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A qualitative theory of gas dynamics

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A Qualitative Theory Of Gas Dynamics

by

Matthew Robert Barry

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Abstract

A QUALITATIVE THEORY OF GAS DYNAMICS

Matthew Robert Barry

Within the realm of research toward the emulation of human intelligence, the problem of how to perform qualitative reasoning with computer programs has received considerable attention. The research field of qualitative physics focuses on the special problems of identifying the basic concepts of nature and the issues related to mechanizing inference about these concepts, so that we might build qualitative models of the physical world. These models are indispensable for providing machines with "common sense" inference capabilities to reason about and describe observations of their environment.

This dissertation contributes a qualitative theory of the gas dynamics domain. The theory comprises a qualitative representation of the dynamic behavior of gases, constructed upon a collection of fundamental process elements. These process elements affect the description of a dynamic scenario by manipulating the qualitative values of physical parameters that change in response to certain physical phenomena. The phenomena covered are those occurring in one- and two-dimensional flows of an ideal gas. Deployed in conjunction with an automated inference mechanism,
the models built from this theory constrain results to those situations validated by natural physical laws. These models provide qualitative reasoning and simulation capabilities to intelligent computer-aided design systems.

The presentation constructs a foundation within the computational framework of Qualitative Process Theory, and shows how an algebra of Qualitative Ratios, introduced herein, provides a convenient representation for describing physical change. The presentation and internal hierarchy of the theory parallels the engineering perspective of this domain, and encourages composition of the fundamental elements into larger, more complex, reasoning components. Several examples demonstrate the utility of the theory during the course of the presentation.
Acknowledgements

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Part I

Qualitative Reasoning

"Experience has shown that science frequently develops most fruitfully once we learn to examine the things that seem the simplest, instead of those that seem the most mysterious."

—Marvin Minsky [109]
Chapter 1

Introduction

One of the most exciting aspects of any scientific research endeavor is the investigation of potential application areas for the results; particularly when these results contribute to the solution of significant problems. The hybrid technologies currently emerging from research into "artificial intelligence" are finding many useful applications in the well-established fields of engineering, which traditionally were considered to be application platforms for the "natural" sciences. While this dissertation does not attempt to debate the issue of whether the pursuit of artificial intelligence is natural, it does condone contemporary efforts to apply the results of the research to engineering problems.

The application selected for consideration herein focuses on certain problems of engineering design and analysis tasks. The technology applied to solve these problems is qualitative reasoning. This dissertation contributes a theory that facilitates the construction of qualitative models for the physical (engineering) domain of gas dynamics. These computational models might enable intelligent computer-aided design systems to "reason" about the dynamic behavior of gases in a qualitative manner, thereby accelerating the design process. Alternatively, they might instill
reasoning capabilities in robotic systems that otherwise would be incapable of carrying out the complex quantitative calculations, and, more importantly, interpreting the results.

The presentation is divided into two Parts. Part I surveys the state of the art in the field of qualitative reasoning, and discusses how the problem-solving capabilities of this technology are being applied in modern computer-aided design and simulation systems. The variety of material presented in Part I is necessary to provide a detailed description of Qualitative Process Theory, which is the basis for the material in Part II. Chapters 1, 2 and 3 introduce the field of Artificial Intelligence (AI), explaining how the problem-solving techniques emerging from AI research relate to interesting problems in the field of engineering design. These Chapters illustrate the necessity for computational systems that reason about problems from a qualitative standpoint. Chapters 4 and 5 show how to perform qualitative reasoning, identifying issues associated with developing models and performing analysis by means of qualitative calculus and qualitative simulation. Chapter 6 presents the significant contributions made by each of several qualitative reasoning research efforts, and describes the three prevalent modeling ontologies. Chapter 7 shows how these (and other) ontologies work in a variety of existing models for physical domains. Chapter 8, which describes the Qualitative Process Theory ontology, provides the imperative disposition of material contained in the preceding Chapters. Finally, Chapter 9 identifies the need for a qualitative model of gas dynamics, and argues
that Qualitative Process Theory provides an adequate foundation for such a model.

Part II adopts the research and application ideas presented in Part I and proposes a new qualitative theory to investigate the potential application of these ideas in a heretofore unexplored domain. This new theory defines a vocabulary and a set of transition rules necessary for describing the dynamic behavior of gases from the perspective of qualitative physics. It identifies several fundamental processes inherent to the dynamic behavior of gases, and composes many of these fundamental processes into larger constructs that describe aggregate behaviors in more complex situations. Interpreting the behavior of the real world, then, is a matter of connecting these processes with observed behaviors. Several examples demonstrate the utility and consequences of the theory for computer-aided design and simulation activities.
Chapter 2

Intelligent Machines

The pursuit of artificial intelligence (AI) is inspired by the challenging proposition that machines can be made to mimic human reasoning. The enigma of the intricate human reasoning system itself poses the quandary. Minsky [108] defines AI as the field of research concerned with making machines do things that people consider to require intelligence.\(^1\) The goal of this research is to make computers intelligent, both to make them more useful and to help us understand the principles of natural intelligence [148].

Many researchers agree that intelligence is the ability to make logical decisions.\(^2\) Other traits generally ascribed to intelligence include

- dealing with abstract concepts,

- learning from experience,

\(^1\)The *Turing Test* [136] proposes a criterion for evaluating intelligent machines. In this test, a human interrogator converses with two entities through separate computer terminals. One entity is another person while the second entity is the machine. The interrogator must determine which entity is the person. The machine's goal, therefore, is to fool the interrogator into believing it is the person. If, after sufficient discourse, the interrogator cannot identify which entity is the person, then the machine passes the test.

\(^2\)Minsky, however, questions whether logic is the fundamental contributor to human intelligence [108].
• adapting to the environment,

• explaining a line of reasoning,

• introspection,

• exhibiting emotion.

A major theme of this thesis, the notion of common sense reasoning, addresses some of the issues associated with realizing many of these capabilities.

Whereas philosophers and psychologists derive comprehensive definitions for the various aspects of intelligence, engineers and scientists engage in efforts to endow machines (in the form of digital computers and robots) with the inclination to exhibit intelligent behavior. Some of these efforts focus on the development of knowledge-based systems to manipulate ill-structured concepts and heuristics, while others create expert systems that help us make decisions. Capturing the common sense nature of real scientific reasoning, however, is an elusive problem that has drawn considerable attention in recent AI research. The desire to amplify the (apparent) intelligence of machines has spawned an initiative to develop deeper models of physical domains, reasoning from the first-principles of nature to generate causal descriptions of how things work.

Expert systems that reason from first principles surmount many of the limitations inherent to those using more “shallow” knowledge. The inability of shallow expert systems to recognize and solve problems outside of their domain of expertise,
for instance, is due in part to their use of strictly-relevant compiled knowledge. Compiled knowledge provides detailed axiomatic statements without the support necessary for interaction, explanation, discovery, and theory formation. These systems usually cannot solve problems that are simpler than the ones they were intended to solve.
2.1 Common Sense Reasoning

"Common sense" embodies a collection of cognitive skills that most people share and consider sound, practical, judgement. Reasoning with common sense requires many cognitive skills, but requires little real knowledge. "Expertise," paradoxically, requires vast amounts of knowledge but only few cognitive skills. One of the focuses of AI research is to determine how this apparent simplicity of common sense emerges from the complex disposition of ideas. As we analyze how it is we solve problems, we may gain insight into how it is our minds work, and thereby identify and understand the fundamental knowledge that underlies physical intuition. Analyzing common sense reasoning involves developing an understanding of how things change and why changes happen.\(^3\) Once we gain insight into these activities, we can reconstruct our understanding with machines. McCarthy [101] provides the measure of satisfaction by stating that "a [machine] has common sense if it automatically deduces for itself a sufficiently wide class of immediate consequences of anything it is told and what it already knows."

The primary motivation behind the development of qualitative reasoning programs is the conjecture that these programs will lead us to a better understanding of the cognitive skills inherent to common sense reasoning. Models of qualitative

\(^3\)An interesting perception to consider for this thesis is that gases seem to have fewer "common sense" qualities than solids or liquids. One seldom observes the behavior of a gas first-hand beyond the air surrounding us, thus limiting our exposure to their fundamental characteristics. Nevertheless, engineers and scientists acquire a "feel" for gases (partially derived from a prior understanding of liquids) that permits reasoning about them in common sense manners.
reasoning attempt to identify and manipulate abstractions of exact quantitative information. A qualitative model of a physical domain enables the identification and simulation of its interesting behaviors. Qualitative reasoning permits an interactive approach to simulation which generates explanations as well as predictions. Furthermore, while the quantitative model can explore the behaviors in minute detail, a qualitative reasoning system can analyze the results.

Many view qualitative reasoning as a complement for quantitative reasoning rather than a replacement [14]. Quantitative techniques require precise computational models, limited initial values, and data that may be expensive (or impossible) to obtain. Qualitative reasoning techniques, on the other hand, are able to produce models that can reason from first-principles, handle incomplete problem definitions, generate all possible solutions, and explain the line of reasoning. Once the interesting outcomes of a problem have been identified, the more expensive quantitative analysis methods (if available) can analyze the detail. The idea is that qualitative reasoning techniques can preclude many iterations of a quantitative analysis that might, in fact, preclude the investigation of extreme situations.

2.2 Qualitative Physics

The goal of research in naïve physics is to provide a common sense representation of the physical world [72]. This research strives to determine what, if any, knowledge representation and reasoning techniques are suitable for solving common sense, or
novice-level, problems involving physical systems. The naive physics effort seeks systems that mimic human thought processes, including those that lead to incorrect results.

Determining the nature of problem solving in physics is itself a difficult problem [119]. Expert problem solving in physics is not so much a matter of searching for the right equations to calculate the value of a desired parameter, but more of a matter of decomposing the problem into smaller subproblems which can be worked forward to the desired states. Naive problem solvers generally work backward from the desired parameter, building equation sequences until the strategy encompasses the known quantities. Experts usually are unable to articulate the processes they employed to arrive at their starting point, which makes the problem of how to represent this expert knowledge an interesting challenge.

A commonly encountered branch of naive physics is the field of qualitative physics. Qualitative physics systems are interested in generating expert-level symbolic reasoning capabilities for the physical world. They strive to identify the characteristic distinctions of physical devices and analyze the laws governing their behavior in order to predict and explain this behavior without recourse to quantitative methods [27,142]. In building computer programs to carry out these qualitative physics simulations, the system requirements present many interesting problems [14,25,101]:

- The system should employ a language capable of representing complex physical devices and their behaviors. This language should account for all of the
functional descriptions that seem natural to engineers and scientists.

- The reasoning system should produce clear causal accounts of the behavior of physical mechanisms. This includes the capability to perform qualitative simulations of these behaviors and to explain the results.

- The granularity of the output should correspond directly to the granularity of the input. That is, the solution to a problem that was specified in intricate detail should itself be intricately detailed. Responses to problems posed at a high level should be rather general.

- The knowledge structures should support reasoning about the various problems encountered by scientists and engineers. Some of these problems lie in the categories of design, diagnosis, operation, theory development, and prediction.

- The reasoning system should be simpler than classical physics (i.e. avoiding differential equations) while retaining all of the important concepts (e.g. state, momentum, oscillation). Whenever necessary to enhance a model, however, the system should be able to incorporate quantitative information.

- The resulting common sense models should provide a foundation for the next generation of expert systems.

Three sorts of systems for reasoning about qualitative physics problems have
emerged. These systems share a basic algorithm for inferring qualitative behavior
[132]:

1. Determine the possible qualitative states, characterized by consistent sets of
   qualitative values for the variables.

2. Calculate the possible state transitions by analyzing the derivatives of the
   qualitative values and adhering to continuity conditions.

3. Identify the possible behaviors, i.e. sequences of state transitions.

These systems also share similar representations for qualitative values and derivatives. The qualitative calculus, described in Chapter 4, provides this representation. The differences between these systems appear in the modeling primitives necessary to describe behavioral relationships among the parameters and thereby calculate state transitions. These characteristic constituents are processes, components, and mathematical constraints.

Qualitative physics systems play an important role in recent applications of AI to engineering problems. These systems, as well as the kinds of engineering problems they address, are described in more detail below.

2.3 Knowledge Representation and Reasoning

In order to write computer programs that exhibit intelligent behavior there must be
some computational means for representing knowledge of the real world and drawing
conclusions from it. Programs manipulate this knowledge to solve problems and to learn new concepts, inferring extensions and revisions to the knowledge base as necessary. Although the human internal representation scheme remains a mystery, a wide variety of pragmatic approaches to solving knowledge representation problems have emerged, each with its own scheme, none of which completely satisfy the requirements for exhibiting intelligent behavior.

A knowledge base has both static and dynamic features [116]. The static features include a formal language whose expressions represent a given body of knowledge and assign meaning to it. The particular semantics chosen determines what knowledge must be explicit as well as what can remain implicit in the representation. The dynamic features include formalisms for reasoning about explicit and implicit knowledge, influencing inference operations, and deriving and discovering new expressions. These components of a knowledge base usually are versatile declarative statements that may be interpreted for use in another domain.

The features of a declarative knowledge base primarily consist of sentences ascribing properties and relations to the basic objects of the system. The basic objects are the individuals which together comprise the chosen domain or universe of discourse. Together with predicates, which designate properties and relations of the individuals, one can devise elaborate representation schemes with complex semantics. Though the representation is complex, the ideal knowledge base remains

---

4 Representation of implicit knowledge is one of the fundamental distinctions between a knowledge base and a data base.
decomposable into independent modules.

A common portrayal of semantic derivations are ordered or hierarchical graphs consisting of nodes and arcs, sometimes referred to as semantic networks. Such schemes simplify interpretation of concepts, defining relationships among classes or sorts of individuals, their place in the graph, and the meaning attributed to the arcs connecting each node. Nodes can describe one individual or it can be a set of sentences forming a complete description of the domain. Arcs may assign objects to be members of certain classes, establish a sequence of events occurring over time, or perhaps identify actions which change the state of the world.

A highly-structured descendant of the semantic network idea is the frame system. A frame is a collection of slots and associated values that describe some entity in the world [107,123]. Each instantiation of the frame to describe a particular entity fills the slots with detailed attributes or establishes links to attributes found in corresponding slots of another frame. In a sense, instantiating a frame to represent an entity in a certain situation attaches both general and specific meanings to that situation, since the frame itself carries certain knowledge (see Figure 2.1). Filling the frame slots with attributes for a particular instance only specializes that knowledge (Figure 2.2). This capability makes frame systems convenient for mechanizing common sense reasoning.

Considering all of these features and possibilities, one sees that part of the knowledge representation problem is domain-specific. One must identify what one
**Figure 2.1:** A simple class frame for representing chemical elements. Certain slots contain default values that may be inherited by specialized instances of the class in the absence of specific information. The contents of a slot can also include heuristic functions to guide value assignment. Slot values which are themselves frames provide another way to determine values through interconnected relationships.
Figure 2.2: A specialization of the Element class frame for the element Krypton. The frame slots have been filled with values specific to this instantiation. With the Stability slot unfilled, any request for this value would be inherited from the corresponding class frame slot.

believes to be the relevant objects, variables and parameters in the domain. Given these concepts, one must also identify the causal and mathematical relationships between them. These abstractions must be general enough to support common sense reasoning, anticipating application of accrued concepts to external domains, while simultaneously providing enough detail to solve complex physical problems [70,81]. This challenging dilemma is one reason that expert systems heretofore have been considered separately from other, more general, reasoning systems.

2.3.1 Logic

The AI community has long maintained that symbolic approaches to knowledge representation and reasoning are considerably more useful than mathematical approaches. The advantage of symbolic inference is that it can manipulate concepts without mathematical theory. Therefore, the most common formal substrate for knowledge representation is symbolic logic [76,118]. Logic is attractive because it
provides structured formal languages, with clearly-defined semantics, that happen to be well-suited for computer program implementation.\textsuperscript{5} Nevertheless, there are many issues regarding just how logic relates to common sense [111].

Logic systems apply a collection of \textit{inference rules} to a set of symbolic expressions, or \textit{facts}, in order to derive additional facts that logically follow from the original set. For example, the formula

$$\forall g. \text{Gas}(g) \rightarrow \text{Material}(g),$$

reads \textit{for any object g in the world, if g is a gas, then g is a material}. If the set of facts also includes the formula \text{Gas}(\text{Ne}) then we may deduce \text{Material}(\text{Ne}) by the \textit{modus ponens} inference rule

$$(\phi \land (\phi \rightarrow \psi)) \rightarrow \psi.$$  

Conversely, if the set also contains the formula

$$\forall l. \text{Liquid}(l) \rightarrow \text{Material}(l),$$

we may \textit{not} draw the conclusion that Ne is a liquid since

$$( (\phi \rightarrow \psi) \land (\varphi \rightarrow \psi) ) \not\Rightarrow (\phi \leftrightarrow \varphi).$$

An examination and comparison of the various forms of logical reasoning used in AI can be found in [66,76,93].

\textsuperscript{5}A rigorous treatment of logic is not warranted in this thesis. Therefore, I presume henceforth that the reader is familiar with symbolic logic syntax, connectors, quantifiers, and set theory. A formal treatment of symbolic logic may be found in [10].
2.3.2 Situation Calculus

The *situation calculus* [101] contributes a simple technique for modeling a dynamic world that employs many of the knowledge representation concepts outlined above. A *situation* is a set of facts that represent the complete state of the world at a given instant. A collection of instants represents the discretization of continuous time. *Actions* provide a means for simulating behavior by specifying what facts follow from certain situations. Actions add or delete facts within a database containing what is presently true in a situation. The sequence of situations obtained by applying actions becomes the record (or prediction) of system behavior.

Situation calculus is attractive because it is a variant of first-order predicate logic. This variant operates on labelled situations, defining *fluent* as those predicates that have situations as arguments. For example, if we let \([p]\) represent the qualitative value of the time derivative of \(p\), then denotes \([\dot{p}](s) = \text{inc}\) states a proposition that \(p\) is increasing in situation \(s\). The predicate *cause* produces changes in the situation through statements of the form

\[
\text{cause}(\pi)(s)
\]

where \(\pi\) is a propositional fluent and \(s\) is a situation. The effect of *cause* is to say

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6McCarthy and Hayes actually specify that the complete state description be *deducible* from a partial description provided as the situation. This consideration relieves the unattainable goal of specifying the complete description of the entire world at each instant.

7McCarthy avoids the second-order calculus interpretation of cause by introducing modal operators into the language [101]. These operators not only recover the first-order interpretation, but they also extend the reasoning capabilities to include concepts such as *necessity* and *possibility*. These concepts are important for filtering out implausible actions as well as permitting reasoning about procedures.
that situation \( s \) will lead to a situation that satisfies the statement \( \pi \). Considering the ideal gas equation as an example,

\[
\forall s \forall p \forall T \forall V \left( [\dot{p}] = \text{inc} \land [\dot{V}] = \text{std} \right) \rightarrow \text{cause}( [\dot{T}] = \text{inc} ) (s)
\]

means that whenever pressure \( p \) is increasing and volume \( V \) is constant, then temperature \( T \) will also increase. In such a situation, the reasoning system would add the assertion \([\dot{T}] = \text{inc}\) to the database of supported facts, noting that it was caused by the situation \( s \) in which \([\dot{p}] = \text{inc} \land [\dot{V}] = \text{std}\).

There are a few unfortunate limitations to situation calculus. Since each situation is spatially unbounded, this representation requires that actions specify axioms for each fact that changes, as well as each fact that does not change.\(^8\) This extra information obscures the important mutations and dependencies in state transitions, thereby complicating explanation or diagnosis efforts. Another limitation lies in the representation of temporal qualities in state variables. Since there is no explicit mechanism for handling time, it is difficult to reason about time-related activities such as duration or delay.

### 2.3.3 Constraint Satisfaction

Problems of **constraint satisfaction** involve discovering a world state that satisfies a given collection of constraints. Computer programs that employ constraint sat-
isfaction methods have been applied successfully to several categories of problems, including computer-aided design, image understanding, equation solving and theorem proving. They can be found in several AI programs, such as the General Problem Solver [117] and various Truth Maintenance systems [28,39] (see Section 2.3.4). Constraint satisfaction techniques are especially useful in qualitative physics systems, where the effect of changing one parameter may propagate a ripple of indirect changes throughout the entire domain.

A constraint is a sentence that expresses a desired relationship between one or more objects. A constraint satisfaction problem consists of a set of constraints in which some of the objects have unknown values. A constraint satisfaction technique determines values of the objects that make all of relationships in the set true [91].

The constraint satisfaction technique usually is part of a problem solving system comprised of rules, which embody the constraints, and a control strategy. The control strategy applies the technique, which usually involves some form of search algorithm, to the set of rules. Ideally, the control strategy is completely independent of the kinds of rules with which it operates.9

One popular control strategy is the Waltz filtering algorithm [138]. This algorithm strives to minimize the number of solution possibilities to be investigated by applying constraints as soon as they are identified. In so doing, it requires

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9 For complex problems, however, this independence has yet to be accomplished. Indeed, the plausibility of this separation of knowledge from inference engine is a key presupposition in common sense reasoning research.
1. Form a queue consisting of all states.

2. Until the queue is empty:
   
   1. Remove the first element from the queue, and call it the current state.
      
      (a) If the current state has never been visited, create a list of values for it consisting of all possible values in the local context. Note that a state change has occurred.
      
      (b) If any value from the current state's list is incompatible with a value in any neighboring state's list, eliminate that incompatible value from the current state's list. Note that a state change has occurred.
   
   2. If a state change has occurred, for each neighboring state that has a list but is not in the queue, add that state to the front of the queue.

Figure 2.3: The Waltz filtering algorithm for determining value assignments with constraint satisfaction. This conditional depth-first search strategy uses information contained at neighboring states to invoke local constraints as soon as they become applicable. This process helps to minimize the number of solution possibilities throughout the search. Waltz originally developed this algorithm to filter labels for the edges and vertices of shapes appearing in simple line drawings of scene analysis problems [138].

minimal backtracking and converges to a solution rapidly. The solution may be unique, wherein only one possibility is consistent with the constraints, or it may be ambiguous, wherein several possibilities are consistent. In qualitative simulation applications, one hopes that filtering will reduce the number of ambiguous situations. Most of the qualitative simulation systems described in Chapter 5, especially the QSIM algorithm, employ filtering algorithms. Figure 2.3 illustrates the complete algorithm.
2.3.4 Nonmonotonic Reasoning

Many aspects of common sense reasoning are considered to be defeasible or non-monotonic [102,107]. A nonmonotonic reasoning system draws tentative conclusions, based upon rationality, which may later become invalidated. Formally, a logic is nonmonotonic if it violates the following condition of monotonicity for any sets of premisses $S$ and $T$:

$$S \subseteq T \rightarrow \{A : S \vdash A\} \subseteq \{A : T \vdash A\},$$

where $A \vdash S$ means that the set of premisses $S$ is derivable from the premisses in $A$. This condition in practice states that contradictions to the premisses (usually referred to in this context as beliefs) in $S$ cannot be derived from $T$. Considering the sets $S$ and $T$ as representations of world states, contradictions occur when a fact that is believed in $S$ is not believed in $T$.

For example, consider the problem of spraying water from the nozzle on an ordinary garden hose connected to a open spigot. If we first assume that a water supply is present at the nozzle, then water should emerge from the nozzle when depressing the trigger. If we observe, however, that water does not emerge from the nozzle, we conclude that our basic assumption that a water supply is present is incorrect. To remedy this situation, we employ a model of the water flow path and retract our assumption that the spigot valve is open. If however we observe that the spigot is indeed open, we instead retract our assumption that a city water
supply to the spigot exists. Alternatively, we might retract an assumption that the hose and valves do not leak, or that the nozzle operates properly.\textsuperscript{10}

Truth Maintenance Systems

One of the most popular techniques for keeping track of these sort of assumptions and conclusions used in deduction processes is known as \textit{truth maintenance}. Truth maintenance systems (TMS) provide practical ways to deal with inconsistencies in knowledge bases. They do this by keeping track of the logical support for each conclusion in the knowledge base. When subsequent information invalidates a previous conclusion, the TMS is able to backtrack the inferencing sequence to that conclusion and start over. Usually, the TMS performs these tasks as an agent cooperating with an external problem solver, providing it with information about what facts it can believe.

There are three common approaches to implementing truth maintenance.\textsuperscript{11} Systems that label sentences in the knowledge base by the hypotheses used to establish their truth value are known as \textit{assumption-based} truth maintenance systems (ATMS) [28,31,33]. Systems that label the sentences according to the axioms used in their derivation are known as \textit{justification-based} truth maintenance systems (JTMS) [39,69]. Finally, systems that maintain the derivation as well as global satisfaction

\textsuperscript{10}The ability to perform this sort of diagnosis and explanation by machine is characteristic of what the qualitative physics endeavor is after.

\textsuperscript{11}While these algorithms technically are for \textit{monotonic} systems, they nevertheless apply to non-monotonic systems as well.
of logical constraints are known as logic-based truth maintenance systems (LTMS) [100]. Each system maintains a special network of nodes establishing intersentence dependencies. Each node represents a potential belief for the problem solver, and as such the TMS determines whether the sentence can be believed and labels it as in or out of the current set of beliefs.

Extensions of the basic TMS mechanisms have been investigated for various reasoning problems, while practical applications have been developed for problems dealing with diagnosis, processes, actions, and so on [29,30,32,40,114]. Chapter 8 describes how the ATMS contained in the Qualitative Process Engine (an implementation of Qualitative Process Theory) manages axioms used to model physical systems.

Formal Approaches

Various other techniques for nonmonotonic reasoning are also being developed in the AI community. A brief overview here of a few of the more formal approaches (default reasoning, circumscription, autoepistemic reasoning, and modal logic) serves to describe interesting ideas for comparison to the TMS approach.

Default reasoning provides reasonable values for inference in the absence of complete information. The reasoning system retracts these rational deductions if contradictory evidence becomes available. Oftentimes statistical occurrence information

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12There has also been some effort expended in attempting to unify these approaches [8,105].
or some other measure of assumed risk can be ascribed to defaults, which allows
the reasoning system to evaluate support for its deductions [4,35,122,120].

Circumscription techniques employ a circumscription axiom in the representa-
tion that is used to extend the database to include all objects that satisfy the axiom.
The circumscription axiom determines the minimal database extension of the logical relation being circumscribed, thereby proving that the objects satisfying the relation are the only ones that do. Circumscription techniques are unique in that they are one of the few classes of nonmonotonic reasoning techniques that employ the language of classical logic [92,103,104].

The basis for autoepistemic reasoning is that complete knowledge of a certain aspect of the world is known. Inferences made using incomplete representations under autoepistemic reasoning cannot be invalidated because of this assumption of complete information. This form of reasoning is nonmonotonic in the context of available deduction rules, rather than derivable facts [64,82,83,112].

Finally, modal logic approaches extend the propositional logic formalism to include reasoning about possibility and necessity. Propositions that are possible are held true in at least one possible world. Propositions that are necessary, on the other hand, must be held true in all possible worlds. This permits assertion of a world in which certain propositions are believed, in order to investigate the consequences of holding that belief [106,113].
Chapter 3

Artificial Intelligence and Engineering

Engineering problems contain a wide spectrum of issues that entails existential components (physical science and mathematics) alongside conceptual components (philosophy and psychology). Dym and Levitt [41] identify two sorts of engineering problems, each containing certain tasks. In classification problems the tasks involve deriving solutions from given facts and data. These problems include diagnosis, in which one generates an explanation for a problematic behavior; interpretation, in which one categorizes observations so as to understand their meaning; and monitoring, which involves tracing and predicting the behavior of a system. In formation problems the tasks involve creating something using guidelines and constraints. These problems include planning, wherein one generates a sequence of actions for achieving some objective; and design, in which one either creates an object or a plan for manufacturing that object.

Since AI research investigates how intelligent agents solve diverse problems, engineering problems provide a logical, useful, and challenging domain for applications.
Forbus [57] explains that engineering problems are interesting to the AI community because understanding the tacit knowledge employed in engineering reasoning may provide insight into common sense theories of space, time, and quantity. Similarly, large engineering problems involve integration of a variety of knowledge from broad domains, exercising competence and completeness issues which might evade attention in smaller problems. Finally, engineering results rely on empirical knowledge and sound theoretical bases, enhancing the potential for developing testable AI systems.

In contrast, Forbus also notes that AI techniques are interesting to the engineering community. For example, as the complexity of design problems increases, the number of applicable constraints increases even more. These sorts of problems are addressed by constraint satisfaction techniques, which have emerged from AI research as efficient methods for adhering to limitations and weighing alternatives (as described earlier in Section 2.3.3). AI techniques for modeling physical systems also play useful roles in intelligent tutoring systems [5], which adapt to problem solving situations and experience while propagating expertise. Furthermore, the problem solving capabilities provided by traditional quantitative reasoning techniques are often inadequate for engineering problems [14,70]. Precise quantitative models are not always available, perhaps because the domain is ill-understood or simply because the relationship between two or more variables is unknown. Even in situations where quantitative models are available, parameter values may not exist
or may be difficult or expensive to obtain, while results of quantitative simulations still require qualitative interpretations. Moreover, quantitative models predict only one of the possible physical behaviors, while qualitative models generally predict all of the possible behaviors.

AI solutions also contribute explanations of their results.¹ These explanations are critical to intelligent tutoring, automated diagnosis, design evaluation, and theory generalization systems. In some AI systems the explanation provided is a complete narrative of the reasoning process. In others, the explanation is an enumerated list of the facts, assumptions and inferences along with supporting rationale for each conclusion.

Generation of a reasonable explanation requires a "deep" model of the system. A deep model contains implicit knowledge about features of the system that are not observable. In contrast, a "shallow" model reasons only about directly observable features. Deep models convey an understanding of the underlying mechanisms responsible for the behavior of the system. This understanding is an integral part of an intuitive explanation of behavior. Shallow models, on the other hand, may be able to convey a certain feeling of expertise, but they are unable to explain how that expertise may have been acquired. Without suitably supported sets of justifications,

¹Numeric differential equation solutions certainly are incapable of explaining their line of reasoning. Not only are the equations themselves prederived (perhaps empirically), but their sequence of application is precompiled without regard to rationale other than program documentation. The equations possess no "knowledge" of their own content, purpose, or application, and therefore are incapable of introspection and explanation.
shallow models cannot perform analogical reasoning, learn new concepts, or adapt to their environment.

The ability of AI systems to generate explanations about how a device works is also important for design and diagnosis tasks. Performing a design task involves predicting the behavior of a proposed mechanism and modifying it such that it matches the required behavior. There is a large amount of qualitative knowledge involved in constructing solutions to design problems and in interpreting results of simulations. Since much of the design process is non-algorithmic, the declarative approaches of AI systems are attractive. Similarly, performing a diagnosis task involves comparing the observed behavior of a device with its intended behavior, then generating hypotheses concerning abnormal behavior of the individual components.

AI research addresses, in one way or another, all of the engineering tasks identified above. The qualitative physics field in particular concerns itself with each of these tasks. Viewed from an engineering perspective, qualitative physics considers how qualitative reasoning techniques can be employed in computer-aided design (CAD) environments. Although the CAD environment is most generally related to design problems, it also provides an environment for considering other engineering problems, such as diagnosis or control.
3.1 Design

Unquestionably a rich environment for AI applications, the problem of engineering design has attracted considerable attention from researchers in the field of qualitative reasoning. Primarily, this is because of the wide variety of interesting subproblems contained therein. Design problems involve materials, geometry, environmental impact, human factors, manufacturing constraints, product lifetimes, candidate evaluation, and a variety of other modeling considerations. Moreover, there are several kinds of models employed in this formation task, including first principles, phenomenological models, analytic models, numerical models, and heuristics [41].

Analyzing how it is that expert engineers synthesize designs, many researchers attempt to model the design process itself [7, 38, 68, 95, 137]. One can think of the design process as having several phases. The early phases involve decomposition of the design task subproblems and frequent iteration based upon the results of constraint applications and simulations. A model of these phases can itself contribute to the design process by leading the user through the usual sequence of considerations. Subsequent phases, such as detail design or analysis, require fewer of the symbolic problem-solving approaches taken by AI while stressing increased numerical detail. Another approach to modeling design is to employ case-based reasoning, which draws analogies between prior design experiences and the current problem, or selects a prototype from a set of generic descriptions.

Coyne et al. [16] offer a phase-independent design model consisting of design
designations, a vocabulary, knowledge, and interpretations. The design descriptions are the results produced from the application of design decisions. The vocabulary defines the manipulated elements comprising the design. The design-specific knowledge relates the vocabulary and interpretations. The interpretations are either goals or requirements (intended interpretations) that drive the design decisions, or an analysis or evaluation of an existing design (actual interpretations).

3.2 Computer-Aided Design

A CAD system is an interactive design, modeling and simulation tool articulated as a collection of computer programs. These sometimes elaborate systems permit interactive problem solving within anticipated categories of problems. The programs therefore require that the engineer determine beforehand which physical laws are important for each category, thereby influencing the results even before the model is complete. When a new problem appears outside of the predetermined categories, we engineers generally develop a new program to model it.

Numerical simulation techniques have become increasingly important while the computer emerges as a significant tool for fluid dynamics research. In fact, computational fluid dynamics models have made possible the solution of problems that previously defied theoretical analysis and experimental simulation [115]. These developments have led to rapid technological progress in the calculation and understanding of multiphase flows, multidimensional compressible viscous flows, turbu-
lence, and chemically reacting flows.

Most phases of the design process, such as detail design, production design, prototyping and testing, benefit from conventional CAD systems [135]. Unfortunately, conventional CAD systems do not support the conceptual phase of the design process. *Intelligent CAD* has emerged as an AI application area intending to develop systems that reduce the time and effort expended on design problems by streamlining tasks such as data generation, specification evaluation, and constraint application, while assisting in the exploration of alternatives, detecting mistakes, and rationalizing design concepts. One of the most interesting AI contributions to CAD will emerge from the *conceptual* design phase, since this phase requires creativity, ingenuity, and certainly common sense. This is the first phase the designer encounters, and it usually accounts for a majority of the design process time.

### 3.3 Intelligent Computer-Aided Design

An intelligent computer-aided design (ICAD) system is, essentially, a CAD system that exhibits many of the attributes ascribed to intelligent machines. Such a system assumes a larger role in the design process than conventional CAD systems, since it understands the design objectives, provides advice, and learns from design experiences.\(^2\) The vision of ICAD system developers is to capture engineering

\(^2\)Considering that these attributes include the original goals of conventional CAD, one may argue whether there is a distinct separation between conventional and intelligent CAD systems, and hence whether certain extents of ICAD have already been achieved [94]. This argument hinges on one's
knowledge (as well as the techniques for manipulating this knowledge) in computer programs [13,57]. This vision accounts for contemporary metaphorical expressions describing systems as a "designer's apprentice," "expert designer" and "intelligent design assistant." These systems package and distribute expertise, amplifying the designers knowledge and contributing to the quality of engineering design work [94].

Given these visions, we can recount the intelligent machine desiderata as an enumeration of desirable attributes for an ICAD system.\(^3\)

- The system should be able to simulate physical system behavior using mixed quantitative and qualitative models, and use this simulation to predict what may happen in a given situation. Similarly, the system should be able to specify how to achieve a desired situation. Ideally, the system should be able to plan an entire course of action for achieving a set of goals.

- The reasoning system should be able to describe situations. This includes interpretation of a current situation, as well as explanation of how a predicted situation may occur or how a previous situation may have transpired.

- An eclectic suite of problem solving techniques should be available to the system, and the system should be able to choose the appropriate technique. This includes combining various quantitative and qualitative reasoning paradigms

\(^{3}\)Many seminal ICAD applications are described in the AI and engineering design literature [16,67,110,131,143].
to exercise the solution technique best suited for a given problem.

- The reasoning system should be able to perform comparative analyses. This involves comparing one situation against another, evaluating each against a set of common criteria, and selecting the situation that best satisfies these criteria.

- The system knowledge should include rich design models as well as tacit information.

- The system should be able to provide information about the teleology of each component in the device, thereby providing diagnostic descriptions in terms of the intended purpose of each component.

- The reasoning system should be able to enhance its own capabilities through learning. This requires such capabilities as identifying important information, conducting designs by drawing from experience, and assimilating new technologies and practices into the knowledge base.

The theories and reasoning ontologies described herein concern all of these attributes except the last.

There are many knowledge representation and reasoning problems encountered in realizing these programs. AI solutions to engineering problems must apply "knowledge" of fundamental principles of physics, chemistry, biology, etc., to a stated problem. The problems require varying levels of abstraction, specification
of physical states in terms of object parameters, modeling changes in dynamic systems, and adhering to real-world constraints. The reasoning system captures this knowledge in a representation that is suitable for manipulation by an inference engine. The knowledge may exist as objects, frames, rules, constraints, graphs, procedures, or whatever capacities an expert might use to develop a solution to the same problem. The approach most amenable to AI applications is to find a representation of the domain that manipulates the lowest-level principles, such as natural physical laws, in a way that seems like common sense. The qualitative reasoning systems described in Section 6.1 present three paradigms for attacking these problems.

\footnote{We refer here only to the \textit{expert system} solutions to engineering problems. In the naïve physics domain one must also consider faulty or incomplete reasoning, so that the computer program makes the same mistakes made by a novice.}
Chapter 4

Qualitative Calculus

The qualitative calculus employed in qualitative reasoning systems is based upon the notion of imprecise quantities. These values comprise a finite set of ordered intervals of the real number space, representing regions of uniform behavior in the physical parameter. Operations upon these qualitative values require an arithmetic of intervals, which, because of the simplifications involved, does not always provide unique results. This Chapter outlines many of the concepts (and a few of the implications) of the qualitative calculus abstraction.¹

4.1 Qualitative Variables

The possible values a qualitative variable can assume are the elements of a finite set of subinterval names dividing the continuous domain of the variable.² In the simplest applications, these subintervals consume the entire domain and are nonoverlapping. The qualitative value of a variable is the name of the subinterval in which the actual quantitative value lies. All quantitative values within a given

¹More detailed discussions of qualitative mathematics may be found in [26,75]. An analytical comparison of the mathematics used in various qualitative reasoning systems is provided in [132].

²The treatment and notation used herein follows that of Struss [132]. This notation is common, though not universal, in the literature.
interval assume the same qualitative value. The collection of intervals is the \textit{quantity space} \( Q \) for the variable. The quantity space provides a finite symbolic lexicon useful for the qualitative representation of ordinal relations, with a mapping from real numbers to qualitative intervals

\[ q : \mathbb{R} \rightarrow Q. \]

The resulting interpretation is a mapping back into the real numbers\(^3\)

\[ p : Q \rightarrow \mathcal{P}(\mathbb{R}) \]

such that all real numbers have an interpretation

\[ \forall x \in \mathbb{R} \ x \in p(q(x)) \]

and that these interpretations are exclusive

\[ \forall a,b \in Q \ a \neq b \Rightarrow p(a) \cap p(b) = \emptyset. \]

The abstraction also requires that the set \( Q \) be (at least partially) ordered, i.e.

\[ \forall x,y \in \mathbb{R} \ x < y \Rightarrow q(x) \leq q(y). \]

For problems containing multiple variables, each with a unique mapping function

\[ q_i : \mathbb{R} \rightarrow Q_i \]

and interpretation function

\[ p_i : Q_i \rightarrow \mathcal{P}(\mathbb{R}), \]

\(^3\mathcal{P}(\mathbb{R})\) represents the power set of \( \mathbb{R} \) (i.e. \( \mathcal{P}(\mathbb{R}) \equiv \{ X \mid X \subseteq \mathbb{R} \} \)).
the mapping of solutions becomes

\[ q' : P(\mathbb{R}^n) \to P(Q^n) \]

and

\[ p' : P(Q^n) \to P(\mathbb{R}^n), \]

where

\[ Q^n \equiv Q_1 \times Q_2 \times \ldots \times Q_n. \]

Each element of \( Q_i \) represents a portion of \( \mathbb{R} \) in which the variable mapped to that element possesses significant attributes or interesting characteristics. Therefore, the behavior of the variable usually changes significantly as the value crosses from one portion to the next. The boundary values separating each interval are called \textit{landmark values}. There may be any number of landmark values within the continuous domain, albeit those states distinguished by significant transitions contain the most intuitive information. Using boundary elements requires limiting \( \mathbb{R} \) to a subset \( B \) such that all intervals are finite:

\[ B \subset \mathbb{R} \cup \{-\infty, \infty\}. \]

The general notation for an interval is

\[ (a, b) \equiv \begin{cases} (a, b) & \text{if } a, b \in B, a < b \\ [a, b] & \text{if } a, b \in B, a = b \end{cases} \]
Selecting intervals delimited by the quantity space $Q$ leads to an ordered subset $E_B$ of $B$ whose elements consist of neighboring intervals and landmark values, i.e.

$$E_B \equiv \{[b, b] \in I_B\} \cup \{(b_i, b_j) \in I_B \mid \neg \exists_{b \in B} b < b < b_j\},$$

where the set of all possible intervals $I_B$ is defined by

$$I_B \equiv \{(a, b) \mid a, b \in B, a < b\} \cup \{[a, b] \mid a \in B \setminus \{-\infty, \infty\}\}.$$ 

The simplest (and most common) quantization of a continuous real-number domain is the special case of a three-element set $E_B$ consisting of the intervals $(-\infty, 0), (0, 0),$ and $(0, +\infty)$. These elements are named negative, zero, and positive, which are often labeled by the symbols $\hat{\imath}$, $\hat{0}$, and $\hat{\dagger}$. When using these symbols the special sign operator $[]$ usually replaces the function $q()$. The variable $[v]$ may be assigned to any of the elements in the quantity space $Q_\Delta$

$$[v] \in Q_\Delta \equiv \{\hat{\imath}, \hat{0}, \hat{\dagger}\}$$

using the mapping

$$[v] \approx \begin{cases} 
\hat{\dagger} & \text{if } v > 0 \\
\hat{0} & \text{if } v = 0 \\
\hat{\imath} & \text{if } v < 0
\end{cases}$$

Another quantization might be defined in the temperature space of a material. In this case, one might assign the qualitative value solid, liquid, or gas to the variable
using a mapping such as
\[
q(v) \approx \begin{cases} 
\text{solid} & \text{if Temp}(v) < \text{Melt}(v) \\
\text{liquid} & \text{if Boil}(v) > \text{Temp}(v) > \text{Melt}(v) \\
\text{gas} & \text{if Temp}(v) > \text{Boil}(v)
\end{cases}
\]
where Temp(v) represents the temperature of v, while Melt(v) and Boil(v) provide the landmark values of v's melting and boiling temperatures, respectively. In this example, the quantity space
\[
Q \equiv \{\text{solid, liquid, gas}\}
\]
corresponds to the set of elementary open intervals
\[
E_B \equiv \{(-\infty, \text{Melt}(v)), (\text{Melt}(v), \text{Boil}(v)), (\text{Boil}(v), \infty)\},
\]
for which the closed intervals \([b, b]\) are not defined.

