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Experimental investigation and constitutive modeling of marine clay

Yu, Shinyuan, Ph.D.
Rice University, 1993
RICE UNIVERSITY

EXPERIMENTAL INVESTIGATION AND
CONSTITUTIVE MODELING OF MARINE CLAY

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE
DOCTOR OF PHILOSOPHY

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Abstract

Experimental Investigation and Constitutive Modeling of Marine Clay

by

Shinyuan Yu

A new cross-anisotropic elasto-plastic constitutive model is developed based on experimental observations of marine clay from the Gulf of Mexico.

Standard triaxial and torsional simple shear tests performed on the aforementioned clay demonstrate that the soil exhibits a cross-anisotropic behavior with lower compressibility in the direction of the deposition than perpendicular to the direction of deposition.

A new general cross-anisotropic model for the stress-dependent elastic moduli is presented. In agreement with experimental evidence, the model considers that the Young’s modulus and the shear modulus of soil depend on the state of stress, while the three Poisson’s ratios are practically constant. The expressions for the stress dependence of the moduli are derived by considering the conservation of energy. Numerical simulations of the undrained elastic reloading demonstrate that the development of pore water pressure depends significantly on the exact representation of the cross-anisotropic elastic parameters.

The plastic model is developed based on the assumptions that the material behavior is time independent and that the interaction between mechanical and thermal processes is negligible. The model, having twelve parameters, consists of a failure function, plastic potential function (non-associated flow rule), yield function and hardening law. To describe the degree of cross-anisotropy at failure, a new four-parameter failure criterion is developed. The plastic potential function, having two parameters, determines the directions of the plastic strain increments which are assumed independent of the stress
path leading to the current state of stress. The potential surface expands along its center line and may translate in the stress space. The yield criteria are associated with and derived from surfaces of constant plastic work. The yield-plastic work relation requires six parameters in which four parameters define the yield function and two parameters define the plastic work equation. For the special case of isotropic soil, all of the functions may be reduced to those of Lade's isotropic elasto-plastic model.

Most of the elastic and plastic parameters can be easily determined by experimental results from standard triaxial tests, but a few parameters need advanced tests such as the torsional simple shear test on hollow cylinder specimens and the cubic triaxial test.

Comparisons between results from computer simulation of tests and actual experimental data showed that the model is satisfactory in predicting the behavior of the tested clay under torsional simple shear and conventional triaxial compression and extension tests.
Acknowledgments

This research was directed by Professor Panos C. Dakoulas, whose guidance, encouragement and friendship is greatly appreciated. The financial support by the Advanced Technology Program of the State of Texas, under grant number 003604-047, is gratefully acknowledged.

I would like to thank Professor Ronald P. Nordgren for the research guidance, and thank Professors John E. Merwin and John B. Cheatham for reviewing and offering helpful suggestions. Special thanks are due to my previous co-worker, Dr. Yuanhui Sun, for his help in the experimental program.

I wish to express my gratitude especially to my wife, Yi-Rong Liou, for her encouragement, support and understanding.

I dedicate this thesis to my parents whose love and support made it all possible.
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\( B \) \hspace{1cm} \text{Pore water pressure parameter, defined by } B = \Delta u / \Delta \sigma_3 \text{ due to isotropic stress change}

\( C \) \hspace{1cm} \text{Parameter of plastic work equation}

\( \text{CTC} \) \hspace{1cm} \text{Conventional triaxial compression test}

\( \text{CTE} \) \hspace{1cm} \text{Conventional triaxial extension test}

\( D \) \hspace{1cm} \text{Parameter of the work-hardening function}

\( d \lambda \) \hspace{1cm} \text{Scalar factor of proportionality in the flow rule}

\( E \) \hspace{1cm} \text{Young's modulus (isotropy)}

\( E_v \) \hspace{1cm} \text{Young's modulus of a cross-anisotropic soil in the vertical direction}

\( E_h \) \hspace{1cm} \text{Young's modulus of a cross-anisotropic soil in the horizontal direction}

\( E_{ur} \) \hspace{1cm} \text{Young's modulus measured at an unloading-reloading loop}

\( e \) \hspace{1cm} \text{Void ratio}

\( f, f_p \) \hspace{1cm} \text{Yield function}

\( G \) \hspace{1cm} \text{Elastic shear modulus (isotropy)}

\( G_{hv} \) \hspace{1cm} \text{Elastic shear modulus of a cross-anisotropic soil in a vertical plane}

\( G_{hh} \) \hspace{1cm} \text{Elastic shear modulus of a cross-anisotropic soil in a horizontal plane}

\( g, g_p \) \hspace{1cm} \text{Plastic potential function}

\( \bar{g}_p \) \hspace{1cm} \text{Modified plastic potential function}

\( h \) \hspace{1cm} \text{Parameter of the yield function}

\( I_1 \) \hspace{1cm} \text{First invariant of the stress tensor}

\( I_2 \) \hspace{1cm} \text{Second invariant of the stress tensor}

\( I_3 \) \hspace{1cm} \text{Third invariant of the stress tensor}

\( I'_1 \) \hspace{1cm} \text{First invariant of the transformed stress tensor}

\( J_2 \) \hspace{1cm} \text{Second invariant of the deviator stress tensor}
\( J^*_{2} \)  Second invariant of the transformed deviator stress tensor

\( K \)  Bulk modulus

\( K_0 \)  Coefficient of earth pressure at rest

\( LL \)  Liquid limit

\( M \)  Parameter of the elastic moduli

\( m \)  Ratio of Poisson's ratios \((= v_{\text{sh}}/v_{\text{sh}})\)

\( m_f \)  Parameter of the failure function

\( n \)  Ratio of Young's moduli \((= E_n/E_v)\)

\( n_f \)  Parameter of the failure function

\( OCR \)  Overconsolidation ratio

\( P \)  Parameter of the plastic work equation

\( PL, I_p \)  Plasticity index

\( PL \)  Plastic limit

\( p_a \)  Atmospheric pressure

\( p' \)  Mean effective stress

\( q \)  Variable of the yield function

\( q' \)  Deviatoric stress

\( S \)  Stress level function

\( TSS \)  Torsional simple shear test

\( t_f \)  Time to failure in soil testing

\( t_{100} \)  Time of 100% consolidation

\( u \)  Pore water pressure

\( W_p \)  Plastic work

\( w \)  Water content

\( w_f \)  Parameter of the failure function

\( \alpha \)  Ratio of shear moduli \((= \sqrt{G_{\text{sh}}/G_{\text{sh}}})\)
\( \alpha_p \) Parameter of the variable \( q \)
\( \alpha_i \) Parameter of the variable \( q \)
\( \Delta \alpha \) Parameter of the variable \( q \)
\( \gamma \) Shear strain
\( \Delta \gamma, d\gamma \) Shear strain increment
\( \varepsilon \) Strain
\( \varepsilon^e \) Elastic strain
\( \varepsilon^p \) Plastic strain
\( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) Principal strain
\( \varepsilon_v \) Volumetric strain
\( \Delta \varepsilon, d\varepsilon \) Strain increment
\( \Delta \varepsilon^e, d\varepsilon^e \) Elastic strain increment
\( \Delta \varepsilon^p, d\varepsilon^p \) Plastic strain increment
\( \Delta \varepsilon_v \) Volumetric strain increment
\( \eta_f \) Parameter of the failure function
\( \lambda \) Parameter of the elastic moduli
\( \mu \) Parameter of the plastic potential function
\( \nu \) Poisson's ratio of isotropic soil
\( \nu_{hv} \) Poisson's ratio of strain in the vertical direction to strain in the horizontal direction due to a horizontal normal stress
\( \nu_{vh} \) Poisson's ratio of strain in the horizontal direction to strain in the vertical direction due to a vertical normal stress
\( \nu_{hh} \) Poisson's ratio of strain in the horizontal direction to strain in an orthogonal horizontal direction due to an orthogonal horizontal normal stress
\( \rho \) Parameter of the work-hardening function
\( \sigma \) Total stress
\( \sigma' \)  
Effective stress \( (\sigma - u) \)

\( \sigma_1', \sigma_2', \sigma_3' \)  
Effective principal stresses

\( \sigma_r, \sigma_\theta \)  
Normal stresses in cylindrical coordinate system

\( \Delta \sigma', d\sigma' \)  
Effective stress increment

\( \Delta u \)  
Excess pore water pressure

\( \tau \)  
Shear stress

\( \Delta \tau, d\tau \)  
Shear stress increment

\( \phi' \)  
Effective friction angle

\( \Psi_1 \)  
Weighting factor of the plastic potential and yield functions

\( \Psi_2 \)  
Parameter of the plastic potential function
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Chapter 1

INTRODUCTION

1.1 Background

One of the major problems in geotechnical engineering is the description of the deformational response of soils under multiple stress conditions. It is usually possible to describe accurately the measured behavior of a given material in a given test (e.g. in a standard triaxial or simple shear test). However, it is much more difficult to predict accurately the behavior of the same material under more complex loading conditions. It is generally recognized that the difficulties and uncertainties encountered in the measurement of actual soil properties in the field and the laboratory, as well as the representation of the strain behavior with a constitutive model, are much greater than the uncertainties found in the currently available computational methods for the analysis of soil-structure interaction problems. For example, the analysis of pile foundations in anisotropic soils can easily be done by using the finite element method once the stress-strain behavior of the soils are modeled properly. In other words, the constitutive model that is employed to predict the material behavior is a very important element of the analysis.

Based on work-hardening concepts, the early work of Drucker et al. (1957) marked the beginning of the development of elasto-plastic models for the mechanical behavior of geomaterials. This first successful synthesis of soil mechanics and the mathematical theory of plasticity led to a steady progress of subsequent refinements in the application of soil plasticity. Later, a fundamental concept of modeling was introduced by the so-called "Cam-Clay" models due to the work of Roscoe et al (1968). Since real geologic materials are much more complex than Cam-Clay in their inelastic response, a wide
variety of plasticity models have attempted to address the various complex aspects of the material behavior by more and more difficult mathematical concepts. The models have been greatly enhanced during the last two decades through research efforts by several investigations such as those by Dafalias and Herrmann (1980, 1982), Krieg (1975), Ghaboussi and Momen (1984), Banerjee and Pan (1986), Faruque and Desai (1985), Hirai (1987), Bannerjee and Yousif (1986), Yong and Mohamed (1988), Lade (1990), Ramsamooj and Alwash (1990), etc. For simplicity, many constitutive models were developed for isotropic soils. However, most of natural soil deposits exhibit some degree of anisotropic behavior in both strength and stress-strain behavior. Anisotropy in soils is caused due to many factors including the manner of deposition, nonuniform initial state of stress leading to preferred orientation of the soil structure, and other complex geologic movements experienced by the soil in their past history. Soil deposits that display fabric anisotropy and exhibit transverse isotropy are called transversely isotropic or cross-anisotropic soil. The yielding and failure characteristics of the anisotropic soils are different than those for isotropic soils and are not yet fully understood or established.

The development of a constitutive model requires the complete knowledge of the deformation characteristics of soils. For soils, which exhibit different stress-strain relations in different directions resulting from the inherent and stress-induced anisotropy, several conventional and advanced tests are required for developing an understanding of their deformational characteristics. Experimental studies proved that cross-anisotropic soils are less compressible and more expandable in the direction of their deposition than in the direction perpendicular to the deposition (Mayne, 1985; Yamada and Ishihara, 1979), and also that the stiffness moduli are significantly different in the vertical direction with in the horizontal directions (Ward et al, 1959; Atkinson, 1975; Gerrard, 1977; Kirkgard and Lade, 1991). Measuring the stiffness modulus only in the vertical direction
and assuming isotropic behavior in geotechnical engineering practice is certainly a risk that should be taken cautiously (Budiman, J. S., Sture, S and Ko, H. Y., 1992).

Some studies (Symes et al, 1984; Towhata and Ishihara, 1985; Hicher and Lade, 1987; Lam and Tatsuoka, 1988) investigated the influence of the three-dimensional behavior of anisotropic soils with the rotation of principal axes using hollow cylinder specimens. Due to limitations of the testing equipment, shear stresses cannot be applied in all planes by using the hollow cylinder specimen testing. The available data are not sufficient yet to explore in depth the stress-strain relations in complex loading conditions and to establish a general constitutive model. Although a few researchers have attempted to develop a constitutive model considering the rotation of principal stresses (Miura et al, 1986), most of the developed constitutive models for anisotropic soils have never considered the difference between rotation and non-rotation of the principal axes. If the yield function is formulated only based on principal stresses in fixed directions, the constitutive model does not consider the effect of principal stress rotation. (For example, the Cam-Clay model and the modified Cam-Clay model use \( p' - q' - e \) coordinates, where \( p' \) mean principal stress, \( q' \) deviatoric stress and \( e \) void ratio, and most of the other models use \( \sigma'_1 - \sigma'_2 - \sigma'_3 \) coordinates. These fixed coordinates are direction-independent and cannot exhibit the effects of principal stress rotation. Instead, either an anisotropic hardening rule is employed to describe the behavior of stress-induced anisotropy or an unsymmetric yield surface with respect to the hydrostatic axis is employed to describe the behavior of inherent anisotropy). Due to the complicated properties of anisotropic soils, a proper anisotropic constitutive model should be formed in a six-dimensional stress space instead of the three-dimensional one.

Although many different advanced models (Bannerjee and Yousif, 1986; Ghaboussi and Momen, 1984; Yong and Mohamed, 1988) for anisotropic soils have been developed,
due to the complexity and lack of sufficient experimental verification there is still a significant gap in our knowledge regarding modeling of anisotropic behavior of soil.

1.2 Objectives

The main objective of this study is to develop an elasto-plastic constitutive model for the behavior of cross-anisotropic soils. The model will be based on experimental data on marine clay from the Gulf of Mexico. In order to investigate the behavior of marine clay, a new state-of-the-art experimental system for advanced testing of soil specimens was developed. The equipment was designed to apply complex combinations of external loads to a soil specimen formed in the shape of a hollow cylinder placed in a cell under pressure. The experimental program started with the selection of representative clay specimens with more or less typical fundamental properties from the Gulf of Mexico, provided by McClelland Engineers, Houston. From this clay, an entirely homogeneous mixture of marine clay was obtained and used to develop identical homogeneous specimens. Although the clay is reconstituted and its behavior is not identical to the behavior of undisturbed marine clay in the field, this difference can be taken into account with appropriate model calibration using "undisturbed" in-situ clay data. A series of experiments was performed in this study which includes general index properties tests, cubic unconfined compression test, consolidation tests, conventional triaxial tests and torsional simple shear tests. Due to limitations of the experimental equipment for soils at Rice, it is impossible to perform some special experiments such as, for example, cubical triaxial tests, for investigating the effects of the different principal stress directions with the anisotropically consolidated soils. Instead, some assumptions are used for the model based on observations from available data in the literature.

The results from the experimental program were analyzed to examine the characteristics of the behavior of the cross-anisotropic soil and develop the basic elements
of the new constitutive model. The presented cross-anisotropic constitutive model may be considered as an extension of Lade's isotropic constitutive model, using a non-associative flow rule. In addition to the new experimental data, existing results from cross-anisotropic clay published in the literature were also used.

The constitutive model includes two parts: an elastic model and a plastic model. A new general model has been developed for the stress-dependency of the elastic moduli for a cross-anisotropic soil. The new model is derived by considering the conservation of energy during a cycle of loading along any arbitrary closed stress path, assuming purely elastic behavior. Furthermore, the effect of anisotropy on the elastic response of a soil element is investigated for various loading and unloading conditions.

The anisotropic plastic model is developed based on the assumptions that the material behavior is time independent and that the interaction between mechanical and the thermal processes is negligible. The model consists of a failure function, plastic potential function (non-associated flow rule), yield function and hardening law, which are formulated based on a few basic postulates and experimental observations. All functions of the plastic model are formulated in a six-dimensional stress space. In addition, the determination of parameters from various experiments is described in this thesis.

To investigate the accuracy of the new constitutive model, a series of computer simulation tests are employed and the computed results are compared with experimental data.
Chapter 2

BACKGROUND AND LITERATURE REVIEW

The material presented in this chapter serves as background and reference material for later chapters. The characteristics of anisotropic soils are reviewed in section 2.1. Section 2.2 describes the basic assumptions and elements in the incremental theories of plasticity for soils. In section 2.3, some existing elasto-plastic models for soils are reviewed. Finally, experimental techniques often used in the establishment of the constitutive modeling of soils are reviewed with emphasis on the hollow cylinder apparatus and the method to determine the suitable rate of strain.

2.1 Experimental Observation of Soil Behavior

It is well established that the deformation characteristics of granular materials such as soils depend on the state of stress. For example, the stress-strain behavior and the maximum shear strength are quite different in compression and extension tests. For inherently anisotropic material, the material properties are different in various directions, and may change due to subsequent stress induced anisotropy. For developing a cross-anisotropic constitutive model, it is very important to have a deep understanding of the soil behavior, including the deformation characteristics of anisotropic soils under specific stress conditions and the influence of the rotation of principal stress directions.

This section presents a review of previous experimental research on the deformation characteristics of anisotropic soils and the influence of principal stress rotation on their behavior.
2.1.1 Deformation Characteristics of Anisotropically Consolidated Soils

The anisotropic consolidation phase in triaxial testing of clay specimens is generally time consuming and expensive, especially if \( K_0 \) conditions are maintained. To establish a constitutive model or determine the parameters of a model, for ease and economy many researchers performed drained or undrained triaxial shear tests using initially isotropic specimens or anisotropically reconsolidated specimens, although the specimen was anisotropically consolidated in the field. Most naturally consolidated soils experience anisotropic stress states (\( K_0 \neq 1 \)). It is important to understand the differences in the deformation characteristics between isotropically and anisotropically consolidated soils.

To examine the differences in the undrained strength and the angle of internal friction between isotropic and anisotropic consolidation clays, Mayne (1985) reviewed available published data from over 40 different clays consolidated under both isotropic and anisotropic conditions before triaxial shear. Two thirds of clays were normally consolidated and one third were overconsolidated. For triaxial compression, the normalized undrained shear strength (i.e. the ratio of undrained shear strength to the effective vertical consolidation pressure) of anisotropically normally consolidated (NC) and overconsolidated (OC) clays typically ranges between 75% and 100% of the normalized strength of isotropically consolidated clays. For triaxial extension, the anisotropically normalized undrained shear strength averages about 60% of the isotropic strength for NC clays. The angle of internal friction of normally consolidated and overconsolidated clays was about 97% of the angle determined from isotropically consolidated clays for both triaxial compression and triaxial extension tests. The results showed that stress anisotropy appears to have little effect on the angle of internal friction \( \phi' \) (effective angle of shearing resistance), and that \( \phi' \) is 1-1.5 times larger in extension than in compression.
Koutsofas (1981) performed a series of undrained shear tests on lean sensitive marine clay in both normally consolidated and overconsolidated states. The types of tests included triaxial compression and extension and direct shear tests on \( K_0 \)-consolidated specimens. They found that the clay is highly anisotropic with respect to its undrained shear strength. The strengths from extension tests are only 55 to 60 percent of the strengths in compression. Direct simple shear tests gave strengths midway between the compression and extension strengths. The clay was also found to exhibit anisotropic behavior with respect to its stress-strain characteristics and undrained moduli. Undrained moduli from compression tests are two to three times greater than moduli determined from extension tests at corresponding incremental shear-stress levels and overconsolidation ratios (OCRs). Direct simple shear tests gave undrained moduli which for normally consolidated specimens were slightly lower than the moduli determined from extension tests, while for overconsolidated specimens were slightly higher than corresponding values from extension tests. In addition, strains at failure were very small in compression particularly for normally consolidated specimens, which exhibit significant strain softening after the peak shear stress is reached in compression.

Yamada and Ishihara (1979) performed a series of drained tests on cubical loose sand specimens prepared by depositing the sand under water. The tests employed different radial stress paths in which the major, intermediate and minor principal stresses were oriented independently from the direction of specimen sedimentation. The test results indicated that when the shear stress was small the deformation of the specimen was highly anisotropic because of the inherent anisotropy of the specimen, but when the shear stress became large enough to produce failure, the effects of inherent anisotropy disappeared. Similar results were also found by Ochiai and Lade (1983) using triaxial compression, plane strain, and cubical triaxial drained tests with Cambria Sand.
Makhlouf and Stewart (1965) conducted a series of triaxial tests on Ottawa sand, and Figure 2-1 shows a loading-unloading-reloading curve obtained from their triaxial compression tests. It shows that strains produced by unloading and reloading are mainly elastic, whereas plastic strains are very small at low stress level, which can be observed from hysteresis loop. At higher stress level, large plastic strains are produced upon unloading and reloading and the hysteresis loop become flatter. The rebound curves become nonlinear and the slope of hysteresis loops decrease. The similar results were also found by Dakoulas and Sun (1992), who performed two isotropic compression tests, including five unloading-reloading loops, on drained solid cylinder specimens prepared at two different relative densities (30% and 75%).

Ko and Scott (1967) performed a series of tests with different stress paths on cubical specimens of Ottawa sand. They found that the primary loading was not affected by previous unloading-reloading loops at small stress levels, i.e. deformations occurring at unloading-reloading loops were independent of the stress history. At higher stress levels, the deformations became increasingly dependent on the stress history and hysteresis loop developed. This behavior is similar to that reported by Markhlouf and Stewart (1965), and Dakoulas and Sun (1991, 1992). Ko and Scott explained the stress-path dependency in terms of the intergranular behavior, associated with the sliding between soil grains. At small stress levels, a portion of the soil grains slips in a state which is essentially recoverable. If the direction of the stress path is changed, some grains will slip following the direction of new stress path and new deformations will be irrecoverable.

Concluding, the experimental evidence indicates that the soil behavior depends significantly on the state of stress. The observation from the aforementioned experimental investigations may be summarized to:
1. The effect of stress path on soil behavior is more significant at higher stress levels than at lower stress levels (Makhlouf and Stewart, 1965; Ko and Scott, 1967).

2. The shear strength from extension test is always lower than the shear strength from compression tests for both isotropic and anisotropic soils (Mayne, 1985; Koutsoftas, 1981; Dakoulas and Sun, 1992).

3. The normalized undrained shear strength in anisotropically consolidated clay is lower than that in isotropically consolidated clay (Mayne, 1985).

4. The angle of internal friction is different between compression tests and extension tests for both isotropic and anisotropic soils, but it is little affected by soil anisotropy (Mayne, 1985).

5. The soil exhibits highly anisotropic behavior when the shear stress is small, but near failure, the effects of inherent anisotropy disappear (Yamada and Ishihara, 1979; Ochiai and Lade, 1983).

6. The cross-anisotropic soil is less compressible and more expandable in the direction of depositions than in the direction perpendicular to the deposition (Yamada and Ishihara, 1979; Mayne, 1985).
Figure 2-1  Loading-unloading-reloading triaxial compression tests on Ottawa Sand (Makhlof and Stewart, 1965)
2.1.2 Influence of Principal Stress Rotation

Principal stress rotation arises when the directions of principal stress increments do not coincide with the directions of current principal stresses, as illustrated in Figure 2-2. The rotation of principal stress directions occurs commonly in the field, as for example in the case of excavation. Figure 2-3 shows a circular arc critical surface ABCD resulting from the excavation. Directions of the major and minor principal stresses at Points A, B, C, and D are significantly different. If the original ground surface before excavation is horizontal, the direction of major principal stress should be vertical at all points due to the weight of soil particles. The excavation causes the soil element to be subjected to a rotation of principal stresses from the vertical direction at point A to the horizontal direction at point D. It can be seen that the degree of rotation of the major principal stress along failure surface varies from 0° at point A to 90° at point D.

For structures subjected to cyclic loading resulting from vibrating machines on the floor, continuous rotation of the principal stress directions may occur in the foundation soil. Another typical example of such stress rotation is the sea-bed deposit by waves passing overhead. The nature of cyclic stress changes occurring in the sea-bed deposit due to wave loading involves a continuous rotation of principal stress directions under the condition of constant deviator stress (as shown in Figure 2-4). Also the shear induced by the traffic loads in the subgrade of road pavement or railways are associated with the rotation of the principal stress rotation.
Figure 2-2 Principal stress rotation in an element (Hight et al, 1983)
**Figure 2.3** Orientation of major stress directions at failure

---

(a) \( \frac{\Delta \sigma_y - \Delta \sigma_h}{2} > 0, \Delta \tau_{vh} = 0 \)  

(b) \( \frac{\Delta \sigma_y - \Delta \sigma_h}{2} = 0, \Delta \tau_{vh} > 0 \)

**Figure 2.4** Changes in state of stress due to propagation of water waves over the seafloor
Since the principal stress rotation cannot be simulated by a conventional triaxial shear apparatus, which suddenly changes the directions of the principal stresses by 90° when the direction of the shear application is reversed, the effects of continuous rotation on the deformation of soil have seldom been investigated. The effects of principal stress rotation may be studied using the torsional simple shear apparatus with hollow cylinder specimens.

The effects of the principal stress rotation on the deformation response of soil under monotonic loading conditions have been investigated recently by many researchers. (Arthur et al, 1981; Symes et al, 1984; Towhata and Ishihara, 1985; Hicher and Lade, 1987; Lam and Tatsuoka, 1988; Dakoulas and Sun, 1991). Their test results showed that the deformation and strength characteristics of the sand are influenced significantly by the rotation of the principal stress directions. Some test results from this literature will be described in detail in the following.

Symes, Gens, and Hight (1984) investigated the anisotropy and effects of principal stress rotation in medium-loose Han River sand, tested under undrained conditions by using a hollow cylinder apparatus. Each sample was prepared by siphoning the saturated sand into a water-filled annular space between inner and outer molds to form the hollow cylinder specimen. To produce an inherently anisotropic soil, a low velocity pouring of the sand grains was achieved by a board fixed below the siphon tube which caused the grains to fall gently through 4 inches of water before reaching the sand surface. Initial anisotropy was determined by shearing with the major principal stress direction fixed at orientations of 0°, 24.5°, and 45° (A0, A2, and A4 tests as shown in Figure 2-5) to the vertical axis of symmetry of the sample. The results of stress paths and the stress-strain relationship are shown in Figure 2-6 and Figure 2-7. It can be seen that specimen A0 without principal stress rotation (α=0°) failed at a $q'/p'$ ratio higher than the other two specimens with 24.5° and 45° rotation respectively. In addition, the specimen loaded
without principal stress rotation was found to be stiffer and to generate less pore pressure during shear.

\[ q' = (\sigma'_1 - \sigma'_3)/2 \]

![Graph showing stress rotation](image)

**Figure 2-5** Tests to define initial anisotropy
**Figure 2-6** Effective stress paths for A0, A2, and A4 (Symes et al., 1984)

**Figure 2-7** Stress-strain behavior in tests A0, A2, and A4 (Symes et al., 1984)
To investigate the effect of principal stress rotation during undrained shear, two tests were also performed in which the direction of major principal stress was rotated from 0° to 45° (R1 test) and from 45° to 0° (R2 test) while the shear stress, \( q' = (\sigma'_1 - \sigma'_2)/2 \), was held constant as shown in Figure 2-8. In the test R1, the specimen was sheared to a level of deviator stress, \( q' = (\sigma'_1 - \sigma'_2)/2 \), equal to 39 kPa, without principal stress rotation, then, held \( q' \) constant and the major principal stress direction was rotated to 45°, and finally the specimen was sheared to failure by increasing \( q' \) with \( \alpha \) fixed at 45°. In the test of R2, the specimen was sheared to a level of \( q' \) equal to 42 kPa with a fixed major principal stress direction of 45°, then, \( q' \) was maintained constant while the major principal stress direction was rotated to 0°, and finally the specimen was sheared to failure by increasing \( q' \) with \( \alpha \) fixed at 0°. Figures 2-9 and 2-10 show the effective stress paths for tests R1 and R2, respectively Figure 2-11 shows the corresponding shear stress-octahedral strain (\( q' - \gamma_{oc} \)) relationships in tests R1 and R2. Figure 2-12 shows the repeatability of tests R1 and R2 during the principal stress rotation. In these figures, there was an increment in pore water pressure (refer to Figures 2-9 and 2-10) and a change in the octahedral shear strain \( \gamma_{oc} \) (refer to Figure 2-12) during the principal stress rotation. Rotation of \( \alpha \) from 0° to 45° which was accompanied by an increase in \( \gamma_{oc} \), resulted in a greater pore water pressure rise than for the rotation from 45° to 0° which was accompanied by a reduction in \( \gamma_{oc} \). The response after the 45° rotation up to failure of the two tests can be compared in Figure 2-9, Figure 2-10 and Figure 2-11. Rotation of \( \alpha \) from 0° to 45° at \( q' \) of 39 kPa brought the effective stress path of test R1 coincident with the effective stress path of test A4, despite their different undrained stress histories. Rotation of \( \alpha \) from 45° to 0° had brought the effective stress path of test R2 well below that of test A0. In addition, the value of \( q'/p' \) at failure was unaffected by the principal stress rotation.
Figure 2-8 Tests to investigate principal stress rotations
Figure 2-9 Effective stress path for test R1 (Symes et al, 1984)

Figure 2-10 Effective stress path for test R2 (Symes et al, 1984)
Figure 2-11  Stress-strain behavior for tests R1 and R2 (Symes et al, 1984)

Figure 2-12  Octahedral shear strains during principal stress rotation in test R1 and R2 (Symes et al, 1984)
Towhata and Ishihara (1985) studied the effect of continuous rotation of the principal stress axes on excess pore water pressure development of the sand during cyclic loading. In their test apparatus, a hollow cylindrical soil specimen is subjected to a simultaneous application of both triaxial and torsional modes of shear stresses, which allows a continuous rotation of the principal stress axes. Test results indicated that the continuous rotation of principal stress axes substantially reduces the resistance of sand to liquefaction by generating a greater amount of excess pore water pressure than in the case without the rotation. However, the angle of internal friction remains unaffected. This result was also found by Lam and Tatsuoka (1988) using triaxial compression, triaxial extension, and plane strain compression drained tests with initially anisotropic sand specimens which were prepared using the air-pluviation method.

