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Singularities of subanalytic sets and energy minimizing maps

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SINGULARITIES OF SUBANALYTIC SETS
AND ENERGY MINIMIZING MAPS

by

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ABSTRACT

SINGULARITIES OF SUBANALYTIC SETS
AND ENERGY MINIMIZING MAPS

Shiah-Sen Wang

This thesis studies some problems derived from differential topology and dif-
ferential geometry by techniques developed from geometric measure theory, vari-
ational calculus, and partial differential equations. It consists of two independent
parts:

Part I: An Isopermetric Type Inequality for Chains on Singular Spaces We find
an isoperimetric type inequality for integral chains with support in a subset of
\( \mathbb{R}^n \), which satisfies some structural conditions but is not in the Lipschitz category.
We also apply this inequality to derive some results in the subanalytic category
for homologically mass minimizing currents.

Part II: Energy Minimizing Sections of a Fiber Bundle We show that a Dirichlet
\( p \)-energy minimizing section of a fiber bundle is Hölder continuous everywhere
except possibly for a closed subset of Hausdorff dimension at most \( m - \lfloor p \rfloor - 1 \),
where \( m \) is the dimension of the base space of the fiber bundle and \( \lfloor p \rfloor \) is the
greatest integer less than or equal to \( p \).
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Finally, I like to dedicate this thesis to my parents. Without their love and support the completion of this thesis would be impossible.
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PART I

AN ISOPERIMETRIC TYPE INEQUALITY
FOR CHAINS ON SINGULAR SETS
CHAPTER I

INTRODUCTION AND
THE STATEMENT OF THE THEOREM

In their celebrated paper [3], F. Federer and W. Fleming constructed the integral homology group of local Lipschitz neighborhood retracts in $\mathbb{R}^n$ by using complexes of integral flat chains. The most significant features of this theory are the isoperimetric inequalities (see [2], 4.4.2, 4.4.3 etc. for details). Such an inequality typically provides, for a given $k-1$ dimensional boundary, then existence of a $k$ chain of mass at most a constant times the $(\frac{k-1}{k})$th power of the boundary mass. These inequalities and the compactness theorem of integral chains (see 4.4.4 of [2]) are the key ingredients in proving the existence of homological mass minimizing integral currents and their lower density bounds (see 5.1.6 of [2]). R. Hardt generalized this homology theory to the subanalytic category in his papers [6] and [7] (This includes sets defined by polynomial equalities and inequalities).

Because a "cusp" can occur on a subanalytic subset of $\mathbb{R}^n$, the local Lipschitz norm of any continuous retraction must be infinity along this "cusp". Thus we cannot expect to have with subanalytic sets the same isoperimetric inequalities as in the local Lipschitz category. For example:

1. Let

$$A_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^3\} \text{ and } A_1 = \{(0, 0, 0)\},$$

and let

$$X_\epsilon = A_2 \cap \{(x, y, z) \in \mathbb{R}^3 \mid z = \epsilon\} \text{ and } Y_\epsilon = \{(x, y, z) \in A_2 \mid 0 \leq z \leq \epsilon\},$$

then

$$\lim_{\epsilon \to 0} \frac{\mathcal{H}^2(Y_\epsilon)}{[\mathcal{H}^1(X_\epsilon)]^2} = \infty.$$
2. Let

\[ A_2 = \mathbb{R}^3 \text{ and } A_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = z^3\}, \]

and let

\[ X_\varepsilon = \{(x, y, z) \in \mathbb{R}^3 \mid x = t\varepsilon, y = 0, z = t\varepsilon^3 \text{ and } t \in (0, 1)\}, \]

then for any "nice" \( Y_\varepsilon \) with

\[ (X_\varepsilon - \text{Bdry}Y_\varepsilon) \cup (\text{Bdry}Y_\varepsilon - X_\varepsilon) = C_\varepsilon \subset A_1, \]

we have

\[ \lim_{\varepsilon \to 0} \frac{\mathcal{H}^2(Y_\varepsilon)}{[\mathcal{H}^1(X_\varepsilon)]^2} = \infty \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\mathcal{H}^1(C_\varepsilon)}{\mathcal{H}^1(X_\varepsilon)} = \infty. \]

In example 2, any continuous retraction of \( A_1 \) fails to be Lipschitz near the \( y \)-axis. However, there is a local Lipschitz retraction from a relative neighborhood of \( y \)-axis in \( A_1 \) onto \( y \)-axis. In example 1, similar behavior occurs in \( A_2 \) near \( A_1 \). We show an "isoperimetric type" inequality for the sets not in the local Lipschitz category, but satisfying some structural conditions generalizing these observations (see the general hypotheses in Chapter I) in two separate cases:

**Theorem.** There are positive constants \( \Gamma_1, \Gamma_2 \) depending on \( m, n, c_i, i = 1, \ldots, 5 \) only, so that the following two statements hold:

1. \( m > 0 \) and \( X \in I_m(\mathbb{R}^n) \) with \( \text{sptX} \subset C \subset A_3, \text{spt}\partial X \subset A_2, \) and \( M(X) \leq \frac{m}{2c_5} \)

   then there is a \( Y \in I_{m+1}(\mathbb{R}^n) \) with \( \text{sptY} \subset A_3, \text{spt}(X - \partial Y) \subset A_2, \) and

   \[ M(Y)^{\frac{m}{m+1}} + M(X - \partial Y) \leq \Gamma_1 [M(X) + r_1 M(\partial X)]. \]

2. \( A_1 \subset C, \ X \in I_m(\mathbb{R}^n) \cap Z_m(A_3, A_2) \cap B_m(U_3, U_2), \) and \( \text{sptX} \subset C, \) then there exists a \( Y \in I_{m+1}(\mathbb{R}^n) \) with \( \text{sptY} \subset A_3, \text{spt}(X - \partial Y) \subset A_2, \)

   \[ M(Y)^{\frac{m}{m+1}} + M(X - \partial Y) \leq \Gamma_2 [M(X) + r_1 M(\partial X)]. \]
Here $I_m(\mathbb{R}^n)$ is the group of integral currents and $Z_m(A,B)$ and $B_m(A,B)$ denote the subgroups of relative cycles and boundaries.

An easy way to picture this theorem: Assume $\mathbb{R}^3 = A_3$, $A_2 \subset A_3$ with a "cusp" $A_1$, and $X$ is a curve of finite length in $C$, a compact subset of $A_3$ with endpoints $\partial X$ on $A_2$, then we can find a surface $Y$ with boundary $X$ and $(\partial Y - X)$ in $A_2$ so that the area of $Y$ and the length of $\partial Y - X$ can be bounded in the above forms.

We prove this theorem in Chapter III. It relies on two important theorems from Federer’s book [2], Theorem 4.2.6 and Theorem 4.4.2. The first one is the Deformation Theorem, which describes how to approximate, both analytically and topologically, a normal current by polyhedral chains. The second one is the isoperimetric inequalities in the homology theory of the local Lipschitz category, which we want to generalize here.

In Chapter IV, we show some applications of this theorem, especially of those in the subanalytic category (see Corollary 2 and Corollary 3).

For basic definitions and notations concerning geometric measure theory we refer to Federer’s book [2] (particularly, the list of notations on pp.670-671); for definitions and properties of subanalytic sets, we refer to the wonderful introductory paper by Bierstone and Milman [1] or Hironaka’s original paper [6].
Suppose that the triples \( \{(A_i, U_i, f_i)\}_{i=1,2,3} \) satisfy the following hypotheses:

(A1) \( U_i \) is a neighborhood of \( A_i \) in \( \mathbb{R}^n \) and \( f_i: U_i \to A_i \) is a continuous retraction for \( i = 1,2,3 \).

(A2) \( A_i \subset A_{i+1}, U_i \subset U_{i+1} \) and \( (1-t)f_i(x) + tx \in U_{i+1} \) for \( x \in U_i, t \in (0,1) \), where \( i = 1,2 \).

(A3) \( f_1 \) and \( f_3 \) are locally Lipschitz.

(A4) for every neighborhood \( V \) of \( A_1 \) in \( U_1 \), \( f_2|_{U_1-V} \) is locally Lipschitz.