### 4.2 Qualitative Arithmetic

Binary operations within \(I_B\) can be defined such that they mimic the corresponding operation in real arithmetic. For example, a qualitative arithmetic operator \(\oplus\) takes two qualitative values into another qualitative value \(\oplus : Q \times Q \rightarrow Q\) by defining
\[
I_1, I_2 \in I_B I_1 \oplus I_2 = (a, b) \oplus (c, d) = (a + c, b + d)
\]

Tables 4.1 through 4.6 provide some arithmetic rules for the simple three-interval qualitative domain. Notice in the tables that some of the results are undetermined.
values, denoted by a ? symbol, indicating that the quantization of the domain introduces ambiguity into the calculus.

The suitability of these equivalent qualitative arithmetic operations is dependent upon the selection of the set of intervals from $I_B$, and even for the restricted set of boundaries $B$ the usual properties of expressions do not normally hold. For example, Struss [134] shows that any algebra of a finite number of intervals can
Table 4.4: Qualitative equality: \([x] \approx [y]\). The values T and F represent the logical values true and false.

\[
\begin{array}{c|cccc}
\approx & \hat{=} & \hat{0} & \checkmark \\
\hline
\hat{=} & T & F & F \\
\hat{0} & F & T & F \\
\checkmark & F & F & T \\
\end{array}
\]

Table 4.5: Qualitative inequality: \([x] > [y]\). The values T and F represent the logical values true and false.

\[
\begin{array}{c|cccc}
> & \hat{=} & \hat{0} & \checkmark \\
\hline
\hat{=} & F & F & F \\
\hat{0} & T & F & F \\
\checkmark & T & T & F \\
\end{array}
\]

Table 4.6: Qualitative negation: \([-x]\).

\[
\begin{array}{c|cc}
[x] & -x \\
\hline
\hat{=} & \checkmark \\
\hat{0} & \hat{0} \\
\checkmark & \hat{=} \\
\end{array}
\]
have no additive inverse, i.e.,

\[ [a] \approx [b] \oplus [c] \not\Rightarrow [a] \ominus [c] \approx [b], \]

so the notion of distributivity does not hold. These seemingly negative features of interval mathematics disturb our expectations for algebraic composition rules in symbolic reasoning systems. That is, rather than having one expression that we manipulate into an alternative form, we might instead be forced to carry around several related expressions that have been carefully rederived for application in foreseeable circumstances.

The expressions of qualitative arithmetic contain only qualitative variables. All constants drop out except for the signs of coefficients. For example, consider the equation

\[ x + cy = z \]

where \( x, y \) and \( z \) are variables. If the constant \( c \) is 0, then the equivalent qualitative equation is simply

\[ [x] \approx [z]. \]

If \( c > 0 \), the sign of \( y \) remains positive, resulting in

\[ [x] \oplus [y] \approx [z]. \]

If instead \( c < 0 \), the qualitative equation becomes

\[ [x] \ominus [y] \approx [z]. \]
Table 4.7: Qualitative equivalent of line equation: $y = mx + b$. For even this simple equation, almost half of the solution possibilities are ambiguous.

As an interesting example of ambiguity, consider the equation for a straight line in a plane

$$y = mx + b.$$ 

This equation has a variety of qualitative equivalents that depend upon the relative values of the constants $m$ and $b$. Table 4.7 shows that there are four ambiguous cases for this simple equation when $m$ and $b$ are non-zero. Since qualitative reasoning systems do not directly consider the magnitude of these constants, they must employ some additional information or reasoning technique to resolve these uncertainties.

The discussions below outline some of these techniques. This is the version of qualitative arithmetic used by the ENVISION application described later. However, for any set of boundary elements other than $B = (-\infty, 0, \infty)$, the above definition of $\oplus$ (and companion operators) carries too many restrictions, such as fixed-length intervals. Instead, for an arbitrary set $B$ it is more convenient to define an operator

$$\ominus' : I_B \times I_B \to I_B$$
by the smallest interval in $I_B$ containing $I_1 \oplus I_2$:

$$\langle a, b \rangle \oplus' \langle c, d \rangle \equiv (\max\{x \in B \mid x \leq a + c\}, \min\{x \in B \mid x \geq b + d\}).$$

This definition permits intervals of arbitrary length, but sacrifices associativity. For example, consider the illustration provided by Struss [132]: If the set of boundary elements is

$$B = \{-\infty, \ldots, 0, 6, 8, 13, 16, 19, 30, \infty\}$$

then the expression

$$( (6, 8) \oplus' (6, 8) ) \oplus' (8, 13) = (8, 16) \oplus' (13, 30)$$

$$= (16, 30)$$

whereas the alternative ordering results in

$$(6, 8) \oplus' ((6, 8) \oplus' (8, 13)) = (6, 8) \oplus' (8, 13)$$

$$= (19, \infty)$$

Despite this penalty (attributable to "rounding" error), this version of qualitative arithmetic is employed by the QSIM and QPT systems described below.

### 4.3 Qualitative Derivatives

In order to evaluate the dynamic behavior of a system it is necessary to reason about how values change with respect to each other. This requires the introduction of qualitative derivatives. The same rules for generating qualitative arithmetic equations apply to generating qualitative differential equations.
Qualitative derivatives generally take on one of the three values $\check{\cdot}, \hat{\cdot}, \check{\check{\cdot}}$, meaning that the parameter is decreasing, steady, or increasing with respect to another parameter. The qualitative derivative of variable $x$ with respect to variable $y$ is written $\left[ \frac{dx}{dy} \right]$. Similarly, the partial derivative is written $\left[ \frac{\partial x}{\partial y} \right]$. Most derivatives in qualitative physics are taken with respect to time $t$, for which we use the special notation $[\dot{x}]$ as a substitution for the fractional form. Using the three-interval quantization introduced above, we can say that over time, $x$ is increasing if $[\ddot{x}] \approx \check{\check{\cdot}}$, steady if $[\ddot{x}] \approx \hat{\cdot}$, or decreasing if $[\ddot{x}] \approx \check{\cdot}$.

The importance of derivatives over time lies in their contribution to prediction problems. If $[x] \approx \check{\check{\cdot}}$ at one time point while $[\dot{x}] \approx \hat{\cdot}$, then we can predict that $[x] \approx \check{\check{\cdot}}$ at the next time point. Time derivatives enable the reasoning system to predict that a parameter will reach certain landmark values, thereby guiding the simulation through new and interesting value changes.

4.4 Ambiguity

Separating the real number domain into qualitative regions introduces ambiguity into qualitative representations. This ambiguity affects the physical state description of each component as well as the relationships between the states of several interacting components. Ambiguous quantities can be introduced as a result of the

\footnote{These three values can describe physical effects in a majority of problems. One can imagine cases, particularly when considering higher-order derivatives, where the quantity space can subdivide further into regions delimiting asymptotes, extrema, inflections, etc.}
application of qualitative arithmetic rules, or as prespecified “unknown” values.

There are three problems that arise when dealing with ambiguity in qualitative simulations. These problems are *simultaneity*, *ambiguous effect*, and *unknown cause* [144]. The problem of *simultaneity* concerns how to determine qualitative values resulting from intercomponent effects which are interdependent. The problem of *ambiguous effect* is the direct result of the quantization of physical space. It may be possible to collect all of the necessary known causes for a rule, but the effect of the rule may not be deducible. The problem of *unknown cause* is the result of propagating a simulation solution forward from cause to effect. After propagation, it is difficult to determine exactly which antecedants combined to result in the current state. Every qualitative simulation described in Chapter 5 considers different ways to reduce ambiguity.
Chapter 5

Qualitative Simulation

Simulations exercise models in various situational contexts to demonstrate and predict physical system behavior. Qualitative simulators carry out these demonstrations using knowledge-based systems approaches. They attempt to describe the behavior of the physical system in terms of qualitative characteristics (increasing, decreasing, oscillating, melting, boiling, and so on) of the variables changing over time. This allows us to use incomplete or “fuzzy” models and view qualitative reports, and to specify problems (and solutions) at various levels of abstraction [46].

The ability to perform qualitative simulation, therefore, is the hallmark of qualitative physics. It permits us to exercise computational models, defined within the bounds of physical paradigms, that predict the behavior of observable phenomena. In so doing, it contributes to the establishment of a theory of common sense.

The fundamental difference between quantitative and qualitative simulations is in the mathematical structure of the models. While quantitative models employ the elaborate calculus of continuous systems, qualitative models employ qualitative calculus. Constructing a model under either paradigm requires considerable effort.
Since there are various ways to model continuous parameters in qualitative calculus, qualitative models carry the additional burdens of selecting applicable quantity spaces for each parameter (i.e. replacing \( \mathbb{R}^n \) with \( Q^n \)) and choosing the interparameter constraint relationships. Nevertheless, the broad application advantage enjoyed by qualitative models warrants that additional attention be paid to definition and implementation of the parameter and concept abstractions. Models with careful and insightful construction can prove to be particularly compliant problem solving tools.

Another difference between these opposite extremes of the simulation spectrum is in the techniques used to describe behavior. Qualitative models assume that describing physical behavior is a matter of specifying interval transitions among characteristic parameters over time. Qualitative simulations therefore proceed on the basis that something interesting happens between successive steps. Usually these steps correspond to the events occurring when a qualitative value reaches (or leaves) a landmark value. Numerical simulations, on the other hand, generally proceed in increments corresponding to the advancement of time, without guarantee that something interesting happens during that interval.

Qualitative simulation is particularly useful in ICAD systems: it provides a way to evaluate designs using only functional requirements; it is able to detect errors in early designs that might prove more expensive to repair after detailed work has already been performed; and it provides a way to perform simulations without having
to assign precise values to all of the parameters. For certain categories of physical systems, programs can provide assistance in developing the models [65]. In some cases, a qualitative simulator collaborates with a quantitative simulator, providing a *hybrid reasoning system* that exploits the unique capabilities of each [124,129]. These hybrid modeling approaches are convenient enhancements to existing numerical simulators, which are difficult or expensive to modify. In some special cases, the qualitative simulation produces a mathematical account of the behavior, which then evolves into the quantitative model [125].

## 5.1 Causal Descriptions

Every qualitative simulation technique presumes that a system goes through a sequence of state transitions, and that recounting or projecting this sequence is adequate for describing the system’s dynamic behavior. The differences between qualitative simulation techniques lie in the representation used to convey this sequence. The similarity lies in the fact that all systems employ a directed graph of nodes (representing states) and arcs (representing transitions). This section describes two of the important representation formalisms, *histories* and *envisionments*.

### 5.1.1 Histories

The frame problem occurring in situation calculus led Hayes to introduce the concept of a *history* to describe changes in an entity over time [73]. Histories are pieces
of the space-time continuum that can be interpreted as a collection of *states* and *episodes* within a directed graph. A state (or *event*) is a particular instant occurring in a history. An episode is an interval of time occurring between these states. The endpoints of an episode are *begin* and *end* points in time. The endpoints of a history are *start* and *finish* states.

The type of histories originally proposed by Hayes have been dubbed *parameter histories*, in which the entity of concern is the value of an object parameter. An *object history* considers an object and all of its associated parameters to be an entity. The object history is therefore composed of the union of its parameter histories [51].

As an example of what a history looks like, consider Figure 5.1. This diagram shows an object history for a gas $g$ flowing subsonic through a converging-diverging nozzle. This simple object history consists of the three parameter histories for entropy $s$, velocity $v$ and pressure $p$. The events occurring in this history correspond to positions along the axis of the nozzle. In the episodes of time prior to reaching the nozzle intake, $p$ and $v$ are unchanging. Between the intake and throat positions, however, $p$ and $v$ enter episodes during which their magnitudes change. Similarly, between the throat and exhaust positions, $p$ and $v$ enter new episodes. Finally, after leaving the nozzle, these parameters return to unchanging episodes. The entropy, meanwhile, is not affected by these events, so it remains in an episode in which it does not change.

Although they appear similar, there are two important differences between the
Figure 5.1: An example object history for the subsonic flow of a gas $g$ through a convergent–divergent nozzle. Time proceeds from the top down. Events are shown as boxes, while episodes are ovals. This object history includes three parameter histories. The parameter histories that intersect share the corresponding events of interaction.
representations of histories and situations. The first difference is the spatial extent. A state describes only the portion of space pertaining to the entity. To account for several entities, one must define several histories. A situation describes the entire space of entities, therefore only one situation is required. The second difference lies in connection semantics. Episodes account for a time interval between states during which something happens; that is, histories are time-dependent. The "connections" between situations, on the other hand, account for a change in an entity but ascribe no temporal significance to that change; situations are time-independent.

A specialization of the notion of history is the notion of a behavior. While a history covers a subsequence of episodes in time, a behavior is path from a particular start node through the history graph to a point of interest. Simulation techniques such as QSIM (described in Section 5.2) are said to be direct behavior generation techniques.

The use of histories seems to solve the frame problem while introducing some new problems. One problem concerns how to generate histories and how to determine which are the relevant objects in a situation. This is referred to as the local evolution problem. Another problem lies in how to determine whether a history intersection represents a real interaction between objects. This is the intersection/interaction problem. Qualitative reasoning systems differ in their treatment of these problems. The Qualitative Process Theory described in Chapter 8 shows how one reasoning system handles them.
5.1.2 Envisionments

Generating the set of all possible qualitative states and the transitions between those states is a means of prediction called envisioning [24,84]. Envisioning produces a causal account of behavior by explaining how effects propagate through a physical system. The product of envisioning is an envisionment. Given an initial configuration, any path through an envisionment identifies a possible behavior (or potential history) of the system. The envisionment, then, represents the problem space in the form of a qualitative state diagram.¹

Envisioning was applied by de Kleer in the first qualitative physics program, NEWTON [22], which solved problems in classical mechanics. The unusual feature of NEWTON was its ability to solve simple problems that could be solved directly by experts, perhaps even by inspection. This program was the first to identify and separate the unique reasoning techniques humans use to solve problems of varying difficulties. NEWTON first employed envisioning and qualitative reasoning techniques to attempt to find a solution, resorting to quantitative and mathematical techniques if all else failed.

¹Envisionments can be viewed as structures containing many histories. A specific history, representing a certain behavior, corresponds to a particular path through a complete envisionment. The interesting problems associated with translating between envisionments and histories are described in [55].
5.2 The QSIM Algorithm

Kuipers describes an algorithm called QSIM that formalizes the mathematics behind qualitative simulation [86]. Exploring the QSIM algorithm provides an excellent illustration of how to realize qualitative simulation from domain models and qualitative calculus. This Section provides a brief overview of the formal development of QSIM (see [86] for more details).

QSIM takes the direct behavior generation approach to simulation. The algorithm propagates qualitative state transitions forward in time, generating a partially-ordered set of possible qualitative states. The result is an acyclic directed graph whose nodes represent the qualitative states. A behavior is any path through the nodes beginning at the initial state. Although Kuipers proves that QSIM generates all possible behaviors, it may also generate behaviors that are not representative of real-world situations [85].

5.2.1 Behavior Generation

The QSIM algorithm assumes that the structural model of the system is already available and presented to it in the form of physical parameters and constraint equations. The physical parameters must be continuously differentiable functions of time, which allows QSIM to assume smooth behavior over the entire qualitative range of the parameter. The constraint equations specify relationships among the physical parameters, including algebraic relationships such as addition (ADD), mul-
tiplication (MULT), and derivative over time (DERIV). Some of the constraints may simply specify that a monotonic relationship exists between two parameters; for example the constraint \( M^+(p, t) \) specifies that \( t \) increases monotonically with \( p \).

The **qualitative state** of a parameter function \( p \) consists of a tuple

\[
< \text{qval}, \text{qdir} >
\]

specifying its ordinal relations with the landmark values (qval) and its direction of change (qdir). The qval specifies that \( p \) is either at a landmark value or is somewhere in the interval between two landmarks. The qdir states whether \( p \) is increasing, decreasing, or steady at time \( t \). Formally, \( p \) is defined in the extended real numbers\(^2\) over the closed time interval \( t \in [a, b] \subseteq \mathbb{R}^* \),

\[
p : [a, b] \rightarrow \mathbb{R}^*.
\]

The landmark values \( l_j \) are an ordered set \( l_1 < \cdots < l_k \) which must include \( p(a) \), \( p(b) \), and 0. Discretization of the time interval to points where \( p \) is at a landmark or boundary defines a finite ordered set of **distinguished time points** \( a = t_0 < t_1 < \cdots < t_n = b \). The qualitative state QS of \( p \) at time \( t \) is

\[
\text{QS}(p, t) = < \text{qval}, \text{qdir} >
\]

\(^2\)The extended real numbers \( \mathbb{R}^* \) allow the points \( \pm \infty \) to be landmark values.
in which

$$\text{qval} = \begin{cases} 
  l & \text{if } p(t) = l \\
  (l, l_{j+1}) & \text{if } p(t) \in (l, l_{j+1})
\end{cases}$$

$$\text{qdir} = \begin{cases} 
  \text{inc} & \text{if } \dot{p}(t) > 0 \\
  \text{std} & \text{if } \dot{p}(t) = 0 \\
  \text{dec} & \text{if } \dot{p}(t) < 0
\end{cases}$$

Similarly, the qualitative state of $p$ during the interval $(t_i, t_{i+1})$ is denoted $QS(p, t_i, t_{i+1})$.

The behavior of $p$ over the interval $[a, b]$ is the sequence of qualitative states

$$QS(p, t_0), QS(p, t_0, t_1), QS(p, t_1), \ldots, QS(p, t_{n-1}, t_n), QS(p, t_n).$$

Given these definitions for a single parameter function $p$, consider a system as a set of $m$ parameter functions $P = \{p_1, \ldots, p_m\}$. The set of distinguished time points for the system is the union of the sets of distinguished time points for each of the parameters. The qualitative state of the system is an $m$-tuple comprised of the qualitative states of each element of $P$:

$$QS(P, t) = < QS(p_1, t), \ldots, QS(p_m, t) > .$$

The behavior of the system then is the sequence of states

$$QS(P, t_0), QS(P, t_0, t_1), QS(P, t_1), \ldots, QS(P, t_{n-1}, t_n), QS(P, t_n).$$

This behavior sequence alternates between points and intervals. This is a representation of how the system progresses from one qualitative state to the next
<table>
<thead>
<tr>
<th>P-transition</th>
<th>QS(f, t_i)</th>
<th>⇒ QS(f, t_i, t_{i+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>&lt; l_j, std &gt;</td>
<td>&lt; l_j, std &gt;</td>
</tr>
<tr>
<td>P2</td>
<td>&lt; l_j, std &gt;</td>
<td>&lt; (l_j, l_{j+1}), inc &gt;</td>
</tr>
<tr>
<td>P3</td>
<td>&lt; l_j, std &gt;</td>
<td>&lt; (l_{j-1}, l_j), dec &gt;</td>
</tr>
<tr>
<td>P4</td>
<td>&lt; l_j, inc &gt;</td>
<td>&lt; (l_j, l_{j+1}), inc &gt;</td>
</tr>
<tr>
<td>P5</td>
<td>&lt; (l_j, l_{j+1}), inc &gt;</td>
<td>&lt; (l_j, l_{j+1}), inc &gt;</td>
</tr>
<tr>
<td>P6</td>
<td>&lt; l_j, dec &gt;</td>
<td>&lt; (l_{j-1}, l_j), dec &gt;</td>
</tr>
<tr>
<td>P7</td>
<td>&lt; (l_j, l_{j+1}), dec &gt;</td>
<td>&lt; (l_j, l_{j+1}), dec &gt;</td>
</tr>
</tbody>
</table>

Table 5.1: The set of point-to-interval transition (P-transition) rules for QSIM [86].

through a qualitative transition. It follows, then, that there are two kinds of transitions: point-to-interval transitions (or P-transitions)

$$\text{QS}(P, t_i) \Rightarrow \text{QS}(P, t_i, t_{i+1})$$

and interval-to-point transitions (or I-transitions)

$$\text{QS}(P, t_i, t_{i+1}) \Rightarrow \text{QS}(P, t_i)$$

Kuipers enumerates the possible transitions in terms of qualitative state values, as shown in Tables 5.1 and 5.2. The legal range of $p$ is the closed interval delimited by landmark values. Legal ranges serve to constrain transitions to interesting regions or to allow a different set of model constraints in different regions. An operating region of the system is a set of legal ranges governed by constraints specifying when the ranges are applicable.

The simulation begins with a given initial state, consisting of an operating region and an assignment of qualitative states to the parameters. From each alternating state and interval, QSIM propagates all of the possible qualitative value changes,
<table>
<thead>
<tr>
<th>I-transition</th>
<th>(QS(f, t_i, t_{i+1}))</th>
<th>(\Rightarrow QS(f, t_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>(&lt; l_j, \text{std} &gt;)</td>
<td>(&lt; l_j, \text{std} &gt;)</td>
</tr>
<tr>
<td>I2</td>
<td>(&lt; (l_j, l_{j+1}), \text{inc} &gt;)</td>
<td>(&lt; l_{j+1}, \text{std} &gt;)</td>
</tr>
<tr>
<td>I3</td>
<td>(&lt; (l_j, l_{j+1}), \text{inc} &gt;)</td>
<td>(&lt; l_{j+1}, \text{inc} &gt;)</td>
</tr>
<tr>
<td>I4</td>
<td>(&lt; (l_j, l_{j+1}), \text{inc} &gt;)</td>
<td>(&lt; (l_j, l_{j+1}), \text{inc} &gt;)</td>
</tr>
<tr>
<td>I5</td>
<td>(&lt; (l_j, l_{j+1}), \text{dec} &gt;)</td>
<td>(&lt; l_j, \text{std} &gt;)</td>
</tr>
<tr>
<td>I6</td>
<td>(&lt; (l_j, l_{j+1}), \text{dec} &gt;)</td>
<td>(&lt; l_j, \text{dec} &gt;)</td>
</tr>
<tr>
<td>I7</td>
<td>(&lt; (l_j, l_{j+1}), \text{dec} &gt;)</td>
<td>(&lt; (l_j, l_{j+1}), \text{dec} &gt;)</td>
</tr>
<tr>
<td>I8</td>
<td>(&lt; (l_j, l_{j+1}), \text{inc} &gt;)</td>
<td>(&lt; l^*, \text{std} &gt;)</td>
</tr>
<tr>
<td>I9</td>
<td>(&lt; (l_j, l_{j+1}), \text{dec} &gt;)</td>
<td>(&lt; l^*, \text{std} &gt;)</td>
</tr>
</tbody>
</table>

Table 5.2: The set of interval-to-point transition (I-transition) rules for QSIM. The \(l^*\) represents a new landmark value such that \(l_j < l^* < l_{j+1}\) [86].

filtering out impossible cases using the model constraints as well as knowledge about the characteristics of the transition rules (such as transition-ordering). If more than one possibility remains after filtering, the state-transition graph branches into separate paths for each possibility. The simulation continues until no further transitions are possible.

The algorithm is worth studying in more detail to gain insight into how the application of constraints filters the possible transitions. In principle, the algorithm starts with an initial state and generates successor states, eliminating transitions that violate consistency within the model. The algorithm maintains a list ACTIVE of states requiring successor state evaluation. To begin, place the initial state in the ACTIVE list. Until the list is empty, perform the following steps:

1. Select a state \(QS\) from the ACTIVE list.

2. For each parameter function \(p_i\) in \(P\), use Tables 5.1 and 5.2 to determine all
of the possible transitions.

3. Substitute the appropriate set of possible transitions for each \( p_i \) appearing as arguments in the constraints, generating a constraint space reflecting all combinations of transitions. Then, eliminate combinations of values that are locally inconsistent with the properties of each constraint.

4. Filter the remaining possible constraint satisfactions against adjacent constraints, using the *Waltz filtering algorithm* (described in Section 2.3.3). This process enforces the requirement that constraints sharing an argument must agree on its transition.

5. Generate a set of global interpretations assigning transition tuples to each \( p_i \). Eliminate each transition value that cannot be included in a global interpretation by using depth-first search among the constraints. Create successor states corresponding to each successful interpretation.

6. Filter the new states against global criteria, such as recurrence of a previously considered state (which identifies a behavior cycle). Add the remaining states to the ACTIVE list.

The QSIM algorithm is unique in several areas. First, it is established in a strong mathematical foundation. Second, the algorithm permits insertion of new landmark values into the quantity space as they are discovered. These new landmarks arise due to application of I-transition rules I8 and I9. Finally, QSIM does not require
precomputation of the possible states. In this sense it is said to be a direct behavior generation technique. Simulation techniques that produce behaviors indirectly, such as ENVISION, generally compute all of the possible states beforehand, then filter the valid transitions.

5.2.2 Behavior Elimination

The QSIM algorithm permits spurious behaviors because it considers constraints only in a local context. Researchers have introduced several heuristic techniques aimed at controlling proliferation of these spurious behaviors, particularly for QSIM applications. Kuipers and Chiu address two approaches to solving this problem [88]. In one approach, they resort to additional mathematical resources in the form of higher-order derivatives derived from the physical constraints. The system determines when these higher-order derivatives are appropriate and applies them only where necessary. In the other approach, a modified QSIM reduces establishment of branches corresponding to unimportant or irrelevant behaviors.

Lee and Kuipers show another approach to branch reduction [90]. This technique considers the global behavior of second-order systems through their phase portraits. Since the phase space trajectory of an autonomous system cannot intersect itself, the modified QSIM algorithm eliminates any behavior that crosses an existing behavior.3 Struss describes similar work that supports the mathematical

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3Research performed by Sacks investigates many qualitative mathematics techniques similar to the phase portrait technique [125,126,127].
foundation of these constraints for global filtering [133]. Kuipers and Berleant show how the reintroduction of incomplete quantitative information can control plausible applications of landmark values [89], resulting in a mixed qualitative/quantitative simulation capability. These approaches, along with many others, attest to the popularity of QSIM as an interesting basis for evaluating qualitative reasoning techniques.
Chapter 6

Qualitative Reasoning

The AI community is pursuing several different ontologies for qualitative reasoning systems. The characteristic similarities between these ontologies lie in their intention to use domain models with first-principle knowledge to perform qualitative simulation. The characteristic differences are found in their unique approaches to solving the knowledge representation problems associated with the domain models.

Each approach includes facilities for describing structural knowledge and behavioral knowledge. Structural knowledge describes the physical configuration of the objects in the domain. Such knowledge provides the reasoning system with physical descriptions of domain elements and the paths of interaction between them. Behavioral knowledge describes the intrinsic functional characteristics of the domain elements and the environment in which they exist. Such knowledge includes qualitative mathematical inequalities, causal relations, and natural physical laws.

Each of the ontologies provides mechanisms for operating on domain models. A domain model is the physical situation representation provided to the reasoning system. From the domain model one can generate scenario models by instantiating domain elements [57]. Composing these elements generates a description of the
active (or inactive) behaviors applicable to the situation. The domain model can be used to generate a variety of scenario models. A pure domain model will not be affected by considerations of how it will be used, since the constructs of the model will support a variety of reasoning styles. Moreover, it may be necessary to switch between domain models and ontologies to solve sophisticated problems.

All solutions to qualitative reasoning problems follow a similar sequence. The reasoning system is presented with domain model knowledge about the structural and behavioral aspects of the system. In most qualitative reasoning systems, this domain model consists primarily of a set of constraints to be satisfied. The system then applies these constraint equations to a given initial state (consisting of one assignment of values to state variables), predicting the possible values of the variables in subsequent states through simulation. The resulting graph of possibilities becomes the basis for a causal explanation of how the physical system behaves in a certain manner.

### 6.1 Reasoning Ontologies

Qualitative reasoning research has focused on three distinct ontologies, centered on the *constraints, components, or processes* that best describe the domain model. Each of the ontologies provides a domain-independent framework that supports general qualitative reasoning about physical systems. It is left to the model builder
to decide which representation best suits the world.\footnote{The "world," as the AI community sees it, contains sufficient elements and the environs necessary to make reasonable conclusions about the domain. Frequently, the world also contains irrelevant facts or insufficient information, just to make the reasoning problems more interesting (and realistic).}

The ontologies differ primarily in their treatments of how to derive constraints and how to present them to the simulation system. The constraint-centered ontology assumes that the constraints have already been specified to a level of detail corresponding to the desired level of abstraction. The device-centered ontology considers the description of system components and their connections to be sufficient for defining structural and behavioral constraints. The process-centered ontology assumes the responsibility for deriving the constraints from a set of active processes that describe what is happening in a situation. Each of these three approaches is described in more detail below.

### 6.1.1 Constraint-Centered Ontology

The constraint-centered ontology approaches physical system modeling by considering qualitative differential equations as constraints. Such a modeling system generates a behavior by propagating variables and constraints throughout the model to generate a collection of value assignments that adhere to the constituent equations. While this ontology places no restriction on the form or function of the constraint equations, Kuipers’ QSIM technique represents the most clearly-defined approach toward working with constraint-centered models.
A constraint-centered model consists of a collection of variables and a set of qualitative differential equations that specify how the physical system behaves. The knowledge engineer creates the domain model by identifying the important elements of the domain in terms of state variables and constraint equations. Each modeling situation consists of selecting among the various elements and defining their interconnections. The model, then, is the union of all of the constraints brought into the scenario by the elements.

The constraint-centered ontology, while the simplest of the three ontologies considered herein, provides the least assistance toward constructing the actual domain model. Since it operates entirely with constraints, the model provides no insight into where the constraints came from, how to derive them, or how to map them back to the physical system. One important research topic is determining how to generate constraint models automatically for real physical mechanisms.

### 6.1.2 Device-Centered Ontology

dé Kleer and Brown [26] describe a model of qualitative physics based upon the interaction of fundamental physical devices, usually regarded as a device-centered, or more generally a component-centered, model. This ontology provides a framework which considers the world to be a collection of devices assembled from a set of elementary components having known functional behaviors. The goal of a device-

\[^2\]One may argue, then, whether this ontology deserves to be labeled as an ontology for performing qualitative physics [25].
centered model is to derive the functional behavior of the composite device by reasoning about the behavioral properties of its components.

This technique employs confluences, or qualitative differential equations, in the description of component behaviors. Each component model contains a set of confluences. A different set of confluences may be assigned to each of the various qualitative states of the component, showing that the device behavior changes between its various states. Prespecified functions of the state variables provide one way to select an applicable set of confluences. For example, consider the relationship for a liquid flow rate $Q$ through an orifice, given by

$$Q = CA\sqrt{\frac{2p}{\rho}}, \quad p \geq 0, A > 0$$

where $A$ is the cross-sectional area, $p$ is the pressure, $\rho$ is the density and $C$ is a constant discharge coefficient. Rewriting this expression using qualitative values results in

$$[Q] = [C][A][\sqrt{2p/\rho}],$$

where

$$[C] = +$$

$$[A] = +$$

$$[\sqrt{2p/\rho}] = [p],$$

which reduces to

$$[Q] = [+][+][p]$$
or the final confluence equation

\[ Q - p = 0. \]

The final confluence equation expresses the desired qualitative relation for constant-area flow: the flow rate is proportional to the pressure.

A combination of qualitative values satisfies a confluence only if every variable has a value and either (1) the equality of the left and right hand sides is true, or (2) the qualitative expression on the left hand side cannot be evaluated. The latter case arises because, according to the qualitative arithmetic tables defined above, certain operations on qualitative values are inconclusive. This condition poses multiple solutions, each of which is considered an interpretation of predicted behavior.

The device-centered model also relies upon the device topology. The device topology is a graph in which the nodes represent pieces of the device and the edges represent connections between the pieces. The connections specify which pieces directly interact as well as how influences from the outside environment enter the device model. The intercomponent conduits must satisfy the constraints of continuity and compatibility. The continuity constraint enforces conservation of information, analogous to a conservation of mass or energy, whereby information entering the conduit results in an equivalent amount of information leaving the conduit. The compatibility constraint specifies that mutually connected conduits share the same information (e.g. current, pressure, or force). These constraints serve to remove most of the functional importance from the conduits, resulting in their primary
purpose as logical connectors.

With the confluences and topology in hand, determining a behavior is a matter of constraint propagation (as described in Section 2.3.3). A solution consists of a set of value assignments to the variables at each state, along with a list of state transitions and their causes. There is one solution for each interpretation of the model.

A key principle for this ontology is that there should be no function in structure: the functional behaviors of the elementary components may not presume functioning of the composite device. This ensures that the results obtained by the model are insensitive to the context. The example of no function in structure provided in [26] describes a model of a switch: if the switch is on, current flows; if the switch is off, current does not flow. This model violates the no function in structure principle because it presumes the potential for current flow through the switch. One can imagine many situations in which current will not flow through a switch regardless of the switch state: unmade contacts, switches in series, open circuits, etc. Consideration of this principle requires careful specification of the elementary component behaviors.\(^3\) One problem with all functional-level representations is that there is always a lower level of detail that remains unspecified. When we modify some facet of a component described only in functional terms, it is impossible to deduce the consequential behavior because the internal structure is unspecified.

\(^3\)A more realistic statement of the no function in structure principle requires definition of class-wide assumptions applicable to various categories of devices [26].
A second principle, *locality*, requires that the behaviors specified for an elementary component not refer to any other component. If a behavior assumed presence of an external component, this would place *a priori* requirements upon the topology of the composite device.

The de Kleer and Brown device-centered model has been implemented and tested in a simulation program called ENVISION, whose purpose is to derive the function of a physical system through a description of its structure. ENVISION uses a topological structure specification and a set of input signals and boundary conditions to predict behaviors of the system in terms of possible states and variable values. The output includes a description of the reasoning steps leading to each prediction, thereby providing a causal explanation of the behavior interpretation.

ENVISION combines constraint propagation with the *generate and test* technique to solve systems of confluence equations. The generate and test algorithm consists of two separate activities: the generate activity enumerates possible solutions, while the test activity evaluates these solutions against acceptance criteria. Once an acceptable solution has been found, the process terminates.

Upon completing a simulation, ENVISION produces an *explanation* of the results. The explanation consists of a logical deduction proof stating assumptions, inferences, and justifications for each reasoning step. The sophisticated deduc-

---

4 This strategy is similar to the *plan-generate-test* technique used in the DENDRAL program [9]. In DENDRAL, a constraint satisfaction process first reduces the problem space to a reasonable size before applying the generate and test process.

5 It is interesting to note that de Kleer and Brown also discuss the undesirable characteristics of
tion system is able to perform both direct and indirect proofs, such as *reductio ad absurdum*. de Kleer and Brown also describe a procedure for creating a *state diagram* depicting the results. The state diagram is a graphical representation of the state conditions and transitions, providing a clear presentation of causality and of behavioral concerns such as oscillation, feedback, or damping.

The ENVISION program assumes that causal effects occur over small time scales. This precludes reasoning about actions and effects which occur over long periods of time, such as indefinite trends, feedback, continuity, memory, and nonlinearity. Causal propagation cannot make predictions about whether (and when) a quantity will move into another region. In [27], de Kleer and Bobrow identify fundamental laws which, though expressed in qualitative terms, govern the behavior of a device over large time scales. These laws require reasoning about higher-order derivatives in order to discern unique qualitative states, particularly extrema and inflection points.

Williams [144] identified additional behaviors which proceed over large time scales, requiring a better model of time. He describes the technique of *Temporal Qualitative Analysis* that provides ambiguity-resolving heuristics to de Kleer and Brown’s basic causal propagation system. This technique uses causal propagation for small-signal analysis, determining causality and influences for local effects, then uses *transition analysis* and *feedback analysis* to determine large-signal effects.
Williams derives these additional analysis techniques from standard quantitative theorems of the calculus of intervals.

6.1.3 Process-Centered Ontology

A process-centered reasoning ontology provides a framework which considers the physical world to be comprised of a collection of interacting ongoing processes. This ontology presumes that all changes occurring among the objects are due to the effect of active processes.

A process is a continuous alteration of the environment through a time-ordered sequence of changes. Physical processes normally are represented by differential equations that characterize the behavior of the parameters of objects over time. A common sense representation of processes necessarily must provide an intuitive characterization of physical change that differential equations cannot provide. This sort of characterization identifies which processes might affect the objects in the domain of discourse, what influences these processes have on the objects, and when the processes exercise these influences.

To perform a simulation based on processes, a reasoning system considers the current state of the objects along with the behavioral attributes of processes available for changing the environment. When objects in the environment change, the kinds of processes affecting the environment may also change. To manage these operations, each process possesses a set of preconditions that act as rules governing
the activation (or deactivation) of that process. When the preconditions are satisfied, the process becomes *active* and affects the pertinent objects. Similarly, when the preconditions become unsatisfied (or if a set of *termination conditions* are met), the process becomes *inactive* and no longer affects the environment. To preclude instantaneous activations, processes may require satisfaction of both initiation and *continuation* conditions [74].

Qualitative simulations using the process-centered approach are comparable to simulations conducted in the other ontologies. Re-evaluation of the active processes occurs when something "interesting" happens, such as a state variable reaching a landmark value in its quantity space. This flexible ontology, however, permits application of additional criteria for selecting among candidate processes, such as achieving a desired effect or through application of a priority scheme.

The seminal example of process-centered modelling activity is the robotic activity planning system developed by Hendrix [74]. This system includes process models containing operators for changing the environment, as well as a *process monitor* feature to control process activity. The process monitor possesses a meta-level interpretation of processes as well as a notion of time and duration. By verifying satisfaction of the conditions under which each process holds, it is able to discern interesting points of time at which significant events occur.

The predominant representation within the process-centered ontology is due to Forbus [48,51]. Forbus' *Qualitative Process Theory* (QPT) formalizes a process-
based language for simulating the behavior of dynamic systems. QPT provides all of the requisite features for modelling process activations and influences. The qualitative theory of the gas dynamics described in Part II uses the QPT framework to construct the domain model. Since this theory is so important to the domain model, Chapter 8 is dedicated to providing a detailed description of the formalism.

One unique characteristic of the process-centered ontology is that it requires a sophisticated model of time. Since processes are time-ordered, it is necessary to define a temporal logic for determining event occurrence relationships. This requires a sense of duration, possibly with open-ended (infinite) initiation and termination points. Hendrix’s system ascribes temporal knowledge to his process monitor function, while Forbus adopts the temporal representation proposed by Allen [1]. Further issues associated with modelling time are discussed in Section 7.1.1.

6.2 Model-Independent Techniques

There are a variety of qualitative reasoning techniques that are virtually independent of the domain model. Two techniques, order-of-magnitude reasoning and comparative analysis, are described below. The descriptions of these techniques serve to provide interesting perspectives from which to compare and contrast qualitative reasoning systems.

Some effort has been made in the community to combine the various ontologies into a reasoning system that exploits the features of each paradigm. Since it is
quite possible that any qualitative reasoning ontology might be applied to a given problem, then there may be multiple models of the same physical system. In other models, it may be convenient to apply primitives from multiple ontologies into a single integrated representation. These issues, as well as those associated with determining how a system might apply metalevel reasoning to decide for itself which qualitative reasoning scheme is best, are discussed in [6].

6.2.1 Order-Of-Magnitude Reasoning

The quantity space representation provides a way to reason about partial orderings of qualitative values in terms of landmark values and inequality relationships. This representation removes some of the ambiguities introduced by qualitative values and calculus. However, while the quantity space provides a partial ordering of landmark values, the structure of the regions between landmark values remains ambiguous. Order-of-magnitude reasoning techniques provide a way to further reduce qualitative ambiguity by enhancing the expressiveness of the quantity space.

Order-of-magnitude reasoning techniques assume that deeper information concerning the partial orderings of quantity spaces can be provided by specifying relations that express degrees of magnitude for comparison purposes. These techniques reduce the high level of abstraction of qualitative values to an intermediate level of abstraction in order to accommodate quantitative information. Two systems that explore explicitly the capabilities of order-of-magnitude reasoning, FOG and O[M],
\[ A \cong B \ (A \text{ is close to } B) \]
\[ A \cong A \]
\[ A \cong B \rightarrow B \cong A \]
\[ A \cong B, B \cong C \rightarrow A \cong C \]
\[ A \cong B, [C] = [A] \rightarrow (A + C) \cong (B + C) \]

\[ A \sim B \ (A \text{ is comparable to } B) \]
\[ A \sim B \rightarrow B \sim A \]
\[ A \sim B, B \sim C \rightarrow A \sim C \]
\[ A \sim B \rightarrow [A] = [B] \]
\[ A \cong B \rightarrow A \sim B \]

\[ A \ll B \ (A \text{ is negligible compared to } B) \]
\[ A \ll B, B \ll C \rightarrow A \ll C \]
\[ A \ll B, B \sim C \rightarrow A \ll C \]
\[ A \ll B \rightarrow \neg A \ll B \]

Table 6.1: Order-of-Magnitude Relations for FOG.

demonstrate how these techniques work.

The formal system for order-of-magnitude reasoning (FOG) defines a technique for comparing expressions and establishing relationships between different objects [121]. The FOG system adds three new order-of-magnitude relations,

\[ A \ll B: \ A \text{ is negligible in relation to } B, \]
\[ A \cong B: \ A \text{ is very close to } B \ (\text{and has the same sign}), \text{ and} \]
\[ A \sim B: \ A \text{ is comparable to } B \ (\text{in both sign and magnitude}) \]

to expand a basic set of relations defined for qualitative calculus [17]. A representative collection of rules for evaluating expressions and the semantics made possible by these relations are shown in Table 6.1.

Consider as an example a simple elastic collision between two objects of differ-
<table>
<thead>
<tr>
<th>$v_{1f}$</th>
<th>$v_{2f}$</th>
<th>$v_{1i}$</th>
<th>$v_{2i}$</th>
<th>$v_{1f} \oplus v_{1i} = v_{2f} \oplus v_{2i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊤</td>
<td>⊤</td>
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<td>? = ⊤</td>
</tr>
</tbody>
</table>

Table 6.2: Ambiguous solutions to elastic collision problem for FOG demonstration.

ent masses, $m_1 \gg m_2$, travelling in opposite directions with velocities $[v_{1i}] = \oplus$, $[v_{2i}] = \ominus$, and $[v_{1f}] = \ominus$, and $[v_{2f}] = -[v_{2i}]$. Using conservation of mass and energy, a qualitative expression describing object velocities $v$ before $(i)$ and after $(f)$ impact may be written

$$[v_{1f}] \oplus [v_{1i}] = [v_{2f}] \oplus [v_{2i}] .$$

Qualitative calculus is unable to provide insight into the solution of this equation for given initial velocities, resulting in several ambiguous cases (Table 6.2). Using $m_1 \gg m_2$ and expressions of the form shown in Table 6.1, FOG can deduce that the cases for which $[v_{1f}] = \ominus$ and $[v_{1f}] = \ominus$ lead to contradictions, therefore the only logical solution is that $[v_{1f}] = \oplus$ and $[v_{2f}] = \oplus$ [121]. This solution agrees with common sense: after the collision the object with much larger mass continues in the same direction it was travelling before the collision, while the smaller mass reverses direction.

