) over \(\mathbb{P}(E \oplus F)\) by

\[
D_{U,s} := \Psi_s^{-1} \circ D_{U_s} \circ \Psi_s,
\]

where we now regard \(\Psi_s\) as a bundle map from \(U\) to \(U_s\). The following result is a corollary of Theorem I.8.14 of [HL].

**Theorem 2.23.**

1. The family \((\overline{D}_s, E)\) \((0 < s < \infty)\) is equivalent over \(\text{Hom}(E, F)\) to the family \((D_{U,s}, U)\). More specifically

\[
D_{U,s} = \Gamma \circ \overline{D}_s \circ \Gamma^{-1} \quad \text{over } \text{Hom}(E, F)
\]

where \(\Gamma : E \to U\) is the graphing map, \(\Gamma(e) := (e, \alpha(e))\), for \(e \in E_\alpha\).

2. Let \(\phi\) be an \(Ad\)-invariant polynomial on \(\mathfrak{gl}(p, \mathbb{C})\). Then the Chern forms \(\phi(\overline{D}_s)\)
(0 < s < \infty) and the transgression forms T_{r,s} (0 < s < r < \infty) of the family 
\left( \overline{D}_s, E \right) \text{ on } \text{Hom}(E, F) \text{ extend smoothly to all of } G_p(E \oplus F) \text{ as the Chern and transgression forms of the family } (D_{u,s}, U) \text{ over } G_p(E \oplus F). \text{ Furthermore}

\begin{equation*}
\frac{d}{dr} T_{r,s} = \phi \left( \overline{D}_r \right) - \left( \overline{D}_s \right) \quad \text{on } G_p(E \oplus F).
\end{equation*}

The corresponding result in the pushforward case can be described as follows. We wish to exploit the natural duality between the approximating pushforward and pullback connections (c.f. [HL: I,4.28]). Let \( Q := E \oplus F / U \) be the rank \( q \) quotient bundle over \( G_p(E \oplus F) \). We begin by defining the family of connections \((D_{Q,s}, Q)\) alluded to above. Generally, given an exact sequence of hermitian bundles

\[ 0 \to U \to V \to Q \to 0 \]

and a connection \( D_V \) on \( V \) recall that we can induce a natural connection \( D_Q \) on \( Q \) as follows. Consider the dual exact sequence

\[ 0 \leftarrow U^* \leftarrow V^* \leftarrow Q^* \leftarrow 0 \]

and the dual connection \( D_{V^*} \) defined on \( V^* \) by the formula

\[ (D_{V^*} v^*)(w) := d(v^*(w)) - v^*(D_V w) \]

for \( v^* \in \Gamma(V^*) \) and \( w \in \Gamma(V) \). Let \( D_{Q^*} \) denote the connection induced on the subbundle \( Q^* \) of \( V^* \) by \( D_{V^*} \). Then the connection \( D_Q \) on \( Q \) is defined to be the dual connection to \( D_{Q^*} \). Actually to define the family of connections \( D_{Q,s} \) on \( Q \) we use some of the extra structure available when working over \( G_p(E \oplus F) \).

**Lemma 2.24.** The bundle isomorphism

\[ \varphi : G_p(E \oplus F) \to G_q(F^* \oplus E^*) \]
defined by sending a $p$-plane $P \in G_p(E \oplus F)$ to its annihilator $\text{Ann}(P) \in G_q(F^* \oplus E^*)$ pulls back the tautological rank $q$ bundle $U_q \to G_q(F^* \oplus E^*)$ to the dual $Q^*$ of the quotient bundle $Q = E \oplus F/U$ over $G_p(E \oplus F)$, i.e.

$$Q^* = \varphi^*(U_q).$$

Furthermore the restriction of the mapping $\varphi$ to the finite part $\text{Hom}(E, F)$ of $G_p(E \oplus F)$ is given by

$$\text{Hom}(E, F) \xrightarrow{\varphi} \text{Hom}(F^*, E^*)$$

$$\alpha \mapsto -\tilde{\alpha}$$

where the dual homomorphism $\tilde{\alpha}$ is defined by

$$\tilde{\alpha}(f^*)(e) := f^*(\alpha(e)).$$

**Proof.** To prove that $Q = \varphi^*(U_q^*)$ note that the fibre of $\varphi^*(U_q^*)$ over a $p$-plane $P \in G_p(E \oplus F)$ is given by $\varphi^*(U_q^*)_P = \text{Hom}(\text{Ann}(P), \mathbb{C})$ and that there is the exact sequence

$$0 \to P \to E_P \oplus F_P \to \text{Hom}(\text{Ann}(P), \mathbb{C}) \to 0$$

$$v \mapsto T_v(w^*) := w^*(v).$$

Finally (2.25) amounts to showing that $\text{Ann} (\text{Graph } \alpha) = \text{Graph } (-\tilde{\alpha})$. □

Let $D_{U_q,s}$ denote the connection defined on $U_q \to G_q(F^* \oplus E^*)$ by equation (2.22).

**Definition 2.27.** The family of connections $D_{Q,s}$ on $Q$ over $G_p(E \oplus F)$ is defined to be the family dual to the family of connections induced on $\varphi^*(U_q)$ by the family...
$D_{U_q,s}$ on $U_q$ over $G_q(F^* \oplus E^*)$.

We are now in a position to state and prove the pushforward analogue of Theorem 2.23.

**Theorem 2.28.**

1. The family $(\overline{D}_s, F)$ ($0 < s < \infty$) is equivalent over $\text{Hom}(E, F)$ to the family $(D_{Q,s}, Q)$. More specifically,

\[ D_{Q,s} = \psi \circ \overline{D}_s \circ \psi^{-1} \quad \text{over } \text{Hom}(E, F) \]

where the bundle isomorphism $\psi : F \to Q$ over $\text{Hom}(E, F)$ is defined by

\[ \psi(f) := [0, f] \quad \text{for } f \in F. \]

Here $[e, f]$ denotes the image of $(e, f)$ in $Q$.

2. Let $\phi$ be an Ad-invariant polynomial on $\mathfrak{gl}(q, \mathbb{C})$. Then the Chern forms $\phi(\overline{D}_s)$ ($0 < s < \infty$) and the transgression forms $T_{r,s}$ ($0 < s < r < \infty$) of the family $(\overline{D}_s, F)$ on $\text{Hom}(E, F)$ extend smoothly to all of $G_p(E \oplus F)$ as the Chern and transgression forms of the family $(D_{Q,s}, Q)$ over $G_p(E \oplus F)$. Furthermore

\[ dT_{r,s} = \phi(\overline{D}_r) - \phi(\overline{D}_s) \quad \text{on } G_p(E \oplus F). \]

**Note.** As suggested in Remark 2.16, Theorems 2.23 and 2.28 also hold for a smooth section $\nu$ of $\mathbb{P}(E \oplus F) \to X$ provided that we replace all spaces, bundles, connections and bundle maps by their pullbacks to $X$ via $\nu$.

**Proof.** By Theorem 2.23 we know that over $\text{Hom}(F^*, E^*)$ the family $(\overline{D}_s, F^*)$ is equivalent to the family $(D_{U_q,s}, U_q)$ and that the isomorphism $\Gamma_{\phi,1} : F^* \to U_q$
setting up this equivalence is given by $\Gamma_{\tilde{\alpha},1}(f^*) = (f^*, \tilde{\alpha}(f^*))$. This data pulls back via the mapping $\varphi : \text{Hom}(E, F) \to \text{Hom}(F^*, E^*)$ of (2.25) to the isomorphism

$$(F^*, \overline{D}_{s,F^*}) \xrightarrow{\varphi^*(\Gamma_{\tilde{\alpha},1})} (\varphi^*(U_q), \varphi^*(D_{U_q,s})) \quad \text{over } \text{Hom}(E, F).$$

Since $\overline{D}_{s,F} = (\overline{D}_{s,F^*})^*$ on $F$ (see [HL, I,4.28]) and

$$(Q, D_{Q,s}) = (\varphi^*(U_q)^*, (\varphi^*(D_{U_q,s}))^*)$$

the dual isomorphism is given by

$$(Q, D_{Q,s}) \xrightarrow{\eta} (F, \overline{D}_s) \quad \text{over } \text{Hom}(E, F),$$

where $\eta := \varphi^*(\Gamma_{\tilde{\alpha},1})$.

To complete the proof of the theorem we just need to identify the map $\eta$ and show that its inverse is the mapping $\psi$ defined in the statement of the theorem. Let $f^* \in F^*$ and $(e, f) \in E \oplus F$ be arbitrary. Recall from the proof of Lemma 2.24 that an element $[e, f] \in Q$ defines a linear functional $T_{[e,f]}$ on $\varphi^*(U_q)$ by

$$T_{[e,f]}(v^*, w^*) = v^*(f) - w^*(e) \quad \text{for } (v^*, w^*) \in \varphi^*(U_q) \subset F^* \oplus E^*.$$

So by the definition of the dual map and using the identification of $F$ with $F^{**}$ we have

$$\eta([e,f])(f^*) = T_{[e,f]}(\Gamma_{\tilde{\alpha},1}(f^*))$$

$$= T_{[e,f]}(f^*, -\tilde{\alpha}(f^*))$$

$$= f^*(f) - \tilde{\alpha}(f^*)(e)$$

$$= f^*(f) - f^*(\alpha(e))$$

$$= (f - \alpha(e))(f^*)$$

and so

$$\eta([e,f]) = f - \alpha(e).$$

Finally it is easy to check that $\eta^{-1}(f) = [0, f] = \psi(f)$. $\Box$
Note. The extensions of the Chern forms $\phi(D_s)$ and $\phi(D_s)$ and the transgression forms $T_{r,s}$ from $\text{Hom}(E, F)$ to all of $G_p(E \oplus F)$ will also be denoted by $\phi(D_s)$, $\phi(D_s)$ and $T_{r,s}$.

Now we return to the case that rank $E = 1$ and rank $F = n$.

Lemma 2.29. [HL: I,9.2]

1. Over $\mathbb{P}(E \oplus F) \sim X$ the smooth family of subbundles $U_s \subset E \oplus F$ for $0 < s < \infty$ extends smoothly to $s = 0$ where $U_0 = p^*L$ is the pullback to $\mathbb{P}(E \oplus F) \sim X$ of the universal bundle $L \to \mathbb{P}(F)$ via the natural map $p : \mathbb{P}(E \oplus F) \sim X \to \mathbb{P}(F)$ defined by (1.7).

2. Over $\text{Hom}(E, F) \subset \mathbb{P}(E \oplus F)$ the smooth family of subbundles $U_s \subset E \oplus F$ for $0 < s < \infty$ extends smoothly to $s = \infty$ where $U_\infty = E$.

Note.

1. The bundle $U_0$ on $\mathbb{P}(E \oplus F) \sim X$ does not extend smoothly across $X$.

2. Even though the bundle $U_\infty = E$ on $\text{Hom}(E, F)$ extends smoothly across $\mathbb{P}(F)$ the smooth family $U_s$ over $\text{Hom}(E, F)$ for $0 < s \leq \infty$ does not extend smoothly across $\mathbb{P}(F)$.

Corollary 2.30.

1. The families $(D_{U,s}, U)$ and $(D_{Q,s}, Q)$ extend smoothly to $s = 0$ over $\mathbb{P}(E \oplus F) \sim X$.

2. The families $(D_{U,s}, U)$ and $(D_{Q,s}, Q)$ extend smoothly to $s = \infty$ over $\mathbb{P}(E \oplus F) \sim \mathbb{P}(F)$.

Combining this corollary with Theorem 2.12 Harvey and Lawson proved

Theorem 2.31. [HL: I,9.12;III,7.6]
Let \( \nu \) be an atomic section of \( \mathbb{P}(E \oplus F) \rightarrow X \).

(1) The smooth form \( \phi(\overline{D}_0) \) on \( X \sim Z \) extends across \( Z \) to a \( d \)-closed \( L^1_{\text{loc}} \) form on all of \( X \), also denoted by \( \phi(\overline{D}_0) \).

(2) The Chern current \( \phi((\overline{D}_0)) \) exists on \( X \) and equals

\[
\phi((\overline{D}_0)) = \phi(\overline{D}_0) + \text{Res}_\phi(\overline{D}_0)[X].
\]

Corollary 2.30 also enables us to show that the Chern current \( \phi((\overline{D}_\infty)) \) exists on \( X \sim P \) and equals \( \phi(D_F) \). The next section is devoted to computing \( \phi((\overline{D}_\infty)) \) on all of \( X \). In order to do this we need to find a local formula for the transgression form \( T_{r,s} \) associated to a section \( \nu \) of \( \mathbb{P}(E \oplus F) \rightarrow X \). To describe this we work universally in a local homogeneous coordinate on \( \mathbb{P}(E \oplus F) \) and pullback the resulting local formula to \( X \) via \( \nu \).

Harvey and Lawson derived the following local formula for the transgression form \( T_{r,s} \). Let \( \alpha : E \rightarrow F \) be the tautological homomorphism over \( \text{Hom}(E, F) \). Let \( e, f \) be local frames for \( E, F \) and define a local fibre variable \( u = (u_1, \ldots, u_n) \) on \( \text{Hom}(E, F) \) by \( \alpha e = u f \). Let \( \alpha^* f = u^* e \) define \( u^* \). Set \( |u|^2 = uu^* \),

\[
Du = du + u \omega_F - \omega_E u \quad \text{and} \quad Du^* = du^* + u^* \omega_F - \omega_F u^*.
\]

Finally let \( u \frac{\partial}{\partial u} \) and \( u^* \frac{\partial}{\partial u^*} \) be the globally defined Euler and co-Euler vector fields on \( \text{Hom}(E, F) \) (c.f. [HL: I,5.24]). Then in the pushforward case we have [HL: III, 3.17] that

\[
T_{r,s} = \int_{X_r}^x \phi \left( \frac{u^* Du}{|u|^2}; \Lambda(x) \right) dx
\]

(2.32)

\[
= -\int_{X_r}^x u^* \frac{\partial}{\partial u^*} \phi(\Lambda(x)) \frac{dx}{x}
\]

(2.33)
where $\chi_s = \frac{|u|^2}{s^2 + |u|^2}$ and

\begin{equation}
\Lambda(x) = \Omega_F + x \left( \frac{u^* \Omega_E u}{|u|^2} - \frac{u^* u}{|u|^2} \Omega_F - \frac{Du^* Du}{|u|^2} \right).
\end{equation}

The formula for the transgression form $T_{r,s}$ in a local homogeneous coordinate on $\mathbb{P}(E \oplus F)$ is obtained as follows. Let $u$ be a local fibre variable on $\text{Hom}(E, F)$ defined as above. Let $(t, u)$ be the local fibre variables on $E \oplus F$ defined by $c = (te, uf)$ where $c : E \oplus F \to E \oplus F$ is the tautological section. We call $[t, u]$ the local homogeneous coordinate on $\mathbb{P}(E \oplus F)$ corresponding to the fibre variable $u$.

Let $\omega$ be a smooth form on $\mathbb{P}(E \oplus F)$. Suppose that over the subset $\text{Hom}(E, F)$ $\omega$ is expressed in terms of a local fibre variable $u$. Let $\omega_{HC} := \pi^* \omega$ be the pullback of $\omega$ to $E \oplus F \sim X$ via the natural projection map $\pi : E \oplus F \sim X \to \mathbb{P}(E \oplus F)$. Then to obtain an expression for $\omega_{HC}$ in the corresponding local homogeneous coordinate $(t, u)$ we simply replace $u$ by $\frac{u}{t}$ in the expression for $\omega$. The resulting expression for $\omega_{HC}$ will be called the expression for $\omega$ in homogeneous coordinates. The following general lemma will be of use in finding expressions for the transgression and Chern forms in homogeneous coordinates.

**Lemma 2.35.** Let $u$ be a local fibre coordinate on $\text{Hom}(E, F)$. Let

\[ C = \left\{ \frac{u^* u}{|u|^2}, \frac{u^* Du}{|u|^2}, \frac{Du^* u}{|u|^2}, \frac{Du^* Du}{|u|^2} \right\} \cup \Omega^*(X), \]

and set $\chi_s = \frac{|u|^2}{s^2 + |u|^2}$. Let $\omega$ be a (possibly $1 \times 1$) matrix valued form on the total space of the bundle $\text{Hom}(E, F) \to X$. Suppose that $\omega$ can be expressed as a polynomial in $\chi_r$, $\chi_s$ and $\frac{\partial \chi}{\partial s}$ with coefficients which are themselves polynomial expressions in (the entries of) the forms belonging to the set $C$. Then the expression for $\omega$ in homogeneous coordinates $[t, u]$ on $\mathbb{P}(E \oplus F)$ is

\begin{equation}
\omega_{HC} = \omega - \frac{dt}{t} \left( u \frac{\partial}{\partial u^*} \omega \right) - \frac{dt}{t} \left( u^* \frac{\partial}{\partial u} \omega \right) - \frac{dt}{|t|^2} \left( u \frac{\partial}{\partial u} u^* \frac{\partial}{\partial u^*} \omega \right).
\end{equation}
where any occurrence of \( \chi_s \) on the R.H.S. is replaced by \( \frac{|u|^2}{s^2|t|^2 + |u|^2} \).

**Proof.** First note that (2.36) holds for each \( \omega \in \mathcal{C} \). For instance

\[
\left( \frac{Du^*Du}{|u|^2} \right)_{HC} = \frac{(Du^* - u^* \frac{dt}{t})(Du - u \frac{dt}{t})}{|u|^2} = \frac{Du^*Du}{|u|^2} + \frac{dt}{t} \frac{Du^*u}{|u|^2} - \frac{dt}{t} \frac{u^*Du}{|u|^2} - \frac{dt}{t} \frac{u^*u}{|u|^2},
\]

which has the required form. Next notice that if \( \omega \) and \( \eta \) satisfy (2.36) then so does \( \omega + \eta \). The proof is completed by checking that if \( \omega \) and \( \eta \) satisfy (2.36) then so does \( \omega \wedge \eta \). \( \Box \)

**Corollary 2.37.** Let \( T_{r,s} \) be the smooth transgression form on \( \mathbb{P}(E \oplus F) \) associated to the family of Chern forms \( \phi(D_s) \). Then the expression for \( T_{r,s} \) in a local homogeneous coordinate system \([t,u]\) on \( \mathbb{P}(E \oplus F) \) is given by

\[
(2.38) \quad (T_{r,s})_{HC} = T_{r,s} - \frac{dt}{t} (u \frac{\partial}{\partial u} \wedge T_{r,s}),
\]

where \( T_{r,s} \) is defined by replacing \( \chi_s \) by \( \frac{|u|^2}{s^2|t|^2 + |u|^2} \) in (2.32) or (2.33).
3. The transgression and Chern currents.

In this section we compute the Chern currents at time infinity,

$$\phi((D_\infty)) := \lim_{r \to \infty} \phi(D_r)$$

and

$$c^{-1}((D_\infty)) := \lim_{r \to \infty} c^{-1}(D_r)$$
on $X$ associated with an atomic section $\nu$ of the bundle $\mathbb{P}(E \oplus F) \to X$. We begin with a discussion of the pushforward case. Let $T_{r,s}$ denote the transgression form induced on $X$ in the pushforward case by a section $\nu$ of $\mathbb{P}(E \oplus F) \to X$ and an invariant polynomial $\phi$.

**Proposition 3.1.** Let $\nu$ be an atomic section of the bundle $\mathbb{P}(E \oplus F) \to X$. Then in the pushforward case the transgression current $T_{\infty,0} = \lim_{s \to 0} T_{r,s}$ converges in $L^1_{\text{loc}}(X)$ and equals

$$T_{\infty,0} = T - \frac{dt}{t} (u \frac{\partial}{\partial u} \perp T)$$

in local homogeneous coordinates $[t,u]$ where

$$T = \int_0^1 \phi \left( \frac{u^*Du}{|u|^2}; \Lambda(x) \right) dx.$$

Here $\Lambda(x)$ is given by (2.34).

**Proof.** Since $\nu$ is atomic we know that $Z \cup P$ has Lebesgue measure zero in $X$ and so by Corollary 2.30 $T_{r,s} \to T_{\infty,0}$ a.e. on $X$ as $r \to \infty$ and $s \to 0$. So to prove that $T_{r,s} \to T_{\infty,0}$ in $L^1_{\text{loc}}(X)$ we just need to show that the Lebesgue Dominated Convergence Theorem applies. This is most easily seen by working locally in the pullback via $\nu$ of the various charts on $\mathbb{P}(E \oplus F)$. Let $[t,u]$ be a local coordinate
expression for $\nu$, defined with respect to a local frame $e, f$ for $E \oplus F$ by $\nu = [te, uf]$. Recall that the coordinate expressions for the transgression form $T_{r,s}$ in the charts $\nu^{-1}(\mathcal{H}) = \{ t \neq 0 \}$ and $\nu^{-1}(\mathcal{U}_j) = \{ u_j \neq 0 \}$ for $j \in \{1, \ldots, n\}$ are given by setting $t = 1$ and $u_j = 1$ for $j \in \{1, \ldots, n\}$ respectively in the formula (2.38) for the expression for $T_{r,s}$ in homogeneous coordinates. Also note that since $\nu$ is atomic the local functions $t$ on $\nu^{-1}(\mathcal{U}_j)$ and $u$ on $\nu^{-1}(\mathcal{H})$ are atomic.

Using the atomicity of $u$ Harvey and Lawson [HL: III, 2.8] have shown that $T_{r,s}$ converges to $T_{\infty,0}$ in $L^1_{\text{loc}}(\nu^{-1}(\mathcal{H}))$. Consulting (2.38) we see that in each of the charts $\nu^{-1}(\mathcal{U}_j)$ the transgression $T_{r,s}$ is a polynomial in $\chi_r$ and $\chi_s$ with coefficients which are either smooth forms on $\nu^{-1}(\mathcal{U}_j)$ or products of a smooth form on $\nu^{-1}(\mathcal{U}_j)$ with the $L^1_{\text{loc}}(\nu^{-1}(\mathcal{U}_j))$ form $\frac{dt}{t}$. So, since $\chi_r$ and $\chi_s$ are bounded and converge to 0 and 1 almost everywhere as $r \to \infty$ and $s \to 0$, the atomicity of $t$ together with the Lebesgue Dominated Convergence Theorem imply that $T_{r,s} \to T_{\infty,0}$ in $L^1_{\text{loc}}(\nu^{-1}(\mathcal{U}_j))$.

Patching these results together we conclude that $T_{r,s}$ converges to $T_{\infty,0}$ in $L^1_{\text{loc}}(X)$. $\square$

**Corollary 3.4.** Let $\nu$ be an atomic section of the bundle $\mathbb{P}(E \oplus F) \to X$. Then the Chern current at time infinity, $\phi(\overline{D}_\infty) := \lim_{r \to \infty} \phi(\overline{D}_r)$, exists on $X$ and is of the form

$$\phi(\overline{D}_\infty) = \phi(D_F) + S,$$

where $S$ is a flat current supported on the pole set $P$ of $\nu$. Furthermore the current equation

$$dT_{\infty,0} = \phi(\overline{D}_\infty) - \phi(\overline{D}_0) \quad \text{on} \; X$$

is the current limit as $s \to 0$ and $r \to \infty$ of the smooth transgression formula

$$dT_{r,s} = \phi(\overline{D}_r) - \phi(\overline{D}_s) \quad \text{on} \; X.$$
Proof. By Corollary 2.30(1) and Proposition 3.1 the Chern current

$$\phi((\overline{D}_\infty)) := \lim_{r \to -\infty} \phi(\overline{D}_r) = \phi(\overline{D}_0) + dT_{\infty,0}$$

exists and is flat on $X \sim Z$, and hence on all of $X$ by Corollary 2.30(2). Finally the difference $S := \phi((\overline{D}_\infty)) - \phi(D_F)$ is a flat current which, by (2.8), is supported on $P$. \hfill \Box

The current $S$ is called the singular part of the Chern current, and has several nice properties which we now describe. First we define the residue form at infinity, $\text{Res}_\phi(\overline{D}_\infty)$. We work universally on $\mathbb{P}(E \oplus F)$. Set $\mathcal{R} := \mathbb{P}(E \oplus F) \sim X$. Choose a hermitian metric on the line bundle $p : \mathcal{R} \to \mathbb{P}(F)$ and let $\rho : S_{\epsilon}(\mathcal{R}) \to \mathbb{P}(F)$ denote the $\epsilon$-circle bundle in $\mathcal{R}$. Note that the restriction of the transgression current $T_{\infty,0}$ to $S_{\epsilon}(\mathcal{R})$ is a smooth form. Finally suppose that the invariant polynomial $\phi$ has pure degree $m$.

**Definition 3.5.** The **residue form at infinity** is the smooth $(m-1)$-form on $\mathbb{P}(F)$ defined by

$$\text{Res}_\phi(\overline{D}_\infty) := \int_{\rho^{-1}} T_{\infty,0} = \rho_*(T_{\infty,0}).$$

Note that the residue form pulls back via the projection map $p$ to a smooth form on $\mathbb{P}(E \oplus F) \sim X$, also denoted by $\text{Res}_\phi(\overline{D}_\infty)$.

**Lemma 3.6.**

1. The expression for the residue at infinity in a local homogeneous coordinate $u$ on $\mathbb{P}(F)$ is

$$\text{Res}_\phi(\overline{D}_\infty)_{HC} = \frac{2\pi}{i} u \frac{\partial}{\partial u} \bigg| T$$
where

\[ T = \int_0^1 \phi \left( \frac{u^* Du}{|u|^2}; \Lambda(x) \right) \, dx. \]

Therefore the residue form is independent of the choice of \( \epsilon \)-circle bundle and hermitian metric on \( \mathcal{R} \).

(2) The residue form \( \text{Res}_\phi(\overrightarrow{D}_\infty) \) is d-closed.

**Proof.**

(1) Let \([t, u]\) be local homogeneous coordinates on \( \mathbb{P}(E \oplus F) \). By (3.2)

\[ (T_{\infty, 0})_{HC} = T + \frac{1}{2\pi i} \frac{dt}{t} \left( \frac{2\pi}{i} u \frac{\partial}{\partial u} \right) T. \]

So

\[ \text{Res}_\phi(\overrightarrow{D}_\infty)_{HC} = \int_{\rho^{-1}} (T_{\infty, 0})_{HC} = \left( \frac{1}{2\pi i} \int_{|t|=\epsilon} \frac{dt}{t} \right) \frac{2\pi}{i} u \frac{\partial}{\partial u} \Lambda T = \frac{2\pi}{i} u \frac{\partial}{\partial u} \Lambda T, \]

since \( t \) can be regarded as a local fibre variable on the bundle \( p : \mathcal{R} \to \mathbb{P}(F) \).

(2) To prove that \( d \text{Res}_\phi(\overrightarrow{D}_\infty) = 0 \) it suffices to show that \( \rho_*(dT_{\infty, 0}) = 0 \). Now

\( S_\epsilon(\mathcal{R}) \subset \text{Hom}(E, F) \sim X \) and so we can apply part (5) of Theorem 2.12 to show that \( dT_{\infty, 0} = \phi(D_F) - \phi(\overrightarrow{D}_0) \) on \( S_\epsilon(\mathcal{R}) \). Now by Lemma 3.7 below

\[ u \frac{\partial}{\partial u} \Lambda \phi(\overrightarrow{D}_0) = 0 \quad \text{and} \quad u^* \frac{\partial}{\partial u^*} \Lambda \phi(\overrightarrow{D}_0) = 0. \]

So by Lemma 2.35 the expressions for \( \phi(D_F) \) and \( \phi(\overrightarrow{D}_0) \) in homogeneous coordinates are of degree zero in the fibre differentials \( dt, d\overline{t} \) of the bundle \( p : \mathcal{R} \to \mathbb{P}(F) \).

This implies that

\[ \rho_*(dT_{\infty, 0}) = \int_{\rho^{-1}} \phi(D_F) - \int_{\rho^{-1}} \phi(\overrightarrow{D}_0) = 0 - 0 = 0. \quad \square \]

**Lemma 3.7.**

\[ u \frac{\partial}{\partial u} \Lambda \phi(\overrightarrow{D}_0) = 0 \quad \text{and} \quad u^* \frac{\partial}{\partial u^*} \Lambda \phi(\overrightarrow{D}_0) = 0. \]
Proof. Harvey and Lawson [HL: III,4.4] prove that, in a local coordinate $u$ on $\text{Hom}(E,F)$, $\phi(\overrightarrow{D}_0) = \phi(\overrightarrow{\Omega}_0)$ where

$$\overrightarrow{\Omega}_0 = \left(1 - \frac{u^*u}{|u|^2}\right) \left(\Omega_F - \frac{Du^*Du}{|u|^2}\right) + \frac{u^*\Omega_Eu}{|u|^2}.$$  

Now $u^*\frac{\partial}{\partial u^*} \perp \phi(\overrightarrow{\Omega}_0) = 0$ since $u^*\frac{\partial}{\partial u^*} \perp \overrightarrow{\Omega}_0 = \left(1 - \frac{u^*u}{|u|^2}\right) \frac{u^*Du}{|u|^2} = 0$. To prove that $u\frac{\partial}{\partial u} \perp \phi(\overrightarrow{\Omega}_0) = 0$ first note that $\overrightarrow{\Omega}_0$ is of the form $\overrightarrow{\Omega}_0 = (1-P) + \alpha P$ where $\alpha$ is a scalar, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $1 \times (n-1)$ block form. Since

$$1 + (1-P)A + \alpha P = \begin{pmatrix} 1 + \alpha & 0 \\ c & 1 + d \end{pmatrix}$$

and

$$1 + A(1-P) + \alpha P = \begin{pmatrix} 1 + \alpha & b \\ 0 & 1 + d \end{pmatrix},$$

$$\det(1 + (1-P)A + \alpha P) = \det(1 + A(1-P) + \alpha P),$$

and so for any invariant polynomial $\phi$,

$$\phi((1-P)A + \alpha P) = \phi(A(1-P) + \alpha P).$$

The result now follows from the fact that

$$u\frac{\partial}{\partial u} \perp \left(\left(\Omega_F - \frac{Du^*Du}{|u|^2}\right) \left(1 - \frac{u^*u}{|u|^2}\right) + \frac{u^*\Omega_Eu}{|u|^2}\right) = 0. \quad \square$$

Finally we come to the main result of the chapter.

**Theorem 3.8.** Let $\nu$ be an atomic section of $\mathbb{P}(E \oplus F) \to X$. Then the singular part of the Chern current $\phi(\overrightarrow{D}_\infty)$ is given by

$$S = \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \ \text{Div}_\infty(\nu).$$

Therefore

$$\phi(\overrightarrow{D}_\infty) = \phi(D_F) + \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \ \text{Div}_\infty(\nu) \quad (3.9)$$

and

$$dT_{\infty,0} = \phi(D_F) + \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \ \text{Div}_\infty(\nu) - \phi(\overrightarrow{D}_0) - \text{Res}_\phi(\overrightarrow{D}_0) \ \text{Div}_0(\nu) \quad (3.10)$$

on $X$. 

