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High throughput optical code-division multiple-access communication systems

Brandt-Pearce, Maïté, Ph.D.

Rice University, 1993

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RICE UNIVERSITY

High Throughput Optical Code-Division Multiple-Access Communication Systems

by

Maïté Brandt-Pearce

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE Doctor of Philosophy

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High Throughput Optical Code-Division Multiple-Access Communication Systems

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Abstract

Optical communication systems offer a bandwidth orders of magnitude larger than any electric or radio-frequency system. Yet, historically, optical communication systems have been designed by adapting existing conventional low frequency systems to operate at light frequencies, thus limiting the rate of information transfer to those previously achievable at lower frequencies. The purpose of this study is to show that it is possible, through the design of the optical communication system according to its own statistical characteristics, to achieve reliable information transfer rates commensurate with the optical bandwidth, that is, orders of magnitude higher than any conventional systems.

Two optical code-division multiple-access systems are proposed, one a conventional time encoded system, and one an innovative spectral amplitude encoded system. The spectral amplitude encoded system provides many advantages including the ability to implement efficient symbol detectors, insensitivity to chromatic dispersion, and availability of large codes. An optimized single-user detector, both hard-decision
and soft-decision multistage detectors, and a local search detector are proposed as alternatives to the weak conventional detector and the complex optimal detector.

A full performance analysis of the two systems with each detector is performed. The main analysis tool is an approximation based on large deviation theory. The approximation to the error probability is verified by comparing the results to values obtained using a characteristic function method for small problems. Some simplifying assumptions are made to evaluate the moment generating function and the validity of these is confirmed by comparing to simulation results. Both random user sequences and well-designed deterministic codes are investigated. The optimized single-user detector performed considerably better than the conventional correlation detector in all cases. All multiuser detectors error probabilities compared favorably to the theoretical lower bound. The soft-decision multistage detector showed significant improvement over the hard-decision counterpart. The local search detector outperformed them all, at the expense of added complexity. The asymptotic multiuser efficiency of the system with each detector was also computed using large deviation theory. All proposed detectors are shown to be robust against multiple-access interference.
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To my parents

Helmut and Josette Brandt
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Chapter 1

Introduction

The main objective of this study is to propose an optical communication system that can transmit information at rates that approach the performance limit of the channel. After motivating the problem and defining some preliminary notions, an optical code division multiple-access communication system that meets this objective is presented. A plan for the mathematical modeling and analysis of this system is then described.

1.1 Preliminaries

In the age of information in which we live, there exists a need for multiple point information transfer, i.e., a communication network. Examples of such networks are the telephone system and local area computer networks. Each person or computer that accesses the network with the intent of transmitting or receiving information is called a user. Each user has to interface with the network in a predetermined organized fashion so that he/she obtains the network access desired and interferes with other users as little as possible. Such network access protocols are named multiple-access communication schemes and can be implemented in a variety of ways. One performance measure of multiple-access schemes is the reliability of the information
transfer for a fixed utilization, such as a fixed number of users accessing the network. Another measure of interest is throughput, which is the total amount of information transfer possible at a given level of reliability.

Digital communication is defined as the successive discrete-time transfer of information chosen from a finite alphabet. Some information is in the form of a digital signal from the onset, such as a computer output, while some information is analog. An analog signal is characterized as a continuous time and amplitude signal, such as a voice signal. A higher degree of reliability in the transfer of analog signals can be achieved by sampling and encoding the signal into a digital signal at the source, transmitting it using a digital communication network, and reconstructing the desired information at the destination. For this reason, many of the communication systems today are adopting a digital format. We are therefore concerned with the reliability and throughput of a digital communication network. The quantity of digital information or data is measured in bits or symbols, yielding a throughput in units of bits per second (bps). The reliability is gauged by the average bit or symbol error probability.

At the signal destination, the user employs a symbol detection algorithm whose only objective is to decide what digital symbol was transmitted, based on a stochastic model of the communication network. The bit error probability is a measure of the likelihood that the symbol detector makes an error in the decision of one bit.

A communication network is composed of one or more information sources, a transmission channel, and one or more destinations. The channel is the physical
medium through which the signal travels from the source to the destination. The
throughput and performance of a communication network are generally limited by
the spectral bandwidth of the transmission medium, which is defined as the width
of the usable part of the frequency response of the channel. This width may be set
by the physical characteristics of the medium, such as the attenuation in a cable or
scattering in free space, or by regulations restricting the use of the channel.

The optical medium provides a physical bandwidth that can be considered virtu-
ally unlimited, compared with the quantity of information that any user can generate.
The goal in designing an optical communication system is to develop a scheme by
which this resource can be fully employed. Since each user's data rate is limited by
the maximum electronic speeds of today (on the order of 1 giga-bps, or Gbps), the
throughput can only be increased by increasing the number of users. A powerful
multiple-access communication scheme is needed to allow many users access to the
channel simultaneously without significant degradation of their performance.

Three main multiple-access schemes exist today. The oldest and most conventional
method is frequency-division multiple-access (FDMA), in which the available band-
width is subdivided into slots and each user is restricted to using one assigned slot. A
more contemporary approach is time-division multiple-access (TDMA), which forces
each user to buffer his/her information and allows the transfer of information only
in a preassigned time slot. The most modern multiple-access scheme is code-division
multiple-access (CDMA), which allows all users access to the entire bandwidth all the time by distinguishing between them through the use of signal encoding.

Of these three methods, TDMA is considered the most restrictive since it requires all users in a network to be fully synchronized. It also requires each user to transmit at the network throughput rate for short periods of time. This method is not appropriate for use on an optical network since network throughput rates are not achievable by any single user. FDMA is generally named wavelength-division multiple-access (WDMA) when referring to optical communications since historically the optical spectrum is described by the wavelength of the light and not its frequency. WDMA depends on each user having an optical source (such as a laser) tuned to a different frequency and each destination having a frequency filter to allow only the desired user signal to pass. Optical CDMA (OCDMA) is a completely asynchronous multiple-access scheme that requires each user to employ a unique code to differentiate that user's signal from that of the other users. The encoding process spreads the user signal in the frequency domain so that each user occupies the entire bandwidth available at all times.

Many networking situations arise which favor the use of OCDMA over WDMA. When the number of users is large or the channel access is bursty, OCDMA offers an advantage over WDMA because OCDMA experiences a graceful degradation as the number of users increases and has constant utilization of the entire bandwidth. In WDMA, the performance of any user suffers from the presence of other users whether or not they are transmitting due to the uniform bandwidth restriction. On
the other hand, a user in an OCDMA network suffers no effect from users that are not currently transmitting. An additional advantage of OCDMA over WDMA is the flexibility in user allocation. As the communication channel is not subdivided into time or frequency slots, CDMA allows random access to an indefinite number of users. As additional users subscribe to the system, they can be given unique codes and then access the channel without concern for any other user. If the number of network subscribers is greater than the number of wavelength slots allotted for WDMA, the network requires a central controller and the user must tune his/her transmitter or receiver to an assigned band, which is a nontrivial task with optical devices. Additional characteristics of CDMA are security against unauthorized users, and antijamming capabilities. Unauthorized users attempting to eavesdrop will find the signal unintelligible unless the codes are known. CDMA systems are also resistant against jamming as a high power jamming signal would have to cover a significant portion of the CDMA bandwidth to create any noticeable degradation. For these and other reasons CDMA is considered the most viable multiple-access scheme for optical communications.

Due to the advantages cited above, CDMA has been the focus of much research in the last fifteen years, primarily in the radio frequency domain. A large portion of this work has concentrated on estimating the performance characteristics of CDMA systems in Gaussian noise employing a suboptimal matched filter symbol detector [47, 28, 12, 30, 24]. Additional studies were performed by Aazhang and Poor [1]
and Enge and Sarwate [9] on CDMA systems under impulsive channel degradations. Deviations from the matched filter approach have proven fruitful in optimal and suboptimal detector analyses performed by Verdú [42], Lupas and Verdú [20], and Varanasi and Aazhang [39, 40].

A less profuse yet equally diverse collection of work has been done in optical CDMA. In the area of coherent OCDMA, Foschini and Vannucci developed a random carrier CDMA system [10], while Tamura et. al. considered a CDMA system using Gold sequences as codes [29]. Current efforts in the area of direct detection optical CDMA have focused on performance evaluations of systems employing a conventional correlation detector (matched filter) or an optimal detector. Hui proposed the use of a code made up of a series of distinct optical pulses, to be demodulated via a correlation detector [16]. Salehi, Brackett, Chung, Wei, and Kumar developed the optical orthogonal codes (OOCs) [33, 34, 6, 7] which yield low interference crosscorrelations and therefore perform well under the correlation detector. Brady and Verdú analyzed the performance of this system in a Poisson channel using computationally efficient arbitrarily tight bounds in [3]. Contributions have also been made by Lam and Hussain who analyzed the correlation detector using moment space bounds and Gaussian approximations to obtain estimates of the probability of error for OCDMA systems using this detector [19]. Verdú considered the optimal demodulation in [43] in which a method of implementing the detector in a suboptimal yet more efficient manner was presented and appropriate performance bounds were derived. Promising
recent work by Mandayam and Aazhang provides a method of simulating an OCDMA system using importance sampling techniques [22].

1.2 Proposed System

To develop a high transfer rate OCDMA system, let us first examine the limitations imposed by the state of electronic and optical technology of today. The degrees of freedom available are the three spatial dimensions and the time-domain real and complex amplitudes, or equivalently, the frequency-domain amplitude and phase variables. Current electronic speeds are limited to at most 1 Gbps, which can be directly translated to a per-user data rate limit. This electronic speed also limits the minimum distinguishable time characteristic. The spatial dimensions suffer similar discernibility constraints. The frequency magnitude range is limited by the optical source bandwidth whose maximum is on the order of several tens of terahertz and the optical fiber characteristics, while the phase is unavailable due to the use of direct detection. Only direct-detection optical communications systems are considered, i.e., only systems relying solely on the photodetection of the light intensity. Coherent optical communication systems are deemed impractical at today’s state of technology due to phase noise of optical sources.

Since the physics of the system and the maturity of the technology limit discernibility for all variables, an ideal scheme allows some modulation of each of the degrees of freedom. The conventional approach to OCDMA is to design a system that re-
lies solely on time discernibility. Such systems time encode short optical pulses and attempt to detect the actual signal or correlate and detect the correlation signal. Practically, this limits the system to either using simplistic demodulation (correlation) or to using no faster than electronic speed encoding. We propose a system based on spectral amplitude encoding for CDMA differentiation and time encoding for successive bit distinction. To be more precise, because time and frequency are not independent variables, the scheme employs a spatial decomposition of the short-time frequency content of the optical beam for CDMA encoding. The time of integration is defined by the bit period so that only the per-user data rate is limited by electronic speeds. The degree of encoding, known as the spectral spreading factor, is limited by the bandwidth of the optical signal and the spatial resolution of the detector.

In Chapter 2, we describe two alternative OCDMA systems, one based on time encoding and one based on spectral amplitude encoding. The degradations considered in these two system models are restricted to the Poisson effects of the photodetection process, additive optical intensity noise, and multiple-access interference. The time encoded system uses a tapped-delay line to encode the user code sequence. The output of the encoder is a series of short laser pulses in preassigned time slots. The signal is detected by integrating the received signal over each time slot and sending this information to the symbol detector. The spectral encoded OCDMA system has the signature sequences of the users on-off encoded on the frequency bands of the optical beam. Two optical sources are considered: an ideal (coherent) mode-locked
laser and a more practical wideband incoherent thermal source. Although the laser is much less noisy than the incoherent source, laser mode locking, which is required to obtain a large bandwidth, is bulky and temperamental and thus limits its own usage. The majority of this study focuses on the coherent optical source system. A physical implementation of the spectral amplitude encoding and decoding and a corresponding mathematical model of each system are given, including the expected statistical variations of the optical source.

The idea of spectral amplitude encoding originated with Weiner, Heritage, and Salehi in a paper on short laser pulse coding and decoding [45]. They further investigated the idea of spectral phase encoding in [35] and Hajela and Salehi analyzed the resulting OCDMA system in [14]. Although this scheme is promising and shows good performance predictions, some severe limitations in its practicality are imposed by both the coherence requirement of phase encoding and by the need for ultrashort pulses. The coherence of the system can be shattered by chromatic dispersion of fibers, which can cause catastrophic phase shifts in such a system if the fibers are long. As the optical beam travels through an optical fiber, the effect of chromatic dispersion, the magnitude of which is a function of the frequency and the length of the fiber, is to delay each frequency a different amount. Also, in the coherent phase encoded system, symbol detection is limited to a simple correlation since no phase ambiguity can be resolved with a photodetector unless heterodyning is used. Spectral amplitude encoding, which requires no coherence, has previously only been mentioned
in the context of pulse shaping [44]. Choosing a spectral amplitude instead of a phase encoding method for CDMA reduces the effects of chromatic dispersion of fibers and enables incoherent detection of the spread signal so that efficient demodulation techniques can be employed.

The level of performance and the number of users that can be supported by an OCDMA system depends strongly on the type of symbol detection algorithm used, which is the subject of Chapter 3. Two types of detection schemes exist, namely single-user and multiuser detectors. Single-user detectors attempt to decipher one user's symbol only, while multiuser detectors decipher all or a subset of all users' symbols simultaneously. The only single-user detectors previously proposed for optical CDMA are the simple correlation detector and the modified correlation detector [2], and the only multiuser detector proposed is the optimal detector. Multiuser detectors require knowing all or a subset of the users' codes, and employ this knowledge to aid in the detection process, thus yielding in general a better performance than the single-user detectors.

The correlation detector has an all-optical implementation that allows full bandwidth utilization, yet it performs poorly in systems with a large number of users, when the codes cannot be optimized for low crosscorrelation. Its performance also degrades significantly when the interfering users' power levels are higher than the desired user's power level. The modified correlation detector is impractical for systems with more than two or three users as it ignores the parts of the received signal where
interference has occurred. This detector will not be discussed further. The optimal
detector suffers a computational complexity that increases exponentially with the
number of users, which forbids its implementation in practical systems. A need exists
for either a more powerful single-user detector or a new multiuser detector tailored to
respond to the cotransmission of many user signals of unequal received power levels,
as an alternative to the highly complex optimal detector.

We propose and compare the performance of an optimized single-user detector,
a local search multiuser detector, a hard-decision multistage detector, and a soft-decision multistage multiuser detector for systems with unequal received powers.
The single-user detector structure is optimized to the statistics of the interference
of other users, an idea first explored by Poor and Verdú for radio domain CDMA
in [27]. The local search detector performs a local optimization instead of a global
one, thus significantly diminishing the computational requirements compared to the
global optimum detector and yet producing a comparable performance. The multi-
stage detector computes a Gauss-Seidel algorithm for finding a local optimum. The
multistage detector has slightly worse performance than the local search detector, yet
benefits from a totally parallelizable implementation. The optical multistage detector
is conceptually similar to the multistage detector developed by Varanasi and Aazhang
[40, 39] for CDMA in Gaussian noise. The soft decision multistage multiuser detector
is a modification of the multistage detector to allow soft decisions to be passed from
stage to stage. The soft-decision version of this algorithm is shown to have better convergence properties and better performance than its hard-decision counterpart.

In Chapter 4, the performance analysis of the detectors described above is accomplished through the use of large deviation theory. Two signal encoding options are considered, one based on random sequences, and the other based on deterministic codes, such as the optical orthogonal codes introduced by Salehi in [33]. For the random sequence case, the error probability for large code lengths is approximated using a large deviation theory approximation similar to Sadowsky and Bahr's in [30] where the radio frequency CDMA problem is addressed. For the deterministic code case, the interdependence of the received statistics and the small number of observations require a large deviation asymptotic approximation using a theorem by Bucklew and Sadowsky in [5]. These performance approximations are compared for small problems to exact error probabilities calculated using a characteristic function method and are found to agree for all low error probability cases. Some approximations needed to characterize the detectors are also verified by comparing the error probabilities to simulation results. As predicted, the multiuser detectors outperformed the single-user detectors. In addition, the optimized single-user detector performance surpassed the correlation detector performance in all cases verified.

A figure of merit of detectors for multiuser systems is the asymptotic multiuser efficiency, introduced by Verdú for radio frequency CDMA systems in [43], and later redefined for optical systems by Brady in [2]. The asymptotic multiuser efficiency
of the OCDMA system using an arbitrary detector is found to be directly proportional to the large deviation theory rate function for the random code case. For the deterministic code case, the asymptotic efficiency can be upper and lower bounded using this rate function. Both the optimized single-user detector and the soft-decision multistage detector show unequal received power robustness. The local search detector and the hard-decision multistage detector using the correlation detector decisions as the initial guesses are less robust to unequal received power than the other two detectors.

Chapter 5 discusses the modifications required to accommodate an incoherent optical source into the framework of the OCDMA system described in the previous chapters. Detection algorithms tailored to the statistics of the received signal are described, followed by a discussion of the corresponding error probability analysis. In all cases, there is a small penalty in performance for using an incoherent source over a coherent one. Again, efficient detectors outperform the detector optimal in single-user transmission.

The conclusion of these analyses, given in Chapter 6, is that efficient symbol detection, i.e., the use of a detection algorithm tailored to the characteristics of the system at hand, increases the throughput and performance of the system by a significant amount. With the use of one of the efficient detectors proposed and spectral amplitude encoding, the OCDMA system surpasses the information transfer capability of any conventional multiuser system.
Chapter 2

System Description

Two methods of encoding the CDMA spreading sequence onto the information symbols are considered. The first, a time encoding scheme, is analogous to the direct sequence CDMA popular in the radio frequency domain. The signature sequence is expressed as a series of strategically placed short optical pulses. The second scheme is a novel idea of spectral amplitude encoding, in which the signature sequences are encoded onto the symbols by modulating the optical beam's spectral characteristics. An exact mathematical model of each system is given, including stochastic descriptions of the received signal and observation.

2.1 Time Domain Multiple-Access Encoding

The OCDMA system model in this study consists of $K$ users, labelled $k = 1, \ldots, K$, with $k = 1$ as the desired user. Each user employs a laser source for signal transmission, and a photodetector (PD) for signal reception, as shown in Figure 2.1. The transmitted field from each user is a sequence of laser pulses, transmitted at pre-assigned intervals. This sequence is determined by the user's assigned code and the symbol that user wishes to transmit. The symbol period is $T$, and the user one symbol
Figure 2.1  Optical code-division multiple-access system. The signal processing at the receiver depends on the type of symbol detection used. The symbol $\hat{b}_k^{(l)}$ represents the detector's decision on the $k$th user's symbol for frame position $l$. 
transmitted in time interval \([lT, (l + 1)T)\) is labelled \(b^{(l)}_k\), for \(l = \ldots, -1, 0, 1, \ldots\). The user bits are numbered so that bit \(l\) of user \(k\) has a positive delay \(\tau_k\) with respect to bit \(l\) of user one, and the users are numbered so that these delays are arranged in increasing order. The delays form a length \(K\) vector \(\tau\) with \(\tau_1 = 0\), since the delays are with respect to user one’s bit transition. The symbols are assumed binary, with \(b^{(l)}_k \in \{0, 1\}\), and they are also assumed statistically independent from all other symbols. Each user can transmit one of two predetermined sequences of optical pulses, represented by the code vectors \(A^{(0)}_k\) and \(A^{(1)}_k\) for user \(k\) symbols \(b^{(l)}_k = 0\) and \(1\), respectively. The code sequences are of length \(J\), with \(A^{(0)}_{k,j} \in \{0, 1\}\), where \(A^{(0)}_{k,j} = 1\) symbolizes an optical pulse in the \(j\)th chip time interval \([lT + (j - 1)T_c + \tau_k, lT + jT_c + \tau_k]\), \(j = 1, \ldots, J\), for \(T_c = T/J\), and \(A^{(0)}_{k,j} = 0\) symbolizes none.