We also assume that

(B) there is a local Lipschitz retraction \( \tilde{r}: U_1 \cap A_2 \to A_1 \) and a locally Lipschitz map \( h: [0,1] \times U_1 \cap A_2 \to A_2 \) so that

\[
\begin{align*}
    h(t,a_1) &= a_1 & \forall a_1 \in A_1 \\
    h(1,a_2) &= a_2 & \forall a_2 \in U_2 \cap A_2 \\
    h(0,a_2) &= \tilde{r}(a_2) & \forall a_2 \in U_1 \cap A_2
\end{align*}
\]

Let \( C \) be a compact subset of \( A_3 \) so that \( C \cap A_1 \) and \( C \cap A_2 \) are compact. Let \( r_1, r_2, r_3 \) be positive constants such that

\[
B_i = \{ x \in \mathbb{R}^n \mid \text{dist}(x, A_i \cap C) < r_i \} \subset U_i \quad \text{for } i = 1,2,3.
\]

Write \( u(x) = \text{dist}(x, A_2 \cap C) \) and \( v(x) = \text{dist}(x, A_1 \cap C) \). Let \( c_1 = \text{Lip}_f|_{B_1}, c_3 = 2n^{2m+2} \) and \( c_3 = \text{Lip}_f|_{B_3} \).

Choose \( c_4 > 0 \) so that \( \mathcal{H}^1(\gamma_x) \leq c_4 v(x) \), where

\[
\gamma_x = \{ y \in \mathbb{R}^n \mid y = h(t,x) \text{ for some } t \in (0,1), x \in B_2 \cap U_1 \}.
\]
and
\[ \| \nabla_x h(t, x) \|_\infty \leq c_4 \quad \forall t \in [0, 1]. \]

Choose \( r' > 0 \) so that
\[ 0 < r' < \frac{r}{c_4} \text{ and } f_2(\{x \in \mathbb{R}^n \mid v(x) < r'\}) \subset \{x \in \mathbb{R}^n \mid v(x) < \frac{r}{c_4}\} = \{v < \frac{r}{c_4}\} \]

Finally, let \( c_2 = \text{Lip}_2 f_2 \mid_{B_2 - \{x \in \mathbb{R}^n : v(x) \leq \frac{r}{c_4}\}} \) and let \( \delta = \min\{\frac{r}{c_2}, r_2\}/(2n + 1) \).
CHAPTER III

PROOF OF THE THEOREM

Suppose that \( \delta \) is given as in Chapter II and that \( X \) be as in (1) or (2). Let \( 0 < \epsilon \leq \delta \). Because \( \text{spt} \partial X \subset \{ x : u(x) = 0 \} \),

\[
\int_0^\epsilon M(\partial(X \setminus \{ u > r \})) \, d\mathcal{L}^1 r \leq M(X)
\]

Choose \( r \) and \( T \) so that \( r \in (0, \epsilon), X \setminus \{ u(x) > r \} = T, \epsilon M(\partial T') \leq M(X) \). This implies that \( T \in I_m(\mathbb{R}^n) \) by the closure theorem (see 4.2.16(2) of [2]).

Define

\[
\Omega = I_m(\mathbb{R}^n) \cap \{ R \mid \text{spt} R \subset U_3 \cap \mu(\mathbb{W}_m') \text{ and } \text{spt} \partial R \subset \mu(\mathbb{W}_{m-1}') \}.
\]

Applying the Deformation Theorem to \( \epsilon \) and \( T \), we have

\[
T = P + Q + \partial S_1,
\]

\[
\text{spt} P \cup \text{spt} S_1 \subset \{ x \mid \text{dist}(x, \text{spt} T') < 2n\epsilon \},
\]

\[
\text{spt} Q \cup \text{spt} \partial P \subset \{ x \mid \text{dist}(x, \text{spt} T') < 2n\epsilon \},
\]

\[
M(P) \leq 2c_3 M(X),
\]

\[
M(Q) \leq c_5 M(X),
\]

\[
M(S_1) \leq \epsilon c_5 M(X),
\]

\[
M(\partial S_1) \leq (3c_5 + 1) M(X).
\]

Since \( \text{spt} \partial T \subset \{ u = r \} \) and \( \epsilon \leq \delta \),

\[
\text{spt} Q \cup \text{spt} \partial P \subset \{ u < (2n + 1)\epsilon \} \subset B_2 \cap B_3,
\]

\[
\text{spt} P \cup \text{spt} S_1 \subset B_3.
\]
Notice that \( X - T + Q = X \mathbb{1}_{\{u < r\}} + Q = X - P + \partial S_1 \). Thus

\[
\text{spt}(X - T + Q) \subset \{u < (2n + 1)\epsilon\} \subset B_2.
\]

In case \( 0 < M(X) \leq \frac{\delta^m}{2c_5} \), choose \( \epsilon > 0 \) so that \( 2c_5 M(X) = \delta^m \), hence \( M(P) \leq \epsilon^m \).

But since \( M(P) \) is an integral multiple of \((2\epsilon)^m\), it follows that \( P = 0 \). Therefore,

\[
X - T + Q = X - \partial S_1
\]

Since \( \int_{r'}^r M((X - \partial S_1, v, s +)) dL^1 s \leq (3c_5 + 2)M(X) \), we may choose \( s \) so that \( \frac{r'}{2} < s < r' \) and

\[
\langle X - \partial S_1, v, s + \rangle \in I(\mathbb{R}^n),
\]

\[
M((X - \partial S_1, v, s +)) \leq 2(3c_5 + 2) \frac{M(X)}{r'}.
\]

Define

\[
S_2 = h_{2\frac{r'}{2}} \left( \left[ 0, 1 \right] \times (X - T + Q) \mathbb{1}_{\{v > s\}} \right).
\]

Then \( \text{spt} S_2 \subset B_3, S_2 \in I_{m+1}(\mathbb{R}^n) \), and

\[
M(S_2) \leq c^m_2(2n + 1)\epsilon M(X - T + Q) \leq c^m_2((c_5 + 1)(2n + 1)\epsilon M(X).
\]

Note that

\[
\partial(X - T + Q) \mathbb{1}_{\{v > s\}} = (\partial X) \mathbb{1}_{\{v > s\}} - \langle X - \partial S_1, v, s + \rangle
\]

and \( \text{spt}(\partial X) \mathbb{1}_{\{v > s\}} \in A_2 \), hence

\[
\partial S_2 = (X - T + Q) \mathbb{1}_{\{v > s\}} - f_{2\frac{r'}{2}} \left( (X - T + Q) \mathbb{1}_{\{v > s\}} \right)
\]

\[
+ h_{2\frac{r'}{2}} \left( \left[ 0, 1 \right] \times \langle X - \partial S_1, v, s + \rangle \right).
\]

We also note that, by the choice of \( r' \), we have

\[
\text{spt} f_{2\frac{r'}{2}}(X - \partial S_1, v, s+) \subset B_1 \cap A_2 \cap \{ v < \frac{r'}{2} \},
\]

\[
\text{spt} h_{2\frac{r'}{2}}(\left[ 0, 1 \right] \times (X - \partial S_1, v, s+)) \subset B_1 \cap \{ v < \frac{r}{2} \}.
\]
Compute

\[ M(f_{2\sharp}(\{(X - T + Q) \leq \{v > s\}) \} \leq c_2^m(c_5 + 1)M(X) \]
\[ M(f_{2\sharp}(X - \partial S_1, v, s+)) \leq 2c_2^{m-1}(3c_5 + 2)\frac{M(X)}{\epsilon} \]
\[ M(\ h_{2\sharp}(\{0, 1\} \times \langle X - S_1, v, s+\}) \leq c_2^{m-1}(2n - 1)\epsilon M(\langle X - \partial S_1, v, s+\}) \]
\[ \leq 2c_2^{m-1}(3c_5 + 2)\epsilon \frac{M(X)}{\epsilon} \]
\[ \leq 2c_2^{m-1}(3c_5 + 2)M(X). \]

Defining

\[ D = h_{2\sharp}(\{0, 1\} \times \{(\partial X) \leq \{v \leq s\} + f_{2\sharp}(X - \partial S_1, v, s+)) \}, \]

we have that \( D \in I_m(\mathbb{R}^m), \text{spt} D \subset B_1 \) and, with \( c_6 = \frac{c_2}{\epsilon}, \)

\[ M(D) \leq c_4^{2m}r_1M(\partial X) + M(f_{2\sharp}(X - \partial S_1, v, s+)) \]
\[ \leq c_4^{2m}[r_1M(\partial X)] + 2c_2^{m-1}c_4^{2m}(3c_5 + 2)c_6M(X). \]