Remark 3.11. Note that the singular part of the Chern current at time infinity decomposes into the product of two terms — the pole divisor which is independent of the invariant polynomial $\phi$, and the smooth form $\nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty))$ on $X \sim Z$ which depends on both $\nu$ and $\phi$. This phenomenon should be compared and contrasted to the decomposition of the singular part of the Chern current at time zero, where the residue form $\text{Res}_\phi(\overrightarrow{D}_0)$ on $X$ is $\nu$-independent.

We will discuss the residue at infinity in much more detail in Section 4. For now note the following two facts.

Corollary 3.12.

\[ \text{Res}_{c_1}(\overrightarrow{D}_\infty) = 1 \]

so that

\[ c_1(\overrightarrow{D}_\infty) = c_1(D_F) + \text{Div}_\infty(\nu) \quad \text{on } X. \]

Proof. By Lemma 3.6, $\text{Res}_{c_1}(\overrightarrow{D}_\infty) = \frac{2\pi}{i} u \frac{\partial}{\partial u} \int T$ where $T = \frac{i}{2\pi} \text{ Tr} \left( \frac{u^* Du}{|u|^2} \right).$

So $\text{Res}_{c_1}(\overrightarrow{D}_\infty) = u \frac{\partial}{\partial u} \int \text{ Tr} \left( \frac{u^* Du}{|u|^2} \right) = \text{ Tr} \left( \frac{u^* u}{|u|^2} \right) = 1. \quad \Box$

Remark 3.13. Case rank $F = 1$. In the case rank $F = 1$ the Chern current formulae are particularly simple. Let $\phi = c_1^m$. By (3.3)

\[ T = \left( \frac{i}{2\pi} \right)^m \int_0^1 m \frac{Du}{u} \left( (1 - x) \Omega_F + x \Omega_E - x \frac{Du^* Du}{|u|^2} \right)^{m-1} dx \]

\[ = \left( \frac{i}{2\pi} \right)^m \frac{Du}{u} \int_0^1 m ((1 - x) \Omega_F + x \Omega_E)^{m-1} dx, \]

since $Du \wedge Du = 0$. Integrating this gives

\[ T = \frac{i}{2\pi} \frac{Du}{u} c_1(D_F)^m - c_1(D_E)^m \]
So by (3.2) and Lemma 3.6

\[(T_{\infty,0})_{HC} = T + \frac{1}{2\pi i} \frac{dt}{t} \text{Res}_m(\overrightarrow{D}_\infty),\]

where

\[\text{Res}_m(\overrightarrow{D}_\infty) = \frac{c_1(D_F)^m - c_1(D_E)^m}{c_1(D_F) - c_1(D_E)}\]

Consequently

\[(3.14)\ dT_{\infty,0} = c_1(D_F)^m - c_1(D_E)^m + (\text{Div}_\infty(\nu) - \text{Div}_0(\nu)) \frac{c_1(D_F)^m - c_1(D_E)^m}{c_1(D_F) - c_1(D_E)}\]

on \(X\). The case \(m = 1\) gives a \(C^\infty\)-generalization of the classical Poincaré-Lelong formula,

\[(3.15)\ dT_{\infty,0} = c_1(D_F) - c_1(D_E) + \text{Div}_\infty(\nu) - \text{Div}_0(\nu).\]

**Proof of Theorem 3.8.** We compute \(S\) by calculating \(dT_{\infty,0}\) in each of the charts \(\nu^{-1}(U_j)\) By (3.2) and Lemma 3.6,

\[T_{\infty,0} = T + \frac{1}{2\pi i} \frac{dt}{t} \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \quad \text{on } \nu^{-1}(U_j)\]

where \(T\) and \(\text{Res}_\phi(\overrightarrow{D}_\infty)\) are smooth forms on \(\nu^{-1}(U_j)\). So

\[dT_{\infty,0} = dT + \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{Div}_\infty(\nu) \quad \text{on } \nu^{-1}(U_j)\]

since \(d\text{Res}_\phi(\overrightarrow{D}_\infty) = 0\). On the other hand we know that

\[dT_{\infty,0} = \phi(D_F) + S - \phi(\overrightarrow{D}_0),\]

where \(\phi(\overrightarrow{D}_0)\) and \(\phi(D_F)\) are smooth forms on \(\nu^{-1}(U_j)\). So

\[S - \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{Div}_\infty(\nu) = \phi(\overrightarrow{D}_0) - \phi(D_F) + dT,\]

and since the L.H.S. is zero on the dense subset \(\nu^{-1}(U_j) \sim P\) and the R.H.S. is smooth on \(\nu^{-1}(U_j)\) we are forced to conclude that both sides of the equation are zero on \(\nu^{-1}(U_j)\). This proves that \(S = \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{Div}_\infty(\nu)\) on \(\nu^{-1}(U_j)\) and hence on all of \(X\). \(\Box\)
Remark 3.16. The pullback case. Analogous results hold in the pullback case, with \( \phi = c^{-1} = (1 + c_1)^{-1} \). In particular
\[
(T_{r,s})_{HC} = T_{r,s} - \frac{dt}{t} (u \frac{\partial}{\partial u} \lrcorner T_{r,s})
\]
where
\[
T_{r,s} := \frac{i}{2\pi} \frac{Duu^*}{|u|^2} \frac{(c(D_E) + \chi_r (c(D_0) - c(D_E)))^{-1} - (c(D_E) + \chi_r (c(D_0) - c(D_E)))^{-1}}{c(D_0) - c(D_E)},
\]
and \( c(D_0) = c(\overline{\Omega}_0) \) where
\[
\overline{\Omega}_0 = \frac{u\Omega_F u^*}{|u|^2} - \frac{Du \left( 1 - \frac{u^* u}{|u|^2} \right) Du^*}{|u|^2}.
\]
The transgression current \( T_{\infty,0} = \lim_{s \to 0} T_{r,s} \) converges in \( L^1_{loc}(X) \) and equals
\[
(T_{\infty,0})_{HC} = T - \frac{dt}{t} (u \frac{\partial}{\partial u} \lrcorner T),
\]
where
\[
T = -\frac{i}{2\pi} \frac{Duu^*}{|u|^2} \frac{c(D_E)^{-1} - c(D_0)^{-1}}{c(D_E) - c(D_0)}.
\]
The Chern current at time infinity exists and is given by
\[
c^{-1}(\langle \overline{D_0} \rangle) = c(D_E)^{-1} + c(D_E)^{-1} \nu^* (c(D_L)^{-1}) \text{ Div}_\infty(\nu),
\]
where \( L \) is the tautological line bundle on \( \mathbb{P}(F) \) with connection \( D_L \) induced from that on \( F \). Furthermore
\[
dT_{\infty,0} = c(D_E)^{-1} + \nu^* \text{Res}_{c^{-1}}(\overline{D_\infty}) \text{ Div}_\infty(\nu) - c^{-1}(\overline{D_0}) - \text{Res}_{c^{-1}}(\overline{D_0}) \text{ Div}_0(\nu),
\]
where
\[
\text{Res}_{c^{-1}}(\overline{D_\infty}) = c(D_E)^{-1} c(D_L)^{-1} \quad \text{on } \mathbb{P}(F)
\]
and
\[
\text{Res}_{c^{-1}}(\overline{D_0}) = c(D_E)^{-1} c(D_F)^{-1} \quad \text{on } X.
\]
Also note that in the case rank \( F = 1 \) equation (3.17) reduces to the negative of equation (3.14).
Note. These results are proved just as in the pushforward case. To prove (3.18) note that \((U_0, \overline{D}_0) = (L, D_L)\) on the projective space \(\mathbb{P}(F)\) at infinity.
4. Residues at infinity.

In this section we aim to compute the residue form at time infinity, $\text{Res}_\phi(\overline{D}_\infty)$, on the projective bundle $\mathbb{P}(F)$ at infinity in $\mathbb{P}(E \oplus F)$. Except for the following fact our results are entirely analogous to those of Harvey and Lawson [HL: III,7]. They were able to prove that the residue form $\text{Res}_\phi(\overline{D}_0)$ on $X$ is a polynomial $\psi(c_1(D_E), c_1(D_F), \ldots, c_n(D_F))$ in the Chern forms of the bundles $(E, D_E)$ and $(F, D_F)$. In contrast to this the residue form at infinity on $\mathbb{P}(F)$ is given by

$$\text{Res}_\phi(\overline{D}_\infty) = \psi(c_1(D_L), c_1(D_E), c_1(D_F), \ldots, c_n(D_F)) + dS_\phi$$

where $L \rightarrow \mathbb{P}(F)$ is the tautological line bundle and $\psi$ is a polynomial in the Chern forms of the bundles $(E, D_E), (F, D_F)$ and $(L, D_L)$. The important point here is that the differential form $S_\phi$ is not closed in general. Therefore we often prefer to compute the residue class, $[\text{Res}_\phi(\overline{D}_\infty)]$, rather than the residue form itself.

The residue class is computable in the sense that for any invariant polynomial $\phi$ there is an explicit formula for the polynomial $\psi$ for which

$$[\text{Res}_\phi(\overline{D}_\infty)] = \psi(c_1(L), c_1(E), c_1(F), \ldots, c_n(F)).$$

Here $c_j(V)$ denotes the $j$-th Chern class of the bundle $V$. Actually for any given invariant polynomial $\phi$ it is also possible to explicitly calculate the form $S_\phi$ in a local homogeneous coordinate system on $\mathbb{P}(F)$. In particular we will prove that $S_{c_1}$ and $S_{c_2}$ are both zero. However this behaviour is far from typical as is suggested by the example of $S_{c_3}$ which is not even a closed form on $\mathbb{P}(F)$.

First we recall the definition of the residue form on $\mathbb{P}(F)$. Let $\mathcal{R} = \mathbb{P}(E \oplus F) \sim X$ and define a mapping

$$(4.1) \quad \pi : \mathcal{R} \rightarrow \mathbb{P}(F)$$

$$[e, f] \mapsto [f].$$
Note that we can view $\mathcal{R}$ as the total space of the tautological line bundle $\text{Hom}(L, E) \to \mathbb{P}(F)$. Let $\rho : S_\epsilon(\mathcal{R}) \to \mathbb{P}(F)$ be the $\epsilon$-sphere bundle and recall that the restriction of the transgression current $T_{\infty,0}$ on $\mathbb{P}(E \oplus F)$ to $S_\epsilon(SR)$ is a smooth form. Then the residue form at infinity on $\mathbb{P}(F)$ is defined by the fibre integral

$$
(4.2) \quad \text{Res}_\phi(\overrightarrow{D}_\infty) := \int_{\rho^{-1}} T_{\infty,0} = \rho_*(T_{\infty,0}).
$$

Note that the current pushforward is equal to the fibre integral because the restriction of $T_{\infty,0}$ to each fibre of $\rho$ is a Lebesgue integral form. Recall that $\text{Res}_\phi(\overrightarrow{D}_\infty)$ is a smooth $d$-closed form on $\mathbb{P}(F)$, well defined independent of $\epsilon$. The following result provides an alternative definition of the residue form, one which is better adapted to the problem of computing the residue.

**Theorem 4.3.** For any invariant polynomial $\phi$,

$$
(4.4) \quad \text{Res}_\phi(\overrightarrow{D}_\infty) = \pi_*(\phi(D_Q)) = \int_{\pi^{-1}} \phi(D_Q),
$$

where $D_Q$ is the connection induced on the quotient bundle $Q = E \oplus F / U$ by $D_{E \oplus F}$ (c.f. Definition 2.7), and $\pi$ is given by (4.1). In particular the cohomology class of the residue on $\mathbb{P}(F)$ is given by the formula

$$
[\text{Res}_\phi(\overrightarrow{D}_\infty)] = \pi_!(\phi(Q))
$$

where $\phi(Q) \in H^*(\mathbb{P}(E \oplus F); \mathbb{R})$ is the $\phi$-characteristic class of $Q$ over $\mathbb{P}(E \oplus F)$.

We say that $\phi$ is **integral** if it corresponds to an integral cohomology class under the canonical identification $\text{I}_{\text{GL}_n(\mathbb{C})} \cong H^*(\text{BGL}_n(\mathbb{C}); \mathbb{R})$ given by the Chern-Weil homomorphism.
Corollary 4.5. If $\phi$ is integral then the residue class $[\text{Res}_\phi(\overrightarrow{D}_\infty)]$ is the image of an integer class on $\mathbb{P}(F)$ under the mapping from $H^*(\mathbb{P}(F) ; \mathbb{Z})$ to $H^*(\mathbb{P}(F) ; \mathbb{R})$.

Proof of Theorem 4.3. By Lemma 2.29 and Theorem 3.8 we have that

$$dT_{\infty,0} = \phi(D_F) + \text{Res}_\phi(\overrightarrow{D}_\infty)[\mathbb{P}(F)] - \phi(\overrightarrow{D}_0)$$

and

$$dT_{1,0} = \phi(D_Q) - \phi(\overrightarrow{D}_0)$$

on $\mathcal{R}$. Subtracting these equations gives

$$\text{(4.6)} \quad \text{Res}_\phi(\overrightarrow{D}_\infty)[\mathbb{P}(F)] = \phi(D_Q) - \phi(D_F) + dT_{\infty,1} \quad \text{on } \mathcal{R}.$$  

Now $\phi(D_Q) \in L^1(\pi^{-1}(p))$ for all $p \in \mathbb{P}(F)$ since it is smooth on all of $\mathbb{P}(E \oplus F)$. Consequently the current push forward $\pi_*(\phi(D_Q))$ is defined and equals

$$\pi_*(\phi(D_Q)) = \int_{\pi^{-1}} \phi(D_Q).$$

Also since $\mathbb{P}(E \oplus F) \to X$ has compact fibres we can use Fubini’s Theorem to show that $T_{\infty,0} \in L^1(\pi^{-1}(p))$ for all $p \in \mathbb{P}(F)$. Therefore $\pi_*(T_{\infty,1})$ is well defined. In fact $\pi_*(T_{\infty,1}) = \int_{\pi^{-1}} T_{\infty,1} = 0$ since, by Corollary 2.37, $T_{\infty,1}$ is of degree $< 2$ in the fibre differentials $dt, dt$. Finally by (4.6), $\text{Res}_\phi(\overrightarrow{D}_\infty) = \pi_*(\text{Res}_\phi(\overrightarrow{D}_\infty)[\mathbb{P}(F)]) = \pi_*(\phi(D_Q))$, since $\pi_*(dT_{\infty,1}) = d\pi_*(T_{\infty,1}) = 0$. □

The following result says that the residue form at time zero on $X$ is the fibre integral of the residue form at time infinity on $\mathbb{P}(F)$.

Proposition 4.7. Consider the bundle map $\eta : \mathbb{P}(F) \to X$. Then

$$\text{Res}_\phi(\overrightarrow{D}_0) = \int_{\eta^{-1}} \text{Res}_\phi(\overrightarrow{D}_\infty).$$
Proof. Recall [HL: III,7.6] that the residue form on $X$ is given by the fibre integral

$$\text{Res}_\phi(\overline{D}_0) = \int_{\mathbb{P}(E \oplus F)_x} \phi(D_Q) = \int_{\mathbb{P}(F)_x} \phi(D_Q) = \int_{\mathbb{P}(F)_x} \int_{\pi^{-1}} \phi(D_Q)$$

$$= \int_{\mathbb{P}(F)_x} \text{Res}_\phi(\overline{D}_\infty) = \int_{\eta^{-1}} \text{Res}_\phi(\overline{D}_\infty).$$

$\square$

Remark 4.8. In the pull back case we have the analogous formulae

$$(4.9) \quad \text{Res}_\phi(\overline{D}_\infty) = \pi_*(\phi(D_U)) = \int_{\pi^{-1}} \phi(D_U),$$

and

$$(4.10) \quad \text{Res}_\phi(\overline{D}_0) = \int_{\eta^{-1}} \text{Res}_\phi(\overline{D}_\infty).$$

Recall (3.18) and (3.19) which said that

$$(4.11) \quad \text{Res}_{c^{-1}}(\overline{D}_\infty) = c(D_E)^{-1}c(D_L)^{-1} \quad \text{on } \mathbb{P}(F)$$

and

$$(4.12) \quad \text{Res}_{c^{-1}}(\overline{D}_0) = c(D_E)^{-1}c(D_F)^{-1} \quad \text{on } X.$$
where $\pi_*$ denotes current pushforward which in this case equals integration over the fibres of $\pi$.

Harvey and Lawson [HL: III,7.14,7.28] have used the topological version of Bott's Theorem to compute the residue form $\text{Res}_\phi(D_0)$. The analogous fact needed to compute the residue at infinity is given by equation (4.11) which we restate in the following proposition.

**Proposition 4.14.** Let $U$ be the tautological line bundle over $\mathbb{P}(E \oplus F)$ and let $\pi$ be given by (4.1). Then

$$\pi_*(c(D_U)^{-1}) = \int_{\pi^{-1}} c(D_U)^{-1} = c(D_E)^{-1} c(D_L)^{-1}. $$

In particular,

$$\pi_*(c_1^k(D_U)) = - \frac{c_1^k(D_L) - c_1^k(D_E)}{c_1(D_L) - c_1(D_E)} = - \sum_{i+j=k-1} c_1^i(D_E)c_1^j(D_L).$$

**Remark 4.15.** In the case $E = \mathbb{C}$ we have

$$\pi_*(c(D_U)^{-1}) = c(D_L)^{-1} \quad \text{and} \quad \pi_*(c_1^k(D_U)) = - c_1^{k-1}(D_L).$$

At this point it might be helpful to recall that the cohomology $H^*(\mathbb{P}(F); \mathbb{Z})$ of a projectivized vector bundle $F \to X$ of rank $n$ is a finitely generated free module over $H^*(X; \mathbb{Z})$ with additive basis $1, t, t^2, \ldots, t^{n-1}$ where $t = - [c_1(L)] \in H^2(\mathbb{P}(F); \mathbb{Z})$ is the first Chern class of the tautological bundle $L$. As an algebra over $H^*(X; \mathbb{Z})$ the cohomology ring $H^*(\mathbb{P}(F); \mathbb{Z})$ is generated by $t$ subject to the relation

$$- t^n = t^{n-1} c_1(F) + t^{n-2} c_2(F) + \cdots + t c_{n-1}(F) + c_n(F).$$

(4.16)
In cohomology the map

$$
\pi_1 : H^*(\mathbb{P}(F); \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})
$$
determined by fibre integration is simply given by

$$
\pi_1(a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}) = a_{n-1},
$$
where $a_0, \ldots, a_{n-1} \in H^*(X; \mathbb{Z})$. Also note that

$$
\pi_1(t^n + k^{-1}) = \text{the degree } 2k \text{ part of } c(F)^{-1}.
$$

Proofs of these facts can be found in [LM] and [HL: III, 7.14]. Note that Bott's Theorem says that these statements actually hold on the level of forms.

The facts outlined above suggest that

$$
[\text{Res}_\phi(\overline{D}_\infty)] = \psi(c_1(L), c_1(E), c_1(F), \ldots, c_n(F))
$$
for some polynomial $\psi$ and that this polynomial can be calculated by

1. Expressing $\phi(Q)$ as a polynomial in $c_1(U)$, $c_1(E)$ and $c_j(F)$ for $j \in \{1, \ldots, n\}$, and
2. Using Proposition 4.14 to find $\pi_1(c_1^k(U))$ in terms of $c_1(E)$ and $c_1(L)$, thereby computing

$$
[\text{Res}_\phi(\overline{D}_\infty)] = \pi_1(\phi(Q)).
$$

First we calculate $[\text{Res}_\phi(\overline{D}_\infty)]$ for the general invariant polynomial $\phi$ and then for some particular invariant polynomials.

Recall that any $\mathfrak{gl}(n, \mathbb{C})$-invariant polynomial can be expressed as a polynomial in the Chern forms $c_1, \ldots, c_n$, and that the residue of a sum of polynomials is the
sum of the residues. Consequently to compute the residue for a general \( \phi \) it suffices to compute it for the polynomials

\[
\phi = \prod_{k=1}^{n} c_k^{j_k}.
\]

Since

\[
0 \rightarrow U \rightarrow E \oplus F \rightarrow Q \rightarrow 0
\]

is exact over \( \mathbb{P}(E \oplus F) \) we have the sequence of formulae

(4.18) \[ c(E \oplus F) = c(U \oplus Q) \]

(4.19) \[ c(E) c(F) = c(U) c(Q) \]

(4.20) \[ c(Q) = c(U)^{-1} c(E) c(F) \]

and

(4.21) \[ c_k(Q) = \{ c(U)^{-1} c(E) c(F) \}_k \]

where \( \{T\}_k \) denotes the degree 2\( k \) part of \( T \). Therefore for any \( \phi \) given by (4.17) we have

\[
\phi(Q) = \prod_{k=1}^{n} \left( \{ c(U)^{-1} c(E) c(F) \}_k \right)^{j_k}
\]

(4.22) \[ = \sum_{l=0}^{\infty} c_l(U) \psi_l(c_1(E), c_1(F), \ldots, c_n(F)) \]

where the polynomials \( \psi_l \) are defined by this last equation. Applying Theorem 4.3 and Proposition 4.14 we have the following result.

**Theorem 4.23.** Let \( \phi = \prod_{k=1}^{n} c_k^{j_k} \). Then

\[ [\text{Res}_\phi(\overline{D}_{\infty})] = -\sum_{l=0}^{\infty} \frac{c_l(L) - c_l(E)}{c_1(L) - c_1(E)} \psi_l(c_1(E), c_1(F), \ldots, c_n(F)) \]

in \( H^*(\mathbb{P}(F); \mathbb{R}) \) where \( \psi_l \) is the polynomial defined by equation (4.22).

For many of the more familiar invariant polynomials it is interesting to try to calculate \([\text{Res}_\phi(\overline{D}_{\infty})]\) without resorting to an expression of the form (4.17) for \( \phi \).
Theorem 4.24.

\[ \text{[Res}_c(\overrightarrow{D}_\infty)] = c(L)^{-1} c(F). \]

Proof. By (4.20) and Proposition 4.14 we have

\[ \text{[Res}_c(\overrightarrow{D}_\infty)] = \pi_1(c(Q)) = \pi_1(c(U)^{-1}) c(E) c(F) \]
\[ = c(E)^{-1} c(L)^{-1} c(E) c(F) \]
\[ = c(L)^{-1} c(F). \quad \square \]

Corollary 4.25. Let \( T = -c_1(L) \). Then

\[ \text{[Res}_c(\overrightarrow{D}_\infty)] = 1 \]

and, for \( k \in \{2, \ldots, n\} \),

\[ \text{[Res}_{c_k}(\overrightarrow{D}_\infty)] = T^{k-1} + T^{k-2} c_1(F) + \cdots + c_{k-1}(F). \]

Recall that the Chern character is defined by \( \text{ch}(\Omega) := \text{tr}(e^{\frac{i}{2\pi} \Omega}) \) and that the \( k \)th trace power, \( b_k \), is given by \( b_k := \text{tr}((\frac{i}{2\pi} \Omega)^k) \). These are related by the formula

\[ \text{ch} = \sum_{k=0}^\infty \frac{b_k}{k!}. \]


\[ \text{[Res}_{ch}(\overrightarrow{D}_\infty)] = \sum_{k=1}^\infty \frac{1}{k!} \frac{c_k^1(L) - c_k^1(E)}{c_1(L) - c_1(E)}. \]

Proof. The formula \( \text{ch}(E \oplus F) = \text{ch}(U \oplus Q) \) implies that

\[ \text{ch}(Q) = \text{ch}(E) + \text{ch}(F) - \text{ch}(U). \]
Therefore
\[
\text{[Res}_{\text{ch}}(D_\infty)] = \pi_1(\text{ch}(Q)) = -\pi_1(e^{c_1(U)})
\]
\[
= - \sum_{k=1}^{\infty} \frac{1}{k!} \pi_1(c_1^k(U))
\]
\[
= - \sum_{k=1}^{\infty} \frac{c_1^k(L) - c_1^k(E)}{c_1(L) - c_1(E)}
\]
by Proposition 4.14. \qed

**Corollary 4.27.**
\[
\text{[Res}_{b_k}(D_\infty)] = \frac{c_1^k(L) - c_1^k(E)}{c_1(L) - c_1(E)}.
\]

**Theorem 4.28.**
\[
\text{[Res}_{c^{-1}}(D_\infty)] = -c(E)^{-1} c(F)^{-1}.
\]

**Proof.** By equation (4.19), \(c(Q)^{-1} = c(U)c(E)^{-1} c(F)^{-1}\). So by Proposition 4.14
\[
\text{[Res}_{c^{-1}}(D_\infty)] = \pi_1(c_1(U)) c(E)^{-1} c(F)^{-1} = -c(E)^{-1} c(F)^{-1}.
\]
\qed

**Theorem 4.29.** Let \(H(c)\) be a multiplicative series of Chern classes associated to the formal power series \(h(x) = 1 + a_1 x + a_2 x^2 + \cdots \in \mathbb{R}[x]\). Then
\[
\text{[Res}_{H}(D_\infty)] = \pi_1(h^{-1}(c_1(U))) H(E) H(F).
\]

In particular if \(h^{-1}(x) = 1 + b_1 x + b_2 x^2 + \cdots\) is the formal power series for the reciprocal of \(h\), then
\[
\text{[Res}_{H}(D_\infty)] = - \sum_{k=1}^{\infty} b_k \frac{c_1^k(L) - c_1^k(E)}{c_1(L) - c_1(E)} H(E) H(F).
\]

**Proof.** Since \(H(E) H(F) = H(U) H(Q)\) and \(H(U) = h(c_1(U))\) we have
\[
\text{[Res}_{H}(D_\infty)] = \pi_1(h^{-1}(c_1(U))) H(E) H(F).
\]
\qed
Remark 4.30. See [LM] for the definition of a multiplicative series of Chern classes. The main examples are given by the Todd series $H = Td$ in which case $h(x) = \frac{x}{1-e^{-x}}$ and the series $H = e^{c_1}$ generated by $h(x) = e^x$.

Remark 4.31. Flat case. Let $T = -c_1(L)$. Then in the flat case we have

\[
[\text{Res}_c(D) = c(L)^{-1}
\]

\[
[\text{Res}_{ch}(D) = Td^{-1}(T)
\]

\[
[\text{Res}_{b_k}(D) = b_{k-1}(L)
\]

\[
[\text{Res}_{c^{l-1}}(D) = 0
\]

\[
[\text{Res}_{H}(D) = -\sum_{k=1}^{n} b_k c_1^{k-1}(L)
\]

and, for any degree $m$ invariant polynomial $\phi$ which is a simple product of the Chern polynomials $c_1, \ldots, c_n$,

\[
[\text{Res}_\phi(D) = T^{m-1}.
\]

In particular

\[
[\text{Res}_\phi(D) = 0 \quad \text{for degree } \phi > n.
\]

Actually the next proposition shows that these formulae hold on the form level.

Proposition 4.32. Let $\phi = \prod_{k=1}^{n} c_k^{l_k}$ be an invariant polynomial of degree $m$. Then in the flat case

\[
\text{Res}_\phi(D) = T^{m-1},
\]

where $T = -c_1(D_L)$. 
Proof. Let $u$ be a homogeneous coordinate on $\mathbb{P}(F)$. First note that

\[ b_k(du^*du) = \text{tr}((\frac{i}{2\pi} du^*du)^k) \]
\[ = (\frac{i}{2\pi} du^*du)^{k-1} \text{tr}(\frac{i}{2\pi} du^*du) \]
\[ = -(\frac{i}{2\pi} du^*du)^k. \]

Then by the Newton identities (and Corollary 4.25),

\[ c_k(-du^*du) = (\frac{i}{2\pi} du^*du)^k \]

and so

\[ \phi(-du^*du) = (\frac{i}{2\pi} du^*du)^m. \]

So by Lemma 3.6

\[ \text{Res}_\phi(\overrightarrow{D}_\infty) = -\frac{1}{m} (\frac{i}{2\pi})^{m-1} u \frac{\partial}{\partial u} \phi \left( \frac{-du^*du}{|u|^2} \right) \]
\[ = (\frac{i}{2\pi})^{m-1} u \frac{\partial}{\partial u} \left( \frac{du^*du}{|u|^2} - \frac{du^*udu^*}{|u|^4} \right)^{m-1} \]
\[ = (\frac{i}{2\pi})^{m-1} \left( \frac{du^*du}{|u|^2} - \frac{du^*udu^*}{|u|^4} \right)^{m-1} \]
\[ = (-c_1(D_L))^{m-1} = T^{m-1} \quad \square \]

We will conclude this chapter by computing the residue form on $\mathbb{P}(F)$ for some special choices of $\phi$, but first we derive a formula which relates $[\text{Res}_{c_1}(\overrightarrow{D}_\infty)]$ to $[\text{Res}_{\psi}(\overrightarrow{D}_\infty)]$. This result is analogous to [HL: III,4.16] which states that $\text{Res}_{c_1}(\overrightarrow{D}_0) = \psi(D_F)$ on $X$.

**Proposition 4.33.** Let $\psi$ be any invariant polynomial. Then

\[ [\text{Res}_{c_1}(\overrightarrow{D}_\infty)] = c_1(F) [\text{Res}_{\psi}(\overrightarrow{D}_\infty)] + \psi(E \oplus L^\perp), \]
where $L^\perp$ is the orthogonal complement of $L$ in $F$ over $\mathbb{P}(F)$.

**Proof.** On the subset $\mathcal{R}$ of $\mathbb{P}(E \oplus F)$ we have

\begin{equation}
(4.34) \quad dT = \phi(D_F) + \text{Res}_\phi(\overrightarrow{D}_{\infty})[\mathbb{P}(F)] - \phi(\overrightarrow{D}_0)
\end{equation}

\begin{equation}
(4.35) \quad dT_1 = \phi(D_Q) - \phi(\overrightarrow{D}_0)
\end{equation}

\begin{equation}
(4.36) \quad d\tau = c_1(D_F) + [\mathbb{P}(F)] - c_1(\overrightarrow{D}_0) \quad \text{and}
\end{equation}

\begin{equation}
(4.37) \quad d\tau_1 = c_1(D_Q) - c_1(\overrightarrow{D}_0).
\end{equation}

Subtracting (4.37) from (4.36) gives

\begin{equation}
(4.38) \quad c_1(D_Q) = c_1(D_F) + [\mathbb{P}(F)] + dr
\end{equation}

where $r = \tau_1 - \tau$, and subtracting (4.34) from (4.35) and setting $\phi = c_1 \psi$ gives

\begin{equation}
(4.39) \quad dR = c_1(D_Q) \psi(D_Q) - c_1(D_F) \psi(D_F) - \text{Res}_{c_1 \psi}(\overrightarrow{D}_{\infty})[\mathbb{P}(F)]
\end{equation}

where $R = T_1 - T$. Plugging (4.38) into (4.39) and simplifying gives

\begin{equation}
(4.40) \quad c_1(D_F) (\psi(D_Q) - \psi(D_F)) + \left(\psi(D_Q) - \text{Res}_{c_1 \psi}(\overrightarrow{D}_{\infty})\right)[\mathbb{P}(F)] = d\tilde{R}
\end{equation}

where $\tilde{R} = R - r\psi(D_Q)$. Taking $\pi_\ast$ of both sides of (4.40) gives

\[ \text{Res}_{c_1 \psi}(\overrightarrow{D}_{\infty}) = c_1(D_F) \text{Res}_\psi(\overrightarrow{D}_{\infty}) + \psi(D_Q) + dS \quad \text{on } \mathbb{P}(F) \]

and the proof is completed by noting that $Q = E \oplus L^\perp$ on $\mathbb{P}(F)$. □
Example 4.41.

\[
[\text{Res}_{c_1c_2}(\overline{D}_\infty)] = c_1(F) [\text{Res}_{c_2}(\overline{D}_\infty)] + c_2(E \oplus L^\perp)
\]

\[
= c_1(F)(c_1(F) - c_1(L)) + c_2(E \oplus L^\perp)
\]

\[
= c_1(F)(c_1(F) - c_1(L)) + c_1(E)c_1(L^\perp) + c_2(L^\perp)
\]

\[
= (c_1(F) + c_1(E))(c_1(F) - c_1(L)) + c_2(F) - c_1(L)(c_1(F) - c_1(L))
\]

\[
= c_1(F)(c_1(F) - c_1(E)) + c_2(F) - c_1(L)(c_1(E) - 2c_1(F)) + c_1^2(L).
\]

Note that if we apply Proposition 4.7 and Theorem 4.14 in the case \( n = 2 \) we recover the fact that \( \text{Res}_{c_1c_2}(\overline{D}_0) = c_1(D_F) \).