Two types of user signature sequences are considered. In the random code scheme, each user sequence is assumed to be a random binary sequence with \(\Pr(A^{(0)}_{k,j} = 1) = \rho\) for \(b^{(l)}_k = 0\) and \(1\). The second sequencing option considered is the use of deterministic codes with weight \(W\) and unit maximum pairwise offset autocorrelation and crosscorrelation. One such code is the optical orthogonal code, which can be constructed as described in [6]. Once a set of codes of the desired length \(J\) which satisfies these conditions is found, each user is assigned one codeword for \(A^{(1)}_k\) and a vector of all zeroes for \(A^{(0)}_k\). This on-off coding is done to obtain a larger number of users since the number of allowed codes is usually very small. Let the user symbol
vector $\mathbf{b}^{(l)} = \{b_1^{(l)}, b_2^{(l)}, \ldots, b_K^{(l)}\}$, and define $\mathbf{A}^{(\mathbf{b}^{(l)})}$ to be the $J \times K$ code matrix composed of the vector $\mathbf{A}_k^{(b^{(l)})}$ as columns for $k = 1, \ldots, K$.

All photodetectors experience a residual current called the dark current when no light is incident on the photosensitive area. This phenomenon is successfully modeled as a constant additive light intensity. The dark current is added to other background radiation and is then represented as an additive intensity labelled $\lambda_d$. The received field is a noncoherent sum of the intensities of the signals transmitted by the $K$ users and this dark current. The average intensity per pulse for user $k$ is $\lambda_k$ given in units of photoelectrons per second. The photodetector transforms a series of incident photons into a series of photoelectrons, with certain efficiency unimportant to this study.

Two extremes in the width of the received optical pulse are investigated. In the first, it is assumed that the optical pulses are narrow enough that they occur entirely within one chip interval irrespective of the delays $\tau_k$, i.e., the value of $T_c$ is much larger than the extent of the optical pulse $\phi(t)$. The chip transitions can thus be assumed common to all users, with each optical pulse always contained entirely within one chip interval. The system appears chip synchronous at the detector, with $\tau_k = 0 \mod T_c$, even though each user is not required to synchronize chip transitions with other users. With these assumptions, the intensity process modeling the received signal can be equivalently modeled as having a rectangular pulse shape of duration $T_c$ instead of the actual pulse shape $\phi(t)$, within each chip containing a pulse. The constant amplitude of the rectangular pulse is determined by the power received from
each user. Regardless of the optical pulse intensity shape, the energy received for user 
k in each chip is either 0 or \( \lambda_k T_c \). This will be referred to as the chip-synchronous case. In the other extreme, the received pulses are assumed to be square pulses of width exactly \( T_c \) and height \( \lambda_k \) for user \( k \). Without loss of generality, the remainder of the study assumes the latter description, since the former can be obtained by simply setting \( \tau_k = j_k T_c \) for some integer \( j_k \) for each \( k \).

Since the main concern of this study is the degradation due to multiple-access interference, the degradations due to additive Gaussian noise, component nonlinearities, and finite bandwidth are not included in the system model. The major degradation caused by the optical channel is the Poisson character of the received process due to photon detection. The sample-function density for a Poisson process depends on the arrival times of photons as well as the total photon count. In practice, the exact arrival times cannot be acquired since the photon arrivals occur faster than any electronic equipment can measure. The only statistics available at the receiver are a series of photon counts accumulated over predetermined intervals. Due to the limited integration speeds available, the receiver statistics for use by the symbol detector are limited to the integrated signal over each users’ chip intervals \( [lT + (j - 1)T_c + \tau_k, lT + jT_c + \tau_k] \) for \( k = 1, \ldots, K, j = 1, \ldots, J \), and \( l = \ldots, -1, 0, 1, \ldots \).

Since the sample path probability distribution of the received signal is that of a conditional Poisson process, the distribution of each resulting integral is that of a Poisson count. The photon count over user \( k \)'s chip \( j \) for bit position \( l \) is labelled \( N_{k,j}^{(l)} \).
as illustrated in Figure 2.2. The intensity of the optical field determines the intensity of the conditional Poisson process. Given $\Lambda_k = \lambda_k T_c$, the intensity integrated over the chip period synchronized to user $k$ corresponding to the observation $N_{k,j}^{(l)}$ is given as

$$r_{k,j}^{(l)} = \Lambda_k A_{k,j}^{(l)} + I_{k,j}^{(l)}, \quad k = 1, \ldots, K, \quad j = 1, \ldots, J,$$

where $I_{k,j}^{(l)}$ is the multiple-access interference to chip $j$ of user $k$ plus the dark current.

For instance, letting $\tau_k = j_k T_c + \tilde{\tau}_k$, $I_{1,j}^{(l)}$ can be written as

$$I_{1,j}^{(l)} = \tilde{\tau}_k \lambda_k A_{k,(j-j_k-1)\text{mod} J}^{(l)} + (1 - \tilde{\tau}_k) \lambda_k A_{k,(j-j_k)\text{mod} J}^{(l)} + \Lambda_d; \quad j = 1, \ldots, J,$$

where $\Lambda_d = \lambda_d T_c$. Each count $N_{k,j}^{(l)}$ is conditionally Poisson distributed with mean value $r_{k,j}^{(l)}$ given $B^{(l-1)}, B^{(l)}, B^{(l+1)}$ and $\tilde{\tau}$.

### 2.2 Frequency Domain Multiple-Access Encoding

The spectral amplitude encoded system consists of $K$ users, again labelled $k = 1, \ldots, K$, each transmitting a continuous stream of binary symbols $b_{k,l}^{(l)} \in \{0, 1\}$ for $l = \ldots, -1, 0, 1, \ldots$. Every user has an optical source transmitting an effective bandwidth $W$ centered at frequency $f_o$ and a symbol rate $T$ common to all users.

Two wideband source types are considered, one a coherent light source, namely a mode-locked laser [45], and one an incoherent light source, an example of which is a superfluorescent fiber source [46]. Both of these sources can achieve bandwidths on the order of 10 terahertz. Without loss of generality, user one is considered the desired user throughout the study, and perfect synchronization to this user is assumed.
Figure 2.2  Labels of chip photon counts for 2 users and $J = 3$ for a chip asynchronous time encoded system.
The user one symbol occupying exactly the time interval \( [lT, (l + 1)T] \) is numbered \( l \), and the corresponding user \( k \) symbol \( b_k^{(l)} \) experiences a positive delay of \( \tau_k \in (0, T] \) relative to user one as in the time encoded system. Each user is assigned one or two predetermined length \( J \) binary sequences, depending on whether random codes or well designed codes are used. The sequence for bit \( b_k^{(l)} \) is denoted by \( A_k^{(b_k^{(l)})} \) with every element \( A_k^{(b_k^{(l)})} \in \{0, 1\} \) for \( j = 1, \ldots, J \).

The physical implementation of the encoding and decoding process, illustrated in Figure 2.3, relies on simple optics. The optical beam is passed through a grating and focused onto a mask. The grating angularly disperses the spectral content of the beam, so that the signal incident on the mask has spatially decomposed frequency content. The masks are length \( J \) sequences of open or closed slits of width \( W_c = W/J \), the pattern of which determines the user's codes. In the case of random codes and two sequences assigned per user, the user symbol drives a switch determining which of two masks the signal will go through. If only one sequence is assigned to a user, the sequence \( A_k^{(0)} = 0 \) for all \( j \) and the user symbol simply on-off modulates the optical beam. The mask for user \( k \) symbol \( b_k^{(l)} = i \) has open slits in frequency locations \( \{ f_0 - W/2 + (j - 1)W_c, f_0 - W/2 + jW_c \} \) for \( A_k^{(i)} = 1 \), and is closed elsewhere. The result is an spread spectrum encoding scheme that relies on on-off modulation of spectral bands. The encoded spectrum is recombined spatially by passing through another grating. At the signal destination, the decoding of the signal is accomplished
Figure 2.3 Optical spectral amplitude encoder and decoder.
by passing the incident optical signal through another grating, and detecting the frequency bands individually by a linear array of photodiodes.

The primary noise contributions considered in this study are the statistical uncertainty in the spectral content of the optical source, the point process character of the direct detection process, and the multiple-access interference. At the signal destination, the energy in each spectral band is accumulated over the bit period in one of the PD array elements. The direct detection process can be defined as a doubly stochastic Poisson point process, whose count statistics dependent on the statistics of the received integrated intensity. Let the complex analytic representation of the transmitted optical field for user $k$ be represented by $E_k(t)$, and further assume that this function is zero outside of the interval $t \in [0, T)$. The grating-mask-grating encoder acts as a bank of narrow bandpass filters $h_j(t)$ for $j = 1, \ldots, J$, each assumed flat within its passband $f \in \{f_0 - W/2 + (j - 1)W_c, f_0 - W/2 + jW_c\}$, and each having ideal frequency cutoff. The output of the encoder can thus be written as

$$
\tilde{E}_k(t) = \sum_{j=1}^J A_{k,j}^{(1)} E_k(t) \ast h_j(t),
$$

where the symbol $\ast$ represents the continuous time convolution operator. The energy content of user $k$'s signal in frequency bin $\{f_0 - W/2 + (j - 1)W_c, f_0 - W/2 + jW_c\}$ is labelled $E_{k,j}(t) = E_k(t) \ast h_j(t)$. As the optical beam travels through an optical fiber, the effect of chromatic dispersion is to delay each frequency a different amount. This delay is a function of the frequency and the length of the fiber. This study assumes that the delay due to chromatic dispersion is insignificant compared to the symbol
time $T$. We also assume that the frequency dependent delay caused by the grating pair at the encoder and the grating at the decoder is short compared to $T$. If the decoder is properly aligned, the signal of user $k$ at the input to the photodetector array element $j$ synchronized to user 1 becomes

$$
\tilde{E}_{k,j}(t) = A_{k,j}^{(l(-1))} E_{k,j}(t - \tau_k + T) + A_{k,j}^{(l(0))} E_{k,j}(t - \tau_k); 0 \leq t \leq T.
$$

(2.4)

The photodetection process can be described as a conditional Poisson process given the light intensity incident upon the photosensitive area. The contribution to the intensity over the bit interval due to dark current for each frequency chip is labelled $\Lambda_d$. Using the bit $l = 0$ of user one as an example, the total intensity received by chip detector $j$ over the bit period $[0, T]$ is

$$
r_{1,j}^{(0)} = A_{1,j}^{(l(0))} \int_0^T |E_{k,j}(t)|^2 dt + \sum_{k=2}^K \left[ A_{k,j}^{(l(-1))} \int_{T-\tau_k}^T |E_{k,j}(t)|^2 dt + A_{k,j}^{(l(0))} \int_0^{T-\tau_k} |E_{k,j}(t)|^2 dt \right] + \Lambda_d.
$$

(2.5)

To simplify notation, let the energy in each frequency band be the same and denote $\Lambda_k(t_1, t_2) = \int_{t_1}^{t_2} |E_{k,j}(t)|^2 dt$, independent of $j$, removing the arguments when $t_1 = 0$ and $t_2 = T$. The intensity contribution onto chip $j$ integrated over the bit period $[lT + \tau_k, (l+1)T + \tau_k)$, i.e., for the observation based on the detector synchronized to the user $k$ symbol period, can now be written as

$$
r_{k,j}^{(0)} = \Lambda_k A_{k,j}^{(l(0))} + I_{k,j}^{(0)}; k = 1, \ldots, K, j = 1, \ldots, J,
$$

(2.6)
where $I_{1,j}^{(0)}$ is again the multiple-access interference to user $k$ in chip $j$. For bit $l = 0$ of user $k = 1$, this interference is

$$I_{1,j}^{(0)} = \sum_{k=2}^{K} \left[ A_{k,j}^{(l=-1)} \Lambda_k(T - \tau_k, T) + A_{k,j}^{(l=0)} \Lambda_k(0, T - \tau_k) \right] + \Lambda_d. \quad (2.7)$$

Each user transmits using its own optical source and thus no coherence between user optical beams is expected. Observations are gathered at each user symbol transition. Figure 2.4 illustrates the notation used. It is assumed that the $r_{k,j}^{(l)}$ are independent for different values of $j$. The photoelectron count $N_{k,j}^{(l)}$ is conditionally Poisson distributed with stochastic parameter $r_{k,j}^{(l)}$.

### 2.2.1 Stochastic Model: Coherent source

The uncertainty in the spectral band energies of a mode-locked laser pulse has not been thoroughly studied since these sources are generally employed for their time domain property of generating well defined ultrashort pulses. On the other hand, the field fluctuations of an incoherent source such as the superfluorescent fiber source can be successfully modeled as that of a thermal light source, i.e., as a complex Gaussian stochastic process. These two models yield very different results.

A stochastic model of the mode-locked laser can be given as [32]

$$E_k(t) = \alpha(t) e^{i2\pi f_d(t)}, \quad (2.8)$$

where $\alpha(t) = \sum_{m=0}^{M} \alpha_m \exp\left(\frac{im-M/2}{T}i2\pi t\right)$ is the complex envelope of the laser output and the $\alpha_m$’s are the stochastic complex amplitudes of the $M$ laser modes. If the
Figure 2.4  Labels of chip photon counts for 2 users and arbitrary $J$ for the spectral amplitude encoded system.
\( \alpha_m \) are assumed independent and identically distributed, which is not likely since they are phase locked, each chip that contains a very large number of modes (on the order of 1000) will have an approximately Gaussian field fluctuation. On the other hand, if the amplitudes \( \alpha_m \) are constant, the intensity in each frequency chip is not random and the integrated intensity \( \Lambda_k \) is deterministic. This deterministic model of the mode-locked laser will be used in this study. The \( k \)'th user's contribution to the photoelectron counts in chip \( j \) is Poisson distributed with its parameter a deterministic function of \( \Delta^{(l-1)}_{k,j} \), \( \Delta^{(l)}_{k,j} \), \( \Delta^{(l+1)}_{k,j} \) and \( \tau_k \). The overall observation \( N^{(l)}_{k,j} \) is therefore also Poisson distributed with parameter \( r^{(l)}_{k,j} \).

2.2.2 Stochastic Model: Incoherent source

The stochastic properties of incoherent or thermal light are much better understood [31, 13]. Let us ignore the delays and the encoding in (2.6) for now and consider the single-user system. Since the field \( E_{k,j}(t) \) is a Gaussian process, the Karhunen-Loève expansion yields

\[
E_{k,j}(t) = \sum_{m=1}^{\infty} \alpha_m \phi_m(t),
\]

where the \( \alpha_m \)'s are independent complex Gaussian random variables and the functions \( \phi_m(t) \) are orthonormal over \((0, T]\), i.e., \( \int_0^T \phi_m^*(t)\phi_l(t)dt = \delta_{m,l} \). Ignoring dark current which is insignificant when light is incident on the detector, the integrated intensity then becomes

\[
r^{(l)}_{k,j} = \sum_{m=1}^{\infty} |\alpha_m|^2.
\]
If the random variable $\alpha_m$ is circularly symmetric in the complex plane, the term $|\alpha_m|^2$ is exponentially distributed with parameter $\sigma_m^2$. For a rectangular spectrum of $E_{k,j}(t)$ of one-sided bandwidth equal to $W/2J$, the first $\mathcal{M} = \pi W_c T + 1$ values of $\sigma_m^2$ are approximately equal and the rest are insignificant [38]. The resulting $r_{k,j}^{(i)}$ is a sum of $\mathcal{M}$ identically exponentially distributed random variables, and is therefore distributed chi squared with $2\mathcal{M}$ degrees of freedom $\chi^2_{2\mathcal{M}}$ (i.e., gamma distributed).

If the incident radiation is unpolarized, twice as many modes exist and $r_{k,j}^{(i)}$ becomes $\chi^2_{4\mathcal{M}}$.

To obtain the probability mass function for the photoelectron counts $N_{k,j}^{(i)}$ for each frequency chip, the statistics on the doubly stochastic Poisson point process with parameter $r_{k,j}^{(i)}$ must be found. This can be done by simply taking the Poisson transform of the probability density of the parameter $r_{k,j}^{(i)}$, yielding in this case a negative binomial probability mass function for $N_{k,j}^{(i)}$

$$
\Pr(N_{k,j}^{(i)} = n) = \binom{n + \mathcal{M} - 1}{n} \left( 1 + \frac{r_{k,j}^{(i)}}{\mathcal{M}} \right)^{\mathcal{M}} \left( 1 + \frac{\mathcal{M}}{r_{k,j}^{(i)}} \right)^{-n}, \quad (2.11)
$$

where $r_{k,j}^{(i)} = E[r_{k,j}^{(i)}]$. When multiple users are present, each with a different received power level, the total statistic for $N_{k,j}^{(i)}$ is that of a sum of negative binomial random variables of different mean values.

Two assumptions are made to simplify the model. The first assumption is that in frequency chips with some incident light the dark current is negligible. If the received field is a sum of thermal light and a coherent term such as dark current, the
overall photoelectron count is Laguerre distributed [11], which is analytically much
less tractable than the negative binomial. Since the average number of photoelectrons
generated by the dark current process is generally less than one per symbol interval,
the assumption is valid. The second assumption is that the probability distribution of
photoelectrons when no light is incident on the photodetector is also negative binomial
with mean $\Lambda_d$. The negative binomial distribution approaches the Poisson whenever
the mean number of photoelectrons is small compared to the number of modes $M$,
which is evidently the case here [13, pages 481–486].

It must be noted that this model is valid for any Gaussian distributed incident
light. In the event that the mode-locked laser behaves unstable enough to generate
independent modes and the resulting field is Gaussian, its statistical description would
be identical to that of the thermal source given above, although possibly of a much
narrower bandwidth. This type of laser light is referred to as pseudothermal light for
this reason [13, pages 145-151].
Chapter 3

Symbol Detection

The performance of the OCDMA system described in Chapter 2 depends on the signal processing or detection that is performed at the signal destination. The symbol detector is an algorithm whose function is to decide which of the binary symbols each user has transmitted from the signal observation. This chapter focuses on presenting several single-user and multiuser detection algorithms for the CDMA system using a coherent optical source, i.e., Poisson distributed observations. The incoherent source case is addressed in Chapter 5.

If the information destinations are collocated, the symbol detector is likely to know exactly what each user's signature sequences are, as well as the users' expected received power levels and relative delays. In this scenario, a multiuser detector that deciphers the symbols of all the users simultaneously can be utilized. If the signal destinations for each users are not collocated and if there is a large number of system subscribers, it is unlikely that each destination has all the information required to detect all the users. In this case, each destination would employ a single-user detector, i.e., a detection algorithm specifically designed to decide on the symbol of one particular user.
3.1 Single-User Detectors

A single-user detector has no a priori knowledge of interfering user signature sequences. We describe two single-user detectors for the CDMA system. The conventional detector is the correlation detector, which ignores the presence of other users. As this detector performs poorly in unequal received power cases, an alternative detector is proposed. The optimized single-user detector assumes concurrent transmission of other signals and optimizes according to the statistics of this interference.

3.1.1 Correlation Detector

The conventional detector employed by OCDMA systems is the matched filter detector, or correlation detector [19, 33, 6, 7, 3, 22]. This detector decides on the desired user's symbol based on statistics corresponding to the correlation of the received signal intensity and a copy of the two expected sequences, $A^{(0)}_i$ and $A^{(1)}_i$, if the desired user is user 1. For the time encoded system, this detector can be physically implemented using two tapped-delay lines as correlators, photodetecting these correlation values, and then comparing their difference to a threshold. For the spectral amplitude encoded system, the incident signal can be passed through a mask identical to the encoding mask and then another grating, detecting the resulting signal and then comparing it to a threshold. The correlation detector for user one is equivalent to

$$\sum_{j=1}^{J} N_{1,j} (A^{(1)}_{i,j} - A^{(0)}_{i,j}) \geq \gamma_r$$ (3.1)
where $\gamma_c$ is the threshold determined from the a priori probabilities of the user symbols and the sequence weights. The detector chooses $b_1^{(l)} = 1$ if the sum in (3.1) is greater than $\gamma_c$, chooses $b_1^{(l)} = 0$ if it is less that $\gamma_c$, and makes a random choice of $b_1^{(l)} = 1$ with probability $\varrho$ if the threshold is achieved exactly, where $\varrho$ is based on the a priori probabilities and the code weights.

The correlation detector became popular in OCDMA systems due to its theoretical optimality in systems with one user and due to the potential for all-optical processing. The optical processing allows the electronic speed to limit only the user bit rate, without limiting the spread spectrum bandwidth. The disadvantage of a correlation detector in a multiuser system is that it ignores the existence of interfering users, so it does not take full advantage of the known structure of the system, yielding a poor performance for networks catering to large numbers of users. This fact suggests a need for examining the characteristics of an optimized detector, which is the topic of the next section.