Hicher and Lade (1987) studied the influence of principal stress rotation on the behavior of $K_0$-consolidated clay to perform monotonic and cyclic torsion shear and cubical triaxial undrained tests. Torsion shear tests were performed with hollow cylinder specimens and the results of these tests were compared with those from cubical triaxial tests in which the specimens experienced similar stress paths, but without the principal stress rotations. The stress-strain and pore pressure relations of monotonic tests are shown in Figure 2-13 and the effective stress paths is shown in Figure 2-14. The results exhibit that the strength and excess pore water pressure development occurred faster in tests without principal stress rotation, which were opposite to the results from Symes' test, but the stress rotation in torsion shear test results in higher pore water pressure than obtained in the corresponding cubical triaxial test without stress rotation under cyclic tests (as shown in Figure 2-15). In addition, the angle of internal friction was not affected by initial $K_0$-consolidation or principal stress rotation.
Figure 2-13 Stress-strain and pore water pressure relations (Hicher and Lade, 1987)
Figure 2-14 Effective stress paths (Hicher and Lade, 1987)
Figure 2-15 Comparison of pore water pressure developed in cyclic torsion shear and cubical triaxial test while cyclic stress ratio equal to 0.68 (Hicher and Lade, 1987)
In summary, soil behavior is significantly affected by the rotation of the principal stress direction. In particular, the following conclusions may be drawn from the experimental evidence:

1. The angle of internal friction is insensitive to the principal stress rotation (Mayne, 1985; Towhata and Ishihara, 1985; Hicher and Lade, 1987; Lam and Tatsuoka, 1988).
2. The maximum deviator stress decreases with increasing the angle of rotation of principal stress axis (Symes et al, 1984; Hicher and Lade, 1987).
3. The generation of excess pore water pressure is clearly affected by the angle of rotation of principal stress axis, but the behavior is different for various soils (Symes et al, 1984; Hicher and Lade, 1987).
2.2 Elasto-Plastic Theory for Constitutive Modeling of Soils

Soil mechanics is a branch of mechanics of solids. A valid solution to any problem in solid mechanics must satisfy the following three basic equations (Chen and Saleeb, 1982; Chen, 1984):

(1) The equations of equilibrium.

(2) The strain-displacement relation and the equations of compatibility.

(3) Material constitutive equations, i.e. the relations between stresses and strains.

(4) Boundary conditions and body forces for the loading.

There are three equations of equilibrium relating gradients of the six components of stress tensor, $\sigma_{ij}$, in a body to the components of body forces, $F_i$. The Cauchy equations relate external surface forces, $T_i$, acting on the boundaries of the body to the stress tensor. There are six equations of kinematics expressing the six components of strain tensor, $\varepsilon_{ij}$, in terms of gradients of the three components of displacements, $u_i$. These are known as the strain-displacement relations. Suppose the strain tensor $\varepsilon_{ij}$ are given, the three unknown components of displacements can be determined from the six equations of kinematics. This is over-determined; thus some restrictions must be placed on $\varepsilon_{ij}$ to ensure the existence of single-valued continuous solutions $u_i$. The restrictions are expressed by the six equations of compatibility on differential forms of $\varepsilon_{ij}$. Both the equations of equilibrium and the strain-displacement relations are independent of the material. The influence of the material is expressed by the constitutive equations, which give the relations between stress components $\sigma_{ij}$ and strain components $\varepsilon_{ij}$ or their increments or rates at any point. Once the material constitutive equations are established, the general formulation for the solution of the soil mechanics problem is completed. The interrelationships of variables encountered in a general formulation are shown schematically in Figure 2-16 for the case of static analysis.
Figure 2-16 Interrelationships of variables in the solution of a solid mechanics problem.
2.2-1 Basic Assumptions

For a long time, the mechanics of deformable solids has been based upon Hook's law of linear elasticity for describing material behavior because of its simplicity. It is well known that most civil engineering materials such as metals, concrete, soil and rock are not linearly elastic for the entire range of loading in some cases. In fact, the actual behavior of materials such as soils is very complicated, as they show a great variety of behavior when subjected to different loading conditions. Drastic idealizations and simplifications are therefore essential in order to model mathematically and approximately the real material behavior for the solution of a practical problem.

No one mathematical model can completely describe the complex behavior of real materials under all conditions. Each material model aims to capture the essential features of a certain class of phenomena, and disregards what is considered to be of minor importance in that class of applications. For example, Hook's law has been used successfully in structural engineering to describe the general behavior of a structure under short-term working load conditions. However, it fails to predict the behavior near ultimate strength conditions, because plastic deformation at this load level attains a dominating influence, whereas elastic deformation becomes of minor importance.

For soils, most of the constitutive models are based on the following two assumptions:

(1) Material behavior is time independent. Therefore, rate sensitivity, creep, and relaxation are not included in such behavior.

(2) Interaction between mechanical and thermal processes is neglected. Thus, effect of temperature on constitutive equations is not considered.
2.2-2 Elasto-Plastic Theory of Soils

For a material which is idealized as time independent, the behavior can be further idealized as elastic behavior and plastic behavior.

For an elastic material there exists a one-to-one correspondence between stress and strain (Cauchy elasticity). In a more restricted sense, some elastic materials satisfy the energy equation of thermodynamics. These elastic materials characterized by this additional requirement are known as hyperelastic (Green elasticity). On the other hand, the minimal requirement for a material to qualify as elastic in any sense is that there exists a one-to-one correspondence between stress increment and strain increment. Thus, an elastic body that returns to its original state of deformation whenever all stress increments are reduced to zero. This reversibility in the infinitesimal sense justifies the use of the term hypoelastic (incremental elasticity) for elastic materials satisfying only this minimal requirement. The incremental constitutive formulations based on hypoelastic models have been increasingly used in recent years by geotechnical engineers for materials such as sand and clay, in which the state of stress is generally a function of the current of strain as well as of the stress path followed to reach that state. The general mathematical expressions for Cauchy elasticity, hyperelasticity and hypoelasticity are described in Appendix A.

The fundamental difference between elasticity and plasticity models lies in the distinction in the treatment of loading and unloading in plasticity theories. This is achieved by introducing the concept of a loading function. In addition, the total strain deformations $\varepsilon_{ij}$ are decomposed into elastic and plastic components $\varepsilon_{ij}^e$ and $\varepsilon_{ij}^p$ by simple superposition: $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$. In the deformation theory of plasticity for work-hardening materials, it is postulated that the state of stress determines the state of strain uniquely as long as plastic deformation continues. In general, the deformation type of
theories cannot lead to meaningful results, and sometimes they lead to contradictions. For example, this type of model does not satisfy the continuity requirement for loading conditions near or at neutral loading. Basically, the difficulty lies in the fact that the deformation theory and the existence of the loading function are incompatible. This has led naturally to the consideration of the second type of formulation based on incremental theory of plasticity. This theory is based on three fundamental assumptions, the shape of an initial yield surface, the evolution of subsequent loading surfaces (hardening rule), and the formulation of an appropriate flow rule. More details about the deformation theory and incremental theory of plasticity are described in Appendix B.

The most commonly used elasto-plastic constitutive models in soil mechanics are based on the incremental theory of plasticity, which usually includes the following basic elements:

(1) The strain increment can be decomposed into the elastic and plastic parts:

\[ \delta \varepsilon^p_{ij} = \delta \varepsilon^e_{ij} + \delta \varepsilon^p_{ij}, \]

(2) The elastic strain increment can be calculated as \( \delta \varepsilon^e_{ij} = C_{ijkl} \delta \sigma_{kl} \), in which the elastic compliance matrix, \( C_{ijkl} \), depends on the stress state.

(3) There exists a yield surface \( f(\sigma_{ij}, W_p) = 0 \), which represents a boundary of all states in the stress space. In the region within this boundary, there is only elastic strain, while outside the boundary both elastic and plastic strains may take place.

(4) When the stress state reaches the yield surface, the plastic strain increments are defined normal to the potential surface \( g_p(\sigma_{ij}) \) (non-associated flow rule), such that

\[ \delta \varepsilon^p_{ij} = d\lambda \frac{\partial g_p}{\partial \sigma_{ij}} \]

or normal to the yield surface \( f(\sigma_{ij}) \) (associated flow rule), such that

\[ \delta \varepsilon^p_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \]

where \( d\lambda \) is a scalar factor of proportionality.
(5) The hardening criterion provides the rule for establishing the subsequent yield surfaces (also called loading surfaces) during loading inducing plastic deformation.

(6) Some types of hardening rules employed for soil constitutive modeling are:

a. Isotropic hardening rule: the yield surface preserves its shape, while its size change is controlled by a single parameter depending on the plastic deformation or plastic work.

b. Kinematic hardening rule: the yield surface does not change its size or its shape but merely translates in the stress space.

c. Mixed hardening rule: the rule is usually a combination of kinematic and isotropic hardening. The yield surface experiences a translation and uniform expansion, but it still retains its original shape.

The above elasto-plastic constitutive model, described in general terms, consists of the plastic potential function (non-associated flow rule), yield function and the hardening rule. In addition, a failure criterion is used to describe the limit of loading criterion based on the assumption of the failure condition, such as maximum effective stress ratio (Lade and Musante, 1978), maximum deviatoric shear stress (Yasufuki et al, 1991; Yong and Mckyes, 1971), no further hardening or softening taken place ($df = 0$) (Ohta and Nishihara, 1985), etc. The model assumes that the material behavior is rate independent.
2.3 Review of Elasto-Plastic Model of Soils

Experimental results have indicated that the nonlinear deformation of soils under different loading conditions are basically inelastic, since upon unloading only a portion of strains is recovered. Therefore, the stress-strain behavior of most soils may be separated into recoverable (elastic) and irrecoverable (plastic) components. The recoverable behavior is treated within the framework of the theory of elasticity, whereas the irrecoverable part is based on the theory of plasticity. Such separation is especially beneficial if cyclic loading and unloading are encountered.

2.3.1 Elastic Model

The elastic models for soils may be classified into linear and nonlinear. A linear elastic model assumes that the Young’s modulus and the shear modulus are independent of the state of stresses (Carter, Booker and Wroth, 1982, Ramsamooj and Alwash, 1990, Faruque and Desai, 1985, Somasundaram and Desai, 1988). A nonlinear elastic model assumes that the Young’s modulus and the shear modulus are functions of the stress state (Duncan and Chang, 1970; Lade and Nelson, 1987; Ghaboussi and Momen, 1984; the cam-clay and modified cam-clay models). Indeed, it has been established experimentally that the elastic moduli are functions of the state of the soil element, i.e. they may be expressed in terms of the soil density or void ratio and the state of stress (Lade and Nelson, 1987).

In the simplest class of hypoelastic models, the incremental stress-strain relations are formulated directly based on the linear elastic form by simply replacing the constant Young’s moduli $E_i$ and Poisson’s ratios $\nu_i$ with a stress-dependent $E_i$ and $\nu_r$. Models of this type are attractive from both computational and practical viewpoint. They are well suited for implementation of finite-element computer programs. The material parameters
involved in this model can be easily determined from standard laboratory tests using well-established procedures.

The most general form for incremental stress-strain relations based on a Cauchy elastic material is given by

\[ d\sigma_y = C_{ijkl} \, de_{ij} \]  

in which \( C_{ijkl} \) = tensor of material elastic constants (or material compliance matrix). Since \( d\sigma_y \) and \( de_{ij} \) are second-order tensors, it follows that \( C_{ijkl} \) is a fourth-order tensor. In general there are 81 constants for such a tensor. However, since \( d\sigma_y \) and \( de_{ij} \) are both symmetric, one has the following symmetry conditions:

\[ C_{ijkl} = C_{ijlk} = C_{jikl} = C_{jilk} \]  

Hence, the maximum number of independent constants is reduced to 36.

For a Green elastic material, the four subscripts of the elastic constants can be considered as pairs \( C_{(ij)(kl)} \), and the order of the pairs can be interchanged, \( C_{(ij)(kl)} = C_{(kl)(ij)} \). As a result, the number of independent constants in compliance matrix is reduced from 36 to 21. In addition, if the elastic material displays orthogonal symmetry, the number of elastic constants is reduced to 9. For transverse isotropy (also called cross-anisotropy), the number is reduced to 5. Finally, if the material is isotropic, then only two elastic constants are needed to describe its behavior.

In the simplest approach to formulating the nonlinear elastic models, the incremental stress-strain relations are developed directly as a simple modification of the linear elastic relations (generalized Hook's law) with the elastic constants replaced by scalar functions associated with the state of stress. This approach has been used in various applications to concrete and soil materials mainly because of its simplicity. Some published stress-dependent elastic moduli are presented in the following.
1. Cam-Clay and modified Cam-Clay models:

The Cam-Clay model and most of the modified Cam-Clay model (Yasufuku et al, 1991, Banerjee and Yousif, 1986, Banerjee and Pan, 1986, Miura and Toki, 1984, Murthy et al, 1991) based on the swelling behavior of soil from the one-dimensional compression test assumed that the elastic volumetric change is expressed by the void ratio of the soil and the mean principal stresses acting on the soil. The typical expression of Cam-Clay elastic model is

\[ E' = \frac{3 \nu \, p'(1-2\nu')}{\kappa} \]  

(2-3a)

(since \( K' = \frac{E'}{3(1-2\nu')} \) and \( \Delta \varepsilon_v^e = \frac{\Delta p'}{K'} \))

or

\[ \Delta \varepsilon_v^e = \frac{\kappa \Delta p'}{\nu \, p'} \]  

(2-3b)

where

- \( E' \) = Young's modulus in drained conditions,
- \( K' \) = Bulk modulus in drained conditions,
- \( \nu' \) = Poisson's ratio in drained conditions,
- \( \Delta \varepsilon_v^e \) = elastic volumetric strain increment,
- \( \nu \) = specific volume,
- \( \kappa \) = slope of the normal consolidation swelling line in a \( \nu \) versus \( \ln p' \) plot, and
- \( p' \) = mean principal stress.

2. Initial tangent moduli

Another widely used modulus value is the initial tangent modulus (Duncan and Chang, 1970), which is dependent only upon the confining pressure, \( \sigma'_c \). Its mathematical form is expressed by
\[ E_{ur} = K_{ur} \left( \frac{\sigma^o}{p_a} \right)^n \]  

(2-4)

in which

- \( E_{ur} \) = the unloading-reloading modulus,
- \( K_{ur} \) = the corresponding modulus number,
- \( n \) = the influence factor of confining pressure, and
- \( p_a \) = atmospheric pressure.

Both the Cam-Clay model and the initial tangent model were developed for isotropic soil, and the principle of conservation of energy was not considered, so that energy may generate or dissipate along a closed stress path.

3. Lade's elastic modulus

Based on the theoretical consideration of the principle of conservation of energy for elastic properties of granular materials, Lade and Nelson (1987) developed an isotropic model for the nonlinear elastic behavior of granular material. In this model, Poisson's ratio is considered as constant and Young's modulus is expressed as a power function involving the first invariant of the stress tensor, \( I_1 \), and the second invariant of the deviatoric stress tensor, \( J_2 \). The stress dependence Young's modulus is expressed by

\[ E = M p_a \left[ \left( \frac{I_1}{p_a} \right)^2 + 6 \frac{1 + \nu}{1 - 2\nu} \frac{J_2}{p_a^2} \right]^\lambda \]  

(2-5)

in which

- \( I_1 = \sigma_x' + \sigma_y' + \sigma_z' \)
- \( J_2 = \frac{1}{6} \left[ (\sigma_x' - \sigma_y')^2 + (\sigma_y' - \sigma_z')^2 + (\sigma_z' - \sigma_x')^2 \right] + \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yx}^2 \)

\( \nu \) is Poisson's ratio, \( M \) and \( \lambda \) are material parameters that can be determined easily from standard triaxial tests, and \( p_a \) is the atmospheric pressure. According to Equation (2-5),
the Young's modulus is constant along rotationally symmetric ellipsoidal surfaces whose long axis coincides with the hydrostatic axis and whose center is located at the origin of the principal stress space.

Figure 2-17 shows cross-sections of the ellipsoidal surface in the triaxial and the octahedral planes for different values of Poisson's ratio. The cross-sections in the octahedral plane are always circular, whereas the cross-sections in the triaxial plane are shaped as ellipses whose aspect ratio depends on the value of Poisson's ratio $\nu$.

Lade's elastic model is a significant improvement with respect to the two aforementioned models in Equations (2-3) and (2-4) for isotropic normal consolidation soils, but generally it cannot be used for anisotropic soils. However, most natural soil deposits display fabric anisotropy due to parallel alignment of particles, exhibiting transverse isotropy or cross-anisotropy. Many experimental investigations (Atkinson, 1975; Yong and Silvestri, 1979; Ghabezghi and Momen, 1984; Kirkgard and Lade, 1991) indicate that natural soils exhibit anisotropic stress-strain behavior and the stiffness in deposition direction is significantly different from the stiffness in the direction orthogonal to the direction of deposition. So far, no nonlinear elastic models for anisotropic soils have been used in the constitutive model of soils, because the elastic behavior of anisotropic soils depends on the direction of applied stress, which results in more elastic parameters and more complicated relations among these parameters.
Figure 2-17 Contours of constant Young's modulus shown in (a) triaxial plane and (b) octahedral plane in the principal stress space (Lade and Nelson, 1987)
2.3.2 Plastic Model

As explained in section 2.2, the incremental theory of plasticity is widely used in constitutive modeling of soils. This theory is based on three fundamental assumptions, the shape of an initial yield surface, the evolution of subsequent loading surfaces (hardening rule), and the formulation of an appropriate flow rule. The following is a review about the functions of incremental plasticity theory which exist in the constitutive model developed for soils.

Failure Criterion

Failure is a (stress) state in which the material is unable to support increased load. The failure criterion is the best known material property for soil which reflects some important features of the strength under various stress states. A number of such criteria have been developed and were classified as one-parameter models, including the Tresca, von Mises, and Lade-Duncan criteria; two-parameter models include the well-known Mohr-Coulomb criterion, the extended Tresca and Drucker-Prager models, and Lade's two-parameter criterion (Chen and Saleeb, 1982; Chen, 1984). Most of the one-parameter criteria are only applied to undrained saturated soils when the analysis is performed in terms of the total stresses. Lade-Duncan's one-parameter criterion is very efficient for cohesionless soils, but it is not suitable for wide range of pressures. The two-parameter failure criteria, shown in Figure 2-18, are used more widely in soil mechanics instead of one-parameter failure criteria. The advantages and the limitations of these models are summarized in Table 2-1. The three-parameter failure criteria have also been developed for cohesive materials such as rocks and concrete (Kim and Lade, 1984).
Figure 2-18 Two-parameter failure criteria of soils
Table 2-1  Comparison of two-parameter failure models

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Limitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Mohr-Coulomb</td>
<td></td>
</tr>
<tr>
<td>Simple</td>
<td></td>
</tr>
<tr>
<td>Its validity is well established for some soils</td>
<td></td>
</tr>
<tr>
<td>2. Drucker-Prager</td>
<td>$\sqrt{J_2} = \alpha l_1 - k$</td>
</tr>
<tr>
<td>Simple</td>
<td></td>
</tr>
<tr>
<td>Smooth</td>
<td></td>
</tr>
<tr>
<td>3. Lade's two-parameter</td>
<td>$\left( \frac{I_3}{I_3} - 27 \right) \left( \frac{I_1}{p_o} \right)^m = n_1$</td>
</tr>
<tr>
<td>Simple</td>
<td></td>
</tr>
<tr>
<td>Smooth</td>
<td></td>
</tr>
<tr>
<td>Curve meridian</td>
<td></td>
</tr>
<tr>
<td>Wider range of pressures than the others</td>
<td></td>
</tr>
</tbody>
</table>

where

$\sigma$ = mean principal stress,
$\tau$ = shear stress defined by $(\sigma'_1 - \sigma'_3)/2$,
$\phi$ = effective friction angle of soil,
$I_1$ = the first invariant of stress tensor,
$I_3$ = the third invariant of stress tensor, and
$J_2$ = the second invariant of deviator stress tensor.
**Yield Criterion**

The yield function defines a surface in a stress space, which is the limit of the region of elastic behavior. Based on the experimental findings about the shape of yield functions, the yield curves seem to be mainly divided into the following four types in $p'$ versus $q'$ plane (Yasufuku *et al.*, 1991), shown in Figure 2-19.

*Type 1:* The yield curve is presented by a straight line at constant stress ratio or the slightly curved line. Such yield curve provides a sufficiently accurate yield characteristic of hard grained sand during shear, particularly at fairly low stress levels. However, they can not fully explain those for the consolidation process.

*Type 2:* The yield curve is roughly given by a flat line perpendicular to the hydrostatic axis. This type of yield curve is often called as volumetric yield curve used to evaluate the yielding behavior due to consolidation. It is introduced on the basis of the fundamental idea that two kinds of yielding due to shear and consolidation take place independently.

*Type 3:* The yield curve is introduced to emphasize the link between consolidation and shear. This type of yield curve is formulated by superimposing plastic strains due to both shear and consolidation. In general, it can be used to model the anisotropic behavior of the soil, but it becomes practically useful only combined with the yield curves of type 1 and type 2.

*Type 4:* It is often called a cap-type yield curve, introduced recently (Bannerjee and Yousif, 1986; Hirai, 1987), and is mainly used to predict the yielding behavior of isotropically preconsolidated clay, and sand which is at lower stress levels. If the characteristics of the shape of the yield curve are reasonably represented, this type is
also available for sand behavior at higher stress levels in which particle crushing occurs due to shear and consolidation.

Only a few experimental investigations have observed the yield curve on isotropic and anisotropic consolidation soils because there has been no generally accepted method to determine the yield point. One of the available methods is to take the point of maximum curvature of the stress-strain curve as a yield point, and it was applied to investigate the yield characteristics of anisotropic consolidation sand by Yasufuku, Murata, and Hyodo (1991). They found that the shape of the yield surface for anisotropically consolidated sand was approximately elliptical and not symmetrical about the consolidation stress path. They also indicated that the yield curves for isotropic and anisotropic consolidation were significantly different from each other, as shown in Figure 2-20.
Figure 2-19  Schematic diagram for typical yield curves (Yasufuku et al, 1991),

\[ p' = \frac{1}{3} (\sigma'_1 + \sigma'_2 + \sigma'_3), \quad q = \sigma'_1 - \sigma'_3 \]
Figure 2-20  Experimental observation of yield curves (Yasufuku et al, 1991)
Plastic Potential Criterion

The plastic potential function is the basis for derivation of the plastic flow rule which establishes the relationship between stresses and plastic strain increments. (The equations for elasto-plastic formulation are presented in Appendix B.) A large number of elasto-plastic constitutive models for initially isotropic and anisotropic soils assume that the yield function and the plastic potential function are identical (associated flow rule) which implies the normality of plastic strain rate to yield surface. Nevertheless, several test observations proved (Graham et al., 1983; Yamada and Ishihara, 1982) that for each type of consolidation the directions of strain increments at failure, assuming no elastic volume change, were not perpendicular to the yield curves (also see Figure 2-20). Similar observations were made and shown in Figure 2-21.

Only a few constitutive models for soils presented expressions for the plastic potential function, and all of them assumed that the plastic strain increment vectors were uniquely determined from the state of stress and independent of the stress path (Hirai, 1987; Lade and Kim, 1988; Yasufuki et al., 1991), as demonstrated by some researchers (Lade and Duncan, 1976). There are three examples of plastic potential surfaces shown in Figure 2-22.
Figure 2-21 Measured strains on the octahedral plane (Yamada and Ishihara, 1982)
(a) Lade's plastic potential surface  
(b) Hirai's plastic potential surface  
(c) Murata's plastic potential surface

**Figure 2-22** Shapes of plastic potential surfaces
2.4 Experimental Techniques

In this section, two major experimental techniques will be presented. The first deals with the influence of strain rate and the method to determine the strain rate in soil testing. The second refers to the torsional simple shear testing using hollow cylinder specimens.

2.4.1 Rate of Strain on Soil Testing

a. Influence of The Rate of Strain

Many experimental observations have already proved that the rate of strain has a significant influence on the undrained shear strength of remolded and undisturbed clays (Casagrande and Wilson, 1951; Richardson and Whitman, 1973; Nakase and Kamei, 1986; Lefebvre and LeBoeuf, 1987). Typical curves relating the undrained shear strength with the rate of strain are summarized in Figure 2-23 (Nakase and Kamei, 1986). As seen in the figure, the shear strength of sand is almost independent of the rate of strain because of the high permeability of sand, but the shear strength of clay is more sensitive to the rate of strain. The results indicate that the shear undrained strength of clay will not be affected by rate of strain, if the rate of strain is slow enough.

Lefebvre and LeBoeuf (1987) performed a series of monotonic and cyclic triaxial tests to study the influence of the rate of strain and load cycles on the undrained shear strength of three undisturbed sensitive clays. For naturally overconsolidated clays, test results show that the pre-failure parts of the curves are essentially linear, and failure occurs at a very low axial deformation of 1% or less. They also show that pore pressure development during shear up to failure is not significantly influenced by the rate of strain, but the peak deviatoric stress at failure increases with the rate of strain. For normally consolidated clays, the stress paths are influenced by the rate of strain from the beginning.
of compression. In a given deviatoric stress, the generated pore pressure is larger at lower rate of strain due to the tendency of the clay skeleton to creep.

![Graph](image)

**Figure 2-23** Typical curves relating the undrained shear strength with the rate of strain (Nakase and Kamei, 1986)
b. A Realistic Method to Determine The Rate of Strain in Soil Testing

In a drained test the rate of strain must be slow enough to allow water to drain out of the specimen so that no appreciable pore pressure can build up. However, a small increment of pore pressure within the specimen is necessary in order to push the drainage water out; the usual requirement is that 95% dissipation of excess pore pressure should have taken place when failure occurs. This criterion enables a suitable rate of strain to be calculated, if the strain at failure can be assumed.

An undrained test can be run at a faster rate of strain than a drained test on similar material because no movement of water throughout the specimen is involved. However, the rate of strain must be slow enough to permit equalization of pore pressure within the specimen. Generally the rate of strain for an undrained test from which only the strength parameters at failure \( (c', \phi') \) are significant, can be nine or more times faster than for a drained test (Head, 1980). If this is applied to undrained tests in which values of the principal effective stress ratio \( \sigma'_i/\sigma'_3 \) are significant, the test period should never be less than about two hours, even if a faster rate is calculated (Head, 1980)

Usually data from the consolidation stage are used to derive a suitable rate of strain for the compression or extension tests.