Now we turn to the question of computing the residue form \( \text{Res}_{c_1}^m(\overline{D}_\infty) \). Note that we could use Theorem 4.23 or Proposition 4.33 to compute the residue class \( [\text{Res}_{c_1}^m(\overline{D}_\infty)] \).

**Proposition 4.42.**

\[
\text{Res}_{c_1}^m(\overline{D}_\infty) = \frac{(c_1(D_F) + c_1(D_E) - c_1(D_L))^m - c_1(D_F)^m}{c_1(D_E) - c_1(D_L)}
\]

as differential forms on \( \mathbb{P}(F) \).

**Proof.** Let \( u \) be a homogeneous coordinate on \( \mathbb{P}(F) \). Either by direct calculation or using [HL: III,7.33] we have

\[
c_1(D_L) = \frac{i}{2\pi} \left( \frac{u\Omega_F u^*}{|u|^2} - \frac{Du(1 - \frac{u^* u}{|u|^2})Du^*}{|u|^2} \right).
\]
Then by Lemma 3.6

\[
\text{Res}_{c_1^m}(\overrightarrow{D}_\infty) \\
= -\left(\frac{i}{2\pi}\right)^{m-1} u \frac{\partial}{\partial u} u^* \frac{\partial}{\partial u^*} \int_0^1 \frac{\text{tr}^m \left( \Omega_F - x \left( \frac{u^* u}{|u|^2} (\Omega_F - \Omega_E) + \frac{D u^* Du}{|u|^2} \right) \right)}{x} dx \\
= -\left(\frac{i}{2\pi}\right)^{m-1} u \frac{\partial}{\partial u} u^* \frac{\partial}{\partial u^*} \int_0^1 \left( \text{tr}(\Omega_F) + x \Omega_E - x \frac{u^* u}{|u|^2} + x \frac{D u^* Du}{|u|^2} \right)^m \frac{dx}{x} \\
= \int_0^1 m(c_1(D_F) + x(c_1(D_E) - c_1(D_L)))^{m-1} dx \\
= \frac{(c_1(D_F) + c_1(D_E) - c_1(D_L))^m - c_1(D_F)^m}{c_1(D_E) - c_1(D_L)}. \quad \square
\]

Theorem 4.43.

(4.44) \quad \text{Res}_{c_2}(\overrightarrow{D}_\infty) = c_1(D_F) - c_1(D_L)

(4.45) \quad \text{Res}_{b_2}(\overrightarrow{D}_\infty) = c_1(D_E) + c_1(D_L)

(4.46) \quad \text{Res}_{b_3}(\overrightarrow{D}_\infty) = c_2^2(D_E) + c_1(D_E) c_1(D_L) + c_1^2(D_L) + ds

(4.47) \quad \text{Res}_{c_3}(\overrightarrow{D}_\infty) = c_2(D_F) - c_1(D_L) c_1(D_F) + c_1^2(D_L) + \frac{1}{3} ds

where

(4.48) \quad S = -\frac{3}{8\pi} \frac{i}{2\pi} \frac{Du (1 - \frac{u^* u}{|u|^2}) \Omega_F u^*}{|u|^2}

in a homogeneous coordinate \( u \) on \( \mathbb{P}(F) \), and

\[ dS \neq 0. \]

The proof of this theorem relies on the following theorem of Harvey and Lawson.
Theorem 4.49. [HL: III, 7.25] Let $\phi$ be a $\mathfrak{gl}(n+1, \mathbb{C})$ invariant polynomial. Then

\begin{equation}
\phi(D_E \oplus D_F) - \phi(D_U \oplus D_Q) = dS_\phi
\end{equation}

on $\mathbb{F}(E \oplus F)$ where the transgression form $S_\phi$ is given as follows. Let $v = [t, u]$ be a local homogeneous coordinate on $\mathbb{F}(E \oplus F)$. Let $P := \frac{v^*v}{|v|^2}$ and $\Omega := \Omega_{E \oplus F}$. Then

\begin{align*}
S_\phi = \int_0^1 \phi \left( \frac{v^*Dv}{|v|^2} (1 - P); \Omega - xP\Omega(1 - P) - x\frac{v^*Dv}{|v|^2} (1 - P) \frac{Dv^*v}{|v|^2} \\
- x(1 - P) \frac{Dv^*Dv}{|v|^2} (1 - P) \right) dx.
\end{align*}

Proof of Theorem 4.43. By equation (4.50) we have

\[ b_k(D_Q) = b_k(D_E) + b_k(D_F) - b_k(D_U) - dS_{b_k} \]

and so

\[ \text{Res}_{b_k}(D_Q) = -\pi_*(b_k(D_U)) - d\pi_*(S_{b_k}) \]

\[ = -\pi_*(c_1^k(D_U)) - d\pi_*(S_{b_k}) \]

\[ = c_1^k(D_L) - c_1^k(D_E) - c_1(D_L) - c_1(D_E) - d\pi_*(S_{b_k}). \]

So to prove (4.45) and (4.46) it suffices to show that

\[ \pi_*(S_{b_2}) = 0 \quad \text{and} \quad \pi_*(S_{b_2}) = S. \]

Note that (4.44) follows from (4.45) and Proposition 4.42 using the Newton identity

\[ c_2 = \frac{1}{2} (c_1^2 - b_2). \]

We will discuss the proof of (4.47) at the end.
First we prove that \( \pi_*(S_{b_2}) = 0 \). By Theorem 4.49

\[
-2\pi^2 S_{b_2} = \int_0^1 \text{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \left( \Omega - x P\Omega(1 - P) - x \frac{v^* Dv}{|v|^2} (1 - P) \frac{Dv^* v}{|v|^2} \right) - x (1 - P) \frac{Dv^* Dv}{|v|^2} (1 - P) \right) dx
\]

\[
= \int_0^1 \text{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \Omega \right) - x \text{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \frac{v^* Dv}{|v|^2} (1 - P) \frac{Dv^* v}{|v|^2} \right) - x \text{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \frac{Dv^* Dv}{|v|^2} (1 - P) \right) dx
\]

and so

(4.51) \[
S_{b_2} = -\frac{1}{2\pi^2} \frac{Dv (1 - P) \Omega v^*}{|v|^2},
\]

since \((1 - P)v^* = 0\) and

\[
\text{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \right) = \frac{Dv v^*}{|v|^2} - \frac{Dv^* v v^*}{|v|^4} = 0.
\]

Next we express \( S_{b_2} \) in terms of a fibre coordinate \( u \) on \( \text{Hom}(E, F) \subset \mathbb{P}(E \oplus F) \) by setting \( v = [1, u] \) in equation (4.51) to give

(4.52) \[
S_{b_2} = -\frac{1}{2\pi^2} \left( \frac{Du \Omega F u^*}{1 + |u|^2} - \frac{Du^*}{1 + |u|^2} \frac{\Omega + u \Omega F u^*}{1 + |u|^2} \right).
\]

Replacing \( u \) by \( \frac{u}{|u|^2} \) in equation (4.52) we obtain the following formula for \( S_{b_2} \) in homogeneous coordinates \([t, u]\) on \( \mathbb{P}(E \oplus F)\),

\[
S_{b_2} = -\frac{1}{2\pi^2} \left( \frac{Du \Omega F u^*}{|u|^2 + |t|^2} - \frac{Du^*}{|u|^2 + |t|^2} \frac{|t|^2 \Omega + u \Omega F u^*}{|u|^2 + |t|^2} \right.
\]

\[
+ \frac{t dt}{(|u|^2 + |t|^2)^2} \left( |u|^2 \Omega - u \Omega F u^* \right) \right).
\]

Finally since \( S_{b_2} \) has no terms of degree 2 in \( dt, d\bar{t} \),

\[
\int_{\pi^{-1}} S_{b_2} = 0,
\]
and the proof is completed by noting that

$$\pi_*(S_{b_3}) = \int_{\pi^{-1}} S_{b_3}.$$  

The proof of the formula for $\pi_*(S_{b_3})$ goes as follows.

$$\begin{align*}
\frac{1}{3} (2\pi)^3 S_{b_3} &= \int_0^1 \mathrm{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \left( \Omega - x P\Omega(1 - P) - x \frac{v^* Dv}{|v|^2} (1 - P) \frac{Dv^* v}{|v|^2} \right) 
- x (1 - P) \frac{Dv^* Dv}{|v|^2} (1 - P) \right)^2 \right) \, dx \\
&= \int_0^1 \mathrm{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \left( \Omega^2 - x \Omega P\Omega(1 - P) - x \Omega \frac{v^* Dv}{|v|^2} (1 - P) \frac{Dv^* v}{|v|^2} 
- x \Omega (1 - P) \frac{Dv^* Dv}{|v|^2} (1 - P) - x \frac{Dv^* Dv}{|v|^2} (1 - P) \Omega 
+ x^2 \frac{Dv^* Dv}{|v|^2} (1 - P) \frac{Dv^* Dv}{|v|^2} (1 - P) \right) \right) \, dx 
\end{align*}$$

and so

$$\begin{align*}
(4.53) \quad S_{b_3} &= 3 \left( \frac{4}{2\pi} \right)^3 \left( \frac{Dv (1 - P) \Omega^2 v^* - Dv (1 - P) \Omega v^* Dv (1 - P) Dv^*}{|v|^4} \right)
\end{align*}$$

since

$$\begin{align*}
\mathrm{tr} \left( P\Omega(1 - P) \right) &= 0 \quad \text{and} \quad \mathrm{tr} \left( \frac{v^* Dv}{|v|^2} (1 - P) \right) = 0.
\end{align*}$$

Setting $v = [1, u]$ in (4.53) gives

$$\begin{align*}
(4.54) \quad S_{b_3} &= 3 \left( \frac{4}{2\pi} \right)^3 \left( \chi \frac{D u \Omega_F u^*}{|u|^2} - \chi^2 \frac{D u^* (\Omega_E^2 + u \Omega_F u^*)}{|u|^4} 
- \chi^2 \frac{D u \Omega_F u^*}{|u|^2} \left( \frac{D u D u^*}{|u|^2} - \chi \frac{D u u^* u D u^*}{|u|^4} \right) 
+ \chi^3 \frac{D u u^* (\Omega_E + u \Omega_F u^*) D u D u^*}{|u|^6} \right)
\end{align*}$$

where $\chi = \frac{|u|^2}{1 + |u|^2}$. Substituting $\frac{4}{i}$ for $u$ in (4.54) we could write down an expression for $S_{b_3}$ in homogeneous coordinates $[t, u]$ on $\mathbb{P}(E \oplus F)$. Rather than doing this
directly we can use the ideas (though not precise statement) of Lemma 2.35 to show that the part of $S_{b_3}$ of degree 2 in $dt, d\bar{t}$ is given by

$$\frac{dt \, d\bar{t}}{|t|^2} u \frac{\partial}{\partial u} u^* \frac{\partial}{\partial u^*} S$$

where $S$ is the expression on the R.H.S. of (4.54) with $\chi$ replaced by $\frac{|u|^2}{|u|^2 + |t|^2}$.

Consequently,

$$\pi_*(S_{b_3}) = -3 \left(\frac{i}{2\pi}\right)^3 \int_{\pi^{-1}} \chi^2(1 - \chi) \left(\frac{u\Omega_F u^* Du u^* - Du\Omega_F u^*}{|u|^4}\right) \frac{dt \, d\bar{t}}{|t|^2}$$

$$= 3 \left(\frac{i}{2\pi}\right)^3 \left(\frac{Du\Omega_F u^*}{|u|^2} - \frac{Du u^* u\Omega_F u^*}{|u|^4}\right) \int_{\pi^{-1}} \frac{|u|^4 \, dt \, d\bar{t}}{(|t|^2 + |u|^2)^3}$$

$$= -\frac{3}{8\pi} \frac{i}{2\pi} \left(\frac{Du(1 - P)\Omega_F u^*}{|u|^2}\right).$$

Finally a tedious calculation using the Bianchi identity $d\Omega_F = \omega_F \wedge \Omega_F - \Omega_F \wedge \omega_F$ shows that

$$dS = \frac{3}{8\pi} \frac{i}{2\pi} \left(\frac{Du(1 - P)\Omega_F(1 - P)Du^*}{|u|^2} - \frac{Du(1 - P)Du^* u\Omega_F u^*}{|u|^4}ight)$$

$$- \frac{u\Omega_F^2 u^*}{|u|^2} + \left(\frac{u\Omega_F u^*}{|u|^2}\right)^2$$

which is non-zero.

To prove (4.47) we use the Newton identity

$$6c_3 = 2b_3 + b_2 - 2b_1 b_2 - b_1^2$$

and show that $\pi_*(S_{b_1 b_2}) = 0$ as follows. First note that

$$b_1 b_2 (A; B) = \frac{d}{dt} \bigg|_{t=0} (b_1 b_2)(B + tA)$$

$$= b_1(A) b_2(B) + b_1(B) b_2(A; B).$$
Let

\[ A = \frac{v^* Dv}{|v|^2} (1 - P), \]

\[ B = \Omega - x P \Omega (1 - P) + x \frac{v^* Dv}{|v|^2} (1 - P) \frac{Dv^* v}{|v|^2} + x (1 - P) \frac{Dv^* Dv}{|v|^2} (1 - P). \]

Then

\[ b_1 b_2 (A; B) = b_1 (\Omega) b_2 (A; B) \]

So

\[ S_{b_1 b_2} = b_1 (\Omega_{E \oplus F}) S_{b_2} \]

and

\[ \pi_*(S_{b_1 b_2}) = b_1 (\Omega_{E \oplus F}) \pi_*(S_{b_2}) = 0. \]
5. Chern currents of atomic undetermined meromorphic sections.

There is a large class of examples which are not atomic sections of the bundle \( \pi : \mathbb{P}(E \oplus F) \to X \) but for which one would like to calculate Chern currents at zero and infinity. The simplest such example is given by the function \( u(z, w) = \frac{z}{w} \) on \( \mathbb{C}^2 \). More generally we would like to compute the Chern currents of a meromorphic section of a holomorphic bundle \( F \to X \).

Following standard definitions, see [G] for example, a **meromorphic section** \( \nu \) of a holomorphic bundle \( F \to X \) is defined to be a holomorphic section \( \nu \) of \( F \) defined on an open dense set \( X \sim P \) of the complex manifold \( X \) so that the closure \( \Gamma_\nu \) of the graph \( \Gamma_\nu^0 := \{ \nu(x) \in F : x \in X \sim P \} \) of \( \nu \) in \( \mathbb{P}(\mathbb{C} \oplus F) \) is a holomorphic subvariety of \( \mathbb{P}(\mathbb{C} \oplus F) \). The **zero set** \( Z \) of \( \nu \) is defined by \( Z := X \cap \Gamma_\nu \) and the **pole set** \( P \) of \( \nu \) is defined by \( P := \pi(\mathbb{P}(F) \cap \Gamma_\nu) \subset X \). The **indeterminacy set** \( I \) is the intersection of the zero and pole sets, \( I := Z \cap P \).

**Remark 5.1.** If the indeterminacy set \( I \) is empty (or if we restrict attention to the manifold \( X \sim I \)) a meromorphic section \( \nu \) of \( F \to X \) defines a holomorphic section of \( \mathbb{P}(\mathbb{C} \oplus F) \to X \), since the holomorphic section \( \nu : X \sim P \to F \) has a smooth and hence holomorphic extension across the pole set \( P \) to a section \( \nu : X \to \mathbb{P}(\mathbb{C} \oplus F) \). Consequently we can calculate the Chern currents of such a "determined" meromorphic section of \( F \).

Many "undetermined" meromorphic sections of a holomorphic bundle \( F \to X \) arise in the following way. Let \( \nu = (\tau, \mu) \) be a holomorphic section of the bundle \( \mathbb{C} \oplus F \to X \). Let \( P := \text{Zero}(\tau) \), \( Z := \text{Zero}(\mu) \) and let \([\tau, \mu] \) denote the induced
mapping

\[(\tau, \mu) : X \sim (Z \cap P) \to \mathbb{F}(E \oplus F).\]

The holomorphic section \(\nu = \frac{\mu}{\tau} : X \sim P \to F\) defines a meromorphic section of \(F\). The problem of computing Chern currents for the meromorphic section \(\nu\) is just the problem of calculating the Chern currents on all of \(X\) of the mapping \([\tau, \mu]\). Thanks to Remark 5.1 we know how to do this on \(X \sim (Z \cap P)\).

The discussion above suggests the following problem. Let \(E, F\) be smooth bundles of ranks 1 and \(n\) over a smooth manifold \(X\) as usual. Let \(\nu = (\tau, \mu) : X \to E \oplus F\) be a smooth section, and let \([\tau, \mu]\) denote the induced mapping from \(X \sim (Z \cap P)\) to \(\mathbb{F}(E \oplus F)\). Find conditions on the section \(\nu\) so that the Chern current formulae of Theorems 2.31 and 3.8 on \(X \sim (Z \cap P)\) extend by zero across \(Z \cap P\) to all of \(X\).

Note that in the case rank \(F = 1\) this problem has been solved by Harvey and Lawson [HL: II,9.10], generalizing the classical Poincaré-Lelong formula. They merely require that \(\tau\) and \(\mu\) be atomic sections of \(E\) and \(F\).

In the case rank \(F > 1\) we require several additional atomic-like conditions to hold in order to compute the Chern currents at zero and infinity on all of \(X\). Rather than giving the details of the computation we explain why these additional atomic hypotheses are needed, write them down, and outline the main steps in the computation of the Chern currents. We will restrict our attention to the pushforward case. Analogous results hold in the pullback case. The atomic hypotheses will be labelled (A1),...,(A6).

(A1) \(\nu = (\tau, \mu) : X \to E \oplus F\) is atomic.

(1). If (A1) holds then the smooth forms \(T_{r,s}\) and \(\phi(D_s)\) on \(X \sim (Z \cap P)\) extend
by zero across $Z \cap P$ to $L^1_{\text{loc}}$ forms on all of $X$.

(A2) $\mu : X \rightarrow F$ is atomic.

(A3) Let $\nu = [t, u]$ be a local coordinate expression for the section $\nu = (\tau, \mu)$. We assume that each such local function $\nu$ satisfies

$$\frac{dt}{t} \frac{du^I}{|u|^p} \wedge \frac{d\bar{u}^J}{|u|^p} \in L^1_{\text{loc}}(X) \quad \text{for all} \ 0 \leq p = |I| + |J| \leq 2n - 1.$$

(2). If (A1), (A2), (A3) hold then $T_{r,s} \rightarrow T_{\infty,0}$ in $L^1_{\text{loc}}(X)$ as $s \rightarrow 0$ and $r \rightarrow \infty$.

(3). If (A2) holds then the smooth form $\phi(\overrightarrow{D_0})$ on $X \sim Z$ has an $L^1_{\text{loc}}$ $d$-closed extension by zero across $Z$ to all of $X$, also denoted by $\phi(\overrightarrow{D_0})$.

(4). If (A1), (A2), (A3) hold then the Chern currents at zero and infinity exist and are of the form

$$\phi(\overrightarrow{D_0}) = \phi(\overrightarrow{D_0}) + S_0 \quad \text{and} \quad \phi(\overrightarrow{D_\infty}) = \phi(D_F) + S_\infty$$

where $S_0$ and $S_\infty$ are $d$-closed flat currents supported on $Z$ and $P$ respectively.

(5). In the universal case on $E \oplus F$

$$\phi(\overrightarrow{D_0}) = \phi(\overrightarrow{D_0}) + \text{Res}_\phi(\overrightarrow{D_0})[E]$$

$$\phi(\overrightarrow{D_\infty}) = \phi(D_F) + \text{Res}_\phi(\overrightarrow{D_\infty})[F]$$

where $\text{Res}_\phi(\overrightarrow{D_\infty})$ is the $d$-closed $L^1_{\text{loc}}$ extension to all of $F$ of the pullback to $F \sim X$ of the residue form at infinity on $\mathbb{P}(F)$.

Note that the example $\nu(z, w) = (z, w)$ on $\mathbb{C} \oplus \mathbb{C}^n$ mentioned above is covered by this case.
(6). If \((A1),(A2),(A3)\) hold then

\[
\begin{align*}
    c_1(\overrightarrow{D}_0) &= c_1(\overrightarrow{D}_0) + \text{Res}_{c_1}(\overrightarrow{D}_0) \text{ Div}(\mu) \quad \text{and} \\
    c_1(\overrightarrow{D}_\infty) &= c_1(D_P) + \text{Div}(\tau) \quad \text{on } X.
\end{align*}
\]

Extra hypotheses are required to ensure that for the general invariant polynomial \(\phi\) the current \(\nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{ Div}(\tau)\) is well defined on \(X\). The problem is that \(\text{Res}_\phi(\overrightarrow{D}_\infty)\) depends on the \(u\)-variables and has singular support on \(Z\) while \(\text{Div}(\tau)\) has singular support on \(P\). So until we make further assumptions \(\nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{ Div}(\tau)\) can only be defined on \(X \sim (Z \cap P)\).

Let \(\text{Reg}(P)\) and \(\text{Sing}(P)\) denote the regular and singular points of \(P = \text{Zero}(\tau)\) (c.f. [HS: 4.1.4.2]), and let \(\{P_j\}_{j=1}^\infty\) be the connected components of \(\text{Reg}(P)\).

(A4) (a) The codimension 2 Hausdorff measure of \(\text{Sing}(P)\) is zero.

(b) \(S := \sum_{j=1}^\infty n_j [P_j]\) has locally finite mass on \(X\).

Then \(S\) defines a current on \(X\). Let \(S = \|S\| \overrightarrow{S}\) be its polar decomposition (c.f. [H]). Then

\[
\text{Div}(\tau) = \sum_{j=1}^\infty n_j [P_j] \quad \text{on } X.
\]

(A5) (a) \(\|S\| (Z) = 0\).

(b) \(\frac{du^I \wedge d\bar{u}^J}{|u|^p} \in L^1_{\text{loc}}(\|S\|, X)\) for all \(p = |I| + |J| \leq 2n - 1\).

(7). Suppose that \((A2),(A3),(A4),(A5)\) hold. Then \(\nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{ Div}(\tau)\) defines a current on \(X\) by

\[
\langle \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \text{ Div}(\tau), \psi \rangle := \sum_{j=1}^\infty n_j \int_X \langle P_j, \nu^*(\text{Res}_\phi(\overrightarrow{D}_\infty)) \wedge \psi \rangle \|P_j\| X
\]
and

\[ dT_{\infty,0} = \phi(D_F) + \nu^*(\text{Res}_\phi(\overline{D}_\infty)) \text{ Div}(\tau) - \phi(D_0) - \text{Res}_\phi(\overline{D}_0) \text{ Div}(\mu) \]
on X.

In order to compute the Chern currents rather than their difference we need one further assumption.

\text{(A6)} For each local coordinate function \( \nu = (t, u) \) we require that

\[
\frac{dt \, \overline{dt} \, d^I u \wedge d^J \overline{u}}{|u|^2 |\nu|^p} \in L^1_{\text{loc}}(X) \quad \text{for } 0 \leq p = |I| + |J| \leq 2n - 2.
\]

Finally we have

\textbf{(9).} If (A1), \ldots, (A6) hold then

\[ \phi(\overline{D}_0) = \phi(D_0) + \text{Res}_\phi(\overline{D}_0) \text{ Div}(\mu) \quad \text{and} \]

\[ \phi(\overline{D}_\infty) = \phi(D_F) + \nu^*(\text{Res}_\phi(\overline{D}_\infty)) \text{ Div}(\tau) \quad \text{on } X. \]

It would be nice to show that (A1), \ldots, (A6) hold for holomorphic (or real analytic?) sections \( \nu = (\tau, \mu) \) for which \( \dim Z, \dim P \) and \( \dim (Z \cap P) \) are not too large.
0. Introduction.

Let $\pi : V \to X$ be an oriented real rank $m$ vector bundle with Riemannian metric $(\cdot, \cdot)$ and metric compatible connection $D_V$. In this chapter we investigate the extent to which the zero and pole sets of a section $\nu$ of the compactification $\mathbb{P}(\mathbb{R} \oplus V) \to X$ of $V \to X$ are related to one another and to the topology of the bundle $V$. Except for a few subtle twists our results are much like those of Chapter I. We build on the work of Harvey and Lawson's Chapter IV on real vector bundles. They constructed a canonical universal family of Thom forms $\tau_s$ for $0 < s < \infty$ on $V$ associated to the metric connection $D_V$ with the following properties.

\begin{align*}
(0.1) & \quad d\tau_s = 0. \\
(0.2) & \quad \tau_s \text{ extends to a smooth } d\text{-closed form on} \\
& \quad \text{the fibrewise compactification } \mathbb{P}(\mathbb{R} \oplus V) \text{ of } V. \\
(0.3) & \quad \int_{\pi^{-1}(x)} \tau_s = 1. \\
(0.4) & \quad \text{The restriction of each of the Thom forms } \tau_s \text{ to the base} \\
& \quad \text{manifold } X \text{ is the Euler form } \chi(D_V) \text{ of } (V, D_V). \\
(0.5) & \quad \tau_s = (\frac{1}{s})^* \tau_1 \quad \text{where } \frac{1}{s} \text{ acts by scalar multiplication} \\
& \quad \text{on the fibres of } V, \text{ and}
\end{align*}
\[ \lim_{\delta \to 0} \tau_\delta = [X] \quad \text{on } V. \]

Properties (0.1) and (0.3) show that the Thom forms represent the Thom class of \( V \), while properties (0.5) and (0.6) say that the Thom forms provide a global geometrization of the notion of an approximate identity (or point mass) in the fibres of \( V \).

Let \( \mu \) be an atomic section of \( V \to X \) and let \( \tau_\delta := \mu^*(\tau_\delta) \) be the pullback of the Thom form \( \tau_\delta \) to \( X \). Harvey and Lawson construct a family of smooth potentials \( \sigma_\delta \) on \( X \) satisfying

\[ \chi(D\nu) - \tau_\delta = d\sigma_\delta \quad \text{on } X. \]

As \( \delta \to 0 \) the potentials \( \sigma_\delta \) converge in \( L^1_\text{loc}(X) \) to an \( L^1_\text{loc} \) potential \( \sigma \). Taking the limit as \( \delta \to 0 \) of (0.7) we have the current equation

\[ \chi(D\nu) - \text{Div}(\mu) = d\sigma \quad \text{on } X. \]

Furthermore for any \( \mathfrak{so}(m) \)-invariant polynomial \( \phi \) we have a current equation of the form

\[ \phi(D\nu) - \phi(D_0) - \text{Res}_\phi(D_0) \text{Div}(\mu) = dT. \]

As in the complex case this equation is the current limit of the smooth transgression formula associated with a smooth family \( D_\delta \) of metric compatible connections approximating a metric compatible singular pushforward connection induced on \( V \) by the section \( \mu \).

Now let \( \nu \) be a section of \( \mathbb{P}(\mathbb{R} \oplus V) \to X \) and let \( \tau_\nu := \nu^*(\tau_\nu) \) be the pullback to \( X \) of the extension to all of \( \mathbb{P}(\mathbb{R} \oplus V) \) of the Thom form \( \tau_\nu \) on \( V \). The main aim of this chapter is to compute the Euler current at time infinity

\[ \tau_\infty := \lim_{r \to \infty} \tau_r \quad \text{on } X. \]
(Note that the Euler current \( \tau_0 = \text{Div}_0(\nu) \) on \( X \) just as in the complex case.) Somewhat surprisingly the nature of the Euler current \( \tau_\infty \) depends on the parity of the rank of the bundle \( V \). If \( V \) has even rank then the Euler current \( \tau_\infty \) is simply the smooth form \( \chi(D_\nu) \) on \( X \). In particular, in stark contrast to the complex case, it does not detect the pole set \( P \) of \( \nu \). Furthermore we have the current equation \[
\chi(D_\nu) - \text{Div}_0(\nu) = d\sigma \quad \text{on } X.
\]

On the other hand if the rank of \( V \) is odd the Euler current \( \tau_\infty \) is of the form \[
\tau_\infty = \nu^* \text{Res} \text{Div}_\infty(\nu) \quad \text{on } X,
\]
and we have the current equation \[
\nu^* \text{Res} \text{Div}_\infty(\nu) - \text{Div}_0(\nu) = d\sigma \quad \text{on } X
\]
relating the zero and pole divisors to each other and to the topology of the bundle \( V \) as it is encoded by the residue \( \nu^* \text{Res} \). Even though this looks much more like the complex case than the even rank case did it differs from it in the following subtle yet important way. The pole divisor is not a current on \( X \) (i.e. a distributional section of the bundle \( T^*X \) of 1-forms on \( X \)), but a distributional section of a certain bundle of twisted 1-forms on \( X \). As a consequence of this the pole divisor can have support on the nonorientable components of the pole set! Similarly the residue is a twisted \( m-1 \) form on \( X \). Note though that the Euler current \( \tau_\infty \), which is the product of these two, is a bona fide current on \( X \).

Basically speaking twisted forms and currents arise in this context because the pole set is the pullback of the nonorientable even dimensional real projective space at infinity, \( \mathbb{P}(\nu) \), in \( \mathbb{P}(\mathbb{R} \oplus V) \). To get a better feeling for these results
we describe the universal case in a little more detail. Let \( \mathcal{O}(\mathbb{TP}(V)) \) be the orientation bundle of the nonorientable manifold \( \mathbb{P}(V) \). Note that \( \mathcal{O}(\mathbb{TP}(V)) \) is isomorphic to the dual, \( L^* \), of the tautological line bundle \( L \) over \( \mathbb{P}(V) \), which is nonorientable when rank \( V > 1 \). We call smooth sections of the bundle \( \mathcal{O}(\mathbb{TP}(V)) \otimes L^*T^*\mathbb{P}(V) \) \( \mathcal{O}(L^*) \)-twisted forms. By design integration of \( \mathcal{O}(L^*) \)-twisted forms over the nonorientable manifold \( \mathbb{P}(V) \) is well defined. This enables us to define \([\mathbb{P}(V)]\) as a distributional section of the bundle of \( \mathcal{O}(L^*) \)-twisted forms on \( \mathbb{P}(\mathbb{R} \oplus V) \). Then in the universal case the Euler current \( \tau_\infty \) is given by

\[
\tau_\infty = \text{Res} [\mathbb{P}(V)],
\]

where \( \text{Res} = \frac{1}{2} \sigma \) is the \( \mathcal{O}(L^*) \)-twisted \( m - 1 \) form on \( \mathbb{P}(V) \) determined by the spherical kernel \( \sigma \) on \( V \sim X \) which was defined by Chern [C].