### 3.1.2 Optimized Single-user Detector

Although the detector has no exact knowledge of users other than the desired user, we assume that a statistical description of the interference is available to the detector and use this information to design an optimized single-user detector. This idea was first developed by Poor and Verdú for radio-frequency CDMA in [27]. Assuming the interference in each symbol period is independent of the interference in other bits,
each symbol decision for user one will depend entirely on the photon counts for that symbol period \((T, (1+1)T)\). Considering the received process as a doubly stochastic Poisson process, the likelihood ratio test for bit \(l = 0\) of user \(k = 1\) results in the detection algorithm [36]

\[
\frac{E[I_1^{(0)}|p_{\Delta_1^{(0)}, I_2^{(0)}}, p_{\Delta_1^{(0)}, I_2^{(0)}}, \Delta_1^{(0)}](n|1,L)]}{E[I_1^{(0)}|p_{\Delta_1^{(0)}, I_2^{(0)}}, p_{\Delta_1^{(0)}, I_2^{(0)}}, \Delta_1^{(0)}](n|0,L)]} \geq _0 \gamma_s, \tag{3.2}
\]

where \(N_1^{(0)}\) is the length \(J\) vector of photon counts \(N_{1,i,j}\) and the vector \(I_1^{(0)}\) is the interference seen by user 1. The threshold \(\gamma_s\) is optimally set to the ratio of the a priori probabilities of the user 1 symbols. This detector defines the optimal detector for independent chip interferences with probability density \(f_{I_1^{(0)}}(I)\), in the sense that it yields the lowest error probability for these statistics.

Since we assume that the chip interference values are independent and since each \(N_{1,i,j}^{(0)}\) is conditionally Poisson given \(r_{k,j}^{(0)}\), the conditional probability mass function \(p_{N_{1,i,j}^{(0)}}[p_{w_{i,j}}^{(0)}](n|\lambda, I)\) is easily calculated yielding a detector of the form

\[
\sum_{j=1}^{J} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left\{ \frac{E[I_1^{(0)}|[I_{2,j}^{(0)} + \Lambda_1]I_{2,j}^{(0)} e^{-(I_{2,j}^{(0)} + \Lambda_1)}}{E[I_1^{(0)}|[I_{2,j}^{(0)}]I_{2,j}^{(0)} e^{-(I_{2,j}^{(0)}}}] \right\} \geq _{e} \gamma_s. \tag{3.3}
\]

Denoting the Poisson transform of \(f(x)\) at \(n\) as \(P_n[f(x)] = \int_{0}^{\infty} e^{-x} n^x f(x) dx\), the optimized single-user detector becomes

\[
\sum_{j=1}^{J} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left\{ \frac{P_{N_{1,j}^{(0)}}[f_{I_2^{(0)}}(I - \Lambda_1)]}{P_{N_{1,j}^{(0)}}[f_{I_2^{(0)}}(I)]} \right\} \geq _{e} \gamma_s. \tag{3.4}
\]

If the probability density of the interferences is multimodal or if the interferences are discrete valued, this likelihood function is likely to be nonmonotonic, as illustrated
Figure 3.1 Likelihood ratio defining the optimized single-user detection algorithm for 5 interferers of equal received user powers. The near-far factor is the ratio of an interfering user power to the desired user power.
in Figure 3.1. In this case, some care must be exercised in choosing this density to match the actual interference. If the codewords for each interfering user are assumed to be random binary sequences, then in each chip the conditional probability mass function of the interference given the delays is

\[ f_{I_{i,j}^{(o)}}(I) = \bigotimes_{k=2}^{K} \rho^2 \delta_{I_{i,k} \Lambda_k} + (1 - \rho)^2 \delta_{I_{i,0}} + \rho(1 - \rho)[\delta_{I_{i,k}(\tau_k)} + \delta_{I_{i,k}(\tau_k - \tau_k)}], \]

for the time encoded system, and

\[ f_{I_{i,j}^{(o)}}^{(f)}(I) = \bigotimes_{k=2}^{K} \rho^2 \delta_{I_{i,k} \Lambda_k} + (1 - \rho)^2 \delta_{I_{i,0}} + \rho(1 - \rho)[\delta_{I_{i,k}(\tau_k)} + \delta_{I_{i,k}(\tau_k - \tau_k)}], \]

for the frequency encoded system, where \( \bigotimes \) represents convolution and \( \delta \) is the Kronecker delta. Assuming independent uniformly distributed delays \( \tau_k \), the total probability density of the interference for both the time and frequency encoded systems is

\[ f_{I_{i,j}^{(o)}}(I) = \bigotimes_{k=2}^{K} \left( \rho^2 \delta_{I_{i,k} \Lambda_k} + (1 - \rho)^2 \delta_{I_{i,0}} + \frac{2\rho(1 - \rho)}{\Lambda_k} I_{[0, \Lambda_k]}(I) \right), \]

where \( I_{[\cdot]}(\cdot) \) is the standard indicator function. On the other hand, if the system is chip-synchronous and interfering codes are random binary sequences of identical energy, \( f_{I_{i,j}^{(o)}}(I) \) is a binomial density. A Gaussian density can be used if there are a large number of independent interference contributions, according to the central limit theorem; since the tail of the density is not the region of interest, the central limit theorem gives an accurate estimate of the density of the interference even for a finite number of users, as low as \( K = 21 \) all transmitting in one chip, according to Figure 3.2. If a Gaussian assumption is not desired, the interfering signal can also be
Figure 3.2  Probability density of the multiple-access interference caused by 10 and 20 users of equal received power.
polled at intermittent times throughout transmission to obtain a histogram estimate of the expected interference probability density.

Unfortunately, the physical implementation or even a computer calculation of the Poisson transform of the function \( f_{l_{1,j}}(I) \) for any but the simplest case is nontrivial. Mandayam and Aazhang have developed a cascade implementation with which the Poisson transform can be calculated for a small number of simultaneous users, usually less than ten [21]. If the number of users is large and the density of the interference is assumed to be a Gaussian with mean \( \mu \) and variance \( \sigma^2 \), the detector becomes

\[
\sum_{j=1}^{J} (A_{l_{1,j}}^{(1)} - A_{l_{1,j}}^{(0)}) \ln \left\{ \frac{E[(X + \Lambda_1)^n|N_{l_{1,j}}^{(0)} = n]}{E[X^n|N_{l_{1,j}}^{(0)} = n]} \right\} \geq \gamma''_s \leq \gamma''_s, \tag{3.8}
\]

where \( X \) is Gaussian with mean \( \mu + \sigma^2 \) and variance \( \sigma^2 \). In complex cases when the density cannot be assumed Gaussian, the Poisson transform can be derived off-line for some values of \( N_{l_{1,j}}^{(0)} \), with a real time calculation relying on interpolated values.

### 3.2 Multiuser Detectors

When all or a subset of all user codes are known, a multiuser detector can be employed. The goal of the multiuser detector is to decipher all user symbols simultaneously, minimizing the error probability of all. Two natural criteria for defining a desirable detector are the minimum error probability criterion and the maximum likelihood criterion, named after the optimal detection algorithms presented by Verdú in [43]. The minimum error probability detector attempts to minimize the error probability

\[
\text{Minimize } P_e \text{ subject to } E[I]\]
of each user independently, while the maximum likelihood detector maximizes the multiuser likelihood function.

This section is divided into two parts. In the first part, we describe the minimum error probability optimal detector and propose an alternate detector, the multistage detector, which has computational complexity much lower than the optimal algorithm. In the second part, we describe the optimal maximum likelihood detector and then consider several suboptimal algorithms for the maximization of the likelihood function that yield lower complexity than the exponential complexity of the optimal solution.

### 3.2.1 Minimum Error Probability Detection

If the main concern of the detector is the error probability of a particular user, say user 1, the most logical approach is to develop an algorithm which minimizes this probability.

**Optimal Detector**

The minimum probability of error detector obeys the maximum likelihood ratio test of one independent user. Suppose the transmission frame is of length $M$. Considering the incident intensity as a random variable, depending on $b_k^{(l)}$ for $l = -M/2 + 1, \ldots, M/2$ and $k = 1, \ldots, K$, the received signal can be described as a doubly stochastic Poisson process for each possible desired user symbol, $b_1^{(k)} = 0$ and $b_1^{(k)} = 1$ for
bit position $l = 0$. The log likelihood function for user 1, $L(b_1^{(0)}): \{0, 1\} \rightarrow \mathbb{R}$, can be described in two ways: by the method of conditioning and by the method of self exciting point processes [36]. Conditioning the point process on the intensity leads to the optimal detector algorithm described by Verdú in [43]. Let $r^{(b_i^{(0)})}(t)$ be the intensity process given $b_i^{(0)}$, i.e., the random process describing the light intensity incident on the photodetector. Describing the signal as a self-excitation point process leads to an optimum detector of the form

$$
\ln L(b_1^{(0)}) = - \int_{(-M/2+1)T}^{MT/2+\tau_K} (\bar{r}(t) - \bar{r}(0)(t)) \, dt + \int_{(-M/2+1)T}^{MT/2+\tau_K} \ln \left( \frac{\bar{r}(t)}{\bar{r}(0)(t)} \right) \, dN_t \geq \gamma_0,
$$

(3.9)

where $\bar{r}(t)$ is defined as

$$
\bar{r}(t) = E[r(t)|N_\nu, -MT/2 + 1 \leq \nu < t] ; i \in \{0, 1\},
$$

and where $N_t$ is the Poisson point process with intensity function $r^{(b_i^{(0)})}(t)$. The actual optical pulse or spectral shape is needed to characterize the intensity process, since the expected value above is a causal estimate. The value of the threshold $\gamma_0$ is the logarithm of the ratio of the a priori probabilities of user one sending $b_1^{(0)} = 0$ and $b_1^{(0)} = 1$. For the symbol asynchronous case the likelihood ratio test depends on the photon arrivals for all symbols in the frame, since the symbols overlap and thus they all affect $\bar{r}(t)$.

The optimal detector has been shown to have complexity at best exponential in $K$, which makes it impractical to use in systems with more than a few users. This
algorithm also requires knowledge of the arrival times that are not generally available. A new detector for optical multiple-access systems that provides a performance comparable to the optimal detector while maintaining linear complexity in the number of users is proposed next.

**Multistage Detector**

The major difficulty in implementing the optimal detector is calculating the conditional expected value of the intensity process. In this section, we propose a detector based on the optimal detector using an estimate of the interference to approximate the received intensity process. This value is then used in lieu of the expected value of the intensity within the optimal detector structure described by (3.9) to obtain a second stage estimate. This process is iterated $S$ times to form an $S$-stage detector. The result of this algorithm is a multiuser detector, i.e., a detector deciphering all user symbols simultaneously.

The proposed detector is similar to the concept introduced by Varanasi and Aazhang in [39, 40] for radio frequency domain CDMA systems. In those studies, an estimate of the interference is obtained and is directly subtracted from the received waveform that is otherwise corrupted only by Gaussian noise. Hypothetically, if the estimate of the interference is exact, the system suffers no multiuser degradation. Such is not the case in the optical system where the nonlinearity of the Poisson process forbids cancellation of the effects of the multiple-access interference.
In the optimal detection method, an estimate of the intensity is calculated using the received photon counts and the arrival times. The multistage detection algorithm is based on using an estimate \( \hat{r}^{(i)}(t) \) of the expected value of \( r^{(i)}(t) \) in (3.9). Since the arrival times are not recorded as an available observation, the estimate is assumed constant over the chip interval of each user. The decisions from the correlation detector presented in Section 3.1.1 form the first stage estimates of the user symbols, denoted as \( \hat{b}^{(i)}_k(1) \). For the detector of user \( k = 1 \) for bit position \( l = 0 \), label the actual interference in chip \( j \) as \( I^{(0)}_{1,j} \) and its first estimate as \( \hat{I}^{(0)}_{1,j}(1) \). With this notation, for the time encoded system, \( \hat{r}^{(i)}(t) \) can be written as

\[
\hat{r}^{(i)}(t) = \frac{1}{T_c} \sum_{j=1}^{J} \left( \Lambda_1 A^{(i)}_{1,j} \Pi_T(t - (j-1)T_c) + \hat{I}^{(0)}_{1,j}(1) \Pi_T(t - (j-1)T_c) \right),
\]

with an interference estimate of the form

\[
\hat{I}^{(0)}_{j}(1) = \sum_{k=2}^{K} \tilde{\tau}_k \lambda_k A_k^{(i)(\frac{j-j_k-1}{j})} + (1 - \tilde{\tau}_k) \Lambda_k A_k^{(i)(\frac{j-j_k-1}{j})} + \Lambda_d; j = 1, \ldots, J.
\]

For the spectral amplitude encoded system, \( \hat{r}^{(i)}(t) \) is given by

\[
\hat{r}^{(i)}(t) = \frac{1}{T} \sum_{j=1}^{J} \left( \Lambda_1 A^{(i)}_{1,j} \Pi_T(t) + \hat{I}^{(0)}_{j}(1) \Pi_T(t) \right),
\]

with interference

\[
\hat{I}^{(0)}_{j}(1) = \sum_{k=2}^{K} \left[ A_k^{(i)(\frac{j-j_k-1}{j})} \Lambda_k(T - \tau_k, T) + A_k^{(0)(1)} \Lambda_k(0, T - \tau_k) \right] + \Lambda_d.
\]

The function \( \Pi_T(t) \) is a unit amplitude square pulse of duration \( T \).
The second stage of the multistage detector obtained by substituting \( \hat{r}^{(i)}(t) \) for \( \hat{r}^{(i)}(t) \) in (3.9) is

\[
\sum_{j=1}^{J} -\Lambda_{1}(A_{1,j}^{(1)} - A_{1,j}^{(0)}) + N_{1,j}^{(0)} \ln \left( \frac{\Lambda_{1}A_{1,j}^{(1)} + \tilde{I}_{1,j}^{(0)(1)}}{\Lambda_{1}A_{1,j}^{(0)} + \tilde{I}_{1,j}^{(0)(1)}} \right) \geq \gamma_{ms_{1}}.
\]

(3.14)

The threshold for user 1 is labelled \( \gamma_{ms_{1}} \). Note that the estimated interference \( \tilde{I}_{1}^{(0)} \) does not depend on the desired user symbol \( b_{1}^{(0)} \) so that the structure for the multistage detector need only include frame position \( l = 0 \).

Further algebraic manipulation results in a more revealing expression for the second stage of the multistage detection algorithm.

\[
\sum_{j=1}^{J} N_{1,j}^{(0)}(A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left( \frac{\Lambda_{1} + \tilde{I}_{1,j}^{(0)(1)}}{\tilde{I}_{1,j}^{(0)(1)}} \right) \geq \gamma_{ms_{1}}',
\]

(3.15)

where \( \gamma_{ms_{1}}' \) is the new threshold. The second stage decision on the user one symbol obtained by this detector is labelled \( \hat{b}_{1}^{(0)}(2) \). This expression of the detector is that of a correlation filter on the photon counts, weighted by a function of the interference predicted in each chip. The form of this detection algorithm illustrates how chips with high predicted interference levels are weighted low compared to chips with low predicted interference levels, as expected. Note that, except for the threshold, the conventional correlation detector is the same as that of the this detector using \( \tilde{I}_{k,j}^{(0)} = \Lambda_{k} \).

A detector analogous to the one above can now be used to obtain a better estimate \( \hat{b}_{k}^{(0)}(2) \) of the symbols transmitted by each of the \( K \) users. These estimates can in turn be used to derive new estimates of the interference. Repeating this process, at
stage $s$ the detector for user $k$ performs

$$\sum_{j=1}^{J} N_{k,j}^{(0)} (A_{k,j}^{(1)} - A_{k,j}^{(0)}) \ln \left( \frac{I_{k,j}^{(0)}(s-1)}{I_{k,j}^{(0)}(s-1)} \right) \begin{cases} \geq & \gamma'_{ms_k}, \\ < & s = 2, 3, \ldots, S, \end{cases} \quad (3.16)$$

where the subscript $k$ on $I_{k,j}^{(0)}(.)$ refers to the fact that the interference perceived by each user is different. The same subscript $k$ on the threshold $\gamma'_{ms_k}$ signifies that the threshold is dependent on the desired user, since the users may have different code weights. The threshold is independent of $s$. Note that each user's receiver employs a different vector $N_k^{(0)}$ due to the symbol delays $\tau_k$. A multiuser multistage detector can be implemented by employing the result of detector (3.16) for all users to form a new estimate of the interference $I_{k,j}^{(0)}(s)$ using formulas like (3.11) and (3.13) for iteration stage $(s + 1)$. The detector can thus attempt to reject the multiple-access interference and yield a lower probability of error for all users. The two-stage detector is illustrated in Figure 3.3.

The computational complexity of the multistage detector for each user is linear in the number of users. Each stage requires $K$ detectors detecting the current frame position user symbols. The previous frame position symbols are also needed, yet they have already been generated for the previous frame position calculation. Thus, for $S$ stages, there are a total of $SK$ comparisons required to estimate the symbol sent by one user. If the receivers are collocated, all users can use the same previous stage detector estimates and the total $K$-user system complexity is $O(SK)$. If the users are not collocated, the total complexity of the system is $O(SK^2)$. Note that there is
Figure 3.3 Two-stage detector diagram for user 1. If all user receivers are collocated, the estimates $\hat{b}_k^{(l)}(1)$ can be shared so duplication is avoided. The process can be iterated for additional stages by replacing all $\hat{b}_k^{(l)}(s - 1)$ by $\hat{b}_k^{(l)}(s)$ and repeating the detection algorithm.
a delay of \((S - 1)T\) in making a final decision on a symbol, since each stage requires an estimate of the next bit for other users.

3.2.2 Maximum Likelihood Detection

A true multiuser detector decides on the bits transmitted by each user simultaneously, which can be done by maximizing the multiuser likelihood function \(L(B) : \{0, 1\}^{MK} \rightarrow \mathbb{R}\), where \(B\) is the \(M \times K\) matrix of user symbols formed by columns \(B^{(l)}\) for \(l = -M/2 + 1, \ldots, M/2\). The optimal detector does just this, with a computational complexity that grows exponentially with the number of users times the length of the frame. Two approaches to the search for a simpler algorithm are (i) to find a "local" optimizer for \(L(B)\), and (ii) to solve an approximate problem. The first requires a definition of a "neighborhood" on the vertices of a \(K\) dimensional cube so that locality can be defined. The second usually involves either a linearization process or an extension/restriction of the original domain. After a description of the optimal algorithm, two suboptimal detectors are proposed illustrating these approaches.

Optimal Maximum Likelihood Detector

With a transmission frame length of \(M\), this optimal detector is equivalent to finding the maximum likelihood over all \(MK\) symbols in the frame when asynchronous transmission is considered. The solution, which is written as \(B^*\), is found by solving

\[
B^* = \arg \max_{B \in \{0, 1\}^{MK}} L(B),
\]

(3.17)
where the likelihood function (or the log-likelihood function) of the user symbols $\mathbf{B}$ is denoted $L(\mathbf{B})$. This type of function is referred to alternatively as a pseudo-Boolean function, i.e., a real valued function with Boolean valued variables, or a nonlinear 0-1 integer function. The NP-hardness of this problem is well known (see, for instance, [25]). Many algorithms have been developed to solve this type of optimization problem, yet, due to the NP-completeness, none is guaranteed to find a solution in polynomial time. To minimize computational complexity, a sequential detection algorithm can be used. A suboptimal soft decision sequential algorithm based on dynamic programming was analyzed by Verdú in [43], which yielded a computational complexity of the order $O(2^K)$ per bit decision, instead of the $O(2^{MK})$ complexity of the optimal algorithm. This complexity is in general still forbidding for problems with a large number of users.

For the system at hand, the likelihood of the observations given the bit matrix $\mathbf{B}$ can be easily written using the following notation. There are $MJ$ observations per user, $J$ for each bit in the frame. Since the accumulation times overlap, the observations for the $K$ users are not statistically independent. Each observation can be subdivided into $K$ subchip counts defined by the user delays such that in each subchip the intensity is constant. These subchip photon counts can be obtained from the observations by simple subtraction of counts in the chip intervals for each user. It is clear that there are less than $(M + 1)JK$ of these photon counts, which are now statistically independent and form a piecewise homogeneous Poisson process.
Labeling these counts as \( N(i), i = 1, \ldots, (M + 1)JK \), the likelihood function has the form

\[
L(B) = \sum_{i=1}^{(M+1)JK} -A(i) + N(i) \ln[A(i)],
\]

(3.18)

where \( A(i) = E[N(i)|B] \) is the constant intensity of light during that subchip interval times the duration of the interval given the symbol matrix \( B \).