Dissipation of Excess Pore Pressure During Test

Let \( s_d \) be the drained compressive strength with 100% dissipation of excess pore pressure on a normally consolidated clay, and let \( s_u \) be the undrained strength with zero dissipation of excess pore pressure. For similar specimens starting at the same initial pore pressure and effective confining pressure, \( s_0 \) will be less than \( s_d \). A similar test allowing partial dissipation of pore pressure will give a measured strength between \( s_d \) and \( s_u \). If the percentage pore pressure dissipation is \( U\% \), the deviatoric stress at failure, \( (\sigma_1 - \sigma_3) \), is given by
\[(\sigma_1 - \sigma_3)_f = 2 \left[ s_u + \frac{U}{100} (s_d - s_u) \right] \]  

(2-4)

The theory of consolidation was applied to the problem of the dissipation of excess pore pressures in triaxial compression by Henkel and Gilbert (1954). They showed that the average degree of dissipation at failure, \( \bar{U}_f \), can be expressed in the form

\[ \frac{\bar{U}_f}{100} = 1 - \frac{L^2}{4 \eta c_v t_f} \]  

(2-5)

where \( L \) is the length of specimen, \( c_v \) the coefficient of consolidation, \( t_f \) the time to failure, and \( \eta \) a factor depending upon drainage conditions at the specimen boundaries. The coefficient of consolidation \( c_v \) can be calculated from the equation

\[ c_v = \frac{\pi D^2}{\lambda t_{100}} \]  

(2-6)

where \( D \) is the diameter of the specimen, and \( \lambda \) is a constant depending on drainage boundary conditions. Values of \( \eta \) and \( \lambda \) are quoted by Bishop and Hankel (1962), and are summarized in Table 2-2 for five drainage conditions.

| Drainage conditions               | \( \eta \) | \( \begin{array}{c|c|c} \text{Values of } \lambda & \text{L/D=2} & \text{L/D=r} \\ \hline \text{drained} & \text{undrained} \end{array} \) |
|-----------------------------------|-----------|----------------------------------|
| from one end                      | 0.75      | \begin{array}{c|c|c} 1 & r^2/4 & 8.5 \\ \hline \text{drained} & \text{undrained} \end{array} |
| from both ends                    | 3.0       | \begin{array}{c|c|c} 4 & r^2 & 8.5 \\ \hline \text{drained} & \text{undrained} \end{array} |
| from radial boundary only         | 32.0      | \begin{array}{c|c|c} 64 & 64 & 12.7 \\ \hline \text{drained} & \text{undrained} \end{array} |
| from radial boundary and one end  | 36.0      | \begin{array}{c|c|c} 80 & 3.2(1 + 2r)^2 & 14.2 \\ \hline \text{drained} & \text{undrained} \end{array} |
| from radial boundary and two ends | 40.4      | \begin{array}{c|c|c} 100 & 4(1 + 2r)^2 & 15.8 \\ \hline \text{drained} & \text{undrained} \end{array} |
Time to Failure in Drained Tests

A theoretical degree of dissipation of 95% of the excess pore water pressure is generally acceptable for deriving the drained strength parameters. Putting $\bar{U}_f = 95\%$ in Equation (2-5), and rearranging, the time required to failure in a drained test is equal to

$$t_f = \frac{L^2}{0.2 \eta c_v}$$  \hspace{1cm} (2-7)

By combining Equations (2-6) and (2-7), the time required for failure, $t_f$, can be calculated directly from $t_{100}$ without first having to determine $c_v$,

$$t_f = \left( \frac{5r^2 \lambda}{\pi \eta} \right) t_{100}$$  \hspace{1cm} (2-8)

For the normal shape of solid cylinder specimen in which $r = 2$, this becomes

$$t_f = \left( \frac{20 \lambda}{\pi \eta} \right) t_{100}$$  \hspace{1cm} (2-9)

Values of $20\lambda/\pi \eta$ are included in Table 2-2 under $t_f/t_{100}$. For other values of $r$ the factor can be calculated from Equation (2-8).

Time to Failure in Undrained Tests

Time required to failure in undrained tests, based on 95% pore water pressure equalization within the specimen, was introduced by Head (1980). The relationship between $t_f$ and $c_v$ depends on whether or not side drains are used.

(i) For tests without side drains and drainage from both ends, the equation is $t_f = 1.6H^2/c_v$ and putting $H = L/2$:

$$t_f = \frac{0.4L^2}{c_v}$$  \hspace{1cm} (2-10)

Substituting for $c_v$ from Equation (2-6),

$$t_f = \lambda t_{100} \frac{0.4L^2}{\pi D^2}$$  \hspace{1cm} (2-11)
and putting $L/D = r$ as above,

$$t_f = 0.127 r^2 \lambda \tau_{100}$$  (2-12)

From Table 2-2, $\lambda = r^2/4$ for drainage from one end; therefore

$$t_f = 0.0318 r^4 \tau_{100}$$  (2-13)

For the normal solid cylinder specimen in which $r = 2$,

$$t_f = 0.127 \times \frac{16}{4} \tau_{100} = 0.508 \tau_{100}$$

(ii) For tests with side drains and drainage from radial boundary only in the consolidation stage, the equation (Head, 1980) is

$$t_f = \frac{0.0175 L^2}{c_v}$$

i.e.  $$t_f = \frac{0.0175 \lambda}{\pi} \frac{L^2}{D^2} \tau_{100}$$  (2-14)

Putting $L/D = r$, and $\lambda = 64$ (from Table 2-2)

$$t_f = 0.3565 r^2 \tau_{100}$$  (2-15)

(iii) For the condition of drainage from one end and the radial boundary during consolidation, it is reasonable to increase the above factor by the ratio 14.2/12.7 (see Table 2-2); therefore for this case

$$t_f = 1.43 \times \frac{14.2}{12.7} \tau_{100} = 1.59 \tau_{100}$$  (2-16)

Similarly for drainage from both ends and the radial boundary, from Table 2-2

$$t_f = 1.43 \times \frac{15.8}{12.7} \tau_{100} = 1.77 \tau_{100}$$  (2-17)

Values of the factor $t_f/\tau_{100}$ for undrained tests, for specimens in which $r = 2$, are included in Table 2-2.
2.4.2 Hollow Cylinder Testing

Principal stress rotation in a soil element is difficult to perform in the laboratory. In most testing devices, such as triaxial, true triaxial with cubical specimen, and plane strain, the principal stresses are fixed in direction and only interchange of principal stress directions can take place. One testing device, which can be used to control the rotation of the principal stress direction, is the hollow cylinder apparatus used in the torsional simple shear testing.

Principles

The hollow cylinder specimen is subjected to axial load, $W$, torque, $M_r$, about a central vertical axis and internal and external radial pressures, $p_i$ and $p_o$ (see Figure 2-24(a)). The torque $M_r$, develops shear stresses, $\tau_{\theta \theta}$ and $\tau_{\theta \phi}$; the axial load, $W$, contributes to a vertical stress, $\sigma_z$. Differences between $p_i$ and $p_o$ establish a gradient of radial stress, $\sigma_r$, across the cylinder wall. The circumferential stress $\sigma_\theta$ is different from $\sigma_r$ by the equation for radial equilibrium

$$\sigma_\theta - \sigma_r = \frac{dr \sigma_r}{dr}$$  \hspace{1cm} (2-18)

The stresses which act on an element in the wall of a hollow cylinder specimen subject to $M_r$, $p_i$, $p_o$, and $W$ are shown in Figure 2-24(b). In testing of a hollow cylinder specimen, both $p_i$ and $p_o$ act through flexible membranes, so that there are no shear stresses on the vertical boundary. Neglecting the effects of end restraint, there are no shear stresses on circumferential surfaces throughout the wall and the radial stress, $\sigma_r$, is always a principal stress. The magnitude and direction of remaining principal stresses can be calculated by the stresses, $\sigma_z$, $\sigma_\theta$, $\tau_{\theta \phi}$, and $\tau_{\theta \phi}$, shown in Figure 2-24(c) and (d). The relative magnitude of the principal stresses is determined by the applied forces and pressures and by the geometry of the specimen.
For the particular case of equivalent of internal and external pressure \((p_i = p_o = p)\), \(\sigma_r\) and \(\sigma_\theta\) are also equal to \(p\).

**Stress Distributions**

If boundary stresses are uniformly applied and there is no end restraint, stress and strain variations can occur across the wall of a hollow cylinder specimen. The definition of average stresses and strains are made by considering the specimen as a single element and calculating values for the average stress and strain using the expression given by (Hight et al, 1983)

\[
\text{Vertical stress } \sigma_z = \frac{W}{\pi(b^2 - a^2)} + \frac{(p_o b^2 - p_i a^2)}{(b^2 - a^2)} \quad (2-19)
\]

\[
\text{Radial stress } \sigma_r = \frac{(p_o b - p_i a)}{(b + a)} \quad (2-20)
\]

\[
\text{Circumferential stress } \sigma_\theta = \frac{(p_o b - p_i a)}{(b - a)} \quad (2-21)
\]

\[
\text{Shear stress } \tau_{o\epsilon} = \frac{3M_r}{2\pi(b^3 - a^3)} \quad (2-22)
\]

\[
\text{Axial strain } e_z = \frac{w}{H} \quad (2-23)
\]
Radial strain \( \varepsilon_r = \frac{(u_o - u_i)}{(b - a)} \) \hspace{1cm} (2-24)

Circumferential strain \( \varepsilon_\theta = -\frac{(u_o + u_i)}{(b + a)} \) \hspace{1cm} (2-25)

Shear strain \( \gamma_{\theta r} = \frac{2\theta(b^3 - a^3)}{3H(b^2 - a^2)} \) \hspace{1cm} (2-26)

The expressions are always valid and independent of the constitutive law of the material, based on assumptions of a linear elastic stress distribution \( (\sigma_r) \), a uniform stress distribution \( (\tau_\theta) \) and a linear variation of radial displacement across the wall \( (\varepsilon_r \text{ and } \varepsilon_\theta) \).

Selection of Specimen Geometry

It is well known that the stress non-uniformities across the wall of hollow cylinder specimen, arising both from curvature and from end restraint, and the differences between real and calculated stress and strain are geometry dependent. To avoid the effects of restraint and non-uniformity, there are some geometric restrictions of the specimen. Saada and Townsend (1981) suggested that the cylinder should have a central zone free from end effects with a length at least equal to the length of the zone influence by the platen. They proposed that the total length, \( H \), of the hollow cylinder should be approximately

\[ H \geq 5.44 \sqrt{r_o^2 - r_i^2} \] \hspace{1cm} (2-27)

in which \( r_o \) and \( r_i \) are the outer and inner radii. Also, to reduce non-uniformity across the wall of the specimen, they suggested that

\[ \frac{r_i}{r_o} \geq 0.65 \] \hspace{1cm} (2-28)
Table 2-3 lists sample dimensions for hollow cylinder specimens reported in published work since 1981. No samples meet the requirement in Equation (2-27), and most of the samples meet or are close to the requirement of Equation (2-28).

Hight *et al* (1983) using finite element analyses investigated the end effects of hollow cylinder specimens, and obtained an accepted dimension of 200 mm i.d. x 250 mm o.d. x 250 mm height which was suitable for observing pre-failure behavior, providing that internal instrument is used over the central zone. The selected height does not meet the proposed criterion in Equation (2-27). Lade (1981), using hollow cylinder specimens of loose Santa Monica Beach Sand with inner diameter of 180 mm and outer diameter of 220 mm, performed two series of tests with heights of 100 mm and 400 mm to investigate the influence of specimen height. He found that the effects of end restraint were negligible in 400-mm-tall heights of loose sand. Although stresses and strains were observed to be nonuniform in the 100-mm-tall specimens, it is likely that sufficiently uniform stress states may be achieved in specimens with the heights considerably lower than 400 mm.

As a slender sample may easily cause a buckling collapse, Equation (2-27) may be conservative, and the height of specimen lower than the value of Equation (2-27) is acceptable by researchers. From Table 2-3 it seems that the ratio of diameters in Equation (2-28) is reasonable, while the height of specimens is greater than the value of the outer diameter.
Figure 2-24 Idealized stress conditions in a hollow cylinder specimen (a) hollow cylinder specimen (b) stresses on an element in the wall (c) principal stresses on an element in the wall (d) Mohr circle representation of stress in the wall (Hight et al, 1983)
Table 2-3  Summary of early hollow cylinder specimens

<table>
<thead>
<tr>
<th>Reference</th>
<th>Sample dimensions: mm</th>
<th>(5.44\sqrt{\frac{r_o^2-r_i^2}{r_i}}) req'd</th>
<th>(r_i/r_o) req'd</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lade (1981)</td>
<td>180</td>
<td>220</td>
<td>100/400</td>
</tr>
<tr>
<td>Hight et al (1983)</td>
<td>200</td>
<td>250</td>
<td>250</td>
</tr>
<tr>
<td>Tatsuoka et al (1986)</td>
<td>60</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Hicher &amp; Lade (1987)</td>
<td>180</td>
<td>220</td>
<td>250</td>
</tr>
<tr>
<td>Pradhan et al (1988)</td>
<td>60</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>Hong &amp; Lade (1988)</td>
<td>180</td>
<td>220</td>
<td>250</td>
</tr>
<tr>
<td>Pradel et al (1990)</td>
<td>60</td>
<td>100</td>
<td>193</td>
</tr>
<tr>
<td>Dakoulas &amp; Sun (1991, 1992)</td>
<td>101</td>
<td>140</td>
<td>203</td>
</tr>
<tr>
<td>Frydman &amp; Talesnick (1992)</td>
<td>50</td>
<td>71</td>
<td>120</td>
</tr>
</tbody>
</table>
Chapter 3

EXPERIMENTAL DEVICES AND TEST SPECIMENS

3.1 Experimental Devices

An entirely new state-of-the-art soil triaxial test system for both solid and hollow cylinder tests has been designed and set-up at Rice University during this research. The computer-automated experimental system for both solid and hollow cylinder testing soils is capable of stress and strain controlled tests. It consists of an MTS axial/torsional servo-hydraulic frame, two MTS electronic control units, a Hewlett-Packard 9000/360 workstation, a HP-3852A data acquisition and control unit, a soil triaxial testing cell with pressure transducers, a pressure control panel and a volume change measurement device, etc (Sun, 1991). The details of the system are provided in Figure 3-1.

Soil Triaxial Device for Hollow Cylinder Specimens

The soil triaxial testing loading cell, designed for hollow cylinder specimens, is made of clear cast acrylic tube and stainless steel base and top (as shown in Figure 3-2). The acrylic tube has a height of 12 inches, an inner diameter of 13 inches and wall thickness of 1.25 inches. Both top and base plates are grooved with O-rings for matching the acrylic tubing and providing a seal for the pressure chamber. Four tie rods of diameter 0.75 inches go through the top plate and screw into the base plate forming a rigid loading cell frame. The loading cell is equipped with appropriate instrumentation. In order to ensure the torque transmission for the hollow cylinder specimens, relatively rough porous stones are used for the specimen pedestal and cap. The upper section of the pedestal and lower section of the cap are used to seal the inner and outer membrane with O-rings so that no special clamps are necessary. A smooth stainless steel rod of 1 inch
diameter is used as a vertical and torque loading piston. The interior of the spacer in the bearing house is grooved for a Teflon O-ring. The function of the O-ring is to seal for the pressure chamber and reduce the friction between the piston and the air bushing. Calibration tests have shown that the friction is less than one percent at any stress level measurement both for the axial and torsional loads. A piston locker is necessary to avoid any disturbance of the specimen when connecting the piston rod to the loading frame. During the test, the piston locker is completely loose from the air bushing in order to eliminate the friction between the piston locker and the piston rod.

*Soil Triaxial Device for Solid Cylinder Specimens*

Two smaller triaxial cells were used for testing of solid cylinder specimens of diameter 1.4 inches (as shown in Figure 3-3). The loading cell, originally designed by Geotest Instrument Inc. for standard triaxial tests, is made of clear cast acrylic tubing with a thickness of 0.75 inches to provide sufficient rigidity and strength for working cell pressure of up to 300 psi. Three tie rods of diameter 5/8 in. go through top of the cell and screw into the base forming a rigid cell frame. The stainless steel piston is guided by a Teflon bushing and sealed with an O-ring. A piston locker is on the top of the Teflon bushing to prevent any disturbance of the specimen when connecting the piston rod to loading frame. For the requirements of this research, two new valves were added to the outlets, with tubing connected to the cap and the pedestal for flushing the pore water during saturation. For triaxial extension tests, the top cap was re-designed to a suction cap device (shown in Figure 3-4), in order to avoid any disturbance of the specimen while connecting the rod piston to the top cap of the specimen.
Pressure Control Panel and Volume Change Measurement

The control panel for applying both cell and back pressures, built with precision instrumentation and operated in a range of 0-400 psi, was designed and set-up for multi-purpose triaxial testing. The control panel can be used to apply the cell and back pressure simultaneously keeping the difference between the cell pressure and back pressure constant. For complete saturation a high value of back pressure is required, typically about 30 psi (205 kPa), in order to resolve the air and allow accurate measurement of the pore water pressure during the test. The volume change measurement device is used in the drained test to measure the volumetric change of the soil sample.

Control Program

A new computer control program was written for this research. The program has mainly the following three functions:

1. To generate digital load (or deformation) histories such as sine waves with 1000 points per period, ramp waves and arbitrary waves and automatically send these loading histories to the Material Testing System (MTS) frame.
2. To receive the response from transducers which are built in the test devices.
3. To interrupt the controller so that the loading type or the control mode can be changed during the test.

The collected data are reduced and plotted in real time on the screen so that researcher can adjust the different loading types to achieve complex stress paths or to protect the test specimen and the device from inadequate control. The flow chart of the computer program is shown in Figure 3-5.
Figure 3-1 The computer-automated experimental system
Figure 3-2 Loading cell for hollow cylinder specimens

Figure 3-3 Loading cell for solid cylinder specimens
Figure 3-4 Suction cap device designed for triaxial extension tests
Figure 3-5 Control program flow chart
3.2 Gulf Marine Clay Specimen Preparation

The soil used was obtained from borings extracted from the Gulf of Mexico, provided by McClelland Engineers, Houston. The clay was pulverized and mixed thoroughly to form a completely uniform material. This material was mixed with water to form a slurry, which was preconsolidated to geostatic stress conditions similar to the in-situ stress conditions. This reconstituted material was reconsolidated and used to obtain homogeneous marine clay specimens. The average specific gravity of mixture is $G=2.76$. The Atterberg limits tests produced a liquid limit (LL) equal to 74\% and a plastic limit (PL) equal to 30\%. The plasticity index (PI) is equal to 44\%. These data show that the marine clay is a highly compressible material. The coefficient of lateral earth pressure $K_0$ is equal to 0.59 (test results described in chapter 4).

3.2.1 Hollow Cylinder Specimen Preparation

Each specimen was prepared from a clay slurry mixed at a water content of double liquid limit and pre-consolidation in a mold which was sunk in water for the entire duration of pre-consolidation. The pre-consolidation mold of the hollow cylinder specimens consists of an exterior diameter of 4 inches cylindrical tubing and an interior diameter of 3 inches cylindrical tubing. The base of the mold consists of a stainless steel ring, covered with a porous stone for water drainage, and the top cap has a similar ring (with a porous stone at its bottom). The interior surface of the exterior tubing and the exterior surface of the interior tubing were lubricated with very thin Vaseline oil to reduce friction between the specimen and the tubing during consolidation. The vertical pre-consolidation pressure was less than 150 kPa, which is much smaller than the final vertical consolidation pressure applied later in the triaxial cell. The process of pre-consolidation requires about 7-10 days. After pre-consolidation, the hollow cylinder
specimen was removed from the mold by fixing the interior and exterior tubing and pushing very carefully with a piston. Due to the low consolidation pressure and to the smoothness of the lubricated tubing surfaces, the friction on the vertical surface of the specimen is small. After the extraction, the specimen preserves the shape of a perfect cylinder, but experiences a small expansion. Then the specimen is transferred into the triaxial loading cell for consolidation to the desired consolidation stress state, which is higher than the pre-consolidation stresses. The $K_0$-anisotropic consolidation of the specimen was achieved by keeping the volumetric strain equal to the vertical strain during the consolidation up to the desired horizontal stress level. The time required for consolidation at the triaxial cell using this procedure is approximately equal to 15 days. The consolidation stress was applied in small stress increments to avoid collapse of the specimen and the last stress increment was kept for three days, so that sufficient secondary compression could take place.

The interior and exterior surface of the pre-consolidated hollow cylinder specimen were partially covered by the saturated filter paper with vertical side drains in order to speed up the consolidation process with a combination of radial and vertical water flow. The filter paper overlapped with the porous stone disc to facilitate drainage. The surface of the specimen was covered with a 0.024 inches thickness rubber membrane (as shown in Figure 3-6).

The hollow cylinder specimens used in this study have the nominal dimensions of an outer diameter of 4 inches, an inner diameter of 3 inches and height about 5.5 inches.

3.2.2 Solid Cylinder Specimen Preparation

Each specimen was prepared from a clay slurry mixed at a water content of double liquid limit and preconsolidated in a mold which was sunk in water for the entire duration of pre-consolidation. The pre-consolidation mold of the solid specimens is a clear plastic
tube with height of 9 inches, inner diameter of 2.5 inches, and thickness of 0.5 inches. The interior surface of the mold was lubricated with very thin Vaseline oil to reduce friction between the specimen and the mold during pre-consolidation. The vertical pre-consolidation pressure was slightly less than the final consolidation pressure applied later in the triaxial cell. After pre-consolidation, the solid specimen was removed from the mold by fixing the mold and pushing very carefully with a piston. After the extraction, the specimen was trimmed to a diameter of 1.4 inches and height of 3 inches.

The pre-consolidated solid cylinder specimens were partially covered by saturated filter paper with spiral side drains in order to speed up the consolidation process with a combination of radial and vertical water flow. The filter paper overlapped with the porous stone disc to facilitate drainage. The exterior surface of the specimen was covered with a 0.024 inches thickness rubber membrane (as shown in Figure 3-7).

3.2.3 Saturation and Specimen Set-up

An important step during saturation is to extract the trapped air between the membrane and the specimen. If such air remains within the specimen, correction of the problem later is practically very difficult and the measurements of pore water pressure during the test will be incorrect. The specimen is saturated by flushing de-aired water from bottom to the top using initially a small vacuum applied for a duration of ten minutes. This helps to significantly reduce the saturation time. The saturation stops when no more air bubbles escape from the transparent tubing connected to the top of the specimen. Due to the very long duration of the consolidation process, air from the pressurized cell water could migrate into the interior of the specimen, penetrating the membrane, and resulting into incorrect measurements of pore water pressure. To avoid this serious problem, de-aired water was used to fill the triaxial cell at a level higher than the top cap of the specimen, and on the top of the de-aired water, a layer of paraffin oil
was placed in order to isolate the pressurized air from the de-aired water. After consolidation, the specimen is ready to be moved to the MTS testing frame. This was done very carefully to avoid any accidental disturbance of the specimen by movement of the piston rod. Before starting the test, the back pressure is increased at a level equal to at least 200 kPa. To ensure that the specimen is fully saturated, a small increment of isotropic consolidation is applied on the clay specimen under undrained conditions and the value of the developed excess pore water pressure is measured. The ratio of the excess pore water pressure over the applied stress increment (B parameter) has to be larger than 0.95 in order to characterize the specimen as fully saturated. The application of the vertical and torsional loads was performed automatically by the computer at the slow strain rate of 1% per hour for undrained tests (tests are described in Chapter 4), to ensure uniform pore water pressure distribution within the soil specimen during tests.
Figure 3-6 Hollow cylinder specimen under $K_0$-consolidation in loading cell
Figure 3-7 Solid cylinder specimen under $K_0$-consolidation in loading cell
Chapter 4

EXPERIMENTAL PROGRAM: RESULTS AND ANALYSIS

The objectives of the experimental program were to provide a set of consistent results from homogeneous marine clay specimens that can be used to select and refine appropriate mathematical expressions and parameters for constitutive modeling. More specifically, the results are used to establish the failure criterion, plastic potential criterion, yield criterion and hardening rule.

A comprehensive experimental program using monotonic load was conducted on reconstituted homogeneous marine clay specimens in order to investigate its mechanical behavior under general loading conditions. The presented results correspond to five different types of tests: (1) general tests for the determination of Atterberg values and the rate of strain, (2) cubic unconfined compression test for Poisson's ratios, (3) a $K_0$-consolidation test, (4) triaxial tests including drained unloading, undrained conventional compression and extension tests, and (5) undrained torsional simple shear tests.

The conventional triaxial compression (CTC) and extension (CTE) tests were performed by solid cylinder specimens which consolidated in different stress levels. The torsional simple shear (TSS) tests were performed by hollow cylinder specimens which also consolidated in different stress levels. The experimental programs in items (4) and (5) are listed in Table 4-1.
Table 4-1  Experimental programs of CTC, CTE and TSS

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Consolidation stress</th>
<th>During testing</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma'_x$</td>
<td>$\sigma'_y$</td>
<td>$\sigma'_z$</td>
</tr>
<tr>
<td>CTC141</td>
<td>468</td>
<td>275</td>
<td>275</td>
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<tr>
<td>CTC142</td>
<td>727</td>
<td>428</td>
<td>428</td>
</tr>
<tr>
<td>CTC143</td>
<td>317</td>
<td>184</td>
<td>184</td>
</tr>
<tr>
<td>CTE146</td>
<td>520</td>
<td>311</td>
<td>311</td>
</tr>
<tr>
<td>CTE147</td>
<td>310</td>
<td>178</td>
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<tr>
<td>CTE148</td>
<td>673</td>
<td>396</td>
<td>396</td>
</tr>
<tr>
<td>TSS150</td>
<td>508</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>TSS151</td>
<td>338</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>TSS155</td>
<td>515</td>
<td>304</td>
<td>304</td>
</tr>
<tr>
<td>TSS156</td>
<td>691</td>
<td>400</td>
<td>400</td>
</tr>
</tbody>
</table>

CTC: Conventional triaxial compression test (solid cylinder specimen).
CTE: Conventional triaxial extension test (solid cylinder specimen).
TSS: Torsional simple shear test (hollow cylinder specimen).
Stress unit: $kN/m^2$
4.1 General Tests

For the determination of the Atterberg limits and the rate of strain, there are two kinds of tests included in this section described by the following.

4.1.1 Determination of the Atterberg Limits

The behavior of cohesive soils is governed by their phase composition and can be broken down to four states: solid, semi-solid, plastic and liquid. The moisture content at the point of transition from one state to the other are defined as the Atterberg limits. The water content at which the soil stops acting as a liquid and starts acting as a plastic solid is called liquid limit, while the water content at which the soil starts behaving as a semi-solid is called plastic limit. The plasticity index is the range of water content in which the soil is plastic; the finer the soil, the greater its plasticity index. The experiments for the Atterberg limits are always performed on remoulded samples of the clay material. Details of the liquid and plastic limit tests are discussed in all soil testing textbooks. The Atterberg limit values of tested marine clay are shown in Figure 4-1. The plasticity chart in Figure 4-2, used for soil classification, suggests that the tested marine clay belongs to high compressibility clays.

4.1.2 Determination of the Rate of Strain

It is well known that the rate of strain has a significant influence on the undrained shear strength of remoulded and undisturbed clays (discussed in section 2.4.1). Generally, the rate of strain must be slow enough to permit equalization of pore water pressure within the specimen, but low rate tests with duration longer than one day pose the problem of having to leave the laboratory unattended overnight. Once shearing has started, it should not be stopped until completion; on the other hand, it is essential that the test equipment must be kept in a good condition to avoid missing critical readings. In
addition, all of the unusual response from the specimen or the device must be recorded in order to decide whether the test data are useful or not. The usual compromise is to reduce the rate of strain to a speed that is slow enough to ensure that the test duration is acceptable for the person performing the test as well as for the test equipment.

To determine a suitable rate of strain for the subsequent triaxial tests, two isotropic consolidation tests were performed in the loading cells. The first test used a solid cylinder specimen to determine the suitable rate of strain for the conventional triaxial compression and extension tests, and the second test uses a hollow cylinder specimen for the torsional simple shear tests. Both tested specimens had the same dimensions and drainage conditions as the actual test specimens. The test results show that the time for 100% consolidation, \( t_{100} \), is 600 minutes for the solid cylinder specimen and 576 minutes for the hollow cylinder specimen (see Figure 4-3 and 4-4). Substituting the value of \( t_{100} = 600 \) minutes into Equation (2-17), the time to failure, \( t_f \), is 17.7 hours for the solid cylinder specimen. Due to the special shape of the hollow cylinder specimen (\( D_i = 3 \) in., \( D_o = 4 \) in. and \( L = 5.5 \) in.), the diameter \( D_h \) may be calculated as the diameter on an equivalent solid cylinder specimen, given by \( D_h = \sqrt{D_o^2 - D_i^2} = 2.65 \). Thus, \( r = L/D_h = 2.07 \), \( \lambda = 105.7 \) and \( \eta = 40.4 \) (refer to Table 2-2). Substituting the values of \( r \), \( \lambda \) and \( \eta \) into Equation (2-8) produces that \( t_f/t_{100} = 17.8 \) for the drained test. For undrained tests, the time to failure of the hollow cylinder specimen is computed as \( t_f = 1.77 \times \frac{17.8}{15.8} = 1.99 t_{100} = 19 \) hrs.