Note that in the case rank \( V = 1 \) \( \mathbb{P}(V) \cong X \) is oriented, the pole divisor is a bona fide current on \( X \) and the residue is 1. So we have the current equation

\[
\text{Div}_\infty(\nu) - \text{Div}_0(\nu) = d\sigma \quad \text{on } X.
\]

The Euler current \( \tau_\infty \) behaves differently in the odd rank case than in the even rank case because the construction of the Thom forms \( \tau_s \) is different in the odd rank case. In the even rank case Harvey and Lawson define the Thom form \( \tau_s \) to be the Chern–Euler form of the approximating connection \( D_s \). However in the odd rank case the Euler class is zero and a more subtle construction is required (see [HL: IV,2] or Section 2 below).

In this chapter we also generalize (0.8), proving that if the rank of \( V \) is arbitrary and \( \phi \) is any \( so \)-invariant polynomial then the \( \phi \)-characteristic current at time infinity \( \phi(D_\infty) = \lim_{r \to \infty} \phi(D_r) \) is simply the smooth form \( \phi(D_\nu) \). This suggests that the nature of the pole set \( P \) of \( \nu \) is not constrained by the topology of the bundle \( V \) — at least so far as it is encoded by the characteristic classes of \( V \).
In the odd rank case the pole divisor and residue are defined in terms of a possibly nonorientable bundle $L^*$ — namely the pullback to $X$ via $\nu$ of the nonorientable bundle $L^* \to \mathbb{P}(V)$. In order to do this we make use of the definitions and results of Section 2 of Chapter IV which deals with divisors and characteristic currents of sections of a nonorientable bundle over an oriented manifold.
1. The even rank case.

We begin by reviewing Harvey and Lawson's construction of the family of Thom forms $\tau_s$ in the even rank case. The material described here will also be required in our discussion of the odd rank case.

We use the following notation. Let $\pi : V \to X$ be an oriented real rank $m$ Riemannian vector bundle with metric compatible connection $D_V$. Let $e = (e_1, \ldots, e_m)$ be a positively oriented orthonormal local frame for $V$. We often regard $e$ as a column vector. Let $\lambda := e_1 \wedge \cdots \wedge e_m$ be the globally defined unit volume element for $V$. Let $\mu$ be the tautological section of the pullback $V$ of $V$ over itself and define a row vector of local fibre variables $u = (u_1, \ldots, u_m)$ for $V$ by the equation $\mu = ue$. The length of the tautological section $\mu$ is given locally by the expression $|u|^2 := uu^t$. Let $\omega_V$ and $\Omega_V$ denote the skew symmetric local gauge and curvature matricies of the connection $D_V$ in the local frame $e$. We use the notation $Du := du + u\omega_V$ and $Du^t = du^t - \omega_V u^t$.

From now on suppose that rank $V = m = 2n$ is even. The Pfaffian of a skew $2n \times 2n$ matrix $A$ is defined by the equation

$$
(1.1) \quad \operatorname{Pf}(A) \lambda = \frac{1}{n!} \left( \frac{1}{2} e^t A e \right)^n,
$$

and the Chern-Euler form of the metric connection $D_V$ is the degree 2n $d$-closed form $\chi(D_V)$ on $X$ defined by

$$
(1.2) \quad \chi(D_V) = \operatorname{Pf}(-\frac{1}{2\pi} \Omega_V) \quad \text{or equivalently} \quad \chi(D_V) \lambda = \frac{1}{n!}(-\frac{1}{4\pi} e^t \Omega_V e)^n.
$$

The bundle $V$ is compactified in the vertical directions by regarding it as a dense open chart in the bundle of projective spaces $\mathbb{P}(\mathbb{R} \oplus V)$ by the natural inclusion
mapping

\[ V \hookrightarrow \mathbb{P}(\mathbb{R} \oplus V) \]
\[ v \mapsto [1, v]. \]

Let \( U \rightarrow \mathbb{P}(\mathbb{R} \oplus V) \) be the tautological line bundle and let \( U_s := \Psi^*_s(U) \), where \( \Psi_s \) is the flow on \( \mathbb{P}(\mathbb{R} \oplus V) \) defined as in (1.2.2). Let \( U_s^\perp \) be the orthogonal complement of \( U_s \) in \( \mathbb{R} \oplus V \). We have the direct sum decomposition

\[ \mathbb{R} \oplus V = U_s \oplus U_s^\perp \quad \text{over} \quad \mathbb{P}(\mathbb{R} \oplus V). \]

The metric connection \( d \oplus D_V \) on \( \mathbb{R} \oplus V \) induces a metric connection \( D_s \) on \( U_s^\perp \).

Note that \( (U_s^\perp, D_{\infty}) = (V, D_V) \) over the subset \( V \) of \( \mathbb{P}(\mathbb{R} \oplus V) \).

**Definition 1.3.** The Thom forms \( \tau_s \) on \( \mathbb{P}(\mathbb{R} \oplus V) \) associated with the metric connection \( D_V \) on \( V \) are defined by \( \tau_s := \chi(D_s) \) for \( 0 < s < \infty \).

**Remark 1.4.** Harvey and Lawson [HL: IV,1.63] prove that the pair \( (U_s^\perp, D_s) \) is bundle isometric over \( V \hookrightarrow \mathbb{P}(\mathbb{R} \oplus V) \) to the pair \( (V, \overline{D}_s) \), where \( \overline{D}_s \) is the family of connections approximating the universal pushforward singular metric connection \( \overline{D} \) on \( V \) defined by [HL: IV,1.7]. Here the real algebraic approximation mode \( \chi(t) = 1 - \frac{1}{\sqrt{1+t}} \) is used to smooth up the singular connection \( \overline{D} \).

**Remark 1.5.** Harvey and Lawson [HL: IV,1.20] derive the following formulae for the Thom forms \( \tau_s \) in a local coordinate \( u \) on \( V \).

\[ \tau_s = (\frac{1}{2\pi})^m \frac{s}{\sqrt{|u|^2 + s^2}} \text{Pf} \left( \frac{Du^tDu}{|u|^2 + s^2} - \Omega_V \right) \]  
(1.6)

\[ \tau_s = \frac{1}{n!} \left( \frac{-1}{4\pi} \right)^n \frac{s}{\sqrt{|u|^2 + s^2}} \left( e^t \Omega_V e - \frac{(Du)^2}{|u|^2 + s^2} \right)^n. \]  
(1.7)
The standard transgression formula says that for $0 < s < r < \infty$,

(1.8) \[ \tau_r - \tau_s = d\sigma_{r,s} \quad \text{on } \mathbb{P}(\mathbb{R} \oplus V), \]

where the transgression form $\sigma_{r,s}$ is given by

(1.9) \[ \sigma_{r,s} = (-1)^n \int_s^r \text{Pf}(\frac{1}{2\pi} \frac{\partial \omega}{\partial s}; \frac{1}{2\pi} \Omega_s) ds, \]

and where $\omega_s$, $\Omega_s$ are local gauge and curvature matrices for the connection $D_s$.

Since the family $(U_s^\perp, D_s) (0 < s < \infty)$ has a smooth extension to $s = \infty$ over the chart $V \subset \mathbb{P}(\mathbb{R} \oplus V)$, we have the equation of smooth forms

(1.10) \[ \chi(D_V) - \tau_s = d\sigma_{\infty,s} \quad \text{on } V. \]

Harvey and Lawson [HL: IV,1.35] prove that $\sigma := \lim_{s \to 0} \sigma_{\infty,s}$ converges in $L^1_{\text{loc}}(V)$ and that

(1.11) \[ \chi(D_V) - [X] = d\sigma \quad \text{on } V \]

is the limiting form of the family of equations (1.10). In particular the current $\tau_0 = \lim_{s \to 0} \tau_s$ exists on $V$ and equals

\[ \tau_0 = [X]. \]

The $L^1_{\text{loc}}$ form $\sigma$ on $V$ is called the spherical kernel. The spherical kernel has top degree part in $du_1, \ldots, du_{2n}$

\[ \sigma_{2n-1} = \text{vol}(S^{2n-1}) \theta \]

where $\theta$ is the solid angle kernel (c.f. I.1).

The following result is a little surprising in light of the results of Chapter I and Section 3 of this chapter.
Proposition 1.12. Let $V$ be a real Riemannian vector bundle of arbitrary rank $m$ with metric compatible connection $D_V$ and let $(U^\perp_s, D_s)$ be given as above. Let $\phi$ be any $\mathfrak{so}(m)$-invariant polynomial. Then the $\phi$-characteristic current at infinity, $\phi(D_\infty) := \lim_{r \to \infty} \phi(D_r)$ exists and is the smooth form

\begin{equation}
\phi(D_\infty) = \phi(D_V) \quad \text{on } \mathbb{P}(\mathbb{R} \oplus V).
\end{equation}

In particular when $V$ has even rank $m = 2n$ the Euler current at infinity is the smooth form

\begin{equation}
\tau_\infty = \chi(D_V) \quad \text{on } \mathbb{P}(\mathbb{R} \oplus V),
\end{equation}

and the current equation

\begin{equation}
\chi(D_V) - [X] = d\sigma_{\infty,0} \quad \text{on } \mathbb{P}(\mathbb{R} \oplus V)
\end{equation}

is the current limit of the standard transgression formula (1.8).

Remark 1.16. When the rank of $V$ is even $\mathbb{P}(\mathbb{R} \oplus V)$ is nonorientable and so smooth forms on $\mathbb{P}(\mathbb{R} \oplus V)$ do not define currents by integration against smooth test forms. Rather they pair with smooth test densities (i.e. $\mathcal{O}(T\mathbb{P}(\mathbb{R} \oplus V))$-twisted forms, see (IV,1.5)) by integration over $\mathbb{P}(R \oplus V)$. So since the currents in (1.14–15) are defined as the limit of smooth forms they act on test densities on $\mathbb{P}(\mathbb{R} \oplus V)$.

Proof. Since $(U_\infty, D_\infty) = (V, D_V)$ over $V \subset \mathbb{P}(\mathbb{R} \oplus V)$ we just need to show that $\phi(D_\infty) = \lim_{r \to \infty} \phi(D_r)$ is smooth in a neighbourhood of $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R} \oplus V)$. Let $0 < s < r < \infty$ and let $T_{r,s}$ be the transgression form on $\mathbb{P}(\mathbb{R} \oplus V)$ satisfying

\begin{equation}
\phi(D_r) - \phi(D_s) = dT_{r,s} \quad \text{on } \mathbb{P}(\mathbb{R} \oplus V).
\end{equation}
By [HL: IV,2.20] we have

\[ T_{r,s} = 2 \int_{\chi_s}^{\chi_r} \phi \left( \frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2} ; A(x) \right) dx \]

in a local coordinate \( u \) on \( V \) where \( \chi_s = \chi(\frac{|u|^2}{s^2}) = 1 - \frac{s}{\sqrt{|u|^2 + s^2}} \) and

\[ A(x) = \Omega_V - x \left( \frac{u^t Du}{|u|^2} \Omega_V + \Omega_V \frac{u^t u}{|u|^2} \right) + 2x \left( 1 - \frac{x}{2} \right) \left( \frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2} \right)^2. \]

Since \( \frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2} \) is skew symmetric the expression for \( T_{r,s} \) in homogeneous coordinates \((t,u)\) on \( \mathbb{P}(\mathbb{R} \oplus V) \) is

\[ (T_{r,s})_{HC} = 2 \int_{\chi_s}^{\chi_r} \phi \left( \frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2} ; A(x) \right) dx \]

and it is easy to show that

\[ T = \lim_{r \to 0} T_{r,s} = -2 \int_0^1 \phi \left( \frac{u^t Du}{|u|^2} - \frac{Du^t u}{|u|^2} ; A(x) \right) dx \]

converges in \( L^1_{\text{loc}}(\mathbb{P}(\mathbb{R} \oplus V)) \). In particular \( T \) is smooth in a neighbourhood of \( \mathbb{P}(V) \subset \mathbb{P}(\mathbb{R} \oplus V) \). Furthermore \( \phi(D_0) \) is smooth in a neighbourhood of \( \mathbb{P}(V) \) since \((U_0^+,D_0)\) is smooth there. So

\[ dT_{r,0} = \phi(D_r) - \phi(D_0) \]

is smooth in a neighbourhood of \( \mathbb{P}(V) \) for \( 0 < r < \infty \). Therefore

\[ \phi(D_\infty) := \lim_{r \to \infty} \phi(D_r) = dT + \phi(D_0) \]

is also smooth in a neighbourhood of \( \mathbb{P}(V) \) and since \( D_\infty = D_V \) on \( V \) we have proved that

\[ \phi(D_\infty) = \phi(D_V) \quad \text{on} \quad \mathbb{P}(\mathbb{R} \oplus V). \]
Remark 1.17. Suppose that the rank of $V$ is even. Let $\nu$ be a smooth section of $\mathbb{P}(\mathbb{R} \oplus V) \to X$. Suppose that the restriction of $\nu$ to the complement of its pole set is an atomic section of $V \to X$. Then the proof of Proposition 1.12 pulls back to $X$ to show that the current limit as $s \to 0$ and $r \to \infty$ of the smooth transgression formula

$$\chi(D_s) - \chi(D_r) = d\sigma_{r,s} \quad \text{on } X$$

is the current equation

$$\chi(D_{\nu}) - \text{Div}_{\nu}(\nu) = d\sigma_{\infty,0} \quad \text{on } X. \quad (1.18)$$

Note that since $X$ is oriented the currents in (1.18) act on bona fide forms on $X$.

More generally if the rank of $V$ is arbitrary and $\phi$ is any $\mathfrak{s}\mathfrak{o}$-invariant polynomial then for any atomic section $\nu$ of $\mathbb{P}(\mathbb{R} \oplus V)$ (as above) there is a current equation of the form

$$\phi(D_{\nu}) - \phi(D_0) - \text{Res}_{\phi}(D_0) \text{ Div}_{\nu}(\nu) = dT_{\infty,0} \quad \text{on } X. \quad (1.19)$$

Curiously enough (1.19) suggests that the nature of the pole set of $\nu$ is not constrained by the topology of the bundle $V$ — at least so far as it is encoded in the characteristic classes of $V$. Nor do the natures of the zero and pole sets appear to influence each other. This behaviour should be compared with that discussed in Theorem 2.29 below where we describe a relationship between the zero and pole divisors of an atomic section of $\mathbb{P}(\mathbb{R} \oplus V) \to X$ in the case that the rank of $V$ is odd.
2. The odd rank case.

Let \( \pi : V \to X \) be an oriented real Riemannian vector bundle of odd rank \( m = 2n - 1 \), endowed with a metric compatible connection \( D_V \). Note that the Pfaffian does not exist in this setting and that it is natural to define the Chern-Euler form of the connection \( D_V \) to be zero, \( \chi(D_V) \equiv 0 \).

Harvey and Lawson [HL: IV,2.1] define the family of Thom forms \( \tau_s \) on \( V \) associated with the metric connection \( D_V \) as follows. Let \( \tilde{V} = \mathbb{R} \oplus V \) be equipped with the direct sum metric and metric connection. Let \( \tilde{\sigma} \) be the spherical potential for the even rank \( 2n \) bundle \( \tilde{V} \). (See (1.11) or [HL: IV,1.35] for the definition of \( \tilde{\sigma} \).) Fix \( s > 0 \). First restrict \(-2\tilde{\sigma}\) as a differential form to the affine subbundle \( \{s\} \times V \subset \tilde{V} \). Second pullback to \( V \) using the obvious identification of \( \{s\} \times V \) with \( V \). The Thom form \( \tau_s \) is defined to be the resulting smooth \( m \) form on \( V \).

Harvey and Lawson show that the Thom forms have properties (0.1)–(0.6) above. In particular \( \tau_s \) extends to a smooth \( d \)-closed form on the compactification \( \mathbb{P}(\mathbb{R} \oplus V) \). Let \( \tau_s \) be the pullback of \( \tau_s \) to \( X \) via a section of \( \mathbb{P}(\mathbb{R} \oplus V) \to X \). The aim of this section is to show that the Euler current \( \tau_\infty := \lim_{r \to \infty} \tau_r \) exists on all of \( X \) and is of the form \( \tau_\infty = \text{Res } \text{Div}_\infty(\nu) \). As usual the approach taken to solve this problem is to define a smooth family of approximate spherical potentials \( \sigma_{r,s} \) on \( X \) for \( 0 < s < r < \infty \) satisfying

\[
d\sigma_{r,s} = \tau_r - \tau_s \quad \text{on } X,
\]

and to show that \( \sigma_{\infty,0} := \lim_{\substack{r \to \infty \\ s \to 0}} \sigma_{r,s} \) exists in \( L^1_{\text{loc}}(X) \).

The family of approximate spherical potentials \( \sigma_{r,s} \) were defined on \( \mathbb{P}(\mathbb{R} \oplus V) \) by Harvey and Lawson [HL: IV,2.3] as follows. Let \( e_0 \) denote the global frame
1 for $\mathbb{R}$. Fix $0 < s < r < \infty$. Let $\mathbb{R}_{r,s}$ denote the submanifold with boundary

\begin{equation}
(2.1) \quad \mathbb{R}_{r,s} = \{te_0 / s \leq t \leq r\},
\end{equation}

and let $\rho_{r,s} : \mathbb{R}_{r,s} \oplus V \to V$ be the natural projection. Define $\sigma_{r,s}$ on $V$ by the fibre integral

\begin{equation}
(2.2) \quad \sigma_{r,s} := -2 \int_{\rho_{r,s}^{-1}(v)} \tilde{\sigma} = -2(\rho_{r,s})_* \tilde{\sigma}.
\end{equation}

**Lemma 2.3.** Let $0 < s < r < \infty$. The approximate spherical potentials $\sigma_{r,s}$ have the following properties.

1. $\sigma_{r,s}$ is a smooth $m - 1$ form on $V$.
2. $\sigma_{r,s}$ has a smooth extension to the compactification $\mathbb{P}(\mathbb{R} \oplus V)$ of $V$, also denoted by $\sigma_{r,s}$, and
3. $d\sigma_{r,s} = \tau_r - \tau_s$ on $\mathbb{P}(\mathbb{R} \oplus V)$.

**Proof.**

1. Let $(s, u) = (s, u_1, \ldots, u_{2n-1})$ be local fibre variables on $\mathbb{R} \oplus V$ corresponding to a positively oriented orthonormal local frame $(e_0, e) = (e_0, e_1, \ldots, e_{2n-1})$ for $\mathbb{R} \oplus V$. Let $\lambda = e_1 \wedge \cdots \wedge e_{2n-1}$ and $\tilde{\lambda} = e_0 \wedge \lambda$ be unit volume elements for $V$ and $\mathbb{R} \oplus V$ respectively. Harvey and Lawson [HL: IV,2.6] have shown that the spherical potential $\tilde{\sigma}$ on $\mathbb{R} \oplus V$ is given by

$$
\tilde{\sigma} = \sum_{p=0}^{n-1} K_{n,p} \frac{(se_0 + ue)(dse_0 + Du e)^{2p+1}}{|u|^2 + s^2} (e^t \Omega_v e)^{n-p-1}
$$

where

$$
K_{n,p} = \frac{(-1)^{n-p}}{\pi^n} \frac{p!}{(n-p-1)!(2p+1)!2^{2n-2p-1}}.
$$
So by (2.2),

\[ (2.4) \]
\[ \bar{\sigma}_{r,s} \tilde{\chi} = \sum_{p=0}^{n-1} 2K_{n,p} \frac{(ue)(Due)^{2p}}{|u|^{2p+1}} \frac{(e^t \Omega \nu e)^{n-p-1}}{(|u|^2 + s^2)^{p+1}} \int_s^r \frac{|u|^{2p+1} ds e_0}{(|u|^2 + s^2)^{p+1}} \]

\[ (2.5) \]
\[ = \sum_{p=0}^{n-1} 2K_{n,p} (ue)(Due)^{2p} (e^t \Omega \nu e)^{n-p-1} \int_s^r \frac{ds e_0}{(|u|^2 + s^2)^{p+1}}. \]

To show that \( \sigma_{r,s} \) is a smooth form on \( V \) it suffices to prove that

\[ \frac{\partial}{\partial u_j} \int_s^r \frac{ds}{(|u|^2 + s^2)^{p+1}} = -(p+1) \frac{\partial |u|^2}{\partial u_j} \int_s^r \frac{ds}{(|u|^2 + s^2)^{p+2}} \]

for all \( 0 < s < r < \infty \) and all non-negative integers \( p \). But this follows by applying the standard differentiation under the integral theorem to the function \( f(x,s) = \frac{1}{(x+s^2)^{p+1}} \) on the domain \([0, \infty) \times (s,r)\).

(2) To show that \( \sigma_{r,s} \) extends to a smooth form on \( P(\mathbb{R} \oplus V) \) we just need to show that the expression \( (\sigma_{r,s})_{HC} \) for \( \sigma_{r,s} \) in the local homogeneous coordinate \( (t,u) \) corresponding to the fibre variable \( u \) is smooth in \( t \) near \( t = 0 \). Replacing \( u \) by \( \frac{u}{t} \) in (2.5) gives the following expression for \( \sigma_{r,s} \) in homogeneous coordinates.

\[ (\sigma_{r,s})_{HC} \tilde{\chi} = \sum_{p=0}^{n-1} 2K_{n,p} \frac{ue((Du - u \frac{dt}{t})e)^{2p}}{t^{2p+1}} \frac{(e^t \Omega \nu e)^{n-p-1}}{(|u|^2 + s^2)^{p+1}} \int_s^r \frac{ds e_0}{(|u|^2 + s^2)^{p+1}} \]

\[ = \sum_{p=0}^{n-1} 2K_{n,p} \frac{ue(ue)^{2p}}{|u|^{2p+1}} \frac{(e^t \Omega \nu e)^{n-p-1}}{(1 + u^2)^{p+1}} \int_{\frac{|u|}{t}}^{\frac{r}{s}} \frac{dv e_0}{(1 + v^2)^{p+1}}, \]

since \( (ue)^2 = 0 \). To see that \( (\sigma_{r,s})_{HC} \) is a smooth function of \( t \) note that

\[ \frac{\partial}{\partial t} \int_{st}^{rt} \frac{dv}{(v^2 + |u|^2)^{p+1}} = \frac{r}{(r^2 t^2 + |u|^2)^{p+1}} - \frac{s}{(s^2 t^2 + |u|^2)^{p+1}} \]

is a smooth function of \( t \) since \( r, s > 0 \) and \( t \) and \( u \) are not simultaneously zero.
(3) Finally, all the terms in the equation $d\sigma_{r,s} = \tau_r - \tau_s$ are smooth and since the equation is valid on the open dense chart $V \subset \mathbb{P}(\mathbb{R} \oplus V)$, (apply Stokes theorem to the fibre $\mathbb{R}_{r,s}$ of $\rho_{r,s}$), it is valid on all of $\mathbb{P}(\mathbb{R} \oplus V)$. □

Let $Z = \nu^{-1}(X)$ and $P = \nu^{-1}(\mathbb{P}(V))$ denote the zero and pole sets of a section $\nu$ of $\mathbb{P}(\mathbb{R} \oplus V) \to X$.

**Definition 2.7.** A smooth section $\nu$ of $\mathbb{P}(\mathbb{R} \oplus V) \to X$ is called **atomic** if the following two conditions hold.

1. The induced section $\nu : X \sim P \to V$ is atomic.
2. $P$ has Lebesgue measure zero in $X$.

**Note.** Of course the tautological section of the pullback $\mathbb{P}(\mathbb{R} \oplus V)$ of $\mathbb{P}(\mathbb{R} \oplus V)$ over itself is atomic.

**Remark 2.8.** Among other things the following comments explain the origin of the second condition in Definition 2.7. Recall that the mapping

$$p : \mathbb{P}(\mathbb{R} \oplus V) \sim X \longrightarrow \mathbb{P}(V)$$

$$[s,v] \mapsto [v]$$

realizes $\mathbb{P}(\mathbb{R} \oplus V) \sim X$ as the total space of a real line bundle over $\mathbb{P}(V)$. In fact this line bundle is isomorphic to the dual $L^* = \text{Hom}(L, \mathbb{R})$ of the tautological line bundle $L$ over $\mathbb{P}(V)$. Over a point $[v] \in \mathbb{P}(V)$ this isomorphism is given by the linear mapping

$$L_{[v]}^* \longrightarrow (\mathbb{P}(\mathbb{R} \oplus V) \sim X)_{[v]}$$

$$\beta \mapsto [\beta(v),v].$$

Note that $L^*$ is nonorientable when rank $V > 1$. 
Let $L^* \to X \sim Z$ be the pullback of $L^* \to \mathbb{P}(V)$ to $X \sim Z$ via the composition $p \circ \nu$. Now following Definition 1.8 and Remark 1.3 of Chapter I we should define a section $\nu$ of $\mathbb{P}(\mathbb{R} \oplus V) \to X$ to be atomic with respect to spatial infinity $\mathbb{P}(V)$ if the induced section $\tilde{\nu}$ of $L^* \to X \sim Z$ is atomic. We would like to define the pole divisor of $\nu$ to be the divisor of the induced section $\tilde{\nu}$. However we will need to modify our definition of divisor to account for the fact that $L^*$ may not be orientable (see Definition 2.21 below).

Returning to the issue of atomicity Harvey and Semmes [HS: 1.28] remark that a smooth function $u : X \to \mathbb{R}$ should be called atomic if its zero set $Z$ has Lebesgue measure zero in $X$. Let $y$ be a coordinate on $\mathbb{R}$ and note that the solid angle kernel $\theta$ on $\mathbb{R}$ is given by $\theta = \frac{\nu}{|y|}$ and that the volume of the unit sphere in $\mathbb{R}$ is $2$. The atomicity of $u$ ensures that $u^*\theta = \frac{u}{|u|}$ is a well defined $L^\infty_{\text{loc}}$ function on $X$. Then the divisor of $u$ is well defined by the current equation

$$\text{Div}(u) = d \left( \frac{1}{2} \frac{u}{|u|} \right).$$

Proposition 2.12. Let $\sigma_{r,s} := \nu^*(\sigma_{r,s})$ denote the pullback to $X$ by an atomic section $\nu$ of $\mathbb{P}(\mathbb{R} \oplus V) \to X$ of the approximate spherical potentials $\sigma_{r,s}$. Then the approximate potentials $\sigma_{r,s}$ converge to the spherical kernel $\sigma_{\infty,0}$ in $L^1_{\text{loc}}(X)$ as $r \to \infty$ and $s \to 0$. Let $[t,u]$ be the local coordinate expression for $\nu$ defined with respect to a positively oriented orthonormal local frame $e$ for $V$ by $\nu = [t1, ue]$. Then the spherical kernel is given by

$$\sigma_{\infty,0} = \frac{|t|}{t} \sigma,$$

where

$$\sigma \lambda = \frac{-1/2}{(n-1)!} \left( \frac{1}{4\pi} \right)^{n-1} \frac{ue}{|u|} \left( e^\lambda \Omega ve - \frac{(Du e)^2}{|u|^2} \right)^{n-1}.$$
Proof. Since \( \nu \) is atomic, we know that \( Z \cup P \) has Lebesgue measure zero in \( X \) and so \( \sigma_{r,s} \to \sigma_{\infty,0} \) a.e. on \( X \) as \( r \to \infty \) and \( s \to 0 \). Also by (2.6) \( |\sigma_{r,s}| \leq |\sigma_{\infty,0}| \) a.e. on \( \mathbb{P}(\mathbb{R} \oplus V) \). So to prove that \( \sigma_{r,s} \to \sigma_{\infty,0} \) in \( L^1_{\text{loc}}(X) \) it suffices to prove that \( \sigma_{\infty,0} \in L^1_{\text{loc}}(X) \). To do this we replace \( u \) by \( \frac{u}{t} \) in (2.4) to obtain

\[
(\sigma_{\infty,0}) \tilde{\lambda} = \sum_{p=0}^{n-1} 2K_{n,p} \frac{|t|}{t} \left[ \frac{ue(Due)^{2p}}{|u|^{2p+1}} \right] (e^t \Omega Ve)^{n-p-1} \int_0^\infty \frac{|t|^p}{(s^2t^2 + |u|^2)^{p+1}} dse_0
\]

which defines an \( L^1_{\text{loc}}(X) \) form since \( P = \{ t = 0 \} \) has measure zero and \( \frac{dY^t}{|u|^{2p}} \in L^1_{\text{loc}}(X) \) for \( p \leq n-1 \) by the atomicity of \( \nu \). Finally (2.14) is obtained from (2.15) using [HL: IV,2.8–9]. \( \Box \)

Let \( \tau_s = \nu^*(\tau_s) \) be the pullback to \( X \) of the Thom form \( \tau_s \) by an atomic section \( \nu \) of \( \mathbb{P}(\mathbb{R} \oplus V) \to X \). The rest of this chapter is devoted to showing that the Euler currents

\[
\tau_\infty := \lim_{r \to -\infty} \tau_r \quad \text{and} \quad \tau_0 := \lim_{s \to 0} \tau_s
\]

exist on \( X \) and to calculate them. We will show that they are of the general form

\[
\tau_\infty = \nu^* \text{Res} \text{ Div}_\infty(\nu) \quad \text{and} \quad \tau_0 = \text{Div}_0(\nu) \quad \text{on} \ X,
\]

and that

\[
d\sigma_{\infty,0} = \nu^* \text{Res} \text{ Div}_\infty(\nu) - \text{Div}_0(\nu) \quad \text{on} \ X.
\]

The zero divisor is defined as follows, just as in the complex case.

**Definition 2.16.** Let \( \nu \) be an atomic section of \( \mathbb{P}(\mathbb{R} \oplus V) \to X \). Then the **zero divisor**, \( \text{Div}_0(\nu) \), is defined to be the extension from \( X \sim P \) to all of \( X \) of the divisor of the induced section \( \nu : X \sim P \to V \) (see I.1.4).
However as we noted in Remark 2.8 the pole divisor is harder to define. Before attempting to do this in the general case we tackle the special case of an atomic section $\nu$ of $\mathbb{P}^r(\mathbb{R} \oplus V) \to X$ where the rank of $V$ is 1.

When $V$ has rank one $X$ sits inside $\mathbb{P}^r(\mathbb{R} \oplus V) \to X$ as the zero section of $V$, denoted by $X_0$, and as the submanifold $\mathbb{P}(V)$ at spatial infinity, denoted by $X_\infty$. Now the bundle $L^* \to \mathbb{P}(V)$ is simply the trivial (oriented!) line bundle $\mathbb{R} \to X_\infty$ and so we can define the pole divisor, $\text{Div}_\infty(\nu)$, to be the divisor of the induced function $\nu : X \sim Z \to \mathbb{R}$ (c.f. Definition I.1.9). Then we have the following result.