**Local Search Detector**

The local search detector is a method of finding a local maximum of the original likelihood function where the neighborhood of a point is defined in the following way.

The \( Q \)-neighborhood of a point \( B^* \) is the set \( S_Q(B^*) = \{B : d(B^*, B)_H \leq Q\} \), where \( d(\cdot, \cdot)_H \) is the standard Hamming distance. As illustrated in Figure 3.4, for \( Q = 1 \) this defines the points linked to \( B^* \) by an edge of the \( K \)-cube, while for \( Q = 2 \) it defines the points laying on the same face of the \( K \)-cube as \( B^* \). A \( Q \)-order detector finds a solution to

\[
L(B^*) \geq L(B) \quad \forall B \in S_Q(B^*).
\]

(3.19)

In this study we develop the \( Q = 1 \) detector.

A local maximum can be found by solving \( MK \) simultaneous inequalities. From the likelihood function in (3.18), \( B \) is a local maximum if it satisfies

\[
(2b_k^{(l)} - 1) \sum_{j=1}^{J}(A_k^{(1)} - A_k^{(0)}) \left( A_k - \sum_{i=1}^{K} N_{k,i}^{(0)}(i) \ln \left( \frac{\hat{A}_k(i) + \hat{I}_{k,j}^{(l)}(i)}{\hat{I}_{k,j}^{(l)}(i)} \right) \right) \leq 0;
\]

\[
\forall k = 1, \ldots, K \text{ and } l = -M/2 + 1, \ldots, M/2,
\]

(3.20)
Figure 3.4  Three dimensional visualization of a $Q = 1$ and $Q = 2$ neighborhood. In each case, all neighbors to the point $B^*$ are encircled.
where $\tilde{N}_{k,j}^{(l)}(i)$ for $i = 1, \ldots, K$ are the $K$ subchip counts forming the observation $N_{k,j}^{(l)}$, $\tilde{A}_k(i)$ is the fraction of $A_k$ in the subchip $i$, and $\tilde{I}_{k,j}^{(l)}(i)$ is the interference as seen by user $k$ in subchip $i$ of chip $j$.

In this study, for simplicity, only the original observations over each chip are employed for each user bit. Although this will not yield an exact local maximum, the computational expense is much less. For $Q = 1$, this is equivalent to finding a $B$ such that

$$
(2^{b_k^{(l)}} - 1) \sum_{j=1}^J (A_{k,j}^{(1)} - A_{k,j}^{(0)}) \left( \Lambda_k - \bar{N}_{k,j}^{(l)} \ln \left( \frac{\Lambda_k + \bar{I}_{k,j}^{(l)}}{\bar{I}_{k,j}^{(l)}} \right) \right) \leq 0; \\
\forall k = 1, \ldots, K \text{ and } l = -M/2 + 1, \ldots, M/2, \quad (3.21)
$$

which is equivalent to performing $MK$ simultaneous single user likelihood ratio tests, one for each of the user symbols.

Unfortunately, for an arbitrary pseudo-Boolean function the problem of finding one local optimum has not been shown to be in the class P [17], i.e., no polynomial time algorithm for finding a solution is known. In fact, the local search problem for some NP-hard problems is also NP-hard [26]. Nevertheless, empirical studies have shown that solutions are generally obtainable in low-order polynomial time [25, 17, 26].

One algorithm for finding a local optimum is the multistage detector, which was introduced in the previous section as a suboptimal algorithm for a minimum error probability detector. The multistage detector of $S$ stages is effectively an iterative Gauss-Seidel algorithm for solving the multiuser maximum likelihood problem that
makes successive estimates of the symbols transmitted by using previous estimates of
the interfering users. Thus at each stage $s$, an estimate $\hat{b}^{(0)}_k(s)$ is made using $\hat{b}^{(0)}_{k'}(s-1)$
for $k' = 1, \ldots, K$ and $l' = -M/2 + 1, \ldots, M/2$. At each step of iteration, each of $K$
simultaneous optimizations are performed, each maximizing the function $L(B)$ with
respect to one $b^{(0)}_k$ with the others fixed at the previous iteration estimate. That is,
at stage $(s + 1)$, the algorithm solves $MK$ equations of the form

$$
\hat{b}^{(0)}_k(s + 1) = \arg \max_{b^{(0)}_k \in \{0, 1\}} L(\hat{B}(s) \setminus \hat{b}^{(0)}_k(s), b^{(0)}_k); \ k = 1, \ldots, K; \ l = -M/2 + 1, \ldots, M/2,
$$

(3.22)

where the notation $A \setminus b$ denotes all elements in the matrix $A$ except element $b$. The
algorithm results in $MK$ detectors performing

$$
\sum_{j=1}^{J} N_{k,j}^{(0)} (A_{k,j}^{(1)} - A_{k,j}^{(0)}) \ln \left(1 + \frac{\Lambda_k}{I_k^{(0)}(s)}\right) \geq \gamma_{\max},
$$

(3.23)

to choose $\hat{b}^{(0)}_k(s + 1) = 0$ or 1, where $\hat{I}_{k,j}^{(0)}(s)$ is the $s$ stage estimate of the actual
interference $I_{k,j}^{(0)}$ according to (3.11) and (3.13). This algorithm has no general con-
vergence properties, yet if a stationary point is reached after many stages, it is a local
maximum. Nonetheless, the algorithm could oscillate between points.

In [15], an iterative method analogous to a steepest descent method in continuous
variable optimization is proposed that guarantees to yield a solution to (3.21). The
algorithm begins with an initial guess and successively replaces only the $b^{(0)}_k$ with the
largest violation to the inequality in (3.21). An algorithm that performs invariably
better than the steepest descent is the circular first-replacement algorithm, which up-
dates its position at the first comparison which "descends" at all, and considers bits
in a circular fashion (round-robin). This algorithm performs the same computation as the multistage detector, yet updates its estimate of the interference at every calculation instead of waiting until all s stage estimates are available. Empirical studies [26, 17] have shown this method finds a local optimum usually in \((MK)^2\) steps or less. The steepest descent can only switch each user symbols once on the average in that time!

Since the multistage detector attempts to compute a local optimum, the performance of the local search detector compares to its performance as the number of stages goes to infinity. The main issue thus becomes complexity. The complexity per frame of a multistage detector with \(S\) stages is \(O(SMK)\), while the local search detector expects to find a local minimum in less than \(O[(MK)^2]\). In spite of this increase in complexity, simulations show that the local search detector performance is superior to the multistage detectors even after only \(SMK\) iterations. The hidden cost of implementing a local search algorithm is the need for constant updating of the interference estimate, which restricts any parallel implementation. When the transmission frame \(M\) is long, a parallel implementation is essential as it reduces the delay from \(M\) bits to \((S - 1)\) bits.

Like any local optimization technique with a sparsely populated neighborhood, the detector based on the \(Q = 1\) neighborhood is extremely sensitive to the initial guess. Although finding all solutions to the local search problem is NP-hard, a set of local optimizers can be obtained by choosing several initial guesses. The one yielding
the largest likelihood function can then be selected. Alternatively, a \( Q \)-neighborhood detector requiring \((MK)^Q\) inequalities can be chosen, which yields a problem much less sensitive to initial conditions. In this way, the quality of the detector can be chosen to fit the computational complexity allowed. This study focuses on the performance of the detector with \( Q = 1 \).

### 3.2.3 Soft-decision Multistage Detector

A heuristic method of solving the problem in (3.19) is to solve the maximum likelihood estimation problem over the solid \( MK \)-cube instead of the decision problem over just the endpoints, and to use this estimate to make a final decision. Consider a modification to the domain of the likelihood function to include the entire \((MK)\)-dimensional interval \([0, 1]^{MK}\). A higher maximum value of the likelihood function can be obtained by considering this extension since it includes the original domain.

Define the real extension \( \hat{L}(\hat{B}) : [0, 1]^{MK} \mapsto \mathbb{R} \) as the log-likelihood function \( L(B) \) with \( \hat{A}_{k,j}^{(l)} = \hat{b}_k^{(l)} \hat{A}_{k,j}^{(1)} + (1 - \hat{b}_k^{(l)}) \hat{A}_{k,j}^{(0)} \) instead of \( A_{k,j}^{(l)} \). This problem has no closed form solution, thus a multistage approach is taken. As before, at each stage \( s + 1 \) the problem of maximizing \( \hat{L} \) is solved by simultaneously solving

\[
\hat{b}_k^{(l)} (s + 1) = \arg \max_{\hat{b}_k^{(l)} \in [0, 1]} \hat{L}(\hat{B}(s) \backslash \hat{b}_k^{(l)} (s), \hat{b}_k^{(l)}); \quad k = 1, \ldots, K; \quad l = -M/2 + 1, \ldots, M/2.
\]

(3.24)

In practice, a finite number of values of \( \hat{b}_k^{(l)} \) would be considered for maximization.

The final stage \( S \) must make a hard decision, i.e., \( \hat{b}_k (S) \in \{0, 1\} \).
This algorithm can be thought of as the standard multistage algorithm with a quantized soft-decisions being passed from one stage to the next. Although the maximum of the likelihood function achieved is at least as high for this problem as for the problem with Boolean argument, the error probability may not be improved. On the other hand, the soft-decision multistage detector has much better convergence properties, since the function \( \tilde{L}(\tilde{B}) \) is convex upward, i.e., its Hessian is negative definite everywhere in \([0, 1]^{MK}\). This is easily shown for the symbol synchronous case.

The second partial derivatives of \( \tilde{L}(\tilde{B}) \) are of the form

\[
\frac{\partial^2 \tilde{L}}{\partial b_k \partial b_{\ell}}(\tilde{B}) = \sum_{j=1}^{J} \frac{N_j (A_{k,j}^{(1)} - A_{k,j}^{(0)}) (A_{\ell,j}^{(1)} - A_{\ell,j}^{(0)}) \Lambda_k \Lambda_{\ell}}{\left[ \sum_{k'=1}^{K} (\tilde{b}_{k'} A_{k',j}^{(0)} + (1 - \tilde{b}_{k'}) A_{k',j}^{(0)}) \Lambda_{k'} + \Lambda_d \right]^2}
\]  

(3.25)

The Hessian \( H \) of \( \tilde{L} \) at \( \tilde{B} \) is a function of \( h \) defined by

\[
H \tilde{L}(\tilde{B})(h) = \frac{1}{2} \sum_{k, \ell=1}^{K} \frac{\partial^2 \tilde{L}}{\partial b_k \partial b_{\ell}}(\tilde{B}) h_k h_{\ell}
\]

(3.26)

In this case, the Hessian simplifies to

\[
\sum_{j=1}^{J} \frac{N_j \left[ \sum_{k'=1}^{K} (\tilde{b}_{k'} A_{k',j}^{(0)} - A_{k,j}^{(0)}) \Lambda h_k \right]^2}{\left[ \sum_{k'=1}^{K} (\tilde{b}_{k'} A_{k',j}^{(0)} + (1 - \tilde{b}_{k'}) A_{k',j}^{(0)}) \Lambda_{k'} + \Lambda_d \right]^2}
\]

(3.27)

which is necessarily negative for all \( h \) and all \( \tilde{B} \).

The negative definiteness of the Hessian of \( \tilde{L} \) guarantees that, until a hard decision is made, the algorithm converges to the global maximum of the likelihood function and that this solution is unique. Note that this is the maximum likelihood for a different problem than the original binary signaling problem, and therefore does not guarantee a better performance measured in error probability.
Chapter 4

Performance Analysis

Some knowledge of the reliability of a communication system is of utmost importance in determining if a system is useful for a particular application. For this reason, some prediction in the reliability of a system is desired. In this chapter, a theoretical analysis of the performance of the coherent laser system described in Chapters 2 and 3 is given. Two performance criteria are presented. The most traditional measure of the reliability of a communication system is the probability of bit error. Another useful performance criterion for multiple user systems is the asymptotic multiuser efficiency, which is a measure of the effectiveness of a detector to combat the degradation caused by high power interfering users.

4.1 Signature Sequence Selection

The performance of the OCDMA system is highly dependent of the types of user signature sequences chosen. These sequences are the codewords assigned to the user for CDMA encoding. Determining which method of analysis is suitable also depends on the types of sequences used. Two alternatives are presented, each appropriate to a different communication network scenario.
4.1.1 Random Codes

The signature sequence selection for a communication network which is accessible to large number of users is limited. Since each user needs one or two codes, depending on whether on-off modulation is used, a very large number of codes is needed. Generally, low crosscorrelation between user codes yields low interference. Low autocorrelation for each code facilitates synchronization. Unfortunately, when many users codes are needed, good autocorrelation and pairwise crosscorrelation properties cannot be enforced without increasing the code length $J$ to unreasonably high numbers. In these case, the common practice is to assign codes from a pseudorandom number generator, taking care to assign unique codes to each user. The performance of this type of system is successfully modeled as having a completely random code assignment. The analysis presented here adopts this assumption, allowing each user two codes to represent letters in the binary alphabet. The main drawback of this assumption is each user does not necessarily have unique codes. A limit on the performance is therefore imposed by the nonzero probability of the two codes for one user being the same and by the probability of distinct users being assigned the same code.

4.1.2 Deterministic Codes

In a scenario where the network is composed of a small number of users, at least compared to the spreading factor, the user code sequences can be designed to minimize the
interference to other users. The optical time and spectral amplitude encoded CDMA system are equivalent to an asynchronous time and frequency-division multiple-access systems, respectively, with intentional overlap and multiuser detection. The quality of the overall system depends strongly on the quality of codes chosen for this reason. In this section we define the characteristics of the optimal codes based on the total number of subscribers for the time encoded system and the spectral amplitude encoded system.

**Time encoded system**

The overlaps of user sequences from asynchronous transmissions are defined by the pairwise linear crosscorrelation. This function should be as small as possible for all delay values so that low levels of multiple-access interference are encountered. For synchronization purposes, the offset autocorrelation function should also be small. A set of codes named the optical orthogonal codes that satisfies fixed crosscorrelation and autocorrelation constraints was designed by Chung, Salehi and Wei in [6]. This study only considers codes with both of these constraints being unity.

A codebook is needed with sufficient codewords to supply each system subscriber with one or two codes. Let the total number of subscribers be $K^{(max)}$. Let $\mathcal{A}$ be the codebook, i.e., the set of codewords $\mathcal{A}_k$. Let us only consider such codes which satisfy $|\mathcal{A}| \geq K^{(max)}$. Salehi determined in [33] a bound on the size of the code based on
unit autocorrelation and crosscorrelation constraints:

\[ |\mathcal{A}| \leq \left\lfloor \frac{J - 1}{W(W - 1)} \right\rfloor. \]  \hspace{1cm} (4.1)

Since this number is very small for practical values of \(J\) and \(W\), only one code \(A_k^{(1)}\) is assigned to each user, letting \(A_k^{(0)} = 0\), for \(k = 1, \ldots, K\).

**Spectral amplitude encoded system**

If the optics of the detector are well aligned, in the spectral amplitude encoded system a spatial shift of the signal from one user to another cannot occur. There is thus no need to consider offset autocorrelation and crosscorrelation functions. The only desirable feature is that the unshifted pairwise crosscorrelation between two codes should be as low as possible. This is equivalent to maximizing the minimum Hamming distance \(d_H(A_{k_1}, A_{k_2})\) for all \(1 \leq k_1, k_2 \leq |\mathcal{A}|\). Trivially, if two codewords are assigned to a user, they should have crosscorrelation of zero so that they are easily differentiated.

Much research has been dedicated to finding the biggest code with fixed length \(J\) and fixed minimum distance, or, equivalently, maximum autocorrelation of \(\theta\). A result by Johnson in [18] states that with these constraints

\[ |\mathcal{A}| \leq \left\lfloor \frac{J}{W} \left\lfloor \frac{J - 1}{W - 1} \right\rfloor \cdots \left\lfloor \frac{J - \theta}{W - \theta} \right\rfloor \cdots \right\rfloor, \]  \hspace{1cm} (4.2)

where \(W\) is the weight of the codewords. One simple method for generating the codes desired is to use all cyclic shifts of the previously defined optical orthogonal codes. Yet
by designing a code without this restriction, a bigger codebook size can be obtained. This result is seen by looking at the table of maximum sizes of codes actually found in [18] and noting that these sizes are not divisible by the codeword length \( J \), as a cyclic code would necessarily be. Although the number of codes available for the spectral encoded system is much larger than that for the time encoded system, the restriction of maximum pairwise crosscorrelation of one and one code per user is still used for simplicity.

4.2 Probability of Bit Error Approximations

The multiple-access interference is a significant cause of bit decision errors for the OCDMA system. To quantify this degradation, exact expressions, estimates, or bounds for the error probability are needed. In some instances, the exact error probability is impossible or extremely time consuming to calculate, thus approximations or bounds are desired. Asymptotic approximations based on large deviation theory are employed in this study to alleviate the need for extensive computation to analyze detector performance. This approximation is then compared for small problems to an exact calculation using a characteristic function method, and further verified via simulation.
4.2.1 Random Codes

The detectors described above have the form

$$Z = \sum_{j=1}^{J} X_j I_{X_j > \theta} \geq 0$$  \hspace{1cm} (4.3)$$

for some random vector $X$ with the decision threshold incorporated in its mean. The random code matrix $A^{(E^{(l)})}$ has independent and identically distributed entries, and let us further assume that the components of $X$ are also independent and identically distributed (this is not the case with chip-asynchronous transmissions in the time encoded system only if pulses occupy consecutive chips, which is a rare occurrence). The desired quantity is the average error probability $\bar{P}_e$. Assuming an equally likely hypothesis case, the problem is symmetric in $b^{(0)}_1$, since the distributions of $A^{(E^{(l)})}$ for $b^{(0)}_1 = 1$ and 0 are the same and the a priori probabilities are also the same. Therefore, without loss of generality, we can assume $b^{(0)}_1 = 0$ which yields

$$\bar{P}_e = \Pr \left( Z > 0 | b^{(0)}_1 = 0 \right) + \frac{1}{2} \Pr \left( Z = 0 \right).$$  \hspace{1cm} (4.4)$$

Using the characteristic function method [12], an exact expression for the error probability of the OCDMA system can be obtained, yet a numerically time consuming inverse FFT is required. An approximation using large deviation theory is therefore sought.
Large Deviation Theory Approximation

Large deviation theory attempts to predict the rate of exponential decay of the probability of an unlikely event as the number of samples of a converging sum increases (see [4]). The central limit theorem accurately predicts the behavior of sums of random variables near the expected value, yet it is not accurate at predicting tail behavior unless the number of samples is nearly infinite. Since the objective here is to find the error probability which is dictated by the behavior near the tail of the probability density of the converging sum of random variables, large deviation theory is employed.

Cramér's Theorem states that, under some limiting conditions, for independent and identically distributed random variables $X_j$, the following holds

$$
\lim_{J \to \infty} \frac{1}{J} \ln[\Pr(\frac{1}{J} Z \geq a)] = -\mathcal{I}(a),
$$

(4.5)

where $\mathcal{I}(a) = \sup_s (as - \ln(M_{X_1}(s)))$ is called the rate function ($X_1$ is chosen arbitrarily). The expression in (4.5) can be interpreted as saying that the rate of exponential decay of the probability of outliers is approximately $\mathcal{I}(a)$. Another way of expressing this result for large $J$ is to say that

$$
\Pr(\frac{1}{J} Z \geq a) \approx K(J)e^{-J\mathcal{I}(a)},
$$

(4.6)
where $K(J)$ is changing slowly in $J$ compared to the exponential. In [4], this function
is approximated for large $J$ to be\footnote{The approximation obtained based on large deviation theory is equivalent to what van Trees calls an approximation based on the Chernoff bound in [38, pages 116–133]. The Chernoff bound is in itself a large deviation theory result.}

$$K(J) \approx \frac{1}{\sqrt{2\pi Js^*}}.$$  \hspace{1cm} (4.7)

where $s^*$ is the value of $s$ achieving the maximum in the rate function\footnote{If $M_{X_1}(s)$ is differentiable everywhere then the supremum in the calculation of $I(\cdot)$ is achieved for some value $s = s^*$.} and where

$$\sigma^2 = \frac{M''_{X_1}(s^*)}{M_{X_1}(s^*)} - a^2.$$  This approximation is obtained by using the central limit theorem on the sum of random variables with a density shifted to a zero mean, which makes it valid even for finite $J$. For a zero threshold, equation (4.6) becomes

$$\Pr(Z \geq 0) \approx \frac{\exp[-J\mu_{X_1}(s^*)]}{\sqrt{2\pi J\mu''_{X_1}(s^*)s^*}},$$  \hspace{1cm} (4.8)

where $\mu_{X_1}(s) = \ln M_{X_1}(s)$ and $s^* = \mu_{X_1}^{-1}(0)$.