If the strain at failure was estimated to 18\%, the rate of strain is 1.018\% for the solid cylinder specimen and 0.95\% for the hollow cylinder specimen.
Liquid limit = 74
Plastic limit = 30
Plastic index $I_p = 44$

**Figure 4-1** Results of the Atterberg limits testing
The first letter:

G: Gravel
S: Sand
M: Silts
C: Clay
O: Organic silts and clays
Pt.: Peat

The second letter:

L (LL<35%): Low compressibility
I (LL 35-50%): Medium compressibility
H (LL>50%): High compressibility

**Figure 4-2** Plasticity chart for soil classification
\( t_f = 1.77 \ t_{100} = 1062 \text{ min.} = 17.7 \text{ hrs} \)

Assume: failure strain = 18 %
Rate of strain = 1.018 %

**Figure 4-3** Isotropic consolidation of a solid cylinder specimen for the determination of the rate of strain
Figure 4.4  Isotropic consolidation of a hollow cylinder specimen for the determination of the rate of strain

$$t_f = 1.99 \quad t_{100} = 19 \text{ hrs}$$

Assume: failure strain = 18 %
Rate of strain = 0.95 %
4.2 Cubic Unconfined Compression Test

According to incremental plasticity theory, a material exhibits purely elastic behavior in the region of the stress space located within the current yield surface. In this region, the elastic strains, which are recoverable, are calculated from Hook's law. In the stress-strain relations of Hook's law, there are three Poisson's ratios, $\nu_{hv}$, $\nu_{vh}$ and $\nu_{hh}$ defined in Chapter 5, for the cross-anisotropic soil.

A cubic specimen of width equal to 1.5 inches in each side was trimmed from a $K_0$-consolidated solid cylinder sample, pre-consolidated to the major (vertical) effective principal stress equal to 200 kPa. Before trimming the pre-consolidated sample to a cubic specimen, the solid cylinder sample of diameter 2.5 inches was kept in 100% humidity for 24 hours to allow for complete expansion of the solid cylinder. In the unconfined test, the cubic specimen was subjected gradually to a uniform load only in the $y$-direction which is perpendicular to the direction of consolidation ($x$-direction) as shown in Figure 4-5. Axial deformations in each side were measured by sensitive dial gauges shown in Figure 4-6. The test results show that the values of the Poisson's ratio are equal to $\nu_{hv} = 0.25$ and $\nu_{hh} = 0.18$ (as shown in Figure 4-7). The third Poisson's ratio $\nu_{vh}$ is determined by CTE tests described in section 5.5.
Figure 4-5 Cubic specimen and loading direction

Figure 4-6 Cubic unconfined compression test device
Figure 4.7 Results from the cubic unconfined compression test
4.3 Consolidation Tests

This test is used to determine the parameters for the plastic work equation of the plastic model presented in Chapter 6.

To simulate the consolidation state of a natural deposition of marine clay in situ, the consolidation tests were performed using the \( K_0 \) condition in which there is no lateral strain in the specimen, known also as the geostatic condition. The test was performed in \( K_0 \)-consolidation test machine with a specimen having a diameter of 2.5 inches and height of 0.75 inches (Figure 4-8). Because of the rigid lateral wall, the specimen is fully in \( K_0 \) condition under the consolidation stresses.

The test results are shown in Figure 4-9, which plots the void ratio, \( e \), versus the mean normal effective stress, \( p' \). The corresponding equations may be expressed by

\[
e = 2.71 - 0.65 \log p'
\]  

(4-1)

The value of \( K_0 \) is obtained from the \( K_0 \)-consolidation in a triaxial loading test, by controlling the axial consolidation stress so that the value of the lateral strain is equal to zero.
Figure 4-8 $K_p$-consolidation test device
Figure 4-9 Results from $K_o$-consolidation test
4.4 Triaxial Tests

In order to simulate the natural deposits of marine clay, all of the specimens in triaxial compression and extension tests were consolidated under $K_0$ conditions.

The compression and extension tests were performed under undrained conditions, with pore water pressure measurements, at different consolidation stress levels (see Table 4-1) for the investigation of the failure criterion, the yield criterion and the plastic potential criterion in the plastic model presented in Chapter 6. An additional unloading test was performed under drained condition with volumetric strain measurements for the determination of the elastic moduli.

4.4.1 Conventional Triaxial Tests

Conventional triaxial tests include three compression tests (CTC) and three extension tests (CTE). All of the tested specimens were consolidated in different stress levels by applying the vertical deformation at slow rate of strain, approximately 1% per hour (determined in section 4.1.2). These tests were performed in undrained condition with pore water pressure measurement. The results from the tests are shown in Figure 4-10 through Figure 4-13.

4.4.2 Drained Unloading Test

The unloading test was performed by reducing the vertical load and keeping the cell pressure constant during the test. For the investigation of the elastic parameters of the marine clay, the test was preferred to the unloading to avoid any plastic deformation response experienced frequently during reloading. The specimen was prepared from a clay slurry mixed at a water content twice the amount of the liquid limit and pre-consolidated in a mold submerged in water. After pre-consolidation, the clay was
consolidated in a conventional triaxial cell under $K_0$ conditions by using small stress increments until $\sigma'_1$ and $\sigma'_3$ reached the values of 176 and 105 kN/m$^2$. Then drained unloading was performed by decreasing vertical load $\sigma'_1$ and keeping $\sigma'_3$ constant with vertical strain and volumetric strain measurement until $\sigma'_1 = 115$ kN/m$^2$. The next measurement of unloading was performed after the specimen was reloaded to $\sigma'_1 = 200$ kN/m$^2$. The stress-strain relationship is shown in Figure 4-14.
Figure 4.10 Effective stress paths in CTC and CTE tests
Figure 4.11: Deviator stress versus major principal strain in CTC and CTE tests.
Figure 4.12 Effective stress ratio versus major principal strain in CTC and CTE tests

Major principal strain, $e_1$, %

Effect stress ratio, $o/o_o$.
Figure 4-13 Excess pore water pressure vs. major principal strain in CTC and CTE tests
Figure 4-14 Stress-strain relation in drained unloading test
4.5 Torsional Simple Shear Tests (TSS)

Four torsional simple shear tests using hollow cylinder specimens were performed by applying the shear strain $\gamma_x$ at a slow rate of approximately 1% per hour (determined by section 4.1.2). All of the tests were performed under undrained conditions with pore water pressure measurement. In two of the tests, TSS150 and TSS151, the normal total stresses were kept constant, i.e. $\sigma_x = \text{constant}$ and $\sigma_y = \sigma_z = K_0 \sigma_x$, during the test. In the other two tests, TSS155 and TSS156, the ratio of normal effective stresses was kept constant, i.e. $\sigma'_y/\sigma'_x = \sigma'_z/\sigma'_x = K_0$, by adjusting the vertical stress $\sigma'_z$ during the test (refer to Table 4-1). Test results from TSS tests are shown in Figure 4-15 through Figure 4-18. A deformed hollow cylinder specimen is shown in Figure 4-19.
Figure 4-15  Effective stress paths in TSS tests

\[ \frac{\sigma'_1 + \sigma'_2 + \sigma'_3}{3} \text{, kN/m}^2 \]
Figure 4.16 Deviator stress versus shear strain in TSS tests
Figure 4-17 Effective stress ratio vs. shear strain in TSS tests
Figure 4.18 Excess pore water pressure versus shear strain in TSS tests
Figure 4-19 Deformed hollow cylinder specimen subjected to torsional simple shear
4.6 Discussion

Failure of a soil specimen is considered the state in which the effective principal stress ratio $\sigma_i^e/\sigma_3^e$ becomes maximum. The plots of stress-strain relations in CTC tests (Figure 4-11) show that in some tests the deviator stress reaches a peak value and then reduces before the failure point is reached. This is more pronounced for specimens with high consolidation stress. The stress-strain relations in CTC and CTE tests also indicate that the $K_o$-anisotropic marine clay experiences larger axial strain in triaxial extension than in triaxial compression. These characteristics are consistent with previous experimental observations (Mayne, 1985; Yamada and Ishihara, 1979; Koutsofias, 1981) presented in section 2.1.

The plot of the effective stress ratio $\sigma_i^e/\sigma_3^e$ versus major principal strain $e_i$ in Figure 4-12 shows that all the CTC tests tend to merge to one curve and all the CTE tests tend to merge to another curve. This indicates that the relationship of the effective stress ratio $\sigma_i^e/\sigma_3^e$ and the major principal strain does not depend significantly on the level of the consolidation stress for these tests. In the TSS tests shown in Figure 4-17, the specimen with low consolidation stress yields a higher value of $\sigma_i^e/\sigma_3^e$ at failure than the specimen with high consolidation stress.

The excess pore water pressure develops rapidly in the CTC tests before the peak deviator stress is arrived. After the peak deviator stress is reached, the development of pore water pressure becomes slower but still a large increase occurs before the specimen fails. In the CTE tests, due to unloading in the vertical direction, negative pore water pressure develops that reaches a minimum value when $\sigma_i^e$ becomes equal to $\sigma_3^e$. After that, the pore water pressure starts to increase (refer to Figure 4-13). The characteristics of the development of pore water pressure are related to the elastic and plastic behavior of the soil specimen. At the beginning of the CTC and the CTE tests, the elastic behavior
plays an important role so that the pore water pressure increases in the CTC test due to loading and decreases in the CTE test due to unloading in the vertical direction. After reaching the peak deviator stress in the CTC test and $\sigma'_d = \sigma'_s$ in the CTE test, the plastic behavior becomes much more important.

The failure envelope plotted in the $\sigma' - \tau$ plot in Figure 4-20 shows that the effective friction angle, $\phi'$, is equal to $22^\circ$ in compression and to $23.5^\circ$ in extension (i.e. $\phi'$ in extension is 7% larger than $\phi'$ in compression). This is in agreement with the findings by Mayne (and the other investigators) who collected 42 research reports for different clays and found that $\phi'$ in extension is 0-50% larger than $\phi'$ in compression (Mayne 1985).

It should be noted that this is only a limited number of successful tests, as significant difficulties were experienced during testing due to the long duration of the consolidation process, which led to the development of undesirable air or gas within the specimen. Consequently, a number of tests had to be abandoned to avoid inaccurate experimental data.

However, in the development of the various criteria of the elasto-plastic model, observations based on more results from the literature have been taken into account.
Figure 4-20 Determination of effective friction angle for CTC and CTE tests
Chapter 5

CONSTITUTIVE MODELING OF CROSS-ANISOTROPIC SOIL: ELASTIC MODEL

A new general nonlinear model for the dependence of the elastic moduli of a cross-anisotropic soil on the state of stress is presented. In agreement with experimental evidence, the model considers that the Young’s modulus, the shear modulus and the bulk modulus of soil depend on the state of stress, while the three Poisson’s ratios are practically constant. The expressions for the stress dependence of the moduli are derived in a rigorous way by considering the conservation of energy during a cycle of loading along any arbitrary closed stress path, assuming purely elastic behavior. A parameteric study elucidates the effects of normal stresses, deviatoric stresses, Poisson’s ratios, and degree of cross-anisotropy on the elastic behavior of soil. Experimental determination of elastic parameters is also introduced in this chapter.

5.1 Introduction

The deformation experienced by a soil element under load consists generally of elastic (recoverable) and plastic (irrecoverable) components. Although in many practical applications the plastic component is the most significant one, the elastic deformation remains a very important element of soil behavior. Hence, it is essential that the elastic behavior is represented properly in elasto-plastic constitutive modeling of soil.

For isotropic soil, the elastic behavior may be expressed with two independent parameters, such as the Young’s modulus, $E$, and Poisson’s ratio, $v$, (or the shear modulus, $G$, and bulk modulus, $B$). Experimental evidence based on several independent investigations shows that the elastic moduli $E$, $G$ and $B$ of a soil element depend
primarily on three factors: (a) the state of stress, (b) the density and (c) the stress history of the material (Hardin & Black 1968, 1969; Hardin 1978; Seed et al. 1984). The shear modulus, $G$, has been expressed generally in the form

$$G = A \ F(e) \ \sigma_o'^{\mu}$$  \hspace{1cm} (5-1)

where $\sigma_o'$ is the mean effective normal stress; $\mu$ is a material constant that is usually taken about 0.5 (but it may range from 0.35 to 0.90 for different soils); $F(e)$ depends on the void ratio, $e$; and the value of $A$ depends on the stress history, angularity of grains, etc. The expression for $G$ proposed by Hardin and Black (1968, 1969) for clays and sands is given by

$$G = 3230 \ \frac{(2.973 - e)^2}{1 + e} \ OCR^k \ \sigma_o'^{0.5}$$  \hspace{1cm} (5-2)

where the mean effective normal stress $\sigma_o'$ is in kN/m$^2$, $OCR$ is the over-consolidation ratio and the exponent $k$ depends on the plasticity index of the soil.

In the above and other similar expressions found in the literature, the dependence of the elastic moduli on the stress state is restricted solely to the mean effective normal stress $\sigma_o' = (\sigma_z' + \sigma_y' + \sigma_z')/3$ where $\sigma_z'$, $\sigma_y'$, and $\sigma_z'$ are the effective normal stresses. For example, a widely used modulus values is the initial tangent modulus (Duncan and Chang, 1970), which is expressed as a power function of the confining pressure, expressed by

$$E = K_w \ p_a \left(\frac{\sigma_o'}{p_a}\right)^n$$  \hspace{1cm} (5-3)

in which $p_a$ is the atmospheric pressure expressed in the same units as $E$ and $\sigma_o'$. For avoiding the influence of non-recoverable plastic deformations, $K_w$, a modulus number, is evaluated by the slope of an unloading-reloading cycle from a triaxial compression test and $n$ is an exponent determining the rate of variation of $E$ with $\sigma_o'$. Both $K_w$ and $n$ are
dimensionless numbers. This simple formulation appears to capture the elastic behavior observed in triaxial and isotropic compression, and the material parameters can be determined from the results of conventional triaxial tests (Lade and Nelson, 1987). However, Zytynski et al. (1978) pointed out that the variation of Young's modulus results in Equation (5-3) violates the principle of conservation of energy, i.e. the model will generate or dissipate energy in violation of the basic premise of elastic behavior.

By considering the principle of conservation of energy, so that, for an elastic isotropic material, there is neither generation nor dissipation of energy in a closed-loop stress path, Lade and Nelson (1987) derived an expression for the elastic moduli given by

$$E = M p_a \left[ \left( \frac{J_1}{p_a} \right)^2 + 6 \frac{1 + \nu}{1 - 2\nu} \frac{J_2}{p_a^2} \right]^\lambda$$

(5-4)

where \(J_2\) is the second deviatoric stress invariant given by

$$J_2 = \frac{1}{6} \left[ (\sigma'_x - \sigma'_y)^2 + (\sigma'_y - \sigma'_z)^2 + (\sigma'_z - \sigma'_x)^2 \right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$

(5-5)

\(\nu\) is Poisson's ratio, \(M\) and \(\lambda\) are material parameters that can be determined easily from standard triaxial tests, and \(p_a\) is the atmospheric pressure. The isotropic shear modulus \(G\) and bulk modulus \(K\) can be computed directly from Equation (5-4) and the relations

$$G = \frac{E}{2(1 + \nu)}$$

(5-6)

and

$$K = \frac{E}{3(1 - 2\nu)}$$

(5-7)

Comparisons with experimental results from triaxial tests and cubical triaxial tests on isotropically consolidated sand with various 2-D and 3-D stress paths demonstrated the ability of Equation (5-4) to capture the true nonlinear elastic behavior of soil (Lade & Nelson 1987). Also, comparisons between results from isotropic unloading and reloading
tests on both loose and dense Fine Ottawa Sands and Equation (5-4) showed excellent agreement for the entire examined stress range (Dakoulas & Sun 1992).

Equation (5-4) is based on the reasonable assumption of the constant value of Poisson's ratio. The theoretical limits of Poisson's ratio for the isotropic material are $-1 \leq \nu \leq 0.5$. These limits are obtained on the basis that Young's modulus and the elastic strain energy are positive for any change of stress. In practice, Poisson's ratio for the elastic behavior of soils is limited to the range from 0 to 0.5. Experimental evidence suggests that for soils, the value of Poisson's ratio at a given void ratio appears to be constant but increases with increasing void ratio (Lade and Nelson, 1987). However, considering the experimental difficulties and the insensitivity of the soil behavior to the value of Poisson's ratio, it is often satisfactory in most applications to assume a constant value (Hardin, 1978; Chen and Saleeb, 1982).

The elastic behavior of a cross-anisotropic material is defined by five independent elastic parameters. Three out of the five parameters can be obtained from standard triaxial tests, while the rest of them require more advanced testing using, for example, a cubical triaxial apparatus and a torsional simple shear device. Descriptions of experimental techniques are given in several published studies of cross-anisotropic soil behavior (Kirkgard & Lade 1991; Graham & Houlsby 1983; Yamada & Ishihara 1979; Atkinson 1975; Saada & Ou 1973).

To avoid the difficulties in determining the anisotropic parameters, it is often convenient in practical applications to ignore the effects of a small anisotropy and, instead, use the much simpler two-parameter isotropic model. However, numerical simulations of the undrained elastic reloading of anisotropically consolidated clay demonstrate clearly that the effect of the elastic anisotropic behavior is important and, hence, the use of an isotropic elastic model in such cases is inadequate (Yu and Dakoulas, 1992). For decreasing the effects of anisotropy, an improved method was
presented by Graham and Houlsby (1983). They reported that for lightly anisotropic natural clays, a simplified anisotropic model based on three rather than five parameters (and two additional assumptions) obtained from standard triaxial tests, reduced the error in the prediction of strains by 30% to 40%, compared to the prediction of the isotropic model using one parameter less.

This chapter presents a new general cross-anisotropic model for the purely elastic behavior of soil and of granular materials in general, while the plastic behavior is treated independently. In accordance with experimental evidence, the model assumes that the elastic Young's moduli and shear moduli are functions of the state of stress, while the three Poisson's ratios are independent of the state of stress. Moreover, the model assumes that, although the elastic Young's moduli in the horizontal and vertical directions, $E_h$ and $E_v$, increase with the confining stress, the ratio $E_h/E_v$ remains constant. This assumption is reasonable for soils or materials with inherent anisotropy, for which there is no significant reorientation of the soil particles or fundamental change in the fabric of the material with stress.

The expressions for the dependence of the elastic moduli on the state of stress are derived in a rigorous way by considering that in a closed-loop stress path there is neither generation nor dissipation of energy. The derivation of the expressions for the elastic moduli is presented in the following.

5.2 Derivation of Stress-Dependent Elastic Moduli

The elastic strain increments can be calculated from the effective stress increments by using Hook's law. The effective normal stress increment $\Delta \sigma'$ is given by $\Delta \sigma - u$ where $\Delta \sigma$ is total normal stress increment and $u$ is the pore water pressure. The effective shear stress increment $\Delta \tau'$ is equal to the total shear stress increment $\Delta \tau$. With the system of coordinate axes $(x,y,z)$, as shown in Figure 5-1, we define that the $x$-axis is
vertical and coincides with the direction of soil deposition. The generalized Hook's law for cross-anisotropic soils can be written by matrix expression as follows (Chen and Saleeb, 1982; Kirkgard and Lade, 1991)

\[
\begin{pmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \varepsilon_z \\
\Delta \varepsilon_{xy} \\
\Delta \varepsilon_{xz} \\
\Delta \varepsilon_{yz}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{E_v} & -\nu_{hv} & -\nu_{hv} & 0 & 0 & 0 \\
-\nu_{hv} & \frac{1}{E_h} & -\nu_{hv} & 0 & 0 & 0 \\
-\nu_{vh} & -\nu_{vh} & \frac{1}{E_h} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2G_{hv}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2G_{hv}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{hv}}
\end{pmatrix}
\begin{pmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \sigma_z \\
\Delta \tau_{xy} \\
\Delta \tau_{xz} \\
\Delta \tau_{yz}
\end{pmatrix}
\]

(5-8)

where

\[E_v = \text{Young's modulus in a vertical (x) direction,}\]
\( E_h \) = Young's modulus in any horizontal \((y, z)\) direction,

\( \nu_{hv} \) = Poisson's ratio of strain in vertical direction to horizontal direction due to a horizontal applied stress,

\( \nu_{vh} \) = Poisson's ratio of strain in horizontal direction to vertical direction due to a vertical applied stress,

\( \nu_{hh} \) = Poisson's ratio of strain in horizontal direction to an orthogonal horizontal direction due to an orthogonal horizontal applied stress,

\( G_{hv} \) = shear modulus in any vertical plane (through x-axes), and

\( G_{hh} \) = shear modulus in any horizontal plane \((y,z)\) plane.

The compliance matrix in Equation (5-8) must be symmetric because of the thermodynamic requirement. Thus,

\[
\frac{\nu_{vh}}{E_v} = \frac{\nu_{hv}}{E_h} \quad (5-9)
\]

Further, due to isotropy in horizontal planes

\[
G_{hh} = \frac{E_h}{2(1 + \nu_{hh})} \quad (5-10)
\]

Equation (5-8) can be rewritten as

\[
\begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \varepsilon_z \\
\Delta \varepsilon_{xy} \\
\Delta \varepsilon_{yz} \\
\Delta \varepsilon_{zx}
\end{bmatrix} =
\begin{bmatrix}
1 & -\nu_{vh} & -\nu_{vh} & 0 & 0 & 0 \\
-\nu_{vh} & E_v & E_v & E_v & 0 & 0 \\
-\nu_{vh} & E_v & E_v & E_v & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2G_{hv}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1 + \nu_{hh}}{E_h} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{hv}}
\end{bmatrix}
\begin{bmatrix}
\Delta \sigma'_x \\
\Delta \sigma'_y \\
\Delta \sigma'_z \\
\Delta \tau_{xy} \\
\Delta \tau_{yz} \\
\Delta \tau_{zx}
\end{bmatrix}
\]
For a general expression of Young's moduli in a cross-anisotropic soil, we consider first an isotropic material, for which the relations between normal stresses and strains may be expressed as

\[
\begin{pmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \varepsilon_z
\end{pmatrix} = \frac{1}{E} \begin{pmatrix}
1 & -\nu & -\nu \\
-\nu & 1 & -\nu \\
-\nu & -\nu & 1
\end{pmatrix}
\begin{pmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \sigma_z
\end{pmatrix}
\] (5-12)

For cross-anisotropic material having identical elastic behavior in the horizontal ($y$, $z$) directions, but different behavior in the vertical ($x$) direction, if we assume that the stiffness in the $y$ and $z$ directions is $n$ times larger than the stiffness in $x$-direction, the compliance matrix will be modified by multiplying with an anisotropic factor $n$ in the first element of the first row. Moreover, the deformation response may be different in $x$ and $y$ directions due to an applied stress in $z$-direction. This is achieved by multiplying an anisotropic factor $m$ to the second and third elements of the first row of the compliance matrix. In order to preserve the symmetry of the compliance matrix, the second and third elements of the first column are multiplied by $m$ to give the form

\[
\begin{pmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \varepsilon_z
\end{pmatrix} = \frac{1}{E^*} \begin{pmatrix}
n & -mv^* & -mv^* \\
-mv^* & 1 & -v^* \\
-mv^* & -v^* & 1
\end{pmatrix}
\begin{pmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \sigma_z
\end{pmatrix}
\] (5-13)

In this equation, the isotropic $E$ and $v$ values have been replaced by modified parameters $E^*$ and $v^*$ for anisotropic soil. When $n > 1$ the material is stiffer horizontally than vertically and for $n < 1$ the material is stiffer vertically than horizontally. By comparing Equations (5-11) and (5-13), we get

\[
E^* = E_h
\] (5-14a)

\[
v^* = v_{hh}
\] (5-14b)
\[ n = \frac{E_h}{E_v} \]  
(5-14c)

\[ m = \frac{E^*}{E_v} \frac{v_{sh}}{v^*} = n \frac{v_{sh}}{v_{sh}} \]  
(5-14d)

From above expression, the value of \( n \) is the ratio of Young’s moduli in the horizontal and vertical directions, and from Equation (5-9) \( n \) is also the ratio of two Poisson’s ratios.

\[ n = \frac{v_{sh}}{v_{sh}} \]  
(5-15)

Generally, the material is subjected to both normal stresses and shear stresses. Let \( \alpha^2 \) be the ratio of shear moduli in the horizontal and vertical plane, i.e.

\[ \alpha^2 = \frac{G_{hh}}{G_{hv}} \]  
(5-16)

The relationship between shear stress and shear strain in vertical plane and horizontal plane is described by

\[ \Delta \varepsilon_{xy} = \frac{1}{2G_{hv}} \Delta \tau_{xy} = \frac{\alpha^2}{2G_{hv}} \Delta \tau_{xy} = \frac{\alpha^2(1 + v_{sh})}{E_h} \Delta \tau_{xy} \]  
(5-17a)

\[ \Delta \varepsilon_{xz} = \frac{1}{2G_{hv}} \Delta \tau_{xz} = \frac{(1 + v_{sh})}{E_h} \Delta \tau_{xz} \]  
(5-17b)

\[ \Delta \varepsilon_{zx} = \frac{1}{2G_{hv}} \Delta \tau_{zx} = \frac{\alpha^2}{2G_{hv}} \Delta \tau_{zx} = \frac{\alpha^2(1 + v_{sh})}{E_h} \Delta \tau_{zx} \]  
(5-17c)

Then, the matrix form, in Equation (5-11), is rewritten as

\[
\begin{bmatrix}
\Delta \varepsilon_x / m \\
\Delta \varepsilon_y \\
\Delta \varepsilon_z \\
\Delta \varepsilon_{xy} / \alpha \\
\Delta \varepsilon_{xz} \\
\Delta \varepsilon_{zx} / \alpha
\end{bmatrix}
= \begin{bmatrix}
n/m^2 & -v_{sh} & -v_{sh} & 0 & 0 & 0 \\
-v_{sh} & 1 & -v_{sh} & 0 & 0 & 0 \\
-v_{sh} & -v_{sh} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 + v_{sh} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + v_{sh} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 + v_{sh}
\end{bmatrix}
\begin{bmatrix}
m \Delta \sigma'_x \\
\Delta \sigma'_y \\
\Delta \sigma'_z \\
\Delta \tau_{xy} \\
\Delta \tau_{xz} \\
\alpha \Delta \tau_{zx}
\end{bmatrix}
\]  
(5-18)

Here, we define a transformed stress tensor by
\[
\begin{bmatrix}
\sigma_y' & \tau_{xy}' & \tau_{yx}' \\
\tau_{xy}' & \sigma_y' & \tau_{yz}' \\
\tau_{yx}' & \tau_{yz}' & \sigma_z'
\end{bmatrix}
= \begin{bmatrix}
m\sigma_y' & \alpha\tau_{xy} & \alpha\tau_{zx} \\
\alpha\tau_{xy} & \sigma_y' & \tau_{yz} \\
\alpha\tau_{zx} & \tau_{yz} & \sigma_z'
\end{bmatrix}
\]

The first invariant and the second deviatoric invariant of the transformed stress tensor are given by

\[
I_1^* = m\sigma_y' + \sigma_y' + \sigma_z'
\]

\[
J_2^* = \frac{1}{6} \left[ (m\sigma_y' - \sigma_y')^2 + (\sigma_y' - \sigma_z')^2 + (\sigma_z' - m\sigma_y')^2 \right] + (\alpha\tau_{xy})^2 + \tau_{yz}^2 + (\alpha\tau_{zx})^2
\]  