**Theorem 2.17.** Suppose that rank $V = 1$ and let $\nu$ be an atomic section of $\mathbb{P}^r(\mathbb{R} \oplus V) \to X$. Let $\sigma_{\infty,0}$ be the $L^1_{\infty}(X)$ spherical kernel associated with $\nu$ as in Proposition 2.12. Then the current equation

$$(2.18) \quad d\sigma_{\infty,0} = \text{Div}_\infty(\nu) - \text{Div}_0(\nu) \quad \text{on } X$$

is the current limit of the smooth "transgression" formula

$$d\sigma_{r,s} = \tau_r - \tau_s \quad \text{on } X.$$

In particular the Euler currents exist and are given by

$$\tau_\infty = \text{Div}_\infty(\nu) \quad \text{and} \quad \tau_0 = \text{Div}_0(\nu) \quad \text{on } X.$$

**Remark 2.19.** Suppose that $X$ is a closed manifold and that rank $V = \dim X$. Suppose further that the zero and pole divisors of the atomic section $\nu$ have locally finite mass. Then by [HS: 4.28]

$$\text{Div}_0(\nu) = \sum_{j=1}^N n_j [z_j] \quad \text{and} \quad \text{Div}_\infty(\nu) = \sum_{k=1}^M m_k [p_k] \quad n_j, m_k \in \mathbb{Z}$$

where $Z = \{z_1,...,z_N\}$ and $P = \{p_1,...,p_M\}$. Pairing (2.18) with the constant function 1 on $X$ gives

$$\# \text{ zeros of } \nu = \# \text{ poles of } \nu.$$
Proof of Theorem. We compute $d\sigma_{\infty,0}$ using a system of positively oriented charts $(\psi_j, U_j)$ for $\mathbb{P}(\mathbb{R} \oplus V)$. Let $e$ be a positively oriented orthonormal local frame for $V \to X$ and let $u$ be the corresponding fibre variable. Let $t$ denote the global fibre variable on $\mathbb{R}$. Define charts $\psi_j : U_j \to \mathbb{P}(\mathbb{R} \oplus V)$ by

$$\psi_0(u) = [1,u] \quad \text{and} \quad \psi_1(t) = [t,-1].$$

It is easy to check that these charts define an orientation — which we declare to be positive — on $\mathbb{P}(\mathbb{R} \oplus V)$.

Now write $\nu = [t,ue]$ where we now regard $[t,u]$ as a local $\mathbb{P}(\mathbb{R}^2)$-valued function on $X$. Then by Proposition 2.12,

$$\sigma_{\infty,0} = -\frac{1}{2} \frac{|t|}{t} \frac{u}{|u|}.$$

On $X \sim P = \nu^{-1}(U_0)$ we set $t = 1$ to get $\sigma_{\infty,0} = -\frac{1}{2} \frac{u}{|u|}$ so that

\begin{equation}
(2.20) \quad d\sigma_{\infty,0} = -\text{Div}(u) = -\text{Div}_0(\nu),
\end{equation}

while on $X \sim Z = \nu^{-1}(U_1)$ we set $u = -1$ to get $\sigma_{\infty,0} = \frac{1}{2} \frac{t}{|t|}$ so that

\begin{equation}
(2.21) \quad d\sigma_{\infty,0} = \text{Div}(t) = \text{Div}_\infty(\nu).
\end{equation}

Combining (2.20) and (2.21) proves the theorem. \hfill \Box

We now return to the problem of defining the pole divisor in the case that rank $V > 1$. Henceforth we will make use of the definitions and results of Section 2 of Chapter IV which deals with divisors and characteristic currents in the nonorientable case. Nevertheless we have endeavoured to keep the exposition as self contained as possible, and have provided specific references to the material in IV.2 when that has not proved possible. The reader could regard IV.2 as an appendix to the rest of this chapter.
Let $Y := X \sim Z$ and let $L^* \to Y$ denote the pullback of the dual of the tautological bundle $L \to \mathbb{P}(V)$ to $Y$ by the mapping $p \circ \nu : Y \to \mathbb{P}(V)$ (c.f. Remark 2.8). We would like to define the pole divisor, $\text{Div}_\infty(\nu)$, on $Y$ to be the divisor of the induced section $\nu$ of $L^* \to Y$. The problem is that the bundle $L^*$ may not be orientable. Let $p : \tilde{Y} \to Y$ be the 2-sheeted covering space corresponding to the principal $\mathbb{Z}_2$-bundle of orientations of $L^* \to Y$ (see IV.2). Let $\tilde{L}^* = p^*L^*$ be the pullback of $L^*$ to $\tilde{Y}$. The point of this construction is that the bundle $\tilde{L}^*$ is oriented by choosing the orientation over a point in $\tilde{Y}$ to be the one determined by that point.

Now the induced section $\tilde{\nu} : \tilde{Y} \to \tilde{L}^*$ is also atomic and its divisor, $\text{Div}(\tilde{\nu})$, is an odd current on $\tilde{Y}$ (IV,2.6). Let $\mathcal{O}(L^*)$ be the orientation bundle of $L^*$ (IV,1.5). Note that the space of $\mathcal{O}(L^*)$-twisted forms $\Omega^*(Y, \mathcal{O}(L^*))$ is in 1-1 correspondence with odd forms on $\tilde{Y}$ (IV,2.2). Let

$$\tilde{\rho}_* : \Omega^\text{odd}_*(\tilde{Y})' \to \Omega^*(Y, \mathcal{O}(L^*))',$$

defined by (IV,2.4), be the map which sets up the 1-1 correspondence between odd currents on $\tilde{Y}$ and distributional sections of $\Omega^*(Y, \mathcal{O}(L^*))$.

**Definition 2.22.** With notation as above the pole divisor, $\text{Div}_\infty(\nu)$, of an atomic section $\nu$ of $\mathbb{P}(\mathbb{R} \oplus V) \to X$ is defined to be the extension to all of $X$ of

$$\text{Div}_\infty(\nu) := \tilde{\rho}_*(\text{Div}(\tilde{\nu})) \in \Omega^*(X \sim Z, \mathcal{O}(L^*))'.$$

**Notes.**

(1) See Propositions IV.2.12 and IV.2.18 for some results concerning the geometric structure of the pole divisor. In particular it can have support on the nonorientable components of the pole set.

(2) If $L^*$ happens to be orientable then the pole divisor is simply the divisor of
the induced section \( \nu \) of \( L^* \to X \sim Z \).

Next we define the residue density as an \( \mathcal{O}(L^*) \)-twisted form on \( X \sim Z \). First we consider the universal case. Let \( S(V) \) be the unit sphere bundle in \( V \) and note that the natural bundle \( p : S(V) \to \mathbb{P}(V) \) can be regarded as the \( \mathbb{Z}_2 \)-bundle of orientations of \( TP(V) \). Thanks to the following lemma this is also the \( \mathbb{Z}_2 \)-bundle of orientations of \( L^* \).

Lemma 2.23.

\[
\mathcal{O}(TP(V)) \cong \mathcal{O}(L^*) \quad \text{over} \quad \mathbb{P}(V).
\]

Proof. Since the total space of \( L^* \) is the orientable manifold \( \mathbb{P}(\mathbb{R} \oplus V) \sim X \) we can apply Lemma IV.1.12 over \( \mathbb{P}(V) \) to show that

\[
\mathbb{R} \cong \mathcal{O}(T(\mathbb{P}(\mathbb{R} \oplus V) \sim X)) \cong \mathcal{O}(L^*) \otimes \mathcal{O}(TP(V))
\]

and so

\[
\mathcal{O}(L^*) \cong \mathcal{O}(TP(V))^* \cong \mathcal{O}(TP(V))
\]

by the same lemma. \( \Box \)

Definition 2.24. Let \( u \) be a coordinate on \( V \). The form \( 2\sigma \) of Proposition 2.12 defines an odd form on \( S(V) \). The residue density, \( \text{Res} \in \Omega^{2n-2}(\mathbb{P}(V), \mathcal{O}(L^*)) \), is defined to be the corresponding \( \mathcal{O}(L^*) \)-twisted form (see IV.2.2).

Remark 2.25. Fix \( u \in S(V) \) and let \( \ell_u^* \in L^*_{[u]} \) be the orthonormal frame defined by \( \ell_u^*(u) = 1 \). Then the residue density is given by

\[
(2.26) \quad \text{Res}_{[u]} = \ell_u^* \otimes 2\sigma(u) \in \Omega^*(\mathbb{P}(V), \mathcal{O}(L^*)). \]

Note that \( \text{Res}_{[u]} \) is well defined since \( \ell_{-u}^* = -\ell_u^* \) and \( \sigma(-u) = -\sigma(u) \).
We use the following general construction to define the residue density on $X \sim Z$.

**Definition 2.27.** Let $f : M \to N$ be a smooth map between manifolds. Let $V$ be a vector bundle over $N$ and let $V = f^*V$. Then we can define a mapping

$$f^* : \Omega^*(N, \mathcal{O}(V)) \to \Omega^*(M, \mathcal{O}(V))$$

as follows (c.f. IV.2.2). Fix $x \in M$ and let $y = f(x)$. Let $e_1, \ldots, e_n$ be an orthonormal local frame for $V_y$ and write $\omega_y \in \Omega^*(N, \mathcal{O}(V))_y$ as

$$\omega_y = (e_1 \wedge \ldots \wedge e_n) \otimes \psi_y$$

where $\psi_y \in \Lambda^* T_y N$.

Then we define

$$(f^* \omega)_x := (e_1 \wedge \ldots \wedge e_n) \otimes (f^* \psi)_x$$

where we now regard $e_1, \ldots, e_n$ as a frame for $V_x$. Furthermore if $V$ is orientable each choice of orientation picks out a choice of form $f^* \omega \in \Omega^*(M)$.

**Definition 2.28.** Let $\nu$ be a smooth section of $\mathbb{P}(\mathbb{R} \oplus V) \to X$, and let $p : \mathbb{P}(\mathbb{R} \oplus V) \sim X \to \mathbb{P}(V)$ be as in (2.9). The residue density, $\nu^* \text{Res}$, is defined to be the $\mathcal{O}(L^*)$-twisted form

$$\nu^* \text{Res} := (p \circ \nu)^* \text{Res} \quad \text{on } X \sim Z$$

where $\text{Res} \in \Omega^{2n-2}(\mathbb{P}(V), \mathcal{O}(L^*))$ is given by (2.24).

**Note.** Let rank $V = 2n - 1$. Then $\nu^* \text{Res} \text{Div}_\infty(\nu)$ defines a codimension $2n - 1$ current on $X$ by

$$(\nu^* \text{Res} \text{Div}_\infty(\nu), \phi) = (\text{Div}_\infty(\nu), \nu^* \text{Res} \wedge \phi),$$

where $\phi \in \Omega^*(X)$. Note that $\nu^* \text{Res} \wedge \phi \in \Omega^*(X, \mathcal{O}(L^*))$.

Finally we are in a position to state and prove the main theorem of this chapter.
Theorem 2.29. Let $V$ be an odd rank real vector bundle and let $\nu$ be an atomic section of the compactified bundle $\mathbb{P}(\mathbb{R} \oplus V) \to X$. Then the Euler currents at time zero and time infinity exist on $X$ and are given by

$$\tau_0 = \text{Div}_0(\nu) \quad \text{and} \quad \tau_\infty = \nu^* \text{Res} \text{Div}_\infty(\nu).$$

Furthermore the current equation

$$d\sigma_{\infty,0} = \nu^* \text{Res} \text{Div}_\infty(\nu) - \text{Div}_0(\nu) \quad \text{on } X$$

is the current limit of the smooth "transgression" formula

$$d\sigma_{r,s} = \tau_r - \tau_s \quad \text{on } X.$$

Remark 2.31. Suppose that $X$ is a closed manifold, that rank $V = \dim X$, that the codimension 1 Hausdorff measure of Sing $P$ vanishes (see IV.2.11) and that the zero and pole divisors of $\nu$ have locally finite mass. Then by [HS: 4.9] and Propositions IV.2.12 and IV.2.18

$$\text{Div}_\infty(\nu) = \sum_{j=1}^M m_j [P_j], \quad m_j \in \mathbb{Z},$$

where $P_j$ are the connected components of Reg $P$ (which may be nonorientable).

Write $\text{Div}_0(\nu) = \sum_{j=1}^N n_j [z_j]$ as in Remark 2.19. Then pairing (2.30) with the constant function 1 on $X$ gives

$$\# \text{ zeros of } \nu = \sum_{j=1}^M \langle [P_j], \nu^* \text{Res} \rangle.$$

Example 2.33.

(1) In the universal case

$$\tau_0 = [X] \quad \text{and} \quad \tau_\infty = \text{Res} [\mathbb{P}(V)].$$
Here $\text{Res} \in \Omega^*(\mathbb{P}(V), \mathcal{O}(L^*))$ is given by (2.24) and $[\mathbb{P}(V)] \in \Omega^*(\mathbb{P}(\mathbb{R} \oplus V), \mathcal{O}(L^*))'$. (2) Let $\nu : S(\mathbb{R} \oplus V) \rightarrow \mathbb{P}(\mathbb{R} \oplus V)$ be the natural projection. The zero set of $\nu$ consists of two disjoint copies of $X$, $X_\pm = (\pm 1, 0)$ inside $S(\mathbb{R} \oplus V)$ and the pole set is the sphere bundle $S(V) \subset S(\mathbb{R} \oplus V)$. Then the Euler currents are given by

$$\tau_0 = [X_+] + [X_-] \quad \text{and} \quad \tau_\infty = 2\sigma [S(V)],$$

where the form $\sigma \in \Omega^{2n-2}(S(V))$ is given by (2.14).

To see that $\tau_0 = [X_+] + [X_-]$ note that in the positively oriented system of charts which includes the two charts

$$\psi_\pm : \mathbb{R}^n \rightarrow S(\mathbb{R} \oplus V)$$

$$\nu \mapsto \frac{\pm 1}{1 + |\nu|^2}(1, \nu e)$$

the coordinate expressions $\nu \circ \psi_\pm : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for $\nu$ are both the identity map.

Proof of Theorem 2.29. First we prove (2.30). On $X \sim P \nu$ induces a section $\nu : X \sim P \rightarrow V$ and $\sigma_{\infty,0} = \sigma$. Now $\sigma$ is a homogeneous form of degree $\leq m - 1$ in the fibre differentials which is $d$–closed on $X \sim Z$ and has residue $-1$ (see [HL: IV, 2.5(5)]). So by the general Residue Theorem 1.10 of [HL: III]

$$d\sigma_{\infty,0} = -\text{Div}_0(\nu) \quad \text{on} \quad X \sim P.$$ 

To check (2.30) on $Y = X \sim Z$ we proceed as follows. On $\mathbb{P}(\mathbb{R} \oplus V) \sim X$, $\frac{1}{2}|\nu|$ and $\text{Res}$ are $\mathcal{O}(L^*)$–twisted forms and the product $\sigma_{\infty,0} = \frac{1}{2}|\nu|$ $\text{Res}$ is a bona fide $L^1_{\text{loc}}$ form. Pulling this back to $Y$ by $\nu$ gives

$$\sigma_{\infty,0} = \frac{1}{2}|\nu| \text{ Res} \quad \text{on} \quad Y$$

where $\nu = [t, ue]$. Let

$$\begin{array}{ccc}
\tilde{L}^* & \xrightarrow{p} & L^* \\
\downarrow & & \downarrow \\
\tilde{Y} & \xrightarrow{p} & Y
\end{array}$$


be as above and let $\tilde{\nu}$ be the induced section of $\tilde{L}^* \to \tilde{Y}$. Now by Lemma IV.2.5(4)

$$p^*\sigma_{\infty,0} = \frac{1}{2} \tilde{t} \nu^* \text{Res} \quad \text{on } \tilde{Y}.$$  

Since $\text{Div}_{\infty}(\tilde{\nu}) = d\left(\frac{1}{2} \tilde{t}\right)$ on $\tilde{Y}$ and $d\sigma = 0$ on $Y$ we have the current equation

(2.34) \hspace{1cm} dp^*\sigma_{\infty,0} = \nu^* \text{Res} \text{ Div}_{\infty}(\tilde{\nu}) \quad \text{on } \tilde{Y}.

Note that $p^*\sigma_{\infty,0}$ is an even form, $\nu^* \text{Res}$ is an odd form and $\text{Div}_{\infty}(\tilde{\nu})$ is an odd current on $\tilde{Y}$. To prove (2.30) we push (2.34) down to $Y$ via the mapping

$$p_* : \Omega^*_{\text{even}}(\tilde{Y})' \to \Omega^*(Y)'$$

defined by $(p_* S, \omega) := \frac{1}{2} (S, p^* \omega)$ (see IV.2.3). So by (2a) and (5) of Lemma IV.2.5 and Definition 2.22 above we have

$$d\sigma_{\infty,0} = dp_*(p^*\sigma_{\infty,0})$$

$$= p_* d(p^*\sigma_{\infty,0})$$

$$= p_* (\nu^* \text{Res} \text{ Div}_{\infty}(\tilde{\nu}))$$

$$= \nu^* \text{Res} \, p_*(\text{Div}_{\infty}(\tilde{\nu}))$$

$$= \nu^* \text{Res} \text{ Div}_{\infty}(\nu)$$

on $X \sim Z$ as required.

Finally we compute the Euler currents. Firstly $\tau_0 = \text{Div}_0(\nu)$ on $X$ since we already know that it is true on $X \sim P$ and $\tau_0 = \lim_{\delta \to 0} \tau_0 = 0$ converges in the $C^\infty$ topology on $X \sim Z$ as can be seen by inspecting the following formula for $\tau_s$ (c.f. [HL: IV,2.7]).

$$\tau_s \lambda = s \sum_{p=0}^{n-1} K_{n,p} \frac{(t(Due)^{2p+1} - (Due)^{2p} udte) (e^t \Omega V e)^{n-p-1}}{(|u|^2 + s^2 |t|^2)^{p+1}}.$$

Finally

$$\tau_\infty = \lim_{r \to \infty} \tau_r = \lim_{r \to \infty} d\sigma_{r,0} + \text{Div}_0(\nu) = \nu^* \text{Res} \text{ Div}_{\infty}(\nu). \quad \square$$
Chapter III
Quatertionic Line Bundles

The theory of Chern currents associated with endomorphisms of smooth quaternionic line bundle maps has strong analogies with that of maps between complex line bundles (see [HL II, V]). Motivated by Harvey and Lawson's Chern current formulae for meromorphic sections of a complex line bundle this chapter is devoted to investigating the Chern current theory of meromorphic maps between quaternionic line bundles. As we shall see the analytic aspects of this case are much more delicate than in the complex case. In fact in order to prove our results we have had to impose some additional conditions on the class of meromorphic maps under consideration. At this stage it is not clear whether or not these additional assumptions are necessary.

We begin by reviewing Harvey and Lawson's results in the complex case. A smooth complex valued function \( g \) on a manifold \( X \) is called atomic if \( \frac{dg}{g} \in L^1_{\text{loc}}(X) \). The divisor of an atomic function \( g \) is defined by the current equation \( \text{Div}(g) := d\left(\frac{1}{2\pi i} \frac{dg}{g}\right) \) on \( X \). Clearly

\[
(1) \quad g, h \text{ atomic } \implies gh \text{ atomic},
\]

and

\[
(2) \quad \text{Div}(gh) = \text{Div}(g) + \text{Div}(h).
\]

Let \( F \to X \) be a complex line bundle. By definition, a meromorphic section \( \nu \) of \( F \) is one which can be expressed in each local frame \( f \) for \( F \) as \( \nu = \frac{a}{b} f \) where \( a, b \) are smooth complex valued functions on \( X \). A meromorphic section \( \nu \) is called atomic if each of the local functions \( a \) and \( b \) are atomic, and the divisor of \( \nu \) is defined by \( \text{Div}(\nu) = \text{Div}(a) - \text{Div}(b) \). These notions are well defined thanks to (1)
and (2). Fix a connection $D_F$ on $F$ and define the local gauge $\omega_F$ by $D_F f = \omega_F f$.

The 1-form

$$\tau := \frac{da}{a} - \frac{db}{b} + \omega_F$$

is a globally defined $L_{loc}^1$ form on $X$ and we have the current equation

$$\frac{1}{2\pi i} d\tau = \text{Div}(\nu) - c_1(D_F)$$

on $X$.

Note that the simplest example of a meromorphic section of $F$ is given by a section of the compactified bundle $\mathbb{P}(\mathbb{C} \oplus F) \to X$ of complex projective spaces. This class of examples was studied in detail in Chapter I.

Now we turn to the quaternionic case. Let $E, F$ be quaternionic line bundles over a manifold $X$. We are interested in studying “meromorphic” quaternionic maps from $E$ to $F$.

**Definition 3.** A meromorphic quaternion-linear bundle map $\nu$ from $E$ to $F$ is a map which can be expressed in each pair of local frames $e$ for $E$ and $f$ for $F$ as

$$\nu e = ab^{-1} f$$

where $a$ and $b$ are smooth quaternion valued local functions on $X$.

Such a mapping $\nu$ is called atomic if each of the local $\mathbb{R}^4$-valued functions $a$ and $b$ are atomic.

We would like to define the divisor of such an atomic map $\nu$ locally by the current equation

$$\text{Div}(\nu) = \text{Div}(a) - \text{Div}(b).$$

The only difficulty in proving the quaternionic analogues of the results described above is to show that the divisor $\text{Div}(\nu)$ is a well defined global current on $X$. 
We will discuss this problem in more generality later, but first we will restrict our
attention to a simple class of atomic meromorphic quaternionic maps from \( E \) to
\( F \) for which it is easy to show that the divisor is well defined. This class consists
of all maps \( \nu \) for which the union \( Z \) of the zero sets of all the local functions \( a \)
defined by (4) is disjoint from the union \( P \) of the zero sets of the local functions \( b \).
If this is the case we say that \( \nu \) is a determined meromorphic map and call \( Z \) and
\( P \) the zero and pole sets of \( \nu \). Note that this class of meromorphic maps includes
all those maps which can be defined by a section of the bundle \( \mathbb{P}_\mathbb{H}(E \oplus F) \to X \)
of quaternionic projective spaces.

Next we state an important lemma and corollary and present the main theorem about the Chern currents associated with determined meromorphic homomorphisms between quaternionic line bundles.

**Lemma 6.** Let \( a, b : X \to \mathbb{H} \) be quaternion valued atomic functions on a manifold \( X \). Let \( A \) and \( B \) be the zero sets of \( a \) and \( b \) and suppose that \( A \cap B = \emptyset \). Then

1. \( ab \) is atomic.
2. Let \( \tau(b) = dbb^{-1}, \tau'(a) = a^{-1} da \) and set

\[
\Omega = \frac{1}{2} \text{Re}(\tau'(a)\tau(b)) = \frac{1}{2} \text{Re}\left( \frac{\overline{a}da \overline{b}b}{|a|^2 |b|^2} \right).
\]

Then the smooth forms \( \Omega \) and \( d\Omega \) on \( X \sim (A \cup B) \) have (unique) \( L^1_{\text{loc}} \)
extensions \( \tilde{\Omega} \) and \( \tilde{d\Omega} \) to all of \( X \). Furthermore

\[
d\tilde{\Omega} = \tilde{d\Omega} \quad \text{in} \quad L^1_{\text{loc}}(X).
\]
3. Let \( \theta(f) := f^{*}\theta \in L^1_{\text{loc}}(X) \) denote the pullback of the solid angle kernel \( \theta \)
on \( \mathbb{H} \) to \( X \) via an atomic function \( f \). Then

\[
\theta(ab) = \theta(a) + \theta(b) + d\tilde{\Omega} \quad \text{in} \quad L^1_{\text{loc}}(X).
\]
(4) $\text{Div}(ab) = \text{Div}(a) + \text{Div}(b)$ on $X$.

(5) (2) holds with $b$ replaced by $b^{-1}$.

(6) The smooth form $\theta(ab^{-1})$ on $X \sim (A \cup B)$ has an $L^1_{\text{loc}}$ extension to all of $X$ and

$$\theta(ab^{-1}) = \theta(a) - \theta(b) + d\tilde{\Omega} \quad \text{in } L^1_{\text{loc}}(X)$$

where $\Omega = \frac{1}{2} \text{Re}(\tau'(a)\tau'(b))$. In particular,

$$\theta(b^{-1}) = -\theta(b).$$

**Corollary 7.** Let $\nu$ be a determined meromorphic quaternion linear bundle map between quaternionic line bundles $E$ and $F$. Suppose that $\nu$ is atomic. Then the divisor, $\text{Div}(\nu)$, of $\nu$ defined by (5) is a well defined current on $X$.

We are now in a position to state a generalization of Harvey and Lawson's main theorem on homomorphisms between quaternionic line bundles, [HL: V, 2.26,2.36]. Endow $E$ and $F$ with metrics with respect to which scalar multiplication by unit quaternions is a pointwise isometry. Choose metric compatible connections $D_E$ and $D_F$ on $E$ and $F$ with respect to which scalar multiplication by any quaternion is parallel. Recall that the instanton class is essentially the only topological invariant of a quaternionic line bundle. The instanton form, $u(D_F)$, defined locally in terms of the curvature $\Omega_F$ of the connection $D_F$ on $F$ by

$$u(D_F) := \frac{1}{16\pi^2} \text{tr}(\Omega^2_F),$$

is the Chern–Weil representative of the instanton class. Note that the instanton form equals the Euler form of the underlying real rank 4 bundle. We use the notation $e := u(D_E)$ and $f := u(D_F)$. 
Theorem 8. Let \( \nu \) be an atomic determined meromorphic quaternionic bundle map from \( E \) to \( F \). Then there is a canonical \( L^1_{\text{loc}} \) form \( T \) on \( X \) called the instanton transgression current such that

\[
f - e - \text{Div}(\nu) = dT
\]

as currents on \( X \). Fix any polynomial \( \phi(u) \in \mathbb{R}[u] \). Then

\[
\phi(f) - \phi(e) - \text{Div}(\nu) \frac{\phi(f) - \phi(e)}{f - e} = dT_{\phi} \quad \text{on } X,
\]

where \( T_{\phi} \) is the \( L^1_{\text{loc}} \) form on \( X \) defined by

\[
T_{\phi} = \frac{\phi(f) - \phi(e)}{f - e} T.
\]

Furthermore the instanton transgression current \( T \) is given by

\[
16\pi^2 T = - \text{tr}(\tau R_E) - \text{tr}(\tau' R_F) + \frac{1}{3} \text{tr}(\tau^3)
\]

(9)

where

\[
\tau = \nu^{-1} \circ D\nu = \nu^{-1} \circ D_F \circ \nu - D_E \quad \text{and} \quad \tau' = D\nu \circ \nu^{-1} = D_F - \nu \circ D_E \circ \nu^{-1}
\]

are \( L^1_{\text{loc}} \) forms on \( X \) with values in \( \text{End}_{\mathbb{H}}(E) \) and \( \text{End}_{\mathbb{H}}(F) \) respectively, and where \( R_E \) and \( R_F \) are the curvature operators on \( E \) and \( F \).

Suppose that the atomic determined meromorphic map \( \nu \) of Theorem 8 is actually a section of the bundle \( \mathbb{P}_{\mathbb{H}}(E \oplus F) \rightarrow X \) of quaternionic projective spaces. Then we can realize the transgression current \( T \) as the current limit of a smooth family of smooth transgression forms \( T_{\tau,s} \) on \( \mathbb{P}_{\mathbb{H}}(E \oplus F) \) and the current equations of Theorem 8 as the current limits of smooth transgression formulae on \( \mathbb{P}_{\mathbb{H}}(E \oplus F) \). This is done as follows. We proceed just as in the complex case (c.f. Chapter I).
Let \( E \xrightarrow{\alpha} F \) be the homomorphism of quaternionic line bundles defined by \( \nu \) over \( X \sim P \). The pullback family of connections on \( E \) is defined over \( X \sim P \) for \( 0 < s < \infty \) by

\[
\widetilde{D}_s := D_E + \chi_s \tau,
\]

where \( \tau = \alpha^{-1} \circ D\alpha \) and \( \chi_s = \chi\left(\frac{\log t}{s}\right) \). Here \( \chi(t) = \frac{t}{1 + t} \) is chosen to be the algebraic approximation mode. Let \( T_{r,s} (0 < s < r < \infty) \) be the smooth transgression form on \( X \sim P \) which satisfies

\[
u(\widetilde{D}_r) - \nu(\widetilde{D}_s) = dT_{r,s} \quad \text{on } X \sim P.
\]

Let \( D_{U,s} (0 < s < \infty) \) be the family of smooth connections defined on the pullback to \( X \) via \( \nu \) of the tautological quaternion line bundle \( U \) over \( \mathbb{P}^2(E \oplus F') \rightarrow X \) just as in (I.2.22). Note that the connections \( D_{U,s} \) are quaternion linear and so the instanton forms \( \nu(D_{U,s}) \) and the corresponding transgression forms \( T_{r,s} \) are well defined forms on \( X \). (Also note that the connections \( D_{U,s} \) are not metric compatible since the mappings \( \Psi_s \) defined as in (I.2.20) are not isometries.) Arguing exactly as in the proof of Theorem I.2.23 we have the following result.

**Proposition 12.**

1. The family \( (\widetilde{D}_s, E) \) is equivalent over \( X \sim P \) to the family \( (D_{U,s}, \nu^*(U)) \).

More precisely

\[
D_{U,s} = \Gamma \circ \widetilde{D}_s \circ \Gamma^{-1} \quad \text{over } X \sim P
\]

where \( \Gamma : E \rightarrow \nu^*(U) \) is the quaternion linear graphing map, \( \Gamma(e) = (e, \alpha(e)) \).

2. Therefore the instanton forms \( \nu(\widetilde{D}_s) (0 < s < \infty) \) and the transgression forms \( T_{r,s} (0 < s < r < \infty) \) of the family \( (\widetilde{D}_s, E) \) over \( X \sim P \) extend smoothly to all of \( X \) as the instanton and transgression forms of the family \( (D_{U,s}, \nu^*(U)) \) over \( X \).
Proposition 13.

(1) Let $\nu$ be an atomic section of $\mathcal{P}_\mathcal{H}(E \oplus F) \to X$. Then the instanton transgression forms $T_{r,s}$ converge in $L^1_{\text{loc}}(X)$ as $r \to \infty$ and $s \to 0$ to the negative of the instanton transgression current defined by (9).

(2) The current limit of the smooth transgression formula

$$u(\overrightarrow{D}_r) - u(\overrightarrow{D}_s) = dT_{r,s} \quad \text{on } X$$

is the current equation

$$e - f + \text{Div}(\nu) = -dT$$

of Theorem 8. Furthermore, set $\text{Div}(\nu) = \text{Div}_0(\nu) - \text{Div}_\infty(\nu)$ where $\text{Div}_\infty(\nu)$ is the part of $\text{Div}(\nu)$ supported on $P$. Then the instanton current at infinity exists and is given by

$$u(\overrightarrow{D}_\infty) := \lim_{r \to \infty} u(\overrightarrow{D}_r) = e - \text{Div}_\infty(\nu),$$

while the instanton current at zero exists and is given by

$$u(\overrightarrow{D}_0) := \lim_{s \to 0} u(\overrightarrow{D}_s) = f - \text{Div}_0(\nu).$$

Proof of Lemma 6.