An estimate of the total error probability as given in (4.4) can now be obtained by using (4.8) as the probability of the decision statistic exceeding or equaling the threshold. One half of the probability of the threshold being achieved exactly, which usually occurs only in the event that the codeword for $\mathcal{A}^{(0)}_1$ is the same as the codeword for $\mathcal{A}^{(1)}_1$, must be subtracted from this approximation. This event has probability $(\rho^2 + (1 - \rho)^2)^J$.

The moment generating function of $X_1$ is calculated by taking the expected value over the user codes and delays. This expected value is difficult to compute directly in
light of the conditional density function obtained from (3.5) and (3.6). On the other hand, best and worst case scenarios can be found. Consider the chip delay $\tau_k$ which minimizes and maximizes the average rate of exponential decay of the probability of error in (4.6). The worst case delay between two users, in the sense described above, is $\tau_k = T/2$ for the spectral amplitude encoded system and any $\tau_k$ with chip asynchronism $\tilde{\tau}_k = T_c/2$ for the time encoded system. This is shown by considering one chip value and equating the derivative of $\mu_{X_1}(s*)$ to zero. Similarly, the value of $\tau_k$ which maximizes the average rate of decay of the error probability is $\tilde{\tau}_k = j_k T_c$ for the time encoded system and $\tau_k = 0$ for the spectral amplitude encoded system. This is illustrated by Figure 4.1 for several values of the error in estimating $I_{1,1}^{(0)}$. For more than two users, the worst case for the desired user is computationally equivalent to having $2(K - 1)$ chip-synchronous (time encoding) or symbol synchronous (spectral amplitude encoding) interferers, pairs with power levels $\Lambda_k/2$. It is interesting to note that a chip-synchronous scenario ($\tilde{\tau}_k = 0$) yielded a worst case performance for the radio domain maximum likelihood detector [41] and multistage detector [39] also. In the Poisson case, it is well known that the optimal signal set consists of placing all the signal energy in a very short pulse. This idea carries directly to the multiuser system noting that the short pulse case is equivalent to the chip synchronous case as discussed in Chapter 2. Since the matrix entries of $A^{(E^{(i)})}$ are identically distributed, the expectation can easily be computed for the chip-synchronous, time encoded system and for the symbol synchronous, spectral amplitude encoded system.
Figure 4.1 Exponent to the moment generating function for one chip for several values of the error in estimating the interference. The delay is normalized to lie within [0, 1].
in the best and worst case. A development of the corresponding moment generating function for all detectors is given in the Appendix.

**Characteristic Function Method**

An exact expression for the probability of bit error based on the characteristic function is obtained for the OCDMA system. Although this method is computationally intensive, it is used to verify the closeness of the large deviation theory asymptotic approximation derived above for finite $J$. We have a test statistic of the form $Z = \sum_{j=1}^{J} X_j$ for some random vector $\mathbf{X}$. The vector $\mathbf{X}$ is a function of the actual interference level $I^{(0)}_{1,j}$ which is in itself random, since $N^{(0)}_{1,j}$ is Poisson with intensity $A_{1,j}^{(0)} + I^{(0)}_{1,j}$.

The calculation of the probability of error in general requires a $J$-fold convolution of the probability densities of the $X_j$'s averaged over all possible interference values if the calculation is done directly. To diminish the computational complexity, the characteristic function can be used, as was first suggested for radio frequency CDMA systems by Geraniotis and Pursley in [12] and later employed by Aazhang and Poor in [1]. The application of this technique to expression (4.4) yields

$$
P_e = \sum_{Z>0} \mathcal{F}^{-1}(\Phi_{Z|b_1=i}(\omega)) + \frac{1}{2} \sum_{Z=0} \mathcal{F}^{-1}(\Phi_Z(\omega)), \tag{4.9}$$

where $\mathcal{F}^{-1}(\cdot)$ is the inverse Fourier transform operator and $\Phi_{Z|b_1=i}(\omega)$ is the characteristic function of $Z$ for $\omega \in [0, 2\pi)$, given the desired user symbol $b_1 = i$.

The main advantage of using the characteristic function method in the case of Poisson based statistics is the ability to derive a closed form expression for the char-
acteristic function. If the observations are independent and identically distributed, as was assumed for the large deviation theory approximation, the total characteristic function is given by

$$\Phi_Z(\omega) = \Phi_{X_j}^{I}(\omega).$$  \hspace{1cm} (4.10)

An expected value over the interference values is needed to calculate the characteristic function of $X_j$, $\Phi_{X_j}$. Once this function is calculated, the primary computational savings arise from the applicability of the fast Fourier transform (FFT) to calculate $\mathcal{F}^{-1}(\Phi_Z)$ to obtain $\bar{P}_e$. The characteristic function of $Z$ for each detector is derived analogously to the moment generating function, which is given in the Appendix.

One note can be made about the use of the characteristic function method for the correlation detector. In cases when $K$ is large, the characteristic function method provides a great computational savings over the direct convolution method. On the other hand, in cases when $2K$ is much smaller than $J$, there is a possibility of performing this convolution directly using a maximum $(2K + 1)$'th order convolution since the random variable $X_j$ takes at most $(2K + 1)$ different distributions, depending on the values for $I_{i,j}^{(0)}$. In this case it would be advantageous to compute the probability distribution directly through convolution.

Results

This section presents the numerical results of the performance analysis of the OCDMA system described in the preceding section. The probabilities of error of all detectors
proposed are shown by comparing their performance to the probability of error of
the optimal detector and the conventional correlation detector. The error probability
of the optimal detector is bounded below by the error probability of the "known-
interference" detector as given in the Appendix and bounded above by the error
probability of all other detectors. The "known-interference" detector is the genie-
aided maximum likelihood detector for one user given all the symbols sent by the
other users. The performance of this detector will be referred to as the lower bound
for the remainder of the study.

Plots showing the agreement between the large deviation theory approximation,
the characteristic function method and simulation results are given in Figures 4.2
and 4.3. for the symbol synchronous case. Monte Carlo simulation [22] results
of the system using an optimized single-user detector, the hard-decision and soft-
decision 2-stage detectors, and the local search detector are presented to confirm
the accuracy of the analytic approximations used in calculating the characteristic
functions and the moment generating functions. These approximations are found
to only slightly overestimate the simulated error probabilities in all cases, allowing
for fluctuations inherent to simulation. Figure 4.3 shows that for \(J > 100\), the
asymptotic approximation based on large deviation theory is very close to the actual
value computed using a characteristic function method.

The OCDMA system using the hard-decision 2-stage detector can support many
users if the codes are sufficiently long, as seen by Figure 4.4. The correlation detector,
Figure 4.2 Performance of the optimized single-user detector, the hard-decision and soft-decision 2-stage detectors, and the local search detector as $J$ increases showing the characteristic function method and simulation. The system parameters are $\Lambda_k = 5$, $K = 15$, $\rho = .05$, and $\Lambda_d = 0.1$. 
Figure 4.3 Performance of the correlation detector, the optimized single-user detector, the hard-decision 2-stage detector, and the lower bound as $J$ increases showing the large deviation theory approximation compared to the characteristic function method. The system parameters are $\Lambda_k = 5$, $K = 5$, $\rho = .05$, and $\Lambda_d = 0.1$. 
Figure 4.4 Performance of the 2-stage detector, the correlation detector, and the lower bound as $J$ increases and for the total number of users $K = 1, 5, \text{and } 15$. The system parameters are $\Lambda_k = 5, \rho = .05, \text{and } \Lambda_d = 0.1.$
the 2-stage detector, and the lower bound are equivalent for $K = 1$, and the error probability decreases exponentially with the code length in this case, as is expected for a Poisson system since the total received energy increases also. Note in particular that for a small number of users ($K = 5$) the error probability for the 2-stage detector is almost the same as the lower bound, which is due to the low error of the initial guess from the correlation detector. The same is true for systems with a large number of users ($K = 15$) but at much longer code lengths.

The advantage of using a well designed detector such as the optimized single-user detector, the local search detector, the multistage detector, or the soft-decision multistage detector, is also apparent from the probability of error curves in Figure 4.5. This plot depicts the best case bit error probability (corresponding to $\tau_k = 0$) as the number of users increases for all detectors. The worst case error, i.e., the error for $\tilde{\tau}_k = T_c/2$ for the time encoded system or equivalently $\tau_k = T/2$ for the spectral amplitude encoded system, is also plotted for the lower bound. The optimized single-user detector using a binomial distribution for the interference performs several orders of magnitude better than the correlation detector, yet it is well outperformed by all multiuser detectors. The soft-decision multistage detector performs surprisingly better than the hard-decision one for the same number of stages. The local search detector performs best of all, which is expected as the local optimum is actually reached. The local search detector performance is very close to the lower bound,
Figure 4.5  Probability of error as a function of the number of users for all detectors and the lower bound for the random code case. All plots are given for the best case and the lower bound is also given for the worst case labelled "tau=1/2". System parameters are $\Lambda_k = 5$, $\Lambda_d = 0.1$, $\rho = 0.05$, and $J = 500$. 


especially for low $K$, as is expected since the correlation detector is used to generate the first stage estimate of the interference levels.

The local search performance is approximated by that of the multistage detector as the number of stages increases. Since both the complexity and the performance of the multistage detector increase with the number of stages, it is necessary to determine how many stages are truly necessary to reach an asymptote, or at least to obtain a significant improvement over the correlation detector. Two or three stages is sufficient in most cases with $K \leq 15$, as evident in Figure 4.6. The conjecture that this asymptote upper bounds the error probability of the local search detector is further validated by the simulation points which are shown to lie exactly on or just below the line. The improvement beyond two stages is minimal for a small number of users, yet becomes slightly higher for $K = 15$, when the interference is the significant contributor to errors and the interference rejection performed by the additional stages aids the detection process. Unless the number of users is much larger than this, the improvement granted by the additional stage does not warrant the additional hardware required. A gap does remain in some cases between the performance of the multistage detector and that of the local search detector.

Consider the challenge of designing a practical OCDMA system. As can be seen from the number of parameters describing each graph, many system variables must be set to define a system and its performance. Figures 4.7, 4.8, and 4.9 provide a method for designing a system with a minimal error probability using a the 2-stage detector.
Figure 4.6 Performance of the multistage detector as the number of stages $S$ increases, for a total number of users $K = 5$ and 15 and for sequence lengths $J = 150$ and 300. The horizontal lines beneath each graph is the lower bound for that case. Simulation of the local search detector error probabilities are shown on the right of each plot. Additional system parameters are $A_k = 5$, $A_d = 0.1$, and $\rho = 0.05$. 
An initial error probability specification requirement leads to a plot such as the one shown in Figure 4.7, which displays the performance of the system for a constant \( \bar{P}_e = 10^{-4} \). The code length required to support \( K \) users at a fixed performance level increases exponentially as \( K \) increases, for both the multistage detector and the lower bound. For the correlation detector, for low \( K \)'s the increase appears almost linear but for many users it is exponential and at a much higher rate than that of the other detectors. A design choice becomes a trade between code length and the number of simultaneous users.

Two other important design parameters are the pulse energy, related to \( \Lambda_1 \), and the average number of pulses per symbol, \( \rho J \). The increase in performance as the laser pulse intensity \( \Lambda_1 \) increases is illustrated in Figure 4.8. As \( \Lambda_1 \) increases, the error probability does not decrease exponentially as expected, even for the one-user case, because the error is dominated by the effects of having random codes. Figure 4.9 depicts the performance of the three detectors as the number of pulses per bit is varied. As the number of pulses increases to nearly half the code length, the interference levels increase also so the average error probability increases. As the number of pulses decreases to zero, the likelihood of error due to low power and random codes increases, thus degrading the performance. It is interesting to note that all three detectors achieve an optimal performance at the same \( \rho \) for a fixed \( K \).

Undeniably, some of the effects in the previous figures are due solely to that fact that random codes are employed. It is clear that the performance of an OCDMA
Figure 4.7  Comparison of the length of the code required to maintain a constant probability of error of $10^{-4}$ for the correlation detector, the two-stage detector, and the known-interference detector. The ratio $\rho$ is 0.1 for this plot. Additional system parameters are $\Lambda_k = 5$ and $\Lambda_d = 0.1$. 
Figure 4.8  Performance of the correlation and multistage detectors and lower bound to the optimal detector as the pulse intensity $\Lambda_k$ increases equally for all users for a total number of users $K = 1$ and 5. The system parameters are $J = 300$, $\Lambda_d = 0.1$, and $\rho = 0.05$. 
Figure 4.9  Performance of the correlation and multistage detectors and lower bound to the optimal detector as the ratio of pulses to chips $\rho$ increases for a total number of users $K = 1, 5, \text{ and } 15$. The system parameters are $J = 300$, $\Lambda_k = 5$, and $\Lambda_d = 0.1$. 
system with random codes is worse on the average than a system with properly
designed codes that do not send pulses in the same chip and that use an optimal
threshold. Using random codes imposes a floor on the error probability due to the
possible equality between codes for symbols 1 and 0. This prompts us to examine the
performance of the system using well designed user sequences.

4.2.2 Deterministic Codes

A OCDMA system based on deterministic codes typically employs a detector of the
form
\[ Z = \sum_{w=1}^{W} X_{j_w} \begin{array}{c} \geq \varepsilon \\ < \varepsilon \end{array} 0 \]  
(4.11)

where \( j_w \) for \( w = 1, \ldots, W \) are the chips with \( A_{k,j_w}^{(1)} = 1 \) and for all other chips
\( A_{k,j}^{(1)} = 0 \). Since the system is not symmetric in the bit \( b_1^{(0)} \) due to on-off modulation,
the probability of error must be written as
\[ \hat{P}_e = \frac{1}{2} \Pr (Z > 0 | b_1^{(0)} = 0) + \frac{1}{2} \Pr (Z < 0 | b_1^{(0)} = 1) + \frac{1}{2} \Pr (Z = 0) , \]  
(4.12)

for the equally likely hypothesis case. For practical values of \( J \) and a reasonable
number of codes, the weight \( W \) must be kept very small, as discussed in Section 4.1.
This forces the sum in (4.11) to have just a few terms. For this reason and due to the
dependence of chip statistics, the large deviation theory approximation above cannot
be employed here. Instead, a different formulation is exploited as follows.
Large Deviation Theory Approximation

A technique similar to one employed by Brady in [2] involves writing the above expression as a large sum of independent and identically distributed random variables. It is a known property of Poisson random variables (called infinitesimally divisible) that \( N_{k,j}^{(0)} \) with intensity \( r_{k,j}^{(0)} \) can be equivalently written as \( \sum_{\ell=1}^{\mathcal{L}} \tilde{N}_{k,j}^{(0)}(\ell) \), where \( \tilde{N}_{k,j}^{(0)}(\ell) \) are Poisson distributed with intensity \( r_{k,j}^{(0)} = r_{k,j}^{(0)}/\mathcal{L} \). If the detection statistics \( X_j \) can be written as \( C_j N_{1,j}^{(0)} \) with \( C_j \) statistically independent of \( N_{1,j}^{(0)} \), the detector can be written as

\[
\sum_{\ell=1}^{\mathcal{L}} \tilde{Z}(\ell) = \sum_{\ell=1}^{\mathcal{L}} \left\{ \sum_{w=1}^{W} \tilde{N}_{1,j,w}^{(0)}(\ell) C_{j,w} \right\} \overset{\mathcal{L}}{\overset{\sim}{\overset{\sim}{\geq}}} 0. \tag{4.13}
\]

It is clear that the expression is a sum of \( \mathcal{L} \) independent and identically distributed random variables, and thus, the large deviation theory asymptotic approximation can be effectively used to yield accurate error probability estimates for large \( \mathcal{L} \). The Appendix discusses how each detector except for the optimized single-user detector can be approximated to fit this form. The calculation of the error probability for the optimized single-user detector can be upper bounded using a Chernoff Bound or calculated exactly using a characteristic function method, as is described in the next section. Large deviation theory gives the probability of an outlier as

\[
\Pr(\sum_{\ell=1}^{\mathcal{L}} \tilde{Z}(\ell) \geq 0) \approx \exp\left[-\mathcal{L} \mu_{\tilde{Z}(\ell)}(s^*)\right], \tag{4.14}
\]

where the moment generating function of \( \tilde{Z}(\ell) \) is \( e^{\mu_{\tilde{Z}(\ell)}(s)} \) and \( s^* = \mu_{\tilde{Z}(\ell)}^{-1}(0) \) as before.
Denoting the exponent to the moment generating function given $b_i^{(0)} = i$ as $\mu_{\tilde{Z}(t)}^{(i)}(s)$, the average error probability in (4.12) becomes

$$P_e \approx \frac{1}{2} \frac{\exp[-\mathcal{L}_\mu_{\tilde{Z}(t)}^{(0)}(s^*)]}{\sqrt{2\pi \mathcal{L}_\mu_{\tilde{Z}(t)}^{(0)}(s^*)^* s^*}} + \frac{1}{2} \frac{\exp[-\mathcal{L}_\mu_{\tilde{Z}(t)}^{(1)}(s^*)]}{\sqrt{2\pi \mathcal{L}_\mu_{\tilde{Z}(t)}^{(1)}(s^*)^* s^*}}. \tag{4.15}$$

The probability that the threshold is achieved exactly is assumed zero since the code sequences are always distinct.

For this large deviation theory approximation to be accurate, the mean value of $\tau_{k,i}^{(0)}$ must be large, which is fortunately the case of interest. The above derivation is actually equivalent to a Gaussian assumption on the random variable left after the exponential decay according to the actual statistics is removed (called a twisted random variable by Bucklew in [4]). It is easy to show that the moment generating function of this twisted random variable approaches $e^{-\frac{s^2}{2}}$ as the signal energies increase, since $Z$ is a weighted sum of Poisson random variables with each approaching a Gaussian in distribution. It is precisely this condition which allows the approximation to hold, as proved in [5].

Again, it can be shown that $\tilde{\tau}_k = 0$ for the time encoded system and $\tau = 0$ for the spectral amplitude encoded system yield a best case decay rate for the probability of error. In the time encoded case an assumption is made that the interference from each user is independent from chip to chip. This assumption is valid since a maximum of one user can have two pulses adjacent and maintain a offset autocorrelation less than or equal to one, and this is the only situation when the chip interferences from one
user are not independent. Similarly, \( \hat{\tau}_k = T_c/2 \) (time encoded system) and \( \tau_k = T/2 \) (spectral amplitude encoded system) give a worst case scenario for a fixed threshold. This case is equivalent to the \( \tau_k = 0 \) case with each interference having half the energy but twice as likely to occur, since twice as many possible chip delays cause interference. Salehi [33] and Brady [3] found the opposite to be true for the correlation detector in a time encoded system. This is because the correlation detector considers the interference contribution for the whole symbol, instead of chip-wise, and the threshold can be optimized. The near-optimal detectors proposed collect the interference at each chip and perform a logarithm before summing them, an operation that benefits more from the large accumulation of interference than the spread out interference present in asynchronous transmission.

For the time encoded system, the moment generating function is found by averaging over all possible instances of each user interference. This can be performed once the joint probability mass function of \( \tau_k \) is obtained for the a chip synchronous system. The derivation is presented here for the equal received power case. Because the maximum crosscorrelation is one, each interfering user sequence can transmit at most one pulse within the set of chips \( j_w, w = 1, \ldots, W \). Define the number of interfering user pulses in chip \( j_w \) as \( \kappa_{j_w} \). Then the joint probability mass function of the length \( W \) vector \( \kappa \) is given by

\[
p_{\kappa_{j_1}, \kappa_{j_2}, \ldots, \kappa_{j_W}}(\alpha_1, \alpha_2, \ldots, \alpha_W) = \frac{K!}{\alpha_1! \alpha_2! \cdots \alpha_W! (K - \alpha_1 - \cdots - \alpha_W)!} \times \\
\left( \frac{W}{2J} \right)^{\alpha_1 + \alpha_2 + \cdots + \alpha_W} \left( 1 - \frac{W}{2J} \right)^{K - W \alpha_1 - (W-1)\alpha_2 - \cdots - \alpha_W}.
\] (4.16)
For a system with equal received user power levels, the joint probability mass function of the interference can be obtained directly from (4.16). For the unequal power case, if the interfering user powers all vary, the joint probability mass function of the actual interference must be calculated explicitly. For the spectral amplitude encoded system, the average is taken over the occurrence of a user sequence, and not the delay. If the user sequences are assigned as cyclic rotations of each other, the resulting expression is equivalent. The moment generating functions for all detectors are then derived as given in the Appendix.