We may rewrite Equation (5-18) for an infinitesimal increment of stresses and strains as

\[
d\varepsilon_x = \frac{1}{E_h} \left[ n m\sigma_z' - m\nu_{ph}(d\sigma_y' + d\sigma_z') \right] \\
= \frac{1}{E_h} \left[ \left( \frac{n}{m} + m\nu_{ph} \right)(m\sigma_z' + m\sigma_y' + d\sigma_y' + d\sigma_z') \right] \\
= \frac{1}{E_h} \left[ \left( \frac{n}{m} + m\nu_{ph} \right)m\sigma_z' - m\nu_{ph} d\varepsilon_z^* \right] 
\]

\[
d\varepsilon_y = \frac{1}{E_h} \left[ -\nu_{ph}(m\sigma_z') + d\sigma_y' - \nu_{ph} d\sigma_z' \right] \\
= \frac{1}{E_h} \left[ (1 + \nu_{ph})d\sigma_y' - \nu_{ph}(m\sigma_z' + d\sigma_y' + d\sigma_z') \right] \\
= \frac{1}{E_h} \left[ (1 + \nu_{ph})d\sigma_y' - \nu_{ph} d\varepsilon_z^* \right] 
\]

\[
d\varepsilon_z = \frac{1}{E_h} \left[ (1 + \nu_{ph})d\sigma_z' - \nu_{ph} d\varepsilon_z^* \right] 
\]

\[
d\varepsilon_{xy} = \frac{\alpha (1 + \nu_{ph})}{E_h} d(\alpha\tau_{xy}) 
\]

\[
d\varepsilon_{yz} = \frac{1 + \nu_{ph}}{E_h} d\tau_{yz} 
\]

\[
d\varepsilon_{zx} = \frac{\alpha (1 + \nu_{ph})}{E_h} d(\alpha\tau_{zx}) 
\]
The increment of elastic work is calculated by

\[ dW = \sigma'_x d\varepsilon_x + \sigma'_y d\varepsilon_y + \sigma'_z d\varepsilon_z + 2 \tau_{xy} d\varepsilon_{xy} + 2 \tau_{xz} d\varepsilon_{xz} + 2 \tau_{zx} d\varepsilon_{zx} \]  

(5-22)

Substituting Equations (5-21) into Equation (5-22), yields

\[ dW = \frac{1}{E_h} \left[ \left( \frac{n}{m^2} + \nu_{hh} \right) (m\sigma'_x) d(m\sigma'_x) - \nu_{hh} (m\sigma'_x) dI'_x \right] \]

\[ + \frac{1}{E_h} \left[ (1 + \nu_{hh}) \sigma'_y d\sigma'_y - \nu_{hh} \sigma'_y dI'_y \right] + \frac{1}{E_h} \left[ (1 + \nu_{hh}) \sigma'_z d\sigma'_z - \nu_{hh} \sigma'_z dI'_z \right] \]

\[ + \frac{2}{E_h} \left[ (1 + \nu_{hh}) \left( \sigma'_y d\sigma'_y \tau_{xy} d\tau_{xy} + (\alpha \tau_{xy}) d(\alpha \tau_{xy}) \right) \right] \]

\[ = \frac{1}{E_h} \left[ \left( \frac{n}{m^2} - 1 \right) (m\sigma'_x) d(m\sigma'_x) - \nu_{hh} l'_x dI'_x + \frac{1}{E_h} \sqrt{J'} \cdot d(\sqrt{J'}) \right] \]

\[ = \left( \frac{n}{m^2} - 1 \right) \frac{m\sigma'_x}{E_h} d(m\sigma'_x) - \frac{\nu_{hh} l'_x}{E_h} dI'_x + 2(1 + \nu_{hh}) \frac{\sqrt{J'}}{E_h} d(\sqrt{J'}) \]  

(5-23)

where

\[ J' = \frac{1}{2} (m\sigma'_x)^2 + \frac{1}{2} \sigma'_y^2 + \frac{1}{2} \sigma'_z^2 + (\alpha \tau_{xy})^2 + \tau_{xz}^2 \]  

(5-24)

Because \( m\sigma'_x, l'_x, \) and \( \sqrt{J'} \) are independent, \( dW \) may be expressed by

\[ dW = \frac{\partial W}{\partial (m\sigma'_x)} d(m\sigma'_x) + \frac{\partial W}{\partial l'_x} dI'_x + \frac{\partial W}{\partial (\sqrt{J'})} d(\sqrt{J'}) \]  

(5-25)

Comparison Equations (5-23) and (5-25) produces

\[ \frac{\partial W}{\partial (m\sigma'_x)} = \left( \frac{n}{m^2} - 1 \right) \frac{m\sigma'_x}{E_h} \]  

(5-26a)

\[ \frac{\partial W}{\partial l'_x} = -\nu_{hh} \frac{l'_x}{E_h} \]  

(5-26b)

\[ \frac{\partial W}{\partial (\sqrt{J'})} = 2(1 + \nu_{hh}) \frac{\sqrt{J'}}{E_h} \]  

(5-26c)
in which Young's modulus $E^*$ is defined as a function of $m\sigma^*_i$, $I^*_i$, and $\sqrt{J^*}$. According to the principle of conservation of energy, the elastic work per unit volume along a closed stress path should be zero

$$W = \int dW = 0 \quad (5-27)$$

Because the elastic work is independent of the stress path, no net work will be generated or dissipated in a closed stress path. The curl of the gradient of elastic work will disappear along the closed path (Stokes' theorem, Wylie and Barrett, 1982), that is

$$\iint_S N \cdot \nabla \times \nabla W \, dS$$

$$= \iint_S \left[ \frac{\partial^2 W}{\partial (m \sigma^*_i) \partial I^*_i} - \frac{\partial^2 W}{\partial I^*_i \partial (m \sigma^*_i)} \right] n_1 + \left[ \frac{\partial^2 W}{\partial \sqrt{J^*} \partial (m \sigma^*_i)} - \frac{\partial^2 W}{\partial (m \sigma^*_i) \partial \sqrt{J^*}} \right] n_2$$

$$+ \left[ \frac{\partial^2 W}{\partial \sqrt{J^*} \partial (m \sigma^*_i)} - \frac{\partial^2 W}{\partial (m \sigma^*_i) \partial \sqrt{J^*}} \right] n_3 \right] \, dS = 0 \quad (5-28)$$

in which $N$ is a unit vector normal to the surface $S$ in space $(m \sigma^*_i, I^*_i, \sqrt{J^*})$, and the closed stress path is lying on the surface $S$ as its boundary curve. From Equations (5-27) and (5-28), the following conditions can be derived.

$$\frac{\partial^2 W}{\partial (m \sigma^*_i) \partial I^*_i} = \frac{\partial^2 W}{\partial I^*_i \partial (m \sigma^*_i)} \quad (5-29a)$$

$$\frac{\partial^2 W}{\partial I^*_i \partial \sqrt{J^*}} = \frac{\partial^2 W}{\partial \sqrt{J^*} \partial I^*_i} \quad (5-29b)$$

$$\frac{\partial^2 W}{\partial \sqrt{J^*} \partial (m \sigma^*_i)} = \frac{\partial^2 W}{\partial (m \sigma^*_i) \partial \sqrt{J^*}} \quad (5-29c)$$

Substituting the derivative of Equations (5-26a) and (5-26b) into Equation (5-29a) produces

$$\left( \frac{n}{m^2} - 1 \right) \frac{1}{I^*_i} \frac{\partial E_h}{\partial I^*_i} = \nu_{bh} \frac{1}{m \sigma^*_i} \frac{\partial E_h}{\partial (m \sigma^*_i)} \quad (5-30)$$
The general solution of the above equation can be expressed by \( E_\eta = E_\eta(X) \), and

\[
X = C_1 \left[ l_i^2 - \frac{1}{v_{hh}} \left( \frac{n}{m^2} - 1 \right) (m/2)^2 \right] + f_i(\sqrt{J^*}) \tag{5-31}
\]

in which \( C_1 \) is a constant and \( f_i \) is a function of \( \sqrt{J^*} \).

Substituting the derivative of Equations (5-26b) and (5-26c) into Equation (5-29b) produces

\[
v_{hh} \frac{1}{\sqrt{J^*}} \frac{\partial E_\eta}{\partial (\sqrt{J^*})} = -2 (1 + v_{hh}) \frac{1}{\Gamma_i} \frac{\partial E_\eta}{\partial \Gamma_i} \tag{5-32}
\]

and

\[
X = C_2 \left[ l_i^2 - \frac{2}{v_{hh}} \left( \frac{1 + v_{hh}}{m/2} \right) \left( \sqrt{J^*} \right)^2 \right] + f_2(m/2) \tag{5-33}
\]

in which \( C_2 \) is a constant and \( f_2 \) is a function of \( m/2 \).

Substituting the derivative of Equations (5-26c) and (5-26a) into Equation (5-29c) produces

\[
2 \left( 1 + v_{hh} \right) \frac{1}{m/2} \frac{\partial E_\eta}{\partial (m/2)} = \left( \frac{n}{m^2} - 1 \right) \frac{1}{\sqrt{J^*}} \frac{\partial E_\eta}{\partial (\sqrt{J^*})} \tag{5-34}
\]

and

\[
X = C_3 \left[ l_i^2 - \frac{2}{v_{hh}} \left( \frac{1 + v_{hh}}{m/2} \right) \left( \sqrt{J^*} \right)^2 \right] + f_3(\Gamma_i) \tag{5-35}
\]

in which \( C_3 \) is a constant and \( f_3 \) is a function of \( \Gamma_i \).

Combining Equations (5-31), (5-33), and (5-35) yields

\[
X = C \left[ l_i^2 - \frac{2}{v_{hh}} \left( \frac{1 + v_{hh}}{m/2} \right) \left( \sqrt{J^*} \right)^2 - \frac{1}{v_{hh}} \left( \frac{n}{m^2} - 1 \right) (m/2)^2 \right] \tag{5-36}
\]

where \( C \) is a constant. By expressing \( J^* \) as

\[
J^* = \frac{1}{6} \Gamma_i^2 + J_2^*
\]

and letting \( C = \frac{3 v_{hh}}{p_2^2 (2v_{hh} - 1)} \), \( X \) can be rewritten in a dimensionless form
\[ X = \left( \frac{J}{p_a} \right)^2 + 6 \frac{1 + v_{hh}}{1 - 2v_{hh}} \frac{J^*_2}{p_a^2} + \frac{3}{1 - 2v_{hh}} (n - m^2) \left( \frac{\sigma_x'}{p_a} \right)^2 \]  \hspace{1cm} (5-38)

(where \( p_a \) = atmospheric pressure)

As in the case of isotropic soil, the Young's modulus, \( E_v \) and \( E_h \) can be expressed as a power function of \( X \) as

\[ E_v = M \cdot p_a \left[ \left( \frac{J}{p_a} \right)^2 + 6 \frac{1 + v_{hh}}{1 - 2v_{hh}} \frac{J^*_2}{p_a^2} + \frac{3(n - m^2)}{1 - 2v_{hh}} \left( \frac{\sigma_x'}{p_a} \right)^2 \right]^\lambda \]  \hspace{1cm} (5-39)

and

\[ E_h = n \cdot M \cdot p_a \left[ \left( \frac{J}{p_a} \right)^2 + 6 \frac{1 + v_{hh}}{1 - 2v_{hh}} \frac{J^*_2}{p_a^2} + \frac{3(n - m^2)}{1 - 2v_{hh}} \left( \frac{\sigma_x'}{p_a} \right)^2 \right]^\lambda \]  \hspace{1cm} (5-40)

where \( M \) and \( \lambda \) are material parameters which can be determined by experiment. Note that the parameter \( M \) is analogous to the terms \( AF(e) \) in Equation (5-1), expressing the dependence of the modulus on the density and stress history of the material. For isotropic material, the values of \( m, n, \) and \( \alpha \) are equal to 1, and Equations (5-39) and (5-40) reduce to Equation (5-4) of an isotropic soil (Lade and Nelson, 1987).
Summary of Elastic Parameters for Cross Anisotropic Soil

1. \( n = \frac{v_{hv}}{v_{hh}} \) \[ \text{or} \quad n = \frac{E_{hh}}{E_v} \] (5-41)

2. \( m = \frac{v_{hv}}{v_{hh}} \) (5-42)

3. \( E_v = M p_a \left[ \left( \frac{I_1^*}{p_a} \right)^2 + 6 \frac{1+v_{hh}}{1-2v_{hh}} \frac{J_2^*}{p_a^2} + \frac{3(n-m^2)}{1-2v_{hh}} \left( \frac{\sigma_z'}{p_a} \right)^2 \right] \) (5-43)

4. \( E_h = n E_v \) (5-44)

5. \( G_{hh} = \frac{E_h}{2(1 + \nu_{hh})} \) (5-45)

6. \( G_{hv} = \frac{G_{hh}}{\alpha^2} \) (5-46)

where

\( I_1^* = m\sigma_z' + \sigma_y' + \sigma_z' \)

\( J_2^* = \frac{1}{6} \left[ (m\sigma_z' - \sigma_y')^2 + (\sigma_y' - \sigma_z')^2 + (\sigma_z' - m\sigma_z')^2 \right] + (\alpha \tau_{xy})^2 + \tau_{yx}^2 + (\alpha \tau_{zz})^2 \)
5.3 Results and Discussion

It is of interest to explore the effect of the three variables $I_1'$, $J_2'$, and $m\sigma_i'$ on $E_v$. Figure 5-2 plots contours of constant $E_v$ in the triaxial and $\pi$ planes in a principal stress space ($\sigma_1'$, $\sigma_2'$, $\sigma_3'$) for $n=m^2$. In this case $E_v$ is given by

$$E_v = M \frac{p_o}{p_s} \left[ \left( \frac{I_1'}{p_o} \right)^2 + 6 \frac{1 + \nu_{hh}}{1 - 2\nu_{hh}} \frac{J_2'}{p_o^2} \right]$$

(5-47)

The results are plotted for $n=0.5$, 1 and 2 in Figs. 5-2(a), 5-2(b) and 5-2(c), respectively, and for six values of Poisson's ratio ($\nu_{hh} = 0$, 0.1, 0.2, 0.3, 0.4 and 0.5). Although the only meaningful part of these surfaces is that lying within the failure surface of the soil, in order to illustrate better their shape, they are plotted even for stress states beyond failure.

Note that for $n=m=1$ in Figure 5-2(b) the results correspond to the isotropic case, in which the contours of constant $E_v$ are ellipsoidal surfaces having an axis of symmetry parallel to the hydrostatic axis, as shown by Lade and Nelson (1987). The interception of these surfaces with the triaxial plane are ellipses whose aspect ratio depends on $\nu_{hh}$, while their interception with the $\pi$-plane are concentric circles. The ellipsoidal surface becomes a sphere for $\nu_{hh}=0$, and degenerates to a line parallel to the hydrostatic axis as $\nu_{hh}$ approaches 0.5.

For a cross-anisotropic soil with $E_h = 0.5E_v$ in Figure 5-2(a), the ellipsoidal surfaces change their aspect ratio and, as $\nu_{hh}$ decreases, rotate their long axis on the triaxial plane towards the direction of $\sigma_i'$. By contrast, for a soil with $E_h = 2E_v$ in Figure 5-2(c), as $\nu_{hh}$ decreases, the ellipsoidal surfaces rotate their long axis towards the horizontal plane. In all cases, for $\nu_{hh} = 0.5$, the ellipsoid degenerates to a line parallel to the direction $(1/m, 1, 1)$. Notice that, if the results in Figures 5-2(a) and 5-2(c) are plotted in the principal stress space ($m\sigma_1'$, $\sigma_2'$, $\sigma_3'$), the shape of the ellipsoids becomes identical to that in Figure 5-2(b).
Figure 5-2 shows the effect on $E_\nu$ of the principal stresses $\sigma'_x$, $\sigma'_y$, and $\sigma'_z$ applied in the directions of cross-anisotropy. For the general stress state shown in Figure 5-1, in which the directions of the principal stresses do not coincide with the directions of anisotropy, the effect of the shear stresses on $E_\nu$ is shown in Figure 5-3. This figure plots contours of constant $E_\nu$ in the shear stress space $(\tau_{xy}, \tau_{xz}, \tau_{yz})$ for $\alpha=0.5$, 1 and 2 and for six values of Poisson’s ratio. For isotropic soil, $\alpha=1$ in Figure 5-3(b), the surfaces of constant $E_\nu$ are concentric spheres, while for cross-anisotropic soil, they become ellipsoids with their long axis parallel the dichotomy of the direction of $\tau_{xy}$ and $\tau_{xz}$ for $\alpha<1$ in Figure 5-3(a) and to the direction of $\tau_{yz}$ for $\alpha>1$ in Figure 5-3(c).

The results in Figures 5-2 and 5-3 illustrate the effect of the ratio of the Young's moduli $n = E_h/E_\nu$ for the special case of $m = \sqrt{n}$. It is of interest to examine the effect of both $n$ and $m$ on $E_\nu$. Figure 5-4 plots contours of constant $E_\nu$ for $n=0.5$, 1 and 1.5, respectively, a constant value of Poisson’s ratio $\nu_{hs}=0.3$, and for $m = 0.5\sqrt{n}$, $\sqrt{n}$, $1.5\sqrt{n}$. The results are plotted in the triaxial plane and in the octahedral plane passing through the same stress point $a$ for all figures. Notice that as $m$ increases the ellipsoidal surfaces of equal $E_\nu$ increase their size, while they also change their aspect ratio. When $n$ increases the ellipsoidal surfaces change their aspect ratio and rotate their long axis on the triaxial plane towards the direction of the horizontal plane. The results for $n=m=1$, plotted with a solid line in Figure 5-4(b), correspond to the isotropic case.
$E_v = \text{CONSTANT}$

$n = m^2 = 0.5$

$\sqrt{2} \sigma'_y = \sqrt{2} \sigma'_x$

(a)

**Figure 5-2** Contours of constant Young's modulus $E_v$ in the triaxial and octahedral planes of the principal stress space for $n = m^2$: (a) $n = 0.5$ (b) $n = 1$ and (c) $n = 2$.

(continued)
\[ E_v = \text{CONSTANT} \]
\[ n = m^2 = 1 \text{ (Isotropy)} \]

Figure 5-2 Contours of constant Young's modulus \( E_v \) in the triaxial and octahedral planes of the principal stress space for \( n = m^2 \): (a) \( n = 0.5 \) (b) \( n = 1 \) and (c) \( n = 2 \).

(continued)
\( E_v = \text{CONSTANT} \)

\( n = m^2 = 2 \)

\[ \sqrt{2} \sigma'_y = \sqrt{2} \sigma'_z \]

**Figure 5-2** Contours of constant Young's modulus \( E_v \) in the triaxial and octahedral planes of the principal stress space for \( n = m^2 \): (a) \( n = 0.5 \) (b) \( n = 1 \) and (c) \( n = 2 \).
Figure 5-3 Contours of constant Young’s modulus $E_v$ in the shear space:
(a) $\alpha = 0.5$ (b) $\alpha = 1$ and (c) $\alpha = 2$  (continued)
$E_v = \text{CONSTANT} \quad , \quad \alpha = 1$

Figure 5-3 Contours of constant Young's modulus $E_v$ in the shear space:
(a) $\alpha = 0.5$ (b) $\alpha = 1$ and (c) $\alpha = 2$ (continued)
$E_\nu = \text{CONSTANT}$, $\alpha = 2$

Figure 5.3 Contours of constant Young’s modulus $E_\nu$ in the shear space:
(a) $\alpha = 0.5$, (b) $\alpha = 1$ and (c) $\alpha = 2$. 
$E_v = \text{CONSTANT}$

$v_{hh} = 0.3, \quad n = 0.5$

---

Figure 5-4 Contours of $E_v$ for three cross-anisotropic soils with (a) $n = 0.5$ (b) $n = 1$ (c) $n = 1.5$

and $m = 0.5 \sqrt{n}, \sqrt{n}, 1.5 \sqrt{n}$ in triaxial plane and octahedral plane at point $a$. 

(continued)
\[ E_v = \text{CONSTANT} \]
\[ \nu_{hh} = 0.3, \quad n = 1 \]

**Figure 5-4** Contours of \( E_v \) for three cross-anisotropic soils with (a) \( n = 0.5 \) (b) \( n = 1 \) (c) \( n = 1.5 \) and \( m = 0.5\sqrt{n}, \sqrt{n}, 1.5\sqrt{n} \) in triaxial plane and octahedral plane at point \( a \).

(continued)
$E_v = \text{CONSTANT}$

$\gamma_{\text{ct}} = 0.3, \quad n = 1.5$

$$m = \frac{0.5 \sqrt{n}}{\sqrt{3}}$$

$$m = \frac{1.5 \sqrt{n}}{\sqrt{3}}$$

Octahedral plane through point $\alpha$

Figure 5.4: Contours of $E_v$ for three cross-anisotropic soils with (a) $n = 0.5$, (b) $n = 1$, (c) $n = 1.5$ and $m = \frac{0.5 \sqrt{n}}{\sqrt{3}}, \frac{1.5 \sqrt{n}}{\sqrt{3}}$ in triaxial plane and octahedral plane at point $\alpha$. 
5.4 Poisson’s Ratio Limits for Cross-Anisotropic Soil

The above analysis is based on effective-stress Poisson’s ratios (i.e. $\nu_{kh}$, $\nu_{nh}$, $\nu_{hv}$), which are practically independent of the confining stress. Based on the thermodynamic requirement that the strain energy of an elastic material should be positive, the compliance matrix in Eq. (5-11) must be positive definite. To satisfy this condition for an elastic cross-anisotropic material, all the principal minors of the compliance matrix must be positive.

\[
\begin{vmatrix} 1 \\ \frac{1}{E_v} \end{vmatrix} = D_1 \geq 0 \tag{5-48}
\]

\[
\begin{vmatrix} 1 & -\nu_{sh} \\ \frac{1}{E_v} & \frac{1}{E_h} \end{vmatrix} = D_2 \geq 0 \tag{5-49}
\]

\[
\begin{vmatrix} 1 & -\nu_{sh} & -\nu_{sh} \\ \frac{1}{E_v} & \frac{1}{E_h} & \frac{1}{E_h} \\ -\nu_{sh} & -\nu_{sh} & \frac{1}{E_n} \end{vmatrix} = D_3 \geq 0 \tag{5-50}
\]

\[
\begin{vmatrix} D_3 \\ \frac{1}{2G_{hv}} \end{vmatrix} = D_4 \geq 0 \tag{5-51}
\]

\[
\begin{vmatrix} D_4 \\ \frac{1}{E_h} \end{vmatrix} = D_5 \geq 0 \tag{5-52}
\]

\[
\begin{vmatrix} D_5 \\ \frac{1}{2G_{hv}} \end{vmatrix} = D_6 \geq 0 \tag{5-53}
\]
Since Young’s moduli and shear moduli are always positive, the limits of Poisson’s ratio $v_{hh}$ can be obtained by Equation (5-52) as

$$v_{hh} \geq -1$$ (5-54)

Although in most soils the value of Poisson’s ratio $v_{hh}$ is between 0 and 0.5, negative values have been indeed reported in certain cases (Gazetas 1982; Gerrard 1977; Hooper 1975). The same requirement of Equations (5-49) and (5-50) limits the value of $v_{vh}$ to be between

$$-\sqrt{\frac{1-v_{hh}}{2}} \leq v_{vh} \leq \sqrt{\frac{1-v_{hh}}{2}}$$ (5-55)

Finally, since $v_{hv} = n v_{vh}$ the value of $v_{hv}$ must satisfy the condition

$$-\sqrt{n \frac{(1-v_{hh})}{2}} \leq v_{hv} \leq \sqrt{n \frac{(1-v_{hh})}{2}}$$ (5-56)

For the new cross-anisotropic elastic model, the values of $v_{vh}$ and $v_{hh}$ can not be zero in order to avoid infinite values of $m$ and $n$ during numerical computations.

5.5 Experimental Determination of Elastic Parameters

The determination of elastic parameters for a cross-anisotropic soil is more difficult than that for an isotropic soil and requires some advanced equipment. The procedure of evaluation for elastic parameters is presented in the following:

1. Poisson’s ratios $v_{hv}$ and $v_{hh}$

The values of $v_{hv}$ and $v_{hh}$ can be directly evaluated from cubical triaxial compression tests (Kirkgaard and Lade, 1991). From the cubical unconfined compression test (described in section 4.2), the average values of $v_{hv}$ and $v_{hh}$ for the marine tested are equal to $v_{hv}=0.25$ and $v_{hh}=0.18$. 
2. Parameter $n$

For an undrained triaxial test, with $\Delta \sigma'_v = \Delta \sigma'_z$, on a saturated specimen, the increment of volumetric strain is zero, i.e.,

$$\Delta \varepsilon_v = \Delta \varepsilon_x + \Delta \varepsilon_y + \Delta \varepsilon_z = 0$$  \hspace{1cm} (5-57)

Substituting the linear strains from Equation (5-11) into Equation (5-57), yields

$$\frac{\Delta \sigma'_x}{\Delta \sigma'_y} = \frac{2v_{sh} - \frac{2}{n} + \frac{2v_{nh}}{n}}{1 - 2v_{sh}}$$  \hspace{1cm} (5-58a)

or

$$\frac{\Delta \sigma'_x}{\Delta \sigma'_y} = \frac{2(v_{hv} + v_{nh} - 1)}{n - 2v_{hv}}$$  \hspace{1cm} (5-58b)

Expressing $\Delta q' = (\Delta \sigma'_x - \Delta \sigma'_y)$ and $\Delta p' = \frac{1}{3}(\Delta \sigma'_x + 2\Delta \sigma'_y)$, the following equation for the slope of the effective stress path is obtained

$$\frac{\Delta q'}{\Delta p'} = 3 \left( \frac{\Delta \sigma'_x}{\Delta \sigma'_y} \right) - 1$$  \hspace{1cm} (5-59)

Substituting Equation (5-58b) into Equation (5-59) produces

$$n = \frac{2}{2r_0 + 3} \left[ r_0 (v_{hv} - v_{sh} + 1) + 3(2v_{hv} + v_{sh} - 1) \right]$$  \hspace{1cm} (5-60)

where $r_0 = \Delta q' / \Delta p'$ is obtained from the initial slope of the effective stress path on the $q' - p'$ diagram.

The initial deformation response during a triaxial extension test may approach purely elastic behavior due to unloading in the vertical direction. Figure 5.5 shows stress paths for conventional triaxial extension tests (CTE) having an average value $r_0 = -15$. By substituting the value of $r_0$ into Equation (5-60), the value of $n$ is found equal to $n = 1.26$. 
3. Poisson's ratio $\nu_{sh}$

The value may be investigated directly from the drained unloading test (described section 4.4.2), computed by

$$\nu_{sh} = -\frac{\Delta\varepsilon_x}{\Delta\varepsilon_z}$$

(5-61)

or can be obtained by $\nu_{sh} = \nu_{hv}/n = 0.2$.

4. Parameter $m$

It may be computed from Equation (5-42). The value of $m$ is equal to 1.

5. Parameters $M$ and $\lambda$ for Young's moduli

To determine the parameters $M$ and $\lambda$ requires measurements of Young's modulus $E_y$ at various stress levels using conventional drained or undrained triaxial tests. The parameters $M$ and $\lambda$ in Equation (5-43) can then be determined by plotting $E_y/p_a$ vs. $X$, in Equation (5-36), in a log-log diagram. The intercept of the best-fitting line with $X=1$ is the value of $M$, and $\lambda$ is the slope of the line.

The Young's moduli of the marine clay were investigated from drained unloading test, as shown in Table 5-1, and the best estimated values of $M$ and $\lambda$ are $M=3.6$ and $\lambda=1.29$ (as shown in Figure 5-6).

6. Parameter $\alpha$

Due to the isotropy in horizontal planes, the shear modulus $G_{hh}$ is obtained from the parameters $E_h$ and $\nu_{hh}$ in Equation (5-10). Like the Young's modulus $E_y$, the shear modulus $G_{hv}$ is also measured at various stress levels. The individual shear moduli of $G_{hv}$, measured from torsional simple shear tests (TSS), are shown in Figure 5-7. Finally, the parameter $\alpha$ is computed from $\alpha = \sqrt{G_{hh}/G_{hv}}$. Table 5-2 lists the measured values of $\alpha$, having an average value of 2.39.
Figure 5-5 The value of \( r_0 \) from conventional triaxial extension tests (CTE)
Table 5-1 Data of the drained unloading test

<table>
<thead>
<tr>
<th>$\sigma'_x$ (kN/m²)</th>
<th>$\sigma'_y = \sigma'_z$ (kN/m²)</th>
<th>$\Delta \sigma'_x$ (kN/m²)</th>
<th>$\Delta \varepsilon_x$ (%)</th>
<th>$E_v$ (kN/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>176</td>
<td>105</td>
<td>-20</td>
<td>-0.001</td>
<td>18180</td>
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<td>-20</td>
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<td>-30</td>
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<tr>
<td>146</td>
<td>105</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5-2 Shear moduli and parameter $\alpha$ for the tested marine clay

<table>
<thead>
<tr>
<th>Tests</th>
<th>Consolidation stress, kN/m²</th>
<th>$G_{hr}$ (kN/m²)</th>
<th>$G_{hb}$ (kN/m²)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSS150</td>
<td>509 $\sigma'_x$ 300 $\sigma'_y = \sigma'_z$</td>
<td>27100</td>
<td>167370</td>
<td>2.48</td>
</tr>
<tr>
<td>TSS151</td>
<td>338 $\sigma'_x$ 200 $\sigma'_y = \sigma'_z$</td>
<td>14000</td>
<td>58260</td>
<td>2.04</td>
</tr>
<tr>
<td>TSS155</td>
<td>515 $\sigma'_x$ 304 $\sigma'_y = \sigma'_z$</td>
<td>27570</td>
<td>172830</td>
<td>2.50</td>
</tr>
<tr>
<td>TSS156</td>
<td>691 $\sigma'_x$ 400 $\sigma'_y = \sigma'_z$</td>
<td>56000</td>
<td>365030</td>
<td>2.55</td>
</tr>
</tbody>
</table>
Figure 5-6 Determination of $M$ and $\lambda$ for vertical Young's modulus, $E_v$. 

\[
\left( \frac{I}{p_a} \right)^2 + 6 \frac{1 + \nu_{kh}}{1 - 2\nu_{kh}} \frac{J_2}{p_a^2} + 3 \frac{n - m^2}{1 - 2\nu_{kh}} \left( \frac{\sigma_x}{p_a} \right)^2
\]
Figure 5-7 Measurement of shear modulus, $G_{hv}$, at various stress levels
Summary of Elastic Parameter Values for Marine Clay

The 6 elastic parameter values required for cross-anisotropically consolidated marine clay are listed in Table 5-3. All parameters are dimensionless. The parameters in Table 5-3 may be used to calculate the elastic strains for any combination of effective stresses during primary loading, neutral loading, unloading, and reloading.