(1) We need to show that

$$\frac{d(ab)^I}{|ab|^p} \in L^1_{\text{loc}}(X) \quad \text{for all } I \text{ such that } p = |I| \leq 3.$$ 

Now each such form is dominated by sums of terms of the form

$$\frac{da^Jdb^K}{|a|^j|b|^k} \quad \text{where } j = |J| \leq 3, \quad k = |K| \leq 3.$$ 

But all of these forms belong to $L^1_{\text{loc}}(X)$ since the zero sets of $a$ and $b$ are disjoint and, in a neighbourhood of the zero set of $a$, $\frac{db^K}{|b|^k}$ is bounded while $\frac{da^J}{|a|^p} \in L^1_{\text{loc}}(X)$. 

(2) An argument similar to the one just given shows that \( \Omega \) and \( d\Omega \) have unique \( L^1_{\text{loc}} \) extensions \( \tilde{\Omega} \) and \( \overline{d\tilde{\Omega}} \) to all of \( X \). Let \( \chi_s = \frac{|a|}{s^2 + |a|^2} \), \( \frac{|b|}{s^2 + |b|^2} \). Using the atomicity of \( a \) and \( b \) to dominate \( d\chi_s \Omega \) by an \( s \)-independent \( L^1_{\text{loc}} \) form and applying the Lebesgue Dominated Convergence Theorem three times gives

\[
d\tilde{\Omega} = d(\lim_{s \to 0} \chi_s \Omega) = \lim_{s \to 0} d\chi_s \Omega + \lim_{s \to 0} \chi_s d\Omega = 0 + \overline{d\tilde{\Omega}},
\]
as required.

(3) Let \( x \) be a coordinate on \( \mathbb{H} \) and set \( \tau = dx x^{-1} \) and \( \tau' = x^{-1} dx \). Harvey and Lawson [HL: V,2.34] showed that the solid angle kernel \( \theta \) on \( \mathbb{H} \) can be written in the form

\[
\theta = -\frac{1}{6} \text{Re}(\tau^3).
\]

Let \( \tau(f) := f^* \tau \) and \( \tau'(f) := f^* \tau' \) for any function \( f : X \to \mathbb{H} \), and note that

\[
\tau(ab) = \tau(a) + a \tau(b) a^{-1}.
\]

Then

\[
\theta(ab) = -\frac{1}{6} \text{Re}(\tau^3(ab)) = \theta(a) + \theta(b) + \frac{1}{2} \text{Re}(d\tau'(a) \tau(b) - \tau'(a) d\tau(b)),
\]

since \( \tau^2 = d\tau \) and \( (\tau')^2 = -d\tau' \). So we have the equation of smooth forms

\[
\theta(ab) = \theta(a) + \theta(b) + d\Omega \quad \text{on} \ X \sim (A \cup B).
\]

Therefore

\[
\overline{d\tilde{\Omega}} = \theta(ab) - \theta(a) - \theta(b) \quad \text{in} \ L^1_{\text{loc}}(X),
\]

since it is true almost everywhere. (3) now follows from (2).

(4) follows immediately from (3). The proofs of (5) and (6) are similar to those of (3) and (4). \( \Box \)
Proof of Corollary 7. Let \( e \) and \( f \) be local frames for \( E \) and \( F \) and write \( \nu e = ab^{-1} f \). Suppose that \( \nu e' = cd^{-1} f' \) is another local expression for \( \nu \), where \( e' = ge \) and \( f' = hf \). We need to show that \( \text{Div}(a) - \text{Div}(b) = \text{Div}(c) - \text{Div}(d) \). Now \( gab^{-1} = cd^{-1} h \) where \( g \) and \( h \) are nonzero. So by (6) of Lemma 6,

\[
\theta(ga) - \theta(b) + d\tilde{\Omega}(ga, b^{-1}) = \theta(c) - \theta(h^{-1}d) + d\tilde{\Omega}(c, h^{-1}d).
\]

Combining this with (4) of Lemma 6 gives \( \text{Div}(a) - \text{Div}(b) = \text{Div}(ga) - \text{Div}(b) = \text{Div}(c) - \text{Div}(h^{-1}d) = \text{Div}(c) - \text{Div}(d) \) as required. \( \square \)

Proof of Theorem 8. Let \( Z \) and \( P \) be the zero and pole sets of the determined meromorphic map \( \nu \). We prove the theorem by checking that the smooth form \( T \) defined on \( X \sim (Z \cup P) \) by (9) has an \( L^1_{\text{loc}} \) extension \( \tilde{T} \) to all of \( X \) and compute \( d\tilde{T} \). Express \( \nu \) locally in the form \( \nu e = ab^{-1} f \) as usual. Then \( T \) can be written as a sum of terms, each of which is the product of an \( L^\infty_{\text{loc}} \) form on \( X \) with one of the forms

\[
\frac{d^{a_l} db^j}{|a|^p |b|^q} \quad \text{where } p = |I|, \ q = |J| \text{ and } p + q \leq 3.
\]

But each of these forms is in \( L^1_{\text{loc}}(X) \) since \( a \) and \( b \) are atomic and have disjoint zero sets. Therefore \( T \) has an \( L^1_{\text{loc}} \) extension \( \tilde{T} \) to all of \( X \). Next note that the smooth form \( dT \) on \( X \sim (Z \cup P) \) has a \( C^\infty \) extension \( d\tilde{T} \) to all of \( X \) since

\[
dT = u(D_F) - u(D_E) \quad \text{on } X \sim (Z \cup P)
\]

by [HL: V,2.10].

Let \( S \) be defined by the current equation \( d\tilde{T} = \tilde{T} + S \) on \( X \). Since \( d\tilde{T} = u(D_F) - u(D_E) \) we just need to check that \( S = -\text{Div}(\nu) \). Let \( T_{\text{top}} \) be that part of \( \tilde{T} \) which is of top degree 3 in the differentials \( da_i, db_j \) and define \( T' \) by the equation \( \tilde{T} = T_{\text{top}} + T' \). Then, since trace equals four times the real part,

\[
T_{\text{top}} = \frac{1}{4\pi^2} \text{tr}(\tau^3(ab^{-1})) = -\frac{1}{2\pi^2} \theta(ab^{-1}),
\]

where \( \tau \) is the Riemann surface.
and so, by (6) of Lemma 6, \( dT_{\text{top}} = -(\text{Div}(a) - \text{Div}(b)) = -\text{Div}(\nu) \). Finally since \( dT' \) is \( L^1_{\text{loc}} \) and \( \bar{d}T + S = dT' - \text{Div}(\nu) \) we are forced to conclude that \( S = -\text{Div}(\nu) \), as required. \( \square \)

**Proof of Proposition 13.** By [HL: V,2.31] the transgression form \( T_{r,s} \) over \( X \sim P \) is given by

\[
16\pi^2 T_{r,s} = (F_r - F_s) \, \text{tr}(\tau R_E) + (\chi_r^2 - \chi_s^2) \, \text{tr}(\tau \alpha^{-1} R_F \alpha) + (G_r - G_s) \, \text{tr}(\tau^3),
\]

where \( \tau = \alpha^{-1} \circ D \alpha \), \( F_r = 2\chi_r - \chi_r^2 \) and \( G_r = -\chi_r^2 + \frac{2}{3} \chi_r^3 \). From here on the proof of the proposition is completely analogous to the proofs of Proposition I.3.1 and Theorem I.3.8. \( \square \)

We know that if \( a \) and \( b \) are any two atomic complex valued functions then

\[
ab \text{ is atomic and } \text{Div}(ab) = \text{Div}(a) + \text{Div}(b).
\]

So far we have only proved the analogous results for quaternion valued functions under the additional assumption that the zero sets of the functions in question are disjoint. Ideally we would like to prove that, for \textit{any} two atomic quaternion valued functions \( a \) and \( b \), (15) is true. Unfortunately we have not been able to do this in full generality. Neither have we been able to find any counterexamples. Instead we conclude this chapter by discussing some weaker additional assumptions, each of which is sufficient to ensure that (15) is true.

Note that we can also prove versions of Theorem 8 in which the atomic determined meromorphic map \( \nu \) is replaced by atomic meromorphic maps which satisfy appropriate modifications of any of the weaker additional assumptions used to prove (15). The precise statements and proofs of these results are left to the reader.
Our first result is motivated by a theorem of Harvey and Semmes [HS, 3.2] (see Theorem I.1.13).

**Definition 16.** Let \( u \) be a smooth \( \mathbb{R}^n \)-valued function on \( X^{\text{open}} \subset \mathbb{R}^m \). We say that \( u \) **vanishes algebraically** if, for each compact set \( K \subset X \), there exist positive constants \( c \) and \( N \) such that

\[
|u(x)| \geq c \text{dist}(x, \text{Zero}(u))^N \quad \text{for all } x \in K.
\]

The zero set of \( u \) is said to have **strong codimension greater than** \( n - 1 \) if for each compact \( K \subset X \) there exists an \( \epsilon > 0 \) such that the upper Minkowski content of \( \text{Zero}(u) \cap K \) in dimension \( m - n + 1 - \epsilon \) is finite.

**Note.** Harvey and Semmes proved that if \( u \) vanishes algebraically and if its zero set has strong codimension greater than \( n - 1 \) then \( u \) is atomic.

**Proposition 17.** Suppose that \( a \) and \( b \) are smooth quaternion valued functions on \( X^{\text{open}} \subset \mathbb{R}^m \).

1. If \( a \) and \( b \) vanish algebraically then so does \( ab \).
2. If the zero sets of \( a \) and \( b \) have strong codimension greater than \( 3 \) then so does that of \( ab \).
3. Therefore if the hypotheses of (1) and (2) hold then \( a, b \) and \( ab \) are all atomic.

**Corollary 18.** If \( a \) and \( b \) are real analytic quaternion valued functions on an open set \( X \) in \( \mathbb{R}^m \) and if each irreducible component of the zero sets of \( a \) and \( b \) have codimension \( \geq 4 \), then \( a, b \) and \( ab \) are all atomic.

**Proof of Proposition 17.**
(1) Suppose that \( a \) and \( b \) vanish algebraically. Let \( A = \text{Zero}(a) \), \( B = \text{Zero}(b) \). Fix a compact \( K \subset X \), and choose \( c, N > 0 \) so that for all \( x \in K \)

\[
|a(x)| \geq c \dist(x, A)^N \quad \text{and} \quad |b(x)| \geq c \dist(x, B)^N.
\]

Then

\[
|ab(x)| = |a(x)||b(x)| \geq c^2 \dist(x, A)^N \dist(x, B)^N \geq c^2 \dist(x, A \cup B)^2 N.
\]

So, since \( A \cup B = \text{Zero}(ab) \), we have shown that \( ab \) vanishes algebraically.

(2) Let \( M^d \) denote upper Minkowski content in dimension \( d \) (see [F]). Choose \( \epsilon_1 \) and \( \epsilon_2 \) so that \( M^{m-3-\epsilon_1}(A) < \infty \) and \( M^{m-3-\epsilon_2}(B) < \infty \). Suppose that \( \epsilon_1 < \epsilon_2 \). Then \( M^{m-3-\epsilon_1}(B) = 0 \) and so by the subadditivity of Minkowski content \( M^{m-3-\epsilon_1}(A \cup B) < \infty \) as required. \( \square \)

**Proposition 19.** Let \( a \) and \( b \) be atomic quaternion valued functions on an \( n \) dimensional manifold \( X \) and let \( A \) and \( B \) denote their zero sets. Suppose that

1. \( ab \) is atomic,
2. \( \mathcal{H}^{n-3}(A \cap B) = 0 \), and
3. \( \Omega = \frac{1}{2} \Re \left( \frac{\bar{a} d\bar{b} + \bar{b} da}{|a|^2 |b|^2} \right) \) has an \( L^1_{\text{loc}} \) extension \( \tilde{\Omega} \) to all of \( X \).

Then

\[
\text{Div}(ab) = \text{Div}(a) + \text{Div}(b) \quad \text{on} \ X.
\]

**Proof.** Since

\[ d\Omega = \theta(ab) - \theta(a) - \theta(b) \quad \text{in} \ C^\infty(X \sim (A \cup B)) \]

and \( a \), \( b \) and \( ab \) are atomic we know that \( d\Omega \) has an \( L^1_{\text{loc}} \) extension \( \partial \tilde{\Omega} \) to all of \( X \). Let \( S = d\tilde{\Omega} - \partial \tilde{\Omega} \). We can combine the fact that \( \tilde{\Omega} \in L^1_{\text{loc}}(x) \) with (2) of Lemma 6 to show that \( S \) is a flat current of dimension \( n - 3 \) supported on \( A \cap B \).
The assumption that $\mathcal{H}^{n-3}(A \cap B) = 0$ enables us to conclude — via the Federer Support Theorem (see [F]) — that $S \equiv 0$. Therefore we have the current equation

$$d\tilde{\omega} = \tilde{d}\omega = \theta(ab) - \theta(a) - \theta(b) \quad \text{on } X,$$

which implies the result of the proposition. □

**Proposition 20.** Let $a$ and $b$ be atomic quaternion-valued functions on an $n$ dimensional manifold $X$ and let $A$ and $B$ denote their zero sets. Suppose that

1. $ab$ is atomic, and
2. $\mathcal{H}^{n-4}(A \cap B) = 0$.

Then

$$\text{Div}(ab) = \text{Div}(a) + \text{Div}(b) \quad \text{on } X.$$ 

**Proof.** As in the proof of Proposition 19 we know that $d\omega \in C^\infty(X \sim (A \cup B))$ has an $L^1_\text{loc}$ extension $\tilde{d}\omega$ to all of $X$ and that $d(\tilde{d}\omega) = 0$ on $X \sim (A \cap B)$. So $d(\tilde{d}\omega)$ is a flat current of dimension $n-4$ supported on the set $A \cap B$ which we have assumed has zero $(n-4)$-dimensional Hausdorff measure. So by the Federer Support Theorem $d(\tilde{d}\omega) = 0$ on $X$. The result now follows from the fact that

$$\tilde{d}\omega = \theta(ab) - \theta(a) - \theta(b) \quad \text{in } L^1_\text{loc}(X).$$ □

**Note.** In contrast to Propositions 19 and 20 note that if $a : X \to \mathbb{H}$ is atomic then $a^n$ is atomic and $\text{Div}(a^n) = n \text{ Div}(a)$ on $X$.

We conclude this chapter by presenting an example of a pair of polynomial functions $a$ and $b$ which do not satisfy the hypotheses of Lemma 6, Proposition 19 or Proposition 20. Nevertheless we can compute directly that $\text{Div}(ab) = \text{Div}(a) + \text{Div}(b)$. 
One obvious way to try to prove that $\text{Div}(ab) = \text{Div}(a) + \text{Div}(b)$ for any pair of atomic functions $a$ and $b$ for which $ab$ is also atomic would be to find a natural current extension $\tilde{\Omega}$ to all of $X$ of the smooth form $\Omega$ on $X \sim (A \cup B)$ and to show that $d\tilde{\Omega} = \theta(ab) - \theta(a) - \theta(b)$. The example below shows that we cannot hope to extend $\Omega$ to an $L^1_{\text{loc}}$ form on all of $X$. We have not been able to find a natural current extension of $\Omega$. Neither have we been able to find a pair of functions $a, b$ for which $\text{Div}(ab) \neq \text{Div}(a) + \text{Div}(b)$.

**Example 21.** Let $a, b : \mathbb{H} \to \mathbb{H}$ be defined by

$$a(x_0, x_1, x_2, x_3) = (x_0, x_1^3, x_2^3, x_3^3),$$

$$b(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, x_3).$$

Then

1. $a, b$ and $ab$ are all atomic,
2. $\mathcal{H}^0(A \cap B) = 1 > 0$,
3. $\Omega \notin L^1_{\text{loc}}(\mathbb{H})$, but
4. $\text{Div}(a) = [0], \text{Div}(b) = [0]$ and $\text{Div}(ab) = 2[0]$, so that

$$\text{Div}(ab) = \text{Div}(a) + \text{Div}(b) \quad \text{on } \mathbb{H}.$$

**Proof.**

1. This follows directly from Corollary 18.
2. This is trivial.
3. The coefficient of $dx_0dx_1$ in $\Omega$ is of the form $\frac{f(x)}{|a|^2 |z|^2}$ where

$$f(x) = x_0x_1 + x_3x_2^3 - x_2x_3^3 + 3x_1^5x_0 - 3x_1^2x_2x_3^3 + 3x_1^2x_2^3x_3,$$

$$g(x) = x_0x_1 + 4x_0x_1^3.$$
and

$$|a|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$  

Let

$$K = B_1(0) \cap \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid 3x_j^2 < x_0^2, \quad j = 1, 2, 3\}.$$  

Then, for each \(x = (x_0, x_1, x_2, x_3) \in K\), \(|x|^2 < 2x_0^2\) and \(|a|^2 < 2x_0^2\). So

$$\int_K \frac{x_0x_j^k}{|a|^2 |x|^2} \, dx > \frac{1}{4} \int_K \frac{x_j^k}{x_0^3} \, dx = C \int_0^1 \frac{k-6}{x_0^3} \, dx_0,$$

which is infinite for \(k \leq 3\). This implies that \(\int_K \frac{a(x) dx}{|x|^2} \) is infinite. Similarly

$$\int_K \frac{f(x) dx}{|x|^2} < \infty.$$  

Therefore \(\Omega \notin L^1_{\text{loc}}(\mathbb{H})\).

(4) Clearly the divisor of the identity function \(b\) on \(\mathbb{H}\) is \([0]\). We can compute the divisor of an atomic function \(f : \mathbb{R}^4 \to \mathbb{R}^4\) with \(\text{Zero}(f) = \{0\}\) as follows. Choose a regular value \(y\) of \(f\). Then

$$\text{Div}(f) = \left( \sum_{x \in f^{-1}(y)} (-1)^{\text{sgn} J(f)(x)} \right) [0],$$

where \(J(f)(x) = \det(Df(x))\) is the Jacobean determinant of \(f\).

Now \(J(a)(x) = 27x_1x_2x_3^2\) and \(a^{-1}(0, 1, 1, 1) = \{(0, 1, 1, 1)\}\), so \((0, 1, 1, 1)\) is a regular value of \(a\), whence \(\text{Div}(a) = [0]\). Finally we show that \(\text{Div}(ab) = 2[0]\).

First note that \((ab)^{-1}(1, 0, 0, 0) = \{\pm (1, 0, 0, 0)\}\). To see this we argue as follows. To solve \(a(x)x = 1\) in \(\mathbb{H}\) it suffices to solve \(a(x) = x^{-1} = \frac{x}{|x|^2}\). But this is equivalent to the system of equations

$$x_0(1 - |x|^2) = 0 \quad \text{and} \quad x_j(1 + x_j^2|x|^2) = 0, \quad j = 1, 2, 3,$$

and it is easy to verify that \((\pm 1, 0, 0, 0)\) are the only solutions of this system.

Finally we check that \(J(ab)(\pm 1, 0, 0, 0) = 2\). \(\Box\)
Chapter IV
Divisors and Characteristic Currents
in the Nonorientable Case

0. Introduction.

In order to define the divisor of an atomic section of a vector bundle and to compute characteristic current formulae Harvey, Semmes and Lawson imposed orientability assumptions on the bundle and on the base manifold. In this chapter we investigate what happens if these orientation assumptions are relaxed. For example, let $\mu$ be a section of a rank $n$ real vector bundle $V$ over an $n$ dimensional manifold $X$, and let $x$ be an isolated zero of $\mu$. Without any orientation assumptions on $V$ or $X$ we cannot define the multiplicity of vanishing of $\mu$ at $x$. However we can define the magnitude of vanishing by making arbitrary choices of orientation on $V$ and $X$ in a neighbourhood of $x$ and computing the absolute value of the multiplicity of vanishing of a local coordinate expression of $\mu$ near $x$. One of the main aims of this chapter is to define the zero divisor of an atomic section of an arbitrary real vector bundle as a distributional section of the appropriate space of twisted forms. The fact that we can only speak of the magnitude and not the multiplicity of vanishing is reflected in the fact that the divisor is forced to act on twisted forms.

This generalized notion of divisor has some nice structure which helps to justify its definition. Firstly, the restriction of the divisor to any oriented open subset of the base manifold $X$ over which the bundle $V$ is also oriented is just the usual divisor of Harvey and Semmes. Secondly, if the zero set of an atomic function happens to be an orientable connected regular submanifold, there is an oriented open subset of $X$ over which $V$ is oriented and on this open subset the divisor is
the usual current of integration over the zero set, with multiplicity. In contrast to
the oriented situation of [HS] we show that the support of the generalized divisor
can be a nonorientable submanifold of $X$. In fact we prove that if $Z = \text{Zero}(\mu)$ is
a nonorientable connected regular submanifold of $X$ then either $\text{Div}(\mu) = 0$ or the
orientation bundle $\mathcal{O}(TZ)$ of $Z$ is given by $\mathcal{O}(TZ) = \mathcal{O}(V) \otimes \mathcal{O}(TX)|_Z$ and there
is an integer $n$ such that $\text{Div}(\mu) = n[Z]$ acting on $\mathcal{O}(V) \otimes \mathcal{O}(TX)$-twisted forms
on $X$. The simplest examples of this phenomenon are found by pulling bundles
back over themselves and considering the divisor of the tautological section. In
the case that $\mu$ vanishes only to first order on $Z$ the results described above follow
immediately from the fact that the restriction of $V$ to $Z$ is isomorphic to the
normal bundle of $Z$ in $X$. In the general case the proofs consist of a sequence of
elementary covering space arguments.

In addition to defining the divisor of an atomic section $\mu$ of an unoriented
bundle $V$ over an unoriented manifold $X$ we derive various characteristic current
formulae associated with the singular pushforward connection induced on $V$ by $\mu$.
These formulae generalize those found by Harvey–Lawson [HL:IV] in the oriented
case. In particular we are able to remove the orientation assumption from the
real version of the Rectifiable Grothendieck–Riemann–Roch Theorem of Harvey–
Lawson [HL: V,3]. Also note that the definitions and results of Section 2 have
already been applied in Chapter II where we discuss the problem of computing
the Thom current at infinity associated with a section of the compactification of
odd rank oriented real vector bundle.

The discussion of the results outlined above is divided into four cases corre-
sponding to the pair $(V,X)$ being (nonorientable, oriented), (oriented, nonori-
entable), (nonorientable, nonorientable) with isomorphic orientation bundles, and
(nonorientable, nonorientable) with non-isomorphic orientation bundles. Note
that the third case, which includes the special case of the tangent bundle of a nonorientable manifold, is somewhat different from the other three in that the divisor acts on bona fide forms and so we can talk about the multiplicity of vanishing of a section. Also note that the results in the first and second cases can be regarded as special cases of the results in the fourth case. Curiously though, in order to prove the results in the fourth case in full generality it seems necessary to invoke those of the first and second cases.

Results which require orientability assumptions are usually extended to the nonorientable case by working on the appropriate double cover and applying the result we wish to extend “upstairs” where everything is oriented. Very often formulating results in terms of objects defined “downstairs” doesn’t yield any fundamentally new information, and to a large extent this is true of the results presented here. In the situation at hand one of the main justifications for formulating results in the downstairs language is that we have applied these results in Chapter II to solve the problem of computing the Thom current at infinity. Another reason for using the downstairs language is that it makes it possible to analyse the fourth case described above. In addition the nice results on the structure of the divisor downstairs add some weight to the argument for working downstairs.

Before embarking on the program outlined above we present a leisurely discussion of the nonorientable Thom isomorphism and show how to adapt Harvey and Lawson’s construction of a family of Thom forms to this situation.
1. Twisted Thom forms and the nonorientable Thom isomorphism.

In their paper Harvey and Lawson give a method of constructing families of Thom forms for complex and real oriented bundles over oriented manifolds. The aim of this section is to show that their method also applies when the orientation assumptions are relaxed, giving a construction of families of twisted Thom forms for unoriented bundles over unoriented manifolds.

We begin by reviewing the oriented and unoriented Thom isomorphism theorems. Our discussion is based upon that of Bott and Tu [BT] and Harvey and Lawson [HL]. Suppose for the moment that \( \pi : V \rightarrow X \) is an oriented real rank \( n \) vector bundle over an oriented manifold. Let \( \Omega^*_{cw}(V) \) denote the complex of differential forms on \( V \) with compact support in the fibre directions. Integration over the fibre of \( V \) is a mapping

\[
\int_{\pi^{-1}} : \Omega^{k+n}_{cw}(V) \rightarrow \Omega^k(X)
\]

which agrees with the current push forward \( \pi_* \) and commutes with the exterior derivative operator \( d \). Consequently the fibre integration map descends to a map on de Rham cohomology,

\[
\pi_1 : H^{k+n}_{cw}(V) \rightarrow H^k(X),
\]

which, according to Thom, is an isomorphism. The Thom class of the bundle \( V \) is that cohomology class in \( H^n_{cw}(V) \) which is mapped to \( 1 \in H^0(X) \) via the Thom isomorphism.

Harvey and Lawson construct a family of Thom forms \( \tau_s \) for \( s > 0 \) on \( V \) having
the following properties.

\begin{align}
(1.1) \quad d\tau_s &= 0 \\
(1.2) \quad \int_{\pi^{-1}} \tau_s &= 1 \\
(1.3) \quad \tau_s &= (\frac{1}{s})^* \tau_1 \\
(1.4) \quad \lim_{s \to 0} \tau_s &= [X]
\end{align}

The first two properties show that the forms $\tau_s$ represent the Thom class of $V$, while properties (1.3) and (1.4) say that the Thom forms provide a global geometrization of the notion of an approximate identity (or point mass) in the fibres of $V$.

The mappings

\[ i_s : \Omega^k(X) \longrightarrow \Omega_{cv}^{k+n}(V), \]
\[ \varphi \longmapsto \pi^* (\varphi) \wedge \tau_s \]

induce a mapping

\[ i_t : H^k(X) \longrightarrow H_{cv}^{k+n}(V), \]

which inverts the Thom isomorphism. Letting $s \to 0$ yields a canonical version of this map

\[ i_0 : \Omega^k(X) \longrightarrow \Omega_{cv}^{k+n}(V)', \]

now into currents with compact support in the fibre directions of $V$ given by

\[ i_0 (\varphi) = \varphi [X]. \]

From now on we drop all orientation assumptions and consider an arbitrary real vector bundle $\pi : V \longrightarrow X$ of rank $n$. The orientation bundle $\mathcal{O}(V) \to X$ is of fundamental importance and is defined as follows.
Definition 1.5. Let \( \{(U_\alpha, e_\alpha)\} \) be a system of local frames \( e_\alpha \) for \( V \) defined over open sets \( U_\alpha \) in \( X \) and let \( g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}) \) be the corresponding transition functions defined by the formula \( e_\alpha = g_{\alpha\beta} e_\beta \). Then the **Orientation Bundle** \( \mathcal{O}(V) \to X \) is the line bundle with transition functions \( \text{sgn}(\det(g_{\alpha\beta})) \).

Note that \( V \to X \) is orientable iff \( \mathcal{O}(V) \to X \) is the trivial bundle.

When \( V \) is unoriented it is straightforward to check that fibre integration is a mapping

\[
\int_{\pi^{-1}} : \Omega^{k+n}_{cv}(V) \to \Omega^{k}(X, \mathcal{O}(V)).
\]

Here \( \Omega^{k}(X, \mathcal{O}(V)) := \Gamma(X, \mathcal{O}(V) \otimes \Lambda^{k}(T^*X)) \) is the space of \( \mathcal{O}(V) \)-twisted \( k \) forms on \( X \). Note that every \( \mathcal{O}(V) \)-twisted \( k \) form \( \omega \) can be expressed as \( \omega = \alpha \otimes \varphi \) where \( \alpha \in \Omega^{0}(X, \mathcal{O}(V)) \) and \( \varphi \in \Omega^{k}(X) \). Since the transition functions of \( \mathcal{O}(V) \) are locally constant we can make \( \Omega^{*}(X, \mathcal{O}(V)) \) into a differential complex and the cohomology \( H^{*}(X, \mathcal{O}(V)) \) makes sense. Note that bundles with locally constant transition functions are called flat vector bundles.

Now we can state the nonorientable Thom isomorphism theorem.

Theorem 1.6. [Bott and Tu]. Let \( \pi : V \to X \) be an arbitrary real vector bundle of rank \( n \). Then the fibre integration map,

\[
\int_{\pi^{-1}} : \Omega^{k+n}_{cv}(V) \to \Omega^{k}(X, \mathcal{O}(V))
\]

induces an isomorphism

\[
\pi_! : H^{k+n}_{cv} \cong H^{k}(X, \mathcal{O}(V)).
\]

This theorem will be proved once we have found a twisted Thom form \( \tau \) on \( V \) with the following properties.
There is a mapping

\[(1.7) \quad i : \Omega^k(X, \mathcal{O}(V)) \longrightarrow \Omega^{k+n}_{cu}(V) \]

\[\alpha \otimes \varphi \longmapsto \pi^*\alpha \otimes \pi^*\varphi \wedge \tau.\]

Note that (1.7) forces

\[\tau \in \Omega^n_{cu}(V, \pi^*(\mathcal{O}(V)^*))\]

which, thanks to Lemma 1.12 below, is equivalent to

\[(1.8) \quad \tau \in \Omega^n_{cu}(V, \pi^*\mathcal{O}(V)).\]

In order that the mapping (1.7) descend to cohomology we require that

\[(1.9) \quad d\tau = 0.\]

Note that this makes sense since \(\pi^*\mathcal{O}(V) \longrightarrow V\) is a flat bundle. Finally we need two more properties to show that the induced mapping,

\[i_! : H^k(X, \mathcal{O}(V)) \longrightarrow H^{k+n}_{cu}(V)\]

inverts the fibre integration map

\[\pi_! : H^{k+n}_{cu}(V) \longrightarrow H^k(X, \mathcal{O}(V)).\]

The first of these is a general fact. Fibre integration can be regarded either as a map

\[\int_{\pi^{-1}} : \Omega^{k+n}_{cu}(V) \longrightarrow \Omega^k(X, \mathcal{O}(V)),\]

or as a map

\[\int_{\pi^{-1}} : \Omega^{k+n}_{cu}(V, \pi^*\mathcal{O}(V)) \longrightarrow \Omega^k(X).\]
The projection formula relates these two fibre integration maps as follows:

\[ (1.10) \quad \int_{\pi^{-1}} \pi^*(\alpha \otimes \varphi) \wedge \tau = \alpha \otimes \varphi \wedge \int_{\pi^{-1}} \tau. \]

Here \( \tau \in \Omega^l(V, \pi^*\mathcal{O}(V)) \) and \( \alpha \otimes \varphi \in \Omega^k(X, \mathcal{O}(V)) \). Secondly we require that the twisted form \( \tau \) should satisfy

\[ (1.11) \quad \int_{\pi^{-1}} \tau = 1. \]

Before constructing the twisted Thom form \( \tau \) we state a general lemma concerning orientation bundles and define the current of integration \([X]\) over the submanifold \( X \) of \( V \).