The second formalism, Orders of Magnitudes (O[M]), addresses some deficiencies of the FOG formalism [97,99]. The improvements offered by O[M] include several
features important for engineering design applications. For example, O[M] permits incorporation of partial quantitative information in relations: if \( A \approx 0.1 \) and \( B \approx 1000 \), O[M] can deduce that \( A \ll B \). For semantic purposes O[M] allows adjustment of a "degree of freedom" to define the tolerance of the new comparators, such as "negligible." O[M] also avoids the restrictive nature of sign and magnitude comparisons.\(^6\)

O[M] adds seven primitive relations to the basic set of qualitative calculus expressions:

\[ A \ll B: \quad A \text{ is much less than } B, \]
\[ A -< B: \quad A \text{ is moderately less than } B, \]
\[ A \sim< B: \quad A \text{ is slightly less than } B, \]
\[ A == B: \quad A \text{ is equal to } B, \]
\[ A \gg B: \quad A \text{ is slightly greater than } B, \]
\[ A >- B: \quad A \text{ is moderately greater than } B, \text{ and} \]
\[ A \gg B: \quad A \text{ is much greater than } B. \]

Multiple primitives can be combined through semantic disjunction to constitute elaborate compound relations. An interesting collection of compound relations and their semantics (from [99]) are defined in Table 6.2.1. The physically meaningful expressions made possible by these compound relations are typical in engineering design applications (and should therefore be inherent to ICAD applications), but

\(^6\)Sign comparisons are especially restrictive. For example, consider values for temperature. When measured in degrees Kelvin, magnitude differences between two values are the same as when measured in degrees Celsius, but in the former system the sign is irrelevant.
Table 6.3: Order-of-Magnitude Relations for O[M]. The notation .. signifies a compound relation with zero or more intermediate primitives between the two specified.

probably are less tangible in natural physics applications [99,98].

The interpretation of the various primitives in O[M] depends upon the size of the ordering intervals selected. The value of the accuracy parameter e specifies the largest amount defining “much smaller than 1.” The range of influence for each primitive relation A ∘ B then falls in certain region relative to 1 (representing A/B) on the number line (see Figure 6.2.1). It is also possible to define these regions so that they either overlap each other (larger ovals) or leave parts of the
Figure 6.1: Relation interpretation regions in O[M]. The accuracy parameter \( e \) defines a series of intervals used to distinguish between the various orders of magnitude allowed by the O[M] primitives.

range uncovered (smaller ovals), requiring heuristic resolution of these ambiguities.

These heuristic approaches allow assertion of quantity assumptions which can be propagated through relationships with other quantities, the supported or retracted as further information arises.

Several types of inference can be made among O[M] expressions. Algebraic assignments assign ranges to variables or relations to other relations. Constraints control which value assumptions hold by specifying strict relationships between quantities. Rules provide a way to provide conditional relationships among quantities, where successful evaluation of the antecedents is cause to assert the consequents. Alternatively, O[M] allows assumption of the truth values of the antecedents in order to assert the consequents as unsupported facts. This latter form of reasoning is controlled by an ATMS (see Section 2.3.4).
6.2.2 Comparative Analysis

Comparative analysis techniques determine how a system will react to changes in its parameters. These techniques generate a description of how a perturbed behavior is different from an unperturbed behavior, as well as an explanation of why the behavior is different. Comparative analysis techniques nicely complement qualitative simulation techniques: qualitative simulation uses a structural description of a system in order to generate its behavior, while comparative analysis uses this behavior to reason about perturbations.7

Weld [140,141,142] describes two theories for performing comparative analysis: differential qualitative (DQ) analysis and exaggeration. Owing primarily to the mathematical foundation of the QSIM algorithm upon which it is built, Weld shows that DQ analysis is sound: when it provides an answer, it can be proven correct. Exaggeration, on the other hand, makes the assumption that parameters behave monotonically. Since this is not always true, exaggeration is not sound: it is heuristic. Though DQ analysis is sound, exaggeration can solve more kinds of problems.8 Each of these theories is explored briefly below.

Differential Qualitative analysis considers how the behavior of a system of physical parameters changes when one or more of the input parameters changes. DQ

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7 The inverse problem, determining how output perturbations change the behavior of the model, has also been investigated [36].

8 Weld [142] presents a hybrid reasoning system that exploits the advantages of each comparative analysis technique. It uses DQ analysis whenever possible, resorting to exaggeration if DQ analysis fails to provide an answer.
analysis describes the behavioral changes in terms of other changing parameters, for example "p increases because V decreases." The DQ analysis inference engine, built upon the QSIM algorithm (described in Section 5.2), propagates the perturbations throughout a qualitative model in order to determine the net effect. The success of this technique is limited to models with clear relationships between the differential equations, though the limiting factors are hard to define [142].

The treatment of time in DQ analysis diverges from the natural treatment afforded by QSIM distinguished time points since the changes occurring in the initial behavior might not coincide temporally with the changes occurring in the perturbed system. Instead, DQ analysis provides a time function that maps state transitions to the distinguished time points at which they occur. State transition events are said to occur at transition points which do not have a particular time label attached to them.

One key step of the DQ analysis process is determining and describing how parameters change from the perspective of another parameter, much like the qualitative analog for a frame of reference. In the elastic collision problem introduced above, we might restate the problem to determine how the mass and velocity parameters of one block change from the perspective of the other. Importantly, DQ analysis is able to provide insight into the behavior of complex systems by shifting among several perspectives and making comparisons.

DQ analysis compares the initial behavior with a perturbed behavior in several
ways. One comparison can be made over the topology of two behaviors, checking whether they contain the same sequence of state transitions (but not necessarily the same parameter values). Another comparison occurs over the relative change values of the parameters from various perspectives, including landmark values and intervals. The time function allows the system to compare interval durations.

Another comparative analysis technique, *exaggeration*, is similar in intention to DQ analysis except that it considers extreme perturbations instead of differential ones [141]. For instance, exaggeration analysis considers what happens if one of the input parameters is taken to a limit, such as assuming that the mass of a block in the collision problem is infinite. Using the behavior resulting from an extreme perturbation, the exaggeration technique scales the parameter changes back to reasonable values in order to answer comparison questions. The limiting assumption in the scaling process is that the parameters exhibit monotonic changes in behavior over the entire range of values. If these changes are not monotonic, exaggeration can make incorrect comparisons and predictions.
Chapter 7

Qualitative Theories of Physics

Investigation into the application of qualitative reasoning techniques to physical domains is well underway. This Chapter describes briefly several evolving qualitative theories, identifying the common sense reasoning and knowledge representation issues they encounter.

7.1 Kinematics and Dynamics

The principles of kinematics and dynamics, along with the more fundamental notions of time and space, are particularly difficult concepts to grasp from a qualitative reasoning standpoint. This Section reviews some of the work performed in these areas. Section 7.1.1 presents a few of the many issues associated with the notion of time. Similarly, Section 7.1.2 presents qualitative approaches to the notions of space and geometry. Section 7.1.3 presents some of the work toward consolidating these concepts into qualitative theories of kinematics.
7.1.1 Time

The nature of time poses special problems to common sense reasoning systems. Reasoning about temporal concepts allows us to anticipate actions, unravel the past, predict the future, and construe relationships between events. Among the common sense concepts are notions of *duration*, *persistence*, *inactivity*, *simultaneity*, and *present moment*. Combining these notions, one can compose theories of actions, plans, and physical dynamics. Several attempts to capture these concepts are described in this section, and addressed further in [128].

A seminal effort toward a common sense representation of time was undertaken by Kahn and Gorry [80]. Their system treats temporal reasoning issues as generic problems that can be divorced from the domain model. To demonstrate their ideas they developed a *time specialist* client program that handles temporal reasoning within a larger problem-solving system.\(^1\) This specialist is able to draw conclusions from a collection of *temporal specifications* that describe the relationship between two or more events, each of which represents a point in time.

Addressing the limitations of situation calculus, attributable to the presumption of *temporal instants*, Williams [145] proposes a new representation that considers each physical state variable individually. This technique establishes for each variable a history describing value changes and temporal relationships among events. Each

\(^1\)This approach is similar to the *process monitor* client defined by Hendrix (see Section 6.1.3), which possesses generic capabilities for reasoning about processes, and to the problem-solving specialists for design delineated by Brown and Chandrasekaran [7].
period of uniform physical behavior is considered a *temporal interval*, which permits an analogy to the representation of time as a quantity space. The technique uses constraint propagation to maintain consistency of each parameter over its temporal intervals, resulting in a system that is able to provide justifications for temporal values.

Allen [1] also proposes that intervals are more useful abstractions of time than are instants. His *general theory of time* argues that intervals are sufficiently flexible to represent events that occur over any period of time, while instants alone do not provide the capability to represent events that happen over a period of time or persist indefinitely. The theory applies constraint propagation to maintain collections of intervals in a partially-ordered quantity space. Assuming allocation of a reference interval, the technique provides a way to infer the logical relationship between two intervals that share the reference. There are exactly 13 interval relationships for time, for which the theory provides the binary predicates Overlaps\((s, t)\), Meets\((s, t)\), Before\((s, t)\) and During\((s, t)\), Equals\((s, t)\), Starts\((s, t)\), and Ends\((s, t)\) to relate an interval \(s\) with an interval \(t\).

Allen [2] then elaborates on the logic of intervals to construct a formalism for reasoning about *actions*. This logic draws a distinction between the static and dynamic components of the world, considering the static components of the world as *properties* and the dynamic components *occurrences*. Occurrences may belong to *events* or *processes*, depending upon whether the activity produces or does not
produce the anticipated results. Using this basis, Allen outlines the differences between knowledge specifying the necessary conditions for an action to occur and the knowledge specifying how an action can be performed. Allen and Hayes [3] further extend the theory by analyzing the beginnings and endings of intervals, which they show to possess instant-like properties.

Galton [63] considers Allen's general theory of time unsuitable for representing situations involving continuous change, and proposes several revisions to remedy this shortcoming. The principal enhancement involves the possibility of reasoning directly about instants in parallel with intervals. This possibility requires explicit statement of whether a property holds throughout an interval, during an interval (yet not necessarily through all of it), or at an instant. Galton diversifies Allen's predicates to pertain to each of these three possibilities. A secondary enhancement is the division of properties into states of position and states of motion, which have differing temporal logics. In this ontology, states of position can hold at certain instants, whereas states of motion cannot.

7.1.2 Space

While one-dimensional time poses certain issues for common sense reasoning, three-dimensional space amplifies them. Representing objects in space requires dealing with notions such as shape, orientation, separation, location, and motion. This requires delineation of the space an object occupies as well as the space that the
object might occupy in the future. Even individuation of objects in space poses problems in delineation.

Most qualitative spatial representations use the various sorts found in Euclidean geometry. They employ terms such as points, measures, and direction to provide mappings between spaces or for comparison among objects. Many systems formalize spatial relationships between individuals as constraint equations specifying locations and relative coordinate orderings. Others use occupancy arrays that assign a set of “pixels” to an object occupying a named region of space. Another approach, commonly found in CAD systems, is the constructive solid geometry (CSG) representation. CSG provides a language of primitive shapes, locations, and operators that approximate complex objects as collections of primitive objects, typically by using set operations [20].

It is tempting to define interval relationships and draw an analogy of between temporal extent and spatial extent. Unfortunately, the 13 relationships for time become $13^3$ relationships for space, few of which have common sense connotations [123]. Instead, qualitative theories of space often regard the world with multiple interacting levels of abstraction (or granularity). Each level applies relations defined over spaces rather than intervals. For example, the predicate $\text{Inside}(x, y, g)$ might relate a space $x$ with a space $y$ at a level of granularity $g$.

Malik and Binford [96] describe a representation formalism for reasoning about both time and space in terms of linear inequalities. In their model, each inequality
provides a constraint describing events and objects in terms of interval endpoints or spatial boundaries. For example, to state that interval $A$ occurs during interval $B$ they propose the endpoint inequality

$$(\text{start}_B \leq \text{start}_A) \land (\text{end}_A \leq \text{end}_B).$$

Inferences are made through standard linear programming techniques, such as the simplex algorithm. To describe relationships between entities, their system uses reference frame transformations to correlate semantically-isolated sets of constraints.

### 7.1.3 Kinematics

Considering motion and geometry together, many researchers consider the field of qualitative kinematics to be a special problem in qualitative physics [44,47,56,70]. Describing the motion of objects through multidimensional space requires an ontological framework that is inherently quantitative, enforcing restrictions that are not applicable to the knowledge representation ontologies described above.

Although qualitative state representations have been applied to problems regarding the motion of objects [47], these representations do not lend themselves well to problems of kinematics. The complexity with which mechanism components interact makes the number of robust simulation rules too extensive for practical application. Instead, a common representation scheme is the configuration space, which isolates components and describes how they interact with other objects in the domain [56,77,78]. Configuration space has as many dimensions as there are objects
of motion in the domain. Unlike the component-centered representation ontology, the configuration space representation requires more than just the functional description of an object to reason about kinematic behavior. Functional descriptions do not support reasoning about changing the shape of an object or possible object interactions.

Most qualitative kinematics frameworks combine qualitative information, in the form of place vocabularies, with weak quantitative information found in metric diagrams. Metric diagrams combine numeric and symbolic information to convey the geometric properties of a situation. A place vocabulary describes the kinematic behavior of a device as a set of contact relationships differentiating the behavior states [44,47,56]. Each relationship represents a possible contact between parts of two objects. Place vocabularies can be computed from metric diagrams, with one place corresponding to each degree of freedom for the system. Computing all of the place vocabularies for a situation results in a set of descriptions describing part interactions, and therefore possible behaviors. Since the place vocabulary permits changes in part dimensions, the set of descriptions may be unnecessarily large when it contains illegal part configurations. To overcome this problem, Faltings et al. define a higher level of abstraction for part interaction in terms of the kinematic topology [45]. This abstraction permits computation of legal configurations of mechanism parts from symbolic descriptions, which might then serve to guide determination of a reduced set of place vocabularies. Without place vocabularies,
the kinematic topology representation sacrifices the capability to compute en-
visionments. Similarly, Joskowicz [79] defines operators from configuration spaces to
categorical abstractions that eliminate irrelevant detail.

Forbus [49] describes an attempt to use Qualitative Process Theory (described in
Chapter 8) for modeling motion. In this model, motion is treated as a process that
causes an unrestrained object to move with a positive velocity in a free direction,
concurrently influencing its spatial position. Similarly, acceleration is a process that
influences the velocity of an object to which a force is applied. While the convenience
of QPT for describing relationships among quantities is apparent, describing the
georometric interaction among objects using quantity spaces and processes remains a
difficult problem.

Joskowicz [78] describes a constraint propagation system that produces en-
visionments for the kinematic behavior. The system uses the configuration space to
simulate kinematic behavior by calculating relative motions between the various
components, then combining these isolated solutions to generate the total behavior.

7.2 Matter

Along with the development of theories about the concepts of time and space, the
has been a significant amount of work toward the development of theories about
matter. This Section describes a few of the interesting approaches to capturing the
qualitative nature of solids and liquids, while leaving the qualitative nature of gases
open for investigation.

7.2.1 Solids

Reasoning about solid objects requires many of the notions of reasoning about space and motion. Objects with various shapes occupy positions in space. Interaction among objects is described in terms of configuration spaces, relative position maps, and Newtonian mechanics. Sorts of object interactions include collision, friction, and support.

Carney and Brown [11] describe a solid object reasoning system implementation that uses a layered approach to determine whether objects fit together qualitatively. Their system groups features on object surfaces in one layer, then describes the pattern of feature groups in terms of a topological structure in another. A third layer determines candidate inter-object fitting orientations and alignments, while the fourth and fifth layers determine surface compatibility.

Davis [20] enumerates many simple axioms for describing solid object dynamics. These axioms permit reasoning about force application, energy conservation, and other principles of physics. He does, however, note that the quantitative theories of Newtonian mechanics and dynamics require reasoning with many concepts that are counterintuitive, thereby questioning their place in common sense theories. Many of the problems associated with formalizing these concepts leave open issues for continued research.
7.2.2 Liquids

Despite the fact that liquids are more complex entities than solids, one of the first qualitative physics models was Hayes' model of liquids [73]. This model in fact proposed two ontologies, a piece-of-stuff ontology, and a contained-liquid ontology (sometimes generalized as a contained-stuff ontology).²

The contained-liquid ontology provides relationships for reasoning about liquids and their containers. Liquids are said to possess a continuous quantity amount-of that may be influenced by various processes, such as evaporation or falling. Liquid objects can be created or destroyed by suitably changing their quantity. Containers are solid objects having geometry that surround a space that can hold a liquid, having faces, portals, and directed surfaces. Under the right conditions, liquids can leave their containers by exiting through the portals. Within these spaces, liquids can be categorized as begin divided or in bulk, and having the characteristics lazy (requiring no driving force) or energetic (requiring a sustaining force). These dimensions permit many classifications, of which only fifteen seem to have some worth to common sense descriptions of liquids (see Figure 7.1). With a seemingly simple foundation, Hayes defines a number of axioms for reasoning about surroundings, immersion, wetting, capacity, etc., that together comprise the theory of contained liquids.

²Hayes' effort on the contained-liquid front led eventually to Forbus' development of Qualitative Process Theory [52].
<table>
<thead>
<tr>
<th>Liquid State</th>
<th>Lazy Still</th>
<th>Lazy Moving</th>
<th>Energetic Moving</th>
</tr>
</thead>
<tbody>
<tr>
<td>On Surface</td>
<td>Wet surface</td>
<td>Flowing down a surface, e.g. a sloping roof</td>
<td>Waves lapping shore (?), jet hitting surface (?)</td>
</tr>
<tr>
<td>In Space</td>
<td>Contained, in container</td>
<td>Flowing along a channel, e.g. river</td>
<td>Pumped along pipeline</td>
</tr>
<tr>
<td>Unsupported</td>
<td>Falling column of liquid, e.g. waterfall</td>
<td></td>
<td>Waterspout, fountain, jet from hosepipe</td>
</tr>
<tr>
<td>On Surface</td>
<td>Dew, drops on a surface</td>
<td></td>
<td></td>
</tr>
<tr>
<td>In Space</td>
<td>Mist filling a valley (?)</td>
<td>Mist rolling down a valley (?)</td>
<td>Steam or mist blown along a tube (?)</td>
</tr>
<tr>
<td>Unsupported</td>
<td>Mist, cloud</td>
<td>Rain, shower</td>
<td>Spray, splash, driving rain</td>
</tr>
</tbody>
</table>

Figure 7.1: The fifteen possible states of a liquid according the Hayes' contained-liquid ontology. The question marks (?) are from [73].
It was for this model that Hayes first introduced the notion of histories. These histories were used to describe how liquids change in different situations. Histories represent the life of an object, occurring in space between two times, and a slice of a history represents an object at a particular time. This scheme allows one to reason about liquids that divide into independent objects or compose into one larger entity: the division or composition event ends the life of some objects and marks the beginning of life for others.

The alternative representation, labelled the piece-of-stuff ontology, reasons about a collection of molecules (representing a piece of a liquid) moving through a system. The collection possesses a fixed mass that travels under the influence of various forces in the system. Assuming conservation of mass, the objects represented by these collections exist indefinitely without being created or destroyed. Since the object is considered to be very small, it can occupy only one point in space at any time, however it is large enough to possess thermodynamic properties. With these assumptions, and presuming individuation by spatial boundaries, one can reason about the indefinite history of a piece of liquid, make common sense decisions about how liquid objects are formed, or even compare it to a contained liquid at coincident episodes in time.

The Collins and Forbus work on molecular collections (MC) is a specialization of the piece-of-stuff ontology [15]. They argue that a description of a fluid domain in terms of contained-stuff is prerequisite to generating MC descriptions. This system
applies a collection of translation rules to a contained-stuff model, described in terms of Qualitative Process Theory (QPT), for generating a collaborative model at the MC-level. With the objective of generating histories for a given MC, they start with the QPT-generated envisionment for the same domain. Selecting an interesting situation history from this envisionment, their system determines possible location and state changes for the MC by knowing something about how active processes influence it. These influences are stated in terms of how the time derivatives of heat, temperature, volume and height of the MC change. Finally, the changes combine into a graph specifying the possible movement of the MC through various locations.

7.2.3 Gases

A robust qualitative theory of gases has not yet been developed. The qualitative theory of gas dynamics presented in this dissertation attempts to complete the top-level set of theories of the dynamics of physical matter.
Chapter 8

Qualitative Process Theory

Qualitative Process Theory (QPT) offers a language and techniques for modeling dynamic systems in common sense terms of processes. This Chapter presents the important concepts of QPT, and shows how the framework supports qualitative modelling and simulation. The discussions and examples herein begin to suggest how the processes of gas dynamics might be protrayed.

Section 8.1 presents the knowledge representation issues along with the required definition of terms and inference operations. Section 8.2 describes how qualitative modeling is carried out using these operations and a few simplifying assumptions. Section 8.3 provides illustrative examples of the mechanics of QPT (and of qualitative reasoning in general). These simple example models show what a qualitative theory of dynamics looks like, and help to convey the fact that the specific content of the models is not a real concern of the theory. Finally, Section 8.4 describes how QPT has been applied to various problems.

The theory presented in Part II makes liberal use of the terminology defined in this Chapter. Much of the information here has been distilled from more comprehensive descriptions of the formalism found in [48,51].
8.1 Knowledge Representation

Qualitative Process Theory rests on the sole mechanism assumption:

Definition 1 (Sole Mechanism Assumption) All changes in physical systems are caused directly or indirectly by active processes.

This assumption explicitly constrains all behaviors to one class of physical abstractions, providing a unified reasoning framework.

8.1.1 Expressions and Notation

QPT can be viewed as the specification of a language for qualitative modeling. This hybrid language is composed of notations from computer programming and from first-order predicate calculus. The programming notation appears in the form of data structures with labelled members, providing templates to be filled with data and logic expressions (suggesting a frame analogy). These data structures are not unique to a particular programming language.

The notation of predicate calculus appears frequently. An individual is one of the basic objects or reasoning elements, loosely equivalent to a logical sort. The collection of individuals comprises the universe of discourse, sometimes referred to as the domain. A situation consists of a collection of individuals and the processes that are occurring. The environment consists of a situation in computational terms, including statements regarding the relationships between individuals, parameters of
the objects, logic expressions, and so on. Sentences about *individuals* and their relationships and properties are given as *argument-expressions*. Argument-expressions consist of named individuals and *predicates* designating the properties and relations of the individuals. For instance, in the sentence `Pressure(XENON)`, the predicate `Pressure` returns the value of "pressure" for its argument, an individual named `XENON`. Argument-expressions may appear in sentences among logical connectives. For example, the sentence

$$\forall g \text{ gas}(g) \Rightarrow (\neg \text{solid}(g) \land \neg \text{liquid}(g))$$

states that every gas $g$ is not a solid and is not a liquid. Other notation will be defined as it is required.

QPT describes *objects* and their *properties* at various times. The *parameters* of an object are those properties for which continuous values can be obtained, which may be interpreted as the object having a *quantity* of some parameter. The predicate `QuantityType` labels the kinds of quantities that an object can have. For example, a gas domain may require the quantities

- `QuantityType(pressure)`
- `QuantityType(temperature)`
- `QuantityType(mass)`

and so on. Generally, the predicate relations or functions are capitalized (e.g. `QuantityType`), object names are upper case (`XENON`), and variables are in lower case (`pressure`). The predicate `HasQuantity` assigns quantities to objects. For example,
a gas object XENON can have the quantities described above:

\[
\begin{align*}
\text{HasQuantity}(\text{XENON}, \text{pressure}) \\
\text{HasQuantity}(\text{XENON}, \text{temperature}) \\
\text{HasQuantity}(\text{XENON}, \text{mass})
\end{align*}
\]

Each quantity has the attributes *amount* and *derivative*, which are numbers that have a sign and magnitude. In QPT all derivatives represent changes in quantities with respect to time. The operators that extract information about numbers from quantities are

\[
\begin{align*}
A_m & \quad \text{the magnitude of the amount,} \\
A_s & \quad \text{the sign of the amount,} \\
D_m & \quad \text{the magnitude of the derivative with respect to time, and} \\
D_s & \quad \text{the sign of the derivative with respect to time.}
\end{align*}
\]

Signs of numbers can have only the values 1, 0 or −1.

The *measure* operator \(M\) relates parts of numbers to objects at particular times, as in

\[
(M \ A_m[\text{pressure}(g)] \ \text{Start}(j)) > 0,
\]

which states that the magnitude of the pressure of \(g\) at the start of a labelled time interval \(j\) is positive. To show that this magnitude is increasing over the same interval, we extract the sign of the derivative:

\[
(M \ D_s[\text{pressure}(x)] \ \text{Start}(j)) = 1.
\]
\[ [v] = \{ \text{stationary, subsonic, sonic, supersonic, hypersonic} \} \]

Figure 8.1: A totally-ordered quantity space for the velocity of a gas. The continuous value of velocity \( v \) has been separated into five regions stationary, subsonic, sonic, supersonic, and hypersonic. Even though the stationary and sonic regions represent only the point at which the Mach number is zero or one, respectively, the quantity space representation can support this sort of zero-width conceptual discretization.

The sign of the derivative is positive, meaning that the magnitude of the associated amount is increasing. Similarly, the modal operator \( T \) denotes the truth of some statement at a particular time. For instance,

\[
(T \text{ Expanding}(\text{XENON}) \text{ INTERVAL})
\]

means that the individual XENON is expanding during the time period INTERVAL. Time \textit{intervals} consist of the time consumed between a start point and a stop point. Time \textit{instants} are considered to be very short intervals.

### 8.1.2 Quantity Spaces

QPT employs the \textit{quantity space} representation of continuous numbers. As Section 4.1 describes, a quantity space is a discretized representation of a continuous quantitative value, described in terms of qualitative names for each of the intervals. Figure 8.1 depicts a typical quantity space, this one for the velocity of a gas.

QPT extends the quantity space concept by permitting \textit{partially-ordered} se-
Figure 8.2: A quantity space for the velocity of a gas flowing through a convergent-divergent nozzle. In this representation, the quantity at the head of the arrow is greater than the quantity at the tail. The partial ordering of this quantity space is dependent upon the conditions determining the nozzle performance mode. As shown, there are three possible quantity spaces for three different operating modes. Additional information from the simulation, such as pressures or mass flow rates, determines which operating mode is appropriate. The graph of the partial ordering indicates that certain quantities may be ordered with respect to another quantity, but not with respect to each other.

quences of qualitative values. While the quantity space still conveys ordinal relationships among many of the values, there may be values among which an ordering is not plausible, as illustrated in Figure 8.2. The active processes determine which elements belong in a particular quantity space, while qualitative calculus determines their ordering. This extension of the quantity space abstraction provides a convenient mechanism for making comparisons, rationalizing changes, and even
maintaining ambiguity if warranted in the physical system.

### 8.1.3 Qualitative Proportionality

The *qualitative proportionality* operator $\propto_Q$ provides a way to describe the functional dependencies between parameters. To state that parameter $x$ is qualitatively proportional to parameter $y$, we write

$$x \propto_Q y.$$  

If the relationship is strictly increasing or decreasing, then we can use the monotonic proportionality operators $\propto_{Q_+}$ or $\propto_{Q_-}$. For example, to show that the pressure $p$ of a constant mass of gas $g$ inside a closed container is directly proportional to the volume $V$ of its container, we may write

$$p \propto_{Q_-} V,$$

which states that whenever the gas volume increases (decreases), the internal pressure decreases (increases).

A *correspondence* relation provides a way to describe commonality between points in different quantity spaces. In a sense, this correspondence connects points in separate quantity spaces through $\propto_Q$ relationships. The syntax of a correspondence expression is

$$\text{Correspondence}(P, Q),$$

where $Q$ is a set of sentences that may be assumed true when the set of sentences $P$
are true. For example, consider a simple gas flow over a flat plate. Using a standard velocity profile, where $v_g$ is the velocity of the gas and $h$ is the height above the plate, we have

$$v_g \propto Q_+ h.$$ 

This provides a relationship between a velocity parameter and a length parameter, but provides no connecting points. We can provide these connecting points by stating constraints, such as the no-slip condition, in the form of correspondences. Assuming equilibrium conditions at the surface of the plate, the correspondence

$$\text{Correspondence}(h = 0, v_g = v_p),$$

constrains the velocity of the gas along the surface to be equal to the velocity of the plate (measured in the same reference frame). This correspondence unites quantities in a velocity space and a length space through a shared physical constraint.

While qualitative proportionality expressions describe relationships between parameters, they fail to provide insight into the magnitudes of proportionality. They are unable to represent the specific manner in which the dependent parameter behaves. These operators specify only that a relationship exists between two parameters, while in reality there may be many parameters that respond to a change in one influential parameter.
8.1.4 Histories and Episodes

QPT employs histories to represent the behavior of an individual over time (as described in Section 5.1.1). Electing to describe time and measure in terms of histories, Forbus defines the function

\[ \text{at}(i,t), \]

which provides the history slice for an individual \( i \) at time \( t \). Assuming applicability throughout all relations, the at function enables elimination of the \( M \) and \( T \) operators; for example, the expressions

\[(T \ \text{Expanding}(\text{XENON}) \ t1)\]

\[(M \ A[\text{pressure}(\text{XENON})] \ t1) > (M \ A[\text{pressure}(\text{XENON})] \ t0)\]

can be written

\[\text{Expanding}(\text{at}(\text{XENON}.t1))\]

\[A[\text{pressure}(\text{at}(\text{XENON}.t1))] > A[\text{pressure}(\text{at}(\text{XENON}.t0))].\]

To collect and reason about multiple individuals simultaneously, Forbus redefines the notion of a situation to mean a collection of history slices among various objects occurring at the same time. Individuals can interact only when their histories intersect. Histories intersect whenever an event ends an episode in one or more parameter histories. Where histories do not intersect, we presume there are no inter-object influences.
In addition, QPT permits two special kinds of histories, both slightly removed from the usual representations for concrete objects and parameters. A process history describes the activity of an abstract process instance over time. The episodes occurring in a process history are delimited by events, just as with object histories. A process episode denotes a period of time in which a particular process is active in a unique instance. Similarly, a view history describes the involvements of an abstract view instance over time (the notions of process instance and view instance and their part in these histories will be described below). The total history for a simulation is the union of all of the object and parameter histories with the process and view histories. Returning to the example of a gas flow through a nozzle, Figure 8.3 shows the result of combining process and parameter histories.

8.1.5 Influences

Influences are expressions stating how one quantity changes in relation to another quantity. Influences can describe direct or indirect relationships. Direct influences contain important causal information about how a process affects quantities. Indirect influences tell us that a quantity changes, but lack the causal traceability necessary for explanations. QPT assumes that a quantity is either directly or indirectly influenced, but not both.

The notation $I_+(Q,n)$ describes a direct influence in which $n$ is a number that has a positive influence on the quantity $Q$. Similarly, a negative influence is shown
Figure 8.3: An example total history for the subsonic flow of a gas $g$ through a convergent-divergent nozzle. This version of the history includes process episodes among the usual parameter episodes. The two processes denote periods of compression and expansion of the gas $g$. The process and parameter histories that intersect share the corresponding events of interaction.
with the \( I_- \) operator, while an unspecified influence is shown with \( I_\pm \). The effects of multiple direct influences are determined through qualitative calculus. For instance, if more than one direct influence acts on a quantity, its qualitative derivatives are the sum of the contributions of each influence. This is called *resolving* the influences. When both positive and negative influences are present concurrently, and additional information cannot resolve this ambiguity, then the simulation must branch to attack each possibility.

Indirect influences occur in quantities that are functions of other changing quantities. This sort of influence arrives through statements that include the qualitative proportionality operator \( \propto_q \). A process indirectly influences a quantity if the process directly influences a different quantity to which the first quantity is qualitatively proportional. Resolving indirect influences is more difficult than resolving direct influences because ambiguous cases cannot be decided by referring to the quantity spaces or by qualitative calculus.

Determining which physical quantity relationships portray the direct influences, and which portray the indirect influences, is one of the most important aspects of building the domain model. This is analogous to deciding which are the independent and which are the dependent variables in the model. Moreover, since all influences expressions in QPT are unidirectional, one must decide what the direction of causal propagation should be, as in the *causal directedness hypothesis*:

**Definition 2 (Causal Directedness Hypothesis)** *Changes in physical situa-
tions which are perceived as causal are due to our interpretation of them as corresponding either to direct changes caused by processes or propagation of those direct effects through functional dependencies.

These determinations are part of the larger problem in discerning and defining the actual processes. Some examples of these considerations, governing how influences and processes operate in domain models, are given in Section 8.3.

### 8.1.6 Individual Views

There are three abstract labellings of individuals in QPT called *individual views*, *processes*, and *encapsulated histories*. Any abstraction can be thought of as a template specifying required attributes for members of that sort [37]. An instantiation of an individual of either type involves filling the template slots with logical expressions characteristic to that individual.

An *individual view* models the dynamic nature of objects and their states within the domain. Each individual view captures a unique behavior of an object. There are four parts to an individual view:

**Individuals** A set of objects that must exist before the individual view can hold.

These objects are the focus of the reasoning activities defined by the other parts of the individual view. This part serves to gather relevant objects and map them to variable names, thereby providing a mechanism for establishing multiple instances of an individual view for different objects. Each instantia-
tion is called a view instance (VI).

**Preconditions** A set of statements constraining the conditions of objects or other related individual views required to hold before this individual view can hold. Establishing these **preconditions** ensures validity of the relationships defined below as **quantity conditions** and **relations**.

**Quantity Conditions** A set of statements constraining inequality relationships between object quantities. These statements also describe requirements governing co-existence of different individual views. For instance, an individual view might require that process $P_1$ be active before process $P_2$ can be active. **Quantity conditions** focus upon the dynamic quantitative relationships between objects.

**Relations** A set of statements that become true whenever the individual view holds. When the individual view ceases to hold, however, these statements do not necessarily become false: additional justification from the **relations** of another individual view may continue to support these statements. These statements become false when all supporting relations have vanished.

These parts of an individual view become labelled members of a computer program data structure definition that represents the template (as in Figure 8.4). Objects described in this manner are useful because they provide implementation-independent
Individual View Liquid

Individuals:
\[ \mathcal{Z} ; \text{a piece of matter} \]

Preconditions:
\[ \emptyset \]

Quantity Conditions:
\[ \neg \text{Solid}(\mathcal{Z}) \]
\[ \neg \text{Liquid}(\mathcal{Z}) \]
\[ \neg (\mathcal{T}_z > \text{BoilingTemp}(\mathcal{Z})) \]
\[ \neg (\mathcal{T}_z < \text{MeltingTemp}(\mathcal{Z})) \]

Relations:
\[ p_\mathcal{Z} \ll Q_- v_\mathcal{Z} \]
\[ \mu_\mathcal{Z} \ll Q_- T_\mathcal{Z} \]
\[ h_\mathcal{Z} \ll Q_+ T_\mathcal{Z} \]
\[ [\rho_\mathcal{Z}] = 0 \]

Figure 8.4: An example of an individual view data structure. The various parts define labelled sections which delimit lists of qualitative mathematical statements. This example shows no preconditions, four quantity conditions, and four relations. The variable names represent pressure \( p \), temperature \( T \), enthalpy \( h \), viscosity \( \mu \), velocity \( v \) and density \( \rho \).

descriptions of expressions and data.\(^1\)

The collection of all individual views is the view vocabulary of the domain. The collection of VI's for which the preconditions and quantity conditions are true in a particular situation is the view structure of that situation.

\(^1\)Forbus sometimes writes the descriptions of qualitative process templates in predicate calculus notation to provide some insight into how the data structures operate. These illustrations will not be provided here, since the notation requires the definition of several supporting operators that are not required elsewhere in this paper. Furthermore, there have been some questions raised within the AI community as to the necessity (or validity) of these attempts to describe QPT in terms of traditional logic (see, for example [14]).
8.1.7 Processes

In QPT, *processes* provide the means for affecting the dynamic nature of objects over the course of time. The active processes determine which events occur by controlling how each parameter episode begins and ends. A *process vocabulary*, which specifies the set of all processes, provides the basic elements of any dynamic physical model of the domain.

There are five parts to a process description, four of which are very much like the corresponding parts in an individual view:

**Individuals** A set of objects upon which the process operates. These objects take part in the determination of satisfaction of preconditions and quantity conditions. Different sets of objects that satisfy the *individuals* bindings may impose unique instantiations of the same process. Each set of objects describes a *process instance* (PI). Process instances may also be among the individuals.

**Preconditions** A set of statements about the individuals that serve to validate the process *quantity conditions* and *relations*.

**Quantity Conditions** A set of inequality statements describing the quantity relationships between the individuals. Unlike individual views, the *quantity conditions* for a process also include statements about the status of processes and individual views. When both the *preconditions* and *quantity conditions* are true, the PI is said to be *active*. Otherwise, the PI is *inactive*. The set of
active PI's is the process structure of a situation.

**Relations** A set of statements describing what impositions the process makes on the parameters of the individuals. *Relations* also include the description of any new individuals created by the process.

**Influences** A set of statements describing what influences the process imposes on the parameters of the individuals. These statements describe only *direct* influences.

Figure 8.5 provides an example of a process definition.

A unique process episode occurs during the period of time when a process is instantiated. The events that begin and end an episode relate to the establishment of conditions satisfying the preconditions and quantity conditions for a collection of individuals. Thus, these individuals and conditions are held constant throughout the episode. When the quantity conditions for an active process are no longer satisfied, that process instantiation becomes inactive and the associated episode terminates.

### 8.1.8 Encapsulated Histories

A *sequential combination* is an irreducible series of situations in which various processes act on the same subset of individuals. This series of situations is a piece of the total history that can be replaced by a new entity in the form of an en-
isentropic change of state process specification

Process IsentropicFlow

Individuals:
IdealGas; an ideal gas individual view
AnyFlow; the generic gas flow processes
AdiabaticFlow; adiabatic flow processes
ReversibleFlow; reversible processes

Preconditions:
∅; no external conditions

Quantity Conditions:
\[ s_2/s_1 \approx 1 \]; constant entropy

Relations:
\[ p_2/p_1 \approx [T_2/T_1] \]
\[ p_2/p_1 \approx (1 \oplus [M_1]) \odot (1 \oplus [M_2]) \]
\[ \rho_2/\rho_1 \approx [T_2/T_1] \]
\[ p_{02}/p_{01} \approx 1 \]
\[ p_2^*/p_1^* \approx 1 \]
\[ \rho_2^*/\rho_1^* \approx 1 \]

Influences:
∅

Figure 8.5: An example of a process data structure. As in an individual view structure, the various parts define labelled sections which delimit lists of qualitative mathematical statements. This example process structure provides qualitative information about the isentropic flow of an ideal gas, requiring several processes and individuals along with the quantity conditions of no entropy change.
capsulated history. While not a process in itself, the new entity represents the net effect of several processes considered together as an abstraction. A good example of an encapsulated history is a harmonic oscillation, in which one cycle of motion repeats many times. The processes involved in progressing from one node to the next can be combined into one piece of history. Substituting the encapsulated history into the total history wherever the cycle occurs reduces the complexity of the total simulation.

Instantiation of an encapsulated history occurs precisely like an individual view or a process. The data structure specifying the characteristics of an encapsulated history contains the same four elements as an individual view, while operating in slightly different manners:

**Individuals** The set of pertinent individuals for the history. These individuals must exist before the encapsulated history can occur.

**Preconditions** A set of statements describing necessary external situation conditions for the encapsulated history, in terms of episodes involving the individuals. Satisfaction of these conditions is prerequisite to evaluating the *quantity conditions*.

**Quantity Conditions** A set of statements describing internal situation conditions, in terms of episodes involving the individuals. Whenever the *quantity conditions* are true for a set of individuals, the history described in the
*relations* portion is instantiated as part of the history of those objects.

**Relations** The *relations* element describes a schematic history for the *individuals*.

An example of the definition of an encapsulated history is shown in Figure 8.6.\(^2\)

### 8.1.9 P-Components

Another abstraction supported by QPT is the *shared parameter combination*. This combination is a set of processes and individual views, known as a *p-component*, that act collectively on an individual. This mutual action can be considered as an independent process consolidating the compound effect of the processes and individual views comprising the p-component. Formally, a process or individual view instance \( P \) is in the same p-component as a process or individual view instance \( Q \) if

- \( P \) influences a quantity mentioned in \( Q \)'s quantity conditions,
- \( P \) influences a quantity influenced by \( Q \),
- the quantity conditions of \( P \) and \( Q \) mention the same parameter, or
- an influence of \( P \) is propagated by an influence of \( Q \).

Reasoning about p-components independently provides a simple solution to the *local evolution* problem described in Section 5.1.1. The set of p-components remains

\(^2\)Cohn points out that encapsulated histories violate the principle of *composability*, since the history does not describe the components of the situation [14]. Furthermore, it remains unclear how the QPT deduction mechanism handles encapsulated histories.
: example encapsulated history

Encapsulated History NozzleFlow(g.i.e)

Individuals:
- IdealGas(g); the gas individual
- State(g,i); the nozzle intake individual
- State(g,e); the nozzle exhaust individual
- State(g,t); the nozzle throat individual
- IsentropicFlow(g,i.e); isentropic process

Preconditions:
\[ \emptyset \]

Quantity Conditions:
- \( M(i) < 1 \); subsonic intake
- \( M(t) < 1 \); subsonic throat
- \( (A(t) < A(i)) \land (A(t) < A(e)) \); throat area is minimum

Relations:
; historical composition of basic processes
- SubsonicExpansion(g,i.t); acceleration before throat
- SubsonicCompression(g,t.e); deceleration after throat
; relations for the special case
- \( M(e) < 1 \); subsonic exhaust.
- \( p_{0i} = p_{0e} \); no loss of stagnation pressure

Figure 8.6: An example of an encapsulated history data structure. The syntax is similar to that of an individual view, containing individuals, preconditions, quantity conditions, and relations. This example suggests an encapsulated history for the subsonic flow of an ideal gas through a convergent-divergent nozzle, wherein the incident flow undergoes an expansion process in the convergent portion, then experiences a compression process in the divergent portion. More complicated histories might have a network of processes and conditional expressions as relations.
constant until the process structure changes, therefore they can be reasoned about independently, propagating their effects until the process structure changes. When the process structure changes, the results of the individual p-component simulations can be recombined to generate a new process structure, and thus a new set of p-components. Similarly, the intersection-interaction problem with histories is solved by considering that episodes can interact only when the processes that control them are part of the same p-component.

8.2 Inferential Operations

Models constructed using QPT support a spectrum of options for behavior analysis. For instance, it is feasible to generate the set of all possible processes and views given a set of individuals (which is analogous to envisioning the behavior). Given this set, along with an assignment of activity statuses to processes, one can generate a process structure that describes the physical system behavior. With the process structure one may predict future possibilities by determining value changes specified by the available time derivatives and influences.

The fundamental algorithms for producing these analyses are described in this Section.
8.2.1 Deduction

QPT deduction activity consists of a cycle of instantiation and limit analysis. Instantiation determines the validity of process and view structures, while limit analysis takes these structures and investigates how situations may change. This cycle continues until no new changes occur.

Instantiation collects the individuals from the domain description and, together with a library of view and process definitions, generates a set of view and process instances. With this set of instances, instantiation investigates the preconditions and quantity conditions to establish the view and process structures, along with the quantity amounts. Within the process structure, the influences resolve to determine the resulting quantity derivatives.