Table 5-3 Summary of elastic parameter values for the tested marine clay

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>$v_{hh}$</td>
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<tr>
<td>$m = v_{hv}/v_{hh}$</td>
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</tr>
<tr>
<td>$n = v_{vh}/v_{hv}$</td>
<td>1.26</td>
</tr>
<tr>
<td>$M$</td>
<td>3.60</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.29</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>2.39</td>
</tr>
</tbody>
</table>
Chapter 6

CONSTITUTIVE MODELING OF CROSS-ANISOTROPIC SOIL: PLASTIC MODEL

The anisotropic plastic model is developed based on the assumptions that the material behavior is time independent and that the interaction between mechanical and the thermal processes is negligible. The model consists of a failure function, plastic potential function (non-associated flow rule), yield function and hardening law, which are formulated based on a few basic postulates and experimental observations. There are twelve material parameters in these functions. To describe the different behavior of isotropic and anisotropic soils at failure, a new four-parameter failure criterion is developed. Unlike existing isotropic failure criteria, the cross-section of the new failure surface with the octahedral plane may not symmetric with respect to the projection of the three principal axes. The plastic potential function, having two parameters, determines the directions of the plastic strain increments which are assumed to be independent of the stress path leading to the state of stress. The plastic potential surface expands along its center line and translates in the stress space depending on the consolidation state and the stress state. The yield criteria are associated with and derived from surfaces of constant plastic work. The yield-plastic work relation requires six parameters in which four parameters define yield function and two parameters define the plastic work equation. For the special case of isotropic soil, all of the functions may be reduced to those of Lade's isotropic elasto-plastic model.
6.1 Introduction

There is a variety of formulations of constitutive models for soils, that in general, can be classified into two basic approaches (Chen and Baladi, 1985):

(1) Finite material characterization in the form of secant stress-strain models. Included in this class of models are those based on nonlinear elasticity and deformation theory of plasticity.

(2) Incremental material descriptions in the form of tangential stress-strain models. The most prominent models of this category are those based on hypoelasticity and flow (incremental) theory of plasticity.

Due to the disadvantages of the deformation theory of plasticity (for example, the models do not satisfy the continuity requirement for loading conditions near or at neutral loading), the flow theory of plasticity will be employed in this study. The stress-strain law for a plastic material reduces to a relation involving the current state of stress and strains and the incremental changes of stress and plastic strain. This relation is generally assumed to be homogeneous and linear in the incremental changes of the components of stress and plastic strain. This assumption precludes viscosity effects and constitutes the time-independent idealization.

The first step towards such a mathematical model is to establish the yield limit of an elastic material. This is known as the yield function which is a certain function of the stress components. A plastic material is called perfectly plastic or work-hardening/softening according to whether the yield function is a fixed surface or it admits changes (expansion or contraction) as plastic strain develops. For moderate strain, mild steel behaves approximately like a perfectly plastic material. However, perfect plasticity is not appropriate for soils. The first work-hardening concepts in the development of
elasto-plastic model for soils were introduced by Drucker et al (1957). Since then, several work-hardening plasticity theories have been developed to describe the stress-strain behavior of soils under monotonic or cyclic loading conditions (Dafalias and Herrmann, 1980 & 1982; Krieg, 1975; Ghaboussi and Momen, 1984; Banerjee and Pan, 1986; Faruque and Desai, 1985; Hiral, 1987; Bannerjee and Yousif, 1986; Yong and Mohamed, 1988; Lade, 1990; Ramsamooj and Alwash, 1990, etc.). Moreover, several hardening rules have been proposed to describe the growth of subsequent yield surfaces for such materials. In general, three types of hardening rules have been commonly utilized, namely,

(1) Isotropic hardening.
(2) Kinematic hardening.
(3) Mixed hardening.

as described in Chapter 2 and Appendix B.

This chapter presents the development of the constitutive equations based on incremental theories of plasticity. These constitutive equations consist of a plastic potential function for the computation of the plastic deformation, yield function and hardening law, and failure criterion defined as the limit of yield function.

6.2 Failure Criterion

In this research, failure is defined as the state at which the effective stress ratio, major principal stress over the minor principal stress $\sigma_1/\sigma_3$, is maximum. The effective stress ratio, taken to indicate failure in soils, often occurs at axial strains of 12 to 15% for all compression tests using isotropic materials and less than 12% using anisotropic materials. Thus, the peak stress difference, $\sigma'_1-\sigma'_3$, in compression tests at high confining pressures are reached much earlier than the stress states corresponding to failure, i.e. the maximum stress difference may not indicate true failure of the granular materials (Lade,
Nelson and Ito, 1988). At failure, very large deformation takes place usually forming of a slip failure surface or zone. For simplicity, several existing constitutive models employ an isotropic failure model for anisotropic soils. Since it is certainly true that some clays exhibit significant anisotropy with respect to their undrained strength, they may require a more general failure criterion instead of an isotropic one. In general, two-parameter failure criteria are widely used for isotropic soil materials and are formulated in the six-dimensional stress space instead of the three-dimensional space of principal stress to take into account the effect of anisotropy.

Failure criteria usually postulate convexity of the failure surface in the principal stress space. This is supported by global stability arguments in plasticity, and all of the presented failure criteria (discussed in Chapter 2) satisfy this assumption. Clearly, there are some questions on the validity of this postulate for some soils, and in fact, there is experimental evidence that the failure envelope for sands over a wide range of hydrostatic pressure is non-convex with respect to the hydrostatic axis (Bishop, 1972). But, for most practical applications, the assumption of convexity is still reasonable especially in a limited range of confining pressures.

Among the existing failure models of soils, the two-parameter isotropic failure criterion by Lade, expressed by six-dimensional stress, has been proven to be quite satisfactory for a wide range of pressures for both sands and normally consolidated clays. It takes into account the curvature of the trace of the failure surface in the meridian plane indicating decreasing friction angle with increasing confining pressures.

In this study, a new four-parameter failure criterion for cross-anisotropic soils is developed. Like Lade's failure criterion, the new anisotropic failure criterion is shaped like a bullet with the pointed apex at the origin of the stress axes, and the shape of the cross section like a triangle with rounded corners. Unlike Lade's failure criterion, the surface of the new failure criterion is formulated by four parameters instead of two. The
new failure criterion is expressed in terms of the stress invariants and transformed stress invariants as:

$$
\left( \frac{I_1}{I_3} - N_f \right) \left( \frac{I_2}{27 I_3^2} \right)^{w_f} \left( \frac{I_3}{P_a} \right)^{m_f} = \eta_f
$$

(6-1)

where

\begin{align*}
I_1 &= \sigma'_z + \sigma'_y + \sigma'_z \\
I'_1 &= k_f \sigma'_z + \sigma'_y + \sigma'_z \\
I'_3 &= k_f \sigma'_z \sigma'_y + 2k_f \tau_{xy} \tau_{x} \tau_{y} - (k_f \sigma'_x \tau_{x}^2 + k_f \sigma'_y \tau_{y}^2 + k_f^2 \sigma'_z \tau_{xy}^2) \\
I''_1 &= \sigma'_z + k_f \sigma'_y + k_f \sigma'_z \\
I''_3 &= k_f^2 \sigma'_z \sigma'_y \sigma'_z \\
p_a &= \text{atmospheric pressure, and} \\
k_f, k_z, w_f, m_f \text{ and } \eta_f &= \text{material constants.}
\end{align*}

The parameter $k_z$ is a factor expressing the difference in the shear strength of the material between the vertical and the horizontal planes. Typical shapes of the failure criterion in the triaxial plane and the octahedral plane for $w_f = 0$ and 5 are shown in Figure 6.1. Notice that the parameter $\eta_f$ controls the size of the failure surface. When $\eta_f$ is equal to zero, the cross-section of the conical surface with the octahedral plane is a triangle with rounded corners, bounded by the $K_0$-line and the hydrostatic axis. When the value of $\eta_f$ increases, the surface expands in stress space. The cross-section of the failure surface with the octahedral plane is in general not symmetric with respect to the projection of the three axes of $\sigma'_z$, $\sigma'_y$, and $\sigma'_x$. To have $\eta_f = 0$ on both $K_0$-line and hydrostatic axis, two stress states, namely, $(\sigma, K_0 \sigma, K_0 \sigma, 0, 0, 0)$ on the $K_0$-line and $(\sigma, \sigma, \sigma, 0, 0, 0)$ on the hydrostatic axis, are substituted into Equation (6-1), yielding

$$
k_f = \frac{2(K_0^{2/3} - K_0)}{1 - K_0^{2/3}}
$$

(6-3)

and
\[ N_f = \frac{(2 + k_f)^3}{k_f} \]  

(6-4)

in which \( K_0 = \text{ lateral/vertical stress ratio at rest, i.e. } K_0 = \sigma_{z_0} / \sigma_{x_0} \) and \( \sigma'_{z_0} \) are minor consolidation stresses, and \( \sigma'_{x_0} \) is the major consolidation stress.

The failure surface in Equation (6-1) is obtained by combining two conical surfaces, expressed by \( I_1^3/I_2^3 - N_f \) and \( I_1^3/(27 I_2^3) \), whose pointed apexes are at the origin of the stress axis and their cross-sections have similar triangular shapes with rounded corners. The value of \( I_1^3/I_2^3 - N_f \) is equal to zero for the conical surface bounded by both \( K_0 \)-line and hydrostatic axis. As its value increases, the surface expands toward the failure surface. On the other hand, the value of \( I_1^3/(27 I_2^3) \) is equal to one on the hydrostatic axis, and as its value increases, the line becomes a conical surface that expands approaching the failure surface. The parameter \( w_f \) controls, as a weighting factor, the expansion rate of \( I_1^3/(27 I_2^3) \) which depends on the difference of the internal friction angles between compression and extension. Apparently, the ratio of compression strength to extension strength in Figure 6.1(a) is greater than that in Figure 6.1(b).

The failure surface is always convex with respect to the hydrostatic axis, and its curvature increases with the value of \( m_f \). If \( m_f = 0 \) the failure surface is straight, and if \( m_f > 1.979 \) the surface becomes concave with respect to the hydrostatic axis (Kim and Lade, 1984). For isotropically consolidated soils, for which \( k_f = 1 \) and \( N_f = 27 \), the failure criterion of Equation (6-1) reduces to Lade's two-parameter failure criterion for \( w_f = 0 \).
Figure 6.1 Typical failure criterion shown in triaxial plane and octahedral plane
(a) \( w_f = 0 \)  (b) \( w_f = 5 \).
6.3 Flow Rule

Based on non-associated flow rule, the plastic strain increments, $de_{yp}$, should be normal to the plastic potential function, $g_p$, at any point in the stress space, i.e.,

$$
(de_{yp})^p = \begin{cases} 
  d\lambda \frac{g_p}{\partial \sigma^r}, & \text{if } f = 0, \dot{f} > 0 \\
  0, & \text{otherwise}
\end{cases}
$$

(6-5)

in which $d\lambda$ is a positive scalar factor of proportionality.

For selecting a suitable plastic potential function, $g_p$, a number of monotonic tests on $K_0$-consolidated marine clay at different stress levels was performed (see Chapter 4). From experimental observations, it is found that the shape of Lade's isotropic plastic potential function, $g_p$, (described in section 2.3) is suitable for stress paths in triaxial compression tests, as well as for stress paths in triaxial extension tests but outside the region defined between the $K_0$-line and the hydrostatic axis (as shown in Figure 6-2). For the region between the $K_0$-line and the hydrostatic axis, the directions of plastic strain increments are toward the hydrostatic axis, while the initially anisotropic soil is subjected to extension loads; however, the normal vector of Lade's plastic potential surface in this region is almost in the opposite direction. To resolve this contradiction, Lade's plastic potential surface is translated in the stress space for the case in which the loading surface expands as the soil yields, but the plastic potential surface contracts. The location of the plastic potential surface in the stress space is defined by $d_{yp}$, and the modified plastic potential function, $\bar{g}_p$, is defined as

$$
\bar{g}_p = \left[ \Psi_1 \left( \frac{1}{I_3} - \frac{1}{I_2} \right) + \Psi_2 \right] \left( \frac{I_4}{P_a} \right)'
$$

(6-6)

in which $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$ are the stress invariants of a modified stress tensor, given by

$$
\bar{I}_i = \bar{\sigma}_i' + \bar{\sigma}_j' + \bar{\sigma}_z'
$$

(6-7a)
\[
\bar{I}_2 = \bar{\tau}_{x}^2 + \bar{\tau}_{y}^2 + \bar{\tau}_{z}^2 - (\bar{\sigma}_x' \bar{\sigma}_y' + \bar{\sigma}_y' \bar{\sigma}_z' + \bar{\sigma}_z' \bar{\sigma}_x') \\
\bar{I}_3 = \bar{\sigma}_x' \bar{\sigma}_y' \bar{\sigma}_z' + 2 \bar{\tau}_{xy} \bar{\tau}_{xz} \bar{\tau}_{zx} - (\bar{\sigma}_x' \bar{\tau}_{xy}^2 + \bar{\sigma}_y' \bar{\tau}_{xz}^2 + \bar{\sigma}_z' \bar{\tau}_{zx}^2)
\]

(6-7b)

(6-7c)

where

\[
\begin{bmatrix}
\bar{\sigma}_x' \\
\bar{\sigma}_y' \\
\bar{\sigma}_z' \\
\bar{\tau}_{xy} \\
\bar{\tau}_{xz} \\
\bar{\tau}_{zx}
\end{bmatrix} = \begin{bmatrix}
\sigma_i' \\
\sigma_j' \\
\sigma_k' \\
\tau_{ij} \\
\tau_{jk} \\
\tau_{ki}
\end{bmatrix} + d_{ij}
\]

(6-8)

and the vector of translation \(d_{ij}\) is defined by

\[
d_{ij} = \begin{bmatrix}
\frac{\sigma_x' + \sigma_y' + \sigma_z'}{3} - \sigma_i' \\
\frac{\sigma_x' + \sigma_y' + \sigma_z'}{3} - \sigma_j' \\
\frac{\sigma_x' + \sigma_y' + \sigma_z'}{3} - \sigma_k' \\
0 \\
0 \\
0
\end{bmatrix}_B,
\]

while \(df_p > 0\) and \(dg_p < 0\)

(6-9)

and

\[d_{ij} = 0, \text{ elsewhere.}\]

in which \(df_p\) is the increment of the value of yield function (described later) and \(dg_p\) is the increment of the value of the isotropic plastic potential function due to the change of the stress state. Figure 6-3 shows a typical example of a triaxial extension test during which the plastic potential function is translated. At the beginning of this test, the initial stress state denoted with point \(A\) in Figure 6-3 is on the \(K_o\)-line and, as the vertical normal stress decreases, the stress point moves into the elastic region until it reaches point \(B\) on the yield surface. At this point, the yield surface starts expanding, while Lade's "isotropic" plastic potential function would be contracting, leading to incorrect
direction of plastic strain increments. The modified plastic potential function defined by Equation (6-6) and (6-9) is a translation of Lade’s “isotropic” plastic potential function by a distance $|\mathbf{d}_p|_1$, equal to the distance of point B from the hydrostatic axis. The subscript B indicates that the stress state in Equation (6-9) corresponds to point B.

The parameter $\Psi_1$ in Equation (6-6) acts as a weighting factor between the triangular shape, due to the $I_3$ term, and the circular shape, due to the $I_2$ term, of the octahedral plane intersection. The parameter $\Psi_2$ controls the intersection of the plastic potential surface with the center line which is parallel to the hydrostatic axis, and $\mu$ determines the curvature of meridians. These parameters can be obtained by conventional triaxial tests. Results from computer simulations of tests demonstrate that the value of $\Psi_1$ is insensitive to the incremental plastic strain angle, $\tan^{-1}(\sqrt{3}(de_x^p - de_y^p)/(2de_x^p - de_y^p - de_z^p))$. Kim and Lade (1988) defined $\Psi_1$ as

$$\Psi_1 = 0.00155 m_f^{-1.27}$$  \hfill (6-10)

in which $m_f$ is a parameter in Equation (6-1).

According to Equation (6-5), for the computation of the direction of the plastic strain increments it is not the plastic potential function itself but its derivatives, which are required. The derivatives of the plastic potential function, $\overline{\sigma}_p$, with regard to the stresses are given by:

$$\frac{\partial \overline{\sigma}_p}{\partial \sigma_{ij}} = \left( \frac{\overline{I}_i}{P_n} \right)^{\mu} \left\{ (\mu + 3)\Psi_1 \frac{\overline{I}_2^2}{\overline{I}_3} - (\mu + 2) \frac{\overline{I}_1}{\overline{I}_2} + \frac{\mu \Psi_2}{\overline{I}_1} \frac{\partial \overline{I}_1}{\partial \sigma_{ij}} + \frac{\overline{I}_2}{\overline{I}_3} \frac{\partial \overline{I}_2}{\partial \sigma_{ij}} - \Psi_1 \frac{\overline{I}_3^3}{\overline{I}_3^3} \frac{\partial \overline{I}_3}{\partial \sigma_{ij}} \right\}$$  \hfill (6-11)

where
\[
\frac{\partial \bar{I}}{\partial \sigma_{ij}} = \begin{pmatrix}
1 & -\sigma_{y}' - \sigma_{z}' \\
1 & -\sigma_{z}' - \sigma_{x}' \\
0 & 2\tau_{xy} \\
0 & 2\tau_{xz}
\end{pmatrix}, \quad \frac{\partial \bar{I}}{\partial \sigma_{ij}'} = \begin{pmatrix}
-\sigma_{y}' - \sigma_{z}' \\
-\sigma_{z}' - \sigma_{x}' \\
2\tau_{xy} \\
2\tau_{xz}
\end{pmatrix}, \quad \text{and} \quad \frac{\partial \bar{I}}{\partial \sigma_{ij}''} = \begin{pmatrix}
\sigma_{y}'\sigma_{z}' - \tau_{yz}^2 \\
\sigma_{z}'\sigma_{x}' - \tau_{zx}^2 \\
\sigma_{x}'\sigma_{y}' - \tau_{xy}^2 \\
2\tau_{xy}\tau_{zx} - 2\sigma_{z}'\tau_{xy}
\end{pmatrix}
\] (6-12)

Substituting Equation (6-12) into Equation (6-11) produces

\[
\begin{pmatrix}
\frac{\partial \bar{G}}{\partial \sigma_{ij}'} \\
\frac{\partial \bar{G}}{\partial \tau_{ij}} \\
\frac{\partial \bar{G}}{\partial \tau_{ij}'} \\
\frac{\partial \bar{G}}{\partial \tau_{ij}''}
\end{pmatrix} = \left( \frac{I_1}{P_u} \right)^\nu \begin{pmatrix}
\bar{G} - (\sigma_{y}' + \sigma_{z}') \frac{I_2}{I_3} - \Psi_1 (\sigma_{y}'\sigma_{z}' - \tau_{yz}^2) \frac{I_3}{I_2} \\
\bar{G} - (\sigma_{z}' + \sigma_{x}') \frac{I_2}{I_3} - \Psi_1 (\sigma_{z}'\sigma_{x}' - \tau_{zx}^2) \frac{I_3}{I_2} \\
2\tau_{xy} \frac{I_2}{I_3} - 2\Psi_1 (\tau_{xy} + \sigma_{y}'\tau_{xy}) \frac{I_3}{I_2} \\
2\tau_{xz} \frac{I_2}{I_3} - 2\Psi_1 (\tau_{xz} + \sigma_{z}'\tau_{xz}) \frac{I_3}{I_2}
\end{pmatrix}
\] (6-13)

in which

\[
\bar{G} = (\mu + 3)\Psi_1 \frac{I_2}{I_3} - (\mu + 2) \frac{I_1}{I_2} + \frac{\mu \Psi_2}{I_1}
\] (6-14)

Once the parameter \(\Psi_1\) is obtained from Equation (6-10), the parameters \(\Psi_2\) and \(\mu\) can be determined using conventional triaxial test data. To obtain \(\Psi_2\) and \(\mu\), the incremental plastic strain ratio is defined as

\[
\nu_p = -\frac{de_{ps}}{de_{ip}}
\] (6-15)

For a conventional triaxial test, \(\sigma_1' = \sigma_2', \ \sigma_3' = \sigma_2' = \sigma_3'\) and \(\tau_{xy} = \tau_{yz} = \tau_{zx} = 0\).

Substituting Equation (6-5) into Equation (6-15), \(\nu_p\) can also be expressed by
\[
\nu_p = - \frac{\left( \frac{\partial \bar{G}}{\partial \sigma'_y} \right)}{\left( \frac{\partial \bar{G}}{\partial \sigma'_y} \right)}
\]

(6-16)

Further, substitution of Equations (6-13) into Equation (6-16) yields

\[
\nu_p = - \frac{\bar{G} - (\sigma'_1 + \sigma'_3) \frac{I^2}{I_2} - \Psi_1 (\sigma'_1 \sigma'_3) \frac{I^3}{I_3}}{\bar{G} - 2 \sigma'_3 \frac{I^2}{I_2} - \Psi_1 (\sigma'_3^2) \frac{I^3}{I_3}}
\]

(6-17)

Moreover, substitution of Equation (6-14) into Equation (6-17) produces

\[
\xi_y = \frac{1}{\mu} \xi_x - \Psi_2
\]

(6-18)

where

\[
\xi_x = \frac{1}{1 + \nu_p} \left[ \frac{I^3}{I_2} (\sigma'_1 + \sigma'_3 + 2\nu_p \sigma'_3) + \Psi_1 \frac{I^4}{I_3} (\sigma'_1 \sigma'_3 + \nu_p \sigma'_3^2) \right]
\]

\[
-3 \Psi_1 \frac{I^3}{I_3} + 2 \frac{I^2}{I_2}
\]

(6-19)

and

\[
\xi_y = \Psi_1 \frac{I^3}{I_3} - \frac{I^2}{I_2}
\]

(6-20)

Thus, \(1/\mu\) and \(-\Psi_2\) in Equation (6-18) can be determined by linear regression between \(\xi_x\) and \(\xi_y\), determined from several data points including tests from conventional triaxial compression and triaxial extension tests.
Figure 6-2 Directions of the plastic strain increments on the stress path and Lade's plastic potential surfaces
\sqrt{2} \sigma'_2 = \sqrt{2} \sigma'_3, \sqrt{2} d\varepsilon_2 = \sqrt{2} d\varepsilon_3

Figure 6.3 Translation and expansion of plastic potential surface in triaxial planes
6.4 Yield Criterion and Work Hardening Rule

In this thesis, the yield surfaces are associated with and derived from surfaces of constant plastic work. Using plastic work contours to express the yield behavior of soils has the advantage of avoiding the difficulties in the determination of yield points on the stress-strain curve. In the development of the yield criterion, the basic assumption is that, if the yield surfaces are defined properly, the amount of plastic work during yielding may be defined uniquely. The yield function, $f_p$, is expressed in terms of the state of stress and plastic work as

$$f_p = f'_p(\sigma') - f''_p(W_p) = 0$$  \hspace{1cm} (6-21)

in which $f'_p$ is a function of the state of stress, $\sigma'$, and $f''_p$ is a hardening/softening function given in terms of the plastic work, $W_p$. For hardening behavior:

$$f''_p = \left( \frac{1}{D} \right)^\mu \left( \frac{W_p}{p_a} \right)^\nu$$  \hspace{1cm} (6-22)

in which $D$ and $\rho$ are parameter constants, and $p_a = \text{atmospheric pressure}$.

For a monotonic proportional increase of the stresses $\sigma'_x$, $\sigma'_y$, and $\sigma'_z$, as in the case of isotropic or $K_0$-consolidation, the plastic work increases monotonically with the pressure and can be modeled by a power function of the first stress invariant, $I_1$, as:

$$W_p = C p_a \left( \frac{I_1}{p_a} \right)^\rho$$  \hspace{1cm} (6-23)

where $C$ and $\rho$ are dimensionless constants, determined by experiments, and $I_1$ is defined in Equation (6-2a).

In a six-dimensional stress space, the function of $f'_p$ may be expressed in terms of stress tensors as:
\[ f_p' = \left( \Psi_1 \frac{\tilde{I}_3^3}{I_3} - \frac{\tilde{I}_2^2}{I_2} \right) \left( \frac{I_1}{I_0} \right)^h e^q \]  \hspace{1cm} (6-24)

where

\[ \tilde{I}_1 = k_y \sigma_y' + \sigma_y + \sigma_z' \]  \hspace{1cm} (6-25a)

\[ \tilde{I}_2 = \tau_{xy}^2 + \tau_{xz}^2 + \tau_{y}^2 - (k_y \sigma_y' + \sigma_y' + \sigma_z' + k_y \sigma_z' \sigma_y') \]  \hspace{1cm} (6-25b)

\[ \tilde{I}_3 = k_y \sigma_y' \sigma_z' + 2 \tau_{xy} \tau_{xz} \tau_{yz} - (k_y \sigma_y' \tau_{xy}^2 + \sigma_y' \tau_{xz}^2 + \sigma_z' \tau_{yz}^2) \]  \hspace{1cm} (6-25c)

in which \( h \) and \( k_y \) are constants determined by the failure points of all tests on the basis that the plastic work is constant along a yield surface. Thus, for two stress points, \( A \) on the \( K_0 \)-line and \( B \) on the failure surface, the following expression is obtained for \( h \):

\[
h = \ln \left( \frac{\Psi_1 \frac{\tilde{I}_{1B}^3}{I_{1B}} - \frac{\tilde{I}_{2B}^2}{I_{2B}}}{\Psi_1 \frac{(k_y + 2K_0)^3}{K_0^2} + \frac{(k_y + 2K_0)^2}{K_0(2k_y + K_0)}} \right) \]  \hspace{1cm} (6-26)

in which \( e \) = the base of natural logarithms, and \( K_0 \) = lateral/vertical stress ratio at rest.