**Lemma 1.12.** Let \( \pi : V \longrightarrow X \) be a real vector bundle. Then

\[ (1.13) \quad \mathcal{O}(V)^* \cong \mathcal{O}(V) \quad \text{on } X. \]

\[ (1.14) \quad \mathcal{O}(TV) \cong \pi^*(\mathcal{O}(V) \otimes \mathcal{O}(TX)) \quad \text{on } V. \]

**Proof.** (1.13) is easy. As for (1.14), let \( \{(U_\alpha, e_\alpha)\} \) be a system of local coordinate charts for \( X \) and let \( g_{\alpha\beta} := \varphi_\alpha^{-1} \circ \varphi_\beta : U_\beta \rightarrow U_\alpha \). Choose local frames, \( e_\alpha = (e^1_\alpha, \ldots, e^n_\alpha) : \varphi_\alpha(U_\alpha) \rightarrow V \), for \( V \) and define transition functions \( h_{\alpha\beta} \) for \( V \) by the formula \( e_\alpha = h_{\alpha\beta} e_\beta \). Then there are local coordinates induced on the manifold \( \text{tot } V \) by

\[ \psi_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow \text{tot } V \]

\[ (p, q) \longmapsto \sum_{i=1}^n q_i e^i_\alpha(\varphi_\alpha(p)), \]
with transition functions

$$
\Psi_\alpha^{-1} \circ \Psi_\beta : \mathcal{U}_\beta \times \mathbb{R}^n \to \mathcal{U}_\alpha \times \mathbb{R}^n
$$

given by

$$
\Psi_\alpha^{-1} \circ \Psi_\beta(p, q) = (g_{\alpha\beta}(p), q h_{\alpha\beta}(\phi_\beta(p))).
$$

Since

$$
D(\Psi_\alpha^{-1} \circ \Psi_\beta)(p, q) = \begin{pmatrix} Dg_{\alpha\beta}(p) & 0 \\ \ast & h_{\alpha\beta}(\phi_\beta(p)) \end{pmatrix}
$$

we see that

$$
\text{sgn}(\det(D(\Psi_\alpha^{-1} \circ \Psi_\beta))(p, q)) = \text{sgn}(\det Dg_{\alpha\beta}(p)) \text{sgn}(\det h_{\alpha\beta}(\phi_\alpha(p))).
$$

That is the transition function of $\mathcal{O}(TV)$ is the product of the transition functions of $\pi^*\mathcal{O}(TX)$ and $\pi^*\mathcal{O}(V)$, as required. □

Let $M$ be an unoriented manifold of dimension $m$. Recall that the integral over $M$ of a compactly supported $\mathcal{O}(TM)$-twisted $m$ form is well defined. Let $\eta$ be a smooth section of a vector bundle $E \to M$. Then $\eta$ defines a continuous linear functional

$$
\eta : \Gamma_{\text{cpt}}(M, E^* \otimes \mathcal{O}(TM) \otimes \Lambda^m(T^*M)) \to \mathbb{R}
$$

$$
\rho^* \otimes \omega \quad \mapsto \quad \int_M \rho^*(\eta) \omega.
$$

Let $\Gamma(E)'$ denote the collection of all continuous linear functionals on $\Gamma_{\text{cpt}}(M, E^* \otimes \mathcal{O}(TM) \otimes \Lambda^m(T^*M))$. Such linear functionals are called distributional sections of $E$.

Let’s suppose for now that the family of twisted Thom forms $\tau_\ast$ exists. (Don’t worry, it does!) Since $\tau_\ast$ is meant to be a global geometrization of the notion
of an approximate point mass in the fibres of $V$ we would hope that there is a
distributional section $[X]$ supported on the submanifold $X$ of $V$ so that

$$[X] = \lim_{s \to 0} \tau_s \text{ in } \Gamma(V, \pi^* \mathcal{O}(V) \otimes \Lambda^n(T^*V))'.$$

In fact we can define $[X]$ as follows.

**Definition 1.15.** Let $\pi : V \to X$ be a real rank $n$ vector bundle over an $m$
dimensional manifold $X$. The current of integration over $X$ is the distributional
section $[X] \in \Gamma(V, \pi^* \mathcal{O}(V) \otimes \Lambda^n(T^*V))'$ defined by

$$[X] : \Gamma(V, \pi^* \mathcal{O}(TX) \otimes \Lambda^m(T^*V)) \to \mathbb{R}$$

$$\pi^*s \otimes \omega \quad \longmapsto \quad \int_X s \otimes i^*\omega.$$  

Here $i : X \hookrightarrow V$ is the inclusion map.

Finally we are in a position to construct a family of twisted Thom forms.
Suppose for now that the bundle $\pi : V \to X$ has even rank $m = 2n$. (The
construction is somewhat different for odd rank bundles.) Endow the bundle $V$
with a Riemannian metric $\langle \cdot, \cdot \rangle$ and metric compatible connection $D_V$. Let $V$
denote the pull back of the bundle $V$ over itself and let $\mu : V \to V$ be the
tautological section. Following Harvey-Lawson [HL,IV] we define the universal
singular push forward connection $\overrightarrow{D}$ on $V$ by the formula

$$\overrightarrow{D}_\nu = D_V \nu - \frac{\langle \nu, \mu \rangle}{|\mu|^2} D_V \mu + \frac{(D_V \nu, \mu)}{|\mu|^2} \mu.$$  

The singular connection $\overrightarrow{D}$ is a metric compatible connection defined on $V$ over
$V \sim X$. Let $\chi(t)$ be a compactly supported approximate one. (i.e. a smooth
function $\chi : [0, \infty] \to [0,1]$ such that $\chi(0) = 0$, $\chi'(t) \geq 0$ and $\chi(t) = 1$ for all
t greater than some $t_0$). Let $\chi_t := \chi(\frac{|\mu|^2}{4\pi^2})$. Then the singular connection $\overrightarrow{D}$ is
smoothed by the family of smooth connections $\overline{D}_s (0 < s \leq \infty)$ on $V$ over $V$ defined by

$$\overline{D}_s \nu = D_V \nu - \chi_s \left( \frac{\nu \mu}{|\mu|^2} \right) D_V \mu + \chi_s \left( \frac{D_V \nu \mu}{|\mu|^2} \right) \mu.$$ 

Note that $\overline{D}_\infty = D_V$.

Let $e$ be a local orthonormal frame for $V$ and define local coordinates $u = (u_1, \ldots, u_n)$ on $V$ by the formula $\mu = u e$. Let $\omega_s$ and $\Omega_s$ denote the corresponding gauge and curvature matricies of the connection $\overline{D}_s$. Note that

(1.16) $e^t \Omega_s e$ is a globally defined section of the bundle $\Lambda^2 V \otimes \Lambda^2 T^* V$ on $V$ and that

(1.17) $\lambda(e) = e_1 \wedge \ldots \wedge e_{2n}$ is a globally defined section of $\mathcal{O}(V) \otimes \Lambda^{2n} V$ on $V$.

Definition 1.18.

(1) The twisted forms $\tau_s$ on tot $V$ defined by

$$\tau_s \lambda(e) = \frac{1}{n!} (-\frac{1}{4\pi} e^t \Omega_s e)^n$$

are called the family of twisted Thom forms associated with the metric connection $D_V$ on $V$. Thanks to (1.16) and (1.17),

$$\tau_s \in \Omega^{2n}_c(V, \mathcal{O}(V)) \quad \text{for} \quad 0 < s < \infty.$$

(2) The form $\tau_\infty =: \chi(D_V) \in \Omega^{2n}(X, \mathcal{O}(V))$ is called the twisted Euler form.

(3) The twisted transgression form $\sigma_s$ on tot $V$ is defined by the equation

$$\sigma_s \lambda = (-1)^n \frac{1}{2^n n! (n-1)! \pi^n} \int_0^\infty (e^t \frac{\partial \omega_s}{\partial s} e)(e^t \Omega_s e)^{n-1} ds.$$ 

Note that $\sigma_s \in \Omega^{2n-1}(V, \mathcal{O}(V))$. 

**Remark 1.19.** The definitions given above can be summarized by saying that the Pfaffian can be regarded as a map from unoriented real rank $2n$ vector bundles $\pi : V \to X$ with metric and metric compatible connection $D_V$ to twisted forms $\text{Pf}(D_V) \in \Omega^{2n}(X, \mathcal{O}(V))$, defined by $\text{Pf}(D_V) \lambda(e) = \frac{1}{n!} \left( \frac{1}{2} e^\dagger \Omega_V e \right)^n$.

The following theorem is proved locally, in exactly the same way as the corresponding theorem of Harvey-Lawson [IV,1.22].

**Theorem 1.20.** The twisted forms $\tau_s$ have the properties of a twisted Thom form in that

1. $\tau_s \in \Omega^{2n}_{c_0}(V, \mathcal{O}(V))$
2. $d\tau_s = 0$
3. $\int_{\pi^{-1}} \tau_s = 1$
4. The restriction to the base manifold $X$ of the twisted Thom form $\tau_s$ is the twisted Euler form $\chi(D_V)$.
5. Also the twisted transgression forms $\sigma_s$ converge in $L^1_{\text{loc}}(V)$ to $\sigma$ and

\[ d\sigma_T = \chi(D_V) - [X] \quad \text{in } \Omega^{2n}(V, \mathcal{O}(V))^{'}, \]

is the limiting form as $s \to 0$ of the equation

\[ d\sigma_s = \chi(D_V) - \tau_s \quad \text{in } \Omega^{2n}(V, \mathcal{O}(V)). \]

6. Therefore

\[ \lim_{s \to 0} \tau_s = [X] \quad \text{in } \Omega^{2n}(V, \mathcal{O}(V))^{'}. \]

7. Furthermore the twisted Thom form $\tau_s$ is given explicitly by

\[ \tau_s \lambda = \frac{1}{n!} \left( \frac{-1}{4\pi} \right)^n (1 - \chi_s) \left( e^\dagger \Omega_V e - 2\chi_s \left( 1 - \frac{\chi_s}{2} \right) \frac{(Du)^2}{|u|^2} \right)^n. \]
\[
+ \frac{2}{(n-1)!} \left( \frac{-1}{4\pi} \right)^n \left( x_s(1-x_s)(1-x_t^s) - \chi^s \frac{|u|^2}{|u|^2} \right) \frac{d|u|^2 (ue)(Due)}{|u|^2} \times \\
\left( e^\Omega v e - 2x_s \left( 1 - \frac{x_t}{2} \right) \frac{(Due)^2}{|u|^2} \right)^{n-1},
\]

and the twisted transgression form \( \sigma_s \) is given by

\[
\sigma_s = \frac{2}{(n-1)!} \left( \frac{-1}{4\pi} \right)^n \frac{(ue)(Due)}{|u|^2} \int_0^{x_s} \left( e^\Omega v e - 2x \left( 1 - \frac{x}{2} \right) \frac{(Due)^2}{|u|^2} \right)^{n-1} dx.
\]

(8) Let \( D_s(V) = \{ \nu \in V : \|\nu\| < s \} \). Then \( \text{spt} \tau_s \subseteq D_s(V) \). The twisted Thom map is given by

\[
i_s : \Omega^k(X, \mathcal{O}(V)) \to \Omega^{k+2n}(V, V \sim D_s(V))
\]

\[
\alpha \otimes \varphi \mapsto \pi^* \alpha \otimes \pi^* \varphi \wedge \tau_s.
\]

This induces the twisted Thom isomorphism,

\[
i_{s!} : H^k(X, \mathcal{O}(V)) \to H^{k+2n}(V, V \sim D_s(V)).
\]

Letting \( s \to 0 \) gives the canonical version of this map

\[
i_0 : \Omega^k(X, \mathcal{O}(V)) \to \Omega^{k+2n}(V, V \sim X)'
\]

\[
\alpha \otimes \varphi \mapsto \alpha \otimes \varphi [X].
\]

**Remark 1.21. Odd rank case.** The odd rank case is treated similarly. The Thom and transgression forms constructed by Harvey and Lawson in the oriented case have twisted counterparts in the unoriented case. In addition the statement of Theorem 1.20 holds in this case. Note though that the Euler form is zero.
Remark 1.22. Complex vector bundles. Let $\pi : F \to X$ be a rank $n$ complex vector bundle over an unoriented manifold $X$ of dimension $m$. Let $F_\mathbb{R}$ denote the underlying real rank $2n$ vector bundle. This bundle possesses a canonical orientation and so the orientation bundle $\mathcal{O}(F_\mathbb{R})$ is trivial. This means that the distributional sections of the bundle $\Lambda^k(T^*F)$ are just the continuous linear functionals on $\Gamma_{\text{cpt}}(\mathcal{O}(TF) \otimes \Lambda^{m+2n-k}(T^*F))$. Note that the Thom isomorphism is the map induced on cohomology by the fibre integration map

$$\int_{\pi^{-1}} : \Omega^{k+2n}_{\text{cu}}(F) \to \Omega^k(X),$$

and that the Thom forms are bona fide differential forms on tot $F$, exactly as in the oriented case.
2. Nonorientable bundles over oriented manifolds.

The material in this section is a reinterpretation and further exploration of the ideas presented in Remark 1.51 of [HL,V]. We begin with some basic facts.

Let \( \pi : V \to X \) be a nonorientable real rank \( n \) vector bundle over an oriented manifold \( X \). Let \( F(V) \to X \) be the principal \( GL(n, \mathbb{R}) \)-bundle whose fibre over a point \( x \in X \) is the set of bases of \( V_x \). The bundle of orientations of \( V \) is the principal \( \mathbb{Z}_2 \)-bundle \( \text{Or}(V) = F(V)/GL^+(n, \mathbb{R}) \), where two bases of \( V_x \) are identified if the matrix transforming one to the other has positive determinant.

We can regard the bundle of orientations as a 2-sheeted covering space over \( X \) which we denote by \( p : \tilde{X} \to X \). The pull back bundle \( \tilde{V} = p^*V \) on \( \tilde{X} \) is oriented by choosing the orientation on \( \tilde{V}_x = V_{p(x)} \) to be the one determined by the point \( \bar{x} \in \tilde{X} \). Note that \( V \) is orientable iff the covering space \( p : \tilde{X} \to X \) is trivial. The deck transformation \( g : \tilde{X} \to \tilde{X} \) lifts to a bundle map,

\[
\begin{align*}
\tilde{V} & \xrightarrow{g^*} \tilde{V} \\
\downarrow & \quad \downarrow \\
\tilde{X} & \xrightarrow{g} \tilde{X}
\end{align*}
\]

which is orientation reversing on the fibres of \( \tilde{V} \).

In the spirit of de Rham [R] a differential form \( \omega \) on \( \tilde{X} \) is called even if \( g^*\omega = \omega \) and odd if \( g^*\omega = -\omega \). Note that every form on \( \tilde{X} \) can be uniquely expressed as the sum of an odd and an even form. There is a 1-1 correspondence between forms on \( X \) and even forms on \( \tilde{X} \) given by the mapping

\[
(2.1) \quad p^* : \Omega^*(X) \longrightarrow \Omega^*_\text{even}(\tilde{X})
\]

\( \omega \mapsto p^*\omega \).

On the other hand the odd forms on \( \tilde{X} \) are in 1-1 correspondence with the \( \mathcal{O}(V) \)-twisted forms on \( X \). To describe this correspondence we make use of the following
description of the orientation bundle \( \mathcal{O}(V) \) of \( V \). Choose a Riemannian metric on the bundle \( \pi : V \to X \). This induces a metric on the line bundle \( \Lambda^n(V) \to X \) which enables us to reduce the structure group of \( \Lambda^n(V) \) from \( \mathbb{R}^* \) to \( O_1 = \{ \pm 1 \} \).

The orientation bundle \( \mathcal{O}(V) \) can be regarded as the line bundle \( \Lambda^n(V) \) with structure group \( O_1 \). For each \( \mathcal{O}(V) \)-twisted form \( \omega \) on \( X \) we construct an odd form \( \tilde{\omega} \) on \( \tilde{X} \) as follows. Fix \( x \in X \) and choose an orthonormal basis \( e(x) = (e_1, \ldots, e_n) \) for \( V_x \). Then we can write

\[
\omega_x = (e_1 \wedge \ldots \wedge e_n) \otimes \psi_x \quad \text{where } \psi_x \in \Lambda^* T_x X.
\]

Let \([e(x)] \in \tilde{X}\) denote the orientation class of \( e(x) \). Define the form \( \tilde{\omega} \) on the fibre of \( \tilde{X} \) over \( x \) by

\[
\tilde{\omega}_[e(x)] := \psi_x
\]

\[
\tilde{\omega}_{e(x)} := -\psi_x.
\]

It is easy to check that \( \tilde{\omega} \) is a well defined smooth odd form on \( \tilde{X} \), and that the mapping

\[
(2.2) \quad \Omega^*(X, \mathcal{O}(V)) \longrightarrow \Omega^*_{\text{odd}}(\tilde{X})
\]

\[
\omega \longmapsto \tilde{\omega}
\]

is a 1-1 correspondence.

Note that the space of \( \mathcal{O}(V) \)-twisted forms on \( X \) can be made into a differential complex as follows. Let \( \omega \in \Omega^*(X, \mathcal{O}(V)) \). Let \( (e_1, \ldots, e_n) \) be any orthonormal local frame for \( V \) and write

\[
\omega = (e_1 \wedge \ldots \wedge e_n) \otimes \psi,
\]

where \( \psi \) is a locally defined smooth form on \( X \). Define

\[
d\omega := (e_1 \wedge \ldots \wedge e_n) \otimes d\psi.
\]
It is easy to check that the differential operator $d$ is well defined and that $d^2 = 0$. Furthermore

$$\tilde{d}\omega = d\tilde{\omega} \quad \text{in } \Omega^*_\text{odd}(X)$$

A current $S$ on $\tilde{X}$ is called even if $g_* S = S$ and odd if $g_* S = -S$. Equivalently a current $S$ is even (odd) iff $\langle S, \omega \rangle = 0$ for all odd (even) forms on $\tilde{X}$. Consequently even (odd) currents on $\tilde{X}$ are precisely the distributional sections $\Omega^*_\text{even}(\tilde{X})'$ ($\Omega^*_\text{odd}(X)'$). Of course every current on $\tilde{X}$ can be uniquely expressed as the sum of an odd and an even current. Just as for forms, the even currents on $\tilde{X}$ are in 1-1 correspondence with currents on $X$, and odd currents on $\tilde{X}$ are in 1-1 correspondence with distributional sections of $\Omega^*(X, \mathcal{O}(V))$. These correspondences are given by the mappings

$$p_* : \Omega^*_\text{even}(\tilde{X})' \longrightarrow \Omega^*(X)'$$

$$S \quad \longrightarrow \quad p_* S$$

where

$$\langle p_* S, \omega \rangle := \frac{1}{2} \langle S, p^* \omega \rangle,$$

and

$$q_* : \Omega^*_\text{odd}(\tilde{X})' \longrightarrow \Omega^*(X, \mathcal{O}(V))'$$

$$S \quad \longrightarrow \quad q_* S$$

where

$$\langle q_* S, \omega \rangle := \frac{1}{2} \langle S, \tilde{\omega} \rangle.$$

It is easy to check that $q_* d = d q_*$. We conclude these preliminary remarks with the following elementary lemma. Note that the orientation on the base manifold $X$ induces a natural orientation
on the manifold $\tilde{X}$ and that the deck transformation $g : \tilde{X} \to \tilde{X}$ preserves the orientation on $\tilde{X}$.

**Lemma 2.5.**

1. Let $\omega \in \Omega^*(\tilde{X}) \subset \Omega^*(\tilde{X})'$. Then

$$g_* \omega = g^* \omega.$$ 

Consequently an even (odd) form on $\tilde{X}$ defines an even (odd) current on $\tilde{X}$.

2a. Let $\omega$ be an $L^1_{\text{loc}}$ form on $X$. Then

$$p_* (p^* \omega) = \omega \quad \text{in } \Omega^*(X)' .$$

2b. Let $\omega$ be an $L^1_{\text{loc}}$ $\mathcal{O}(V)$-twisted form on $X$. Then

$$\tilde{p}_* (\tilde{\omega}) = \omega \quad \text{in } \Omega^*(X, \mathcal{O}(V))'.$$

3. Let $\omega \in \Omega^*(\tilde{X})$ and $S \in \Omega^*(\tilde{X})'$. Then

$$g_* (\omega \wedge S) = g^* \omega \wedge g_* S .$$

4. Let $\omega, \eta$ be $\mathcal{O}(V)$-twisted forms on $X$. Then

$$p^* (\omega \wedge \eta) = \tilde{\omega} \wedge \tilde{\eta} .$$

5. Let $\omega$ be an $\mathcal{O}(V)$-twisted form on $X$ and $S$ an odd current on $\tilde{X}$. Then

$$p_* (\tilde{\omega} \wedge S) = \omega \wedge \tilde{p}_*(S) .$$
Proof. (1) follows from the change of variables formula and the fact that $g^2 = \text{Id}$. (2)-(5) are exercises in the definitions. □

Now we are in a position to define the divisor of an atomic section $\mu$ of a nonorientable bundle $V$ over an oriented manifold $X$. The lift $\tilde{\mu} = \mu \circ p$ of $\mu$ from $V$ to $\tilde{V}$ is also atomic and its divisor $\text{Div}(\tilde{\mu})$ is an odd current on $\tilde{X}$,

\[(2.6) \quad g_* \text{Div}(\tilde{\mu}) = -\text{Div}(\tilde{\mu}).\]

So we define the divisor of $\mu$ by the equation

\[\text{Div}(\mu) = \tilde{p}_* \text{Div}(\tilde{\mu}).\]

Note that $\text{Div}(\mu) \in \Omega^*(X, \mathcal{O}(V))'$. We will discuss the structure of this divisor at the end of this section.

From now on we assume that $\pi : V \to X$ is a real rank $2n$ Riemannian vector bundle with metric compatible connection $D_V$. Let $\mu$ be an atomic section of $V$ and let $\overrightarrow{D}(\mu)$ denote the metric compatible pushforward singular connection on $V$ defined by an atomic section $\mu$ of $V$, [HL,IV]. Let $\phi$ be an $\mathfrak{so}(2n)$-invariant polynomial. Our aim is to make sense of the characteristic equation

\[\phi(D_V) - \phi(D_0) - \text{Res}_\phi(\overrightarrow{D}(\mu)) \text{Div}(\mu) = dT \quad \text{on } X.\]

Recall that each $\mathfrak{so}(2n)$-invariant polynomial is a polynomial in the Pontrjagin polynomials $p_1, \ldots, p_n$ and the Pfaffian $\chi$ and that $\chi^2 = p_n$. So it suffices to consider the following two cases.

**Case 1.** $\phi = \psi(p_1, \ldots, p_n)$, where $\psi(p_1, \ldots, p_n)$ is a polynomial in $p_1, \ldots, p_n$.

**Case 2.** $\phi = \chi \psi(p_1, \ldots, p_n)$
Case 1. Since the invariant polynomials $p_1, \ldots, p_n$ are defined independent of a choice of orientation, the characteristic form $\phi(D_V)$ is a well defined form on $X$ and the $L^1_{loc}$ transgression form $T$ and $L^1_{loc}$ form $\phi(D_0)$ on $X$ are defined as usual. However, since $V$ is a nonorientable bundle over $X$, the residue is a twisted form $\text{Res}_\phi(\vec{D}(\mu)) \in \Omega^*(X, \mathcal{O}(V))$, whose value at a point $x \in X$ is given by

$$\text{Res}_\phi(\vec{D}(\mu))_x := -e_1 \wedge \ldots \wedge e_n \otimes \int_{S(V_x)} T,$$

where $(e_1, \ldots, e_n)$ is an orthonormal basis for $V_x$ and the $e$-sphere $S(V_x)$ is oriented by the basis $(e_1, \ldots, e_n)$. Note that $\text{Res}_\phi(\vec{D}(\mu)) \text{ Div}(\mu) \in \Omega^*(X)^\circ$.

**Proposition 2.7.** Let $\phi = \psi(p_1, \ldots, p_n)$ be a polynomial in the Pontrjagin polynomials $p_1, \ldots, p_n$ and let $\mu$ be an atomic section of a nonorientable vector bundle $V$ over an oriented manifold $X$. Then we have the current equation

$$\phi(D_V) - \phi(D_0) - \text{Res}_\phi(\vec{D}(\mu)) \text{ Div}(\mu) = dT$$

acting on $\Omega^*(X)$.

**Proof.** Let $D_{\vec{V}}$ denote the pull back of the connection $D_V$ on $V$ to $\vec{V}$. The pull back of the singular connection $\vec{D}(\mu)$ on $V$ is just the singular connection $\vec{D}(\vec{\mu})$. Note that $\phi(D_{\vec{V}}) = p^* \phi(D_V)$, $\phi(\vec{D}_0) = p^* \phi(D_0)$ and $T(\vec{\mu}) = p^* T(\mu)$ are even forms and currents and that $\text{Res}_\phi(\vec{D}(\vec{\mu})) = \text{Res}_\phi(\vec{D}(\mu))$ is an odd form. Since $\vec{V} \rightarrow \vec{X}$ is oriented we can apply the results of [HL,IV] to obtain the even current equation

$$\phi(D_{\vec{V}}) - \phi(\vec{D}_0) - \text{Res}_\phi(\vec{D}(\vec{\mu})) \text{ Div}(\vec{\mu}) = dT.$$

Pushing this formula down to $X$ by $p_*$ and invoking Lemma 2.5 proves the proposition. \qed
Case 2. Let $\phi = \chi \psi(p_1, \ldots, p_n)$. As we have seen in Section 1, $\chi(D_V) \in \Omega^*(X, \mathcal{O}(V))$. So $\phi(D_V)$ and $T \in \Omega^*(X, \mathcal{O}(V))'$. Also recall [HL,IV:1.8,2.21] that $\phi(D_0) = 0$ and $\text{Res}_\phi(\overline{D}(\mu)) = \psi(D_V) \in \Omega^*(X)$. Arguing as in the proof of Proposition 2.7 we have

**Proposition 2.8.** Let $\phi = \chi \psi(p_1, \ldots, p_n)$ and let $\mu$ be an atomic section of a nonorientable real rank $2n$ Riemannian vector bundle $V$ over an oriented manifold $X$. Then we have the following equation between distributional sections of $\Omega^*(X, \mathcal{O}(V))$

$$\phi(D_V) - \text{Res}_\phi(\overline{D}(\mu)) \text{ Div}(\mu) = dT,$$

acting on $\mathcal{O}(V)$-twisted forms on $X$.

These ideas also enable us to remove the orientation hypothesis from the real version of the Rectifiable Grothendieck–Riemann–Roch Theorem, [HL: V,3].

**Corollary 2.9.** Let $V$ be an nonorientable real rank $2n$ vector bundle with metric compatible connection over an oriented manifold $X$ and let $E$ be a complex vector bundle with connection $D_E$ over $X$. For each atomic section $\mu$ of $V$ we have the following current equation

$$(2.10)$$

$$\text{ch}(D_{A^\text{even}(V) \otimes E}) - \text{ch}(D_{A^\text{odd}(V) \otimes E}) = (-1)^n \text{ch}(D_E) \chi(D_V) \hat{A}^{-2}(D_V) \text{ Div}(\mu) + dT$$

acting on $\Omega^*(X)$, where $T$ is the $L^1_{\text{loc}}$ form $T = \text{ch}(D_E) \chi(D_V) \hat{A}^{-2}(D_V) \mu^*(\sigma)$.

**Proof.** Let $\tilde{E} = p^*E$ and note that $\hat{A}(D_{\tilde{V}}) = p^*\hat{A}(D_V)$. Up on $\tilde{X}$ we have the equation of even currents [HL: V,3.43],

$$\text{ch}(D_{A^\text{even}(\tilde{V}) \otimes \tilde{E}}) - \text{ch}(D_{A^\text{odd}(\tilde{V}) \otimes \tilde{E}}) = (-1)^n \text{ch}(D_{\tilde{E}}) \chi(D_{\tilde{V}}) \hat{A}^{-2}(D_{\tilde{V}}) \text{ Div}(\mu) + dT$$

which pushes down to (2.10) via $p_*$. □
Finally we turn to the question of the structure of the divisor, \( \text{Div}(\mu) \in \Omega^*(X, \mathcal{O}(V))' \). Of course all of the beautiful results of [HS: 4] apply to the divisor of the pull back section \( \tilde{\mu} \) of \( \tilde{V} \). The question is what else can be said about \( \text{Div}(\mu) \)? For instance how is it related to the current of integration over the regular points of the zero set \( Z \) of \( \mu \), and can it have support on the nonorientable components of \( Z \)?

Let

\[
\text{Reg} \ Z = \{ x \in X \mid Z \text{ is a codimension } n \text{ Lipschitz submanifold near } x \} 
\]

denote the set of regular points of \( Z \) and

\[
(2.11) \quad \text{Sing} \ Z = Z \sim \text{Reg} \ Z
\]

denote the set of singular points. Let \( \tilde{Z} = p^{-1}(Z) = \text{Zero}(\tilde{\mu}) \). Obviously \( \text{Reg} \tilde{Z} = p^{-1}(\text{Reg} \ Z) \) and \( \text{Sing} \tilde{Z} = p^{-1}(\text{Sing} \ Z) \). Let \( \{ \tilde{Z}_j \} \) denote the family of connected components of \( \text{Reg} \tilde{Z} \). By [HS,IV:4.3] we know that there are integers \( n_j \in \mathbb{Z} \) so that

\[
\text{Div}(\tilde{\mu}) = \sum n_j [\tilde{Z}_j] \quad \text{on } X \sim \text{Sing} \ Z.
\]

If \( n_j = 0 \) then \( \tilde{Z}_j \) need not be orientable and if \( n_j \neq 0 \) we can choose the orientation on \( \tilde{Z}_j \) so that \( n_j > 0 \).

Now suppose that \( U \) is an open subset of \( X \) such that \( V \to U \) is an orientable bundle. This orientation gives us a way to identify \( \Omega^*(U, \mathcal{O}(V)) \) with \( \Omega^*(U) \). Under this identification the divisor \( \text{Div}(\mu) \in \Omega^*(U, \mathcal{O}(V))' \) is identified with the usual divisor of the section \( \mu \) of the oriented bundle \( V \to U \), and all the results of [HS: 4] apply. From now on let's suppose - for simplicity of exposition - that \( \text{Sing} \ Z = \emptyset \) and that \( Z = \text{Reg} \ Z \) is connected.
Proposition 2.12. Suppose that \( Z = \text{Zero}(\mu) \) is a connected regular submanifold of \( X \). Suppose that \( Z \) is orientable. Then either \( \text{Div}(\mu) = 0 \) or

there is a neighbourhood \( U \) of \( Z \) in \( X \) so that \( V \to U \) is orientable.

Furthermore, for either choice of orientation on \( V \) over \( U \), there is an integer \( n > 0 \) such that

\[
(2.13) \quad \text{Div}(\mu) = n[Z]
\]

under the identification between \( \Omega^*(U, \mathcal{O}(V))' \) and \( \Omega^*(U)' \). Note that if we choose the opposite orientation on \( V \) over \( U \) the orientation induced on \( Z \) by (2.13) changes.

Proof. Either choice of orientation on \( Z \) induces an orientation on the submanifold \( \tilde{Z} = p^{-1}(Z) \subset \tilde{X} \) with respect to which the deck transformation \( g : \tilde{X} \to \tilde{X} \) is orientation preserving. Suppose for the sake of contradiction that \( \tilde{Z} \) is connected.