Given the representation of the situation after instantiation, the limit analysis algorithm determines which quantities change in subsequent states. This is accomplished by selecting neighboring values and reevaluating the conditions under which the views and processes hold. In terms of its use of quantity spaces and landmark values, limit analysis is one of the most fundamental techniques in qualitative reasoning. The technique of limit analysis proceeds as follows:

1. Select a quantity and determine the neighboring values in all directions.

2. For each direction,

   (a) If there is no neighboring value, then any change in value in that direction
is uninteresting to the qualitative simulation since the view and process structures will not change.

(b) If there is a neighboring value, determine how a change to that value will change the partial orderings in quantity space. The set of possible orderings is the \textit{quantity hypothesis} for the new situation:

i. If the quantity hypothesis set is empty, the view and process structures will not change.

ii. If the quantity hypothesis set is not empty, and the set contains orderings that change the view or process structure, then the set is also the \textit{limit hypothesis} for the new situation.

Certain combinatorial problems are likely to occur during a limit analysis of a complex system. Resolving these problems is largely a matter of including domain-dependent knowledge in the model in order to reduce the size of the search space. Since there may be many active processes, some of which might have multiple influences, there may be several parameters changing values in any situation. This may result in an interpretation that is inconsistent, requiring knowledge about which changes are mutually exclusive or globally unimportant to resolve the inconsistency. The simulation must also consider the assumptions about continuous changes of the parameter and its derivatives, and which changes preclude other changes. Search space complexity (as well as the ambiguity resulting from qualitative calculus) might also be resolved with other reasoning techniques. Hybrid reasoning systems that
confront these problems might combine quantitative techniques [22, 62], teleological relationships [23], probabilities [34], default or observed values [50, 53, 54], or order-of-magnitude relationships [18] with the qualitative reasoning system.

The result of the instantiation and limit-analysis cycle is a set of process structures that comprise the nodes of an envisionment. Branching in the envisionment occurs, as in every other qualitative reasoning ontology, due to ambiguity. Ambiguity in QPT arises directly from influence resolution and processes that influence multiple quantities, and indirectly from the partial ordering of the quantity spaces.

Summarizing, the basic deduction algorithm proceeds as follows:

1. Form a list \( S \) consisting of the initial qualitative state.

2. Until the list is empty, perform the following steps:

   (a) Remove an element of the list, and call it the current situation \( s \).

   (b) Determine the view and process structures for \( s \).

   (c) Resolve indirect influences to determine the derivative values for each quantity.

   (d) Perform limit analysis to determine what sort of transitions are possible from \( s \).

   (e) Create a state for each possible transition. For each state that has not already been visited, add that state to \( S \).
This algorithm propagates an attainable envisionment from a given initial state. This sort of envisionment is suitable for the conceptual phase engineering design. A total envisionment is the envisionment for all possible initial states. Total envisionments are necessary for diagnosis and measurement interpretation problems, in which the initial state is often unknown. Generating total envisionments requires a slightly different algorithm and raises a number of computational efficiency concerns. Section 8.2.4 briefly addresses these issues.

8.2.2 Heuristics

Heuristics provide convenient knowledge applicable to the problem of ambiguity reduction. They help to reduce the complexity of reasoning problems by suggesting normal paths of deduction or by ruling out unlikely hypotheses. Commonly, model developers acquire heuristics from persons having considerable experience in a certain field. Appropriate use of this heuristic knowledge usually is prerequisite to declaring that an agent is an "expert" decision maker.

A QPT model may employ several heuristics concerning processes. The ineffectuality heuristic assumes that a process is not active if its results are not distinguishable [50]. A distinguishable result is one that affects a value by more than some small tolerance level. For example, an evaporation process affecting a quantity of liquid is likely indistinguishable in the presence of a pouring process. Another heuristic is the equality change law.
Definition 3 (Equality Change Law)  *With two exceptions, a process structure lasts over an interval of time. It lasts for an instant only when either (1) a change from equality occurs, or (2) a change to equality occurs between quantities that were influenced away from equality for only an instant [51].*

The thrust of this law is to state that changes to equality take a finite amount of time, while changes from equality occur instantaneously. This law does not, however, prohibit envisionment cycles of consecutive instants, which Forbus calls *stutter cycles*.

### 8.2.3 Closed-World Reasoning

The *closed-world assumption (CWA)* is the assumption of complete knowledge about which positive facts are true in the world [43]. Closed-world reasoning accepts positive facts as the base set of beliefs, and sanctions negative conclusions whenever a statement does not logically follow from the base set of beliefs [66]. Conclusions therefore can be drawn from the presence of supporting evidence in the absence of contradictory evidence.

QPT necessarily adopts the CWA as a consequence of the sole mechanism assumption. Under the CWA, we presume that process vocabularies for a given domain are complete. No other objects from outside of the domain enter into the model, and no additional processes can occur.\(^3\) This implies that the inferences

\(^3\)The assumption that all of the domain objects can be named using all of the object and function
we make about processes, objects and quantities are sound. As a consequence, we assume that we know all of the influences acting on a parameter, therefore we are able to determine all of the potential ways a situation can change. Without the CWA, qualitative physical models would be unable to simulate dynamic behavior. Inferences made by the reasoning system might be inconsistent, incomplete, or impossible. Therefore, unfortunately, the CWA does not preclude ambiguous results.

QPT actually considers the constituent views and processes as open-world model fragments, which, when composed to represent a situation and its collection of influences, becomes a closed world. Closing the world allows the system to consider uninfluenced quantities as constant and to combine multiple influences into an aggregate effect. The net result allows us to treat closed-world influences as constraint equations.

### 8.2.4 Qualitative Process Engine

Forbus' recent implementation of QPT is the Qualitative Process Engine (QPE), a problem solver that collaborates with an ATMS to perform qualitative simulation [59,61]. Although the explanation of QPE requires a much deeper implementation discussion than is provided above, it is interesting to discuss, even at a shallow

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constants in the language is sometimes referred to as the domain closure assumption [66]. This assumption is significant because it normally allows the removal of quantifiers from logic sentences. While we assume that no other objects enter into QPT models from outside of the domain, this does not rule out creating or destroying objects using processes, so quantifiers remain in the sentences.
depth, how the ATMS manages the assumptions required for history generation.

QPE uses a domain model to generate an envisionment for a given scenario. The envisionment consists of connected situations, each of which is an ATMS environment. An environment is the set of assumptions governing the believe (or disbelief) of a certain fact in that situation. The set of possible environments asserting belief of a fact is the label for a node. The status of the node (IN or OUT), derived from its justifications and environments, dictates whether the belief of the associated fact is held in the situation. Justifications enable propagation of environments to other nodes, thereby determining the consequences of assumptions for a particular context. These consequences serve to define the possibility of a fact holding in a certain situation, which in turn identifies the need for an ATMS to manage the assumptions pertaining to that fact.

To avoid having to restate persistent assertions in each new situation, there is a certain economy found in assuming a many-worlds database, which degrades the notion of time to an implicit representation. The many-worlds database permits the same justification to partake in many situations, while sacrificing the ability to compare values in different situations. For example, the explicit temporal representation

\[(\text{StaticPressure (at Inlet S1)})\]

becomes the implicit representation

\[(\text{StaticPressure Inlet}).\]
This sort of assumption is especially important in models that do not consider
temporal changes, such as those used for steady-flow problems.

8.3 QPT Examples

Given the exposition of QPT fundamentals, along with the discussion of some of
the automated reasoning considerations from Section 8.2, we now consider a few
simple examples of Qualitative Process Theory in action.

Example 1 (Influence Diagrams) Consider the simple situation shown in Fig-
ure 8.7, in which a tank expels a gas through a pipe into free space. For this sit-
uation we assume two individual views, ContainedGas (Figure 8.9) and FreeSpace
(Figure 8.10), and a GasExpel process view (Figure 8.11) activated by a pressure
difference across the pipe. The region of free space is so immense that the mass
of gas expelled into it from the tank changes neither the pressure nor the mass of
free space (somewhat like a simple propulsion device). We therefore might select the
trivial quantity spaces

\[ p_S \in \{\hat{0}\} \]
\[ p_T \in \{\hat{0}, \hat{+}\} \]
\[ m_S \in \{\infty\} \]
\[ m_T \in \{\hat{0}, \hat{+}\} \]

for this problem, where \( S \) identifies the free space object and \( T \) identifies the supply
tank object. Assuming the initial mass in the tank \( m_T = \hat{+} \), we may instantiate the
ContainedGas and FreeSpace individual views. This instantiation asserts the corresponding influences on the world. With these influences from the “open” world, one may construct a diagram showing both direct and indirect relationships between the parameters. Figure 8.8 suggests one approach, showing both the positive (+) and negative (−) influences each parameter has upon any other parameter in the model; for example, if the pressure in space ps increases, it has a negative effect on the mass flow rate mw, causing it to decrease (with all else held constant). Once the influences have been asserted, we “close” the world, thereby stating that the current individuals and processes are the only ones involved in the model. In the closed world, the combined influences become the constraints for the model. With these constraints and the initial conditions in hand, we can activate the processes to generate an (attainable) envisionment. The initial conditions comprise the state that is the first element of the list of states to be investigated. Activating the processes involves carrying out a limit analysis, determining which values change for subsequent states in the envisionment. If more than one value may change independently, the envisionment branches to address each value individually. If we allow cycles in the envisionment, then it is necessary to determine whether the new states are identical to ones generated earlier; if so, the envisionment graph may point back to the previous state rather than adding a new (repeated) state. Only new states need to be added to the list of states yet to be investigated. Once this list of states has been depleted, the envisionment is complete. In this example, when the tank is empty,
Figure 8.7: A simple gas expansion system. The compressed gas inside the tank exhausts through a pipe into free space until the gas is depleted. The consequence of this action is negligible to space.

Figure 8.8: An influence diagram for the parameters of the tank expulsion example. Combining the active individuals and processes enables construction of a diagram of both direct and indirect influences in the model.

_The condition_ $\text{Contains}(T, \text{GAS})$ _is no longer satisfied. This requires something like an encapsulated history, activated on the condition of no mass or pressure in the tank, to describe depletion. Alternatively, we might be satisfied with deactivation of the GasExpel process once the inequality condition $p_T > p_S$ fails._

**Example 2 (Transition Graphs)** The QPT deduction process may be depicted as a directed graph showing the possible transitions between various states. Each
Individual View ContainedGas(G,T)

Individuals:
Gas(G)
Container(C)

Preconditions:
Contains(C,G) ; gas is inside container

Quantity Conditions:
m_G > 0 ; the mass of gas is positive

Relations:
p_G \propto q_+ m_G ; pressure increases with mass
p_G \propto q_- V_G ; pressure decreases with volume
Correspondence(p_G = 0, m_G = 0)

Figure 8.9: A simple individual view template ContainedGas describing conditions and relations for a gas inside a container.

Individual View FreeSpace(S)

Individuals:
Region(S) ; some geometrical description

Preconditions:
InfiniteSink(S) ; region absorbs all objects

Quantity Conditions:
0

Relations:
p_S = 0 ; vacuum region
m_S = \infty ; infinite mass

Figure 8.10: A simple individual view template FreeSpace describing relations for an infinite sink.
simple gas expulsion process
alternative IV perspective
Process GasExpel(GAS.T.S,P)

Individuals:
   ContainedGas(GAS.T); the gas involved
   Space(S); free space
   Connection(P,T,S)

Preconditions:
   OpenPath(P)

Quantity Conditions:
   \( p_T > p_S \); driving pressure difference

Relations:
   \( \dot{m}_T \propto q_+ (p_T - p_S) \)

Influences:
   \( I_-(m_T,\dot{m}_T) \)

Figure 8.11: A simple process template for a gas expulsion driven by a pressure gradient.

state situation consists of an individual structure (IS) and process structure (PS),
while the limit hypothesis (LH) characterizes the transition between these states.
Consider a problem of three open liquid tanks (T1, T2, and T3), interconnected by
two pipes (P12 and P23), as illustrated in Figure 8.12 (from [51]). Initially, the
quantity of liquid is different in each tank, though there is no flow between them
because the valves are closed. After opening the valves, as our intuition suggests,
the liquid will flow through the pipes until a force equilibrium is reached between the
tanks. The process view LiquidFlow describing the basic transfer process is shown in
Figure 8.13. This process activates for a flow through a pipe between two contained
liquids src and dst. The ContainedLiquid concept (due to Hayes [73]) describes
matter (or stuff) existing in a container and satisfying the requirements of a liquid
individual view, and therefore possessing certain properties and a functional measure of how much of the container the liquid consumes. The precondition OpenPath ensures that the pipe under consideration provides an unobstructed path between the source and destination tanks; e.g. any valves in this pipe must be open,

\[ \forall v \in \text{valves}(\text{path}) \text{Open}(v), \]

where valves(x) returns the set of all valves associated with the path x. The basic relation held in the LiquidFlow process is that the mass flow rate through the connecting pipe is qualitatively proportional to the pressure difference across it. The direct influences state that an increasing mass flow rate between the source and destination objects positively influences (increases) the mass of liquid at the destination, and negatively influences (decreases) the mass of liquid at the source. After instantiating the liquid in the tanks as ContainedLiquids and opening the valves, we impose these influences as constraints and close the world, thereby eliminating all other processes except LiquidFlow. Through the limit analysis procedure and the resulting transition graph, we determine that stutter cycles are possible between adjacent situations (Figure 8.14). These graph cycles are characterized by the equality change law as stutter i (happening only for an instant) rather than real physical oscillations that we cannot describe with this simple qualitative model.

Example 3 (Specifications) Forbus [51] suggests several interesting individual and process view specifications for various object, problems and situations. A few
Figure 8.12: A three-tank and two-pipe system for an equilibrium liquid flow model.

```plaintext
; simplified flow of a liquid

; Process LiquidFlow(src, dst, path)

  Individuals:
  ContainedLiquid(src): the flow source
  ContainedLiquid(dst): the flow destination
  Connection(path, src, dst): pipe connecting src and dst

  Preconditions:
  OpenPath(path): pipe is unobstructed

  Quantity Conditions:
  \( p_{src} > p_{dst} \): driving pressure difference

  Relations:
  \( \dot{m}_{path} \propto Q_{+} \left( p_{src} - p_{dst} \right) \)

  Influences:
  \( I_{+}(m_{dst}, \dot{m}_{path}) \)
  \( I_{-}(m_{src}, \dot{m}_{path}) \)
```

Figure 8.13: A QPT process template for a simple LiquidFlow, used in the transition graph demonstration of behavior "stutter."
Figure 8.14: A transition graph resulting from limit analysis on the three-tank two-pipe liquid equilibrium model. The node with the empty process structure $PS = \emptyset$ represents the equilibrium state. The transitions labelled $i$ show transitions that occur in an instant, representing stutter in the qualitative model.
of them are provided here as examples of how QPT modeling specifications might be written. Figure 8.15 shows an individual view Elastic for an elastic object. Such an object might also be described by more specific individual views that specify quantity conditions for the state of the object, such as Relaxed or Stretched. Figure 8.16 shows an encapsulated history Explode that describes how an object may cease to exist under certain conditions. In this example, the pressure inside a rigid container increases over some interval \( e \), finally reaching the burst pressure of the container. The end of the episode is an event that Terminates the individual involved. Finally, Figure 8.17 shows another encapsulated history Collide that describes a collision between two bodies, raising interesting questions about how to model geometry, interaction, and time. This history uses the process Motion (Figure 8.18) to convey the geometrical information about velocity and direction.

### 8.4 Applications

Qualitative Process Theory has been applied to a variety of problems, primarily by Forbus, to demonstrate its utility. Some of these applications, and the qualitative reasoning issues they address, are described here.

Forbus [50] shows how to apply QPT to the problem of interpreting measurements and observations in physical systems. This application assumes that a domain-dependent method for converting numerical measurement data into qualitative values exists, and that the envisionment is computable. The input consists
Figure 8.15: An individual view for an elastic object, specifying the properties of that object and certain correspondences between those properties.

Figure 8.16: An encapsulated history suggesting termination of a container when the internal pressure reaches the burst pressure.
Encapsulated History Collide(A,B,dir,e)

Individuals:
  Object(A)
  Mobile(A) ; A is movable
  Object(B)
  ~Mobile(B) ; B is not movable
  Direction(dir)
  Episode(e); any episode

 Preconditions:
  (T contact(A,B,dir) start(e))
  (T direction-of(dir,velocity(A)) start(e))

 Quantity Conditions:
  (T Motion(B,dir) start(e))

 Relations:
  (M A[velocity(A)] start(e)) = -(M A[velocity(A)] end(e))
  (M A[velocity(B)] during(e)) = 0
  duration(e) = 0
  (T contact(A, B, dir) end(e))

Figure 8.17: An encapsulated history describing the result of a collision event between a mobile and an immobile object.
of a time-referenced sequence of measurements for quantities which are to appear in the envisionment. The application collects the periods of time into segments that delineate unique local behaviors for each quantity. Global segments cut across all of the segments for the individual parameters, representing an ordered sequence of unique global behaviors. Each interpretation of a global segment consists of one or more qualitative states from the envisionment. Waltz filtering reduces the set of possible interpretations by checking whether adjacent global segments have successor qualitative states. Ambiguities can be reduced by adding different measurements or domain-dependent heuristics. The final result is a set of interpretations which describe what is happening in the observed situation.

Forbus [60] describes how action-augmented envisionments can incorporate knowl-
edge of an agent's actions for reasoning about planning and procedure-generation problems. Relations within each state of the envisionment may carry with them an action for changing their value, such as a procedure OpenValve(V, t0, t1) for changing the value of ValveClosed(at(V, t0)) to ValveOpen(at(V, t1)). These capabilities permit development of instructions for achieving a goal, such as "increase the water level in this container," based on knowledge of actions and of how processes achieve a desired effect. The action-augmented envisionment provides the search space for determining a suitable course of action.

D'Ambrosio [19] extends the qualitative mathematics behind QPT to better apply it toward control system applications. Adding support for the linguistic variables of fuzzy logic into QPT, he shows how this enhancement provides a way to reduce the ambiguity of qualitative calculus. In particular, linguistic variables allow quantity measurements and relationships to be represented in terms of uncertainty and likelihood. D'Ambrosio also addresses some of the issues with reasoning about actions that are addressed independently by Forbus in [60].
Chapter 9

Summary

This dissertation is concerned with the development of a unique approach to modeling the domain of gas dynamics. Theories of dynamics describe the cause and effect of forces in the environment. In fluid dynamics, these forces can be characterized as processes that bring about change over time. Fluid dynamics models describe how liquids and gases behave in terms of processes characterized as laminar flow, turbulent flow, compressible flow, viscous flow, and so on. Rather than specifying analytical or empirical equations to describe dynamic behavior, this theory is developed under the notion of qualitative reasoning. This notion provides certain abstractions of the physical world that enable contemporary problem-solving techniques provided by research in the field of Artificial Intelligence. Certain misguided postulations that existing qualitative models of liquid behaviors are representative of the fluid dynamics domain unfortunately are improper generalizations. Accordingly, the qualitative model of gas dynamics developed herein does not purport to encompass the broad spectrum of fluid dynamics. Only after the individual special cases of gas and liquid dynamics have been defined can we attempt to generalize a theory of fluid dynamics.
Qualitative representation and reasoning ontologies fall into two categories: *explicit-mechanism* frameworks are those which presume a specific agent for change, while *implicit-mechanism* frameworks make no assumptions about how changes occur in a situation. Developing a model within an explicit-mechanism framework, such as QPT, is the more difficult task because it requires the model builder to state the physical mechanisms in terms of a specific agent for change [70]. In QPT, this agent is a physical *process*.

Qualitative Process Theory provides an excellent framework in which to construct physical domain models of the explicit-mechanism nature. The fundamental intentions behind any qualitative model are (1) to specify the vocabulary necessary for qualitative state representations, and (2) to express the interparameter relationships necessary to determine state transitions and thereby domain behaviors. The particular interpretation is not important, so long as it is understood. QPT provides a suitable vehicle for satisfying these intentions. Using this well-established framework, upon which theories from other domains already have been constructed, it should be possible to leverage the development of axioms in related domains into more powerful (common sense) theories of physical dynamics. Furthermore, once a complete set of physical models is available, then further abstractions can be identified to handle decomposition of complex dynamic systems into smaller problems that can be solved independently by the appropriate modeling technique.

A qualitative theory of gas dynamics is necessary to bring AI and the prac-
tice of qualitative reasoning into the engineering world to solve problems in design, monitoring, diagnosis, and training. Gas dynamics is more amenable to a representation based on processes that it is to representations based on constraints or components. One normally pursues gas dynamics problems by considering process notions, such as expansion or diffusion, not components or constraints. In keeping with the mission of common sense reasoning, the gas dynamics theory strives to avoid unwarranted complications. The theory should be grounded in the fundamental laws of nature presented in terms of a few frequently encountered (observable) measurements: common sense things like pressure and velocity instead of Reynolds Number or bulk modulus. The theory should exploit the sole mechanism assumption and the causal directedness hypothesis. The elements should not consider definitive event timings, as the notion of time is of little significance to a majority of gas dynamics problems. Consequently, the QPT notion of indirect influence seems misconstrued in this domain, warranting a new representation of change based more closely on partial derivatives and reference states. Part II of this dissertation describes a theory of gas dynamics that addresses these requirements, along with the various issues and considerations generally associated with qualitative physics applications identified in the preceding Chapters, and defines many elements from which to build qualitative models of this complex domain.
Part II

A Qualitative Theory of Gas Dynamics

"Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone."

—Albert Einstein [42]
Chapter 10

Overview

Part I of this dissertation describes the importance of qualitative reasoning to problems encountered both in engineering and in artificial intelligence research. It compares several techniques for performing qualitative reasoning, and presents some of the issues involved in capturing and invoking "common sense" knowledge about physical systems. It places particular emphasis on how various modeling ontologies might be used to carry out qualitative simulations. One of these ontologies, *Qualitative Process Theory*, is argued to be a useful formalism for representing the domain of gas dynamics.

Part II of this dissertation proposes a qualitative theory of gas dynamics. This theory identifies and organizes the important concepts of the gas dynamics domain so that models of gas behavior may be constructed for simulation and problem solving. It presents these concepts within the ontology characterized by Qualitative Process Theory; that is, physical changes are associated with active processes. While developing these process concepts from a general foundation laid in the fundamental laws of nature, the theory also incorporates knowledge specific to the domain of gas dynamics, which provides the *expertise* necessary to eliminate cer-
tain ambiguities arising from qualitative approximations.

The qualitative theory covers an amount of material roughly equivalent to that included in a senior or first-year graduate course in gas dynamics. It discusses the role of conservation laws, state equations, and thermodynamics, examining the behavior of gases from a macroscopic and Eulerian perspective. The theory considers applications of the governing equations to both internal (one-dimensional) and external (two-dimensional) flows, primarily under steady-state conditions. It exploits certain simplifications of the general case that, when combined with the ideal gas assumption, make idealized problems solvable. Along the way, the discussion weighs the impacts of these assumptions and simplifications when applied to qualitative reasoning problems. The technical material covered herein originates from [12] unless otherwise stated.

To begin the development, Chapter 11 proposes a new twist to a well-established knowledge representation technique. This technique, called qualitative ratios, is suitable for implementation in most qualitative physics systems, though it proves extraordinarily convenient for qualitative models of gas dynamics because of the frequent use of reference parameters in the ancestral quantitative theories. Importing these ratios from the quantitative theories makes the qualitative presentation more natural to an engineer, thereby promoting assimilation into computer-aided design and simulation environments.

Employing this new representation technique, Chapter 12 provides a brief qual-
itative reduction of the most general governing equations of gas dynamics. The primary equations include the laws of conservation of mass, conservation of energy, conservation of momentum, and the second law of thermodynamics, along with their one-dimensional simplifications. The secondary equations include state equations for various situations commonly encountered in gas dynamics problems. Chapter 13 elaborates on these general equations and derives qualitative models for internal flows. This Chapter also describes certain applications of these models and provides some illustrative examples of their utility in qualitative reasoning. Chapter 14 proposes a similar set of constructs for external flows, but argues that a complete qualitative representation cannot be devised without significant advances in other qualitative reasoning disciplines. Chapter 15 then considers several example envisionments and simulations for real gas dynamics problems, while also considering some of the modeling issues faced by qualitative reasoning systems. Finally, Chapter 16 summarizes the development of this theory, discussing what has been accomplished and identifying several research and application paths which might build upon these accomplishments.
Chapter 11

Qualitative Ratios

Problems in the field of gas dynamics usually describe interesting parameters and interparameter relationships in terms of ratios. These ratios provide a way to state a relationship with respect to some reference value, such as the relationship between a dynamic pressure ($p$) and stagnation pressure ($p/p_0$) or the pressure achieved in the critical state ($p/p^*$). Alternatively, the ratios may describe a relationship between similar measurements at different physical locations in the flow ($p_2/p_1$). In either case, the ratio provides a measure of a relationship between the value of physical parameter and its equivalent value in another (perhaps imaginary) flow state. The result is a measure of relative change between two quantities. In qualitative physics systems, determining and propagating these relative changes is the primary method for simulating the behavior of physical systems. For the gas dynamics domain, we will use qualitative ratios to represent these relationships.

One frequently encountered example of such a ratio is the Mach number $M$: the ratio of the local gas velocity to the sonic velocity. Qualitatively, one may separate the information provided by the Mach number into four important flow regimes according to whether the value indicates a stationary, subsonic, sonic, or
supersonic flow. The interval selection inequalities for this decomposition are $M = 0, 0 < M < 1, M = 1,$ and $M > 1$. This simple decomposition is made possible by the fact that the parameter is a ratio and because the physical parameters are non-negative.\footnote{By convention, we describe all measurements with respect to non-negative bases, absolute zero, and sequential states in the downstream flow direction. We assume the few measurements that have a real meaning when described as a negative value will either be interpreted in the opposite sense or as a non-ratio value. For example, one may describe a wall force as being applied either \textit{by the wall on the fluid} or \textit{by the fluid on the wall}. Under the intervals selected here for describing ratios, only one interpretation of a physical parameter (such as wall force) will permit consistent interpretation within the models.} Furthermore, it transpires that almost \textit{all} ratios of interest in gas dynamics problems have the same set of intervals! The four intervals delimited by the inequalities are the \textit{values} of the qualitative ratios comprising the Mach number quantity space.

Conveniently, qualitative ratios also import the additional luxuries of \textit{non-dimensionality} from the quantitative to the qualitative representation. Non-dimensional parameters permit a significant reduction in the number of qualitative state spaces that might have been required in a dimensional representation, because most non-dimensional ratios have the same state space. Using these non-dimensional parameters in equations (or confluences) reduces our worry over matching dimensions on both sides of the equation, and in most cases adherence to this standard eliminates annoying proportionality constants.

In order to make use of these ratios in the qualitative theory, we first define some of their algebraic properties. While doing this we also see how a mixed, or \textit{hybrid qualitative-quantitative}, algebra works to exploit the advantages of each sys-
tem. This hybrid algebra for ratios is based loosely on the SRI algebra developed by Williams [147], which has been developed in much greater detail and formality than is pursued here. While this hybrid algebra of ratios has not yet been implemented, Williams' algebras have been implemented in a qualitative symbolic algebra program called Minima [146,147].

11.1 Qualitative Ratio Spaces

Recall from Section 4.1 the simple abstraction of the sign of a quantity as its quantity space. In this abstraction, a physical value takes on one of the three values denoting the sign of that value through our use of the $\lfloor \rfloor$ operator. The possible values are elements of the set $Q_\Delta \equiv \{-, 0, +\}$. Specifically, the elements of this set represent any value within the intervals $(-\infty, 0]$, $[0, 0]$, and $(0, +\infty)$ respectively. To abstract a positive physical quantity $p \in \mathbb{R}$ to this set then we use the expression $[p] = +$, so that $[p] \in Q_\Delta$. For convenience, let us also add to $Q_\Delta$ an element ? that spans all of the other intervals, $(-\infty, +\infty)$, so that now $Q_\Delta \equiv \{-, 0, +, ?\}$. The element ? will be used to denote ambiguity, as this interval contains all of the others. Using the sign operator and the set $Q_\Delta$ we can draw upon the various mathematical operators from Section 4.2 to solve basic confluence equations.

Now let us diverge from convention and change the representation of the domain such that there are no values $r$ less than zero, or $r \in \mathbb{R}^+ \equiv \{x : x \in \mathbb{R} \land x \geq 0\}$. Such is the case when dealing with ratios of non-negative numbers, as we use frequently
in gas dynamics. Here the interesting set of intervals becomes \([0, 0], (0, 1], [1, 1]\) and \((1, +\infty)\), which we represent by the symbols \(\hat{\mathcal{O}}, \hat{L}, \hat{I}, \hat{G}\) respectively. Again, we use the symbol \(\hat{?}\) to denote overlapping intervals, and define the *qualitative ratio quantity space* as the set

\[
Q_R = \{\hat{\mathcal{O}}, \hat{L}, \hat{I}, \hat{G}, \hat{?}\}.
\]

This quantity space \(Q_R\) is similar to \(Q_\Delta\) but it has two landmarks, 0 and 1, and reaches infinity only in one direction. If we also bend the abstraction operator \([\ ]\) somewhat to map elements of \(\mathbb{R}^+\) to elements of \(Q_R\) so that if \(p \in \mathbb{R}\) then \([p] \in Q_R\) and \(p \in [p]\). Thus the symbol assignments are as follows:

\[
|r| = \begin{cases} 
\hat{\mathcal{O}} & \text{if } r = 0 \\
\hat{L} & \text{if } 0 < r < 1 \\
\hat{I} & \text{if } r = 1 \\
\hat{G} & \text{if } r > 1 
\end{cases}
\]

Since the nature of the physical domain guarantees that we will not encounter a negative ratio in any consistent interpretation, we have eliminated values less than zero from the number line while maintaining a fully-ordered set of elements with meaningful quantizations. For some physical parameters, of course, one or more of these intervals may be divided into two or more subintervals, providing further quantization of important physical behaviors. For example, the space of qualitative values for the ratio of dynamic pressure to stagnation pressure might be defined as

\[
[p/p_0] \in \{\hat{\mathcal{O}}, (0, r_{p_0}), [r_{p_0}, r_{p_2}]; (r_{p_2}, r_{p_3}); [r_{p_3}, r_{p_5}]; [r_{p_5}, r_{p_6}]\}.
\]
\[(r_{p_{c2}}, r_{p_{c1}}), [r_{p_{c1}}, r_{p_{c1}}], (r_{p_{c1}}, 1), \hat{i}, \hat{G}\],

where \(r_{p_{c1}}, r_{p_{c2}}, \) and \(r_{p_{c3}}\) are the so-called critical pressure ratios delimiting meaningful ordered subdivisions of the qualitative interval \(\hat{L}\). We will address the benefits of these decompositions as we uncover their utility in problem solving.

Although we have eliminated an infinity by introducing another interval boundary, we shall see below that we must still deal with ambiguity in the various algebraic combinations of these intervals. In fact, the algebraic composition of these elements is just as ambiguous as the elements in \(Q_{\Delta}\). Therefore, aside from one technical observation to be discussed in Section 11.4, the introduction of the \(Q_R\) quantity space is merely for convenience in the translation between quantitative and qualitative interpretations. Furthermore, despite the wide occurrence of ratios in gas dynamics, some of the source expressions encountered in this theory involve abstractions of terms that are more appropriately mapped into the \(Q_\Delta\) space than the \(Q_R\) space. We will address these terms individually and argue that the abstraction process can fairly eliminate them from the final qualitative expressions.

### 11.2 Qualitative Ratio Algebra

Given the qualitative ratio quantity space \(Q_R\), we can define a hybrid algebraic system \(RR1\) governing their application in symbolic expressions. Like \(SR1\), which combines the qualitative signs with real numbers, \(RR1\) combines qualitative ratios with real numbers. Either system is a kind of interval algebra, permitting various
mathematical operations to be made in a restricted (finite) domain of real-number intervals. The algebra $RR1$ is specified as the system

$$RR1 = (Q_R \cup \mathbb{R}, \oplus, \otimes, [])$$

where $Q_R \cup \mathbb{R}$ is the domain and $\oplus$, $\otimes$, and $[\ ]$ are operators on that domain. The operators $\oplus$ and $\otimes$ are equivalent to $+$ and $\times$. The desired $\oplus$ and $\otimes$ operators corresponding to $-$ and $/$ can be defined in terms of the $\oplus$ and $\otimes$ operators.

Section 4.2 described some of the limitations of interval algebras; most notably the lack of an additive inverse. This limitation applies, however, only to interval terms of non-zero width. So while we cannot rearrange the expression

$$[a] \oplus [b] \approx [c] \oplus [d] \not\approx [a] \oplus [b] \otimes [c] \oplus [d] \approx \hat{0}$$

for the general case of unknowns $[a],[b],[c],[d] \in Q_R$, we are still able to rearrange the terms of zero-width intervals $\hat{0}$ and $\hat{1}$ in qualitative expressions, such as

$$[a] \oplus [b] \otimes \hat{1} \approx \hat{0} \Rightarrow [a] \cong [b] \approx \hat{1}.$$ 

Although the algebra of signs in the $Q_\Delta$ space has a multiplicative inverse $\otimes$, the algebra of ratios unfortunately loses this property along with the additive inverse. For qualitative equality we continue to use the $\approx$ symbol, which will identify expressions that are not transitive, such as

$$[a] \oplus [b] \approx [c], [c] \approx [d] \not\Rightarrow [a] \oplus [b] \approx [d],$$

while the hybrid algebra still permits quantitative equality $=$ between real numbers $(c = d)$ and between identical elements of $Q_R$ ($[a + b] = [c]$).
\[\begin{array}{|c|c|c|c|}
\hline
\oplus & \hat{0} & \hat{L} & \hat{1} & \hat{G} \\hline
\hat{0} & \hat{0} & \hat{L} & \hat{1} & \hat{G} \\hline
\hat{L} & \hat{L} & ? & \hat{G} & \hat{G} \\hline
\hat{1} & \hat{1} & \hat{G} & \hat{G} & \hat{G} \\hline
\hat{G} & \hat{G} & \hat{G} & \hat{G} & \hat{G} \\hline
\hat{?} & \hat{?} & \hat{?} & \hat{?} & \hat{?} \\hline
\end{array}\]

Table 11.1: Qualitative addition operations \(\oplus\) in \(RR1\) on the elements of \(Q_R\).

\[\begin{array}{|c|c|c|c|}
\hline
\ominus & \hat{0} & \hat{L} & \hat{1} & \hat{G} \\hline
\hat{0} & \hat{0} & \hat{U} & \hat{U} & \hat{U} \\hline
\hat{L} & \hat{L} & ? & \hat{U} & \hat{U} \\hline
\hat{1} & \hat{1} & \hat{L} & \hat{0} & \hat{U} \\hline
\hat{G} & \hat{G} & ? & ? & ? \\hline
\hat{?} & \hat{?} & \hat{?} & \hat{?} & \hat{?} \\hline
\end{array}\]

Table 11.2: Qualitative subtraction operations \(\ominus\) in \(RR1\) on the elements of \(Q_R\). The symbol \(U\) stands for undefined.

Tables 11.1 and 11.2 show the results of the qualitative binary operators \(\oplus\) (qualitative ratio addition) and \(\ominus\) (qualitative ratio subtraction) in \(RR1\). These axioms are given without proof since in every case their combinations coincide with our intuition. Similarly, Tables 11.3 and 11.4 show the results of the qualitative binary operators \(\otimes\) (qualitative ratio multiplication) and \(\oslash\) (qualitative ratio division) in \(RR1\). From these tables we can extract the interesting axioms of a symbolic algebra involving \(\langle Q_R, \oplus, \ominus \rangle\):

**Identity** The additive identity element, as expected, is \(\hat{0}\). Surprisingly, there are two multiplicative identities. The first is the element \(\hat{1}\), where \([x] \oslash \hat{1} = [x]\).
<table>
<thead>
<tr>
<th>$\otimes$</th>
<th>$\hat{0}$</th>
<th>$\hat{L}$</th>
<th>$\hat{i}$</th>
<th>$\hat{G}$</th>
<th>$\hat{?}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
</tr>
<tr>
<td>$\hat{L}$</td>
<td>$\hat{0}$</td>
<td>$\hat{L}$</td>
<td>$\hat{L}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{i}$</td>
<td>$\hat{0}$</td>
<td>$\hat{L}$</td>
<td>$\hat{i}$</td>
<td>$\hat{G}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{G}$</td>
<td>$\hat{0}$</td>
<td>$\hat{?}$</td>
<td>$\hat{G}$</td>
<td>$\hat{G}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{?}$</td>
<td>$\hat{0}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
</tr>
</tbody>
</table>

Table 11.3: Qualitative multiplication operations $\otimes$ in RR1 on the elements of $Q_R$.

<table>
<thead>
<tr>
<th>$\ominus$</th>
<th>$\hat{0}$</th>
<th>$\hat{U}$</th>
<th>$\hat{i}$</th>
<th>$\hat{G}$</th>
<th>$\hat{?}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{0}$</td>
<td>$\hat{U}$</td>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
<td>$\hat{0}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{L}$</td>
<td>$\hat{U}$</td>
<td>$\hat{?}$</td>
<td>$\hat{L}$</td>
<td>$\hat{L}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{i}$</td>
<td>$\hat{U}$</td>
<td>$\hat{G}$</td>
<td>$\hat{i}$</td>
<td>$\hat{L}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{G}$</td>
<td>$\hat{U}$</td>
<td>$\hat{G}$</td>
<td>$\hat{G}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
</tr>
<tr>
<td>$\hat{?}$</td>
<td>$\hat{U}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
<td>$\hat{?}$</td>
</tr>
</tbody>
</table>

Table 11.4: Qualitative division operations $\ominus$ in RR1 on the elements of $Q_R$. The symbol $U$ stands for undefined.
The second is the element itself, since $[x] \otimes [x] = [x]$ for any $[x] \in Q_R$. This leads to the following assignments for exponentiation: if $[x] = \hat{G}$ then

$$[x^n] = [x]^n = \begin{cases} \hat{1} & \text{if } n = 0 \\ \hat{G} & \text{if } n > 0 \\ \hat{L} & \text{if } n < 0 \end{cases}$$

If $[x] = \hat{L}$ then

$$[x^n] = [x]^n = \begin{cases} \hat{1} & \text{if } n = 0 \\ \hat{L} & \text{if } n > 0 \\ \hat{G} & \text{if } n < 0 \end{cases}$$

If $[x] = \hat{0}$, then

$$[x^n] = [x]^n = \begin{cases} \hat{1} & \text{if } n = 0 \\ \hat{0} & \text{if } n > 0 \\ U & \text{if } n < 0 \end{cases}$$

Finally, if $[x] = \hat{1}$, then $[x^n] = [x]^n = \hat{1}$, and if $[x] = \hat{?}$, then $[x^n] = [x]^n = \hat{?}$.

**Inverse** There is neither an additive nor multiplicative inverse. Table 11.2 shows that $[x] \ominus [x] = \hat{?}$ if $[x] \in \{\hat{L}, \hat{G}\}$, i.e., for non-zero-width intervals. Tables 11.3 and 11.4 show that $[x] \otimes [x]^{-1} = \hat{?}$ for $[x] \neq \hat{0}$, except for $[x] = \hat{1}$. As a result of this lack of inverses, we cannot move addends or multiplicands between sides of an equation.

**Commutativity** From the off-diagonal symmetry of Tables 11.1 and 11.3 we conclude that expressions involving $\ominus$ and $\otimes$ are commutative under both addi-
tion and multiplication,

\[ [x] \oplus [y] = [y] \oplus [x] \]
\[ [x] \otimes [y] = [y] \otimes [x] \]

**Associativity** This algebra is also associative in both addition and multiplication,

\[ ([x] \oplus [y]) \oplus [z] = [x] \oplus ([y] \oplus [z]) \]
\[ ([x] \otimes [y]) \otimes [z] = [x] \otimes ([y] \otimes [z]) \]

**Distributivity** Finally, this algebra is distributive,

\[ [x] \otimes ([y] \oplus [z]) = [x] \otimes [y] \oplus [x] \otimes [z] \]

Williams [147] provides detailed descriptions of the algebraic simplification and composition rules used in SR1, along with algorithms for translating (abstracting) quantitative expressions into equivalent qualitative expressions. Although the quantity space has been modified slightly, an equivalent formal account of these algorithms for RR1, having been inspired by SR1, should follow a similar development. Since this paper develops the necessary abstractions, simplifications and compositions as part of the theory of gas dynamics, the rigorous specification of the hybrid algebra and its implementation is left as consideration for future work.
11.3 Qualitative Ratio Derivatives

Construction of a qualitative ratio \([a/b]\) from the component terms \(a\) and \(b\) follows the simple rules

\[
\begin{align*}
[a/b] &= \begin{cases} 
\hat{G} & \text{if } a > b \\
\hat{I} & \text{if } a = b \\
\hat{L} & \text{if } a < b
\end{cases}
\end{align*}
\]

where \(a > 0\) and \(b > 0\), which is a reasonable assumption for the majority of physical parameters of interest in the gas dynamics domain. We might also construct the ratio \([a/b]\) from the qualitative terms \([a]\) and \([b]\) using the \(\odot\) operator, albeit risking the possibility of ambiguity.

We can also use these rules to determine how a particular parameter changes between two states. Let the differential \(\Delta p\) represent the change in \(p\) between states 1 and 2, i.e. \(\Delta p = p_2 - p_1\). This expression might also be written

\[
\Delta p = p_2 - p_1 = p_1 \left( \frac{p_2}{p_1} - 1 \right).
\]

Rearranging to solve for the fraction, we obtain

\[
\frac{p_2}{p_1} = \frac{\Delta p}{p_1} + 1.
\]

Applying the \([\ ]\) operator to both sides we create the qualitative equivalent expression for a change in value represented as a qualitative ratio

\[
\begin{bmatrix} p_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} \Delta p \\ p_1 \end{bmatrix} + 1.
\]
The two terms on the right-hand side of this equation may be separated from the enclosing \([\ ]\) operator because

\[
\left[ \frac{\Delta p}{p_1} + 1 \right] = \left[ \frac{\Delta p}{p_1} \right] + \hat{i}.
\]

Thus, for equations involving differentials we can make the substitution

\[
\left[ \frac{\Delta p}{p_1} \right] = \left[ \frac{p_2}{p_1} \right] - \hat{i}.
\]

Treating the ratio as a quotient (and introducing ambiguity), and multiplying both sides by \([p_1]\),

\[
[\Delta p] \approx [p_1] \otimes \left( \left[ \frac{p_2}{p_1} \right] - \hat{i} \right). \quad (11.1)
\]

Now we observe that since \([p_1] = \hat{+}\) it has no effect on the outcome of the equation! Given the mappings for change in value \([\Delta p]\) from \(Q_\Delta\) we can make a similar mapping for change in value \([p_2/p_1]\) from \(Q_R\),

\[
[\Delta p] = \hat{0} \Rightarrow [p_2/p_1] = \hat{i}
\]

\[
[\Delta p] = \hat{+} \Rightarrow [p_2/p_1] = \hat{G}
\]

\[
[\Delta p] = \hat{\sim} \Rightarrow [p_2/p_1] = \hat{L}
\]

we see that each side of equation 11.1 always evaluates to the same value from \(Q_\Delta\).