The value of \( q \) varies with stress level from zero at a conical surface bounded by the \( K_0 \)-line and the hydrostatic axis to unity at the failure surface, defined by

\[ q = \frac{\alpha S}{1 - (1 - \alpha) S} \]  \hspace{1cm} (6-27)

in which \( S \) is a function of the stress state defined below and \( \alpha \) is a function of the angle \( \theta \), defined in the octahedral plane shown in Figure 6-4. The parameter \( \alpha \) reflects the influence of the rotation of principal stresses and is defined by

\[ \alpha = \alpha_1 + \Delta \alpha \left( \frac{1 - \cos \theta}{2} \right) \]  \hspace{1cm} (6-28)
in which \( \alpha_i \) and \( \Delta \alpha \) are constants that can be obtained by conventional triaxial compression (\( \theta = 0 \)) and extension tests (\( \theta = \pi \)). The mathematical expression of \( \cos \theta \) is written by

\[
a = \begin{bmatrix}
\sigma'_z - \sigma' \\
\sigma'_r - K_0 \sigma' \\
\sigma'_\xi - K_0 \sigma' \\
\tau_{xy} \\
\tau_{yx} \\
\tau_{\xi \eta}
\end{bmatrix}
\]

\[
b = \begin{bmatrix}
\sigma' \\
\frac{1}{2} \sigma' \\
\frac{1}{2} \sigma' \\
\frac{1}{2} \sigma' \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\cos \theta = \frac{a \cdot b}{|a||b|} = \sqrt{\frac{1}{3}} \frac{(\sigma'_z - \sigma') - \frac{1}{2}(\sigma'_r - K_0 \sigma') - \frac{1}{2}(\sigma'_\xi - K_0 \sigma')}{\sqrt{(\sigma'_z - \sigma')^2 + (\sigma'_r - K_0 \sigma')^2 + (\sigma'_\xi - K_0 \sigma')^2 + \tau_{xy}^2 + \tau_{yx}^2 + \tau_{\xi \eta}^2}}
\]

(6-29)

and

\[
\sigma' = \frac{\sigma'_z + \sigma'_r + \sigma'_\xi}{1 + 2 K_0}
\]

(6-30)

Like \( q \), the stress level \( S \) also varies from zero at the same conical surface bounded by the \( K_0 \)-line and the hydrostatic axis to unity at the failure surface. The definition of \( S \) is

\[
S = \frac{1}{\eta_f} \left( \frac{I_3^3}{I_3} - N_f \right) \left( \frac{I_1^3}{27 I_3^3} \right)^{w_f} \left( \frac{I_1}{P_a} \right)^{n_f}
\]

(6-31)
where $I'_1$, $I''_1$, $I'_3$ and $I''_3$ are given by Equations (6-2a)-(6-2c). Material parameters $k$, $w$, $m$ and $\eta$ are the parameters of the failure criterion expressed by Equation (6-1).

Typical contours of $S$ on triaxial plane and octahedral plane are shown in Figure 6-5.

As the values of $D$ and $\rho$ in Equation (6-22) are constants for a given material, $f_p''$ varies only with plastic work. Considering that for $K_y$-anisotropic consolidation test, $\sigma'_y = \sigma'_z = K_0 \sigma'_x$ and $\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$, Equation (6-24) becomes

\[
f_p'' = \left( \frac{\psi_1 (k_y + 2 K_0)^3}{k_y K_0^2} + \frac{(k_y + 2 K_0)^2}{K_0 (2 k_y + K_0)} \right) \left( \frac{I_1}{p_a} \right)^h
\]  

(6-32)

Substituting Equations (6-22) and (6-32) into Equation (6-21), we obtain

\[
W_p = D \rho p_a \left( \frac{\psi_1 (k_y + 2 K_0)^3}{k_y K_0^2} + \frac{(k_y + 2 K_0)^2}{K_0 (2 k_y + K_0)} \right)^\rho \left( \frac{I_1}{p_a} \right)^{\rho h}
\]  

(6-33)

Comparison of Equations (6-23) and (6-33), yields values of $D$ and $\rho$ equal to

\[
\rho = \frac{p}{h}
\]  

(6-34)

and

\[
D = \frac{C}{\left( \frac{\psi_1 (k_y + 2 K_0)^3}{k_y K_0^2} + \frac{(k_y + 2 K_0)^2}{K_0 (2 k_y + K_0)} \right)^\rho}
\]  

(6-35)

The anisotropic yield surfaces are shaped as twisted waterdrops in their intersection with the triaxial plane (Figure 6-6) and have smoothly rounded triangular cross-sections on the octahedral plane. As the plastic work increases, the yield surface expands until the current stress point reaches the failure surface. For isotropic soils, the yield function and work hardening rule, expressed by Equations from (6-21) to (6-35), are identical to Lade's definition, if $k_y = 1$ and $\Delta \alpha = 0$. 
Figure 6-4 Definition of angle $\theta$ in the octahedral plane
Figure 6-5 Typical contours of S on (a) Triaxial plane and (b) Octahedral plane
\[ \sqrt{2} \sigma_2 = \sqrt{2} \sigma_3 \]

Figure 6-6  Yield surface shown in triaxial plane
6.5 Experimental Determination of Plastic Parameters

Modeling of the cross-anisotropic plastic behavior of the investigated marine clay involves 12 constant plastic parameters. These values of parameters are determined using results from a $K_o$-anisotropic consolidation test, conventional triaxial compression tests (CTC), conventional triaxial extension tests (CTE), and triaxial torsional simple shear tests (TSS), as shown in Chapter 4.

Failure Criterion

The soil parameters required in the failure criterion given by Equation (6-1) were determined from a log-log plot of the mathematical expression

$$\log \left[ \left( \frac{l_3^3}{I_3^3} - N_f \right) \left( \frac{l_3^3}{27 I_3^3} \right)^{w_f} \right] = \log \eta_f + m_f \log \left( \frac{P_a}{I_t} \right)$$

(6-36)

The parameters $m_f$ and $\eta_f$ are determined by using the least square method for the values of $w_f$ and $k_s$ that produce the least error. More specifically, the error from the least square method is expressed by

$$E_r = \sum_{i=1}^{n} [Y_i - f(X_i)]^2$$

(6-37)

where

$$Y = \log \left[ \left( \frac{l_3^3}{I_3^3} - N_f \right) \left( \frac{l_3^3}{27 I_3^3} \right)^{w_f} \right]$$

(6-38)

$$f(X) = \log \eta_f + m_f \log(X)$$

(6-39)

$$X = \frac{P_a}{I_t}$$

(6-40)
and \( n \) = the total number of tests. The contours of the error \( E \), for different combinations of \( w_f \) and \( k_s \) for the tested marine clay are shown in Figure 6-7, and the least error is occurred in \( w_f = 4.1 \) and \( k_s = 1.12 \).

Based on these values of \( w_f \) and \( k_s \), the values of the parameters \( m_f \) and \( n_f \) are determined by Equation (6-36) in a log-log diagram as shown in Figure 6-8. The best fitting line on this diagram corresponds to \( m_f = 0.89 \) and \( n_f = 208 \).
Figure 6-7 Contours of the error \( E \), for different combinations of \( w_f \) and \( k_s \) for the tested marine clay
\[
\left( \frac{I'_3^3}{I'_3} - N_f \right) \left( \frac{I''_3}{27 I'_3} \right)^{\eta_f}
\]

\[\eta_f = 208\]
\[m_f = 0.89\]

Figure 6-8 Determination of parameters \(m_f\) and \(\eta_f\) for the tested marine clay
Plastic Potential Function

The plastic potential function given in Equation (6-6) requires the determination of \( \Psi_2 \) and \( \mu \) from the experimental data including triaxial compression and triaxial extension tests. These parameters are determined using Equation (6-18). Note that to ensure that the plastic potential surface is convex with respect to its center line and that the direction of plastic strain increments is always outward and normal to the plastic potential surface, the limit of \( \Psi_2 \) is

\[
\Psi_2 > \Psi_1 \frac{I_3^3}{I_3^2} - \frac{I_2}{I_3} \tag{6-41}
\]

For the tested marine clay, it was found that \( \Psi_2 > -3.0481 \).

Figure 6-9 shows that all points of CTC and CTE tests from the marine clay fall almost on one straight line, and the best choice of these parameters are \( \Psi_2 = -3.048 \) and \( \mu = 2.26 \).

Yield Criterion and Work-Hardening Law

Six parameters are required to determine the yield-plastic work relation: \( k_y, h, \alpha_1 \), and \( \Delta \alpha \), defining the yield function, and \( C \) and \( p \), defining the plastic work equation for \( K_0 \)-anisotropic consolidation. The work-hardening relation along \( K_0 \)-line, expressed in Equations (6-22) and (6-23), is determined first, because the values of \( C \) and \( p \) are required for the determination of \( k_y, h, \) and \( q \) for the yield criterion.

Figure 6-10 shows the plastic work generated from Equation (4-1) plotted versus the value of \( I_1/p_a \) on dimensionless form. This relation is modeled by Equation (6-24) to

\[
\log \left( \frac{W^p}{p_a} \right) = \log C + p \log \left( \frac{I_1}{p_a} \right) \tag{6-42}
\]
In the log-log coordinates, \( C \) is the intercept at \( I_i/p_a = 1 \), and \( p \) is the slope of the straight line. For the tested marine clay, the best fitting values are \( C = 0.023 \) and \( p = 1.19 \).

The yield criterion in Equation (6-24) requires four parameter values. The value of \( h \) is determined from Equation (6-26) based on the computed value of \( k_y \) in advance, using failure points from all of test data including conventional triaxial compression tests, conventional triaxial extension tests, and torsional simple shear tests. The selected values of \( k_y \) and \( h \) those corresponding to the least scatter in the \( k_y-h \) diagrams, as shown in Figure 6-11. For the tested marine clay, \( k_y = 0.6 \) and \( h = 1.65 \). Note that the value of \( k_y \) is close to the value of \( K_o = 0.59 \). For simplicity, the value of \( k_y \) may be taken equal to the value of \( K_0 \), and the value of \( h \) can be obtained by Equation (6-26), which is modified to

\[
h = \frac{\ln\left(\frac{\psi_1 \frac{I_1^3}{I_{3B}} - \frac{I_2^3}{I_{2B}}}{27\psi_1 + 3}\right)}{\ln\frac{I_{1A}}{I_{1B}}}
\]

(6-43)

The values of the constants \( \rho \) and \( D \) are then computed from Equations (6-34) and (6-35). To determine the variation of the exponent \( q \) in Equation (6-27) with the stress level \( S \), the values of \( q \) are calculated from

\[
q = \ln\left(\frac{W_p}{D \cdot p_a}\right)^{\psi_1^{\psi_0}}
\]

(6-44)

and values of \( S \) are obtained from Equation (6-31). The relationship between \( q \) and \( S \) is shown in Figure 6-12 and is described by Equation (6-27). The parameter \( \alpha_i \) in Equation (6-28) is evaluated from conventional triaxial compression tests (\( \theta = 0 \)), while the parameter \( \Delta \alpha \) is the difference of the value of \( \alpha \) obtained from conventional triaxial
compression tests and conventional triaxial extension tests. For the tested marine clay, $\alpha = 0.1$ in conventional triaxial compression tests and $\alpha = 4$ in conventional triaxial extension tests. The value of $\Delta \alpha$ is equal to 3.9.

\[ \Psi_2 = 3.048 \]

\[ 1/\mu = 0.443 \]

\[ \Rightarrow \mu = 2.26 \]

**Figure 6-9** Determination of the parameters $\Psi_2$ and $\mu$ of the plastic potential function for the tested marine clay
Figure 6-10 Determination of the parameters $C$ and $p$ of the work-hardening relation for the tested marine clay
Figure 6-11 Relations of the parameters $k_y$ and $h$ based on the failure points from various tests for the tested marine clay
Figure 6-12 Determination of the parameter $\alpha$ of the yield criterion for the tested marine clay
Summary of plastic parameter values for marine clay

The 12 plastic parameter values required for the cross-anisotropic marine clay are listed in Table 6-1. These parameters may be used to calculate plastic strains for any combination of effective stresses during primary loading.

<table>
<thead>
<tr>
<th>Model component</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Failure criterion</td>
<td>$k_s$</td>
<td>1.12</td>
</tr>
<tr>
<td></td>
<td>$w_f$</td>
<td>4.1</td>
</tr>
<tr>
<td></td>
<td>$m_f$</td>
<td>0.89</td>
</tr>
<tr>
<td></td>
<td>$\eta_f$</td>
<td>208</td>
</tr>
<tr>
<td>Plastic potential</td>
<td>$\mu$</td>
<td>2.26</td>
</tr>
<tr>
<td></td>
<td>$\psi_2$</td>
<td>-3.048</td>
</tr>
<tr>
<td>Yield criterion</td>
<td>$k_y$</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>$h$</td>
<td>1.65</td>
</tr>
<tr>
<td></td>
<td>$\alpha_i$</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>$\Delta\alpha$</td>
<td>3.9</td>
</tr>
<tr>
<td>Hardening function</td>
<td>$C$</td>
<td>0.023</td>
</tr>
<tr>
<td></td>
<td>$p$</td>
<td>1.19</td>
</tr>
</tbody>
</table>
Chapter 7

COMPUTER SIMULATIONS OF TEST RESULTS

Rigorous nonlinear inelastic analysis of the stress-strain behavior in soil-structure interaction problems is generally very complicated. With the present state of development of computational methods such as the finite element method, an almost unlimited range of solutions can be obtained once the stress-strain behavior of the soils is modeled properly. This chapter mainly evaluates the accuracy of the new constitutive model by comparing the test and computer simulation results. In order to understand the effect of the elastic anisotropic behavior, numerical simulations of the undrained elastic reloading are analyzed in section 7.1. In section 7.2, two basic techniques for implementing the constitutive model into a computer program are described. Finally, the comparisons of test data with simulation results for CTC, CTE and TSS tests are shown in section 7.3.

7.1 Effect of Anisotropy on the Purely Elastic Behavior

The importance of anisotropic behavior may be illustrated clearly by comparing the undrained response of soil elements during elastic reloading for different degrees of anisotropy. To obtain the value of elastic modulus, a drained unloading test was performed in solid cylinder specimen. The specimen was consolidated in a conventional triaxial cell under $K_0$ conditions by using small stress increments until $\sigma'_1$ and $\sigma'_3$ reached the values of 176 and 105 kN/m$^2$. Then drained unloading was performed by decreasing vertical load $\sigma'_1$ and keeping $\sigma'_3$ constant until $\sigma'_1 = \sigma'_3 = 105$ kN/m$^2$. Figure 7-1a shows the stress path during the $K_0$-consolidation and unloading in $q'-p'$ plot, where $q' = \sigma'_1 - \sigma'_3$ and $p' = (\sigma'_1 + 2\sigma'_3)/3$. Due to the
dependence of the Young’s modulus on the stress state, each unloading step produced a different value of \( E_v \). The average value of Poisson’s ratio \( \nu_{\text{sh}} \) measured directly during unloading is approximately equal to 0.24.

If the values of \( n \) and \( m \) are known, the parameters \( M \) and \( \lambda \) can be calculated by plotting in a log-log diagram \( E_v/p_a \) versus \( X \), defined in Equation (5-38), and performing regression analysis. However, as explained below, the measurement of \( n \) and \( m \) requires more advanced testing equipment. Thus, in many practical situations where only conventional triaxial test results are used, it is necessary to make additional assumptions with respect to the degree of anisotropy of the soil. Here, we attempt to examine the importance of such assumptions on the prediction of soil behavior. The actual values \( n \) and \( m \) of the tested marine clay are ignored and three different cases are considered: (a) isotropic clay \((n = m = 1)\); (b) cross-anisotropic clay with \( n = 0.5 \) and (c) cross-anisotropic clay with \( n = 2 \). For all three cases the value of \( E_v \) at every stress state is equal to the measured value during unloading, while the (unknown) value of \( \nu_{\text{sh}} \) is assumed equal to the (measured) value of \( \nu_{\text{sh}} = 0.24 \). The computed parameters \( m, M \) and \( \lambda \) are given in Table 7-1.

Figure 7-1a illustrates the effects of the three assumptions regarding the degree of anisotropy on the stress paths of simulated undrained triaxial compression tests starting from the same initial isotropic stress state in which \( \sigma'_1 = \sigma'_3 = 105 \text{ kN/m}^2 \). In these tests the total vertical stress \( \sigma_1 \) was increased, while keeping the total horizontal stress \( \sigma_3 \) constant, until the deviatoric stress \( q' = 60 \text{ kN/m}^2 \) was reached. As expected, the stress path for isotropic behavior \((n = 1)\) is a vertical line \( ac \). For cross-anisotropic behavior, the stress path \( ad \) for \( n = 0.5 \) turns to the right, while the stress path \( ab \) for \( n = 2 \) turns to the left, approaching rapidly the failure surface (Figure 7-1a). Figure 7-1b plots the deviatoric stress, \( q' \), versus the axial strain, \( \varepsilon_1 \). Notice that the stress-strain relationships are not straight lines due to the dependence of the
elastic moduli on the effective stress level. The figure shows that the axial strain for \( n=1 \) is only about 4% smaller than that for \( n = 0.5 \) and 9% greater than that for \( n = 2 \). However, the development of excess pore water pressure, \( \Delta u \), occurs at quite different rates for the three different assumptions, as shown in Figure 7-1c. Thus, for the same value of \( q'=60 \text{ kN/m}^2 \), the value of excess pore water pressure for the isotropic assumption is 2 times larger than the value assuming \( n = 0.5 \), and 2 times smaller than the value assuming \( n = 2 \). This substantial difference in the pore water pressure development is also reflected in the significant deviation among the three stress paths \( ab, ac, \) and \( ad \) in Figure 7-1a. The results in Figure 7-1 demonstrate clearly that the effects of anisotropy may be very important in the development of excess pore water pressures, and, when appropriate, should be taken into account.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( n = 0.5 )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( M )</td>
<td>0.0208</td>
<td>0.4476</td>
<td>1.948</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>3.522</td>
<td>2.125</td>
<td>1.419</td>
</tr>
</tbody>
</table>
Figure 7-1 Undrained elastic reloading in triaxial compression tests of three soils of the same $E_v$ and $n = 0.5, 1$ and 2: (a) stress path (b) stress-strain relationship (c) excess pore water pressure vs. vertical axial strain.
7.2 Numerical Implementation of Work-Hardening Model

There are two basic techniques for implementing the presented elasto-plastic material model into a computer program. The first is the force method, controlled by stresses, and the second is the displacement method, controlled by strains.

In the force method, the strain increments are directly obtained from the amount of plastic work increment and the stress increments, instead of producing an elasto-plastic flexibility matrix during calculation. Because the method does not require the derivatives of the yield function, which may be very complicated, for the elasto-plastic flexibility matrix, it is much easier to write to a computer program which performs the necessary calculation for the prediction of the deformation response. The force method is seldom used for practical applications, but it is still a good method to treat a small element of soil for comparing deformation response from test data and the results from computer simulations.

The displacement method is based on the solution of the stress increment from the elasto-plastic tangent stiffness matrix and the strain increment vector. The nonlinear stress-strain relation, which involves a proper procedure for satisfying the required constitutive equations, is widely used in a subroutine of the finite element method. Numerically, it is convenient to consider small strain increments so that linear approximations may be used.

7.2.1 Force Method

According to the elasto-plastic theory, the strain increment can be calculated by combining elastic and plastic strain increment as

\[ de_{ij} = de_{ij}^e + de_{ij}^p \]  

(7-1)
The elastic strain increment, $de_{ij}^e$, is directly obtained from Hook's law shown in Equation (5-8). Note that, the Young's moduli, $E_v$ and $E_h$, are stress dependent. The plastic strain increment, $de_{ij}^p$, is calculated by Equation (6-5). The procedure is presented in the following.

Using the expression for the plastic potential in Equation (6-5), the relation between the plastic work increment and the proportionality constant $d\lambda$ may be obtained by multiplying the stress vector on both sides of Equation (6-5) as

$$d\lambda = \frac{dW_p}{\sigma' \cdot \frac{\partial g_p}{\partial \sigma'}}$$  \hspace{1cm} (7-2)

in which $dW_p = \sigma' \cdot de_{ij}^p$. Usually, the increment of plastic work can be determined by differentiation of the hardening/softening equations. For hardening, Equation (6-22) yields

$$dW_p = D \cdot p_a \cdot \rho \cdot f'_p \cdot df'_p$$  \hspace{1cm} (7-3)

Combining Equations (7-2) and (7-3) and substituting this and Equation (6-13) into Equation (6-5) produces the expression for the incremental plastic strain increments.

For the simulation of undrained tests, the prediction of pore water pressure and soil behavior is based on the condition that no volume change occurs in the soil for any load increment, i.e.

$$\Delta\varepsilon_v = \Delta\varepsilon_v^e + \Delta\varepsilon_v^p = 0$$  \hspace{1cm} (7-4)

in which $\Delta\varepsilon_v^e$ = elastic volumetric strain increment; and $\Delta\varepsilon_v^p$ = plastic volumetric strain increment for a given load increment. The computation procedure is given in a flow chart shown in Figure 7-2.
SUBROUTINE

Input $\{\sigma''\}$, $\{\Delta \sigma\}$, $f_p''$

A $\Delta u = 0$ no 

Undrained?

yes

Assume a series of pore water pressure increment $\Delta u_i, i = 1, M$

Compute effective stresses

$\{\sigma''\}_i = \{\sigma''\} + \{\Delta \sigma\} - \Delta u_i$

Compute $f_{p,i}'$ and $df_p' = f_{p,i}' - f_p''$

where $f_{p,i}'$ is obtained from $\{\sigma''\}_i$

$dW_p = D \cdot p_a \cdot \rho \cdot (f_{p,i}')^{\phi - 1} \cdot df_p'$

Compute $\frac{\partial g_p}{\partial \sigma'}$ from $\{\sigma''\}_i$

C

Figure 7-2 Force method flow chart (continued)
\[
\{\Delta \varepsilon^e\}_i = -\frac{dW_p}{\{\sigma^e\}_i^T \left[ \frac{\partial g_p}{\partial \sigma^e} \right]} \left[ \frac{\partial g_p}{\partial \sigma^e} \right]
\]

Compute elastic stiffness matrix \([C^e]\) from \(\{\sigma^e\}_i\)

\[
\{\Delta \varepsilon^e\}_i = [C^e]_i \{\Delta \sigma^e\}_i
\]

Strain increment \(\{\Delta \varepsilon\}_i = \{\Delta \varepsilon^e\}_i + \{\Delta \varepsilon^p\}_i\)

Total volumetric strain

\[
\varepsilon_{v,i} = (\varepsilon_{v,i})_{\text{previous}} + \Delta \varepsilon_{v,i}
\]

Use a numerical method such as cubic-spline method to search the pore pressure increment \(\Delta u\) so that \(\varepsilon_v = 0\)

\[
\{\sigma^{i+1}\}_i = \{\sigma^i\}_i + \{\Delta \sigma\}_i - \Delta u
\]

**Figure 7-2** Force method flow chart (continued)
Figure 7-2 Force method flow chart
7.2.2 Displacement Method

To establish the stress-strain relation from the constitutive model, it is necessary to develop an incremental form of the model,

\[
\{d\sigma^\prime\} = [C^{EP}] \{de\}
\]  \hspace{1cm} (7-5)

where \([C^{EP}]\) is the elasto-plastic tangent stiffness matrix. \(\{d\sigma^\prime\}\) is the stress increment vector and \(\{de\}\) is the strain increment vector.

Based on the elasto-plastic theory, the strain increment can be divided into elastic strain increment, \(de^e_{ij}\), and plastic strain increment, \(de^p_{ij}\). The yield function, \(f_p\), is always equal to zero during any increment in stress or strain, expressed in Equation (6-21). The derivative of \(f_p\) is

\[
df_p = \frac{\partial f_p}{\partial \sigma^\prime_{ij}} \cdot d\sigma^\prime_{ij} + \frac{\partial f_p}{\partial W_p} \cdot dW_p
\]

\[
= \frac{\partial f_p}{\partial \sigma^\prime_{ij}} \cdot d\sigma^\prime_{ij} + \frac{\partial f_p}{\partial W_p} \cdot dW_p = 0
\] \hspace{1cm} (7-6)

in which

\[
d\sigma^\prime_{ij} = C^E_{ijkl} \cdot de^e_{kl} = C^E_{ijkl} \cdot (de_{kl} - de^p_{kl}) = C^E_{ijkl} \cdot (de_{kl} - d\lambda \cdot \frac{\partial g_p}{\partial \sigma^\prime_{ij}})
\] \hspace{1cm} (7-7)

and

\[
dW_p = \sigma^\prime_{ij} \cdot de^p_{ij} = \sigma^\prime_{ij} \cdot d\lambda \cdot \frac{\partial g_p}{\partial \sigma^\prime_{ij}}
\] \hspace{1cm} (7-8)

Substituting Equations (7-7) and (7-8) into (7-6) produces

\[
\frac{\partial f_p}{\partial \sigma^\prime_{ij}} \cdot C^E_{ijkl} \cdot (de_{kl} - d\lambda \cdot \frac{\partial g_p}{\partial \sigma^\prime_{ij}}) + \frac{\partial f_p}{\partial W_p} \cdot \sigma^\prime_{ij} \cdot d\lambda \cdot \frac{\partial g_p}{\partial \sigma^\prime_{ij}} = 0
\] \hspace{1cm} (7-9)

or
\[ d\lambda = \frac{\partial f'_p}{\partial \sigma'_y} \cdot C_{ijkl}^E \cdot d\epsilon_{kl} \]

\[ d\lambda = \frac{\partial f'_p}{\partial \sigma'_y} \cdot C_{ijkl}^E \cdot \frac{\partial g_p}{\partial \sigma'_i} - \frac{\partial f''_p}{\partial W_p} \left( \sigma'_y \cdot \frac{\partial g_p}{\partial \sigma'_y} \right) \]

Therefore, substituting Equation (7-10) into (6-5) yields

\[ d\epsilon_{k} = \frac{\partial f'_p}{\partial \sigma'_y} \cdot C_{ijkl}^E \cdot d\epsilon_{kl} \]

\[ d\epsilon_{k} = \frac{\partial f'_p}{\partial \sigma'_y} \cdot C_{ijkl}^E \cdot \frac{\partial g_p}{\partial \sigma'_i} - \frac{\partial f''_p}{\partial W_p} \left( \sigma'_y \cdot \frac{\partial g_p}{\partial \sigma'_y} \right) \]

and

\[ d\sigma'_y = C_{ijkl}^E \cdot d\epsilon_{kl} - C_{ijkl}^E \cdot d\epsilon_{mn} \]

\[ = C_{ijkl}^E \cdot d\epsilon_{kl} - \frac{\partial f'_p}{\partial \sigma'_y} \cdot C_{ijkl}^E \cdot \frac{\partial g_p}{\partial \sigma'_i} - \frac{\partial f''_p}{\partial W_p} \left( \sigma'_y \cdot \frac{\partial g_p}{\partial \sigma'_y} \right) \]

\[ = C_{ijkl}^E \cdot d\epsilon_{kl} - \frac{\partial f'_p}{\partial \sigma'_y} \cdot C_{ijkl}^E \cdot \frac{\partial g_p}{\partial \sigma'_i} - \frac{\partial f''_p}{\partial W_p} \left( \sigma'_y \cdot \frac{\partial g_p}{\partial \sigma'_y} \right) \]

\[ = \left[ \begin{array}{c} d\epsilon_{k} \\ \vdots \\ d\epsilon_{k} \\ \vdots \\ d\epsilon_{k} \end{array} \right] = \left[ \begin{array}{c} C_{ijkl}^E \cdot \frac{\partial g_p}{\partial \sigma'_i} \cdot \frac{\partial f'_p}{\partial \sigma'_y} - C_{ijkl}^E \cdot \frac{\partial g_p}{\partial \sigma'_i} \cdot \frac{\partial f''_p}{\partial W_p} \left( \sigma'_y \cdot \frac{\partial g_p}{\partial \sigma'_y} \right) \end{array} \right] \]

The incremental stiffness matrix for the elasto-plastic material model then takes the form

\[ [C^{EP}] = [C^E] - \left[ C^E \right] \left[ \frac{\partial g_p}{\partial \sigma'} \right] \left[ \frac{\partial f'_p}{\partial \sigma'} \right]^T \left[ C^E \right] \]

\[ - \left[ \frac{\partial f''_p}{\partial W_p} \left( \sigma'_y \cdot \frac{\partial g_p}{\partial \sigma'_y} \right) \right] \]
Because the elasto-plastic stiffness matrix, \( [C^{EP}] \), is developed based on the derivatives of yield function in Equation (7-6) instead of the yield function in Equation (6-21), in general \( f_p(\sigma', \sigma'_y, W_p) = f'_p(\sigma'_y) - f''_p(W_p) = \Delta f \neq 0 \) and a small departure from the yield surface is obtained. As such departures accumulate, a stress correction must be made to restore stresses to the correct yield surface in each time step, so that Equation (6-21) can be satisfied. The method of stress correction, presented by Nayak and Zienkiewicz (1972), is achieved assuming that the stress change is in the direction of the normal to the yield surface. Thus if the stress correction is
\[
\delta \sigma'_y = c \frac{\partial f_p}{\partial \sigma'_y}
\]  
(7-14)
where \( c \) is a scalar and yield function correction
\[
\delta f_p = -\Delta f = \left( \frac{\partial f_p}{\partial \sigma'_y} \right)^T \delta \sigma'_y = c \left( \frac{\partial f_p}{\partial \sigma'_y} \right)^T \left( \frac{\partial f_p}{\partial \sigma'_y} \right) \]  
(7-15)
we have
\[
\delta \sigma'_y = - \left[ \frac{\Delta f}{\left( \frac{\partial f_p}{\partial \sigma'_y} \right)^T \left( \frac{\partial f_p}{\partial \sigma'_y} \right)} \right] \left( \frac{\partial f_p}{\partial \sigma'_y} \right) \]  
(7-16)
The procedure of the displacement method is described by the flow chart in Figure 7-3.
SUBROUTINE

input \([\sigma^i], W_p^i, [\Delta\varepsilon^r+\Delta\varepsilon^i]\)

Compute \([C_{\varepsilon,i}^{E,i}] \) from \([\sigma^i] \)

\([\Delta\sigma_{\varepsilon,i}^{r,i}+\Delta\sigma^i] = [C_{\varepsilon,i}^{E,i}][\Delta\varepsilon_{\varepsilon,i}^{r,i}\Delta\varepsilon^i]\)

Estimate new state of stress by

\([\sigma^i'] = [\sigma^i] + [\Delta\sigma_{\varepsilon,i}^{r,i}+\Delta\sigma^i]\)

\(F = f_p(\sigma^i', W_p^i) = f_p^i(\sigma^i') - f_p^i(W_p^i)\)

UNLOADING yes

\(F \leq 0\) yes

no

LOADING

Figure 7-3 Displacement method flow chart (continued)
Figure 7-3 Displacement method flow chart (continued)
Figure 7-3 Displacement method flow chart
7.3 Comparisons of Test Data and Computer Simulations

The results from the theoretical predictions obtained from the new developed cross-anisotropic constitutive model are compared with results from test measurements using the force method. For the undrained tests of marine clay, the pore water pressure is estimated by the cubic spline method (or bi-section method) based on the conditions that no volume change occurs in the soil for any load increment. The error limit used for the total volumetric strain (not the volumetric strain increment for each load step) is ±0.001%. The predicted pore water pressure and soil behavior is not significantly affected by using error limits smaller than ±0.001%.