Define

\[ Z = (X - \partial S_1) \leq \{v \leq s\} - h_{2\sharp}(\{0, 1\} \times \langle X - \partial S_1, v, s+\}) - D. \]

Then \( Z \in I_m(\mathbb{R}^n), \text{spt} Z \subset B_1 \subset B_3 \), and \( \partial Z \subset A_1 \), because \( f_2(x) = x \), if \( x \in \text{spt} \partial X \cap \{v = s\} \); and

\[ \partial Z = (\partial X) \leq \{v \leq s\} + \langle X - \partial S_1, v, s+\) \]
\[ - \langle X - \partial S_1, v, s+\) + f_{2\sharp}(X - \partial S_1, v, s+) \]
\[ - (\partial X) \leq \{v \leq s\} - f_{2\sharp}(\langle X - \partial S_1, v, s+\}) \]
\[ - \tilde{r}_{2\sharp}(\ (\partial X) \leq \{v \leq s\} + f_{2\sharp}(\langle X - \partial S_1, v, s+\)) \]
\[ = \tilde{r}_{2\sharp}(\partial X) \leq \{v \leq s\} + f_{2\sharp}(\langle X - \partial S_1, v, s+\)). \]
\[ M(Z) \leq (3c_5 + 2)M(X) + 2c_2^{m-1}(3c_5 + 2)c_8 M(X) \]
\[ + c_4^{2m} r_1 M(\partial X) + 2c_2^{m-1}c_4^{2m}(3c_5 + 2)c_6 M(X) \]
\[ = c_7 M(X) + c_8 r_1 M(\partial X), \]

with appropriate choices of \( c_7 \) and \( c_8 \).

Let \( X' = f_3 f(Z) \). Then \( \text{spt} X' \subset f_3(\overline{B}_1) \), which is compact and contained in \( A_3 \); also \( \text{spt} \partial X' = \text{spt} f_3(\partial Z) = \text{spt} \partial Z \subset A_1 \). Compute

\[ M(X') \leq c_3^m M(Z) \leq c_3^m c_7 M(X) + c_3^m c_8 r_1 M(\partial X) \]

Notice that \( h_1([0, 1] \times Z) \in I_{m+1}(\mathbb{R}^n) \), \( \text{spt} h_1([0, 1] \times Z) \subset U_3 \), and that

\[ \partial h_1([0, 1] \times Z) = Z - f_1 f(Z); \]

hence

\[ X' = f_1 f(Z) + \partial f_3 h_1([0, 1] \times Z), \]

which implies

\[ X' \in R_m(\mathbb{R}^n) \cap Z_m(A_3, A_1) \cap B_m(U_3, U_1). \]

Applying the isoperimetric inequality of local Lipschitz category (see 4.4.2(2) of [2]) to \( X', A_3, A_1, f_3(\overline{B}_1), U_3 \), and \( U_1 \) in place of \( X, A, B, K, U \), and \( W \) correspondingly, we have

\[ Y_3 \in I_{m+1}(\mathbb{R}^n), \text{spt} Y_3 \subset A_3, \text{spt} (\partial Y_3 - M(X')) \subset A_1, \]

and

\[ M(Y_3)^{\frac{m+1}{m}} + M(X' - \partial Y_3) \leq \sigma M(X'), \]

where \( \sigma \) is the isoperimetric constant in 4.4.2(2) of [2].
Choose \( Y = f_3(S_1) + f_3(S_2) + Y_3 \). Then \( \text{spt} Y \subset A_3 \) and
\[
M(Y) \leq c_3^{m+1} \varepsilon M(X) + c_3^{m+1} c_7^n (c_5 + 1)(2n + 1) \varepsilon M(X) + [\sigma M(X')]^{m+1}.
\]
Plugging in \( \varepsilon = [2c_5 M(X)]^{\frac{1}{m}} \), we have
\[
M(Y) \leq c_3^{m+1}(2c_5)^\frac{1}{m} M(X) - \frac{m+1}{m} + c_3^{m+1} c_7^n (c_5 + 1)(2n + 1)(2c_5)^\frac{1}{m} M(X) - \frac{m+1}{m}

+ \sigma^{\frac{m+1}{m}} (c_3^n c_8 M(X) + c_3^n c_8 [r_1 M(\partial X)])^{\frac{m+1}{m}}

= c_9 M(X) - \frac{m+1}{m} + c_{10} [r_1 M(\partial X)]^{\frac{m+1}{m}}
\]
with appropriate choices of \( c_9 \) and \( c_{10} \). Hence
\[
M(Y) \leq c_{11} [M(X) + r_1 M(\partial X)]^{\frac{m+1}{m}}, \quad \text{where } c_{11} = \max(c_9, c_{10}, 1).
\]

Compute
\[
\partial Y = f_3(\partial S_1) + f_3(\partial S_2) + \partial Y_3 - X' + X'

= X - f_2([X - T + Q] \cap \{v > s\}) + D + X' - \partial Y_3,
\]
so \( \text{spt}(X - \partial Y) \subset A_2 \) and
\[
M(X - \partial Y) \leq M(f_2([X - T + Q] \cap \{u > s\})) + M(D) + M(X - \partial Y)

\leq c_2^m (c_5 + 1) M(X) + c_3^m [r_1 M(\partial X)]

+ 2c_2^{m-1} c_4^m (3c_5 + 2) c_6 M(X)

+ \sigma (c_3^m c_7 M(X) + c_3^m c_8 (r_1 M(\partial X))

= c_{12} M(X) + c_{13} [r_1 M(\partial X)],
\]
with appropriate \( c_{12} \) and \( c_{13} \). Hence,
\[
M(X - \partial Y) \leq c_{14} [M(X) + r_1 M(\partial X)], \quad \text{where } c_{14} = \max(c_{12}, c_{13}).
\]
Choosing $\Gamma_1 = \max(c_{11}^{m+1}, c_{14})$ completes the proof of (1).

For the second case (2), we may assume $2c_5M(X) > \delta^m$ and choose $\delta = \varepsilon$. Let

$$\Phi = \Omega \cap B_m(U_3, U_2).$$

Notice that $\Phi$ and $\Omega$ are free abelian groups, generated over $\mathbb{Z}$ by finitely many $\mathbb{R}$-independent elements.

For each $R \in \Phi$ there exist

$$F \in \mathcal{R}_{m+1}(\mathbb{R}^n) \text{ with } \text{spt}F \subset U_3, \text{ spt}(R - \partial F) \subset U_2,$$

$$G \in \mathcal{R}_m(\mathbb{R}^n), H \in \mathcal{R}_{m+1}(\mathbb{R}^n) \text{ with } R - \partial F = G + \partial H, \text{spt}G \cup \text{spt}H \subset U_2,$$

hence $F + H \in \mathcal{I}_{m+1}(\mathbb{R}^n)$, spt$(F + H) \subset U_3$, and spt$(R - \partial(F + H)) \subset U_2$.

Therefore, we can construct a homomorphism

$$\Lambda: \Phi \rightarrow \mathcal{I}_{m+1}(\mathbb{R}^n)$$

such that spt$\Lambda(R) \subset U_3$ and spt$[R - \partial \Lambda(R)] \subset U_2, \forall R \in \Phi$. Observing that $\Phi$ spans a finite dimensional subspace of $\mathcal{P}_m(\mathbb{R}^n)$ normed by $M$, and that $\Lambda$ has a linear extension mapping this subspace into $\mathcal{N}_{m+1}(\mathbb{R}^n)$, we obtain, for $\theta \in (0, 1)$, a positive number $c_{15}$ so that when $s \in (\frac{r'}{2}, r')$ and $R \in \Phi$,

$$\begin{align*}
M \left( f_{3_2} \{ \Lambda(R) + h_{2_4}[[0, 1]] \times (R - \partial \Lambda(R)) \} \right) &\leq c_{15} M(R) \\
M \left( f_{3_2} \{ h_{2_4}[[0, 1]] \times (R - \partial \Lambda(R), v, s+) \} \right) &\leq c_{15} M(R) \\
M(f_{2_4}((R - \partial \Lambda(R)) \cup \{v > s\})) &\leq c_{15} M(R) \\
M((R - \partial \Lambda(R), v, s+)) &\leq \frac{\theta}{2} M(R - \partial \Lambda(R)) \\
M(R - \partial \Lambda(R)) + M(f_{2_4}(R - \partial \Lambda(R), v, s+)) &\leq c_{15} M(R) \\
\end{align*}$$

(*)

with the exception of a set of $\mathcal{L}^1$-measure at most $\frac{\theta r'}{2}$.