**Characteristic Function Method**

The characteristic function method is even less practical for the deterministic code case than for random codes, yet it is presented for completeness. For very small problems, it can be used to verify the accuracy of the large deviation theory approximation.

For a detection statistic of the form $Z = \sum_{u=1}^{W} X_{j,u}$ where the $X_{j,u}$ are not independent, the only advantage served by the characteristic function method is the use of an FFT over the convolution operator. Once the characteristic function of each $X_{j,u}$ for a particular realization of the delays $\tau$ is found, the characteristic function of $Z$ for that $\tau$ is given by

$$
\Phi_{Z|\tau}(\omega) = \prod_{u=1}^{W} \Phi_{X_{j,u}|\tau}(\omega).
$$

(4.17)
The average characteristic function can be directly obtained using the probability density of the interferences given in (4.16). The error probability in (4.12) is computed by taking the inverse FFT of the characteristic function under each hypothesis and averaging the result.

Results

The various approximations made in the evaluations of the probability of error are verified in Figure 4.10, which shows the large deviation theory approximation to the error probability and simulated values using deterministic codes as the pulse intensity $\Lambda_k$ increases. The simulation is performed using a Monte Carlo method. Even for low $\Lambda_k$'s the approximation closely matches the actual value of the probability of error. Two additional points are illustrated by Figure 4.10. First, the error probability of the "known interference" detector decreases exponentially as the pulse intensity increases, thus proving the effects of random codes on the error probability plot in Figure 4.8. This is not the case for the correlation detector and the multistage detector with a finite number of stages, whose performance eventually becomes multiple-access interference limited. Second, the proposed optimized single-user detector and all multiuser detectors provide a better system performance than the correlation detector for well designed codes.

The advantage of using either the optimized single-user detector, the local search detector, or the hard-decision or soft decision multistage detectors instead of the
Figure 4.10  Error probabilities for all detectors and the lower bound as the pulse intensity $\Lambda_k$ varies equally for all $k$. The system parameters are $J = 16$, $K = 7$, $W = 2$, and $\Lambda_d = .1$. 
correlation detector is further illustrated in Figure 4.11, where the error probability is plotted as the number of users increases. The result of this plot exactly parallels the random code case, with the optimized single user detector outperforming the correlation detector, and all multiuser detectors achieving close to the lower bound. The performance of the multiuser detectors is indistinguishable from the lower bound. The major distinction between this plot and the one in Figure 4.5 is the relative insensitivity of the system with deterministic codes to an increase in the number of users, as long as there are enough low crosscorrelation codes available.

4.3 Asymptotic Multiuser Efficiency

To make a fair comparison between the detectors described above, the asymptotic multiuser efficiency can be used as a measure of the robustness of a detector against high power interfering users. Asymptotic multiuser efficiency is defined as the limit as all energies approach infinity of the ratio of energy needed to transmit in single-user mode to the energy required to transmit multiple user symbols. The effect of multiple-access interference is isolated by using the asymptotic behavior as the quantum noise (Poisson noise) and the dark current vanish [2]. Since in direct detection OCDMA the interference always degrades performance [13], the best performance a detector could have is to require the same energy than for single-user operation, which

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3An alternative definition was proposed by Nelson and Poor in [23] comparing the energy of a given detector to that of the known interference detector for fixed performance. The former definition is addressed here.
Figure 4.11  Probability of error as a function of the number of users for all detectors and the lower bound for the deterministic code case. All plots are given for the best case and the lower bound is also given for the worst case. System parameters are $\Lambda_k = 10$, $\Lambda_d = 0.1$, $W = 3$, and $J = 1000$. 
would yield an asymptotic efficiency of one. The lower the asymptotic efficiency, the more the interference affects the signal detector.

4.3.1 Random Codes

The asymptotic efficiency for optical CDMA systems is derived for on-off coding in [2]. For arbitrary encoding schemes where both \( A_k^{(0)} \) and \( A_k^{(1)} \) are allowed to be nonzero, both the derivation and the final expressions are similar to the on-off case. The asymptotic behavior as the symbol energies increase forces the dark current \( \Lambda_d \) to become insignificant, so the following assumes \( \Lambda_d = 0 \). This assumption causes all decision errors to be made when nonzero expected energy chips yield zero photoelectrons and thus allows a simple expression for the probability of error.

Consider a single-user transmission of user one with symbol sequences \( A_i^{(0)} \) and \( A_i^{(1)} \). Label the set \( J_0 \) the set of nonzero chips of sequence \( A_i^{(0)} \) which are also non-one chips of sequence \( A_i^{(1)} \). Similarly, call \( J_1 \) the set of all one-valued chips of \( A_i^{(1)} \) which are not one-valued chips of \( A_i^{(0)} \). Then place all one-valued chips on both sequences in \( J_{0,1} \). Allowing \( J \) to be the whole set, the set \( (J - J_1) - J_0 - J_{0,1} \) cannot have any photon counts almost surely, since these chips have Poisson distributions with mean 0. Suppose sequence \( A_i^{(0)} \) is transmitted, then the chips belonging to \( J_1 \) have 0 photon counts a.s.. If optimal detection is employed, errors are encountered only if all chips in \( J_0 \) have zero photon counts, when a randomized choice must be made as to what symbol was transmitted. Assuming an equal a priori probability of symbol
one or zero, the total error probability is

$$P_e = \frac{1}{4} [\Pr(N_{\mathcal{J}_0} = 0) + \Pr(N_{\mathcal{J}_1} = 0)],$$

(4.18)

where $N_{\mathcal{J}_i}$ is the vector consisting of all counts for chips in $\mathcal{J}_i$ for $i = 0$ and 1. Letting the cardinality of $\mathcal{J}_i$ be $\tilde{w}_i$, each chip having equal intensity $\Lambda_1$, the error probability becomes an invertible function of $\tilde{w}_0 + \tilde{w}_1$. The average energy per symbol, for user one in single-user transmission mode, given by $\mathcal{E}_1^{(S)} = \frac{1}{2} (\tilde{w}_0 + \tilde{w}_1) \Lambda_1 + \tilde{w}_{0,1} \Lambda_1$ can be written as

$$\mathcal{E}_1^{(S)} = -\frac{1}{2} \ln[4P_e] + \tilde{w}_{0,1} \Lambda_1. \quad (4.19)$$

For a true measure of efficiency, the minimum energy required to achieve the given error rate is desired so the number of coincident chips $\tilde{w}_{0,1}$ is set to 0. If the energy required to transmit with error probability $P_e$ in multiuser mode is $\mathcal{E}_1^{(M)}$, the asymptotic multiuser efficiency for user one can be written as

$$\eta_1 = \lim_{\Lambda_1 \to \infty} \frac{\mathcal{E}_1^{(S)}}{\mathcal{E}_1^{(M)}} = \lim_{\Lambda_1 \to \infty} \frac{-\ln[4P_e]}{2\mathcal{E}_1^{(M)}}. \quad (4.20)$$

This expression is found to be equivalent to the expression derived in [2] for the on-off coded case, since $\mathcal{E}_1^{(M)}$ increases linearly with $\Lambda_1$ and the term $\lim_{\Lambda_1 \to \infty} \frac{-\ln[4P_e]}{2\mathcal{E}_1^{(M)}}$ vanishes. The same expression for the asymptotic multiuser efficiency as in [2] can therefore be employed for the random code case with $\mathcal{E}_1^{(M)} = \rho J \Lambda_1$.

The asymptotic multiuser efficiency derived above can be calculated using large deviation theory. In [2], Brady calculates the asymptotic efficiency for a two user optimal detector, and the derivation presented here is an extension of that work.
to the more general case, allowing any single or multiuser detector and an arbitrary number of users. Letting $\Lambda_k = \nu \Lambda_1$ for $k = 2, \ldots, K$, define a new sequence of random variables $Z_{\Lambda_1} = \sum_{j=1}^J X_j(\Lambda_1)$, where $X_j(\Lambda_1)$ is the decision statistic for chip $j$ with $\Lambda_1$ as the integrated intensity of an optical pulse for user 1. The value of $\nu$ is called the near-far factor. Denoting $\mu_{Z_{\Lambda_1}}(s) = \frac{1}{\Lambda_1} \ln M_{Z_{\Lambda_1}}(s)$, the Gärtner-Ellis Theorem of large deviation states that if $\mu_{Z_\infty}(s) = \lim_{\Lambda_1 \to \infty} \mu_{Z_{\Lambda_1}}(s)$ exists and is differentiable for all real $s$, then

$$\lim_{\Lambda_1 \to \infty} \frac{1}{\Lambda_1} \ln \Pr[Z_{\Lambda_1} \geq 0] = \lim_{\Lambda_1 \to \infty} \frac{1}{\Lambda_1} \ln \Pr[Z_{\Lambda_1} > 0] = -\inf_{\gamma \geq 0} \mathcal{I}_{Z_\infty}^{\infty}(\gamma), \quad (4.21)$$

where $\mathcal{I}_{Z_\infty}^{\infty}(\gamma) = \sup_s [s\gamma - \mu_{Z_\infty}(s)]$. The existence and differentiability of $\mu_{Z_\infty}(s)$ for all four detectors presented in Chapter 3 is shown in the Appendix. This expression from large deviation theory differs from the one previously used in (4.5) in the limit taken that in this case is over a power variable $\Lambda_1$ rather than the number of terms in a large sum.

As $\Lambda_1$ becomes large the probability of error is equal to $\Pr[Z_{\Lambda_1} \geq 0|b^{(0)}_1 = 0]$ from (4.4), since $\lim_{\Lambda_1 \to \infty} 1/\Lambda_1 \ln \Pr[Z_{\Lambda_1} = 0] = 0$. Thus $\lim_{\Lambda_1 \to \infty} \frac{1}{\Lambda_1} \ln [P_e] = -\inf_{\gamma \geq 0} \mathcal{I}_{Z_\infty}^{\infty}(\gamma)$. Also since $E[Z_{\Lambda_1}|b^{(0)}_1 = 0] < 0$, it is evident from the convexity of $\mathcal{I}_{Z_\infty}^{\infty}(\gamma)$ that $\inf_{\gamma \geq 0} \mathcal{I}_{Z_\infty}^{\infty}(\gamma) = \mathcal{I}_{Z_\infty}^{\infty}(0)$. The asymptotic multiuser efficiency in this case

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4The theory actually states results on the lim sup and lim inf of the probability of closed and open sets. In this case, $\mathcal{I}_{Z_\infty}^{\infty}(\gamma)$ is continuous at 0 and $(0, \infty)$ is an open convex set, thus the limit exists and the equalities hold (see [37, page 85] and [8, page 37]).
can be written as

\[ \eta_1 = \lim_{\Lambda_1 \to \infty} -\frac{\ln[P_e]}{2\mathcal{E}_1^{(M)}} = \frac{-1}{2\rho J} \lim_{\Lambda_1 \to \infty} \frac{\ln[P_e]}{\Lambda_1} \]

\[ = \frac{1}{2\rho J} \mathcal{I}_Z(0). \]

(4.22)

The efficiency is directly proportional to the large deviation rate function. The average asymptotic multiuser efficiency thus becomes

\[ \bar{\eta}_1 = \frac{1}{2\rho J} \sup_s \mu(s). \]

(4.23)

The asymptotic multiuser efficiency for the four detectors is given in Figure 4.12 as the near-far factor, which is the ratio of interfering user power to desired user power, increases. The optimized single-user detector and the soft-decision multistage detector are shown to be near-far robust for the example chosen, and both exhibit a dip in efficiency at a low near-far factor. This effect was previously seen by Brady for the optimal detector in [2], as was the lack of near-far robustness of the correlation detector. The 2-stage detector and the local search detector based on the result of the correlation detector as initial guess failed to show near-far robustness. This is due to the fact that, as the energy increases, the correlation detector performance does not increase even for the single user system in the random code case as seen in Figure 4.8. The reason the efficiency does not reach unity even for low near-far factors is that the single user performance is valid for \( \Lambda_d = 0 \) but these detectors are not defined for \( \Lambda_d = 0 \) (singular case), so a very low \( \Lambda_d \) is chosen for the calculations instead.
Figure 4.12 Asymptotic efficiency as a function of the near-far factor for all detectors proposed and the known interference detector for the random code case. All plots are for the best case $\tau_k = 0$ (time encoded system) or $\tau_k = 0$ (spectral amplitude encoding). System parameters are $\rho = 0.05$ and $K = 5$. 
4.3.2 Deterministic Codes

For the deterministic code case, since the codewords are on-off modulated by the user symbol, the same expression for the asymptotic efficiency as in [2] can be used with $E_i^{(M)} = \frac{1}{2} W \Lambda_1$. The probability of error must be written as $P_e = \frac{1}{2} \Pr[Z_{\Lambda_1} > 0|b_1^{(0)} = 0] + \frac{1}{2} \Pr[Z_{\Lambda_1} < 0|b_1^{(0)} = 1]$ for large $\Lambda_1$, since the problem is not symmetric in $b_1^{(0)}$ as in the random code case. The large deviation result follows that of Section 4.3.1 with

$$
\lim_{\Lambda_1 \to \infty} \frac{1}{\Lambda_1} \ln \Pr[Z_{\Lambda_1} > 0|b_1^{(0)} = 0] = -\inf_{\gamma > 0} T_{Z_{\infty}^0}^{(0)}(\gamma) = I_{Z_{\infty}^0}(0) \quad \text{and} \quad \lim_{\Lambda_1 \to \infty} \frac{1}{\Lambda_1} \ln \Pr[Z_{\Lambda_1} < 0|b_1^{(0)} = 1] = -\inf_{\gamma < 0} T_{Z_{\infty}^1}^{(1)}(\gamma) = I_{Z_{\infty}^0}(0),
$$

(4.24)

where $T_{Z_{\infty}^i}^{(i)}(\gamma) = \sup_s \{s \gamma - \mu_{Z_{\infty}^i}(s)\}$ is the rate function for the random sequence $Z_{\Lambda_1}$ given $b_1^{(0)} = i$. The asymptotic multiuser efficiency can now be written as

$$
\eta_1 = \lim_{\Lambda_1 \to \infty} -\frac{\ln[P_e]}{2E_i^{(M)}} = \lim_{\Lambda_1 \to \infty} -\frac{\ln[\frac{1}{2} P_e|0] + \frac{1}{2} P_e|1]}{2E_i^{(M)}}
$$

(4.25)

where $P_e|i$ is the error probability given $b_1^{(0)} = i$. Since $P_e \leq \max(P_e|0, P_e|1)$ and $P_e \geq \min(P_e|0, P_e|1)$, $\eta_1$ can be upper and lower bounded using the large deviation rate functions from (4.24) as

$$
\frac{1}{W} \min_{i=0,1} T_{Z_{\infty}^i}(0) \leq \eta_1 \leq \frac{1}{W} \max_{i=0,1} T_{Z_{\infty}^i}(0).
$$

(4.26)

The bounds for the average efficiency are calculated as

$$
\frac{1}{W} \min_{i=0,1} \sup_s \mu_{Z_{\infty}^i}(s) \leq \bar{\eta}_1 \leq \frac{1}{W} \max_{i=0,1} \sup_s \mu_{Z_{\infty}^i}(s).
$$

(4.27)
The asymptotic multiuser efficiency for all detectors described is shown by Figure 4.13, where the asymptotic efficiency is plotted against the near-far factor. The upper and lower bounds are given for the single-user detectors, while only the lower bound is given for the multiuser detectors. All detectors proposed are shown to be robust against near-far problems.
Figure 4.13  Bounds to the asymptotic efficiency as functions of the near-far factor for all detectors and the known interference detector for the deterministic code case. The lower bound is given for all detectors, and the upper bound is given also only for the two single-user detectors. All plots are for the best case $\tau_k = 0$ (time encoded system) or $\tau_k = 0$ (spectral amplitude encoding). System parameters are $W = 3$, $J = 1000$, and $K = 5$. 
Chapter 5

Incoherent Optical Source

The optical spectral amplitude encoded CDMA system can use either a coherent optical source, such as a laser, or a broadband incoherent source, such as a superfluorescent fibers source. As discussed in Section 2.2.2, if an incoherent source is used, the statistics of the photon counts on the received signal can be assumed to be a sum of negative binomial random variables. The detector design and system analysis of Chapters 3 and 4 have assumed Poisson distributed photon statistics, i.e., a coherent optical source. In this chapter we consider the modifications needed to design a system compatible with the expected negative binomial statistics of the received signal from an incoherent optical source. The analysis is also valid for laser systems in which the source experiences severe phase fluctuations. An additional assumption is made that all received power levels are equal and that the user transmissions are fully synchronized. The derivation for the asynchronous case is similar except that the number of coherence modes in each observation for each user depends on the relative delays. Since we are only considering synchronous transmission, let $A_{k,j}^{(i,d)} = A_{k,j}^{(i)}$, $N_{k,j}^{(i)} = N_j$ and $I_{k,j}^{(i)} = I_{1,j}$. 
5.1 Detector Design

According to Figure 2.3, the photodetector observes the number of photoelectrons received in each chip and sends that information to the symbol detector, which is then responsible for deciding which symbol was sent by each user. Since the symbol synchronous system is considered, only $J$ observations are used per bit decision. In all cases the observations are assumed statistically independent\(^1\). The following sections describe the minimum error probability single-user detector and options for a multiuser detector.

5.1.1 Single-user Detectors

The crudest attempt at a detector for a multiuser system is to use the detector optimal for a system composed of only one user, i.e., a single user system which ignores the presence of any other user. For the coherent optical source case, this detector is a correlation detector. The incoherent source version of this detector is shown to be a weighted correlator also. Since the performance of the simplistic single-user detector is usually poor for a system catering to a large number of users, an alternative detector is sought that acknowledges and compensates for the presence of simultaneous users. An optimized single-user detector analogous to the one described for the Poisson statistics model is described.

\(^1\)If a single spatial mode is angularly dispersed onto the photodetector array, the uncertainty due to the incoherence of the source is diminished since coherence exists between the chip observations, and the assumption of statistical independence may be violated.
Correlation Detector

The optimal detector for a single user system is obtained from the likelihood ratio test. Two assumptions on the statistics of the received photon counts are described in Chapter 2: that the intensity contribution due to dark current is negligible in the presence of intentional transmission, and, that the probability distribution of the photon count under just dark current intensity is well modeled by a negative binomial. The resulting expression is a ratio of two negative binomial probability mass functions with parameter $M$ and mean value $A_1$, written as $NB(\cdot, M, A_1)$, times a ratio of another two negative binomial mass functions with parameter $M$ and mean $A_d$,

$$\frac{\prod_{j=1}^{A_1(j)} NB(N_j, M, A_1) \prod_{j=0}^{A_1(0)} NB(N_j, M, A_d)}{\prod_{j=1}^{A_1(0)} NB(N_j, M, A_1) \prod_{j=0}^{A_1(j)} NB(N_j, M, A_d)} \xrightarrow{\approx} \gamma,$$  

(5.1)

where $\gamma$ is the ratio of a priori probabilities of the two symbols for user 1. This expression simplifies to a correlation detector, leaving a final detector expression

$$\sum_{j=1}^{J} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) N_j \ln \left( \frac{(A_d + M) A_1}{(A_1 + M) A_d} \right) \xrightarrow{\approx} \gamma_c,$$  

(5.2)

where the threshold $\gamma_c$ is

$$\gamma_c = \ln(\gamma) - \sum_{j=1}^{J} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) M \ln \left( \frac{(A_d + M)}{(A_1 + M)} \right).$$

If the integrated intensities for one user are not equal for all chips, the values of $A_1$ can simply be replaced by $A_{1,j}$, resulting in a weighted correlator as a detection algorithm. In the limit as $\frac{\Lambda}{M}$ approaches 0, the statistics of $N_j$ resemble those of a
Poisson process and this detector resembles the one in Chapter 3 equation (3.1) with \( \Lambda_1 + \Lambda_d \approx \Lambda_1 \).