Thus we have the equivalent substitution of qualitative differentials for qualitative ratios,

\[
[\Delta p] \approx \left[ \frac{p_2}{p_1} \right] - \hat{i}.
\]
Finally, for the kinds of relationships and results we want from a qualitative reasoning system we assume that $\partial p \approx \Delta p$ and that

$$\left[ \frac{\partial p}{\partial x} \right] \approx \left[ \frac{\Delta p}{\Delta x} \right].$$

A short digression regarding the qualitative derivative notation described in Section 4.3 now can be made. Recall that the conventional notation $[\Delta p]$ (or $[\partial p]$) in other systems is meant to relate the change in $p$ with respect to a change in time $t$, i.e.

$$\partial p \Rightarrow \frac{\partial p}{\partial t} \approx \left[ \frac{\Delta p}{\Delta t} \right].$$

Since we prefer to think of time as always increasing, then $[\Delta t] = \hat{+}$ and the expression can be written equivalently as

$$\left[ \frac{\Delta p}{\Delta t} \right] \approx [\Delta p].$$

In many fluid dynamics problems, however, changes in physical quantities with respect to time are not interesting. For instance, in a steady flow problem one might be concerned with change in pressure $p$ with respect to geometric position $x$ along a duct or wall: the quantity space for $x$ is simply the position of each station $[x] \in \{[x_1],[x_2],\ldots,[x_n]\}$. Since proper definition of this basis guarantees that $[\Delta x] = \hat{+}$, then

$$\left[ \frac{\Delta p}{\Delta x} \right] \approx [\Delta p] \in Q_\Delta.$$

Therefore, time is not the only quantity to have this qualitative property. To avoid ambiguity, we will always specify the full derivative notation in the remaining
discussion.

11.4 Qualitative Ratios vs. Qualitative Proportionalities

Given the representational convenience provided to us by the hybrid algebra RR1, one may compare this system to the qualitative proportionality representation employed by Qualitative Process Theory. Forbus employs the qualitative proportionality scheme in QPT to provide a convenient way to relate various quantities without regard to the specific mechanism of their interaction.

Consider two physical characteristics of a gas, pressure $p$ and temperature $T$. Let the qualitative value of pressure change over time $[\frac{\partial p}{\partial t}]$ have a quantity space $Q_p$, with SI units of $\frac{N}{\text{sec-m}^2}$, and let the qualitative value of temperature change $[\frac{\partial T}{\partial t}]$ have a different quantity space $Q_T$, with units of $\frac{K}{\text{sec}}$. Since they have different quantity spaces, a relationship between changes in either parameter can be made only by a proportionality constant or operator. The qualitative proportionality operator $\alpha_Q$ provides this capability:

$$
\left[ \frac{\partial p}{\partial t} \right] \propto_Q \left[ \frac{\partial T}{\partial t} \right].
$$

Expressions such as this provide indirect influence relationships for active processes, stating that with all else equal, a change in pressure indirectly causes a corresponding proportional change in temperature. This equation is semantically equivalent
to

\[
\left[ \frac{\partial p}{\partial t} \right] = c \otimes \left[ \frac{\partial T}{\partial t} \right],
\]

where \(c > 0\). Since we have seen that the positive coefficient has no effect on the qualitative outcome of the equation, then we can eliminate it, along with the \(\partial t\) divisors on each side, leaving

\[
[\partial p] \approx [\partial T],
\]

an expression which makes no presumptions about time but limited in use by the qualitative equality. If instead the changes physical parameters were stated as ratios, then

\[
\left[ \frac{(p_2/p_1) - 1}{(t_2/t_1) - 1} \right] = \left[ \frac{(T_2/T_1) - 1}{(t_2/t_1) - 1} \right].
\]

Eliminating the change in time and cancelling the 1's across \(=\) (as is permitted with zero-width intervals), this leaves

\[
\left[ \frac{p_2}{p_1} \right] = \left[ \frac{T_2}{T_1} \right].
\]

Since both sides of the equation have the same quantity space \(Q_R\), they permit the stronger result attributable to \(=\).

One might conclude, therefore, that there is no need for the qualitative proportionality operator when using state-difference ratios to represent physical change! Consequently, the notion of indirect influences is not required within the representation system. Certainly the notion of direct influence carries more intuitive physical support, such that any relationship that can be defined in this manner should be,
and it will be more apparent that the consequences of applying the relationship are more in agreement with nature. Without indirect influence relationships, then, the corresponding slots in the QPT process templates might be removed along with all of their attendant computational mechanisms. For consistency and comparison purposes, however, these indirect influence slots will simply remain unfilled in the gas dynamics theory.
Chapter 12

Governing Equations

Definition 4 A fluid is any substance that deforms continuously under the application of a shearing stress.

A fluid experiencing application of a shearing stress will attain dynamic equilibrium while undergoing motion. An elastic solid experiencing application of a shearing stress will remain in static equilibrium. Therefore it is reasonable that we prefer to discuss the equilibrium behavior of fluids under the category of fluid dynamics, since we are in fact concerned with their motion. If we further restrict the discussion to those fluids that are gases, we retitle the domain gas dynamics.

This Chapter examines the physical properties of fluids and the fundamental laws describing their behavior. Section 12.1 identifies the fluid properties and reviews some of the basic thermodynamic behaviors of these properties while undergoing change. Then we examine the four basic physical laws comprising the governing equations of fluid dynamics:

- the principle of conservation of mass,
- the principle of conservation of energy,
• the principle of conservation of linear momentum, and

• the second law of thermodynamics.

Sections 12.2.1 through 12.3 present each of these fundamental laws and develop their qualitative equivalents.

The development approach for this qualitative theory of gas dynamics resembles the engineering approach to the quantitative description of this field. First, we assume continuous behavior and consider the gross motion of the fluid treated as a continuous mass, instead of considering the microscopic behavior of individual molecules, which requires statistical analysis. Then for additional simplifications, frequently we will assume that the fluid is perfect, that is, nonviscous and nonconducting. Finally, for interior flows (primarily encompassing flows through ducts) we assume one-dimensional steady flow, and for exterior flows (such as flows around solid bodies) we assume two-dimensional steady flow. These few assumptions place several restrictions upon the flow field, however the limiting restrictions, which are reasonable enough for engineering applications, certainly are reasonable enough for qualitative model applications.

One manifestation of a perfect fluid for the gas dynamics domain is the concept of an ideal gas. This simplifying assumption enables specification of simple state equations that in turn significantly reduce the complexity of the governing equations. Section 12.4.1 reviews the conditions under which this assumption holds.

The following descriptions of the fundamental laws of gas dynamics demonstrate
the complexity reduction made possible by the ideal gas assumption. Although no real gas strictly satisfies the requirements for an ideal gas, it transpires that the ideal gas assumption is an excellent contributor to the development of qualitative models of gas dynamics. This assumption is quite reasonable for commonly encountered gases, and in particular it is an excellent assumption for air. The assumption allows good approximations of real behavior within the physical conditions encountered in most elementary gas dynamics problems. Most importantly, the few real-world effects that we cannot describe under the ideal gas assumption are among those that we choose to ignore for the purposes of qualitative modeling. As we shall see below, this assumption is the most powerful contributor to the goal of attaining qualitative modeling and simulation capabilities for gas flows.

12.1 Thermodynamics

This section introduces several fluid properties and describes a few of the physical behaviors from which we can produce fundamental qualitative relationships. Section 12.1.2 demonstrates some of the considerations involved in developing qualitative expressions from the quantitative statements of real physical observations. Then, using these relationships as examples, section 12.1.4 compares qualitative proportionality operators with the qualitative ratio technique.
<table>
<thead>
<tr>
<th>Physical Symbol</th>
<th>Description</th>
<th>Qualitative Value</th>
<th>Landmark Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>dynamic viscosity</td>
<td>$[\mu]$</td>
<td>${0, \infty}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
<td>$[\rho]$</td>
<td>${0, \rho^*, \rho_0, \infty}$</td>
</tr>
<tr>
<td>$C$</td>
<td>heat capacity</td>
<td>$[C]$</td>
<td>${0, \infty}$</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
<td>$[p]$</td>
<td>${0, p^*, \rho_0, \infty}$</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
<td>$[T]$</td>
<td>${0, T_0, T^*, \infty}$</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
<td>$[v]$</td>
<td>${0, v^*, \infty}$</td>
</tr>
</tbody>
</table>

Table 12.1: Several fluid properties and their qualitative equivalent symbols. The appropriate quantity spaces for the qualitative values can be determined by including both the ordered set of landmark values and appropriate values for the intervals between these landmarks. The definitions of specific landmark values are provided later.

### 12.1.1 Qualitative Fluid Properties

Table 12.1 lists the basic fluid properties and their equivalent qualitative notations.

With these properties one can define some of the appropriate attributes for QPT using the predicate `HasQuantity` and an arbitrary gas $g$:

```
HasQuantity(g, dynamic-viscosity).
HasQuantity(g, density).
HasQuantity(g, heat-capacity).
```

and so on. So long as we do not depart from intuition at this level of thermodynamic specification, these statements really are important only to the computer program implementation, and will not be continued herein. Figure 12.1 shows the first individual view structure of the theory, defining a `Gas` as having certain physical properties. All instances of a gas then inherit these properties from this individual view.
Individual View Gas(g)

Individuals:
- $g$; a piece of matter

Preconditions:
- $\text{HasQuantity}(g, \mu)$; the dynamic viscosity of $g$
- $\text{HasQuantity}(g, \rho)$; the density of $g$
- $\text{HasQuantity}(g, C)$; the heat capacity of $g$
- $\text{HasQuantity}(g, p)$; the pressure of $g$
- $\text{HasQuantity}(g, T)$; the temperature of $g$
- $\text{HasQuantity}(g, v)$; the velocity of $g$

Quantity Conditions:
- $\neg \text{Solid}(g)$
- $\neg \text{Liquid}(g)$

Relations:
- $\emptyset$

Figure 12.1: The individual view structure for a gas with specified thermodynamic properties. The Solid and Liquid tokens identify Boolean functions indicating whether the material satisfies the conditions of existence for a solid or liquid.
12.1.2 Fundamental Qualitative Relationships

Using these fundamental properties, we may state several basic thermodynamic relationships from which to derive the companion set of qualitative expressions providing the foundation for our qualitative models. As an example of how this translation works, consider one of the properties: the fluid density $\rho$. The density provides the fluid mass $m$ per unit volume $V$ so that

$$m = \rho V. \tag{12.1}$$

If we define the quantity spaces for $m$ and $V$ as $\{0, (0, \infty), \infty\}$, assuming that there are no interesting landmark values between 0 and $\infty$, then we can define the qualitative expression

$$[m] \approx [\rho] \otimes [V].$$

Now, given the non-negative quantity spaces for all of the attendant values and the ambiguity of qualitative multiplication, it transpires that this expression does not provide much information beyond than the constraint $[m] = \hat{0}$ whenever $[\rho] = \hat{0}$ or $[V] = \hat{0}$, or both. By differentiating and applying the qualitative ratio representation rules, however, equation 12.1 provides a powerful constraint over changes in the values, without loss of generality. To simplify the differentiation we take the natural logarithm of both sides and use the rules of combination for logarithms to simplify:

$$m = \rho V,$$

$$\ln m = \ln \rho V,$$
\[
\ln m = \ln \rho + \ln V.
\]

Now, differentiating this expression we obtain

\[
\frac{1}{m} \, dm = \frac{1}{\rho} \, d\rho + \frac{1}{V} \, dV.
\]

For qualitative reasoning purposes it is plausible to assume that \( dm \approx \Delta m \), \( d\rho \approx \Delta \rho \), and \( dV \approx \Delta V \). Furthermore, we can introduce the desired qualitative ratio by first applying the substitution expression

\[
dm \approx \Delta m = m_2 - m_1 = m_1 \left( \frac{m_2}{m_1} - 1 \right).
\]

Substituting for the appropriate values in equation 12.2 and rearranging we obtain the desired qualitative expression

\[
\begin{bmatrix}
  m_2 \\
m_1
\end{bmatrix} \approx \begin{bmatrix}
  \rho_2 \\
\rho_1
\end{bmatrix} \oplus \begin{bmatrix}
  V_2 \\
V_1
\end{bmatrix} \ominus \hat{1},
\]

where

\[
\begin{bmatrix}
  \rho_2 \\
\rho_1
\end{bmatrix}, \begin{bmatrix}
  m_2 \\
m_1
\end{bmatrix}, \begin{bmatrix}
  V_2 \\
V_1
\end{bmatrix} \in \mathbb{Q}_R.
\]

Examining this expression we see that given the values for two of the ratios, we can determine the third. For example, if the density decreases between states while the volume remains constant, then

\[
\begin{bmatrix}
  m_2 \\
m_1
\end{bmatrix} \approx \hat{L} \odot \hat{1} \ominus \hat{1},
\]

\[
\approx \hat{L},
\]
meaning that the mass also decreases. Examination also reveals that ambiguity can arise under several situations. For instance, if the volume increases and the mass decreases, then

\[ \dot{L} \oplus \dot{1} \approx \dot{G} \approx \left[ \frac{\rho_2}{\rho_1} \right] \oplus \dot{G}, \]

for which the only solution is \([\rho_2/\rho_1] = \hat{?}\). A simulation encountering this situation must branch to explore each of the possible solutions encompassed by \(\hat{?}\) individually. Similarly, if we are given only one of the three values then the equation reduces to a constraint equation on the remaining two values, which the simulation might also branch to explore.

Continuing the development of the fundamental relationships, we focus on the property of heat capacity, which describes how a quantity of heat changes \((dQ)\) in response to a temperature change \((dT)\),

\[ C = \frac{dQ}{dT}. \]

Since we know that heat and temperature changes have the same signs, then we know \(C\) is a positive number. This relationship becomes

\[ [\Delta Q] \approx C \otimes [\Delta T], \]

or better yet in terms of ratios

\[ \left[ \frac{Q_2}{Q_1} \right] \approx \left[ \frac{T_2}{T_1} \right]. \]

Similarly, the heat capacity at constant pressure \(C_p\) and the heat capacity at constant volume \(C_V\) specify the same relationship for processes constrained to those
conditions,

\[ C_p = \left( \frac{\partial H}{\partial T} \right)_p \]

and

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V, \]

where \( H \) is the enthalpy and \( U \) is the internal energy of the fluid. Using the fundamental thermodynamic relationship

\[ \Delta U = \Delta Q + \Delta W \]

and assuming that there is no work done on the fluid, then

\[ \Delta U = \Delta Q - p \Delta V. \]

Now we introduce the more commonly encountered specific heat capacities \( c_p \) and \( c_V \), expressed on a per-unit-mass basis,

\[ c_p = \frac{C_p}{m} = \left( \frac{\partial h}{\partial T} \right)_p \]

\[ c_V = \frac{C_V}{m} = \left( \frac{\partial u}{\partial T} \right)_V, \]

where \( h \) and \( u \) are the specific enthalpy and specific internal energy, respectively, and

\[ \partial h = \partial q + V \partial p, \]

\[ \partial u = \partial q - p \partial V, \]

with \( q \) representing the specific heat. Before ratio substitution, we have the constraint equations

\[ [\Delta h] \approx [\Delta q] \otimes [V] \otimes [\Delta p], \]
\[ [\Delta u] \approx [\Delta q] \otimes [p] \otimes [\Delta V]. \]

Recognizing that \([V] = \hat{\imath}\) and \([p] = \hat{\imath}\), we can eliminate these terms because the result of the multiplication does not affect the result:

\[ [\Delta h] \approx [\Delta q] \oplus [\Delta p], \]
\[ [\Delta u] \approx [\Delta q] \oplus [\Delta V]. \]

Making the ratio substitutions we obtain the equivalent expressions under \(Q_R\):

\[
\begin{bmatrix}
\frac{h_2}{h_1} \\
\frac{u_2}{u_1}
\end{bmatrix}
\approx
\begin{bmatrix}
\frac{q_2}{q_1} \\
\frac{p_2}{p_1}
\end{bmatrix}
\oplus
\hat{\imath},
\]

\[
\begin{bmatrix}
\frac{u_2}{u_1}
\end{bmatrix}
\approx
\begin{bmatrix}
\frac{q_2}{q_1} \\
\frac{V_2}{V_1}
\end{bmatrix}
\oplus
\hat{\imath}.
\]

Both sets of equations provide the desired results with only two sources of ambiguity in each expression: when the terms on the right-hand side reflect opposing changes in the equation for \([h_2/h_1]\), and when the terms reflect similar change in the equation for \([u_2/u_1]\).

### 12.1.3 Fundamental Qualitative Processes

With the thermodynamic properties and a few fundamental expressions in hand, we can begin to define some of the processes appearing in gas dynamics problems. The QPT templates for these processes will be developed incrementally, installing new components to the templates for each new addition to the theory. The simplest and most general flow process, which we call AnyFlow, requires as its individuals a
Process AnyFlow(g.1.2)

Individuals:
  Gas(g); the gas undergoing change
  State(g.1); the initial state
  State(g.2); the final state

Preconditions:
  \emptyset

Quantity Conditions:
  \emptyset

Relations:
  \emptyset

Influences:
  \emptyset

Figure 12.2: The beginnings of a QPT process specification AnyFlow describing any change of state.

fluid and two numbered states, where the fluid starts in state 1 and undergoes some physical processes to achieve state 2. Figure 12.2 introduces a process template for this basic situation. This template will be enhanced in later sections to include the several fundamental relationships describing all flow fields. Hence the template serves as an ancestor individual view that is inherited by the each of the more specific flow processes. Since the general case problems are the hardest to solve, there will be few instances in which the AnyFlow process is the only active process. Many more interesting flow problems can be solved with certain restrictions (or assumptions) asserted as flow conditions, and we will later specify process templates for each of these situations. However, each of the specific processes will inherit the properties of the AnyFlow process.
; adiabatic change of state process specification
; (preliminary)
Process AdiabaticFlow(g.1.2)

Individuals:
  AnyFlow(g.1.2); general flow relations

Preconditions:
  ∅

Quantity Conditions:
  \[ Q_2/Q_1 = 1 \] ; no heat added or removed

Relations:
  ∅

Influences:
  ∅

Figure 12.3: The beginnings of a QPT process specification for an adiabatic change of state.

For example, a process in which there is no heat transfer \( \Delta Q \) is an adiabatic process. Figure 12.3 shows the first requirements for the corresponding process template AdiabaticFlow. This process is a specialization of the general case, therefore it requires the AnyFlow process specification as an individual. The state achieved by bringing a flow to rest \( (v = 0) \) by means of an adiabatic process is the stagnation state, which we denote by the subscript 0; e.g., the temperature in the stagnation state is labelled \( T_0 \). As alluded to earlier in the discussion in the usefulness of ratios, the stagnation state is important for reference and comparison purposes.

A flow process which occurs in the presence of friction \( f \) is irreversible. For the purposes herein, we consider a reversible process to be a flow that occurs without friction. A simple QPT process template ReversibleFlow representing this condition is shown in Figure 12.4.
Process ReversibleFlow(g.1.2)

**Individuals:**
*AnyFlow(g.1.2)*; general flow relations

**Preconditions:**
\[ \emptyset \]

**Quantity Conditions:**
\[ [f] = 0; \text{no friction} \]

**Relations:**
\[ \emptyset \]

**Influences:**
\[ \emptyset \]

**Figure 12.4:** A QPT process specification for a reversible change of state.

A process in which there is no entropy change is an *isentropic* process. For an isentropic process, \( \Delta s = 0 \) so therefore \([s_2/s_1] = \hat{1}\). Figure 12.5 shows the beginnings of a specification IsentropicFlow for this process. An isentropic process can be active only if a flow is both adiabatic and reversible, so these processes comprise the individual requirements for activation of the isentropic process template. This kind of process will be important in the discussions on the second law of thermodynamics and some simplifying assumptions of flow behaviors.

### 12.1.4 Comparison of Qualitative Expressions

Comparing these results to what we can derive from the qualitative proportionality approach, we can identify some important differences between the two techniques.
Figure 12.5: A first attempt at a QPT process specification for an isentropic change of state.

Returning to the basic relationship

\[ m = \rho V, \]

we can state two relationships about density simply by inspection,

\[ ([\partial \rho] \propto_{Q^+} [\partial m])_V \]

and

\[ ([\partial \rho] \propto_{Q^-} [\partial V])_m. \]

The first relationship states that under constant volume, density will increase when mass increases. The second states that under constant mass, density will decrease when volume increases. Recalling that application of the qualitative proportionality operators implies that all else is equal, we can drop the subscripts on these
relationships, leaving

\[ [\partial p] \propto_{Q_+} [\partial m] \]

and

\[ [\partial p] \propto_{Q_-} [\partial V]. \]

Operating under these limiting assumptions, however, we see that the application of the qualitative proportionality operators requires specification of two constraint equations (with separate quantity conditions), whereas the qualitative ratio approach requires only one (with no quantity conditions),

\[
\begin{bmatrix}
  m_2 \\
  m_1
\end{bmatrix}
\approx
\begin{bmatrix}
  \rho_2 \\
  \rho_1
\end{bmatrix}
\oplus
\begin{bmatrix}
  V_2 \\
  V_1
\end{bmatrix}
\ominus \hat{1}.
\]

Similarly, for the specific heat relationships, we might define the list of qualitative proportionality results

\[ [\partial q] \propto_{Q_+} [\partial T], \]

\[ [\partial u] \propto_{Q_+} [\partial T], \]

\[ [\partial h] \propto_{Q_+} [\partial T], \]

\[ [\partial h] \propto_{Q_+} [\partial p], \]

\[ [\partial h] \propto_{Q_+} [\partial q], \]

\[ [\partial u] \propto_{Q_-} [\partial V], \]

and

\[ [\partial u] \propto_{Q_+} [\partial q], \]

whereas in the qualitative ratio representation just two constraint equations suffice.
In fact, using the qualitative proportionality operators we have *lost* some important information about how to interrelate these two constraints. The presumptions of *all else being equal*\(^1\) and *states change only over time* lead to broader branching in the envisionment because one must fix all other parameters in order to pursue a change in the parameter of interest. The QPT limit hypothesis algorithm invokes these branchings. In the qualitative ratio technique, however, we can prune many possible branches because the more detailed constraint relationships eliminate certain outcomes due to physical impossibility. This is especially important because we do not make the assumption in this theory that *time* is the independent variable causing state transitions. Hence, wherever an independent variable appears in the model, the detailed constraint equations afford additional implicit preconditions on their applicability that do not have to be stated explicitly in the process templates.

Unfortunately, not all of the governing principles in the gas dynamics domain are as simple as the density relationship. The translation to qualitative expressions cannot always be accomplished, forcing us to make some assumptions and introduce various degrees of uncertainty into the models. Nevertheless, qualitative reasoning systems are meant to handle this kind of uncertainty in favor of modeling flexibility and adaptive representations. Through the next several Sections it will be shown that, despite the complexity of the physical domain, we can develop some very

---

\(^1\)One notable exception to this QPT statement is that *change in time* is the general mechanism for *change in a property*. Therefore the statement actually should read *all else but time being equal*. 
powerful qualitative models of the dynamic behavior of gases.

12.2 Conservation Laws

Three of the four fundamental principles governing fluid flow concern the conservation of some quantity: these are mass, energy, and momentum. Derivation of these conservation laws requires use of a control volume $\mathcal{V}$ enclosing the flow field of interest. Fixing the control volume both in dimension and spatial position allows us to reason about fluid entering and leaving $\mathcal{V}$ through its surface $\mathcal{S}$. As suggested in Figure 12.6, the unit normal vector $\vec{N}$ points outward from an element of surface area $dA$. Heat $\dot{Q}$ enters and work $\dot{W}$ leaves the control volume through $\mathcal{S}$. The vector $\vec{F}$ represents the forces acting on $\mathcal{V}$. With these definitions in place, we examine each of the three conservation laws from the perspective of the control volume, progressing rapidly through the simplifications necessary to reduce these
complex equations into forms the qualitative theory can accommodate.

### 12.2.1 Conservation of Mass

**Definition 5 (Conservation of Mass)** *In the absence of sources and sinks, mass is conserved for any real process that is not concerned with relativistic changes.*

Our qualitative theory of gas dynamics is not concerned with relativistic changes, and per the assumptions regarding the control volume concept there are no known sources or sinks. Therefore, we account only for the mass entering and leaving the control volume.

In quantitative terms, the *continuity equation* relates the time rate of change of mass in $\mathcal{V}$ to the net rate of outflow of mass across the boundaries:

$$- \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \, d\mathcal{V} = \int_{\mathcal{S}} \rho (\vec{v} \cdot \vec{N}) \, dA.$$  \hspace{1cm} (12.3)

For conservation of mass at a point, this equation reduces to the *Eulerian continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0.$$  \hspace{1cm} (12.4)

For steady, one-dimensional flow between two points 1 and 2 in a flow, equation 12.3 reduces to

$$\rho_2 A_2 v_2 - \rho_1 A_1 v_1 = 0.$$

Introducing the mass flow rate of the fluid $\dot{m}$, this equation becomes

$$\dot{m} = \text{constant} = \rho A v,$$
which holds at any point in the flow. Differentiating logarithmically with respect to the position along the flow \(x\),

\[
\ln(\rho A v) = \ln(\dot{m})
\]
\[
\ln(\rho) + \ln(A) + \ln(v) = \ln(\dot{m})
\]
\[
\frac{1}{\rho} \frac{d\rho}{dx} + \frac{1}{A} \frac{dA}{dx} + \frac{1}{v} \frac{dv}{dx} = 0
\]
\[
\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v} = 0.
\]

This result translates conveniently to the qualitative ratio constraint equation

\[
\left[ \frac{\rho_2}{\rho_1} \right] \oplus \left[ \frac{A_2}{A_1} \right] \oplus \left[ \frac{v_2}{v_1} \right] \approx 3.
\]

Since we are interested in moving fluids, such that \(\dot{m}\) is always positive, the alternative form of the conservation equation could be stated as

\[
\left[ \frac{\dot{m}_2}{\dot{m}_1} \right] \approx 1.
\]

As governing fundamental relationships, both of these qualitative expressions are suitable for the Relations slot of the AnyFlow process (Figure 12.7). These are the first contributions to the qualitative theory.

### 12.2.2 Conservation of Energy

**Definition 6 (The First Law of Thermodynamics)** Energy can be neither created nor destroyed, but only converted from one form to another.

The principle of conservation of energy, like the two other conservation principles, provides a strict accounting of heat and work energies occurring in the control
Figure 12.7: A modification of the AnyFlow process specification to include the relations provided by the law of conservation of mass.

volume \( \mathcal{V} \). This principle states that the rate of heat transfer across the surface \( \mathcal{S} \) (\( \dot{Q} \)) is equal to the sum of the rate of energy accumulated within \( \mathcal{V} \) (\( \dot{U}_\mathcal{V} \)), the net rate of energy flux through \( \mathcal{S} \) (\( \dot{U}_\mathcal{S} \)), and the rate of work done on the surrounding fluid (\( \dot{W} \)). In quantitative form,

\[
\dot{Q} = \frac{\partial}{\partial t} \int_\mathcal{V} (u + \frac{v^2}{2}) \rho \, d\mathcal{V} + \int_\mathcal{S} (u + \frac{v^2}{2}) \rho \vec{v} \cdot \vec{N} \, dA + \\
\int_\mathcal{S} p \vec{N} \cdot \vec{v} \, dA + \dot{W},
\]

(12.5)

which accounts for specific internal (\( u \)) and kinetic (\( \frac{v^2}{2} \)) energies in a perfect fluid. The potential and chemical energy contributions normally are negligible in gas dynamics problems, so they will not be accounted for here.
For steady flow cases, equation 12.5 reduces to

\[ \dot{Q} = \int_S \left( u + \frac{v^2}{2} \right) \rho \vec{v} \cdot \vec{N} \, dA + \int_S p \vec{N} \cdot \vec{v} \, dA + \dot{W}. \]

Now using the definition of enthalpy \( h \),

\[ h = u + \frac{p}{\rho}, \]

and the rate substitutions

\[ Q = \frac{\dot{Q}}{m}, \quad W = \frac{\dot{W}}{m}, \]

the energy equation for steady flow reduces further to

\[ Q - W = \left( h_2 + \frac{v_2^2}{2} \right) - \left( h_1 + \frac{v_1^2}{2} \right). \]

If \( Q = 0 \), the flow is said to be adiabatic. If we also know that no work enters or leaves \( \mathcal{V} \), then we have the condition

\[ h + \frac{v^2}{2} = \text{constant}, \]

where \( h \) and \( \frac{v^2}{2} \) represent the thermal and mechanical energies of the flow, respectively. If the fluid in such a state is brought to rest \( (v = 0) \) adiabatically, then all of the remaining energy is the stagnation enthalpy

\[ h_0 = h + \frac{v^2}{2}. \]
Now we desire a qualitative statement of the balance between the total thermal and mechanical energies between any two flow states. Translating the stagnation enthalpy into a qualitative ratio expression, we obtain

\[
\frac{h_{02}}{h_{01}} \approx \frac{h_2}{h_1} \oplus \frac{v_2}{v_1} \oplus \hat{i}.
\]

Obviously, if a flow is adiabatic between states then \(h_{02} = h_{01}\) and \([h_{02}/h_{01}] = \hat{i}\), so we may rewrite the energy equation for adiabatic flow as

\[
\frac{h_2}{h_1} \oplus \frac{v_2}{v_1} \approx 2.
\]

Working backwards and reconsidering the total heat and work in terms of qualitative ratios, we have

\[
[Q] \approx \frac{Q_2}{Q_1} \oplus \hat{i},
\]

\[
[W] \approx \frac{W_2}{W_1} \oplus \hat{i},
\]

which enables reconstruction of a general energy equation for steady flow in terms of qualitative ratios

\[
\frac{v_2}{v_1} \oplus \frac{h_2}{h_1} \oplus \frac{Q_2}{Q_1} \oplus \frac{W_2}{W_1} \approx 2.
\]

This relation is added to the AnyFlow process template to represent conservation of energy (Figure 12.8).

### 12.2.3 Conservation of Momentum

**Definition 7 (Newton's Second Law of Motion)** *At any instant in time the rate of change of linear momentum of a material volume is equal to the resultant
; general change of state process specification
; (including conservation of energy)

Process AnyFlow(g.1.2)

Individuals:
  IdealGas(g) ; the gas undergoing change
  State(g.1) ; the initial state
  State(g.2) ; the final state

Preconditions:
  0

Quantity Conditions:
  0

Relations:
  \[ \frac{m_2}{m_1} = \hat{1} \]
  \[ \frac{\rho_2}{\rho_1} \oplus \frac{A_2}{A_1} \oplus \frac{v_2}{v_1} \approx 3 \]
  \[ \frac{v_2}{v_1} \oplus \frac{h_2}{h_1} \oplus \frac{Q_2}{Q_1} \oplus |W_2/W_1| \approx 2 \]

Influences:
  0

Figure 12.8: A modification of the AnyFlow process specification to include the relations provided by the law of conservation of energy.
force acting on the volume.

Using the control volume \( \mathcal{V} \) and the previous definition of the conservation of mass, wherein mass enters and leaves the control volume, we account similarly for the momentum entering and leaving the same fixed volume.

Quantitatively, the conservation of momentum requires that the resultant force acting on \( \mathcal{V} \) be equal to the sum of the time rate of change of momentum within \( \mathcal{V} \) and the net flux of momentum through \( \mathcal{S} \), or

\[
\sum \vec{F} = \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \vec{u} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \vec{u} (\vec{v} \cdot \vec{N}) \, d\mathcal{A}. \tag{12.6}
\]

There are two sorts of forces to account for: body forces and surface forces. Body forces are those that act on the control volume from a distance, such as gravity and electromagnetic effects. For the kind of gas dynamics problems pursued here, these forces are negligible. Surface forces result from the normal and shear fluid stresses,

\[
\vec{t} = \vec{t}_s + \vec{t}_N.
\]

The static pressure is the only normal stress if we neglect the normal viscous stress, so that

\[
\vec{t} = \vec{t}_s - p \vec{N},
\]

then equation 12.6 without body forces reduces to

\[
\vec{F}_f - \int_{\mathcal{S}} p \vec{N} \, d\mathcal{A} = \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \vec{u} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \vec{u} (\vec{v} \cdot \vec{N}) \, d\mathcal{A}, \tag{12.7}
\]
where the contribution of frictional forces $F_f$ is

$$
F_f = \int_S t_s^* \, d\mathcal{A}.
$$

For conservation of linear momentum at a point, we assume that the fluid is inviscid so that equation 12.7 reduces to

$$
\int_V \frac{\partial}{\partial t} (\rho \vec{v}) \, dV + \int_S \rho \vec{v} (\vec{v} \cdot \vec{N}) \, d\mathcal{A} + \int_S p \vec{N} \, d\mathcal{A} = 0.
$$

Applying the continuity equation 12.4 this becomes

$$
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} + \frac{1}{\rho} \nabla p = 0.
$$

For steady flow cases, the momentum equation reduces to

$$
F_f - \int_S p \vec{N} \, d\mathcal{A} = \int_S \rho \vec{v} (\vec{v} \cdot \vec{N}) \, d\mathcal{A},
$$

and for general one-dimensional steady flow cases we obtain the further simplification

$$
F_w = \bar{N}_2 A_2 (p_2 + \rho_2 v_2^2) - \bar{N}_1 A_1 (p_1 + \rho_1 v_1^2).
$$

This statement of the general one-dimensional steady flow case can be rewritten as specific equations for special situations in one-dimensional flow problems. For steady flow in straight, symmetrical ducts the unit normal vector disappears, leaving

$$
F_w = A_2 (p_2 + \rho_2 v_2^2) - A_1 (p_1 + \rho_1 v_1^2).
$$

This expression does not translate well to the desired qualitative ratio syntax, however we will see later that the problem-specific equations of state enable substi-
tutions which will allow us render the general momentum equation in qualitative form.

Considering further direct simplifications, if we restrict the duct to be a passage of constant cross-sectional area, then

\[ F_w = A[(p_2 + \rho_2 v_2^2) - (p_1 + \rho_1 v_1^2)]. \]

Neglecting friction, the wall force term \( F_w \) drops out and this equation reduces to

\[ p + \rho v^2 = \text{constant}. \]

Differentiating we obtain the quantitative expression for changes

\[ dp + 2\rho v \, dv + v^2 \, d\rho = 0, \]

which, after some heuristic simplification of the coefficients to provide reasonable results, translates into the qualitative ratio constraint equation

\[ \left[ \frac{p_2}{p_1} \right] \oplus \left[ \frac{\rho_2}{\rho_1} \right] \oplus 2 \otimes \left[ \frac{v_2}{v_1} \right] = 4. \]

From here we can generate the general qualitative expression for the one-dimensional constant-area momentum equation, similar to our qualitative derivation of the energy equation, by including the force-change ratio

\[ [F_w] = \left[ \frac{F_2}{F_1} \right] - \hat{1}, \]

so that

\[ \left[ \frac{p_2}{p_1} \right] \oplus \left[ \frac{\rho_2}{\rho_1} \right] \oplus 2 \otimes \left[ \frac{v_2}{v_1} \right] \oplus \left[ \frac{F_2}{F_1} \right] = 3, \quad (12.8) \]
general change of state process specification
(including conservation of momentum)

**Process AnyFlow(g.1.2)**

**Individuals:**
- IdealGas(g); the gas undergoing change
- State(g.1); the initial state
- State(g.2); the final state

**Preconditions:**
- \(\emptyset\)

**Quantity Conditions:**
- \(\emptyset\)

**Relations:**
\[
\begin{align*}
\frac{\dot{m}_2}{\dot{m}_1} &= 1 \\
\frac{\rho_2}{\rho_1} &\oplus \frac{A_2}{A_1} &\oplus \frac{v_2}{v_1} &\approx 3 \\
\frac{v_2}{v_1} &\oplus \frac{h_2}{h_1} &\oplus \frac{Q_2}{Q_1} &\oplus \frac{|W_2|}{|W_1|} &\approx 2 \\
\frac{A_2}{A_1} &\oplus \frac{p_2}{p_1} &\oplus \frac{\rho_2}{\rho_1} &\oplus 2 &\oplus \frac{v_2}{v_1} &\oplus \frac{F_2}{F_1} &\approx 4
\end{align*}
\]

**Influences:**
- \(\emptyset\)

Figure 12.9: A modification of the AnyFlow process specification to include the relations provided by the law of conservation of linear momentum.

which reduces to the frictionless constant-area momentum equation when \(\frac{F_2}{F_1} = 1\). Working backwards even further, we may reconsider area changes qualitatively and notice that we can obtain the desired results simply by adding the area change term \(\frac{A_2}{A_1} - 1\), leaving

\[
\begin{align*}
\frac{A_2}{A_1} &\oplus \frac{p_2}{p_1} &\oplus \frac{\rho_2}{\rho_1} &\oplus 2 &\oplus \frac{v_2}{v_1} &\oplus \frac{F_2}{F_1} = 4
\end{align*}
\]

Again, we add this result as a basic element of the AnyFlow process (Figure 12.9).
12.3 The Second Law of Thermodynamics

Definition 8 (The Second Law of Thermodynamics) The entropy change for an irreversible process which transfers heat at constant temperature is greater than the entropy change for a reversible process.

The Second Law of Thermodynamics states that entropy always increases during an irreversible process. Stated as the Clasius Inequality,

\[
\frac{\partial Q}{T} \leq dS.
\]

This principle relates the change in heat quantity (\(\partial Q\)) across the surface with the sum of the time rate of change of the fluid entropy within the control volume and the rate at which the fluid crossing the surface convects entropy into the volume.

For a fluid with specific entropy \(s\) and heat flux vector \(\vec{q}\) in the fixed volume \(\mathcal{V}\), the Second Law takes the form

\[
\int_{_{\mathcal{S}}} \frac{\vec{q} \cdot \vec{N}}{T} \, dA \leq \frac{\partial}{\partial t} \int_{_{\mathcal{V}}} s \rho \, dV + \int_{_{\mathcal{S}}} s \rho \vec{u} \cdot \vec{N} \, dA. \tag{12.9}
\]

The equality condition is valid only for reversible processes.

For ideal gases, equation 12.9 reduces to the differential equation

\[
ds \geq c_p \frac{dT}{T} - R \frac{dp}{p},
\]

where \(c_p\) is the specific heat at constant pressure and \(R\) is the specific gas constant.

If we translate this directly into a qualitative expression, then we have

\[
[\Delta s] \geq ([c_p] \otimes [\Delta T] \otimes [T]) \otimes ([R] \otimes [\Delta p] \otimes [p]),
\]
where \([c_p] = \hat{\delta}, [R] = \hat{\delta}, [T] = \hat{\delta}, \) and \([p] = \hat{\delta}. \) Since multiplication over the \( \hat{\delta} \) terms does not affect the qualitative result of the expression, we can eliminate these terms,

\[
[\Delta s] \geq [\Delta T] \otimes [T] \otimes [\Delta p] \otimes [p].
\]

Performing ratio substitutions of the form

\[
[\Delta x] = [x_1] \otimes \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \oplus \hat{\delta}
\]

reduces the Second Law expression to the simple statement

\[
\begin{bmatrix} s_2 \\ s_1 \end{bmatrix} \geq \begin{bmatrix} T_2 \\ T_1 \end{bmatrix} \otimes \begin{bmatrix} p_2 \\ p_1 \end{bmatrix} \oplus \hat{\delta}.
\]

This expression is added to the AnyFlow process view in Figure 12.10.

If we specify that the flow is adiabatic, the second law of thermodynamics provides the further constraint that the entropy of the system never decreases. Given the fact that the intervals in \(Q_\Delta\) and \(Q_R\) are fully ordered (disregarding \(\hat{\delta}\)), \(\Delta s \geq 0\) provides the constraints \([\Delta s] \in \{0, \hat{\delta}\}\) or \([s_2/s_1] \in \{\hat{1}, \hat{G}\}\). With this constraint we may generate a truth table for the possible temperature and pressure ratio pairs satisfying this constraint, as in Table 12.2. From this table we see that whenever \([T_2/T_1] > [p_2/p_1]\) we satisfy the constraint, as well as when both ratios are \(\hat{1}\). The unfortunate ambiguity of \([x] \otimes [y]\) leaves two undetermined places in the table, preventing one from concluding satisfaction of the second law whenever \([T_2/T_1] \geq [p_2/p_1]\).

We can, however, employ the non-decreasing entropy constraint as a quantity condition for the activation of adiabatic processes.
Table 12.2: Qualitative solutions to the second law of thermodynamics for an ideal gas undergoing an adiabatic change of state.

<table>
<thead>
<tr>
<th>$s_2/s_1$</th>
<th>$p_2/p_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{G}$</td>
<td>? T T</td>
</tr>
<tr>
<td>$\hat{i}$</td>
<td>F T T</td>
</tr>
<tr>
<td>$\hat{L}$</td>
<td>F F ?</td>
</tr>
</tbody>
</table>

; general change of state process specification (including second law of thermodynamics)

Process AnyFlow(g.1.2)

Individuals:
  IdealGas(g) ; the gas undergoing change
  State(g.1) ; the initial state
  State(g.2) ; the final state

Preconditions:
  $\emptyset$

Quantity Conditions:
  $\emptyset$

Relations:
  $\dot{m}_2/\dot{m}_1 = \hat{1}$
  $[p_2/p_1] \oplus [A_2/A_1] \oplus [v_2/v_1] \approx 3$
  $[v_2/v_1] \oplus [h_2/h_1] \oplus [Q_2/Q_1] \oplus [W_2/W_1] \approx 2$
  $[p_2/p_1] \oplus [p_2/p_1] \oplus [v_2/v_1] \oplus [F_2/F_1] \approx 2$
  $[s_2/s_1] \geq [T_2/T_1] \oplus [p_2/p_1] \oplus \hat{1}$

Influences:
  $\emptyset$

Figure 12.10: A modification of the AnyFlow process specification to include the relations provided by the second law of thermodynamics.
Individual View State\((g,n)\)

**Individuals:**
- \(\text{Gas}(g)\) : \(g\) is a subset of all gases
- \(p\) : the pressure of \(g\) at \(n\)
- \(T\) : the temperature of \(g\) at \(n\)
- \(\rho\) : the density of \(g\) at \(n\)
- \(v\) : the velocity of \(g\) at \(n\)
- \(s\) : the entropy of \(g\) at \(n\)

**Preconditions:**
- \(\emptyset\)

**Quantity Conditions:**
- \(\emptyset\)

**Relations:**
- \(\emptyset\)

Figure 12.11: An individual view structure State defining the gas properties required to determine the state of a gas \(g\).

### 12.4 State Equations

A complete state specification requires determination of the velocity, pressure, temperature and density of the fluid. Unfortunately, the four governing equations alone are not enough to solve for the flow state in most gas dynamics problems. One more relation, derived from the *thermodynamic specification* and the *equation of state*, provides the additional mechanism to determine the complete state. Regardless of the specification technique, however, we can define the basic individual view structure for the state of a gas State, as shown in Figure 12.11.