Because $P = X - (X - T + Q) - \partial S_1 \in \Phi$,

$$X - \partial \Lambda(P) - \partial S_1 = (P - \partial \Lambda(P)) + (X - T + Q)$$
and thus $\text{spt}(X - \partial \Lambda(P) - \partial S_1) \subset U_2$.

Since
\[
\int_{\frac{r'}{2}}^{r'} M((X - T + Q, v, s+)) + M((P - \partial \Lambda(P), v, s+)) \, d\mathcal{L}^1 s \\
\leq M(X - T + Q) + M(P - \partial \Lambda(P)) \\
\leq (c_5 + 1)M(X) + 2c_{15}c_5M(X) = c_{16}M(X)
\]
with appropriate $c_{16}$; hence we may choose an $s \in (\frac{r'}{2}, r')$, so that
\[
\langle X - T + Q, v, s+ \rangle \in \mathbf{I}_{m-1}(\mathbb{R}^n),
\]
\[
\langle P - \partial \Lambda(P), v, s+ \rangle \in \mathbf{I}_{m-1}(\mathbb{R}^n)
\]
and (*) hold, with
\[
M((X - T + Q, v, s+)) + M((P - \partial \Lambda(P), v, s+)) \leq 2c_{16} \frac{M(X)}{r'}.
\]
Thus
\[
M(f_{2\xi}^*(X - \partial \Lambda(P) - \partial S_1, v, s+)) \leq 2(c_2^{m-1} + c_{15})c_{16} \frac{M(X)}{r'}.
\]
Define
\[
S_2' = h_{2\xi} \left( [0, 1] \times \{(X - T + Q) \sqsubset \{v > s\} \} \right),
\]
so $S_2' \in \mathbf{I}_{m+1}(\mathbb{R}^n)$, with $\text{spt} S_2' \subset B_3$ and
\[
M(f_{2\xi}(S_2')) \leq c_{17}M(X)\delta,
\]
where $c_{17} = c_3^{m+1}c_22^m(c_5 + 1)(2n + 1)$.

Check that
\[
\partial S_2' = (X - T + Q) \sqsubset \{v > s\} - f_{2\xi}((X - T + Q) \sqsubset \{v > s\}) \\
+ h_{2\xi} \left( [0, 1] \times \partial[(P - \partial \Lambda(P)) \sqsubset \{v > s\}] \right) \\
+ h_{2\xi} \left( [0, 1] \times (X - \partial S_1 - \partial \Lambda(P), v, s+) \right).
\]
Define $D' = h_1 \left( [0,1] \times \{(\partial X) \upharpoonright_{v \leq s} + f_{2\frac{m}{2}}((X - \partial A(P) - \partial S_1, v, s+))\} \right)$. Thus $D' \in I_m(\mathbb{R}^n)$, spt$D' \subset \{ v < \frac{r_1}{c_4} \} \cap A_2$, and

$$M(D') \leq c_4^{2m} [r_1 M(\partial X)] + 2c_4^{2m} c_16(c_2^{m-1} + c_15)M(X).$$

Define

$$Z' = (X - T + Q) \upharpoonright_{v \leq s} - h_1 \left( [0,1] \times (X - \partial A(P) - \partial S_1, v, s+) \right) - D' + [P - \partial A(P)] \upharpoonright_{v \leq s}.$$

Then spt$Z' \subset U_1$ and spt$\partial Z' \subset A_1$. Compute

$$M(f_{3\frac{m}{2}}(Z')) \leq c_3^m(c_5 + 1)M(X) + c_3^m \delta c_2^{m-1}(2n + 1)(c_5 + 1)M(X)$$

$$+ 2c_15c_3 M(X) + 2c_2^{2m} c_16(c_2^{m-1} + c_15)c_3^m M(X)$$

$$+ c_4^{2m} c_3^m [r_1 M(\partial X)]$$

$$= c_18 M(X) + c_19[r_1 M(\partial X)],$$

with appropriate $c_{18}$ and $c_{19}$.

By the same argument as in (1) applied to $Z'$ in place of $Z$, we have $Y_3' \in I_{m+1}(\mathbb{R}^n)$ such that spt$Y_3' \subset A_3$, spt$[\partial Y_3' - f_{3\frac{m}{2}}(Z')] \subset A_1$ and

$$M(Y_3')^{\frac{m}{m+1}} + M(\partial Y_3' - f_{3\frac{m}{2}}(Z')) \leq \sigma M(f_{3\frac{m}{2}}(Z')).$$

Take

$$Y = f_{3\frac{m}{2}}(S_1 + S_2) + f_{3\frac{m}{2}} \left( A(P) + h_1 \left( [0,1] \times [P - \partial A(P)] \upharpoonright_{v \geq s} \right) \right) + Y_3'.$$

Then $Y \in I_{m+1}(\mathbb{R}^n)$, spt$Y \subset A_3$ and

$$\partial Y = X + [\partial Y_3' - f_{3\frac{m}{2}}(Z')] + f_{2\frac{m}{2}} \left( (X - T + Q) \upharpoonright_{v \geq s} \right) - f_{3\frac{m}{2}} \left( (P - \partial A(P)) \upharpoonright_{v \geq s} \right) - D'. $$
so \( \text{spt}(X - \partial Y) \subset A_2 \). It remains to estimate \( M(Y) \) and \( M(X - \partial Y) \).

\[
M(X - \partial Y) \leq \sigma c_{18} M(X) + c_{19}[r_1 M(\partial X)] + c_2^m (c_5 + 1) M(X) + 2c_5 c_{15} M(X) \\
+ 2c_4^2 c_{16} (c_2^{m-1} + c_{15}) M(X) + c_4^2 [r_1 M(\partial X)] \\
= c_{20} M(X) + c_{21} M[r_1 M(\partial X)],
\]

with appropriate \( c_{20} \) and \( c_{21} \). Also

\[
M(Y) \leq c_3^{m+1} c_5 \delta M(X) + c_{17} \delta M(X) + 2c_{15} c_5 M(X) \\
+ [\sigma c_{18} M(X) + c_{19}[r_1 M(\partial X)] \frac{m+1}{m} \\
\leq c_3^{m+1} c_5 (2c_5)^\frac{1}{m} M(X) \frac{m+1}{m} + c_{17} (2c_5)^\frac{1}{m} M(X) \frac{m+1}{m} \\
+ 2c_{15} \frac{c_5}{\delta} (2c_5)^\frac{1}{m} M(X) \frac{m+1}{m} \\
+ (2\sigma c_{18}) \frac{m+1}{m} M(X) \frac{m+1}{m} + (2c_{19}) \frac{m+1}{m} [r_1 M(\partial X)] \frac{m+1}{m} \\
= c_{22} M(X) \frac{m+1}{m} + c_{23} [r_1 M(\partial X)] \frac{m+1}{m},
\]

by choosing proper \( c_{22} \) and \( c_{33} \), where the second inequality is from our assumption that \( 2c_5 M(X) > \delta^m \). Now we can conclude the proof by taking \( \Gamma_2 = \max(c_{20}, c_{21}, c_{24}^m) \).
CHAPTER IV
APPLICATIONS AND REMARKS

Corollary 1. In case \( \partial X \) or \( (\partial X)\big|_{\{v<v'\}} = 0 \), we have

\[
M(Y) \frac{\pi^2}{\mathfrak{m}^2} + M(X - \partial Y) \leq cM(X)
\]

in either (1) or (2) of the Theorem, for some \( c \) independent of \( X \).

Proof. The assertions follow from the constructions of higher dimensional currents in the proof of the theorem.

Corollary 2. If \( A_i, f_i, \bar{r}, h, \) and \( X \) are in the subanalytic category, then so is \( Y \) for both cases of the Theorem.

Proof. Note that the distance functions \( u \) and \( v \) are subanalytic functions, and that affine homotopy of subanalytic functions is still a subanalytic functions. Hence, every step of our constructions in the proof of the Theorem is still in the subanalytic category, the conclusion follows.

Corollary 3. If \( A_3, f_3 \) are subanalytic, \( A_1 \) is the codimension one subanalytic submanifold of the subanalytic manifold \( A_2 \) with singularity \( A_1 \), and \( (A_2 - A_1, A_1) \) is Whitney (b)-regular along \( A_1 \), then the hypotheses (A1)-(A4) and (B) hold.