This detector can be used in a multiuser system, after some adjustment in the threshold. As in the Poisson based system, the performance of this single-user detector is poor when many users are accessing the network. This motivates us to consider a single-user detector which takes into account the multiple-access interference.

**Optimized Single-user Detector**

The optimized single-user detector maximizes the likelihood ratio while considering the multiple-access interference as an additive noise with known distribution. Let the number of interferers which are transmitting a signal in chip \( j \) as seen by user \( k \) be denoted \( \kappa_{k,j} = I_{k,j}/\Lambda_k \). The probability mass function of the received signal is

\[
p_N(n) = \prod_{j=1}^{J} \left( \frac{n_j + (A_{1,j}^{(b)} + \kappa_{1,j})M - 1}{n_j} \right)^n_j \left( \frac{\Lambda_1}{M + \Lambda_1} \right)^n_j \left( \frac{M + \Lambda_1}{M} \right)^{(A_{1,j}^{(b)} + \kappa_{1,j})M}.
\]

(5.3)

The detector defined by (3.2) becomes

\[
\sum_{j=1}^{J} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) \left\{ \ln E_{\kappa_{1,j}} \left[ \frac{(N_j + (1 + \kappa_{1,j})M - 1)!}{((1 + \kappa_{1,j})M - 1)!} \left( \frac{M}{M + \Lambda_1} \right)^{(1+\kappa_{1,j})M} \right] - \ln E_{\kappa_{1,j}} \left[ \frac{(N_j + \kappa_{1,j}M - 1)!}{(\kappa_{1,j}M - 1)!} \left( \frac{M}{M + \Lambda_1} \right)^{\kappa_{1,j}M} \right] \right\} \overset{\geq}{\leq} \gamma_2.
\]

(5.4)

Since the probability distribution of the interference \( \kappa_{1,j} \) can be modeled as a binomial in the equal received energy synchronous system, the detector algorithm is fully
defined. Note that the physical implementation of this algorithm is nontrivial because of the expectation operators and the factorials.

### 5.1.2 Multiuser Detectors

Multiuser detectors attempt to demodulate all the user symbols simultaneously. In the coherent source case, we developed a set of multiuser detection algorithms that were shown to perform much better than the single-user detectors. For the incoherent source case, we will only develop the detection algorithm analogous to the multistage detector for Poisson observation statistics. The algorithms for the local search detector and the soft-decision multistage detectors can easily be extrapolated from these results.

The multistage detector performs a Gauss-Seidel algorithm to find the maximum likelihood estimate of all user symbols. The method is essentially to derive a decision on each symbol independently using a previously obtained estimate of the other user symbols. The optimal algorithm for deciding on the symbol for user $k = 1$ given an estimate $\hat{\kappa}_{1,j}$ of $\kappa_{1,j}$ defined previously can easily be derived. After some algebraic manipulation, this algorithm becomes

$$
\sum_{j=1}^{J} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left( \frac{(N_j + (1 + \hat{\kappa}_{1,j})M - 1)!((\hat{\kappa}_{1,j}M - 1)!)}{(N_j + \hat{\kappa}_{1,j}M - 1)!((1 + \hat{\kappa}_{1,j})M - 1)!} \right) \geq e^{-\gamma_{m_{s1}}}. \tag{5.5}
$$

The factor $\frac{(\hat{\kappa}_{1,j}M - 1)!}{((1 + \hat{\kappa}_{1,j})M - 1)!}$ does not seem to depend on the observations, yet indirectly is does since $\hat{\kappa}_{1,j}$ does; it is therefore included in the detector and not the threshold $\gamma_{m_{s1}}$. 
The multistage detector uses the decisions from the single-user detector defined in Section 5.1.1 to form an initial guess for \( \kappa_{1,j} \). The second stage of the detector makes a decision on the symbol transmitted by user 1 based on the algorithm (5.5). A similar detector can be user on the other user symbols, thus completing one stage of detection. Additional stages use estimates from the previous stage. The implementation of this detector can thus be accomplished as in Figure 3.3.

5.2 Probability of Error Analysis

The most important performance criterion of a digital communication system is the bit error probability. In this section, the approach taken to determine the performance of the incoherent source OCDMA system is a large deviation theory approximation equivalent to that derived in Section 4.2.1. The known interference detector, that is the optimal detector for user one given the symbols sent by all the other users is known, provides a lower bound on the error probability of the other detectors.

The decision algorithm for the detectors considered all have the form

\[
Z = \sum_{j=1}^{J} X_j \overset{\geq}{\sim} 0,
\]

where the threshold has been incorporated into the mean of \( X_j \). The large deviation theory derivation in Section 4.2.1 can be directly applied to this problem. The main difficulty in this approach is that the exponent to the moment generating function is not analytically tractable. Although the moment generating function of a negative binomial is trivial to compute, the moment generating function of the decision
statistics for the optimized single-user detector and the multistage detector are not. Therefore, a numerical approach to the evaluation of the moment generating function is adopted. The brute force evaluation of the moment generating function, i.e., the computation of $E[e^{Zc}]$ directly, can be done for low values of $\Lambda_1$ and $M$ by truncating the sum in the expectation to a few times the mean of the distribution.

The performance of well-designed detectors again proves superior to that of the simplistic single-user detector. The error probability for the correlation detector, the optimized single-user detector, the 2-stage detector, and the known interference lower bound are plotted in Figure 5.1 as the number of users varies. In addition, the performance of the OCDMA system using a coherent source for the same system parameters is given for comparison. It is evident, and expected, that a penalty is incurred by using an incoherent source instead of a coherent source. This is due to the fact that a doubly stochastic Poisson point process necessarily has larger variance than a Poisson process with deterministic intensity. Note that this effect diminishes as the number of interferers increases. Modeling the interference as a random process forces even the system using a coherent light source to behave as a doubly stochastic point process, thus overshadowing the effect of the incoherent light.

The number of significant modes of coherence $M$ in the OCDMA system is fixed by the bandwidth used and by the time of observation $T$. The penalty to the error probability incurred from using an incoherent source diminishes as the number of modes increases, as illustrated in Figure 5.2. This is attributed to the property of the
Figure 5.1 Error probability for all detectors considered for the incoherent source case as well as for the coherent source case as the number of user $K$ increases. The system parameters are $\Lambda_k = 5$, $\Lambda_d = .1$, $\rho = 0.05$, $M = 10$, and $J = 500$. 
Figure 5.2  Error probability for all detectors considered for the incoherent source as a function of the number of modes $M$. The system parameters are $\Lambda_k = 5, \Lambda_d = .1, K = 5, \rho = 0.05$, and $J = 500$. 
negative binomial distribution of approaching a Poisson distribution as the ratio of the number of arrivals to the number of modes decreases.

In summary, the performance of the OCDMA system using an incoherent source parallels that of the coherent source system. Using a detector designed to match the statistics of the system, including the effects of the multiple-access interference, provides a gain in performance of significant amount. Multiuser detectors in general perform better than the single-user detectors. The penalty incurred by using an incoherent source becomes small for systems accessed by a large number of users.
Chapter 6

Conclusions

A need exists for a multiple-access communication network that is capable of supporting a large number of users at high data rates and with high reliability. The optical medium with its broad bandwidth is a prime candidate for such a system. The objective of this study is to describe and analyze the performance of an optical code-division multiple-access system (OCDMA) specifically designed to fulfill this need.

6.1 Summary

The OCDMA framework provides an opportunity to utilize the vast bandwidth available on an optical channel. Two system configurations are proposed, namely a time encoded system and a spectral amplitude encoded system. Each system is composed of $K$ users transmitting binary encoded information simultaneously through an optical channel, the main difference being the way the user signature sequences are encoded onto the optical beam. The method of time encoding is the conventional approach to OCDMA. Spectral amplitude encoding is a new idea we propose for its ease in implementation, its suitability to well-designed detectors, the fewer restrictions it
imposes on what defines a good user sequence, and its insensitivity to chromatic dispersion. Both a coherent laser source and an incoherent optical source are described and analyzed. The multiple-access interference, Poisson characteristic of the optical detection, and constant dark current noise are the primary performance degradations considered.

The optical direct detection CDMA system can benefit from the implementation of an efficient symbol detector, particularly in the presence of unequal received user powers. The main contribution of this study is to define several single-user and multiuser detectors and devise a general method for acquiring both the error probability and the multiuser asymptotic efficiency. The conventional correlation detector and a new formulation of the optimal detector are given. Four detectors are then introduced as alternatives to provide a lower complexity than the optimal detector yet a better performance than the correlation detector. An optimized single-user detector is described which uses a statistical characterization of the expected interference. The multiuser detectors proposed are the local search detector, the multistage detector and the soft-decision multistage detector. These detectors are shown to have computational complexity linear or quadratic in the number of users, as compared to the optimal detector whose complexity is exponential in the number of users.

The bit error probability of these detectors for both random codes and well-designed deterministic codes is obtained using a large deviation theory approximation. A separate analysis is performed for a system using a laser source and an incoherent
optical source. By using the moment generating function of the decision statistic, an approximation for large code lengths is obtained for the random code case. An approximation for large received energies is found for the deterministic code case. The detectors proposed are shown to perform exceptionally well in the presence of a large number of interfering users. The optimized single-user detector compares favorably to the correlation detector in all cases examined. Of the multiuser detectors, the local search detector has the best performance, yet it is the most complex. A significant gain in performance is achieved from the multistage detector by using a soft decision multistage detector. The accuracy of the approximations to the error probabilities is verified by comparing them to exact values found using a characteristic function method and further confirmed by comparing to simulation results. The performance of the OCDMA system using an incoherent source is found to suffer some degradation over the system using a laser source.

In addition to the probability of error analysis, a general formulation for obtaining the multiuser asymptotic efficiency by means of the large deviation theory rate function is given. The asymptotic efficiency is found to be proportional to the rate function of large deviation theory in the random code case. In the deterministic code case, it can be upper and lower bounded using the rate function. The optimized single-user detector and the soft-decision multistage detector are found to the robust against multiple-access interference in a near-far scenario. The multistage detector
and the local search detector are robust to unequal received powers only in the deterministic code case.

6.2 Practical Considerations and Design Issues

The primary goal in designing a multiuser communication system is to allow a large amount of information to transfer from source to destination in a reliable manner. The challenge facing a design engineer is to construct a system which provides sufficient throughput at sufficiently low error rates, i.e., high reliability, and for the least amount of hardware complexity possible so as to minimize cost. This study shows that by using spectral amplitude encoding and a near-optimal symbol detection algorithm, a large number of users and lower error probability can be achieved. It is also shown that a direct trade exists between the reliability/throughput maximization and the cost in complexity.

The accessibility of a multiuser communication system is determined by two factors: the total number of users that have unrestricted access to the system and the maximum number of users simultaneously accessing the channel. A time-encoded OCDMA system suffers more from large user pools than a spectral encoded OCDMA system since the codes that are assigned have progressively worse mutual interference properties as the number of subscribers gets large. The spectral encoded system reaches a performance limit as the number of simultaneous users increases, yet allows
a much larger subscriber pool than a time encoded system does before any severe degradation is encountered.

Once a system that can support the appropriate number of users for a particular application is defined, the performance of this system is set by many factors including the optical source power and the type of symbol detection algorithm chosen. This study shows that for any source power level, a gain in performance can be achieved by using a detector designed to combat the multiple-access interference instead of the conventional detector that ignores the presence of interference. In particular, if a single-user detector is necessary, the optimized single-user detector performance is superior to the correlation detector performance. If multiuser detection is possible, the wisest choice is to implement either a soft-decision multistage detector or a local search detector, depending on the permissible complexity.

The expected difference between received powers for the separate users also helps the system designer choose a symbol detector. In a fiber optic network, it is unlikely that the received power levels vary significantly, since the loss in the fiber is minimal. Two scenarios which would yield large differences in received power are wide area networks where the attenuation in the fiber becomes significant, and systems in which each user employs a source with a different transmit power. In free space optical transmission, the losses due to the separation between the transmitter and receiver increase as the square of the distance, and thus the difference in received power can again be significant. In such situations it is advisable that a detector robust against
the near-far problem be employed. If well-designed codes are used, all proposed multiuser detectors satisfy this requirement. The optimized single-user detector is near-far robust under some constraints also.

The optical spectral amplitude encoded system allows the practical implementation of all of these recommended detectors without forcing a decrease in the individual user data rates. The time-encoded system requires a decrease in the data rate and spectral spreading factor so that is it possible to acquire the necessary observations for these detectors within the limits of electronic speeds. The only valid alternative for the time encoded system to support high data rates and a large number of users is to use a simple correlation detector and allow the performance to degrade. A fair comparison then is one involving a spectral amplitude encoded CDMA system using either an optimized single-user detector or a multiuser detector, and a time encoded CDMA system using a correlation detector. This defines the nature of the design trade between hardware complexity and performance.

The last design issue addressed here is the choice between implementing a system using mode-locked laser source or an incoherent laser source in the case of spectral amplitude encoding. It is shown that the laser source is inherently much less noisy and yields a higher performance than the incoherent source. The penalty in using this type of optical source is that the apparatus needed is bulky and often extremely sensitive to alignment problems and vibrations. The cost is also often inhibiting. An incoherent source implemented using a superfluorescent fiber, on the other hand, is
physically small and much less troublesome according to preliminary research. The choice of which light source to use depends on the application and the operational environment of the system.

6.3 High Throughput Example

To illustrate the capability of the optical CDMA system and some of the design trade-offs, a high data rate, high utilization system is described. Consider a system consisting of one thousand simultaneous users ($K = 1000$) each transmitting binary symbols at the rate of 1 Gbps. The total throughput is then 1 tera bps (Tbps). Obviously no radio frequency system can provide sufficient bandwidth for such throughput requirements. Suppose a error probability no higher than $10^{-9}$ is desired. Both time encoding and spectral amplitude encoding are investigated.

With spectral amplitude CDMA, near-optimal detectors such as the soft-decision multistage detector or the local search detector can be utilized. With a code length of $J = 1000$, which is deemed a practical limit for photodetector arrays, to support $K = 1000$ simultaneous users, the weight of each code word must be at most $W = 30$ for a maximum crosscorrelation of one chip and one code per user, according to equation (4.2). For a system with these characteristics using a mode-locked laser and a local search detector, the number of photoelectrons required per chip to achieve an error probability of $10^{-9}$ is $\Lambda_k = 5$. 
Time encoding is accomplished via a tapped delay line, which also restricts the code lengths to approximately $J = 1000$. The CDMA system relying on time encoding is restricted to using a correlation detector at data rates of one Gbps since chip photon accumulation at such rates is impossible with today's technology due to limited electronic gate speeds. If one code is used per user and for a maximum crosscorrelation and offset autocorrelation of three chips, the weight of each codeword is limited to $W = 33$ according to bounds presented in [33]. With these constraints, even for an arbitrarily large number of photoelectrons per chip, the error probability remains at approximately $10^{-2}$ for the equal received energy case. The system performance is multiple-access interference limited. To achieve error rates of $10^{-9}$, either the length of the code must be increased, i.e., decreasing the data rate, or the number of users must be decreased. In other words, there is no physically realizable time-encoded system which can support 1000 users at data rates of 1 Gbps with error probabilities of $10^{-9}$.

### 6.4 Future Work

The primary focus of the continuation of this project is to develop a proof on concept experiment for the spectral amplitude encoded system. Through such an experiment, some data on the statistics of the optical source can be obtained. The statistical properties of the spectral content of mode-locked lasers is not well understood, as these sources are primarily studied for their time-domain characteristics. With the aid of
experimental data, refinements on the model can be made. The ideal filter assumption for the amplitude mask should be replaced be a more accurate filter description, taking into account the characteristics of the optical beam. Several assumptions such as the independence of observation between chip counts must be validated. The effects of misalignment or broadening of the beam can also be modeled. Finally, a working multiuser communication prototype can be constructed to verify the reliability of the system and compare the bit error rate to the predicted error probability.

Additional contributions to this study can be made by considering the issues concerning the practical implementation of near-optimal symbol detectors. In practice, some combination of a single-user detector and a multiuser detector may be required, where some nearby user codes are known but some background intensity from unknown users must also be accounted for. It is unreasonable to assume that, with today's technology, each signal destination can manipulate Tbps information so as to acquire the desired Gbps signal. Some additional processing is recommended to discard the least necessary information as quickly as possible.
Appendix A

Moment Generating Function

The following is a development of the moment generating function for both the random code case and the optical orthogonal code case for the coherent optical source system. The moment generating function and the characteristic function differ only in the argument. Once the moment generating function with argument $\zeta$ is acquired, the characteristic function can be obtained by directly substituting the imaginary argument $i\omega$ for the argument $\zeta$. We consider the correlation detector, the optimal detector, the optimized single-user detector, the multistage detector, the local search detector, and the soft-decision multistage detector. Only the chip synchronous case is considered for the time encoded system and the symbol synchronous case for the spectral amplitude encoded system.

Under the random code assumption, the interference is a random vector depending on the energies received for each user. In the case of equal energy received for each user, the distribution of the interference is binomial. Under the deterministic code assumption, a random allocation of the single interference possible from each user is accomplished by considering a equal probability of the user delays amongst all chip for the time encoded case. For the spectral amplitude encoded system, a random
allocation of this chip interference is justified by considering a random usage of the channel by users assigned cyclically shifted versions of the same codes. In this way the interference values for the deterministic codes are as derived in (4.16)). We therefore only need consider the moment generating function given the interference vector.

### A.1 Correlation Detector

The decision statistic for the OCDMA system using a correlation detector has the form

$$Z = \sum_{j=1}^{J} X_j,$$

(A.1)

with

$$X_j = -c + N_{i,j}^{(0)} (A_{i,j}^{(1)} - A_{i,j}^{(0)}),$$

(A.2)

where the threshold $c$ is set to $\frac{\Lambda_1 (A_{i,j}^{(1)} - A_{i,j}^{(0)})}{\ln(1+\Lambda_1/\Lambda_4)}$ in the single-user case, and is adjusted to match the number of interfering users in the multiuser case.

The moment generating function $M_{X_j}(c) = E[e^{cX_j}]$ can be written as

$$M_{X_j|I_{i,j}^{(0)}}(c)$$

$$= E_{N_{i,j}^{(0)}} \left[ \exp \left( -c + c (A_{i,j}^{(1)} - A_{i,j}^{(0)}) I_{i,j}^{(0)} \right) \right]$$

$$= \exp \left\{ -c + (\Lambda_1 A_{i,j}^{(0)} + I_{i,j}^{(0)}) (e^{c (A_{i,j}^{(1)} - A_{i,j}^{(0)})} - 1) \right\}.$$  

(A.3)

A near optimal threshold for multiuser transmission can be found by minimizing the moment generating function with respect to $c$. 

...
A.2 Optimized Single-user Detector

Neither the characteristic function nor the moment generating function of $X_j$ for the OCDMA system using an optimized single-user detector can be derived explicitly, due to the structure of the detector which will not allow the random variable $N_{i,j}^{(0)}$ to be isolated. For this detector, if the threshold $\gamma_s$ is 0, $X_j$ has the form

$$X_j = -\Lambda_1(A_{1,j}^{(1)} - A_{i,j}^{(0)}) + \ln \left\{ \frac{\sum_I (\Lambda_1 A_{1,j}^{(1)} + I)^{N_{i,j}^{(0)}} e^{-I p_{i,j}^{(0)}(I)}}{\sum_I (\Lambda_1 A_{1,j}^{(0)} + I)^{N_{i,j}^{(0)}} e^{-I p_{i,j}^{(0)}(I)}} \right\}. \quad (A.4)$$

The exact moment generating function can be explicitly calculated as $E[e^{\lambda X_j}]$,

$$M_{X_j|I_{1,j}^{(0)}}(\lambda) = e^{-\lambda A_1(A_{1,j}^{(1)} - A_{i,j}^{(0)})} \times \sum_{n=0}^{\infty} \left( \frac{\sum_I (\Lambda_1 A_{1,j}^{(1)} + I)^n e^{-I p_{i,j}^{(0)}(I)}}{\sum_I (\Lambda_1 A_{1,j}^{(0)} + I)^n e^{-I p_{i,j}^{(0)}(I)}} \right)^n p_{N_{i,j}^{(0)}, I_{1,j}^{(0)}}(n|I_{1,j}^{(0)}), \quad (A.5)$$

where

$$p_{N_{i,j}^{(0)}, I_{1,j}^{(0)}}(n|I_{1,j}^{(0)}) = \frac{[\Lambda_1 A_{1,j}^{(0)} + I_{1,j}^{(0)}]^n e^{-(\Lambda_1 A_{1,j}^{(0)} + I_{1,j}^{(0)})}}{n!}, \quad (A.6)$$

and $I_{1,j}^{(0)}$ depends on whether a time encoded system or a spectral amplitude encoded system is considered. The infinite sum may be truncated to a small factor times $\sum_{k=1}^{K} \Lambda_k$ and still maintain accurate results.