The comparisons for conventional triaxial compression (CTC), conventional triaxial extension (CTE) and torsional simple shear (TSS) tests are shown in Figure 7-4 through Figure 7-23, which plot the effective stress paths and the deviatoric stress $\sigma'_1 - \sigma'_2$, the effective stress ratio $\sigma'_1/\sigma'_2$ and excess pore water pressure $\Delta u$ versus the major principal strain $\varepsilon_1$ (for CTC and CTE) or shear strain $\gamma_\alpha$ (for TSS). The comparisons show that the overall soil behavior is predicted with good accuracy by the model. Because the parameter values were initially derived from the results of these and similar tests at various consolidation pressures, good agreement could be expected. The prediction of excess pore water pressure in some of the tests is not as good as that of the stress-strain behavior. One possible explanation for the discrepancy may not be related to the model, but to the fact that due to the long consolidation period under air pressure, air may have penetrated the soil membrane or gas may have developed from decaying organic constituents within the marine clay specimen. Such air or gas could affect the accuracy of the pore water pressure measurements.
Figure 7-4 Comparison of measured and predicted deviator stress versus the major principal strain for the CTC141 test
Figure 7-5 Comparison of measured and predicted effective stress ratio versus the major principal strain for the CTC141 test
Figure 7-6 Comparison of measured and predicted excess pore water pressure versus the major principal strain for the CTC141 test
Figure 7-7 Comparison of measured and predicted effective stress path for the CTC141 test
Figure 7-8 Comparison of measured and predicted deviator stress versus the major principal strain for the CTC142 test
Figure 7-9  Comparison of measured and predicted effective stress ratio versus the major principal strain for the CTC142 test
Figure 7-10 Comparison of measured and predicted excess pore water pressure versus the major principal strain for the CTC142 test
Figure 7-11  Comparison of measured and predicted effective stress path for the CTC142 test.
Figure 7-12  Comparison of measured and predicted deviator stress versus the major principal strain for the CTE146 test
Figure 7-13  Comparison of measured and predicted effective stress ratio versus the major principal strain for the CTE146 test
Figure 7-14 Comparison of measured and predicted excess pore water pressure versus the major principal strain for the CTE146 test
Figure 7-15 Comparison of measured and predicted effective stress path for the CTE146 test
Figure 7-16  Comparison of measured and predicted deviator stress versus the major principal strain for the CTE147 test
Figure 7-17 Comparison of measured and predicted effective stress ratio versus the major principal strain for the CTE147 test
Figure 7.18 Comparison of measured and predicted excess pore water pressure versus the major principal strain for the CTE147 test.
Figure 7-19 Comparison of measured and predicted effective stress path for the CTE147 test
\( \sigma'_1 - \sigma'_3 \), kN/m²

**Figure 7-20** Comparison of measured and predicted deviator stress versus shear strain for the TSS151 test
Figure 7-21 Comparison of measured and predicted effective stress ratio versus shear strain for the TSS151 test
Figure 7-22 Comparison of measured and predicted excess pore water pressure versus shear strain for the TSS151 test
Figure 7-23 Comparison of measured and predicted effective stress path for the TSS151 test
Chapter 8

SUMMARY AND CONCLUSIONS

To investigate the characteristics of the anisotropic marine clay, a series of experiments using solid cylinder and hollow cylinder specimens were performed under monotonic loads. The results from these tests combined with other results from the literature were used to develop constitutive equations for cross-anisotropic soil.

A new general nonlinear model for the dependence of elastic moduli of a cross-anisotropic soil on the stress state has been developed. The model considers that the Young's moduli and shear moduli of soil depend primarily on its state of stress, density and stress history, while Poisson's ratios are practically independent of the confining stress, in agreement with the experimental evidence. The expressions for the stress dependence of the elastic moduli are derived in a rigorous way by considering the conservation of energy during a cycle of loading along any arbitrary closed stress path, assuming purely elastic behavior. The new model requires only one more parameter to express the dependence of the elastic moduli on the state of stress, in addition to the five independent parameters which define a cross-anisotropic elastic model with constant moduli. The importance of the stress-dependence of the elastic moduli of an anisotropic soil is elucidated through a parameteric study, which examines the effects of normal stresses, deviatoric stresses, Poisson's ratios and degree of cross-anisotropy. The computer simulation for undrained tests proved that the elastic parameters play an important role in the prediction of pore water pressure which affects the directions of stress paths in the stress space. The approximation of the elastic behavior of a strongly cross-anisotropic soil with an isotropic model, adopted often for simplicity in practice, may lead to substantial errors in the prediction of the soil response. The presented model
offers a general method for considering the nonlinearity in the elastic soil behavior, caused by the dependency of the elastic moduli on the state of stress for cross-anisotropic soils.

The development of the plastic model is more complex than the nonlinear elastic cross-anisotropic model described above. The plastic model consists of the following three elements: (a) a failure criterion; (b) a plastic potential function for the computation of the plastic deformation; (c) a yield criterion and hardening rule. A new plastic model has been developed for cross-anisotropic clay. The model consists of a new set of generalized mathematical expressions for the failure, yielding and hardening rules and a new approach to describe the development of plastic deformation. The model may be used to describe the mechanical response of the soil under any monotonic external loading. For a given arbitrary load history, the constitutive model may be used to predict the deformation characteristics, the excess pore water pressure, and the amount of yielding and failure of the material. The mathematical expressions that form the constitutive model are general and depend on a number of material constants that must be determined through soil testing. Although the development of yield and failure criteria require a number of complex tests, once the expressions for the criteria are known, the determination of the soil constants for practical problems can be achieved based on a few simple conventional tests. This is a significant advantage, because it makes the method very practical and easy to use.

The new constitutive model for cross-anisotropic marine clay has been implemented into a computer algorithm which was used to verify its validity through simulation of actual tests on marine clay specimens subjected to various loading conditions. Comparisons between the experimental results and the computed response using the new model for the behavior of tested marine clay showed a good agreement except for the prediction of the excess pore water pressure in certain tests. The problems in the
prediction of the excess pore water pressure may be related to air or gas developing in the tested specimens. Although initially all specimens are fully saturated, some air may migrate from the cell water through the membrane into the specimen during the long consolidation period. Alternatively, organic constituents of the marine clay specimen may decay with time producing gas with the specimen. Any air or gas may result in inaccurate pore water pressure measurement for some of the tests.

The new elasto-plastic constitutive model for the behavior of cross-anisotropic materials is found to give good overall predictions of the response of the marine clay subjected to various types of monotonic loading. Although the presented results show good overall agreement with the experimental data, more experimental data are required for a more thorough verification of the model, using the tested marine clay, as well as other soils. This task will be the subject of future investigation.
Chapter 9

REFERENCES


Appendix A

GENERAL ELASTICITY-BASED MODELS

Three commonly used elasticity-based models for soil materials are introduced in the following sections. These are the Cauchy elastic, Green hyperelastic, and incremental (hypoelastic) types of formulation (Chen and Saleeb, 1982; Chen, 1984).

A.1 Cauchy Elasticity

For this type of elastic model, the current state of stress depends only on the current state of deformation. The constitutive relations for a Cauchy elastic material is written by

\[ \sigma_y = F_y(\varepsilon_y) \]  \hspace{1cm} (A-1)

where \( F_y \) is the elastic response function of the material and \( \sigma_y \) and \( \varepsilon_y \) are the stress and strain tensors.

The behavior of such material is both reversible and path independent in the sense that stresses are uniquely determined by the current state of strain (or vice versa). In general, although stresses are uniquely determined from strains (or vice versa), the converse is not necessarily true. The Cauchy type of elastic models may generate energy for certain loading-unloading cycle. That is, the model may violate the laws of thermodynamics, which is not acceptable on physical grounds.

A.2 Green Elasticity (Hyperelasticity)

This is based on the assumption of the existence of a strain energy density function \( W \) (or complementary energy density function \( \Omega \)) such that
\[ \sigma_y = \frac{\partial W}{\partial \varepsilon_y}, \quad \varepsilon_y = \frac{\partial \Omega}{\partial \sigma_y} \quad (A-2) \]

in which \( W \) and \( \Omega \) are functions of the current components of the strain or stress tensors. Equation (A-2) yields an one-to-one relation between actual state of stress and strain. In addition to the reversibility and path independence of stresses and strains in hyperelastic type of elastic models, thermodynamics laws are always satisfied, and no energy can be generated through load cycles.

For an isotropic linear elastic material, both Cauchy elastic and Green hyperelastic types of formulation reduce to the familiar generalized Hook's law with only two independent material constants. However, for an anisotropic linear elastic material, the Cauchy type of formulation has 36 material constants, whereas only 21 constants are needed in the Green formulation owing to the additional symmetry requirements imposed by Equation (A-2).

Differentiation of Equation (A-2) results in the following incremental stress-strain relations:

\[ \dot{\sigma}_y = \frac{\partial^2 W}{\partial \varepsilon_y \partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} = H_{ijkl} \dot{\varepsilon}_{kl} \quad (A-3) \]

and

\[ \dot{\varepsilon}_y = \frac{\partial^2 \Omega}{\partial \sigma_y \partial \sigma_{kl}} \dot{\sigma}_{kl} = H'_{ijkl} \dot{\sigma}_{kl} \quad (A-4) \]

where \( \dot{\sigma}_y \) and \( \dot{\varepsilon}_y \) are the stress and strain increment tensors. The incremental relationships are suitable for use in incremental nonlinear finite element analyses of soil mass. Thus, the hyperelastic model yields a constitutive equation which is incapable of describing load history-dependence and rate-dependence. Material instability occurs when
\[ \text{det} |H_{ijkl}| = 0 \text{ or } \text{det} |H'_{ijkl}| = 0 \] (A-5)

Despite their shortcomings, hyperelasticity type models have been utilized as non-linear constitutive relations for soil.

A.3 Incremental Elasticity (Hypoelasticity)

This type of formulation is often utilized to describe the mechanical behavior of a class of materials in which the state of stress depends on the current state of strain as well as on the stress path followed to reach that state. In general form, the incremental constitutive relations for time-independent materials are written as

\[ F(\sigma_{ij}, \dot{\sigma}_{kl}, \varepsilon_{mn}, \dot{\varepsilon}_{pq}) = 0 \] (A-6)

The above equation is general, but it is not possible to indicate the relations between the total or incremental stresses and strains because of its complexity. Therefore, for simplicity, special cases of the general law are frequently used. On the other hand, the minimal requirement for a material to qualify as elastic in any sense is that there exists a one-to-one correspondence between stress and strain increment tensors. The simplest type of such materials are those for which the strain increments, \( \dot{\varepsilon}_{ij} \), are linearly related to the stress increments, \( \dot{\sigma}_{ij} \), through material response moduli. Four special cases of this type of formulation are given by

\[ \dot{\sigma}_{ij} = C_{ijkl}(\sigma_{mn}) \dot{\varepsilon}_{kl}, \quad \dot{\varepsilon}_{ij} = C_{ijkl}(\varepsilon_{mn}) \dot{\varepsilon}_{kl} \] (A-7a)

or

\[ \dot{\varepsilon}_{ij} = D_{ijkl}(\varepsilon_{mn}) \dot{\sigma}_{kl}, \quad \dot{\varepsilon}_{ij} = D_{ijkl}(\sigma_{mn}) \dot{\sigma}_{kl} \] (A-7b)

The above equation satisfies the reversibility requirement only in the incremental sense. These incremental stress-strain relations provide a natural mathematical model for materials with limited memory. For example, this can be seen by integration of Equation (A-7a).
\[
\sigma_\nu = \sigma_\nu^0 + \int_0^t C_{ijkl}(\sigma_{mn}) \frac{\partial E_{kl}}{\partial \tau} \, d\tau
\]  

(A-8)

The integral expression clearly indicates the path-dependency and irreversibility of the process. In the linear case for which \( C_{ijkl}(\sigma_{mn}) \) is a constant, the hypoelasticity degenerates to hyperelasticity, which corresponds to the history independent secant modulus formulation.

There are two problems associated with the hypoelasticity model (Chen, 1984). The first problem is that, in the non-linear range, the models exhibit stress induced anisotropy. This anisotropy implies that the principal axes of stress and strain are different. As a result, a total of 21 material moduli for general triaxial conditions have to be defined for every point of the material loading path. This is a difficult task for practical application. The second problem is that, under multiaxial stress conditions, the hypoelastic formulation provides no clear criterion for loading or unloading. Thus a loading in shear may be accompanied by an unloading in some of the normal stress components. Therefore, additional assumptions are needed for defining the loading-unloading criterion.
Appendix B

THEORY OF PLASTICITY

An external load causes strains and stresses in a body. When the external load is removed, the body may or may not return to its original configuration. If it returns to its original configuration when the load is removed, the behavior of the material is called *elastic*. If it retains some irrecoverable deformations upon unloading, the behavior of the material is called *inelastic* or *plastic*.

The theory of plasticity was initially developed in order to describe the behavior of metals. The development of the soil plasticity was influenced by the theory of metal plasticity (Chen and Baladi, 1985). Unlike metals, the shear strength of soils is strongly influenced by the compressive normal stress acting on the shear plane and therefore by the hydrostatic pressure. Since soils have little or no tensile strength, the tension test cannot be applied to them. A typical stress-strain curve of a soil sample under drained triaxial compression ($\Delta \sigma_1 > 0$ and $\Delta \sigma_2 = \Delta \sigma_3 = 0$) is shown in Figure B-1. When the load is increased gradually, the soil behaves elastically up to point A, and regains the original state if the load is removed. If the specimen is stressed beyond point A to point B, and then unloaded, there will be some permanent or irrecoverable deformations in the body. If the specimen is reloaded from point C, it will often behave elastically until the stress level reaches point D. The reloading portion between points C and D is approximately like the unloading portion. When the specimen is loaded from A to B, both elastic and plastic deformations will occur. The increase of stress with plastic deformation is called *hardening*. 
Many engineering materials exhibit a work-hardening behavior. Increasing the stress beyond the initial yield surface produces both elastic and plastic deformations. At each stage of plastic deformation, a new yield surface, called the subsequent loading surface, is established. If the state of stress moves inside the yield surface in the stress space, the behavior of the material is again elastic, and no plastic deformations take place. The stress-strain behavior related to loading or unloading from the new yield surface is loading-path dependent.

In developing constitutive equations for soils, two types of the rate-independent plasticity theories have commonly been used: the deformation theory and the incremental theory (Chen, 1984; Chen and Baladi, 1985; Lubliner, 1990).

B.1 Deformation Theory

This theory, by the form of the total stress-strain relation, assumes that the state of stress determines the state of strain uniquely as long as the plastic deformation continues.
This is identical with the nonlinear elastic stress-strain relation as long as unloading does not occur. Thus, the general form of this theory during loading may be written as

\[ \varepsilon^p_y = \varepsilon_y - \varepsilon^e_y = f(\sigma_y) \]  

(B-1)

The above equation indicates a loading-path independent behavior. It cannot adequately describe the phenomena associated with loading and unloading near the yield surface along a neutral loading path. Nevertheless, such theories have been used extensively in practice for the solution of elastic-plastic problems because of their comparative simplicity. However, the total stress-strain relation based on deformation theory is only valid in the case of proportional loading.

### B.2 Incremental Theory or Flow Theory

The formulation of this theory relates the increment of plastic strain components \( d\varepsilon^p_y \) to the state of stress, \( \sigma_y \), and the stress increment, \( d\sigma_y \). Basic assumptions used in the development of the incremental theory of work-hardening plasticity include:

(a) The existence of an initial yield surface which defines the elastic limit of the material in a multiaxial state of stress.

(b) The hardening rule which describes the evolution of subsequent yield surfaces.

(c) The flow rule which is related to a plastic potential function and defines the direction of the incremental plastic strain increment vector in strain space.

This theory accounts for loading, unloading, and reloading and is suitable for describing the complete stress-path-dependent behavior of a work-hardening soils. The basic concepts of the incremental theory of plasticity for work-hardening soils are reviewed in the following.
B.2-1 Loading Surface and Loading Criterion

Loading surface is the subsequent yield surface, which defines the boundary of the current elastic region. If a stress point lies within this region, no additional plastic deformation takes place. If the state of stress is on the boundary of the elastic region and tends to move out of the current loading surface, additional plastic deformations will occur, accompanied by a configuration change of the current loading surface. In the simplest models of plasticity the internal variables are taken as the plastic strain components \( \varepsilon_y^p \) themselves and the hardening variable \( \kappa \) (Lubliner, 1990). When the yield function is taken to have the form

\[
f(\sigma_y, \varepsilon_y^p, \kappa) = 0
\] (B-2)

both isotropic and kinematic hardening can be described as shown in Section B.2-2.

The concepts of "loading" and "unloading" must be clearly specified. The loading surface itself is an essential part of defining loading and unloading. To express the loading and unloading precisely, a unit vector \( n^f \) is introduced normal to the loading surface in stress space (Figure B-2) whose components are given by

\[
n^f_y = \frac{\left( \frac{\partial f}{\partial \sigma_y} \right)}{\sqrt{\frac{\partial f}{\partial \sigma_y} \frac{\partial f}{\partial \sigma_{kl}}}}
\] (B-3)

If the angle between the vector \( d\sigma_y \) and \( n^f_y \) is acute (Figure B-2), additional plastic deformation will occur. Thus, the criterion for loading is

\[
\text{if } f = 0 \text{ and } \left( \frac{\partial f}{\partial \sigma_y} \right) d\sigma_y > 0, \text{ then } d\varepsilon_y^p \neq 0
\] (B-4)
If the two vectors $d\sigma_y$ and $n^f_y$ form an obtuse angle, unloading will occur. Thus, the criterion for unloading is

$$\text{if } f=0 \text{ and } \left( \frac{\partial f}{\partial \sigma_y} \right) d\sigma_y < 0, \text{ then } d\varepsilon^p_y = 0 \quad (B-5)$$

In the neutral loading case, the additional load vector $d\sigma_y$ is perpendicular to the normal vector $n^f_y$, and no plastic deformation will occur. The criterion for neutral loading is

$$\text{if } f=0 \text{ and } \left( \frac{\partial f}{\partial \sigma_y} \right) d\sigma_y = 0, \text{ then } d\varepsilon^p_y = 0 \quad (B-6)$$

For a perfectly plastic material, there is no neutral loading case such as satisfied by Equation (B-6).

\[\text{Figure B-2 Loading criterion for a work-hardening material}\]
B.2-2 Hardening Rules

A number of hardening rules have been proposed. The most widely used rules are those of isotropic hardening, kinematic hardening, and a combination of both, i.e., the so-called mixed hardening. The general form of the loading function of Equation (B-2) can be written as

\[ f(\sigma_y, \varepsilon_0^p, \kappa) = F(\sigma_y, \varepsilon_0^p) - k^2(\kappa) = 0 \]  \hspace{1cm} (B-7)

in which the hardening function \( k^2 \) represents the size of the yield surface, while the function \( F(\sigma_y, \varepsilon_0^p) \) defines the shape of that surface. Here, the function \( k^2 \) is expressed as a function of \( \kappa \). The value of \( \kappa \) depends on the loading history or the plastic strain path.

(i) Isotropic Hardening

The work-hardening rule is based on the assumption that the initial yield surface expands uniformly without distortion and translation as plastic flow occur. The general form is expressed by

\[ f = F(\sigma_y) - k^2(\kappa) = 0 \]  \hspace{1cm} (B-8)

For a perfectly plastic material, the equation for the fixed yield surface has the form \( F(\sigma_y) - k^2 = 0 \), where \( k \) is a constant. It is well known that isotropic hardening rule applies mainly to monotonic loading while for cyclic loadings, kinematic hardening rule would be more appropriate.

(ii) Kinematic Hardening

It assumes that during plastic deformation, the loading surface translates as a rigid body in the stress space, maintaining the size, shape, and orientation of the initial yield surface. The equation of the loading surface has the general form

\[ f(\sigma_y, \varepsilon_0^p) = F(\sigma_y - \alpha_y) - k^2 = 0 \]  \hspace{1cm} (B-9)
where \( k \) is a constant and \( \alpha_y \) is translation vector, which changes with the plastic deformation.

(iii) Mixed Hardening

A combination of kinematic and isotropic hardening would lead to the more general mixed hardening rule:

\[
f(\sigma_y, \varepsilon_y^p, \kappa) = F(\sigma_y - \alpha_y) - k^2(\kappa) = 0
\]

(B-10)

In this case, the loading surface experiences a translation defined by \( \alpha_y \) and a uniform expansion measured by \( k^2 \), but it still retains its original shape. With the mixed hardening rule, different degrees of the Bauschinger effect can be simulated, by simply adjusting the two hardening parameters, \( \alpha_y \) and \( k^2 \).

B.2.3 Flow Rules

The connection between the loading function and the stress-strain relation for a work-hardening material, such as soils, will be made by a flow rule. When the current yield surface \( f \) is reached, the material is in a state of plastic flow upon further loading. The flow rule is defined by

\[
d\varepsilon_y^p = d\lambda \frac{\partial g}{\partial \sigma_y}
\]

(B-11)

where \( g \) is a plastic potential function and \( d\lambda > 0 \) is a scalar factor that will vary throughout the history of the straining process. The gradient of the plastic potential surface \( \partial g / \partial \sigma_y \) defines the direction of the plastic strain increment vector \( d\varepsilon_y^p \), which is normal to the plastic potential surface. If the plastic potential surface has the same shape as the current yield or loading surface \( f \), it is called associated flow rule, i.e.

\[
d\varepsilon_y^p = d\lambda \frac{\partial f}{\partial \sigma_y}
\]

(B-12)
If the plastic surface has the different shape with the current or loading surface, it is called non-associated flow rule, expressed by Equation (B-11). In general, there is very little experimental evidence on plastic potential function for soils. Although some researchers have already proved that the plastic strain increments of soils were not perpendicular to the yield surface (Graham et al, 1983; Yamada and Ishihara, 1982), the associated flow rule is widely applied to soils.

B.2-4 Drucker's Stability Postulate

A more restricted definition of work-hardening was formulated by Drucker. If a unit volume of an elastic-plastic specimen under uniaxial stress is initially at stress $\sigma$ and plastic strain $\varepsilon^p$, and if an "external agency" slowly applies an incremental load resulting in a stress increment $d\sigma$ and subsequently slowly removes it, then $d\sigma d\varepsilon = d\sigma (d\varepsilon^e + d\varepsilon^p)$ is the work performed by the external agency in the course of incremental loading, and $d\sigma d\varepsilon^p$ is the work performed in the course of cycle consisting of the application and removal of the incremental stress. Since $d\sigma d\varepsilon^e$ is always positive, and for a work-hardening material $d\sigma d\varepsilon^p \geq 0$, it follows that for such a material $d\sigma d\varepsilon > 0$. Drucker accordingly defines a work-hardening (or "stable") plastic material as one in which the work done during incremental loading is positive, and the work done in the loading-unloading cycle is non-negative; this definition is generally known in the literature as Drucker's postulate (Lubliner, 1990).

Having defined hardening in terms of work, Drucker naturally extends the definition to general three-dimensional states of stress and strain, such that

$$d\sigma_{ij} d\varepsilon_{ij} \geq 0 \quad \text{and} \quad d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \quad (B-13)$$

the equality holding only if $d\varepsilon^p = 0$. For perfectly plastic ("neutrally stable") materials Drucker's inequalities are $d\sigma_{ij} d\varepsilon_{ij} \geq 0$ and $d\sigma_{ij} d\varepsilon_{ij}^p = 0$. It can be seen that the equality

$$\dot{\sigma}_{ij} \dot{\varepsilon}_{ij}^p \geq 0 \quad (B-14)$$
sometimes known simply as Drucker's inequality, is valid for both work-hardening and perfectly plastic materials. If the initial stress \( \sigma^* \) is inside the elastic region, an external agency adds stresses to \( \sigma \) on the yield surface, then releases the external agency and returns the state of stress back to \( \sigma^* \). As the elastic deformations are fully recoverable and independent of the path from \( \sigma^* \) to \( \sigma \) and back to \( \sigma^* \), all the elastic energy is recoverable. The plastic work done by the external agency on this loading and unloading cycle is the scalar product of the stress vector \( \sigma - \sigma^* \) and the plastic strain increment vector \( d\epsilon^p \). Drucker's postulate, consequently, implies

\[
(\sigma_0 - \sigma_0^*) \dot{\epsilon}_0^p \geq 0
\]  

(B-15)

The postulate leads to the following consequences for work-hardening materials:

Convexity: The initial yield and all the subsequent loading surface must be convex.

Normality: The plastic strain increment vector must be normal to the yield or loading surface at a smooth point (associated flow rule).

B.2-5 Stability Conditions for Granular Materials

Drucker's stability postulate for rate-independent materials provides a sufficient condition for stability and ensures uniqueness in dynamic and static problems. The postulate is satisfied when associated flow rule is employed in the construction of constitutive models. For the constitutive models employing non-associated flow rule, the plastic strain increment vector must be normal to the plastic potential surface. A typical isotropic yield surfaces and a plastic potential surface at point A are shown in Figure B-3. The shaded wedge between the yield surface and the plastic potential surface defines a region in which Inequality (B-14) is not satisfied for the material with non-associated flow. Since a stress increment vector from point A located inside the wedge and the plastic strain increment vector form an obtuse angle, the scalar product of these two vectors is negative. According to Drucker's stability postulate, the material should
exhibit unstable behavior if the stress increment lies in the shaded region. Lade, Nelson and Ito (1987) performed a series of tests to explore stability in the behavior of granular materials. The tests indicated that the material is perfectly stable at stress points where the normal to the yield surface points in the outward direction of the hydrostatic axis. For this condition the deviator stress can be safely increased to produce further plastic shear strains (work-hardening). In other words, the material can sustain higher loads and behave in an inelastic manner without undergoing any instability or collapse (Lade, Nelson and Ito, 1988). Their conclusion is that Drucker’s stability is sufficient to guarantee stability. However, this is not a necessary condition for stability of granular or frictional materials, i.e., stability may be obtained even when Inequality (B-14) is violated. Stable behavior for granular material exists when

\[ \dot{\sigma}_y \dot{\varepsilon}^p_y < 0 \quad \text{(B-16)} \]

and

\[ \frac{\partial f}{\partial \sigma_y} \cdot \delta_y > 0 \quad \text{(B-17)} \]

where \( \delta_y \) is a vector on the hydrostatic axis. Further, stability is also obtained when Inequality (B-16) holds and both

\[ \frac{\partial g}{\partial \sigma_y} \cdot \delta_y \leq 0 \quad \text{(B-18)} \]

and

\[ \frac{\partial f}{\partial W_p} \geq 0 \quad \text{(B-19)} \]

are satisfied. Experiments to support this stability condition were presented in Lade, Nelson and Ito (1987).
Figure B-3 Pattern of yield surfaces for isotropic granular materials
(Lade, Nelson and Ito, 1988)