Proof. Only need to check that the existence of \( \bar{r} \) and \( h \), which we can deduce from Pawłucki's theorem in his paper [8].

Remarks:
1. By Corollary 1, boundary regularity of homological mass minimizing integral currents away from \( A_1 \) is readily available from Lin's thesis [7], where he proved
the boundary of such a minimizer can be as smooth as $A_2$ for the case when $A_1 = \emptyset$.

2. The hypothesis $A_1 \subset C$ in (2) of the Theorem cannot be reduced by using our method here because we have no control over $\text{spt} F \cup \text{spt} G$ in our proof, i.e. it might be too close to $A_1 - C$ where we can not control the local Lipschitz norm of any continuous retraction.
PART II

ENERGY MINIMIZING SECTIONS OF A FIBER BUNDLE
CHAPTER V

INTRODUCTION

Interior partial regularity for minimizers of functionals having nonquadratic growth between Riemannian manifolds has been extensively studied. See [5], [7], [8] and references therein for details. Here we study sections of a Riemannian fiber bundle $X$ that locally minimize the $L^p$ norm of the gradient among all $L^{1,p}_{loc}$ sections when $p \in (1, \infty)$. We show that such a local minimizing section is Hölder continuous everywhere except a closed subset $Z$ of the base manifold $M$, and that the set $Z$ has Hausdorff dimension at most $m - \lfloor p \rfloor - 1$, where $m$ is the dimension of $M$.

It is a well-known topological fact that there is no continuous unit tangent vector field on an even-dimensional sphere; thus continuity of a local minimizing section on all of $M$ may be impossible by topological obstructions. In the trivial bundle case, i.e. $X = M \times N$ with $N$ as the fiber, and $p = 2$, the problem studied here can be easily reduced to study minimizing harmonic maps from $M$ to $N$; therefore, continuity of local minimizing sections may be impeded by energy considerations (see [6]), even without the topological obstructions.

In contrast with harmonic sections (see [1], 2.39), we do include the "horizontal" energy in the energy functional. This causes a major problem in proving the partial regularity for minimizing sections of the simplest form of functionals having nonquadratic growth discussed here because we have to deal with the map constraint—the projection map $\pi$ of the fiber bundle.

The methods used to prove the results are described as follows:

In Chapter VI, first we locally associate each $L^{1,p}$ section $\tilde{v}$ with a map $v \in$
$L^{1,p}(\Omega, N)$ for some bounded open subset $\Omega$ of $M$ by the local trivialization property of the bundle, then we construct a new functional $\mathcal{G}$ defined on $L^{1,p}(\Omega, N)$ from the original one—the $L^p$ norm of the gradient. Via this reformulation, we can study $\mathcal{G}$-minimizers with submanifold $N$ constraint instead of $p$-energy minimizers with the mapping constraint $\pi$.

In Chapter VII, we show that if the normalized $p$–Dirichlet energy of a $\mathcal{G}$-minimizer $u$ is small, then $u$ is Hölder continuous using the De Giorgi blowing up argument outlined in Luckhaus’ paper [8]—where he studies general functionals with nice blow-ups. The key ingredients of the proof are Lemma VII.2 and Lemma VII.3. We show the blow-up functional $\mathcal{F}$ of $\mathcal{G}$ is nice (in fact, our blow-up functional $\mathcal{F}$ is nicer than the one studied in Luckhaus’ paper) in Lemma VII.2 by applying Tolksdorff’s results on system of degenerate elliptic p.d.e.’s (see [12]). Then we show energy decay inequality in Lemma VII.3 by the De Giorgi blowing up argument. In order to use this argument, we also use Luckhaus’ comparison map lemma (see Lemma VII.1) and rescale both the domain and the target manifolds as in Proposition 1 of [8]. Once the energy decay inequality is established, we can iterate this inequality to get Morrey’s growth estimate, and so the Hölder continuity follows. Also, by standard covering argument and the partial Hölder continuity result, we see that the singular set $Z$ defined by

$$Z = \{x \in M \mid \Theta(x) = \lim_{r \to 0} \sup r^{p-m} \int_{B_r(x)} |\nabla u|^p > 0\},$$

is a closed subset of $\Omega$ and has Hausdorff dimension at most $m - p$.

In Chapter VIII, we show that the Hausdorff dimension estimate on $Z$ can be improved. By rescaling the domain near a point in $Z$, we show that the blow-up map $\hat{u}$ of $u$ minimizes a functional $\mathcal{D}$, and a monotonicity formula hold for $u_\rho$, obtained from $u$ by rescaling the domain; and thus $\hat{u}$ is radial homogeneous of
order 0, i.e. $\partial_r \hat{v} = 0$. Hence Federer's dimension reduction argument can be applied here, so the assertion on the Hausdorff dimension of $Z$ follows.

Concerning higher regularity of a $\mathcal{G}$-minimizer $u$ where $u$ is Hölder continuous, we can quote the results in Giaquinta and Modica's paper in case $p \geq 2$ (see [5] in which they study maps between coordinate neighborhoods) by the fact that we already establish Hölder continuity. Unfortunately, we are not able to extend this result to the case when $p \in (1, 2)$ at this moment because some technical inequalities are not true when $p \in (1, 2)$ (see Section 2 of [5]).
CHAPTER VI

PRELIMINARY SETUP AND NOTATIONS

Suppose $B$ is a fiber bundle consisting of:

1) a base space $M$ — an $m$-dimensional $C^2$ Riemannian manifold;
2) a fiber space $N$ — a closed $n$-dimensional $C^2$ submanifold of some Euclidean space $\mathbb{R}^k$;
3) a total space $X$ — an $(m+n)$-dimensional $C^2$ Riemannian manifold;
4) a projection map $\pi: X \to M$ — a $C^2$ submersion from $X$ onto $M$ so that $N_x = \pi^{-1}\{x\}$ is $C^2$ diffeomorphic to $N$ for all $x \in M$.

Let $N_\tau = \{y \in \mathbb{R}^k \mid \text{dist}(x, N) < \tau\}$ for some $\tau > 0$, be a neighborhood of $N$ so that the unique nearest point projection $\xi: N_\tau \to N$ is well-defined and let $\Gamma_1$ be a positive constant depending on $N$ only so that

$$\|\nabla(\xi(y) - \Lambda_{\xi(y)})\| \leq \Gamma_1|y - \xi(y)|,$$

where $\Lambda_{\xi(y)}$ is the orthogonal projection of $\mathbb{R}^k$ onto $\text{Tan}(N, \xi(y))$.

For a point $a$ in $M$, let $\Omega \subseteq M$ be a neighborhood of $a$, and let $h: \Omega \times \mathbb{R}^k \to X$ be a $C^2$ map so that $g = h \big|_{\Omega \times N}$ is a $C^2$ diffeomorphism from $\Omega \times N$ onto $V = g(\Omega \times N)$, a neighborhood of $N_a$ in $X$. Let $\pi_2: \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ be the standard coordinate projection.

For $p \in (1, \infty)$, define

$$SL^{1,p}(\Omega, X) = \{\tilde{w} \in L^{1,p}(\Omega, X) \mid \pi \circ \tilde{w} = x \ a.e. \ x \in \Omega\},$$

and

$$L^{1,p}(\Omega, N) = \{w \in L^{1,p}(\Omega, N) \mid w(x) \in N \ a.e. \ x \in \Omega\}.$$
Define \( \Phi: L^{1,p}(\Omega, N) \to SL^{1,p}(\Omega, X) \) by

\[
\Phi(w)(x) = g(x, w(x)), \quad \forall x \in \Omega \text{ and } w \in L^{1,p}(\Omega, N).
\]

Clearly, \( \Phi \) is bijective with inverse map defined by

\[
\Phi^{-1}(\tilde{w})(x) = \pi_2 \circ g^{-1} \circ \tilde{w}(x) \quad \forall x \in \Omega \text{ and } \tilde{w} \in SL^{1,p}(\Omega, X).
\]

Define \( \mathcal{E}: SL^{1,p}(\Omega, X) \to R \) by

\[
\mathcal{E}(\tilde{w})(x) = \int_{\Omega} |\nabla \tilde{w}|^p \quad \forall \tilde{w} \in SL^{1,p}(\Omega, X),
\]

and define \( \mathcal{G}: L^{1,p}(\Omega, N) \to R \) by

\[
\mathcal{G}(w) = \int_{\Omega} G(x, w, \nabla w)dx \quad \forall w \in L^{1,p}(\Omega, N),
\]

with

\[
G(x, y, \eta) = |A(x, y) + B(x, y)\eta|^p \quad \forall (x, y, \eta) \in \Omega \times \mathbb{R}^k \times M(k, m),
\]

where

\[
A(x, y) = D_x h(x, y) \in M(m + n, n),
\]

\[
B(x, y) = D_y h(x, y) \in M(m + n, k),
\]

\( \forall (x, y) \in \Omega \times \mathbb{R}^k \). Notice that if \((x, y) \in \Omega \times N\), then

\[
A(x, y)\big|_{\text{Tan}(\Omega, x)} \in Gr(m + n, m), \text{ and } B(x, y)\big|_{\text{Tan}(N, y)} \in Gr(m + n, m).
\]

This fact will be used in proving Lemma VII.2. Observe that \( u \) is a \( \mathcal{G} \)-minimizer in \( L^{1,p}(\Omega, N) \) iff \( \Phi(u) \) is a \( \mathcal{E} \)-minimizer in \( SL^{1,p}(\Omega, X) \) because \( \mathcal{G}(u) = \mathcal{E}(\Phi(u)) \).