The thermodynamic specification applies flow field assumptions to the problem, restricting the results to certain thermodynamic conditions such as *isentropic,*
isothermal, or adiabatic flow. The equation of state relates the minimum number of properties necessary to determine all of the other properties. For gas dynamics problems this equation normally is separated into two equations, a thermal equation of state and a caloric equation of state. The thermal equation of state is of the form

\[ f(p, \rho, T) = 0, \]

while the caloric equation of state is usually of the form

\[ f(h, p, T) = 0. \]

Sections 12.4.1 and 12.4.2 identify specific forms of these equations for application in gas dynamics problems.

12.4.1 Ideal Gases

An ideal gas obeys the thermal equation of state

\[ p = \rho RT, \]

where \( R \) is the specific gas constant, and the caloric equations state

\[ c_V = \text{constant}, \]

\[ c_p = \text{constant}. \]

Knowing that the specific heats are constant we conclude that the ratio of these specific heats \( k \) is also constant,

\[ k = \frac{c_p}{c_V} = \text{constant}. \]
Differentiating the thermal equation of state, knowing that for an ideal gas

\[ R = c_p - c_v = \text{constant} \]

from the caloric equations of state, we obtain the relationship for change in pressure

\[ \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}. \]

From this quantitative expression we derive the qualitative ratio expression

\[ \left[ \frac{p_2}{p_1} \right] \approx \left[ \frac{\rho_2}{\rho_1} \right] \oplus \left[ \frac{T_2}{T_1} \right] \ominus \hat{1}. \]

Evaluation of this expression provides most of the desired results: for the pressure to increase, there must be either a density increase or temperature increase, or both. The pressure may remain constant only if both the density and temperature also remain constant, while the case of increasing density and decreasing temperature (or vice versa) cannot be evaluated qualitatively with only this information. Similarly, since

\[ h = \frac{kRT}{k - 1} \]

we may derive the following qualitative expressions for changes in enthalpy

\[ \left[ \frac{h_2}{h_1} \right] \approx \left[ \frac{T_2}{T_1} \right] \]

and from quantitative substitution

\[ \left[ \frac{p_2}{p_1} \right] \approx \left[ \frac{h_2}{h_1} \right] \oplus \left[ \frac{\rho_2}{\rho_1} \right] \ominus \hat{1}. \]
Finally, by definition the velocity of sound is

\[ c = \sqrt{\frac{dp}{d\rho}} \]

which reduces to

\[ c = \sqrt{kRT} \]

for an ideal gas, providing another qualitative dependency on temperature change arises as

\[
\begin{bmatrix}
    c_2 \\
    c_1
\end{bmatrix} \approx \begin{bmatrix} T_2 \\ T_1 \end{bmatrix}.
\]

If we are interested in a gas whose actual state lies within the (qualitative) constraints levied by the equations of state, then the ideal gas assumption is a powerful convenience for solving problems. The assumptions entering the definition of the thermal equation of state are that the influences of intermolecular forces and molecular size are negligible, which are not valid assumptions at low temperatures (TempLow) and high pressures (PressureHigh). The assumptions entering the definition of the caloric equations of state are that the influences of molecular rotational and vibrational excitations are negligible, which are not valid assumptions at high temperatures (TempHigh). In order to satisfy these assumptions we specify inequality constraints as preconditions in the individual view structure, as in Figure 12.12.

Similar reasoning leads to other important expressions for an ideal gas. Since \( h = c_p T \), we can derive a relationship between enthalpy changes and temperature
Individual View $\text{IdealGas}(g)$

**Individuals:**

$\text{Gas}(g) : g$ is an element of the set of all gases

$\text{State}(g, \text{dynamic}) :$ the dynamic flow state

**Preconditions:**

$[T] < \text{TempHigh}$

$[T] > \text{TempLow}$

$[p] < \text{PressureHigh}$

**Quantity Conditions:**

$R = \text{constant}$

$c_v = \text{constant}$

$c_p = \text{constant}$

**Relations:**

$\emptyset$

Figure 12.12: The first elements of individual view structure for an ideal gas $g$ that inherits behaviors from the individual views Gas and State.

changes,

$$
\begin{bmatrix}
  h_2 \\
  h_1
\end{bmatrix} \approx 
\begin{bmatrix}
  T_2 \\
  T_1
\end{bmatrix}.
$$

According to expectation, we can substitute the enthalpy equation into the state equation, giving

$$
\begin{bmatrix}
  h_2 \\
  h_1
\end{bmatrix} \approx 
\begin{bmatrix}
  p_2 \\
  p_1
\end{bmatrix} \odot 
\begin{bmatrix}
  \rho_2 \\
  \rho_1
\end{bmatrix} \odot \hat{1}.
$$

Using the sonic velocity $c$, we may restate the enthalpy relationship as

$$
h = \frac{c^2}{k - 1},
$$

or in qualitative terms

$$
\begin{bmatrix}
  h_2 \\
  h_1
\end{bmatrix} \approx 
\begin{bmatrix}
  c_2 \\
  c_1
\end{bmatrix}.
The energy equation for an ideal gas then reduces to

\[
Q = \left( \frac{c_2^2}{k-1} + \frac{v_2^2}{2} \right) - \left( \frac{c_1^2}{k-1} + \frac{v_1^2}{2} \right)
\]

\[
= h_{02} - h_{01}
\]

\[
= \frac{kR}{k-1} (T_{02} - T_{01})
\]

\[
= \frac{kR}{k-1} \Delta T_0,
\]

which translates into the qualitatively equivalent expressions

\[
\begin{bmatrix}
Q_2 \\
Q_1
\end{bmatrix}
\approx
\begin{bmatrix}
h_{02} \\
h_{01}
\end{bmatrix}
\approx
\begin{bmatrix}
T_{02} \\
T_{01}
\end{bmatrix}.
\]

For adiabatic flow \( Q = 0 \) or \( |Q_2/Q_1| = \hat{1} \) and we have the qualitative solutions

\[
\begin{bmatrix}
h_{02} \\
h_{01}
\end{bmatrix}
\approx
\begin{bmatrix}
T_{01} \\
T_{02}
\end{bmatrix}
\approx \hat{1}.
\]

From this result we derive an alternative form of the adiabatic energy equation

\[
\frac{c^2}{k-1} + \frac{v^2}{2} = \text{constant},
\]

which defines an inverse relationship between the thermal and mechanical energies of the flow, stated qualitatively as

\[
\begin{bmatrix}
c_2 \\
c_1
\end{bmatrix} \oplus \begin{bmatrix}
v_2 \\
v_1
\end{bmatrix} \approx 2.
\]

When the local sonic and flow velocities are equal, the energy state is known as the \textit{critical state}, denoted by a * superscript, as in \( v = c = v^* = c^* \). For the ideal gas
the critical temperature can be determined from the critical velocity,

\[ T^* = \frac{c^*}{kR}, \]

which then provides a direct relationship between temperature change and specific energy change

\[ \begin{bmatrix} T_2^* \\ T_1^* \end{bmatrix} \approx \begin{bmatrix} c_2^* \\ c_1^* \end{bmatrix} \approx \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}. \]

Carrying out various substitutions for measures of specific energy and translating into qualitative form, we derive the set of equivalent relations

\[ \begin{bmatrix} h_2 \\ h_1 \end{bmatrix} \oplus \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} \ni \begin{bmatrix} h_{02} \\ h_{01} \end{bmatrix} \oplus \hat{1}, \]

\[ \ni \begin{bmatrix} T_{02} \\ T_{01} \end{bmatrix} \oplus \hat{1}, \]

\[ \ni \begin{bmatrix} c_{02} \\ c_{01} \end{bmatrix} \oplus \hat{1}, \]

\[ \ni \begin{bmatrix} T_2^* \\ T_1^* \end{bmatrix} \oplus \hat{1}. \]

For an adiabatic flow we capture the equivalent relations \([T_{02}/T_{01}] = [c_{02}/c_{01}] = [T_2^*/T_1^*] = \hat{1} \). Using all of these results, along with the inequality constraint on entropy change provided by the second law of thermodynamics, we can now assemble the process view AdiabaticFlow (Figure 12.13).

Using the sonic velocity as a reference, we define the ratio of the local flow velocity to the local sonic velocity as the Mach number,

\[ M = \frac{v}{c}. \]
\[
\text{Process AdiabaticFlow(g.1.2)}
\]

**Individuals:**
AnyFlow(g.1.2) ; fundamental relationships

**Preconditions:**
\( \emptyset \)

**Quantity Conditions:**
\[
\frac{Q_2}{Q_1} = \hat{1} \\
\frac{s_2}{s_1} \in \{\hat{1}, \hat{G}\}
\]

**Relations:**
\[
\frac{h_2}{h_1} \oplus \frac{v_2}{v_1} \approx 2 \\
\frac{c_2}{c_1} \oplus \frac{v_2}{v_1} \approx 2 \\
\frac{T_2^*}{T_1^*} \approx \hat{1} \\
\frac{T_{o2}}{T_{o1}} \approx \hat{1} \\
\frac{h_{o2}}{h_{o1}} \approx \hat{1} \\
\frac{c_{o2}}{c_{o1}} \approx \hat{1} \\
\frac{c_2^*}{c_1^*} \approx \hat{1}
\]

**Influences:**
\( \emptyset \)

Figure 12.13: An enhanced specification of the AdiabaticFlow process view to include state relations for an ideal gas.
Introducing this ratio into the energy equation discussion above, we obtain convenient relationships among the various flow parameters. The Mach number is especially interesting because of its qualitative description of the flow characteristics: subsonic, sonic, or supersonic, corresponding directly to the situations in which $[M] = \hat{L}$, $[M] = \hat{1}$, and $[M] = \hat{G}$. Indeed, we will see the utility of this ratio and the convenience provided by it, which supports the decision to employ the qualitative ratio algebra. For example, the definition itself provides the qualitative relationship

$$
\begin{bmatrix}
v_2 \\
v_1
\end{bmatrix} \approx \begin{bmatrix}
c_2 \\
c_1
\end{bmatrix} \oplus \begin{bmatrix}
M_2 \\
M_1
\end{bmatrix} \oplus \hat{1}.
$$

Additionally, since

$$
\frac{T_0}{T} = 1 + \frac{k - 1}{2} M^2
$$

we have

$$
\begin{bmatrix}
T_0 \\
T
\end{bmatrix} \approx \hat{1} \oplus [M] \geq \hat{1}.
$$

Since the ratio $T_{02}/T_{01}$ represents the heat added to the system, we may derive a relationship between the temperature and Mach number changes between states,

$$
\frac{T_2}{T_1} = \frac{T_2/T_{02}}{T_1/T_{01}} = \frac{1 + \frac{k - 1}{2} M_1^2}{1 + \frac{k - 1}{2} M_2^2}.
$$

Unfortunately, the qualitative substitution does not work well since

$$
\frac{T_2/T_{02}}{T_1/T_{01}} = \frac{\hat{1} \oplus [M_1]}{\hat{1} \oplus [M_2]} \approx \frac{\hat{G}}{\hat{G}} \approx \hat{?}
$$

for all non-zero Mach numbers. Instead of the algebraic derivation, therefore, we apply our expertise and observe that the qualitative relationship we desire is con-
ditional:

\[
\frac{T_2/T_{02}}{T_1/T_{01}} \approx \begin{cases} 
\hat{G} & \text{if } |M_1| > |M_2| \\
\hat{L} & \text{if } |M_1| < |M_2| \\
\hat{i} & \text{if } |M_1| = |M_2|
\end{cases}
\]

For adiabatic flow,

\[
\frac{T_2}{T_1} \approx \begin{cases} 
\hat{G} & \text{if } |M_1| > |M_2| \\
\hat{L} & \text{if } |M_1| < |M_2| \\
\hat{i} & \text{if } |M_1| = |M_2|
\end{cases}
\]

These expressions correctly convey the relationship that a change in temperature is inversely proportional to the change in Mach number, and it is able to provide results other than \( \hat{?} \).

Considering further thermodynamic specifications for an ideal gas, we return to the second law of thermodynamics. For isentropic flow the entropy change \( \Delta s \) is 0, so equation 12.9 reduces to

\[
c_p \ln \left( \frac{T_2}{T_1} \right) = R \ln \left( \frac{p_2}{p_1} \right).
\]

Given \( c_p \) and \( R \) for an ideal gas, this equation becomes

\[
\frac{c_p}{R} \ln \left( \frac{T_2}{T_1} \right) = \ln \left( \frac{p_2}{p_1} \right),
\]

\[
\ln \left( \frac{T_2}{T_1} \right)^{k/(k-1)} = \ln \left( \frac{p_2}{p_1} \right),
\]

\[
\left( \frac{T_2}{T_1} \right)^{k/(k-1)} = \frac{p_2}{p_1}.
\]

Since \( (T_2/T_1) \) is positive, we drop the exponent in the qualitative translation, leaving

\[
\frac{T_2}{T_1} \approx \left[ \frac{p_2}{p_1} \right] .
\]
Figure 12.14: The enhancement of the IdealGas view structure template to include stagnation temperature relationships.

Substituting for \( \frac{p_2}{p_1} \) from the state equation, we then similarly obtain another satisfying result

\[
\frac{T_2}{T_1} \approx \left( \frac{p_2}{p_1} \right).
\]

Figures 12.14 through 12.17 show the enhancement of the QPT templates to include these new relationships.

**Example 4** *Air is flowing in a straight symmetric duct between points 1 and 2. If we know \( p_1, A_1, v_1 \) and \( T_1 \) at the first point, and we know that the gas temperature decreases \( (T_2 < T_1) \) while the velocity increases \( (v_2 > v_1) \), what constraints apply to the dynamic pressure \( p_2 \) and cross-sectional area \( A_2 \) with respect to these values at point 1?*
Process  AnyFlow(g.1.2)

Individuals:
IdealGas(g); the gas undergoing change
State(g.1); the initial state
State(g.2); the final state

Preconditions:
∅

Quantity Conditions:
∅

Relations:
\[ \frac{m_2}{m_1} = \hat{1} \]
\[ \frac{\rho_2}{\rho_1} \oplus \frac{A_2}{A_1} \oplus \frac{v_2}{v_1} \approx 3 \]
\[ \frac{v_2}{v_1} \oplus \frac{h_2}{h_1} \oplus \frac{Q_2}{Q_1} \oplus \frac{W_2}{W_1} \approx 2 \]
\[ \frac{p_2}{p_1} \oplus \frac{\rho_2}{\rho_1} \oplus \frac{v_2}{v_1} \oplus \frac{F_2}{F_1} \approx 2 \]
\[ \frac{p_2}{p_1} \approx \frac{\rho_2}{\rho_1} \oplus \frac{T_2}{T_1} \oplus \hat{1} \]
\[ \frac{h_2}{h_1} \approx \frac{T_2}{T_1} \]
\[ \frac{c_2}{c_1} \approx \frac{T_2}{T_1} \]
if \[ |M_1| > |M_2| \] then \[ \frac{T_2}{T_1} \frac{T_{02}}{T_{01}} \approx \hat{G} \]
if \[ |M_1| < |M_2| \] then \[ \frac{T_2}{T_1} \frac{T_{02}}{T_{01}} \approx \hat{L} \]
if \[ |M_1| = |M_2| \] then \[ \frac{T_2}{T_1} \frac{T_{02}}{T_{01}} \approx \hat{1} \]

Influences:
∅

Figure 12.15: A modification of the AnyFlow process specification to include the Mach number relation provided by the stagnation temperature ratio.
; adiabatic change of state process specification
; (enhanced)
**Process** AdiabaticFlow(g.1.2)

**Individuals:**
AnyFlow(g.1.2); fundamental relationships

**Preconditions:**
∅

**Quantity Conditions:**
\[
\frac{Q_2}{Q_1} = \hat{1} \\
\frac{s_2}{s_1} \in \{\hat{1}, \hat{G}\}
\]

**Relations:**
\[
\frac{h_2}{h_1} \oplus \frac{v_2}{v_1} \approx 2 \\
\frac{c_2}{c_1} \oplus \frac{v_2}{v_1} \approx 2 \\
\frac{T_2^*}{T_1^*} \approx \hat{1} \\
\frac{T_{02}}{T_{01}} \approx \hat{1} \\
\frac{h_{02}}{h_{01}} \approx \hat{1} \\
\frac{c_{02}}{c_{01}} \approx \hat{1} \\
\frac{c_2^*}{c_1^*} \approx \hat{1}
\]

if \([M_1] > [M_2]\) then \([T_2/T_1] \approx \hat{G}\)
if \([M_1] < [M_2]\) then \([T_2/T_1] \approx \hat{L}\)
if \([M_1] = [M_2]\) then \([T_2/T_1] \approx \hat{1}\)

**Influences:**
∅

**Figure 12.16:** Enhancement of the AdiabaticFlow process template to include Mach number relations.
From the given information we know that \([T_2/T_1] \approx \dot{L}\) and \([v_2/v_1] \approx \dot{G}\).

We assume that no work is done on the fluid, so that \([W_2/W_1] \approx \dot{I}\).

We assume also that air behaves as an ideal gas, therefore we activate the \textit{IdealGas} individual view and the \textit{AnyFlow} process. Activating the \textit{IdealGas} view provides the result

\[
\begin{bmatrix}
  h_2 \\
  h_1
\end{bmatrix} \approx \dot{L}
\]

and the contraint equation between pressure and density

\[
\begin{bmatrix}
  p_2 \\
  p_1
\end{bmatrix} \approx \begin{bmatrix}
  \rho_2 \\
  \rho_1
\end{bmatrix} \odot \dot{L} \odot \dot{I}.
\]

The \textit{AnyFlow} view energy relationship provides

\[
\dot{G} \odot \dot{L} \odot \begin{bmatrix}
  Q_2 \\
  Q_1
\end{bmatrix} \odot \dot{I} \approx 2
\]
\[ \hat{G} \otimes \begin{bmatrix} Q_2 \\ Q_1 \end{bmatrix} \approx \hat{1}, \]

which unfortunately leads to ambiguity in determining the heat transfer \([Q_2/Q_1] \approx \hat{?}\). The given information alone is not enough to solve the problem, so for qualitative simulation purposes we assume certain (parametric) conditions and determine the consequences. In this case we branch the solution and assume each of the three possible cases for pressure change, i.e. (1) \([p_2/p_1] = \hat{1}\), (2) \([p_2/p_1] = \hat{L}\), and (3) \([p_2/p_1] = \hat{G}\), then attempt to determine the resulting constraint conditions applying to the cross-sectional area.

1. Assume \([p_2/p_1] = \hat{1}\). From the ideal gas assumption we conclude that \([\rho_2/\rho_1] \approx \hat{G}\). From continuity we obtain the constraint \([A_2/A_1] \approx \hat{L}\). This is one possible solution.

2. Assume \([p_2/p_1] = \hat{L}\). In this case the state equation provided by ideal gas assumption provides the ambiguous result \([\rho_2/\rho_1] \approx \hat{?}\). Branching again on the possible values of density ratio, we assume each of the three possible cases:

   (a) Assume \([\rho_2/\rho_1] = \hat{1}\). Conservation of mass provides the result \([A_2/A_1] \approx \hat{L}\). This is a possible solution.

   (b) Assume \([\rho_2/\rho_1] = \hat{G}\). Conservation of mass again provides the result \([A_2/A_1] \approx \hat{L}\). This, too, is a possible solution.
(c) Assume \( \rho_2 / \rho_1 = \hat{L} \). Conservation of mass does not provide a result since \( A_2 / A_1 \approx \hat{?} \). Therefore, no interesting solution is provided.

3. Assume \( p_2 / p_1 = \hat{G} \). From the ideal gas assumption we conclude that \( \rho_2 / \rho_1 \approx \hat{G} \). From continuity we obtain the constraint \( A_2 / A_1 \approx \hat{L} \). This is another possible solution.

If instead of branching the solution on the pressure ratio we further assume adiabatic flow, then \( Q_2 / Q_1 = \hat{1} \) and we activate the AdiabaticFlow process. Then the second law of thermodynamics can be satisfied \( (s_2 / s_1) \in \{\hat{1}, \hat{G}\} \) only if \( p_2 / p_1 \approx \hat{L} \). This precludes the case-by-case exploration of pressure ratio possibilities, but still requires exploration of the density ratio possibilities. If we then further assume IsentropicFlow, this immediately provides the result \( \rho_2 / \rho_1 \approx \hat{L} \). Interestingly, this is the combination that provides the ambiguous result for area ratio \( A_2 / A_1 \approx \hat{?} \).

### 12.4.2 Non-Ideal Gases

For gas flows that do not obey the constraints of the ideal gas assumptions, we must specify unique relationships for the equations of state. Once identified, we can insert these relationships into a new individual view structure that accounts for the behaviors occurring in that situation. Such situations may occur in chemically-reacting
flows, during gas ionization or dissociation, in the presence of high temperatures, and many other real-world cases. Unfortunately, these relationships are difficult to define and usually must be generated empirically.

Since an investigation of the behavior of real (non-ideal) gases requires the introduction of empirical formulae into the equations of state, these problems do not, in the general case, seem to be good candidates for qualitative reasoning systems. However, in these instances a temporary assumption of ideality for the purposes of generating an initial simulation may certainly prove worthwhile, if only to assist in selecting the most appropriate of the available empirical models. In the interest of broadest applicability and "common sense" representations, the theory developed herein considers only ideal gases.
Chapter 13

Interior Flows

This Chapter investigates one-dimensional flow through passages usually referred to as ducts. The category of ducts includes various shapes of channels, nozzles, and diffusers.

The one-dimensional flow assumption permits us to reason about one spatial direction of the velocity vector $\vec{v}$, so that we assume the flow velocity profile is uniform at any given cross-section (see Figure 13.1). The steady flow assumption permits us to remove all dependencies on time from the governing equations, so that “states” refer to physical positions along a duct rather than points in time. Furthermore, it transpires that the downstream direction usually is independent of magnitude; that is, if we consider the $x$-axis of a spatial coordinate system to lie inside the duct and aligned with the direction of flow, then the distance between states along the $x$-axis rarely enters the problem.\(^1\)

Using these assumptions, which were prescribed earlier in the derivation of relationships for each of the conservation laws, we are able to apply the results to describe physical phenomena. Sections 13.1 and 13.2 describe certain special cases

\(^1\)Flow in the presence of friction, described in Section 13.4.1, is the exception.
Figure 13.1: One-dimensional velocity profiles for flow in ducts. Figure (a) shows a realistic profile indicating viscous stresses at the duct wall. Figure (b) shows the uniform profile assumption that neglects these effects.

of steady one-dimensional duct flows, including isentropic steady flow, adiabatic steady flow, and normal shock waves. With these cases in hand, we first apply the results to problems involving ducts of varying cross-sectional area, such as nozzles and diffusers (Section 13.3), then we apply them to problems involving ducts of constant cross-sectional area (Section 13.4). The results are summarized in Section 13.5.

13.1 Isentropic Steady Flow

This section concerns steady flow of an ideal gas through ducts wherein the entropy change between any two points is zero. For this assumption to be valid, the flow must first be adiabatic and reversible. In particular, this assumption proves
quite powerful for solving problems concerning ducts of varying cross-sectional area. Many of the interesting concepts of isentropic steady flow through nozzles and diffusers are captured by a single relationship between the change in cross-sectional area along the direction of flow and the corresponding change in velocity. The assumption of isentropy reduces the Second Law of Thermodynamics to

\[ dh = \frac{dp}{\rho} \Rightarrow \left[ \frac{h_2}{h_1} \right] \approx \left[ \frac{p_2}{p_1} \right], \]

so that the energy equation reduces to

\[ v \, dv + \frac{dp}{\rho} = 0 \]

or

\[ \left[ \frac{v_2}{v_1} \right] \oplus \left[ \frac{h_2}{h_1} \right] \approx 2. \]

Combining these results with the continuity equation yields the Hugoniot Equation:

\[ \frac{dA}{A} = \frac{dv}{v} \left[ M^2 - 1 \right]. \]

We now explore the various interpretations of this special result.

13.1.1 Nozzles and Diffusers

By convention, we say that a passage that accelerates a flow is a nozzle, while a passage that decelerates a flow is a diffuser. From the Hugoniot Equation it can be seen that a subsonic flow \((M < 1)\) will be accelerated \((dv > 0)\) in a convergent passage \((dA < 0)\), while a supersonic flow \((M > 1)\) will be accelerated in a divergent
passage ($dA > 0$). Contrarily, a subsonic flow will be decelerated in a divergent passage, while a supersonic flow will be accelerated in a convergent passage. If the flow is sonic, then $dA = 0$ and the passage is neither convergent nor divergent, but at a minimum cross-sectional area (or throat).\footnote{The other case of $dA$ reaching a local maximum requires an infinite cross-sectional area.} So if $M = 1$ somewhere in the passage, that point must be the throat; however, at the throat we do not require that $M = 1$ (e.g. subsonic flow throughout the passage). Various substitutions of the continuity and energy equations uncover relationships similar to the Hugoniot Equation for the change in density with respect to cross-sectional area,

$$\frac{dA}{A} = -\frac{d\rho}{\rho} \frac{[M^2 - 1]}{M^2},$$

and for the change in pressure with respect to cross-sectional area,

$$\frac{dA}{A} = -\frac{dp}{\rho v^2} [M^2 - 1].$$

These expressions show that nozzle passages require a decrease in both pressure and density, while diffuser passages require an increase in both pressure and density. Processes that cause an increase in pressure are compression processes, while processes that cause a decrease in pressure are expansion processes. Therefore, in summary, nozzles accelerate and expand a flow, while diffusers decelerate and compress a flow.

These results translate into the qualitative formulation rather easily. Translating
the Hugoniot Equation directly into a qualitative expression, we derive

\[ [dA] \approx [dv] \otimes ([M] \otimes \hat{1}), \]

which is generally useful but less intuitive than it might be if we prespecify the Mach number regime and continue the reduction: if \([M] \approx \hat{1}\), then \([dA] \approx \hat{0}\) and \([dv] \approx \hat{?}\), or \([A_2/A_1] \approx \hat{1}\) and \([v_2/v_1] \approx \hat{?}\); if \([M] \approx \hat{L}\), then \([dA] = -[dv]\) or \([A_2/A_1] \otimes [v_2/v_1] \approx 2\); and if \([M] \approx \hat{G}\), then \([dA] = [dv]\) and \([A_2/A_1] \approx [v_2/v_1]\).

This interpretation is rather cumbersome unless we separate the situations into two general processes from which four different applications may occur. We will call the general processes IsentropicExpansion and IsentropicCompression. Regardless of the incident Mach number, isentropic flow through a nozzle will result in \([v_2/v_1] \approx \hat{G}\), \([\rho_2/\rho_1] \approx \hat{L}\), and \([p_2/p_1] \approx \hat{L}\). Similarly, isentropic flow through a diffuser will result in result in \([v_2/v_1] \approx \hat{L}\), \([\rho_2/\rho_1] \approx \hat{G}\), and \([p_2/p_1] \approx \hat{G}\). Assuming the pressure difference and some area change as the quantity conditions, these processes are shown in Figures 13.2 and 13.3.

An important issue raised by the specification of quantity conditions in these two process templates concerns the nature of the problems being solved. There are two classes of problems generally encountered in gas dynamics problems involving internal flows:

**Design Problems** Given the required flow performance characteristics, find the passage geometry necessary to satisfy these requirements;
; isentropic expansion process specification

Process IsentropicExpansion(g.1.2)

Individuals:
  IsentropicFlow(g.1.2) ; fundamental relationships

Preconditions:
  \emptyset

Quantity Conditions:
  \begin{align*}
  \frac{p_2}{p_1} & \approx \hat{L} \\
  \frac{A_2}{A_1} & \neq \hat{L}
  \end{align*}

Relations:
  \begin{align*}
  \frac{v_2}{v_1} & \approx \hat{G} \\
  \frac{\rho_2}{\rho_1} & \approx \hat{G}
  \end{align*}

Influences:
  \emptyset

Figure 13.2: A process template for isentropic flow through a nozzle. The pressure change is assumed as the quantity condition, though any of the relations might have been specified as well. Activation of the process also requires that the passage area not be constant.

; isentropic compression process specification

Process IsentropicCompression(g.1.2)

Individuals:
  IsentropicFlow(g.1.2) ; fundamental relationships

Preconditions:
  \emptyset

Quantity Conditions:
  \begin{align*}
  \frac{p_2}{p_1} & \approx \hat{G} \\
  \frac{A_2}{A_1} & \neq \hat{L}
  \end{align*}

Relations:
  \begin{align*}
  \frac{v_2}{v_1} & \approx \hat{L} \\
  \frac{\rho_2}{\rho_1} & \approx \hat{G}
  \end{align*}

Influences:
  \emptyset

Figure 13.3: A process template for isentropic flow through a diffuser.
Performance Problems Given the passage geometry, determine the resulting flow performance characteristics.

The selection of the problem class to be pursued governs our selection of quantity conditions for the various process templates. Indeed, one might develop a unique set of process templates to address each problem class. In the specification of the IsentropicExpansion and IsentropicCompression processes, the pressure change was used as the quantity condition (which must be satisfied along with affirmed instantiation of the individuals) before that process becomes active. Depending upon the problem to be solved, we might have chosen to specify the density or velocity ratios as the quantity conditions for these two processes. Identification of the appropriate quantity conditions is particularly important for a computer program implementation, since the program will select processes based only on the prespecified conditions. Alternatively, we might permit several possible quantity conditions by specifying a logical expression to test for satisfaction, such as

\[
\left( \frac{p_2}{p_1} \approx \bar{L} \lor \left[ \frac{p_2}{\rho_1} \right] \approx \bar{L} \lor \left[ \frac{v_2}{v_1} \right] \approx \bar{G} \right) \land \left[ \frac{A_2}{A_1} \right] \approx \hat{1}.
\]

The performance problem is the more common of the two cases, and is particularly interesting for qualitative reasoning systems because of the wide variety of parameters to be determined. Since there are relatively few geometries to select from, and given the spatial constraints of qualitative physics, it is rather easy to specify a geometry and calculate the corresponding performance characteristics. So the model process templates developed herein will be specified from the standpoint
of performance problems, although it is understood that the quantity conditions and relations might be rearranged within the templates to accommodate design problems.

Considering each of the four cases (subsonic or supersonic, converging or diverging) as a flow performance process, we simply apply the necessary geometry and incident Mach number preconditions and state the ratio results directly as relations. For the performance problem, the SubsonicExpansion process requires \([M] \approx \hat{L}\) and \([A_2/A_1] \approx \hat{L}\); SubsonicCompression requires \([M] \approx \hat{L}\) and \([A_2/A_1] \approx \hat{G}\); SupersonicExpansion requires \([M] \approx \hat{G}\) and \([A_2/A_1] \approx \hat{G}\); and SupersonicCompression requires \([M] \approx \hat{G}\) and \([A_2/A_1] \approx \hat{L}\). These conditional requirements serve as the quantity conditions for the basic isentropic flow performance problem processes provided in Figures 13.4 through 13.7.
Process SubsonicCompression(g.1.2)

Individuals:
IsentropicFlow(g.1.2) ; fundamental relationships

Preconditions:
∅

Quantity Conditions:

$\frac{|M|}{\hat{L}} \approx \frac{A_2}{A_1}$

Relations:

$\frac{\nu_2}{\nu_1} \approx \frac{\hat{G}}{\hat{G}}$

$\frac{\rho_2}{\rho_1} \approx \frac{\hat{G}}{\hat{G}}$

Influences:
∅

Figure 13.5: A process template for subsonic isentropic flow of through a diffuser.

Process SupersonicExpansion(g.1.2)

Individuals:
IsentropicFlow(g.1.2) ; fundamental relationships

Preconditions:
∅

Quantity Conditions:

$|M| \approx \hat{G}$

$\frac{A_2}{A_1} \approx \hat{G}$

Relations:

$\frac{\nu_2}{\nu_1} \approx \hat{G}$

$\frac{\rho_2}{\rho_1} \approx \hat{L}$

$\frac{p_2}{p_1} \approx \hat{L}$

Influences:
∅

Figure 13.6: A process template for supersonic isentropic flow of through a nozzle.
13.1.2 The Stagnation State

As defined in Section 12.2.2, the stagnation enthalpy $h_0$ represents the amount of thermal energy in a flow brought to rest adiabatically. The temperature achieved through this action is the stagnation temperature $T_0$. These parameters and conditions, unfortunately, are not sufficient to define the entire stagnation state. If, however, we add the thermodynamic specification of isentropic flow of an ideal gas, we may define relationships sufficient to determine another property which, along with the temperature, provides enough information to define the stagnation state.

From the isentropic ideal gas relations we know

$$\frac{p_f}{p_i} = \left(\frac{T_f}{T_i}\right)^{k/(k-1)} = \left(\frac{\rho_f}{\rho_i}\right)^k$$
and the qualitative equivalents

\[
\begin{bmatrix}
\frac{p_2}{p_1} \\
\frac{\rho_2}{\rho_1}
\end{bmatrix} \approx \begin{bmatrix}
\frac{T_2}{T_1}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\frac{p_0}{p} \\
\frac{\rho_0}{\rho}
\end{bmatrix} \approx \begin{bmatrix}
\frac{T_0}{T}
\end{bmatrix}.
\]

If we let the final flow state refer to the stagnation state, then the ratios may be rewritten

\[
\begin{bmatrix}
\frac{p_0}{p} \\
\frac{\rho_0}{\rho}
\end{bmatrix} \approx \begin{bmatrix}
\frac{T_0}{T}
\end{bmatrix}.
\]

These new relations can be added to the IdealGas view template, contributing expressions in terms of an individual view reference state State 0, which we define as the stagnation state.

Using the stagnation state in the Second Law of Thermodynamics we derive further relationships for isentropic flow. Restated in terms of the stagnation temperature and pressure,

\[
\Delta s = \Delta s_0 = c_p \ln \frac{T_{02}}{T_{01}} - R \ln \frac{p_{02}}{p_{01}}.
\]

Since \(\Delta s = \Delta s_0\) by definition, then

\[
\begin{bmatrix}
\frac{s_2}{s_1}
\end{bmatrix} \approx \begin{bmatrix}
\frac{s_{02}}{s_{01}}
\end{bmatrix} \approx 1.
\]

For an adiabatic flow, \(T_{02}/T_{01} = 1\) and the Second Law reduces to

\[
\Delta s = -R \ln \frac{p_{02}}{p_{01}} \geq 0,
\]
which in turn reduces to a requirement on the stagnation pressure ratio

\[
\frac{p_{02}}{p_{01}} \leq 1.
\]

Qualitatively, this translates into the obvious relation

\[
\left[ \frac{p_{02}}{p_{01}} \right] \leq \hat{1}.
\]

The inequality condition \( [p_{02}/p_{01}] \approx \hat{L} \) applies to adiabatic flows, while the equality condition \( [p_{02}/p_{01}] \approx \hat{1} \) applies to isentropic flows. As expected, these relationships state that there is no loss of stagnation pressure between states in an isentropic flow, but that stagnation pressure will always be lost in a flow that is adiabatic but not isentropic.

We may also derive isentropic flow relations from the Second Law in terms of Mach number. The quantitative equations

\[
\begin{align*}
\frac{p_0}{p} &= \left( 1 + \frac{k - 1}{2} M^2 \right)^{k/(k-1)}, \\
\frac{\rho_0}{\rho} &= \left( 1 + \frac{k - 1}{2} M^2 \right)^{1/(k-1)},
\end{align*}
\]

translate into the simple qualitative equations

\[
\begin{align*}
\left[ \frac{p_0}{p} \right] &\approx \hat{1} \oplus [M] \geq \hat{1}, \\
\left[ \frac{\rho_0}{\rho} \right] &\approx \hat{1} \oplus [M] \geq \hat{1}.
\end{align*}
\]

Like the temperature ratio \([T_0/T]\), the ratios \([p_0/p]\) and \([\rho_0/\rho]\) are \(\hat{G}\) for all non-zero Mach numbers. Regardless of the flow regime identified by the Mach number,
the temperature, pressure and density of an ideal gas are greater in the stagnation state than in the dynamic state. Similar to what was done earlier for the dynamic temperature ratio, we may write an expression for the dynamic pressure ratio in terms of only the Mach number,

\[
\frac{p_2}{p_1} = \frac{p_2}{p_02} \frac{p_{01}}{p_1}.
\]

Just as before, substitution of the qualitative expressions \([p_0/p] \approx \hat{G}\) provides an ambiguous result, so we resort to expertise and write the intermediate qualitative expressions by condition:

\[
\frac{p_2/p_{02}}{p_1/p_{01}} \approx \begin{cases} 
\hat{G} & \text{if } [M_1] > [M_2] \\
\hat{L} & \text{if } [M_1] < [M_2] \\
\hat{i} & \text{if } [M_1] = [M_2]
\end{cases}
\]

For an isentropic flow, wherein \(p_{02} = p_{01}\), we obtain the ratio of dynamic pressures as a function of Mach number only,

\[
\frac{p_2}{p_1} \approx \begin{cases} 
\hat{G} & \text{if } [M_1] > [M_2] \\
\hat{L} & \text{if } [M_1] < [M_2] \\
\hat{i} & \text{if } [M_1] = [M_2]
\end{cases}
\]

**Example 5** An ideal gas is flowing in a duct between points 1 and 2. If \(M_1 = 1.40\), what are the values of \(p_2\) and \(T_2\) relative to \(p_1\) and \(T_1\) if is desired that (a) \(M_2 = 0.4\), or (b) \(M_2 = 2.5\)?

Given the ideal gas and two flow states, we activate the IdealGas view
and the AnyFlow process. We also know $[M_1] \approx \hat{G}$ from the given information.

(a) Considering the $M_2 = 0.4$ case, $[M_2] \approx \hat{L}$. From the AnyFlow process we calculate

$$\begin{bmatrix} \frac{T_2}{T_{02}} \\ \frac{T_1}{T_{01}} \end{bmatrix} \approx \hat{G}.$$  

Unfortunately, that is all that can be determined without further assumptions or information. Therefore, we branch the solution and consider further assumptions. If we assume the flow to be adiabatic, then $[Q_2/Q_1] \approx \hat{I}$ and we activate the AdiabaticFlow process. This provides the relation

$$\begin{bmatrix} T_2 \\ T_1 \end{bmatrix} \approx \hat{G}$$

which implies that $T_2$ will be greater than $T_1$. For the pressure change, we require $[p_2/p_1]$, but cannot determine a reasonable value. Even though we know $[p_{01}/p_1] \approx \hat{G}$ and $[p_{02}/p_2] \approx \hat{G}$ from the IdealGas view, and we know $[p_{02}/p_{01}] \approx \hat{L}$ from the AdiabaticFlow process, the combination equation fails due to the ambiguity of qualitative multiplication and division,

$$\begin{bmatrix} p_2 \\ p_1 \end{bmatrix} \approx \begin{bmatrix} p_{01} \\ p_1 \end{bmatrix} \odot \begin{bmatrix} p_{02} \\ p_2 \end{bmatrix} \odot \begin{bmatrix} p_{02} \\ p_{01} \end{bmatrix},$$

$$\approx \hat{G} \odot \hat{G} \odot \hat{L},$$

$$\approx ?.$$
Similarly, although the AdiabaticFlow process requires $[s_2/s_1] \in \{\hat{i}, \hat{G}\}$, the Second Law solution is ambiguous,

\[
\begin{bmatrix}
T_2 \\
T_1
\end{bmatrix} \odot \begin{bmatrix}
p_2 \\
p_1
\end{bmatrix} \odot \hat{i} \in \{\hat{i}, \hat{G}\},
\]
\[
\hat{G} \odot \begin{bmatrix}
p_2 \\
p_1
\end{bmatrix} \odot \hat{i} \in \{\hat{i}, \hat{G}\},
\]
\[
\hat{G} \odot \begin{bmatrix}
p_2 \\
p_1
\end{bmatrix} \in \{\hat{i}, \hat{G}\},
\]

which means $[p_2/p_1] \approx \hat{G}$ due to the ambiguity of subtraction. If we further assume that the flow is isentropic, we add the fact that $[s_2/s_1] \approx \hat{i}$ and activate the IsentropicFlow process. This process provides the relation $[p_2/p_1] \approx [T_2/T_1]$, which now sets the value $[p_2/p_1] \approx \hat{G}$, so that $p_2$ is greater than $p_1$.

(b) Considering the $M_2 = 2.5$ case, this gives $[M_2] \approx \hat{G}$ and $[M_1] < [M_2]$.

From the AnyFlow process we calculate

\[
\begin{bmatrix}
T_2/T_02 \\
T_1/T_01
\end{bmatrix} \approx \hat{L},
\]

which represents the limits of our knowledge with the current assumptions. Again, we branch to consider additional possibilities. If we assume that the flow is adiabatic, we activate the AdiabaticFlow process to provide the relation

\[
\begin{bmatrix}
T_2 \\
T_1
\end{bmatrix} \approx \hat{L}.
\]

As in case (a), the only information known about the dynamic pressures at this point is $[p_{01}/p_1] \approx \hat{G}$, $[p_{02}/p_2] \approx \hat{G}$, and $[p_{02}/p_{01}] \leq \hat{i}$. If the
consider the flow to be isentropic, then we add the relation

\[
\begin{bmatrix}
    p_2 \\
    p_1
\end{bmatrix} \approx \hat{L},
\]

along with the constraint that \([p_2/p_1] \approx [T_2/T_1]\). Therefore we have the result \(p_2 < p_2\) and \(T_2 < T_1\).

### 13.1.3 The Critical State

The stagnation state is one of the two important reference states for the gas dynamics domain. The other is the critical state, the state held by a gas wherein the velocity \(v\) is equal to the local sonic velocity \(c\).

To define the critical state we require specification of another parameter to accompany \(T^*\), which was defined under the condition of adiabatic flow in Section 12.4.1. In terms of the Mach number, the ratio of the critical temperature \(T^*\) (at which point \(M^* = 1\)) to the local temperature \(T\), we have

\[
\frac{T^*}{T} = \frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}.
\]

To define another parameter in the critical state we apply the thermodynamic specification for isentropic flow, and write the same ratio for the critical pressure \(p^*\) as

\[
\frac{p^*}{p} = \left(\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right)^{k/(k-1)}.
\]

Together the values \(T^*\) and \(p^*\) are enough to define the critical state. For convenience, we also write the ratio for the critical density \(\rho^*\) as

\[
\frac{\rho^*}{\rho} = \left(\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right)^{1/(k-1)}.
\]
Owing to the various constants, translating any of these expressions into their qualitative counterparts requires inspection instead of algebraic derivation. Inspecting the equation for the temperature ratio, we first conclude that whenever $M = 1$, we have $T^*/T = 1$ as expected. Whenever $M < 1$, we determine $T^*/T < 1$, and whenever $M > 1$, we determine $T^*/T > 1$. Since the right-hand side exponents will not affect the outcome of the qualitative ratio results, we conclude the same Mach number relationship for the critical pressure and density ratios. Qualitatively, this analysis is represented by the simple equations

\[
\begin{align*}
\begin{bmatrix} T^* \\ T \end{bmatrix} & \approx [M] \\
\begin{bmatrix} p^* \\ p \end{bmatrix} & \approx [M] \\
\begin{bmatrix} \rho^* \\ \rho \end{bmatrix} & \approx [M]
\end{align*}
\]

for $[M] > 0$, and

\[
\begin{align*}
\begin{bmatrix} T^* \\ T \end{bmatrix} & \approx \hat{L} \\
\begin{bmatrix} p^* \\ p \end{bmatrix} & \approx \hat{L} \\
\begin{bmatrix} \rho^* \\ \rho \end{bmatrix} & \approx \hat{L}
\end{align*}
\]

for $[M] = 0$. Of course, since the equations are simply a special case ($M_2 = 1$) of the isentropic flow relationships, we might have proceeded directly to the AdiabaticFlow process relations and evaluated $[T_2/T_1] \approx (\hat{1} \oplus [M_1]) \otimes (\hat{1} \oplus [M_2])$ for $[M_2] = \hat{1}$ and $[T_2] = [T^*]$. This substitution also provides the desired result $[T^*/T] \approx [M]$ for $[M] > 0$. Similarly, the IsentropicFlow process relations provide the desired result.
for \( [p^*/p] \). Interestingly, this substitution would also have worked in the derivation of the stagnation state ratios, since for that case \( |M_2| = \hat{0} \) and division by \( \hat{1} \) provides the result \( [T_0/T] \approx \hat{G} \) for any \( |M_1| \).