A.3 Optimal Detector

A complication exists in the exact calculation of the probability of error associated with the optimal detector, since the expression depends on the expected value of the received intensity for all chips. The calculation of this expected value is of exponential
complexity. Nonetheless, the probability of error may easily be bounded, which is the approach developed in this section.

An upper bound to the error probability of the optimal detector is trivial since any other detector that relies on no additional information, such as the correlation detector and the multistage detector, suffers an error probability higher than that of the optimal detector. A lower bound to the optimal detector error probability can be found by considering what is referred to here as the “known-interference” detector. The “known-interference” detector is defined by (3.9) except that the estimated intensity levels \( \bar{r}(i)(t) \) are replaced by the actual intensity levels \( r(i)(t) \) for both \( i = 0 \) and \( i = 1 \). This detector can obviously not be implemented since the actual user symbols are not known at the detector, yet its probability of error lower bounds the probability of error of the optimal detector. This is shown in the following way.

The likelihood function given that the exact interference is known is equivalent to the likelihood function of a single user optical system with dark current equal to \( I_{1,j}^{(0)} + A_d \) in each chip interval. The likelihood ratio for detecting symbol \( b_j^{(0)} \) in the time encoded case yields

\[
\ln \Lambda = - \int_0^T (r(1)(t) - r(0)(t)) \, dt + \sum_{j=1}^J N_{1,j}^{(0)} \ln \left( \frac{\int_0^{T_e} r(1)(t) \, dt}{\int_0^{T_e} r(0)(t) \, dt} \right) > \equiv \eta, \quad (A.7)
\]

and in the spectral amplitude encoded case

\[
\ln \Lambda = - \int_0^T (r(1)(t) - r(0)(t)) \, dt + \sum_{j=1}^J N_{1,j}^{(0)} \ln \left( \frac{\int_0^T r_j(1)(t) \, dt}{\int_0^T r_j(0)(t) \, dt} \right) > \equiv \eta, \quad (A.8)
\]
where \( r_j^{(i)}(t) \) is the intensity in frequency chip \( j \) given \( b_1^{(0)} = i \). Note that since \( r^{(0)}(t) = r^{(1)}(t) \) outside the interval \([0, T]\), the detection algorithm considers only the counts corresponding to the symbol in question. The detector defined by the likelihood ratio determines the minimum probability of error detector, and therefore yields a lower error probability than a detector using \( r^{(i)}(t) \).

This lower bound for the optimal detector error probability is useful since it is easily calculated. The "known-interference" detector simplifies to an expression identical to (3.15) with \( I_{1,j}^{(0)} \) instead of \( \tilde{I}_{1,j}(0) \). Therefore, (A.1) may be used with

\[
X_j = -\gamma_{lb} + N_{1,j}^{(0)}(A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left( \frac{\Lambda_1 + I_{1,j}^{(0)}}{I_{1,j}^{(0)}} \right),
\]

where the threshold \( \gamma_{lb} = -\Lambda_1(A_{1,j}^{(1)} - A_{1,j}^{(0)}) \). The moment generating function of \( X_j \) in this case is given by

\[
M_{X_j|I_{1}^{(0)}}(s) = \exp \left\{ -\gamma_{lb} + (\Lambda_1 A_{1,j}^{(i)} + I_{1,j}^{(0)}) \times \right. \\
\left[ \exp \left( s(A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left( \frac{\Lambda_1 + I_{1,j}^{(0)}}{I_{1,j}^{(0)}} \right) \right) - 1 \right] \right\}. \quad (A.9)
\]

Note that the large deviation theory approximation and the characteristic function method for the OCDMA system with this detector can be employed since statistical independence exists between the samples \( X_j \) given \( I_{1}^{(0)} \).

### A.4 Local Search Detector

For the chip synchronous case in the time encoded system or the symbol synchronous case in the spectral amplitude encoded system, the local search detector performs
an operation similar to the multistage detector but with constant updates to the interference estimate. This algorithm necessarily finds a local maximum to the likelihood function. The local search detector error probability can therefore be tightly upper bounded by the probability of error of the multistage detector in the limit as the number of stages increases, since the multistage detector iterates to find a local maximum and either finds one or oscillates between several points. This conjecture is verified in Figure 4.6 by comparing simulated local search detector error probabilities with the asymptote as the number of stages increases of the multistage detector error probabilities.

A.5 Multistage Detector

It is easier to examine the statistics of the hard-decision multistage detector of Section 3.2.1 if one defines an arbitrary multistage detector as

\[
\sum_{j=1}^{J} N_{1,j}^{(0)} (A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left( \frac{\Lambda + I_{1,j}^{(0)} + \Delta_j(s)}{I_{1,j}^{(0)} + \Delta_j(s)} \right) > \gamma_{m_2}, \quad (A.10)
\]

where the \(J\)-dimensional random vector \(\Delta(s) \in [-I_1^{(0)} + \Lambda_d, K - I_1^{(0)} + \Lambda_d]\) is the only parameter changing between iterations. For the multistage detector defined by (3.15), the random vector \(\Delta(s)\) used is the difference between the estimated interference and the actual interference, written as

\[
\Delta(s) = \hat{I}_1^{(0)} (s - 1) - I_1^{(0)}. \quad (A.11)
\]
The probability of error of the arbitrary multistage detector is a function of several random variables, namely \( \mathcal{N}_1^{(0)}, \Delta(s), \) and \( I_1^{(0)} \). Throughout this section, the notation \( P_e(\mathcal{N}_1^{(0)}, \Delta(s), I_1^{(0)}) \) signifies the probability of an error of the \( s \)-stage of detection as a function of the arguments, and averages of the probability of error are indicated by an overbar and the omission from the \( P_e \) argument list of the random variable used in averaging. The notation \( n, \hat{s}(s), \) and \( \bar{I} \) symbolizes the occurrences of the random vectors \( \mathcal{N}_1^{(0)}, \Delta(s), \) and \( I_1^{(0)} \), respectively.

The probability of error of the proposed multistage detector is nearly impossible to compute exactly due to the dependence of \( \Delta(s) \) in (A.11) on \( \mathcal{N}_1^{(0)} \). Given a particular combination of codes used, the exact probability of error for the \( s \)'s stage of the actual multistage detector can be written as

\[
\bar{P}_e(\mathcal{I}) = \sum_{\mathcal{N}_1^{(0)}(s)} P_e(n, \hat{s}(s), \mathcal{I}) p_{\mathcal{N}_1^{(0)}|\mathcal{I}}(n|\mathcal{I}); \ s = 2, 3, \ldots, S, \tag{A.12}
\]

where \( P_e(n, \hat{s}(s), \mathcal{I}) \) is the probability of an error if \( \Delta(s) = \hat{s}(s) \) is used as the interference estimate error in detector (A.10), if the interference is actually \( \mathcal{I} \) and the observed counts are \( n \). Given \( I_1^{(0)} = \mathcal{I} \), the random vector \( \Delta(s) \) as defined in (A.11) is uniquely determined by the counts \( \mathcal{N}_1^{(0)} \) since \( \mathcal{N}_1^{(0)} \) determines the symbol decisions on the previous stage giving rise to \( I_1^{(0)}(s - 1) \). The probability of error can therefore also be written as

\[
\bar{P}_e(\mathcal{I}) = \sum_{\mathcal{N}_1^{(0)}} \sum_{\hat{s}(s)} P_e(n, \hat{s}(s), \mathcal{I}) p_{\Delta(s)|\mathcal{N}_1^{(0)}|\mathcal{I}}(\hat{s}(s)|n, \mathcal{I}) p_{\mathcal{N}_1^{(0)}|\mathcal{I}}(n|\mathcal{I}), \tag{A.13}
\]
where the vector \( \delta(s) \) is in \([-L + \Lambda_d, K - 1 - L + \Lambda_d]\) and the probability

\[
p_{\Delta(s) | \mathcal{N}_1^{(0)} \Delta_1^{(0)}(\delta(s)|\mathcal{N}, L)} = \begin{cases} 1 & \delta(s) = \hat{L}(s - 1) - L \\ 0 & \text{otherwise.} \end{cases} \tag{A.14}
\]

An approximation for the probability of error of the multistage detector can be obtained by artificially imposing statistical independence between the vector \( \Delta(s) \) and the vector \( \mathcal{N}_1^{(0)} \). This can be done by simply using the marginal probability \( p_{\Delta(s) | \mathcal{N}_1^{(0)} \Delta_1^{(0)}}(\delta(s)|\mathcal{N}, L) = \sum_{\mathcal{N}} p_{\Delta(s) | \mathcal{N}_1^{(0)} \Delta_1^{(0)}(\delta(s)|\mathcal{N}, L)} p_{\mathcal{N}_1^{(0)} \Delta_1^{(0)}(\mathcal{N})} \) instead of the conditional probability \( p_{\Delta(s) | \mathcal{N}_1^{(0)} \Delta_1^{(0)}}(\delta(s)|\mathcal{N}, L) \) in (A.13),

\[
\bar{P}_e(L) \approx \sum_{\delta(s)} \sum_{\mathcal{N}} P_e(\mathcal{N}, \delta(s), L) p_{\mathcal{N}_1^{(0)} \Delta_1^{(0)}}(\mathcal{N}|L) p_{\Delta(s) \Delta_1^{(0)}}(\delta(s)|L) = \bar{P}_e(L). \tag{A.15}
\]

Although it cannot be said that such an assumption of independence is in any way valid, the effect averaged over all possible counts is very close to that of the actual error encountered, as is shown when this approximation is compared to simulation results in Sections 4.2.1 and 4.2.2. Intuitively this approximation is expected to overestimate the error probability in general since the actual detector uses the information in the counts \( \mathcal{N}_1^{(0)} \) to determine \( \Delta(s) \) instead of making a random choice as calculated in the approximation.

The resulting approximation to the probability of error can be computed using the large deviation theory approximation or the characteristic function method since the chip counts and the weighting factors of the detector are now independent given the interference levels \( \mathcal{L}_1^{(0)} \). The moment generating function is given by

\[
M_{X_1 \mathcal{L}_1^{(0)}}(\zeta) \approx E_{\Delta(s) | \mathcal{L}} \exp \{-\zeta \Lambda_1 (A_{i,j}^{(1)} - A_{i,j}^{(0)}) + \}
\]
\[(A_1 A_{1,j}^{(i)} + I_{1,j}^{(0)}) \left\{ \exp \left( \gamma (A_{1,j}^{(1)} - A_{1,j}^{(0)}) \ln \left( \frac{A_{1,j}^{(0)} + \Delta_j(s)}{I_{1,j}^{(0)} + \Delta_j(s)} \right) - 1 \right) \right\} \right]. \quad (A.16)\]

The calculation of \( p_{\Delta(s)\mid \xi}(\delta(s)\mid L) \) in general requires a convolution of probability densities of the errors encountered by each interfering user. For the equal received interferer power case, these densities can be immediately derived from the error probabilities of the previous stage of the multistage detector as follows.

Let \( \kappa_{1,j}^{(0)} = I_{1,j}^{(0)}/A_1 \) be the number of interfering users transmitting in chip \( j \). The chip interferences \( I_{1,j}^{(0)} \) are random and independent from each other, and thus the \( \kappa_{1,j}^{(0)} \) are also. The estimate errors \( \Delta_j(s) \) are not independent from the interference values so conditional marginals are desired. To derive \( p_{\Delta(s)\mid \xi}(\delta(s)\mid L) \), it is sufficient to consider \( p_{\Delta_j(s)\mid \kappa_{1,j}^{(0)}}(\delta_j(s)\mid \kappa_j) \) for an arbitrary chip \( j \). Let \( \Delta_j(s) = \Delta^+ - \Delta^- \), where \( \Delta^+ \) is the sum of all positive error contributions, \( \Delta^- \) is the sum of all negative error contributions. The conditional marginal of \( \Delta_j(s) \) for stage \( s \) can be written as

\[
p_{\Delta_j(s)\mid \kappa_{1,j}^{(0)}}(\delta_j(s)\mid \kappa_j) = \sum_{\alpha} \Pr(\Delta^+ = \delta_j(s) + \alpha \mid \kappa_j) \Pr(\Delta^- = \alpha \mid \kappa_j); \]

\[
\delta_j(s) = -\kappa_j, \ldots, K - 1 - \kappa_j, \quad (A.17)
\]

where \( \alpha \in \{\max(0, -\delta_j(s)), \ldots, \min(\kappa_j, K - 1 - \kappa_j - \delta_j(s))\} \). For the random code case, the densities of \( \Delta^+ \) and \( \Delta^- \) are shifted binomials, yielding the conditional probability marginal

\[
\sum_{\alpha} \left( \begin{array}{c} K - 1 - \kappa_j \\ \delta_j(s) + \alpha \end{array} \right) \times \left( \begin{array}{c} \kappa_j \\ \alpha \end{array} \right) \times \left[ \hat{P}_e(s-1) \rho \right]^{\alpha} \times \left[ 1 - \hat{P}_e(s-1) \rho \right]^{\kappa_j - \alpha} \times
\]
\[
\left[ \hat{P}_e^{(s-1)} (1 - \rho) \right]^{\delta_j(s) + \alpha} \times \left[ 1 - \hat{P}_e^{(s-1)} (1 - \rho) \right]^{K - \kappa_j - \delta_j(s) - \alpha};
\]
\[
\delta_j(s) = -\kappa_j, \ldots, K - 1 - \kappa_j, \tag{A.18}
\]

where \( \hat{P}_e^{(s-1)} \) is the approximation to the average error probability for the previous stage. For the deterministic code case, the densities of \( \Delta^+ \) and \( \Delta^- \) are also shifted binomials but with parameter \( \hat{P}_e^{(s-1)} \frac{W}{2J} \). In the above derivation one simplifying assumption is made, that the error probability of the previous stage \( \hat{P}_e^{(s-1)} \) is independent of any \( \kappa^{(0)}_{1,j} \), which is so for large \( J \).

### A.6 Soft-Decision Multistage Detector

The exact error probability of the OCDMA system using a soft-decision multistage detector is virtually impossible to calculate exactly since the soft-decision estimates of the bits depend on \( \Delta^{(0)}_1 \). Instead, an approximation to the error probability can be obtained using the average probability of the estimate falling on a particular quantized value for the previous stage. If the soft-decision multistage detector has quantization \( q \), an approximation to the average probability of bit estimate \( \hat{b}_k(s) = \frac{i}{q} \) for \( i = 0, \ldots, q \) can be easily obtained as follows.

Consider the detector which decides on the symbol sent based on the following algorithm

\[
\sum_{j=1}^{J} -\frac{1}{q} A_1(A_1^{(1)} - A_1^{(0)}) + N_1^{(0)} \ln \left[ \frac{\left( \frac{i+1}{q} A_1^{(1)} + (1 - \frac{i+1}{q}) A_1^{(0)} \right) A_1 + \hat{b}^{(0)}_1(s)}{\left( \frac{i}{q} A_1^{(1)} + (1 - \frac{i}{q}) A_1^{(0)} \right) A_1 + \hat{b}^{(0)}_1(s)} \right] > \gamma. \tag{A.19}
\]
The error probability associated with this detector is equal to the probability that stage $s$ of detection chooses $\hat{b}_1^{(0)} (s) = \frac{i}{q}$ rather than $\hat{b}_1^{(0)} (s) = \frac{i+1}{q}$ for $b_1^{(0)} = 1$, and visa versa for $b_1^{(0)} = 0$. For example, if $q = 1$ for the first stage, the expression in (A.19) is that of a correlation detector whose error probability is the probability that $\hat{b}_1^{(0)} (1) = 0$ given $b_1^{(0)} = 1$ or, equivalently, that $\hat{b}_1^{(0)} (1) = 1$ given $b_1^{(0)} = 0$. The algorithm in (A.19) extrapolates this idea to quantization $q$.

These probabilities for $i = 0, \ldots, q - 1$ can be used to estimate the moment generating function of the decision statistic for the soft-decision multistage detector. The moment generating function is calculated by using an assumption similar to that made for the 2-stage detector. It is assumed that the estimate $\hat{b}_k^{(0)} (s)$ is independent of the photon counts, so that the average can be taken over all quantizations. The final expression for the moment generating function for the 2nd stage is

$$M_{X_1 | L_1^{(0)}} (\zeta) \approx \sum_{i=0}^{q} M_{X_1 | L_1^{(0)}, \hat{b}_k^{(0)} (1) = \frac{i}{q}} (\zeta) \Pr \left( \hat{b}_k^{(0)} (1) = \frac{i}{q} \right), \quad (A.20)$$

where

$$\left( X_1 | L_1^{(0)}, \hat{b}_k^{(0)} (s) = \frac{i}{q} \right) = \sum_{j=1}^{J} -(A_{1,j}^{(1)} - A_{1,j}^{(0)}) \Lambda_1 + N_{1,j}^{(0)} \ln \left[ \frac{A_{1,j}^{(1)} + \hat{I}_{1,j}^{(0)} (s, i)}{A_{1,j}^{(0)} + \hat{I}_{1,j}^{(0)} (s, i)} \right], \quad (A.21)$$

and $\hat{I}_{1,j}^{(0)} (s, i)$ is the $s$-stage’s estimate of the interference for the case that the soft estimate of each user symbol is $i$. The interference estimate is thus a sum of contributions that assume a form $\frac{i}{q} A_{k,j}^{(1)} + \left( 1 - \frac{i}{q} \right) A_{k,j}^{(0)}$ for each user $k$. The expression $M_{X_1 | L_1^{(0)}} (\zeta)$ can be calculated using an equation as in (A.9) with $\hat{I}_{1,j}^{(0)} (s, i)$ instead of $\hat{I}_{1,j}^{(0)}$. 


A.7  Existence and Differentiability of Moment Generating Functions

For the derivation of the asymptotic multiuser efficiency, the existence and differentiability of $\mu_{Z_{\Lambda_1}}(\varsigma) = \frac{1}{\Lambda_1} \ln M_{Z_{\Lambda_1}}(\varsigma)$ is needed. The correlation detector, the approximate description to the multistage detector and the soft decision multistage detector, and the “known interference” detector (yielding the lower bound) have $\mu_{Z_{\Lambda_1}}(\varsigma)$ all of the form

$$\mu_{Z_{\Lambda_1}}(\varsigma) = \sum_{j=1}^{J} -(A_{1,j}^{(1)} - A_{1,j}^{(0)})\varsigma + (A_{1,j}^{(i)} + \kappa_{1,j}^{(0)}) \left( e^{(A_{1,j}^{(1)} - A_{1,j}^{(0)})\varsigma \ln[1+1/(\hat{\kappa}_{1,j}^{(0)})]} - 1 \right),$$  \hspace{1cm} (A.22)

where $\kappa_{1,j}^{(0)} = I_{1,j}^{(0)}/\Lambda_1$ and $\hat{\kappa}_{1,j}^{(0)}$ is its estimate which depends on which detector is considered. Since the intensity of all users is increased, the value of $\kappa_{1,j}^{(0)}$ depends on the near-far ratio and not on the value of $\Lambda_1$. The accuracy of the estimate $\hat{\kappa}_{1,j}$ is a function of $\Lambda_1$, but it is bounded to lie within $[0, \sum_{k=2}^{K} \Lambda_k/\Lambda_1]$ which is a function of the near-far ratio and not of $\Lambda_1$. Therefore and as is seen from (A.22), the expression for $\mu_{Z_{\Lambda_1}}(\varsigma)$ except for $\hat{\kappa}_{1,j}$ is not a function of $\Lambda_1$ and the limit $\mu_{Z_{\infty}}(\varsigma)$ must therefore exists; $\mu_{Z_{\infty}}(\varsigma)$ is also differentiable for all real $\varsigma$ since it is a sum of a linear and an exponential term. For the optimized single-user detector, using the separation theorem of detection [36], the decision statistic can also be written as above, with some value for the estimate $\hat{\kappa}_{1,j}$. Since this estimate is again bounded, the same argument as for the other detectors follows.
Bibliography