Let \( \Gamma_2 \) and \( \Gamma_3 \) be two positive numbers so that

\[
(1) \quad \Gamma_2^{-1} \leq \|Dh(x, y)\|, \|Dg^{-1}(z)\| \leq \Gamma_2 \quad \forall (x, y) \in \Omega \times N, \forall z \in g^{-1}(\Omega \times N),
\]
and

\begin{equation}
\Gamma_3^{-1} \leq \|A\big|_{\text{Tan}(\Omega, x)}\|, \|B\big|_{\text{Tan}(N, y)}\| \leq \Gamma_3 \quad \forall x \in \Omega, \forall y \in N
\end{equation}

Next, we make some observations to simplify our exposition of the following two sections.

1. The compactness assumption on \( N \) can be replaced by the hypothesis that the image of a small ball for an \( G \)-minimizer is contained in \( \tilde{N} \subseteq N \) (compare [11], Theorem I).

2. It can be easily checked that the integrand \( G \) of the functional \( G \) defined above satisfies all the growth conditions studied by Luckhaus in [8] and [9], also studied by Giaquinta and Modica in [5] for the case when \( p \geq 2 \) and when the maps are between coordinate neighborhood. However, we will carry out most of the computation in the proof, since some of them are simpler here without referring to the general hypotheses of \( G \) these papers imposed, \( i.e. \) here we provide an example that these hypotheses on \( G \) are "natural".

3. We will confine our study to the case that \( B_2(0) \subseteq \Omega \subseteq \mathbb{R}^k \) with the standard Euclidean metric, because the general case can be easily modified by first shrinking \( \Omega \) if necessary so that the general metric is \( C^1 \) close to the Euclidean one and then by rescaling the Euclidean metric as in the minimizing \( p \)-harmonic case studied by Hardt and Lin in [7], Section 7.

We complete this section with a discussion on scaling:

For \( w \in L^{1,p}(B_r(a), \mathbb{R}^k) \) with \( a \in B_1(0) \) and \( r \in (0, 1) \), the expression

\[ w_{r,a}(x) = w(rx + a) \quad (= w_r(x), \text{ when } a = 0) \quad \forall x \in B_1(0) = B \]

defines a map in \( L^{1,p}(B, N) \).
If $u$ is a $G$–minimizer in $L^{1,p}(B, N)$, then $u_{r,a}$ is minimizes the functional $G_{r,a}$ among maps in $L^{1,p}(B, N)$, defined by

$$G_{r,a}(w) = \int_B G_{r,a}(x, w, \nabla w)dx, \quad w \in L^{1,p}(B, N),$$

where

$$G_{r,a}(x, y, \eta) = G(rx + a, y, \frac{\eta}{r}), \quad \forall (x, y, \eta) \in \Omega \times \mathbb{R}^k \times M(k, m).$$

For $w \in L^{1,p}(B_r(a), \mathbb{R}^k)$, write

$$E_{r,a}(w) = r^{p-m} \int_{B_r(a)} |\nabla w|^p \quad (= E_r(w), \text{ when } a = 0).$$

Note that

$$E_{\rho}(w_{r,a}) = E_{r\rho,a}(w), \quad \rho \in (0, 1].$$
CHAPTER VII

SMALL ENERGY IMPLIES HÖLDER CONTINUITY

The following lemma extends Lemma 4.3 of [11], and is due to Luckhaus (see [8], Lemma 1).

Lemma VII.1. For $\beta \in \left(\frac{p-1}{p}, 1\right)$, there is a positive constant $c_1 = c(m, k, p, \beta)$ so that if $0 < t \leq \frac{1}{2}, 0 < \rho, \epsilon < 1, a \in B$, and $v_1, v_2 \in L^{1,p}(S_\rho(a), N)$, then there is a map $w \in L^{1,p}(B_\rho(a) - B_{(1-t)\rho}(a), \mathbb{R}^k)$ satisfying

$$w(x) = \begin{cases} v_1(x) & x \in S_\rho(a); \\ v_2(\frac{x-a}{1-t} + a) & x \in S_{(1-t)\rho}(a). \end{cases}$$

and

$$\text{dist} \ (w(x), N) \leq r \quad x \in B_\rho(a) - B_{(1-t)\rho}(a),$$

and

$$\int_{B_\rho(a) - B_{(1-t)\rho}(a)} |\nabla w|^p \leq c_1 K^p (1 + (\frac{\rho}{r})^p)t,$$  \hspace{1cm} (3)

where $r = c_1 K e^{1-\rho} \frac{(p-1)(p-1)}{p}$, $K^p = \int_{S_\rho(a)} (|\nabla \tan v_1|^p + |\nabla \tan v_2|^p + \frac{|v_1 - v_2|^p}{\epsilon^p})$.

The next lemma shows that the blow-up function is nice (actually Hölder continuity is sufficient to apply Luckhaus' results, see Hypothesis (A2) in [8]).

Lemma VII.2. There is a positive constant $c_2 = c(m, k, p, \Gamma_3)$ such that if $(x_0, y_0) \in \Omega \times N, v \in L^{1,p}(B, \mathbb{R}^k)$ with $v(B) \subset \text{Tan}(N, y_0) (\simeq \mathbb{R}^n)$, and $v$ minimizing $\mathcal{F}$ among maps in $L^{1,p}(B, \text{Tan}(N, y_0))$, then $v \in C^{1,\gamma}$ for some $\gamma \in (0, 1)$ and

$$\|v\|_{L^{\infty}(B_{2R}(a))} \leq c_2 \int_{B_{2R}(a)} (1 + |\nabla v|^p)^{1/\gamma}$$  \hspace{1cm} (4)
whenever $B_3 R(a) \in B$, where

$$\mathcal{F}(w) = \int_B F(w) \quad \forall w \in L^{1,p}(B, \text{Tan}(N, y_0)),$$

with

$$F(\eta) = |B(x_0, y_0) \eta|^p \quad \eta \in M(m + n, k).$$

**Proof.** Since $v$ minimizes the functional $\mathcal{F}$ among maps in $L^{1,p}(B, \text{Tan}(N, y_0))$, it satisfies the following degenerate elliptic system of equations

$$\int_B |B(x_0, y_0) \nabla v|^p - 2(B(x_0, y_0) \nabla v, B(x_0, y_0) \nabla \varphi) = 0 \quad \forall \varphi \in L^{1,p}_0(B, \text{Tan}(N, y_0)).$$

Hence, the conclusions follow immediately from Theorem 5.1 and Theorem 6.1 of [12].

**Lemma VII.3.** For $\alpha \in (0, 1)$, there are positive constants $c_3 > 1$, $\epsilon_0$ depending only on $m, p, k, \Gamma_{i=1,2,3}$, and $\alpha$, such that if $u$ is a $G$-minimizer and $r^{\alpha p} < E_{r,a}(u) < \epsilon_0^p$ with $0 < r < 1, a \in B$, then we have

$$E_{c_3a,\alpha}(u) \leq c_3^{-\alpha p} E_{r,a}(u).$$

**Proof.** Were the conclusion false, there would exist a sequence of balls $\{B_{r_i}(a_i)\}_i$, for some constant $c_3$ to be chosen later, so that

$$\begin{cases} 
E_{r_i,a_i}(u) = \epsilon_i^p \\
E_{s_i,a_i}(u) \geq c_3^{-\alpha p} \epsilon_i^p,
\end{cases}$$

where $s_i = \frac{r_i}{c_3}$ and that as $i \to \infty$, $\epsilon_i \to \infty$.