Although the gas flow may not actually pass through it, the critical state represents a reference state for standardized comparison of parameters. It describes the state the gas would achieve if its velocity were adjusted such that \( v = c \), which we can evaluate at any point in a flow. For comparison purposes, we consider how the critical state values change. If the flow between two states is adiabatic, we know that \( T_1^* = T_2^* \), or \( |T_2^*/T_1^*| \approx \hat{1} \). If the flow is also isentropic, we add the conclusions \( p_1^* = p_2^* \), or \( |p_2^*/p_1^*| \approx \hat{1} \), and \( \rho_1^* = \rho_2^* \), or \( |\rho_2^*/\rho_1^*| \approx \hat{1} \).

All of these new relations can be added to the various process templates, contributing expressions in terms of a reference state referred to as the critical state. Within the **IdealGas** view template we label this new individual State *.

### 13.1.4 Mass Flow Relations

Given an isentropic process and the principle of mass conservation, the mass flow rate anywhere in a flow is equivalent to the mass flow rate in the critical state, i.e.

\[
\dot{m} = \dot{m}^*
\]

\[
\rho Av = \rho^* A^* c^*.
\]

Since we have expressions for the ratio of critical density to the local density, dividing the right-hand side of the last equation by the left-hand side yields an expression
in Mach number providing a ratio of the area at any point in the flow to the area
at the point where \( M = 1 \), which we call the critical area \( A^* \):

\[
\frac{A}{A^*} = \frac{1}{M} \left( 1 + \frac{k-1}{2} M^2 \right)^{(k+1)/2(k-1)}.
\] (13.1)

This equation describes a function with a minimum \( A/A^* = 1 \) at \( M = 1 \), such that
\( A/A^* > 1 \) when either \( M < 1 \) or \( M > 1 \). Therefore, qualitatively, we state this
relationship as

\[
[A/A^*] \geq \hat{1}.
\]

As with the other parameters in the critical state, the area \( A^* \) is only a reference
point which need not exist physically, but if a flow between two points is isentropic,
then the points share the same reference point \( A_1^* = A_2^* \), or \( [A_2^*/A_1^*] \approx \hat{1} \).

For the isentropic flow case in a design problem, the ratio of the areas between
the two points may be determined as a function only of the local Mach numbers,

\[
\left[ \frac{A_2}{A_1} \right] \approx \left[ \frac{A_2}{A_2^*} \right] \odot \left[ \frac{A_1}{A_1^*} \right].
\]

Unfortunately, this provides the ambiguous result \( \hat{?} \) since both \( [A_2/A_2^*] \) and \( [A_1/A_1^*] \)
are \( \hat{G} \) everywhere \( A \neq A^* \).

For performance problems we want to determine the other parameters given
the area, so we return to the Hugoniot relationships and the Expansion and Diffu-
sion processes which require geometry information as quantity conditions. These
processes now inherit the constraint \( [A/A^*] \geq \hat{1} \) which limit the size of their small-
est cross-sectional area to \( A^* \). This raises the point that if \( A < A^* \) somewhere
in the flow, then an isentropic solution is not possible for the given inlet conditions. Without the IsentropicFlow process active, none of the Hugoniot processes we have defined could be active because the Individuals conditions are not satisfied, and therefore the qualitative theory would not provide a behavior model. This is a desireable feature.

An important measure of performance in one-dimensional systems is the mass flow rate per unit cross-sectional area, or mass flow density, which can be provided from the continuity equation,

\[ \frac{\dot{m}}{A} = \rho v. \]

To express this mass flow density only in terms of the reference states and the local critical area ratio, we perform several substitutions,

\[ \frac{\dot{m}}{A} = \rho^* v^* A^* \]
\[ = \sqrt{\frac{k}{R}} \frac{p^*}{\sqrt{T^*}} A^* \]
\[ = \sqrt{\frac{k}{R}} \frac{p_0}{\sqrt{T_0}} \left( \frac{2}{k+1} \right)^{(k+1)/(2(k-1))} A^* \frac{A^*}{A}. \]

Since \( A/A^* \) has a minimum at \( M = 1 \), this equation shows that \( \dot{m}/A \) has a maximum value \( \dot{m}/A^* \) at \( A = A^* \) or \( M = 1 \). Consequently, \( \dot{m}/A \leq \dot{m}/A^* \) everywhere, and we have a qualitative result similar to the area ratio,

\[ \frac{\dot{m}/A^*}{\dot{m}/A} \geq \hat{i}. \]

The qualitative evaluation of the change in mass flow density derives from the
continuity equation,
\[
\begin{bmatrix}
\dot{m}_2 / A_2 \\
\dot{m}_1 / A_1
\end{bmatrix} \approx \begin{bmatrix}
\rho_2 \\
\rho_1
\end{bmatrix} \oplus \begin{bmatrix}
v_2 \\
v_1
\end{bmatrix} \otimes \hat{i},
\]
which applies to the AnyFlow process. For isentropic flow, we may substitute equation 13.1 to determine the mass flow density as a function of the Mach number,
\[
\frac{\dot{m}}{A} = \sqrt{\frac{k}{R}} \frac{p_0}{\sqrt{T_0}} M \left(1 + \frac{k-1}{2} M^2\right)^{-(k+1)/2(k-1)}.
\]
(13.2)

Stating the mass flow density ratio in terms of Mach number, we regain the same qualitative relationship as before
\[
\begin{bmatrix}
\dot{m} / A^* \\
\dot{m} / A
\end{bmatrix} \geq \hat{i},
\]
which has a minimum of \(\hat{i}\) at \([A/A^*] \approx \hat{i}\), and has the value \(\hat{G}\) for \([M] \approx \hat{L}\) or \([M] \approx \hat{G}\). The maximum mass flow density value then may be calculated by setting \(M = 1\) in equation 13.2,
\[
\frac{\dot{m}}{A^*} = \frac{p_0}{\sqrt{T_0}} \sqrt{\frac{k}{R}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)}.
\]
(13.3)

For an isentropic flow, \(p_{02} = p_{01}\) and the maximum mass flow density does not change. However, for a flow that is adiabatic but nonisentropic, this relation provides important insight into the change in critical area,
\[
\frac{A_1^*}{A_2^*} = \frac{p_{02}}{p_{01}} \leq 1,
\]
or
\[
\begin{bmatrix}
A_2^* \\
A_1^*
\end{bmatrix} \geq \hat{i}.
\]
This result shows that the minimum area through which a certain mass flow rate will pass increases in an adiabatic process.

Finally, to enforce the correspondence between $A = A^*$ and $M = 1$ anywhere in an isentropic flow, we define a new process IsentropicChokedFlow which activates on the quantity condition $|M| \approx \hat{1}$. Since choked flow occurs under different conditions in other kinds of flows, we first define a general ChokedFlow process view that simply specifies that the flow occurs at the maximum mass flow rate (Figure 13.8). The IsentropicChokedFlow process requires the IsentropicFlow and the ChokedFlow process views, while providing correlation between the local flow parameters and their values in the critical state along with the cross-sectional geometry constraint. To define the choking point for the isentropic flow process, we add a Correspondence relation that correlates $M \geq 1$ to $\dot{m} = \dot{m}_c$. The process definition for Isentropic-ChokedFlow is given in Figure 13.9.

### 13.1.5 Wall Force Relations

The force applied by the moving gas to the passage containing it is of interest in both design and performance problems. The principle of conservation of momentum (outlined in Section 12.2.3) provides a means for calculating this force. Reiterating the ideal gas result for a straight, symmetrical duct, we derived

$$F_w = A_2(p_2 + \rho_2v^2 \ast 2) - A_1(p_1 + \rho_1v_1^2).$$
; choked flow process specification
;
Process ChokedFlow(g)

Individuals:
   IdealGas(g) ; the gas undergoing change

Preconditions:
   0

Quantity Conditions:
   $[\dot{m}/\dot{m}_c] \approx \hat{1}$

Relations:
   0

Influences:
   0

Figure 13.8: The ChokedFlow process definition, activated by achieving the maximum mass flow rate for a given situation.

; isentropic choked flow process specification
;
Process IsentropicChokedFlow(g.1.2)

Individuals:
   ChokedFlow(g) ; the general choked flow conditions
   IsentropicFlow(g.1.2) ; isentropic flow relations

Preconditions:
   0

Quantity Conditions:
   0

Relations:
   $[A^*/A_{\min}] \approx \hat{1}$

Influences:
   0

Figure 13.9: The IsentropicChokedFlow process definition, specializing the combined conditions for choked flow and isentropic flow.
From this result we identify a function for calculating the force at a particular point,

\[ F = A(p + \rho v^2), \]

and use the difference between the force values at various points to determine the total force \( F_w = F_2 - F_1 \). For an ideal gas we may write this function in terms of the Mach number,

\[ F = Ap(1 + kM^2), \]

or at the critical state

\[ F^* = A^*p^*(1 + k). \]

Since \( A^* \) and \( p^* \) are constant for an isentropic flow, we conclude \( F^*_2 = F^*_1 \), or

\[ \left[ \frac{F^*_2}{F^*_1} \right] \approx \hat{1}. \]

Taking the ratio of the two equations and performing the Mach number substitutions for \( A/A^* \) and \( p/p^* \) we obtain

\[ \frac{F}{F^*} = \frac{1}{M} \frac{1 + kM^2}{2(k + 1) \left( 1 + \frac{k-1}{2}M^2 \right)^{1/2}}, \]

which is yet another relation that has a minimum value \( F/F^* = 1 \) at \( M = 1 \). The qualitative translation therefore is simple,

\[ \left[ \frac{F}{F^*} \right] \geq \hat{1}, \]

which provides another relation for the isentropic flow process view.
Individual View IdealGas(g)

Individuals:
- Gas(g) : g is an element of the set of all gases
- State(g, dynamic) : the dynamic flow state
- State(g, 0) : the stagnation flow state
- State(g, *) : the critical flow state

Preconditions:
- \[ T < \text{TempHigh} \]
- \[ T > \text{TempLow} \]
- \[ p < \text{PressureHigh} \]

Quantity Conditions:
- \( R = \text{constant} \)
- \( c_v = \text{constant} \)
- \( c_p = \text{constant} \)

Relations:
- \[ \frac{T_0}{T} \geq \hat{1} \]
- \[ \frac{p_0}{p} \geq \hat{1} \]
- \[ \frac{\rho_0}{\rho} \geq \hat{1} \]
- if \( |M| > \hat{0} \) then \( \frac{T^*}{T} \approx |M| \) else \( \frac{T^*}{T} \approx \hat{L} \)
- if \( |M| > \hat{0} \) then \( \frac{p^*}{p} \approx |M| \) else \( \frac{p^*}{p} \approx \hat{L} \)
- if \( |M| > \hat{0} \) then \( \frac{\rho^*}{\rho} \approx |M| \) else \( \frac{\rho^*}{\rho} \approx \hat{L} \)
- \[ \frac{m^*/A^*}{m/A} \geq \hat{1} \]

Figure 13.10: An enhancement of the IdealGas view structure template to include relations provided by definition of the stagnation state.

Figures 13.10 through 13.13 update the quantity conditions and relations now available for the IdealGas individual view and the AnyFlow, AdiabaticFlow and IsentropicFlow process views.
; general change of state process specification

; Process AnyFlow(g.1.2)

Individuals:
  IdealGas(g) ; the gas undergoing change
  State(g.1) ; the initial state
  State(g.2) ; the final state

Preconditions:
  \emptyset

Quantity Conditions:
  \emptyset

Relations:
  \frac{\dot{m}_2}{\dot{m}_1} = \hat{i}
  \frac{\rho_2}{\rho_1} \oplus \frac{A_2}{A_1} \oplus \frac{v_2}{v_1} \approx 3
  \frac{\rho_2}{\rho_1} \oplus \frac{h_2}{h_1} \oplus \frac{Q_2}{Q_1} \oplus \frac{W_2}{W_1} \approx 2
  \frac{A_2}{A_1} \oplus \frac{p_2}{p_1} \oplus \frac{p_2}{p_1} \oplus 2 \oplus \frac{v_2}{v_1} \oplus \frac{F_2}{F_1} \approx 4
  \frac{p_2}{p_1} \approx \frac{p_2}{p_1} \oplus \frac{T_2}{T_1} \oplus \hat{i}
  \frac{h_2}{h_1} \approx \frac{T_2}{T_1}
  \frac{c_2}{c_1} \approx \frac{T_2}{T_1}

if \ [M_1] > [M_2] \ then \ \frac{T_2}{T_1} \frac{T_0_2}{T_0_1} \approx \hat{G}

if \ [M_1] < [M_2] \ then \ \frac{T_2}{T_1} \frac{T_0_2}{T_0_1} \approx \hat{L}

if \ [M_1] = [M_2] \ then \ \frac{T_2}{T_1} \frac{T_0_2}{T_0_1} \approx \hat{i}

if \ [M_1] > [M_2] \ then \ \frac{p_2}{p_0_2} \frac{p_1}{p_0_1} \approx \hat{G}

if \ [M_1] < [M_2] \ then \ \frac{p_2}{p_0_2} \frac{p_1}{p_0_1} \approx \hat{L}

if \ [M_1] = [M_2] \ then \ \frac{p_2}{p_0_2} \frac{p_1}{p_0_1} \approx \hat{i}

\frac{m_2}{m_1} \approx \frac{\rho_2}{\rho_1} \oplus \frac{v_2}{v_1} \oplus \hat{i}

Influences:
  \emptyset

Figure 13.11: A modification of the AnyFlow process specification to include the Mach number relation provided by the stagnation pressure ratio.
Process AdiabaticFlow(g.1.2)

Individuals:
   AnyFlow(g.1.2) ; fundamental relationships

Preconditions:
   0

Quantity Conditions:
   \[ \frac{Q_2}{Q_1} = \hat{i} \]
   \[ \frac{s_2}{s_1} \in \{\hat{\theta}, \hat{\gamma}\} \]

Relations:
   \[ \frac{h_2}{h_1} \oplus \frac{v_2}{v_1} \approx 2 \]
   \[ \frac{c_2}{c_1} \oplus \frac{v_2}{v_1} \approx 2 \]
   \[ \frac{T_2^*}{T_1^*} \approx \hat{i} \]
   \[ \frac{T_{o2}}{T_{o1}} \approx \hat{i} \]
   \[ \frac{h_{o2}}{h_{o1}} \approx \hat{i} \]
   \[ \frac{c_{o2}}{c_{o1}} \approx \hat{i} \]
   \[ \frac{c_2^*}{c_1^*} \approx \hat{i} \]
   if \[ [M_1] > [M_2] \] then \[ \frac{T_2}{T_1} \approx \hat{\gamma} \]
   if \[ [M_1] < [M_2] \] then \[ T_2/T_1 \approx \hat{L} \]
   if \[ [M_1] = [M_2] \] then \[ T_2/T_1 \approx \hat{i} \]
   \[ \frac{p_{o2}}{p_{o1}} \approx \hat{L} \]
   \[ \frac{A_2^*}{A_1^*} \geq \hat{i} \]

Influences:
   0

Figure 13.12: Enhancement of the AdiabaticFlow process template to include the stagnation pressure loss.
isentropic change of state process specification
(enhanced)

Process IsentropicFlow(g.1.2)

Individuals:
- AdiabaticFlow(g.1.2) : the adiabatic process
- ReversibleFlow(g.1.2) : the reversible process

Preconditions:
\[ \emptyset \]

Quantity Conditions:
\[ \frac{s_2}{s_1} \approx \hat{1} \]
\[ \frac{s_{02}}{s_{01}} \approx \hat{1} \]

Relations:
\[ \frac{p_2}{p_1} \approx \frac{T_2}{T_1} \]
if \[ M_1 > M_2 \] then \[ \frac{p_2}{p_1} \approx \hat{G} \]
if \[ M_1 < M_2 \] then \[ \frac{p_2}{p_1} \approx \hat{L} \]
if \[ M_1 = M_2 \] then \[ \frac{p_2}{p_1} \approx \hat{1} \]
\[ \frac{\rho_2}{\rho_1} \approx \frac{T_2}{T_1} \]
\[ \frac{p_{02}}{p_{01}} \approx \hat{1} \]
\[ \frac{p_2^*}{p_1^*} \approx \hat{1} \]
\[ \frac{\rho_2^*}{\rho_1^*} \approx \hat{1} \]
\[ \frac{A_2^*}{A_1^*} \approx \hat{1} \]
\[ \frac{F_2^*}{F_1^*} \approx \hat{1} \]
\[ \frac{A/A^*} \geq \hat{1} \]
\[ F/F^* \geq \hat{1} \]

Correspondence(\[ |M| \geq \hat{1}, |\dot{m}/\dot{m}_c| \approx \hat{1} \])

Influences:
\[ \emptyset \]

Figure 13.13: An enhanced QPT process specification for an isentropic flow of an ideal gas. The correspondence relation maps the choking point to a particular Mach number.
13.2 Normal Shock Waves

One way for a gas flow to adjust to its environment, in order to guarantee satisfaction of the conservation laws, is through the phenomenon known as a shock wave. Shock disturbance waves occur through the coalescence of weak disturbance waves owing to the relative motion between the fluid and the duct. The effect of the wave is to spawn an irreversible internal adjustment for the flow to conform to the non-linear situation. As such, one may think of the shock wave as an adjustment process that acts upon the flow to change its behavior.

For one-dimensional (interior) flows, we model this irreversible disturbance as a discontinuity, wherein the error attributable to this approximation is no more than the error already introduced by the linear and one-dimensional flow assumptions. Consequently, the real phenomenon is reduced to a collection of expressions describing how the various flow properties change when a gas experiences a shock wave normal to the direction of flow. This occurrence may be stationary at a certain position, or may be travelling along the duct with the gas.

Owing to the treatment of the normal shock wave as a discontinuity, we conclude that the thickness of the disturbance is small enough to be negligible in both the quantitative and qualitative models. To apply the conservation equations to this situation, we define the control volume Ω as infinitesimally thin, as suggested in Figure 13.14. The upstream and downstream states maintain their customary designations, however they now denote points that are infinitesimally close together.
Figure 13.14: An infinitesimally-thin control volume $\mathcal{V}$ for analysis of a normal shock wave in an interior flow. The stations 1 and 2 denote the locations upstream and downstream of the shock wave, although for this control volume the cross-sectional areas $A_1$ and $A_2$ are equivalent.

Now since $A_1 = A_2$, this precludes effects contributed by the passage wall (such as friction) and reduces the problem to investigation of a discontinuity in an adiabatic, frictionless, constant-area duct flow.\(^3\)

For constant-area flow of an ideal gas, the continuity, energy, and momentum equations reduce to

\[
\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}},
\]

\[
T_1 \left(1 + \frac{k-1}{2}M_1^2\right) = T_2 \left(1 + \frac{k-1}{2}M_2^2\right),
\]

\(^3\)Apparently, we continue to ignore the one spatial dimension of the flow. In fact, because of our disregard for spatial dimension, the qualitative model will permit shock waves of any thickness, so long as the points of interest (state positions) bracket the wave.
and

\[ p_1(1 + kM_1^2) = p_2(1 + kM_2^2), \]

respectively. In terms of ratios, we rewrite these expressions as

\[ \frac{M_2}{M_1} = \frac{p_1}{p_2} \sqrt{\frac{T_2}{T_1}}, \]

\[ \frac{T_2}{T_1} = \frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2}, \]

and

\[ \frac{p_2}{p_1} = \frac{1 + kM_1^2}{1 + kM_2^2}. \]

For the discontinuity case the states 1 and 2 must be different on each side of the control volume. From the ratio expressions it is clear that once \( M_1 \) and \( M_2 \) are known, the ratios \( p_2/p_1 \) and \( T_2/T_1 \) provide enough information to define the complete state. If \( M_1 \) is given, then the problem reduces to one of determining whether the governing equations permit \( M_2 \neq M_1 \), and if so, calculating its value.

Combining the ratio expressions listed above yields the quadratic equation

\[ \frac{M_1^2}{M_2^2} = \left[ \frac{1 + kM_1^2}{1 + kM_2^2} \right]^2 \frac{1 + \frac{k-1}{2}M_2^2}{1 + \frac{k-1}{2}M_1^2}. \]

The two solutions to this equation are

\[ M_2 = M_1, \]

and

\[ M_2 = \left[ \frac{\frac{2}{k-1} + M_1^2}{\frac{2k}{k-1}M_1^2 - 1} \right]^{1/2}, \quad (13.4) \]
which imply that a fluid flowing under the conditions listed above may or may not experience a discontinuous change in Mach number. Since the first solution $M_2 = M_1$ also means that $p_2/p_1 = 1$ and $T_2/T_1 = 1$, this is not the discontinuous solution: the flow continues unchanged. The second solution, however, provides a distinct $M_2$ for each $M_1$. Substituting this solution into the conservation equations and state equations for an ideal gas, we obtain expressions for the change in pressure, temperature, and density across a normal shock wave in terms of the incident Mach number $M_1$:

\[
\frac{p_2}{p_1} = \frac{2k}{k+1}M_1^2 - \frac{k-1}{k+1},
\]

\[
\frac{T_2}{T_1} = \frac{2(k-1)}{(k+1)^2M_1^2} \left(1 + \frac{k-1}{2}M_1^2\right) \left(\frac{2k}{k-1}M_1^2 - 1\right),
\]

\[
\frac{\rho_2}{\rho_1} = \frac{k+1}{2} \frac{M_1^2}{1 + \frac{k-1}{2}M_1^2}.
\]

The flow was assumed to be adiabatic but irreversible, so it is not isentropic. Therefore, the flow will not experience a change in stagnation temperature

\[
\frac{T_{02}}{T_{01}} = 1
\]

or

\[
\left[\frac{T_{02}}{T_{01}}\right] \approx \hat{1},
\]

but it will experience a stagnation pressure change

\[
\frac{p_{02}}{p_{01}} \neq 1.
\]

Before deriving the companion qualitative results, one important point must be made about the values of the incident and resultant Mach numbers. Considering the
critical Mach number $\bar{M} = v/c^*$, the relationship between the local Mach number and the critical Mach number is

$$M = \bar{M} \left[ \frac{2}{1 - \frac{k-1}{k+1} \bar{M}^2} \right]^{1/2}.$$

Substituting this result into equation 13.4 yields the simple relationship

$$\bar{M}_1 \bar{M}_2 = 1.$$

This result is important because it defines the qualitative relationship between the incident critical Mach number $\bar{M}_1$ and the resultant Mach number $\bar{M}_2$ across a normal shock wave. Furthermore, since $M < 1$ when $\bar{M} < 1$ and $M > 1$ when $\bar{M} > 1$, we may state the following more tenable relationships: when the incident flow is supersonic, $M_1 > 1$, then the resultant flow is subsonic, $M_2 < 1$; and when the incident flow is subsonic, $M_1 < 1$, then the resultant flow is supersonic, $M_2 > 1$. Qualitatively, this suggests

$$M_2 \approx \begin{cases} \hat{L} & \text{if } [M_1] \approx \hat{G} \\ \hat{G} & \text{if } [M_1] \approx \hat{L} \end{cases}$$

Applying the Second Law of Thermodynamics to determine the change in stagnation pressure for each case, where

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_{01}} \frac{p_1}{p_1},$$

or in terms of Mach number

$$\frac{p_{02}}{p_{01}} = \left[ \frac{k+1}{1 + \frac{k-1}{k+1} M_1^2} \right]^{k/(k-1)} \left[ \frac{2k}{k+1} \frac{M_1^2}{k+1} - \frac{k-1}{k+1} \right]^{1/(k-1)}, \quad (13.8)$$
we determine that for \( M_1 > 1 \) there is a loss of stagnation pressure \( \frac{p_{02}}{p_{01}} < 1 \), and for \( M_1 > 1 \) there is a gain of stagnation pressure \( \frac{p_{02}}{p_{01}} > 1 \). For an adiabatic process, the Second Law requires that

\[
\Delta s = -R \ln \left( \frac{p_{02}}{p_{01}} \right) \geq 0.
\]

From this we conclude that the case in which \( M_1 < 1 \) and \( M_2 > 1 \), also known as a rarefaction shock, is impossible for an adiabatic process, since \( \Delta s < 0 \). Therefore, the qualitative normal (compression) shock process need only define expressions for the case where \( |M_1| \approx \hat{G} \).

The qualitative translation process draws heavily upon this result to simplify the resulting expressions. First, with equation 13.5 and presuming \( M_1 > 1 \) and \( k > 1 \), we have

\[
\begin{bmatrix}
  k - 1 \\
  k + 1
\end{bmatrix} \approx \hat{L}
\]

so

\[
\begin{bmatrix}
  p_2 \\
  p_1
\end{bmatrix} \approx \hat{G} \odot \hat{G} \odot \hat{L},
\]

which further reduces to

\[
\begin{bmatrix}
  p_2 \\
  p_1
\end{bmatrix} \approx \hat{G} \odot \hat{L}.
\]

Although this \( \odot \) operation is ambiguous, we may resort to expertise to eliminate the ambiguity knowing that the desired result is to show an increase in dynamic pressure across the shock wave

\[
\begin{bmatrix}
  p_2 \\
  p_1
\end{bmatrix} \approx \hat{G}.
\]
Similarly, the translation of the three multiplicands of equation 13.6 yields

\[
\frac{T_2}{T_1} \approx \hat{L} \otimes \hat{G} \otimes \hat{G},
\]

which may be reduced to

\[
\frac{T_2}{T_1} \approx \hat{L} \otimes \hat{G}.
\]

Again, this result is ambiguous and we resort to our knowledge of the desired result \(T_2/T_1 > 1\) to determine

\[
\frac{T_2}{T_1} \approx \hat{G}.
\]

Equation 13.7 has the same translation problem, and we resort instead to translating the desired result \(\rho_2/\rho_1 > 1\) into qualitative form

\[
\frac{\rho_2}{\rho_1} \approx \hat{G}.
\]

Finally, instead of translating equation 13.8 into qualitative form, we obtain the desired result directly from the Second Law of Thermodynamics, which requires \(p_{02}/p_{01} < 1\) or

\[
\frac{p_{02}}{p_{01}} \approx \hat{L}.
\]

Consequently, as shown in Section 13.1.4 the minimum area \(A^*\) through which the flow will pass without further adjustment increases,

\[
\frac{A_2^*}{A_1^*} \approx \hat{G}.
\]

To assemble and apply these results we define a new process NormalShock which requires that the previously-defined AdiabaticFlow process be active, that the incident Mach number \([M_1]\) be greater than one, and that the resultant Mach number
$[M_2]$ be less than one. All of the attributes of the AdiabaticFlow process are inherited by NormalShock, such that redundant results (such as $[A_2^* / A_1^*] \approx \hat{G}$) need not be specified. The process view for NormalShock is shown in Figure 13.15. Since the presence of a shock wave causes irreversibilities in the flow stream, the ReversibleFlow process must also account for this effect in its activation conditions. Rather than creating a new variable to signal the presence of a shock, we simply conjoin the NormalShock process name with the existing quantity condition, assuming that the process is logically “true” if active and “false” if inactive. Figure 13.16 shows the new ReversibleFlow process with this test in the quantity conditions.

Although the NormalShock process might be considered applicable in any supersonic flow, we restrict its applicability by requiring the resultant Mach number to be subsonic. In this way we avoid activating the NormalShock process for all supersonic adiabatic flows, but we admit activation for those in which a normal shock is the likely cause for a subsonic flow downstream. This additional precondition may require that additional limit hypotheses be pursued for unknown downstream Mach numbers, but the restrictive preconditions are justified by the dependency of flow adjustments like shock waves on the external conditions not being considered between the two points of interest. These external conditions, such as the flow conditions further downstream than point 2, may in fact eliminate some of the cases in the limit hypothesis for $M_2$, thereby strengthening the simulation. Another approach might be to identify several different circumstances under which
normal shock wave process specification

Process NormalShock(g.1.2)

Individuals:
   AdiabaticFlow(g.1.2) ; the adiabatic process

Preconditions:
   \emptyset

Quantity Conditions:
   [M_1] \approx \hat{C}
   [M_2] \approx \hat{L}

Relations:
   \frac{p_2}{p_1} \approx \hat{C}
   \frac{T_2}{T_1} \approx \hat{C}
   \frac{\rho_2}{\rho_1} \approx \hat{C}
   \frac{s_2}{s_1} \approx \hat{C}

Influences:
   \emptyset

Figure 13.15: A QPT process specification for the flow of an ideal gas undergoing a normal shock wave.

normal shock waves occur, including external factors, and to list these considerations as part of the activation conditions for processes that are more specific than the general NormalShock process.
; reversible change of state process specification
; includes shock wave test
Process ReversibleFlow(g.1.2)

Individuals:
   AnyFlow(g.1.2) ; general flow relations

Preconditions:
   \emptyset

Quantity Conditions:
   \ [f] = \emptyset \land \neg NormalShock(g.1.2)

Relations:
   \emptyset

Influences:
   \emptyset

Figure 13.16: A QPT process specification for a reversible change of state.
13.3 Varying-Area Duct Flow

Section 13.1.1 presented the introductory qualitative discussion regarding nozzles and diffusers, leading to the specification of the processes IsentropicNozzle and IsentropicDiffuser. While these basic processes capture the essence of expansion and compression flows, combining them in models to simulate practical operating performance of nozzles and diffusers leads to many possible process scenarios and broad envisionments. These results can be attributed to the locality of the basic processes: the processes are activated considering only the states directly influenced by the process (and identified as Individuals) and the quantity conditions made available by selecting those states.

Although these envisionments certainly are worth pursuing for certain qualitative reasoning problems, we can provide better models for engineering simulations by further considering specific operating characteristics of these devices, then defining new process views that reduce the size of the envisionment while enhancing its usefulness for analyzing engineering problems. To do this, we employ certain global influences which act from outside of the two states which we view as “containing” the active process. Section 13.3.1 identifies such a global influence, and proposes new process views for describing nozzle operations, including both converging and converging-diverging kinds of nozzles. Section 13.3.2 then proposes new process views for describing diffuser operations. These devices comprise what we consider to be the family of varying-area ducts: Section 13.4 will consider problems in the
family of constant-area ducts.

13.3.1 Nozzle Operation

The performance problem for nozzles involves determining the flow characteristics given a fixed nozzle geometry and a set of operating (or boundary) conditions. Operating conditions include specification of the flow state at the inlet, usually assumed to be an infinite reservoir at constant pressure and temperature, and the back pressure \( p_b \) of the duct exhaust region. The reservoir assumption allows us to reason that the inlet state is the same as the stagnation state, since the reservoir contains the gas at zero velocity. The back pressure usually is varied to "drive" the nozzle through the entire range of operating conditions, and represents an ambient (stationary) exhaust region. Since the value of the back pressure can affect the performance of the nozzle upstream of the nozzle exit plane when the flow is subsonic, we may consider it to have a global influence on the flow at the various stations along the nozzle.

Considering nozzle performance at a macro scale, we may define qualitative processes that describe all of the the various internal (and external) flow characteristics in terms of exit plane conditions and back pressure. For each case, we assume an ideal gas is supplied at \( p_0 \) and \( T_0 \) with a known nozzle geometry, and wish to determine the mass flow rate and pressure distribution within the nozzle for the range of back pressures \( p_b \) under adiabatic flow conditions. To identify the
qualitatively-unique performance regimes we decrement the back pressure \( p_b \) from \( p_0 \) to 0, describing the internal and external flows of the nozzle for each different situation. In the exit plane, the pressure and Mach number are \( p_e \) and \( M_e \), however for evaluation and comparison purposes we will use the exit plane pressure ratio \( p_e/p_0 \).

The flow internal to the nozzle passage is analyzed just as before, according to the passage geometry. The flow external to the nozzle, however, has no guiding walls but may be analyzed as having a virtual geometry according to the principles of expansion and compression. If \( p_e < p_b \), then the exhausting gas must continue some external compression process until \( p_e = p_b \), which might be accomplished with a virtual convergent passage. Similarly, if \( p_e > p_b \) the gas must undergo and external expansion process until \( p_e = p_b \), which might be accomplished with a virtual divergent passage. We perform this evaluation for both kinds of nozzles: the simple convergent nozzle and the convergent-divergent (De Laval) nozzle.

For the convergent nozzle, we assume the situation depicted in Figure 13.17. Since the incident Mach number is zero, there can be no supersonic flow attained within the convergent passage because the flow will be accelerated toward Mach one, which can be reached only when the throat area \( A_t = A_e \) is the minimum cross-sectional area \( A_e \) for the given stagnation conditions. Consequently, there can be no normal shock waves within the nozzle and we may assume isentropic flow. When \( p_b/p_0 = 1 \), there is no flow since there is no pressure gradient across the nozzle.
Figure 13.17: The general geometry of a simple convergent nozzle. The upper panel depicts the nozzle operation as flow from a reservoir (representing the stagnation state $p_0$ and $T_0$) into a convergent passage exhausting into free space maintained at variable back pressure $p_b$. The lower panel depicts the various operating pressure profiles along the $x$ axis of the nozzle.
boundaries. This corresponds to line (a) in Figure 13.17. As \( p_b \) is decreased below \( p_0 \), as shown by line (b) in the figure, the flow starts through the nozzle, accelerating and expanding through the convergent passage as described by the IsentropicNozzle and SubsonicExpansion processes. So long as \( p_b/p_0 > p^*/p_0 \), \( M_e \) is subsonic and \( p_b/p_0 = p_e/p_0 \), so that the mass flow rate \( \dot{m} \) is less than the mass flow rate for choked flow \( \dot{m}_c \), which is known from equation 13.3. This also provides the ratio of the exit plane area \( A_e \) to the minimum area \( A^* \), since

\[
\frac{\dot{m}}{\dot{m}_c} = \frac{A^*}{A_e} < 1.0.
\]

Now when \( p_b/p_0 \) decreases to exactly \( p^*/p_0 \), as shown by line (c), the nozzle achieves the maximum mass flow density for the given stagnation conditions, such that \( A_e = A_t \) and \( \dot{m} = \dot{m}_c \). Moreover, the exit plane Mach number \( M_e = 1 \). Further decreasing the back pressure such that \( p_b/p_0 < p^*/p_0 \) now has no affect on the upstream conditions in the nozzle, since the maximum mass flow density has been achieved. Therefore, to accomodate the lower back pressure there must be an expansion adjustment \textit{external} to the nozzle, as suggested by line (d) in Figure 13.17. This external adjustment will occur for all \( p_b < p^* \).

These few interesting regions of operating pressure values suggest a special quantity space for those values. Just as the quantitative description above reduced the back pressure through regions of unique behaviors for the nozzle, these regions should correspond to an interval of qualitatively-consistent performance. In the interval \( p^*/p_0 < p_b/p_0 < 1 \), the flow is subsonic throughout the nozzle \((M_e < 1)\)
with \( \dot{m} < \dot{m}_e \). When \( p^*/p_0 = p_b/p_0 \), a special situation exists such that \( M_e = 1 \) and \( \dot{m} = \dot{m}_e \). In the interval \( 0 \leq p_b/p_0 < p^*/p_0 \), the maximum mass flow rate has been achieved \( \dot{m} = \dot{m}_e \), the exit plane Mach number \( M_e = 1 \), and an external expansion adjustment occurs to match the back pressure. With these landmark values, the quantity space for the qualitative operating pressure ratio \([p_b/p_0]\) is

\[
\left[ \frac{p_b}{p_0} \right] \in \left\{ \left[ 0, \frac{p^*}{p_0} \right], \left[ \frac{p^*}{p_0}, \frac{p^*}{p_0} \right], \left( \frac{p^*}{p_0}, 1 \right), [1, 1] \right\}.
\]

Since all of the interesting landmark values are less than one, it would be difficult to apply the current qualitative ratio algebra to this quantity space. However, we assume that the fully-ordered set of elements for this quantity space may be reasoned about in terms of their inequality relationships so that we may define meaningful quantity conditions for new processes that describe these nozzle performance behaviors. This quantity, in fact, may be deployed in such a manner that it does not appear in algebraic expressions, but rather serves only as a means for testing satisfaction of quantity conditions.

From this discussion, we may conclude that there are essentially two interesting qualitative behaviors for the convergent nozzle. The first describes the performance of the nozzle while \( p_b/p_0 > p^*/p_0 \), in which the exit plane flow is subsonic. The second describes the performance of the nozzle while \( p_b/p_0 \leq p^*/p_0 \), in which the exit plane flow is sonic, the mass flow is a maximum, and there may be an external adjustment to the back pressure. As alluded to earlier, the operating pressure inequality provides the global influence necessary to describe a nozzle performance
process between the inlet and outlet, which we usually prefer to think of as states 1 and 2, respectively. The back pressure actually is measured downstream of the exit plane, but it still may be referenced as a quantity condition. For the subsonic flow performance case we define an encapsulated history ConvergentSubsonic that captures the qualitative performance of a nozzle without choked flow (Figure 13.18). The history includes the isentropic flow process and its attendant states, for which we define a correspondence between the stagnation and exit states, as well as the operating pressure ratio \( \frac{p_e}{p_0} \). In the preconditions slot, we include the specification that the reservoir maintain the stagnation conditions at constant values. The only quantity condition is the ordering relationship on the operating pressure, which must be greater than the critical pressure ratio. The relations provided by activating this compound process include the isentropic flow process throughout the nozzle, with specific variable values provided to limit the history. These state that the mass flow rate is less than the critical mass flow rate, the exit area is greater than the critical exit area, the exit plane pressure ratio is equal to the operating pressure ratio, and the exit flow is subsonic.

Similarly, for the sonic flow performance case we define a new encapsulated history ConvergentSonic that captures the qualitative performance of a nozzle in which the operating pressure causes choked flow to occur (Figure 13.19). In this history, the quantity condition requires that the operating pressure ratio be less than or equal to the critical pressure ratio, so that the relations describe sonic flow
; convergent nozzle performance with subsonic outflow

Encapsulated History ConvergentSubsonic(g.1.2,r_p)

Individuals:
  IdealGas(g) ; ideal gas view
  State(g.1) ; the inlet conditions
  State(g.2) ; the outlet conditions
  State(g.0) ≡ State(g.1) ; inlet is stagnation state
  State(g.e) ≡ State(g.2) ; outlet is exit state
  r_p ≡ [p_e/p_0] ; the operating pressure ratio

Preconditions:
  State(g.0) = constant ; constant inlet conditions

Quantity Conditions:
  [A_2/A_1] ≈ \hat{L}
  [p_e/p_0] > [p_v/p_0]

Relations:
  IsentropicFlow(g.1.2) ; the isentropic flow process
  \[ \dot{m}/\dot{m}_e \approx \hat{L} \]
  \[ [A_e^*]/A_e \approx \hat{L} \]
  \[ [M_e] \approx \hat{L} \]
  \[ [p_e/p_0] \approx [p_v/p_0] \]

Figure 13.18: A QPT encapsulated history specification for the flow of an ideal gas in a convergent nozzle less than the maximum mass flow density. This special process history inherits all of the relations attributed to both the IdealGas and IsentropicFlow views, while defining correspondences between the stagnation state and state 1, and between the exit state and state 2.
Encapsulated History ConvergentSonic(g, 1.2, r_p)

Individuals:
- IdealGas(g) ; the ideal gas view
- State(g.1) ; the inlet state
- State(g.2) ; the outlet state
- State(g.0) ≡ State(g.1) ; inlet is stagnation state
- State(g.e) ≡ State(g.2) ; outlet is exit state
- \( r_p \equiv \frac{p_b}{p_0} \) ; the operating pressure ratio

Preconditions:
- State(g.0) = constant ; constant inlet conditions

Quantity Conditions:
- \( \frac{A_2}{A_1} \approx \frac{L}{\dot{m}} \)
- \( \frac{p_b}{p_0} \leq \frac{p_e}{p_0} \)

Relations:
- IsentropicChokedFlow(g, 1.2) ; the choked flow process
- \( \frac{A^*}{A_e} \approx \frac{1}{1} \)
- \( M_e \approx 1 \)
- \( \frac{p_e}{p_0} \geq \frac{p_b}{p_0} \)

Figure 13.19: A QPT encapsulated history specification for the flow of an ideal gas in a convergent nozzle at the maximum mass flow density. This template provides a history for isentropic choked flow through the nozzle, reaching specific outlet conditions.

at the exit plane, and the maximum mass flow density and minimum exit plane area.