Now since \( \text{Div}(\mu) \neq 0 \) there is a non-zero integer \( n \) such that

\[
\text{Div}(\mu) = n[\tilde{Z}].
\]

On the one hand we know that

\[
(2.14) \quad g_* \text{Div}(\mu) = -\text{Div}(\mu)
\]

and on the other, by the change of variables formula

\[
(2.15) \quad g_* [\tilde{Z}] = [\tilde{Z}],
\]

since \( g \) is orientation preserving on \( \tilde{Z} \). Taken together (2.14) and (2.15) imply that \( n = 0 \), a contradiction. Consequently \( \tilde{Z} \) must have two connected components which implies that the bundle \( V \to Z \) is orientable. Finally we can use the tubular neighbourhood theorem to find an open set \( U \subset X \) containing \( Z \) so that \( V \) is orientable over \( U \). \( \square \)
Lemma 2.16. Let $V$ be a nonorientable bundle over a manifold $X$ and let $p : \tilde{X} \to X$ be the bundle of orientations of $V$. Let $Z$ be a nonorientable submanifold of $X$ and let $\tilde{Z} = p^{-1}(Z)$. Then

$$\tilde{Z} \text{ is orientable} \iff \mathcal{O}(TZ) = \mathcal{O}(V) \mid _Z.$$

Proof. If $\tilde{Z}$ is orientable then the principal $\mathbb{Z}_2$-bundles of orientations of $V$ and $TZ$ would be equal,

$$(2.17) \quad \text{Or}(TZ) = \text{Or}(V) \mid _Z.$$  

Now for any vector bundle $E$ over $X$, $\mathcal{O}(E) = \text{Or}(E) \times_\rho \mathbb{R}$ where $\rho : \mathbb{Z}_2 \to \mathbb{R}^*$ is the natural inclusion homomorphism. (See [LM] for the definition of the associated fibre bundle $P \times_\rho F$ of a principal bundle $P$.) Therefore (2.17) forces $\mathcal{O}(TZ) = \mathcal{O}(V) \mid _Z$. The converse is even easier. $\square$

Proposition 2.18. Suppose that $Z = \text{Zero}(\mu)$ is a nonorientable connected regular submanifold of $X$. Then either $\text{Div}(\mu) = 0$ or

$$\mathcal{O}(TZ) = \mathcal{O}(V) \mid _Z$$

and there is an integer $n$ so that

$$(2.19) \quad \text{Div}(\mu) = n[Z] \quad \text{in } \Omega^*(X, \mathcal{O}(V))',$$

where the current $[Z]$ of integration of $\mathcal{O}(V)$-twisted forms over $Z$ is defined as in (1.15).

Proof. If $\tilde{Z}$ is nonorientable then $\text{Div}(\tilde{\mu}) = 0$ and so $\text{Div}(\mu) = 0$. On the other hand if $\tilde{Z}$ is orientable then by Lemma 2.16 $\mathcal{O}(TZ) = \mathcal{O}(V) \mid _Z$. (2.19) follows by pushing the odd current equation $\text{Div}(\tilde{\mu}) = n[\tilde{Z}]$ on $\tilde{X}$ down to $X$ via $\tilde{\nu}_*$ and checking that $[Z] = \tilde{\nu}_*[\tilde{Z}]$ (independent of choice of orientation on $\tilde{Z}$). $\square$
Example 2.20. Let $X$ be a nonorientable manifold and consider the pull back of the tangent bundle over itself:

\[
\begin{array}{ccc}
TX & TX & \\
\downarrow & \downarrow & \\
TX & \xrightarrow{\pi} & X.
\end{array}
\]

Recall that the total space of the tangent bundle is an oriented manifold (Lemma 1.12). Then it is straightforward to verify that the divisor of the tautological section $c$ of $TX$ is

\[
\text{Div}(c) = [X] \quad \text{in } \Omega^*(TX, \mathcal{O}(TX))',
\]

where $[X]$ is defined by (1.15).

Example 2.21. Let $L \to \mathbb{P}(\mathbb{R}^4)$ be the tautological line bundle over the oriented real projective space $\mathbb{P}(\mathbb{R}^4)$. Being non-trivial the line bundle $L$ is nonorientable. Recall that sections of $L$ are in 1-1 correspondence with homogeneous functions of degree $-1$ on $\mathbb{R}^4$. Let $\mu$ be the atomic section of $L$ defined by the function

\[
u(x_1, x_2, x_3, x_4) = \frac{x_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}.
\]

The zero set of $\mu$ is the nonorientable connected submanifold $\mathbb{P}(\mathbb{R}^3)$ of $\mathbb{P}(\mathbb{R}^4)$. Under the natural projection map,

\[p : S^3 \to \mathbb{P}(\mathbb{R}^4),\]

the line bundle pulls back to the normal bundle to $S^3$ in $\mathbb{R}^4$, which is trivial and thus orientable. The section $\mu$ pulls back to the function

\[
\tilde{\mu}(x_1, x_2, x_3, x_4) = x_4 \quad \text{on } S^3,
\]
which has non-zero divisor \( \text{Div}(\mu) = [S^2] \). So by the proof of Proposition 2.18,

\[
\mathcal{O}(\text{T}P^3) = \mathcal{O}(L) \quad \text{on } \mathcal{P}(\mathbb{R}^3)
\]

and

\[
\text{Div}(\mu) = [\mathcal{P}(\mathbb{R}^3)] \quad \text{in } \Omega^*(\mathcal{P}(\mathbb{R}^4), \mathcal{O}(L))'.
\]
3. Oriented bundles over nonorientable manifolds.

In this section we define the divisor of a section $\mu$ of an oriented vector bundle over a nonorientable manifold $X$, discuss its structure and analyse the characteristic currents of the push forward singular connection induced on $V$ by $\mu$. As we shall see there are close analogies between the material in this section and that of Section 2.

Let $p : \tilde{X} \to X$ be the oriented double cover of the nonorientable manifold $X$. Most of the general discussion presented at the beginning of Section 2 also holds when $V = TX$ is the tangent bundle. Note that $T\tilde{X} = \tilde{T}{X}$ and that the deck transformation $g$ on $\tilde{X}$ is orientation reversing. Therefore for each form $\omega$ on $X$ we have

\begin{equation}
\tag{3.1}
g_\ast \omega = -g^* \omega.
\end{equation}

Consequently even (odd) forms on $\tilde{X}$ define odd (even) currents. Put another way, smooth forms downstairs on $X$ define distributional sections of $T^*X$ (i.e. elements of $\Omega^*(X)'$) by integration against $\mathcal{O}(TX)$-twisted forms.

Let $f : X \to \mathbb{R}^n$ be an atomic function on $X$ and let $\tilde{f} = f \circ p$ be the pull back of $f$ to $\tilde{X}$. Thanks to (3.1) the divisor of $\tilde{f}$ is an odd current on $\tilde{X}$,

$$g_\ast \text{Div}(\tilde{f}) = -\text{Div}(\tilde{f}).$$

We define the divisor of the function $f$ on $X$ by

$$\text{Div}(f) := p_\ast(\text{Div}(\tilde{f})) \in \Omega^*(X)'.$$

Note that Div($f$) acts on $\Omega^*(X, \mathcal{O}(TX))$.

Now let $V$ be an oriented rank $n$ vector bundle over $X$, and let $\mu$ be an atomic section of $V$. Let $\tilde{V}$ and $\tilde{\mu}$ denote the pull backs of $V$ and $\mu$ over the oriented
manifold $\tilde{X}$. Note that $\tilde{V}$ inherits an orientation from $V$. Just as in the trivial case the divisor of $\mu$ is defined by

$$\text{Div}(\mu) := \tilde{p}_*(\text{Div}(\mu)) \in \Omega^*(X)'.$$ 

Let $D_V$ be a connection on $\pi : V \to X$ and let $\overrightarrow{D}(\mu)$ be the pushforward singular connection on $V$ induced by $\mu$. Let $\phi$ be any invariant polynomial. Note that the residue form $\text{Res}_\phi(\overrightarrow{D}(\mu))$ is a smooth form on $X$ and that the current $\text{Res}_\phi(\overrightarrow{D}(\mu)) \text{Div}(\mu)$ acts on $\mathcal{O}(TX)$-twisted forms on $X$. The following proposition is proved just like Proposition 2.7.

**Proposition 3.2.** Let $\mu$ be an atomic section of an oriented bundle $V$ over a nonorientable manifold $X$. Then

$$\phi(D_V) - \phi(D_0) - \text{Res}_\phi(\overrightarrow{D}(\mu)) \text{Div}(\mu) = dT$$

in $\Omega^*(X)'$, acting on $\mathcal{O}(TX)$-twisted forms on $X$.

We conclude with a brief discussion on the structure of the divisor of an atomic function on a nonorientable manifold.

**Proposition 3.3.** Let $f : X \to \mathbb{R}^n$ be an atomic function on a nonorientable manifold $X$. The divisor of $f$, $\text{Div}(f) \in \Omega^*(X)'$, has the following properties.

1. Let $U$ be an orientable open subset of $X$. Choose an orientation on $U$. Then under the induced identification between $\Omega^*(U, \mathcal{O}(TX))'$ and $\Omega^*(U)'$ the restriction of $\text{Div}(f)$ to $U$ is the usual divisor of the atomic function $f : U \to \mathbb{R}^n$.

2. Suppose that $Z = \text{Zero}(f)$ is a connected regular submanifold of $X$.
   (a) If $Z$ is orientable then either $\text{Div}(f) = 0$ or there is an orientable
neighbourhood $U$ of $Z$ in $X$. Furthermore for each choice of orientation of $U$ there is an integer $n > 0$ such that

$$\text{Div}(f) = n[Z]$$

under the identification of $\Omega^*(U, \mathcal{O}(TX))'$ with $\Omega^*(U)'$.
(b) If $Z$ is nonorientable then either $\text{Div}(f) = 0$ or

$$\mathcal{O}(TZ) = \mathcal{O}(TX) \big|_Z$$

and there is an integer $n$ so that

$$\text{Div}(f) = n[Z] \quad \text{in } \Omega^*(X)'$$

acting on $\mathcal{O}(TX)$-twisted forms.

Proof. The proof of (1) is trivial. Let $V := TX$. Then the proofs of (2a) and (2b) are identical to those of Propositions 2.12 and 2.18. $\square$

Example 3.4. Let $V$ be an oriented bundle over a nonorientable manifold $X$. Then the tautological section $c$ of the pull back of $V$ over itself has divisor

$$\text{Div}(c) = [X] \quad \text{in } \Omega^*(V, \mathcal{O}(TX))'.$$
4. Nonorientable bundles over nonorientable manifolds.

Let $V$ be a nonorientable Riemannian vector bundle over a nonorientable manifold $X$. Suppose for now that $\mathcal{O}(V) = \mathcal{O}(TX)$. Let $\mu$ be an atomic section of $V$ over $X$ and let $\tilde{D}(\mu)$ be the metric compatible push forward singular connection induced by $\mu$. Let $p: \tilde{X} \rightarrow X$ be the oriented double cover of $X$. Since $\mathcal{O}(V) = \mathcal{O}(TX)$ the pull back bundle $\tilde{V} = p^*V$ is also orientable, and since the deck transformation $g: \tilde{X} \rightarrow \tilde{X}$ reverses the orientation on both $X$ and $V$ we have that

$$g_*\omega = -g^*\omega$$

for all forms $\omega \in \Omega^*(X) \subset \Omega^*(X)'$, and that

$$g^*\chi(D_{\tilde{V}}) = -\chi(D_{\tilde{V}}).$$

Consequently the divisor of the pull back section $\tilde{\mu}$ is an even current,

$$g_* \text{Div}(\tilde{\mu}) = \text{Div}(\tilde{\mu}).$$

So we can define the divisor of $\mu$ to be the usual current push forward (with a factor of a half) of the divisor of $\tilde{\mu}$,

$$\text{Div}(\mu) := p_* \text{Div}(\tilde{\mu}).$$

Note that $\text{Div}(\mu) \in \Omega^*(X, \mathcal{O}(TX))'$ acts on bonafide forms on $X$. The following proposition is proved just like Propositions 2.7 and 2.8.

**Proposition 4.1.** Let $\mu$ be an atomic section of $\pi: V \rightarrow X$ with $\mathcal{O}(V) = \mathcal{O}(TX)$ non-trivial. Then

$$(1) \quad \chi(D_{\tilde{V}}) - \text{Div}(\mu) = d\mu^*\sigma \quad \text{in} \ \Omega^*(X, \mathcal{O}(TX))'$$
acting on forms on $X$.

(2) Let $\phi = \psi(p_1, \ldots, p_n)$ be a polynomial in the Pontrjagin polynomials. Then

$$\text{Res}_\phi(\tilde{D}(\mu)) \in \Omega^*(X, \mathcal{O}(TX))$$

and

$$\phi(D_V) - \phi(D_0) - \text{Res}_\phi(\tilde{D}(\mu)) \text{ Div}(\mu) = \varpi^*T \quad \text{in } \Omega^*(X)'$$

acting on $\mathcal{O}(TX)$-twisted forms on $X$.

(3) Let $\phi = \chi \psi(p_1, \ldots, p_n)$. Then

$$\text{Res}_\phi(\tilde{D}(\mu)) = \psi(D_V) \in \Omega^*(X)$$

and

$$\phi(D_V) - \text{Res}_\phi(\tilde{D}(\mu)) \text{ Div}(\mu) = \varpi^*T \quad \text{in } \Omega^*(X, \mathcal{O}(TX))'$$

acting on forms on $X$.

The following result concerning the structure of the divisor should be compared to the results on the structure of the divisor presented in sections 2 and 3.

**Proposition 4.2.** Let $\mu$ be an atomic section of a nonorientable bundle $V$ over a nonorientable manifold and suppose that $\mathcal{O}(V) = \mathcal{O}(TX)$. Let $Z = \text{Zero}(\mu)$ and let $\{Z_j\}$ be the family of connected components of $\text{Reg} Z$ which are orientable. Then there are integers $n_j \geq 0$ so that

$$\text{Div}(\mu) = \sum n_j [Z_j] \quad \text{in } \Omega^*(X \sim \text{Sing} Z, \mathcal{O}(TX))'$$

acting on forms on $X \sim \text{Sing} Z$. In particular the divisor does not have support on any of the nonorientable components of $Z$.

**Proof.** Without loss of generality $Z$ is a connected regular submanifold of $X$. First suppose that $Z$ is oriented. Then $\tilde{Z} = \text{Zero}(\tilde{\mu})$ is also oriented. If $\tilde{Z}$ is
disconnected then there is an orientable open subset $U$ of $X$ containing $Z$ so that $V \to U$ is oriented and the standard theory shows that $\text{Div}(\mu) = n[Z]$ acting on forms with support in $U$. On the other hand if $\widetilde{Z}$ is connected then $\text{Div}(\widetilde{\mu}) = n[\widetilde{Z}]$ for some integer $n$ and since $[Z] = p_*[\widetilde{Z}]$ it follows that $\text{Div}(\mu) = n[Z]$ too.

Next suppose that $Z$ is nonorientable. We need to show that $\text{Div}(\mu) = 0$. Suppose, for contradiction, that $\widetilde{Z}$ is orientable. Then it must be connected which implies that $\text{Div}(\widetilde{\mu}) = n[\widetilde{Z}]$ for some integer $n$ which we can assume to be nonzero. Since $\text{Div}(\widetilde{\mu})$ is an even current we must have that $g_*[\widetilde{Z}] = [\widetilde{Z}]$ which forces $Z$ to be oriented, a contradiction. Thus $\widetilde{Z}$ is nonorientable and so $\text{Div}(\widetilde{\mu}) = 0$ which forces $\text{Div}(\mu) = 0$, as required. \hfill \Box

We conclude this discussion with the two basic examples of bundles $\pi : V \to X$ with $\mathcal{O}(V) = \mathcal{O}(TX)$, non-trivial.

**Example 4.3.** Let $V$ be a nonorientable bundle over an oriented manifold $X$ and let $V$ be the pull back of $V$ over itself. Then by Lemma 1.12, $\mathcal{O}(TV) = \pi^* \mathcal{O}(V) = \mathcal{O}(V)$, and the divisor of the tautological section $c : V \to V$ is given by

$$\text{Div}(c) = [X]$$

acting on smooth forms on $V$.

**Example 4.4.** Let $X$ be a nonorientable compact manifold without boundary and consider the tangent bundle $TX \to X$. Let $\mu$ be an atomic vector field on $X$ with zero set $\{p_1, \ldots, p_N\}$. By Proposition 4.2 there are integers $n_j$ such that

$$\text{Div}(\mu) = \sum_{j=1}^{N} n_j [p_j]$$
acting on smooth functions on $X$. Note that the integer $n_j$ is the well defined
multiplicity of vanishing of the vector field $\mu$ at $p_j$. We can pair the equation

$$\chi(D_{TX}) - \text{Div}(\mu) = d\mu^*(\sigma)$$

with the constant function 1 on $X$ to obtain the classical Poincaré-Hopf theorem,

$$\chi(X) = \sum_{j=1}^{N} n_j.$$ 

Recall that the Euler characteristic $\chi(X)$ of $X$ is half that of its oriented double cover.

Finally we come to the last of the four cases described in the introduction to
this chapter. Let $V$ be a nonorientable bundle over a nonorientable manifold $X$
and suppose that $\mathcal{O}(V) \neq \mathcal{O}(TX)$. Thanks to Lemma 2.16 we know that the pull
back bundle $\tilde{V} = p^*V$ over $\tilde{X}$ is nonorientable.

Let $g : \tilde{X} \to \tilde{X}$ be the deck transformation. Note that $g$ reverses the orientation
on $X$. Our next task is to define a natural mapping,

$$g^* : \Omega^*(\tilde{X}, \mathcal{O}(\tilde{V})) \to \Omega^*(\tilde{X}, \mathcal{O}(\tilde{V})).$$

First note that $\mathcal{O}(\tilde{V}) = p^*\mathcal{O}(V)$. Let $\omega \in \Omega^*(\tilde{X}, \mathcal{O}(\tilde{V}))$ and fix $\tilde{x} \in \tilde{X}$. We define
$g^*(\omega)_{g(\tilde{x})}$ as follows. Let $(e_1, \ldots, e_n)$ be an orthonormal basis for $V_{p(\tilde{x})}$. Since
$\mathcal{O}(\tilde{V})_{\tilde{x}} = \mathcal{O}(\tilde{V})_{g(\tilde{x})} = \mathcal{O}(V)_{p(\tilde{x})}$ we can write

$$\omega_{\tilde{x}} = e_1 \wedge \ldots \wedge e_n \otimes \psi_{\tilde{x}}$$

where $\psi_{\tilde{x}} \in \Lambda^* T_{\tilde{x}} \tilde{X}$ and define

$$\omega_{g(\tilde{x})} := e_1 \wedge \ldots \wedge e_n \otimes g^* \psi_{\tilde{x}}.$$
An $\mathcal{O}(\tilde{V})$-twisted form $\omega$ on $\tilde{X}$ is called even (odd) if $g^*\omega = \omega$ ($g^*\omega = -\omega$).

Just as in Section 1 there are natural 1-1 correspondences

$$p^*: \Omega^*(X, \mathcal{O}(V)) \longrightarrow \Omega^*_{\text{even}}(\tilde{X}, \mathcal{O}(\tilde{V}))$$

and

$$\Omega^*(X, \mathcal{O}(V) \otimes \mathcal{O}(TX)) \longrightarrow \Omega^*_{\text{odd}}(\tilde{X}, \mathcal{O}(\tilde{V})).$$

Also note that since $g$ reverses the orientation on $X$,

(4.5) \quad \quad g^*\omega = -g^*\omega \quad \text{in} \quad \Omega^*(\tilde{X}, \mathcal{O}(\tilde{V}))'$$

for any $\mathcal{O}(\tilde{V})$-twisted form $\omega$.

Now let $\tilde{\mu} = \mu \circ p$ be the pull back section. By the results of Section 2,

$$\text{Div}(\tilde{\mu}) \in \Omega^*(\tilde{X}, \mathcal{O}(\tilde{V}))',$$

and thanks to (4.5),

$$g_* \text{Div}(\tilde{\mu}) = - \text{Div}(\tilde{\mu}).$$

So we define the divisor $\text{Div}(\mu) \in \Omega^*(X, \mathcal{O}(V))'$ by

$$\langle \text{Div}(\mu), \omega \rangle := \frac{1}{2} \langle \text{Div}(\tilde{\mu}), \tilde{\omega} \rangle$$

acting on $\mathcal{O}(V) \otimes \mathcal{O}(TX)$-twisted forms, $\omega$.

**Proposition 4.6.** Let $\mu$ be an atomic section of a nonorientable Riemannian vector bundle $V$ with metric compatible connection $D_V$ over a nonorientable manifold $X$. Suppose that $\mathcal{O}(V) \neq \mathcal{O}(TX)$. Let $\overrightarrow{D}(\mu)$ be the metric compatible singular push forward connection induced on $V$ by $\mu$. Then

(1) \quad \quad \chi(D_V) - \text{Div}(\mu) = d\mu^* \sigma \quad \text{in} \quad \Omega^*(X, \mathcal{O}(V))'$
acting on $\mathcal{O}(V) \otimes \mathcal{O}(TX)$-twisted forms.

(2) Let $\phi = \psi(p_1, \ldots, p_n)$ be a polynomial in the Pontrjagin polynomials. Then

$$\text{Res}_\phi(\overrightarrow{D}(\mu)) \in \Omega^*(X, \mathcal{O}(V))$$

and

$$\phi(D_V) - \phi(D_0) - \text{Res}_\phi(\overrightarrow{D}(\mu)) \text{Div}(\mu) = d\mu^*T \quad \text{in } \Omega^*(X)$$

acting on $\Omega^*(X, \mathcal{O}(TX))$.

(3) Let $\phi = \chi \psi(p_1, \ldots, p_n)$. Then

$$\text{Res}_\phi(\overrightarrow{D}(\mu)) = \psi(D_V) \in \Omega^*(X)$$

and

$$\phi(D_V) - \text{Res}_\phi(\overrightarrow{D}(\mu)) \text{Div}(\mu) = d\mu^*T \quad \text{in } \Omega^*(X, \mathcal{O}(V))$$

acting on $\mathcal{O}(V) \otimes \mathcal{O}(TX)$-twisted forms.

Proof. Push the current formulae on $\tilde{X} \times X$ given in Propositions 2.7 and 2.8 down to $X$ by the appropriate map. □

**Proposition 4.7.** Let $\mu$ be an atomic section of $\pi : V \to X$ with $\mathcal{O}(V) \neq \mathcal{O}(TX)$, both non-trivial. Then

1. Let $U$ be an orientable open subset of $X$ over which $V$ is also orientable.

   Choose orientations on $U$ and $V \to U$. Then under the induced identification between $\Omega^*(U, \mathcal{O}(V) \otimes \mathcal{O}(TX))$ and $\Omega^*(U)$ the restriction of the divisor of $\mu$ to $U$ is the usual divisor of the atomic section $\mu$ of $V \to U$.

2. Suppose that $Z = \text{Zero}(\mu)$ is an orientable connected regular submanifold of $X$. 
Then either

(a) $\operatorname{Div}(\mu) = 0$

or

(b) There is an orientable neighbourhood $U$ of $Z$ in $X$ such that $V \to U$ is also oriented. Also for each choice of orientation on $U$ and $V \to U$ there is an integer $n > 0$ so that

$$\operatorname{Div}(\mu) = n[Z] \quad \text{on } U$$

under the identification between $\Omega^*(U, \mathcal{O}(V) \otimes \mathcal{O}(TX))'$ and $\Omega^*(U)'$.

or

(c) There is a neighbourhood $U$ of $Z$ in $X$ so that $\mathcal{O}(V) = \mathcal{O}(TU)$ on $U$ and so

$$\operatorname{Div}(\mu) = n[Z] \quad \text{in } \Omega^*(U, \mathcal{O}(TU))'$$

acting on forms on $U$.

\textbf{Proof.} Suppose $\operatorname{Div}(\mu) \neq 0$. Now $\tilde{Z} = \text{Zero}(\tilde{\mu})$ is orientable. If $\tilde{Z}$ is disconnected then $TX \to Z$ is orientable and so by the tubular neighbourhood theorem we can choose an oriented open subset $U$ of $X$ containing $Z$. Then $V \to U$ is an unoriented bundle over an oriented manifold and we can use Proposition 2.18 to conclude that (b) is true.

On the other hand if $\tilde{Z}$ is connected then by Proposition 2.12 there is an oriented neighbourhood $\tilde{U}$ of $\tilde{Z}$ in $\tilde{X}$ so that $\tilde{V} \to \tilde{U}$ is oriented. Let $U$ be the image of $\tilde{U}$ in $X$. If $U$ is orientable we can apply Proposition 2.12 to conclude that (b) is true, while if $U$ is nonorientable we can use Lemma 2.16 to show that $\mathcal{O}(V) = \mathcal{O}(TU)$ on $U$ and so we can apply Proposition 4.2 to conclude that (c) is true. $\Box$
Proposition 4.10. Let $\mu$ be an atomic section of $\pi : V \to X$ with $\mathcal{O}(V) \neq \mathcal{O}(TX)$, both nontrivial and suppose that $Z = \text{Zero}(\mu)$ is a nonorientable connected regular submanifold of $X$. Then either $\text{Div}(\mu) = 0$ or

$$\mathcal{O}(TZ) = \mathcal{O}(V) \otimes \mathcal{O}(TX) \big|_Z$$

and there is an integer $n$ such that

$$\text{Div}(\mu) = n[Z] \quad \text{in } \Omega^*(X, \mathcal{O}(V))^\prime$$

acting on $\mathcal{O}(V) \otimes \mathcal{O}(TX)$-twisted forms on $X$.

Example 4.11. Let $\pi : V \to X$ be a nonorientable bundle over a nonorientable manifold $X$ and suppose that $\mathcal{O}(TX) \neq \mathcal{O}(V)$. Let $V$ be the pull back of $V$ over itself. Note that $\mathcal{O}(V) \neq \mathcal{O}(TV)$. A fiddly little argument (much like that given below) shows that the divisor of the tautological section $c : V \to V$ is given by

$$\text{Div}(c) = [X] \quad \text{in } \Omega^*(V, \mathcal{O}(V))^\prime$$

acting on $\mathcal{O}(V) \otimes \mathcal{O}(TV)$-twisted forms (since $\mathcal{O}(V) \otimes \mathcal{O}(TV) = \pi^*\mathcal{O}(TX)$). In particular the support of the divisor, $\text{Div}(c)$ is nonorientable. Note that this universal setting is precisely the one dealt with in Section 1.

Proof of Proposition 4.10. First suppose that $\widetilde{Z}$ is orientable. Then $\widetilde{Z}$ must be connected and by Proposition 2.12 either $\text{Div}(\widetilde{\mu}) = 0$ (which implies that $\text{Div}(\mu) = 0$) or there is an oriented neighbourhood $\widetilde{U}$ of $\widetilde{Z}$ in $\widetilde{X}$ so that $\widetilde{V} \to \widetilde{U}$ can be oriented and $\text{Div}(\widetilde{\mu}) = n[\widetilde{Z}]$ acting on $\Omega^*(U, \mathcal{O}(\widetilde{V})) \cong \Omega^*(\widetilde{U})$. Let $U$ be the image of $\widetilde{U}$ in $X$. If $U$ is nonorientable then Lemma 2.16 forces $\mathcal{O}(V) = \mathcal{O}(TU)$ on $U$ and so by Proposition 4.2, $\text{Div}(\mu) = 0$ since $Z \subset U$ is nonorientable. On the other
hand if $U$ is orientable then $V \to U$ is an unoriented bundle over an orientable
manifold and we can apply Proposition 2.18 to show that either $\text{Div}(\mu) = 0$ or
$\mathcal{O}(TZ) = \mathcal{O}(V) |_Z$ and $\text{Div}(\mu) = n[Z]$ in $\Omega^*(U, \mathcal{O}(V))'$, acting on $\mathcal{O}(V)$-twisted
forms on $U$. But this implies the result of the proposition since $\mathcal{O}(TU) = \mathbb{R}$.

Secondly we suppose that $\tilde{Z}$ is nonorientable. By Proposition 2.18 either
$\text{Div}(\tilde{\mu}) = 0$ (which implies that $\text{Div}(\mu) = 0$) or $\mathcal{O}(T\tilde{Z}) = \mathcal{O}(\tilde{V}) |_{\tilde{Z}}$ and

$$\text{(4.12)} \quad \text{Div}(\tilde{\mu}) = n[\tilde{Z}]$$

acting on $\mathcal{O}(\tilde{V})$-twisted forms on $\tilde{X}$. Then by Lemma 4.13 below either $\mathcal{O}(TZ) = \mathcal{O}(V) |_Z$ or $\mathcal{O}(TZ) = \mathcal{O}(V) \otimes \mathcal{O}(TX) |_Z$. In the second case the current $[Z] \in \Omega^*(X, \mathcal{O}(V))'$ is defined as in (1.15) and $[Z] = \tilde{p}_*[\tilde{Z}]$ is easy to check. Applying
$\tilde{p}_*$ to both sides of (4.12) yields the required result. Finally we treat the case
$\mathcal{O}(TZ) = \mathcal{O}(V) |_Z$. Let $q : X' \to X$ be the double cover corresponding to the
bundle of orientations $\text{Or}(V)$ over $X$ and let $V' = q^*(V)$ be the oriented pull
back of $V$ over $X'$. Then by Lemma 2.16 $Z' = q^{-1}(Z)$ is orientable and therefore
connected. So by Proposition 3.3 either $\text{Div}(\mu') = 0$ (which forces $\text{Div}(\mu) = 0$)
or there is an oriented neighbourhood $U'$ of $Z'$ in $X'$ so that $\text{Div}(\mu') = n[Z']$
in $\Omega^*(U', \mathcal{O}(TX'))' \cong \Omega^*(U')'$. Let $U$ be the image of $U'$ in $X$. The same
argument given above in the case that $\tilde{Z}$ is orientable finishes the proof of the
proposition. □

Lemma 4.13.

(1) Let $f : M \to N$ be a smooth map between manifolds and let $V$ be any
vector bundle defined over $N$. Then

$$\mathcal{O}(f^*V) = f^*(\mathcal{O}(V)).$$

Note that the triviality of $\mathcal{O}(f^*V)$ does not imply that of $\mathcal{O}(V)$. 
(2) Let \( p : \tilde{X} \to X \) be the oriented double cover of a nonorientable manifold \( X \) and let \( L_1 \) and \( L_2 \) be two line bundles with structure group \( O_1 \) on \( X \). Suppose that
\[
p^*L_1 = p^*L_2 \quad \text{on} \quad \tilde{X}.
\]
Then either
\[
L_1 = L_2 \quad \text{or} \quad L_1 = L_2 \otimes O(TX).
\]

Proof.

(1) This follows from the fact that \( O(V) \) is just \( \text{Det}(V) \) with structure group reduced to \( O_1 \).

(2) Since the \( O_1 \)-line bundles form a group with respect to the tensor product it suffices to prove that if \( L \) is nontrivial and \( p^*L \) is trivial then \( L = O(TX) \). But this follows immediately from Lemma 2.16 and the fact that \( L = O(L) \). \( \square \)
Bibliography