Let $\overline{y}_i = \int_{B_{r_i}(a_i)} u$. Then we have

$$\text{dist } (\overline{y}_i, N) \leq c_4 \epsilon_i^p,$$
for some $c_4 = c(m, p, k)$ by the Poincaré inequality. Hence $y_i = \xi(\bar{y}_i)$ is well-defined, when $i$ is sufficiently large.

For these $i$'s, let

$$v_i = \frac{u(r_i x + a_i) - y_i}{\epsilon_i}, \text{ and } u_i = \epsilon_i v_i + y_i$$

By the Poincaré inequality again, we have

$$\|v_i\|_{L^1, p(B, \mathbb{R}^k)} \leq c_5 \quad \text{for some } c_5 = c(m, p, k, \Gamma_1).$$

Note that $N$ is compact. Passing to subsequences without changing notations, we may assume that

$$\begin{cases}
  a_i \to x_0 \in B, \\
  y_i \to y_0 \in N, \\
  v_i \to v \quad \text{strongly in } L^p \text{ norm and pointwise a.e. on } B, \\
  \nabla v_i \to \nabla v \quad \text{weakly in } L^p(B, \mathbb{R}^k), \\
  \alpha_i^{-p} G(a_i, v_i, \alpha_i \eta) \to F(\eta) \quad \text{for all } \eta \in \mathbb{R}^k, \text{ where } \alpha_i = \frac{\epsilon_i}{r_i}.
\end{cases}$$

**Claim:** $v$ satisfies the hypothesis of Lemma VII.2 (and so its conclusion), and for each $\rho \in (0, 1),$

$$\epsilon_i^{-p} \int_{B_{\rho r_i}(a_i)} G(x, u, \nabla u) dx \to \int_{B_{\rho}(0)} F(v). \quad (7)$$

Assume the Claim is true for the moment. Then by Lemma VII.2 (4) with $a = 0, R = \frac{1}{3}$, and requiring $c_3 \geq \frac{3}{2}$, we have

$$E_{\frac{1}{c_3}}(v) < c_6, \quad \text{with } c_6 = (m, k, p, \Gamma_3).$$

It follows that

$$c_3^{-p} \int_{B_{\frac{1}{c_3}}} F(v) < c_7 c_3^{-p} \quad \text{where } c_7 = c(m, k, p, \Gamma_3).$$
Thus by (7), we see that
\[
\lim_{i \to \infty} \frac{s_i^{p-m} \int_{B_{r_i}(a_i)} G(x, u, \nabla u) \, dx}{r_i^{p-m} \int_{B_{r_i}(a_i)} G(x, u, \nabla u) \, dx} < c_7 c_3^{-p}.
\]
By taking \( i \) sufficiently large, one has
\[
s_i^{p-m} \int_{B_{s_i}(a_i)} G(x, u, \nabla u) \, dx \leq 2c_7 c_3^{-p} r_i^{p-m} \int_{B_{r_i}(a_i)} G(x, u, \nabla u) \, dx,
\]
so
\[
s_i^p + E_{s_i,a_i}(u) \leq c_8 c_3^{-p}(r_i^p + E_{r_i,a_i}(u)) \quad \text{for some } c_8 = c(m, p, \Gamma_2, \Gamma_3)
\]
Hence by (6) and the definition of \( s_i \), we have
\[
c_9 c_3^{-p} e_i^p \geq c_3^{-\alpha p} e_i^p \quad \text{for some } c_9 = (m, p, \Gamma_{i=1,2,3}).
\]
This leads to a contradiction to (6) by further requiring that \( c_8 c_3^{(\alpha-1)p} \leq 1 \).

Proof of the Claim: The proof here is due to Luckhaus (see pp.358 of [8]). Let \( \hat{v} \) be any comparison function coinciding with \( v \) in \( B - B_{1-\lambda} \) for some \( \lambda \in (0, 1) \). By Fatou’s lemma and Fubini’s theorem, we may assume that there is a \( \hat{\rho} \in (1-\lambda, 1) \) such that
\[
\int_{S_{\hat{\rho}}} |v_i - \hat{v}|^p \to 0, \text{ as } i \to \infty, \quad \int_{S_{\hat{\rho}}} (|\nabla v_i|^p + |\nabla \hat{v}|^p) \leq c_{10} < \infty.
\]
Furthermore, choose a sequence of positive numbers \( R_i \to \infty \) as \( i \to \infty \) such that
\[
\|(\text{Id} - \xi)|_{B_{2R_i}(y_i) \cap (\text{Tan}(N,y_i)+y_i)}\|_{L^\infty} = o(\varepsilon_i)
\]
\[
\|\nabla(\text{Id} - \xi)|_{B_{2R_i}(y_i) \cap (\text{Tan}(N,y_i)+y_i)}\|_{L^\infty} = \frac{o(\varepsilon_i)}{\varepsilon_i}.
\]
Define
\[
\hat{v}_i = \frac{R_i \hat{v}}{\max(|v_i|, R_i)}, \quad \hat{u}_i = \xi(y_i + \varepsilon_i \hat{v}_i), \quad u_i = y_i + \varepsilon_i v_i,
\]
and apply Lemma VII.1 to $\tilde{u}_i, u_i, \lambda_i$ and $\hat{\rho}$ in place of $v_1, v_2, t$ and $\rho$. We find a map $\tilde{w}_i$ such that

$$
\tilde{w}_i(x) = \begin{cases} 
\tilde{u}_i(\frac{x}{1-\lambda_i}) & \text{for } |x| < \hat{\rho}(1 - \lambda_i), \\
u_i(x) & \text{for } |x| > \hat{\rho},
\end{cases}
$$

$$
\int_{B_{\hat{\rho}} - B_{\hat{\rho}(1 - \lambda_i)}} |\nabla \tilde{w}_i|^p \leq c_{11} \lambda_i \varepsilon_i^p,
$$

$\lambda_i \to 0$ and $\operatorname{dist} (y, N) \to 0$ uniformly as $i \to \infty$.

Thus

$$
\int_{B_{\hat{\rho}}} F(\nabla \hat{\vartheta}) = \lim_{i \to \infty} \alpha_i^{-p} \int_{B_{\hat{\rho}}} G(a_i, y_i, \alpha_i \nabla \hat{\vartheta}_i)
$$

$$
= \lim_{i \to \infty} \varepsilon_i^{-p} \int_{B_{\hat{\rho}}} G(a_i + r_i x, \xi(\tilde{w}_i)), \frac{\varphi(\tilde{w}_i)}{r_i}
$$

$$
\geq \lim_{i \to \infty} \varepsilon_i^{-p} \int_{B_{\hat{\rho}}} G(a_i + r_i x, u_i, \frac{\varphi u_i}{r_i})
$$

$$
\geq \int_{B_{\hat{\rho}}} F(\nabla u),
$$

where the last inequality follows from the lower semicontinuity of $\mathcal{F}$. Hence the \textbf{Claim} holds by taking $\hat{\vartheta} = v$ in the above inequalities.

\textbf{Theorem VII.4. Under the same assumptions as in Lemma VII.3, there is a positive constant $c_7$, depending on $m, k, p, \alpha$ and $\Gamma_i$, for $i = 1, 2, 3$, such that}

$$
E_{B_{r'}(b)}(u) \leq c_{12} \left(\frac{r'}{r}\right)^{\alpha p} \forall b \in B_r(A) \text{ and } r' \in (0, r)
$$

and $u$ is $C^\alpha$ on $B_{\frac{r}{2}}(a)$.

\textbf{Proof.} The first assertion is from iterating $(*)$, and the second one is from the first one and Morrey's growth estimate Lemma (see [10], 3.5.2).
Corollary VII.5. Any $G-$minimizer is Hölder continuous on $M - Z$, where $Z$ is defined as in Chapter V. Moreover, $Z$ is relatively closed in $M$ and has Hausdorff measure at most $m - p$.

Proof. The closedness of $Z$ and Hölder continuity of $u$ follow immediately from Theorem VII.4, and the Hausdorff dimension estimate on $Z$ is from the standard covering argument.
CHAPTER VIII

IMPROVEMENT ON THE HAUSDORFF DIMENSION
OF THE SINGULAR SET

In this chapter we will assume that \( p \in (1, m) \). By Theorem VII.4, we know that \( u \) is regular at a point in \( M \) if the normalized \( p \)-energy \( E \) tends to 0 when the radius tends to 0. Hence we only need to study the case when \( a \in Z \), where \( Z \) is defined as in Chapter V and

\[
E_{r,a}(u) \geq \varepsilon_0 \quad \forall r \in (0, 1).
\]

For simplicity, we assume that \( a = 0 \in Z \). Define the functional \( \mathcal{D} : L^{1,p}(B, N) \to R \) by

\[
\mathcal{D}(w) = \int_B |B(0, w)\nabla w|^p \quad w \in L^{1,p}(B, N),
\]

(compare hypotheses (A3) in [8]). We write \( \mathcal{D}_t(w) = \int_{B_t} |B(0, w)\nabla w|^p, \forall t \in (0, 1) \).

**Theorem VIII.1.** There exists a sequence \( u_{r_i} \in L^{1,p}(B, N) \) such that

\[
\begin{align*}
    u_{r_i} &\to u_0 \in L^{1,p}(B, N) \text{ strongly in } L^p \text{ norm}, \\
    \nabla u_{r_i} &\to \nabla u_0 \text{ weakly in } L^p(B, N), \\
    r_i^{m-p} \int_{B_{r_i}} G(x, u, \nabla u) &\to \mathcal{D}(u_0),
\end{align*}
\]

as \( i \to \infty \). Furthermore, \( u_0 \) minimizes \( \mathcal{D} \) with respect to a fixed trace on \( S \), and \( \partial_r u_0 \equiv 0 \)

**Proof.** To show that \( u_{r_i} \) converges to some \( u_0 \) weakly in \( L^{1,p}(B, N) \), we notice that by (2)

\[
\Gamma_3^{-1} \int_{B_t} |\nabla u_r|^p \leq \mathcal{D}_t(u_r) \leq \Gamma_3 \int_{B_t} |\nabla u_r|^p, \quad \forall t, r \in (0, 1].
\]
Hence, it is sufficient to show that $\mathcal{D}(u_r)$ is bounded by Rellich compactness theorem.

Choose a constant $c_{13} = c(m, p, k, N, \Gamma_{i=1,2,3}) > 1$ such that

$$|r^p G_{r,0}(x, y, \eta) - |B(0, y)\eta|^p| \leq c_{13} r^p (1 + |\eta|^p) \quad \forall (x, y, \eta) \in B \times N \times M(k, m).$$

Let $t \in (0, 1)$ and define

$$u_{r,t} = \begin{cases} u_r(x) & \text{for } |x| > t; \\ u_r(t \frac{x}{|x|}) & \text{for } |x| \leq t. \end{cases}$$

Thus, by the homogeneity on the gradient variable of the integrand for $\mathcal{D}$ and Fubini's theorem, we have

$$\int_{B_t} |B(0, u_{r,t}) \nabla u_{r,t}|^p = \frac{t}{m - p} \int_{S_t} |B(0, u_r) \nabla u_r|^p. \tag{10}$$

Note that by (1), (2), and (10), we have

$$\begin{align*}
(1 - c_{14} r^p) r^p G_{r,0}(x, u_{r,t}, \nabla u_{r,t}) &\leq |B(0, u_{r,t}) \nabla u_{r,t}|^p + c_{14} r^p, \\
(1 - c_{14} r^p) |B(0, u_r) \nabla u_r|^p &\leq r^p G(x, u_r, \nabla u_r) + c_{14} r^p, \tag{11, 12}
\end{align*}$$

where $c_{14} = \max(c_{13}, c_{13} \Gamma^p_2, c_{13} \Gamma^p_3)$.

By (11), (12), and $\mathcal{G}_{r,0}$-minimality of $u_r$, we have

$$(1 - c_{15} r^p) \int_{B_t} |B(0, u_r) \nabla u_r|^p \leq \frac{t}{m - p} \int_{S_t} |B(0, u_r) \nabla u_r|^p + c_{15} r^p t^m,$$

where $c_{15} = c_{14}^2 \mathcal{L}^m(B)$. By (8) and taking larger $c_{15}$, the last term in the above inequality can be absorbed into the left-hand side to get

$$(1 - c_{15} r^p) \mathcal{D}_t(u_r) \leq \frac{t}{m - p} \int_{S_t} |B(0, u_r) \nabla u_r|^p. \tag{13}$$

Hence we obtain the following monotonicity inequality

$$\partial_t \left( \log[t^{p-m} \mathcal{D}_t(u_r)] + c_{15} r^p \log t \right) \geq 0.$$
Apply (13) to the sequences \( r_i = e^{-i}, t_i = \frac{r_{i+1}}{r_i} = e^{-1} \) in place of \( r, t \) respectively. Write \( u_i = u_{r_i} \). Note that \( e^{m-r} \mathcal{D}_{e^{-1}}(u_i) = \mathcal{D}(u_{i+1}) \) and that

\[
- \sum_i r_i^p \log \left( \frac{r_{i+1}}{r_i} \right) = \sum_i r_i^p < \infty.
\]

Therefore \( \lim_{i \to \infty} \mathcal{D}(u_i) \) exists, \( \mathcal{D}(u_i) \) is bounded and

\[
\lim_{i \to \infty} r_i^{p-m} \mathcal{G}_{r_i,0}(u) = \lim_{i \to \infty} \mathcal{D}(u_i).
\]

Hence, by passing to a subsequence without changing notation, we have that

\[ u_i \to u_0 \text{ strongly in } L^p(B, N), \text{ and } u_i \to u_0 \text{ weakly in } L^{1,p}(B, N). \]

This completes our proof of the first assertion.

To show \( u_0 \) minimizes \( \mathcal{D} \), let \( v \in L^{1,p}(B, N) \) such that \( u_0 - v = 0 \) on \( B - B_{1-t} \) for some \( t \in (0,1) \). Choose another subsequence of \( u_i \) and \( \rho \in (1-t,1) \), if necessary by Fatou's lemma and Fubini's theorem, we may assume that

\[
\int_{S_p} (|\nabla u_i|^p + |\nabla v|^p) \leq c_{16} < \infty, \text{ and } \int_{S_p} |u_i - v|^p \to 0 \text{ as } i \to \infty.
\]

By Lemma VI.1, we have a sequence \( w_i \in L^{1,p}(B, N) \), so that

\[
w_i(x) = \begin{cases} 
  v \left( \frac{x}{1 - \lambda_i} \right) & \text{for } x \in B_{(1-\lambda_i)\rho}, \\
  u_i(x) & \text{for } x \in B - B_{\rho},
\end{cases}
\]

\[
\int_{B_{\rho} - B_{(1-\lambda_i)\rho}} |w_i|^p \leq c_{18} \lambda_i,
\]

where \( \lambda_i \to 0 \) as \( i \to \infty \). By (10), the lower semicontinuity of \( \mathcal{D} \), and the \( \mathcal{G}_{r_i,0} \)-minimality of \( u_i \), we obtain

\[
\mathcal{D}(u_0) \leq \lim_{i \to \infty} \mathcal{D}(u_i) \leq \lim_{i \to \infty} \mathcal{D}(w_i) = \mathcal{D}(v).
\]

Hence \( u_0 \) minimizes \( \mathcal{D} \).
Notice that (13) also implies that \( \partial_r r^{n-m} D_r(u_0) = 0 \), and thus

(14) \[ r^{n-m} D_r(u_0) = D(u_0). \]

Since \( u_0 \) minimizes \( D \), compare with

\[
u_{0,r}(x) = \begin{cases} u_0(x) & \text{for } x \in B - B_r; \\ u_0(r \frac{x}{|x|}) & \text{for } x \in B_r,
\end{cases}\]

we have

\[ D_r(u_0) \leq \frac{r}{m-p} \int_{S_r} |B(0, u_0) \nabla \tan u_0|^p. \]

This inequality and (14) imply that \( \partial_r u_0 \equiv 0 \).

Corollary VIII.2. Suppose that \( u \in L^1_{\text{loc}}(M, N) \) locally minimizes the functional \( G \), then its singular set \( Z \) has Hausdorff dimension at most \( m - [p] - 1 \). In particular, \( Z \) is discrete if \( m = [p] + 1 \).

Proof. The proof is the same as that of 4.5 of [7] or [2].
REFERENCES

PART I


PART II