For the convergent-divergent nozzle, we refer to the situation shown in Figure 13.20 and apply the same sort of analysis we performed for the convergent nozzle. In this case the incident Mach number still is zero, however the geometry of passage provides a minimum cross-sectional area which may correspond to the critical area, therefore a supersonic flow may be established under the appropriate conditions. Since supersonic flow may occur, the isentropic flow relations may not
Figure 13.20: The general geometry of a convergent-divergent nozzle. The upper panel depicts the nozzle operation as flow from a reservoir (representing the stagnation state $p_0$ and $T_0$) into a convergent-divergent passage exhausting into free space maintained at variable back pressure $p_b$. The lower panel depicts operating pressure profiles along the $x$ axis of the nozzle.
be applicable under those situations in which a normal shock develops in the throat or divergent section of the nozzle. The nozzle geometry is fixed with throat area $A_t$ and exit area $A_e$. Again, the inlet conditions are considered to be the stagnation state, exhausting into a region maintained at the back pressure $p_b$. As before, with $p_b/p_0 = 1$ there will be no flow through the nozzle (line (a) in Figure 13.20). When $p_b$ has been decreased such that the pressure ratio at the throat $p_t/p_0$ remains greater than the critical flow pressure ratio $p^*/p_0$, as described by line (b), the maximum mass flow rate has not been achieved and flow occurs subsonically throughout the nozzle. Since the exit flow is subsonic, we require $p_e/p_0 = p_b/p_0$, which may be obtained through isentropic flow. This performance is maintained until the back pressure is reduced to the point where the dynamic pressure at the throat is the critical pressure, $p_t/p_0 = p^*/p_0$, so that choked flow is achieved and the throat Mach number $M_t = 1$ (line (c)). This operating pressure ratio is called the first critical ratio, labelled $r_{p,t1}$. All operating conditions less than this pressure ratio will encounter choked conditions $\dot{m} = \dot{m}_c$ at the throat, where $A_t = A^*$. Since there are both supersonic and subsonic isentropic solutions corresponding to these throat conditions, there is also an operating pressure ratio condition corresponding to the supersonic solution. This condition is the third critical ratio $r_{p,t3}$ and is described by line (g) in Figure 13.20. For the first critical conditions, the divergent section of the passage serves as a diffuser to compress the flow; for the third critical conditions, the divergent section serves as a nozzle to expand the flow exactly to
\( p_b/p_0 = p_e/p_0 \). Between these two isentropic solutions there exists a region of back pressures for which the flow assumes supersonic conditions in the divergent section before encountering a normal shock wave, which correctly adjusts the downstream subsonic flow such that the exit plane pressure equals the back pressure. The presence of the shock wave voids the isentropic flow assumption, though the adiabatic flow conditions remain valid. Such a situation is suggested in lines (d) and (e), where line (e) depicts the shock standing in the exit plane. In this case, the dynamic pressure achieved just downstream of the shock wave is exactly equal to the back pressure. The operating condition for the situation represented by line (e) is known as the second critical ratio \( r_{p,e} \). Above the second critical ratio the exhaust flow is subsonic and the exit plane pressure is equal to the imposed back pressure. If the operating pressure \( p_b/p_0 \) decreases to a value between \( r_{p,e} \) and \( r_{p,e3} \), a normal shock can no longer stand in the exit plane, but instead the flow experiences a compression adjustment external to the nozzle, as suggested by line (f). Similarly, if the operating pressure decreases below \( r_{p,e3} \), the flow experiences an expansion adjustment external to the nozzle, as suggested by line (h).

Repeating the observation made for the convergent nozzle, there are a few regions of qualitatively-similar behaviors delimited by certain operating pressure ratios. The the convergent-divergent nozzle, the landmark pressure ratios correspond exactly to the critical pressure ratios. When \( r_{p,e1} < p_b/p_0 < 1 \), the flow is subsonic throughout the nozzle, with the mass flow rate less than the maximum mass
flow rate. When $r_{pc2} < p_b/p_0 < r_{pc1}$, the flow is choked at the throat (at the maximum mass flow rate) but experiences a normal shock in the divergent section which causes the exhaust velocity to be subsonic. When $r_{pc3} < p_b/p_0 < r_{pc2}$ the flow is choked with supersonic outflow and an external compression adjustment. When $p_b/p_0 = r_{pc3}$ exactly, the exit flow is perfectly expanded to achieve the back pressure, and no external adjustment is required. This condition corresponds to the optimum design performance condition. Finally, when $p_b/p_0 < r_{pc3}$, the flow is choked with supersonic outflow and an external expansion adjustment. These regions of behavior correspond to the quantity space

$$\frac{p_b}{p_0} \in \{[0, r_{pc3}), [r_{pc3}, r_{pc1}), (r_{pc1}, r_{pc2}), [r_{pc2}, r_{pc3}), [r_{pc1}, r_{pc3}), (r_{pc1}, 1), [1, 1]\}.$$

The six interesting performance situations for a qualitative reasoning system rely, as they did for the convergent nozzle, on this quantity space to provide an inequality ordering for quantity condition satisfaction tests. We may assume critical pressure ratios $r_{pc1}$, $r_{pc2}$, and $r_{pc3}$ to have qualitative values $[r_{pc1}]$, and so on, so long as we relax the definition of the $[\ ]$ operator, allowing it to map, in this case, values within the new operating pressure quantity space. Since these critical pressure ratios will not appear in the relations (and hence algebraic computations), we will use the operator simply to imply that a qualitative ordering relationship is possible.

Each of the six convergent-divergent nozzle performance situations corresponds to a unique process history. Each history requires the convergent-divergent nozzle geometry, which is cumbersome (but sufficient) to specify as the two inequality
conditions \([A_1/A_t] \approx \hat{G}\) and \([A_2/A_t] \approx \hat{G}\). There are no constraints on the relative sizes of areas \(A_1\) and \(A_2\). Using the abbreviation CD to represent "convergent-divergent," we define these new performance views as follows:

**CDSubsonic** For the operating pressure \(p_b/p_0 > r_{pe1}\), the flow is subsonic throughout so \(M_t < 1\), \(M_e < 1\), \(p_b \approx p_e\), \(A_t > A^*\), and \(\dot{m} < \dot{m}_c\). Since there are no irreversible occurrences in this case, we may apply the IsentropicFlow results as part of the flow process. Figure 13.21 shows this simple encapsulated history description.

**CDSubsonicChoked** For the operating pressure \(p_b/p_0 = r_{pe1}\), the flow is subsonic everywhere but at the throat, where the area corresponds to the critical area for the given conditions. Therefore, \(M_t = 1\), \(M_e < 1\), \(p_b = p_e\), \(A_t = A^*\), and \(\dot{m} = \dot{m}_c\). Again, since there are no irreversible occurrences in this case, we may apply the IsentropicFlow results within the history, however in this case we may also apply the results of the general ChokedFlow process at the nozzle throat, thereby reducing the relations required in the CDSubsonicChoked history. Figure 13.22 shows this encapsulated history description.

**CDNormalShock** When the operating pressure ratio lies between the first and second critical ratios, a normal shock positions itself in the divergent section of the nozzle such that the subsonic flow downstream of the shock compresses just enough to equal the back pressure upon reaching the exit plane. At
Encapsulated History CDSubsonic(g.1.2.b)

Individuals:
IdealGas(g); the ideal gas view
State(g.1); the inlet state
State(g.2); the outlet state
State(g.b); the exhaust region conditions
State(g.0) = State(g.1); inlet is stagnation state
State(g.e) = State(g.2); outlet is exit state
State(g.t); the conditions at the nozzle throat

Preconditions:
State(g.0) = constant; constant inlet conditions

Quantity Conditions:
\[ \frac{A_2}{A_1} \approx \hat{G} \]
\[ \frac{A_1}{A_t} \approx \hat{G} \]
\[ \frac{r_{p,e}}{r_{p,t}} < \frac{p_b}{p_0} < 1 \]

Relations:
IsentropicFlow(g.1.2); the isentropic flow process
\[ \frac{\dot{m}}{\dot{m}_c} \approx \hat{L} \]
\[ \frac{A^*/A_t} \approx \hat{L} \]
\[ M_e \approx \hat{L} \]
\[ M_t \approx \hat{L} \]
\[ \frac{p_e}{p_0} \approx [p_b/p_0] \]

Figure 13.21: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent nozzle with subsonic flow throughout the passage.
Encapsulated History CDSubsonicChoked(g.1.2,b)

Individuals:
- IdealGas(g) ; the ideal gas view
- State(g.1) ; the inlet state
- State(g.2) ; the outlet state
- State(g.e) ; the exhaust region conditions
- State(g.0) ≡ State(g.1) ; inlet is stagnation state
- State(g.e) ≡ State(g.2) ; outlet is exit state
- State(g.t) ; the conditions at the nozzle throat

Preconditions:
- State(g.0) = constant ; constant inlet conditions

Quantity Conditions:
\[
\begin{align*}
\left[ A_2/A_1 \right] & \approx \hat{G} \\
\left[ A_1/A_2 \right] & \approx \hat{G} \\
\left[ p_t/p_0 \right] & = [r_{P_0}]
\end{align*}
\]

Relations:
- IsentropicChokedFlow(g.1.2) ; the choked flow process
\[
\begin{align*}
\left[ M_2 \right] & \approx \hat{M} \\
\left[ p_t/p_0 \right] & \approx \left[ p_t/p_0 \right]
\end{align*}
\]

Figure 13.22: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent nozzle at the maximum mass flow rate but subsonic flow throughout the divergent section.
the second critical ratio the shock stands in the exit plane itself, so that no further compression is necessary. These situations requiring an internal normal shock adjustment are similar enough to combine into the same qualitative representation, which we define as the process CDNormalShock. In this case the isentropic flow relations do not apply since the shock itself is non-isentropic. However, the flow remains adiabatic, so the process history includes the general results of the AdiabaticFlow process. Furthermore, the results of the NormalShock and ChokedFlow processes are inherited to describe the loss of stagnation pressure, the subsonic Mach number at the exit plane (where a correspondence with state 2 has been established in the individuals slot), and the critical mass flow and throat area. With the quantity condition \( r_{p_2} \leq p_b/p_0 < r_{p_1} \), this encapsulated history is shown in Figure 13.23.

**CDOverExpansion** With operating pressures between the second and third critical pressure ratios, the flow is supersonic throughout the divergent section of the nozzle and requires an external compression adjustment to match the back pressure. This history definition admits the IsentropicFlow relations because the NormalShock process is not active. Furthermore, the exit plane relations reflect the requirement for external adjustment. The conditions for external adjustment are defined in the OverExpansion process (Figure 13.24). The CDOverExpansion encapsulated history that applies these conditions is shown in Figure 13.25.
Encapsulated History CDNormalShock(g.1.2.b)

Individuals:
- IdealGas(g) ; the ideal gas view
- State(g.1) ; the inlet state
- State(g.2) ; the outlet state
- State(g.b) ; the exhaust region conditions
- State(g.0) ≡ State(g.1) ; inlet is stagnation state
- State(g.e) ≡ State(g.2) ; outlet is exit state
- State(g.t) ; the conditions at the nozzle throat

Preconditions:
- State(g.0) = constant ; constant inlet conditions

Quantity Conditions:
- \[ \frac{A_2}{A_d} \approx \hat{G} \]
- \[ \frac{A_1}{A_d} \approx \hat{G} \]
- \[ r_{p.e2} \leq \left[ \frac{p_b}{p_0} \right] < \left[ r_{p.e1} \right] \]

Relations:
- ChokedFlow(g) ; choked flow conditions
- NormalShock(g,t,2) ; normal shock between throat and exit
- \[ \left[ \frac{p_b}{p_0} \right] \approx \left[ \frac{p_e}{p_0} \right] \]

Figure 13.23: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent nozzle with a normal shock in the divergent section.
; over expansion of duct flow
;
Process OverExpansion(g.e.b)

**Individuals:**
- AnyFlow(g.e.b) ; valid for any flow situation
- ChokedFlow(g) ; the maximum mass flow density is achieved
- State(g.b) ; exhaust region conditions
- State(g.e) ; exit plane conditions
- State(g.*); critical state conditions

**Preconditions:**
- $\emptyset$

**Quantity Conditions:**
- $[p_b/p^*] \approx \hat{L}$

**Relations:**
- $[p_e/p_b] \approx \hat{L}$

**Influences:**
- $\emptyset$

*Figure 13.24:* A QPT process specification for overexpansion of an interior flow into an exhaust region, requiring external compression adjustment.
convergent-divergent nozzle performance
with an external compression adjustment

Encapsulated History CDOverExpansion(g.1.2.b)

Individuals:

IdealGas(g) ; the ideal gas view
State(g.1) ; the inlet state
State(g.2) ; the outlet state
State(g.0) ≡ State(g.1) ; inlet is stagnation state
State(g.e) ≡ State(g.2) ; outlet is exit state
State(g.t) ; the conditions at the nozzle throat
State(g.b) ; the exhaust region conditions

Preconditions:
State(g.0) = constant ; constant inlet conditions

Quantity Conditions:

\[
\frac{A_2}{A_1} \approx \hat{G} \\
\frac{A_1}{A_t} \approx \hat{G} \\
[r_{p.2}] < [p_b/p_0] < [r_{p.2}]
\]

Relations:
IsentropicChokedFlow(g.1.e) ; choked flow conditions at the throat
OverExpansion(g.e.b) ; external compression adjustment
\[M_e \approx \hat{G} ; \text{ supersonic exit}\]

Figure 13.25: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent nozzle requiring an external compression adjustment.
Process PerfectExpansion(g.e.b)

Individuals:
   AnyFlow(g.e.b) ; valid for any flow situation
   ChokedFlow(g) ; the maximum mass flow density is achieved
   State(g.b) ; exhaust region conditions
   State(g.e) ; exit plane conditions
   State(g.* ) ; critical state conditions

Preconditions:
   ∅

Quantity Conditions:
   \[ \frac{p_b}{p^*} \approx \hat{L} \]

Relations:
   \[ \frac{p_e}{p_b} \approx \hat{1} \]

Influences:
   ∅

Figure 13.26: A QPT process specification for perfect expansion of an interior flow into an exhaust region.

CDPerfectExpansion When the operating pressure ratio is exactly equal to the third critical ratio \( r_{PC3} \), the flow is perfectly expanded with supersonic exhaust velocity, hence requiring no internal or external adjustment. The perfect expansion condition is suitable for implementation as a process itself, so that it might be deployed for other problems. Such a process definition is shown in Figure 13.26. The convergent-divergent nozzle encapsulated history incorporating these results is shown in Figure 13.27.

CDUnderExpansion Finally, when the operating pressure ratio is reduced below the third critical pressure ratio, the supersonic flow in the exit plane requires
; convergent-divergent nozzle performance
; with perfect supersonic expansion
;
Encapsulated History CDPerfectExpansion(g.1,2,b)

Individuals:
 IdealGas(g) ; the ideal gas view
 State(g.1) ; the inlet state
 State(g.2) ; the outlet state
 State(g.0) ≡ State(g.1) ; inlet is stagnation state
 State(g.e) ≡ State(g.2) ; outlet is exit state
 State(g.t) ; the conditions at the nozzle throat
 State(g.b) ; the exhaust region conditions

Preconditions:
 State(g.0) = constant ; constant inlet conditions

Quantity Conditions:
 [A2/A1] ≈ \dot{G}
 [A1/A4] ≈ \dot{G}
 [p_t/p_0] ≈ [r_{pe}] 

Relations:
 IsentropicChokedFlow(g.1,2) ; the choked flow process
 PerfectExpansion(g.e,b) ; ideal exhaust conditions
 [M_e] ≈ \dot{G} ; supersonic exit

Figure 13.27: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent nozzle with perfect supersonic expansion.
### Process UnderExpansion(g.e.b)

**Individuals:**
- `AnyFlow(g.e.b)`; valid for any flow situation
- `ChokedFlow(g)`; the maximum mass flow density is achieved
- `State(g.b)`; exhaust region conditions
- `State(g.e)`; exit plane conditions
- `State(g.*)`; critical state conditions

**Preconditions:**
- $\emptyset$

**Quantity Conditions:**
- $[p_b/p^*] \approx \bar{L}$

**Relations:**
- $[p_e/p_b] \approx \bar{C}$

**Influences:**
- $\emptyset$

**Figure 13.28:** A QPT process specification for insufficient expansion of an interior flow into an exhaust region.

Further expansion to match the imposed back pressure. This encapsulated history definition (Figure 13.29) is similar to the CDOverExpansion history except that the exit plane pressure relation reflects the under-expanded flow conditions. Similarly, it takes advantage of a generic UnderExpansion process that may apply to other problems (Figure 13.28).

### 13.3.2 Diffuser Operation

The performance problem for diffusers is similar to that of nozzles. It involves determining the flow characteristics given a fixed geometry and a set of operating conditions. Normally, we measure diffuser performance by the pressure increase
; convergent-divergent nozzle performance
; with an external expansion adjustment

Encapsulated History CDUnderExpansion(g.1.2.b)

Individuals:
IdealGas(g) ; the ideal gas view
State(g.1) ; the inlet state
State(g.2) ; the outlet state
State(g.0) ≡ State(g.1) ; inlet is stagnation state
State(g.e) ≡ State(g.2) ; outlet is exit state
State(g.t) ; the conditions at the nozzle throat
State(g.b) ; the exhaust region conditions

Preconditions:
State(g.0) = constant ; constant inlet conditions

Quantity Conditions:
\[ \frac{A_2}{A_1} \approx \hat{G} \]
\[ \frac{A_1}{A_4} \approx \hat{G} \]
\[ \frac{P_0}{P_0} < [\tau_{\text{Pco}}] \]

Relations:
IsentropicChokedFlow(g.1.2) ; the choked flow process
UnderExpansion(g.e,b) ; external expansion adjustment
\[ [M_e] \approx \hat{G} ; \text{supersonic exit} \]

Figure 13.29: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent nozzle with an external expansion adjustment.
achieved by the flow as it passes through the device. The inlet conditions, then, along with a required pressure increase, represent the operating conditions. This evaluation policy is somewhat different than the one applied to the evaluation of nozzle operation, since in that case the various degrees of pressure decrease were used to determine the flow characteristics, and the pressure could not have increased greater than the stagnation pressure. In the diffuser case, we assume that the inlet conditions pertain to a moving fluid representing a flow feed from a duct or the free-stream conditions as the diffuser moves through the fluid. Given the variety of geometries and inlet conditions available, one might choose to repeat the analysis approach used for nozzles, changing the inlet boundary conditions such that the flow through the device may indeed expand or compress, depending on the global behavior of the flow. Instead, let us pursue definition of only those qualitative process histories that perform the intended compression of a flow.

Requiring the pressure increase $p_2 > p_1$ is simply a matter of specifying the quantity condition $[p_2/p_1] \approx \hat{G}$. The operating pressure quantity space for $p_b/p_0$ is not important in this context. Furthermore, without incorporating order-of-magnitude reasoning, we cannot perform relative comparisons of pressure increases (as a measure of efficiency) under various inlet conditions and passage geometries. Therefore our approach will be simply to provide a way to recognize diffuser-like processes given only very general conditions. To do this, we require only the cross-sectional area change and the inlet Mach number to specify unique diffuser processes.
If the inlet Mach number is subsonic, \( M_1 < 1 \), then the diffuser requires a simple divergent geometry, \( A_2/A_1 > 1 \) to provide \( p_2/p_1 > 1 \). Since the Mach number downstream decreases, there will be no shock waves and isentropic flow may be assumed. In a less specific manner, the IsentropicCompression process already defined in Figure 13.5 satisfies these requirements. Adding the inlet condition \( |M_1| \approx \hat{L} \) specializes this process for application to diffuser performance problems, while encouraging "reverse" application to diffuser design problems. The SubsonicCompression process of Section 13.1.1 satisfies the more specific requirements, including geometry and inlet Mach number conditions.

If the inlet Mach number is supersonic, \( M_1 > 1 \), then the diffuser requires only a convergent passage \( A_2/A_1 < 1 \) to provide the pressure increase. The SupersonicCompression process from Figure 13.7 activates on these requirements and conditions, so long as the flow remains isentropic. If a normal shock occurs in the passage, however, the IsentropicFlow conditions are no longer satisfied and this process cannot be activated. Ideally, a convergent-divergent passage would be used to first decelerate the flow in the convergent section, then transition through the throat at Mach 1 to subsonic velocity and further compression in the divergent section. With this geometry we cannot state a direct relationship between \( A_1 \) and \( A_2 \), so long as both are greater than \( A_t \). Nevertheless, we may apply the compression condition \( p_2/p_1 > 1 \) and the inlet and outlet conditions \( M_1 > 1 \) and \( M_2 < 1 \). The optimum diffuser would require that no shock waves appear in the passage, however without
Encapsulated History CDSupersonicCompression(g.1.2)

**Individuals:**
- IdealGas(g) ; the ideal gas view
- State(g.1) ; the inlet state
- State(g.2) ; the outlet state
- State(g.*) ; the critical state
- State(g.t) ; the conditions at the nozzle throat

**Preconditions:**
- State(g.1) = constant ; constant inlet conditions

**Quantity Conditions:**
- $[A_2/A_4] \approx \hat{G}$
- $[A_1/A_4] \approx \hat{G}$
- $[A^*/A_4] \geq \hat{I}$
- $[p_2/p_1] \approx \hat{G}$
- $[M_1] \approx \hat{G}$ ; supersonic inlet

**Relations:**
- AdiabaticFlow(g.1.2) ; the adiabatic flow process
- $[M_2] \approx \hat{L}$ ; subsonic exit
- $[M_4] \approx \hat{I}$ ; sonic throat

Figure 13.30: A QPT encapsulated history specification for the flow of an ideal gas in a convergent-divergent diffuser with supersonic inlet conditions.

A measure of efficiency we obtain the same qualitative results when we permit a normal shock to perform the transition to subsonic flow in the divergent section. With this allowance we lose the isentropic flow characteristics and resort to adiabatic flow characteristics, but since the boundary pressures and velocity relationships are prespecified, there is no real loss of reasoning capability. Figure 13.30 shows an encapsulated history template CDSupersonicCompression that captures this mode of diffuser operation.
13.4 Constant-Area Duct Flow

In the above discussion of the nature of interior flows, we attributed physical changes in the fluid flow primarily to changes in the duct geometry. For example, a decreasing cross-sectional area was shown to act as either a nozzle or diffuser, depending upon additional characteristics of the flow. To make the problems easier to solve, many external or real-world effects were ignored in the ideal one-dimensional scenarios of interior flow.

If we restrict the passage geometry to a constant cross-sectional area, then we may reintroduce some of these real-world and external influences into the one-dimensional analysis of interior flows. The effect of these influences, like changes in passage geometry, is to accelerate or decelerate the flow within the limits of certain physical phenomena. Section 13.4.1 discusses the effects of wall friction, and identifies processes for constant-area duct flows which experience changes in the physical characteristics due simply to the presence of friction. Section 13.4.2 then considers the same sort of changes owing instead to the addition of heat into the flow stream.

A flow in an ideal constant-area duct that experiences no external influences will continue unchanged. Therefore, we cannot activate or deactivate qualitative processes in this situation, because doing so would imply that some change takes place due to that process. Both of the situations analyzed in the following sections introduce qualitative processes that are very similar to those identified for varying-
area duct flows.

13.4.1 Flow With Friction

Our initial assumption of a uniform velocity profile at any cross section of the flow precludes evaluation of flow losses due to frictional effects because we are unable to determine the viscous stresses at the passage boundaries. Rather than introducing more spatial dimensions to enable reasoning about these stresses, we remain within the realm of one-dimensional techniques and employ the Fanning friction factor $f$,

$$ f = \frac{\tau_0}{\frac{1}{2} \rho v^2}, $$

where $\tau_0$ is the shear stress at the duct wall. This friction factor normally is tabulated for various materials with approximate roughness $\epsilon$ against the flow Reynolds number $N_{RE}$ at the mean profile velocity.

Qualitatively, all we need to know in order to activate a process is that friction is present, i.e. $f \neq 0$; we do not need to know the magnitude of the frictional effect as we might in a quantitative analysis. Unlike the varying-area flow situations, discrimination of the various processes in constant-area flow situations requires knowledge (or assumption) of the passage length $l$. Using the Fanning friction factor, the pressure change due to friction over some length of duct is

$$ \left( \frac{dp}{dl} \right)_f = -f \frac{1}{R_h} \frac{1}{2} \rho v^2, $$

where $R_h$ is the hydraulic radius of the duct, $R_h = A/C$ and $C$ is the wetted perimeter. Obviously, this expression requires a pressure loss due to friction along
the duct, so that the qualitative statement is simply

\[
\begin{bmatrix}
p_2 \\
p_1
\end{bmatrix}_f \approx \hat{L},
\]

where we assume that \( dl \) is positive in the direction of flow. Accordingly, the wall force representation of the friction along the wall is

\[
\left( \frac{dF_w}{dl} \right)_f = -\frac{f}{R_h} \frac{1}{2} \rho v^2 A,
\]

and so the qualitative change in force functions is

\[
\begin{bmatrix}
F_2 \\
F_1
\end{bmatrix}_f \approx \hat{L}.
\]

Unfortunately, the total pressure change for a compressible flow is dependent upon friction and the momentum changes, so that we cannot fix \([p_2/p_1] \approx \hat{L}\) as we might for a liquid. Therefore, in the presence of friction, we continue to apply the constant-area momentum equation (Equation 12.8) as specified in the AnyFlow process without simplification. The energy equation also remains unchanged. We may, however, simplify the continuity equation by eliminating the constant \([A_2/A_1] = \hat{1}\), leaving

\[
\begin{bmatrix}
\rho_2 \\
\rho_1
\end{bmatrix} \oplus \begin{bmatrix}
v_2 \\
v_1
\end{bmatrix} \approx 2.
\]

With this single reduction of the general problem we are still faced with many difficult problems and uncertainties, both for qualitative and quantitative models. Moreover, since the flow is irreversible, the conveniences provided by the isentropic flow relations are not available for these problems. So, as before, we will pursue the solution of constant-area flow with friction under special cases.
A rather convenient qualitative representation of adiabatic flow with friction is the Fanno line (Figure 13.31). The Fanno line describes a curve of constant total energy and mass flow density in the $h - s$ plane for steady, adiabatic flows, generated by combining the adiabatic energy and continuity equations for constant area with some general equation of state for the gas. Since the specified flow is adiabatic, the Second Law of Thermodynamics requires that sequential flow states lie along this line and move only toward the right, i.e. with increasing entropy. As suggested by the figure, there exists a point on this curve at which the entropy no longer increases, therefore this point establishes a limit point for the flow under the given conditions. It transpires that this limit point occurs when $M = 1$. For subsonic flows, which have increasing enthalpy from the point of zero kinetic energy ($h_0$), the flow states must lie along the upper portion of the Fanno line, accelerating until at the limit point $M = 1$. On the contrary, supersonic flows experience decreasing enthalpy in successive flow states, decelerating toward the limit point at $M = 1$. This limit point, therefore, represents the choked flow condition for constant-area flows with friction.

Given the existence of a limit point for choked flow, we must determine whether the flow reaches this condition after travelling a certain distance through the passage. In order to evaluate the changes in physical parameters between two points in the flow, it is convenient to relate their values to those achieved at the choked flow condition. This reference state is the critical state, and the length of duct necessary
Figure 13.31: The Fanno line in the $h - s$ plane. Flows with increasing entropy tend toward $M = 1$. 
to accelerate or decelerate a flow to \( M = 1 \) is the critical duct length \( l^* \). For an ideal gas, this critical duct length may be determined as a function of the friction factor, hydraulic radius, and local Mach number from the non-dimensional equation

\[
\frac{f l^*}{R_h} = \frac{1}{k} \left\{ \frac{k + 1}{2} \ln \frac{k+1/2M^2}{1 + \frac{k-1/2M^2}{M^2}} + \frac{1}{M^2} - 1 \right\}.
\]

Any two points along the duct will have the critical state in common, so that the distance between the points \( l \) is the difference in the critical lengths \( l = l_1^* - l_2^* \), knowing from the Fanno line that all flows tend toward the critical state. Equation 13.9 then provides a way to calculate downstream conditions (or distance) at point 2 for given initial conditions at point 1,

\[
\frac{f l_2^*}{R_h} = \frac{f l_1^*}{R_h} - \frac{f l}{R_h}.
\]

For qualitative analysis, it is simple to state a relationship between the critical lengths at any two points as

\[
\begin{bmatrix} f l_1^* \\ R_h \end{bmatrix} > \begin{bmatrix} f l_2^* \\ R_h \end{bmatrix},
\]

although the \( Q_R \) quantity space is not meaningful for these ratios. Alternatively, we may state the dimensionless relationship as \( [l_2^*/l_1^*] \approx \hat{L} \), which is constant regardless of the flow conditions, so that the calculations involving the friction factor and Mach number are unnecessary. All that is necessary for frictional flow, in fact, is a set of qualitative process specifications describing the Fanno line. We separate the line into two regions according to the initial conditions, one each for supersonic flow and subsonic flow. The adiabatic flow process continues to hold, so that these new Fanno
line processes simply reduce the number of possibilities for a constant-area flow with friction. Although the critical state for these problems is not achieved isentropically, there is no confusion with the isentropic critical state we have already defined since the IsentropicFlow process that provides these relationships is not active.

Just as with the varying-area passages we will identify the various behaviors of flows in constant-area passages by using the operating pressure conditions. A fixed-geometry passage will be fed from a supply at known conditions and exhaust into a region of back pressure \(p_b\) that we reduce from \(p_0\) in order to control the flow.

First we shall consider flow that is not choked, so that \(0 < \dot{m} < \dot{m}_c\), or \(\dot{m}/\dot{m}_c \approx \hat{L}\). The back pressure is reduced from \(p_0\) but above \(p_*\), so that subsonic flow exists throughout the passage (line (b) in Figure 13.32). If \([M_1] \approx \hat{L}\), then \([M_2] \approx \hat{L}\) while \([M_1] \otimes [M_2] \approx \hat{L}\). The enthalpy decreases, so that \([h_2/h_1] \approx \hat{H}\), while the entropy increases \([s_2/s_1] \approx \hat{S}\). The energy equation for adiabatic flow provides \([T_2/T_1] \approx \hat{L}\). The continuity equation provides \([\rho_2/\rho_1] \approx \hat{L}\) since \([v_2/v_1] \approx \hat{G}\). The AnyFlow process then provides \([p_2/p_1] \approx \hat{L}\) and \([F_2/F_1] \approx \hat{L}\). Since the exit plane Mach number is subsonic, the exit pressure \(p_2\) must equal the back pressure \(p_b\). These results show that subsonic flow with friction is analogous to a subsonic isentropic expansion through a nozzle; due to the quantity conditions of the SubsonicExpansion process, however, we cannot reuse that process template for this application. Instead, we create a new process history FannoSubsonic that applies the appropriate quantity conditions and summarizes the interesting results (see Fig-
Figure 13.32: The general geometry of a performance problem for subsonic flow through a constant-area duct with friction. The upper panel shows the passage being fed from a reservoir (representing the stagnation state $p_{01}$ and $T_0$) and accelerated through an isentropic converging nozzle. The nozzle connects the reservoir with a duct of constant cross-sectional area $A$ with friction $f$. The constant-area passage exhausts into a region of free space maintained at variable back pressure $p_b$. The lower panel depicts the various operating pressure profiles along the length of the duct.
ure 13.33). If the back pressure is further reduced such that \( p_\theta = p^* \), the flow cannot be accelerated further and the resulting flow is choked at \( M_2 = 1 \) (line (c) in Figure 13.32). Further reductions in back pressure such that \( p_\theta < p^* \) remain choked with \( p_\theta = p^* \), and the flow undergoes an external expansion in the exhaust region (line (d)). The process history definitions FannoPerfectExpansion (Figure 13.34) and FannoUnderExpansion (Figure 13.35) control these behaviors.

To pursue the supersonic internal flow cases, we change the passage geometry such that the constant-area duct is supplied through a convergent-divergent nozzle. The nozzle continues to behave as before, so that a normal shock may occur in the divergent section, thereby providing subsonic flow to the duct at point 1. If there are no shocks in the nozzle, the flow velocity at point 1 may indeed be supersonic. According to the Fanno line, this flow will be decelerated by the presence of friction, to the limit point where \( M_2 = 1 \), so long as the actual duct length is less than the critical length, \( l < l^*_1 \). If the actual length is greater than the critical length, the flow will require some internal adjustment.

Without presenting elaborate detail, the discussion of the supersonic case parallels the subsonic case. As the back pressure \( p_\theta \) decreases from the stagnation pressure \( p_{01} \), subsonic flow exists throughout the duct. Even when a normal shock exists in the nozzle, the duct flow remains subsonic, as described in the process FannoSubsonic. As the back pressure decreases, the normal shock positions itself within the duct and the flow downstream of the shock accelerates but never reaches Mach
; constant-area subsonic flow with friction
;
Encapsulated History FannoSubsonic(g.1.2.b)

Individuals:
  IdealGas(g) ; the ideal gas view
  State(g.1) ; the inlet state
  State(g.2) ; the outlet state
  State(g.*) ; the critical state
  State(g.b) ; the exhaust region conditions
  State(g.t) ; throat conditions

 Preconditions:
  ∅

 Quantity Conditions:
  $[f] \neq 0$
  $[M_1] \approx \hat{L}$
  $[p_0/p^*] \approx \hat{G}$
  $[A_2/A_1] \approx \hat{1}$
  $[A_1/A_d] \approx \hat{1}$

 Relations:
  AdiabaticFlow(g.1.2) ; the adiabatic process
  $[M_2] \approx \hat{L}$ ; subsonic exit
  $[M_2/M_1] \approx \hat{G}$
  $[p_2/p_1] \approx \hat{L}$
  $[s_2/s_1] \approx \hat{G}$
  $[F_2/F_1] \approx \hat{L}$
  $[m/m_c] \approx \hat{L}$

Figure 13.33: A QPT encapsulated history specification for the adiabatic flow of an ideal gas in a constant-area duct with friction ($[f] \neq 0$).
Encapsulated History FannoPerfectExpansion(g.1.2.b)

Individuals:
- IdealGas(g) ; the ideal gas view
- State(g.1) ; the inlet state
- State(g.2) ; the outlet state
- State(g.b) ; the exhaust region conditions
- State(g.t) ; throat conditions

Preconditions:
- \( f \neq 0 \)
- \( A_2/A_1 \approx \hat{1} \)
- \( A_1/A_2 \approx \hat{1} \)

Quantity Conditions:
- AdiabaticFlow(g.1.2) ; the adiabatic process
- PerfectExpansion(g.2.b) ; no external adjustment
- \( M_2 \approx 1 \) ; sonic exit
- \( s_2/s_1 \approx \hat{G} \)

Figure 13.34: A QPT encapsulated history specification for the perfectly expanded flow of an ideal gas in a constant-area duct with friction (\( |f| \neq 0 \)). This process is valid for either subsonic or supersonic upstream flows.
Figure 13.35: A QPT encapsulated history specification for an underexpanded flow of an ideal gas in a constant-area duct with friction ([f] \neq 0). This history is valid for either subsonic or supersonic upstream flows, testing the initial Mach number and duct length to determine the appropriate exit plane Mach number.
1. The exit plane pressure is equal to the imposed back pressure. This situation is captured in the FannoSupersonic process history (Figure 13.36), which provides only one new relation not inherited through other processes. As the back pressure continues to drop, the shock moves out of the duct and the flow is supersonic at the exit plane, $M_2 > 1$. The FannoPerfectExpansion and FannoUnderExpansion histories cover these situations (regardless of the upstream Mach number), using conditional relationships among the process relations to cover both subsonic and supersonic cases. These histories also cover the supersonic case where $l > l_1^*$, wherein the exit plane Mach number is sonic. The one remaining case lacking a process definition is the supersonic flow for $l < l_1^*$ with overexpanded conditions at the exit plane. For this case we add the process history FannoOverExpansion (Figure 13.37) to complete the set of histories for flow with friction.

### 13.4.2 Flow With Heat Addition

The second interesting analysis of real-world one-dimensional flow problems involves the frictionless flow of a gas through a constant-area duct with heat addition. Like the analysis in Section 13.4.1 of flow with friction, we continue to employ the standard conservation laws in their one-dimensional forms while noticing a special extension of their application toward special problems. In Section 13.4.1, we uncovered special qualitative relationships applicable to ideal gas flows with friction; in this section we uncover similar special relationships for ideal gas flows with heat
; constant-area supersonic flow with friction

Encapsulated History FannoSupersonic(g.1.2.b)

Individuals:

IdealGas(g) ; the ideal gas view
State(g.1) ; the inlet state
State(g.2) ; the outlet state
State(g.b) ; the exhaust region conditions
State(g.t) ; throat conditions
State(g.*) ; the critical state

Preconditions:

∅

Quantity Conditions:

\[ f \neq 0 \]
\[ \frac{t}{t_1} \approx \hat{t} \]
\[ M_1 \approx \hat{G} \]
\[ M_2 \approx \hat{L} \]
\[ \frac{p_t}{p^*} \approx \hat{G} \]
\[ \frac{A_2}{A_1} \approx \hat{I} \]
\[ \frac{A_1}{A_1} \approx \hat{I} \]

Relations:

NormalShock(g.1.2) ; the normal shock process
ChokedFlow(g) ; the maximum mass flow density process
\[ \frac{p_t}{p_2} \approx \hat{I} \]

Figure 13.36: A QPT encapsulated history specification for supersonic flow of an ideal gas in a constant-area duct with friction (\[ f \neq 0 \]).
; constant-area choked flow with friction and overexpansion

Encapsulated History FannoOverExpansion(g.1.2.b)

Individuals:
  IdealGas(g) ; the ideal gas view
  State(g.1) ; the inlet state
  State(g.2) ; the outlet state
  State(g.b) ; the exhaust region conditions
  State(g.t) ; throat conditions
  State(g.*) ; the critical state

Preconditions:
  $\emptyset$

Quantity Conditions:
  $[f] \neq 0$
  $[A_2/A_1] \approx 1$
  $[A_1/A_t] \approx 1$
  $[M_1] \approx \hat{G}$

Relations:
  NormalShock(g.1.2) ; a normal shock occurs in duct
  OverExpansion(g.2.b) ; exhaust requires external adjustment
  $[M_2] \approx 1$ ; sonic exit
  $[s_2/s_1] \approx \hat{G}$

Figure 13.37: A QPT encapsulated history specification for an over-expanded flow of an ideal gas in a constant-area duct with friction ($[f] \neq 0$). This history is valid for supersonic upstream flows.
addition.

For this special extension to one-dimensional flows, we assume that heat of a known amount is added to a gas travelling in a constant-area duct under a reversible process (i.e. without friction). The heat added $Q$ increases the stagnation enthalpy, $Q = h_{02} - h_{01}$, or $[h_{02}/h_{01}] \approx \hat{G}$. With this heat addition we eliminate the possibility of using the AdiabaticFlow process to simplify the general equations of AnyFlow. Nevertheless, the few assumptions made for this problem permits certain simplifications of the general equations. The continuity equation for constant-area flow reduces to

$$\rho v = \text{constant}$$

as in the previous section. The momentum equation for constant-area and frictionless flow reduces to

$$p + \rho v^2 = \text{constant}.$$

Assuming some relationship between the thermodynamic properties of the gas, it transpires that all sequential state points for this special flow lie along a single curve in the $h - s$ plane. This curve is known as the Rayleigh line, and it is in many respects similar to the Fanno line. As a subsonic flow is heated it accelerates, increasing in entropy until the limit point $M = 1$ at which $ds = 0$. As a supersonic flow is heated it decelerates, while increasing entropy until reaching the same limit point at $M = 1$. If heat is removed instead of added, the opposite velocity changes occur, while remaining on the same Rayleigh line contour.
One significant qualitative difference between the Fanno and Rayleigh lines is the double-valued enthalpy for subsonic flows. That is, rather than a experiencing strictly-decreasing enthalpy (starting from the stagnation enthalpy $h_0$) as a Fanno flow increases entropy, a Rayleigh flow will experience both an enthalpy increase and decrease as entropy increases. Considering the correspondence between enthalpy and temperature for an ideal gas $h = c_p T$, this situation is shown in Figure 13.38 in a $T - s$ diagram. From this diagram the physical meaning of the upper (subsonic) contour is clear: as the gas is heated it will increase in temperature and accelerate until some maximum temperature, at which point the gas will decrease in temperature while continuing to accelerate. Obviously, the flow cannot round the $M = 1$ corner without decreasing entropy, therefore there exists a choked-flow condition for excess heat addition.

Avoiding elaborate derivation, the Rayleigh line for an ideal gas is stated as the relationship

$$\frac{dT}{ds} = \frac{T}{c_p} \frac{kM^2 - 1}{M^2 - 1}.$$ 

From this equation the point at which $dT/ds = 0$ is easily found to occur when $M = 1/\sqrt{k}$. The point at which $dT/ds = \infty$ occurs when $M = 1$. Solving the conservation equations for these special conditions and using the ideal gas state equations, we obtain the following relationships in terms of the Mach numbers:

$$\frac{p_1}{p_2} = \frac{1 + kM_2^2}{1 + kM_1^2},$$
Figure 13.38: The Rayleigh line in the $T-s$ plane. Subsonic flows of an ideal gas experience maximum temperature at $M = 1/\sqrt{k}$. Both supersonic and subsonic frictionless flows with increasing entropy tend toward $M = 1$ when heat is added.
\[
\frac{T_1}{T_2} = \frac{M_1^2(1 + kM_2^2)^2}{M_2^2(1 + kM_1^2)^2},
\]
\[
\frac{p_{01}}{p_{02}} = \frac{1 + kM_2^2}{1 + kM_1^2} \left[ \frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2} \right]^{k/(k-1)},
\]
\[
\frac{T_{01}}{T_{02}} = \frac{M_1^2(1 + kM_2^2)^2}{M_2^2(1 + kM_1^2)^2} \left( \frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2} \right).
\]

If given the quantity of heat addition \(Q\) one may determine \(T_{02}/T_{01}\), and since we assume \(M_1\) is known we may solve the fourth equation for \(M_2\), which we then use to solve the first three equations. To provide a reference state for comparison purposes, we again use the critical state.\(^4\)

Applying the qualitative results as process views, we may define several processes that are similar to those defined for isentropic nozzle flow and adiabatic flow with friction. For subsonic flows with heat addition, \([M_1] \approx \hat{L}\) and \([M_2] \approx \hat{L}\) although we know \([M_1] \otimes [M_2] \approx \hat{L}\). Since heat is added we know that \([T_{02}/T_{01}] \approx \hat{G}\). If \([M_{1,2}] < 1/\sqrt{k}\), we know that \([T_2/T_1] \approx \hat{G}\), while if \([M_{1,2}] > 1/\sqrt{k}\) we know that \([T_2/T_1] \approx \hat{L}\). Furthermore, we know \([p/p^\ast] \approx \hat{G}\) and \([p_0/p_0^\ast] \approx \hat{G}\), so that \([p_2/p_1] \approx \hat{L}\) and \([p_{02}/p_{01}] \approx \hat{L}\). There is a maximum stagnation temperature change \((T_{02}/T_{01})_{max} = T_{01}^\ast/T_{02}\) that can be accomodated by the heat addition. If the heat addition per unit mass exceeds this limit, there is an internal reduction of the mass flow rate such that the flow never exceeds this maximum limit. The process history RayleighSubsonic (in Figure 13.39) applies these conditions and relations in the familiar situation where flow is fed from a reservoir and exhausts into a region of

\(^4\)Since this critical state is not attained isentropically, it is different than the critical state achieved through isentropic flow in Section 12.4.1 The \(^\ast\) notation refers only to the state in which \(M = 1\).
back pressure $p_b$, where $p_b > p^*$. Carrying the analogy to Fanno flow further, there are several slightly different cases for supersonic flow with heat addition, wherein the flow is decelerated and choked, allowing for the possibility of normal shock waves in the duct and various external adjustments to the imposed back pressure. These histories are RayleighSupersonic (Figure 13.40), RayleighOverExpansion (Figure 13.41), RayleighPerfectExpansion (Figure 13.42), and RayleighUnderExpansion (Figure 13.43).

13.5 Interior Flow Summary

This section collects the individual and process views defined to this point in the discussion of interior flows. It summarizes each of the individual and process dependencies in order to better examine the hierarchy of conceptual objects. Figure 13.44 depicts this hierarchy.

The following three subsections each provide a list of views: one for individual views, one for elementary process views, and one for special process views. These lists attempt to categorize the various components of the qualitative model for gas dynamics into application areas.

13.5.1 Individual View Summary

Gas The basic definition of the form of matter applicable to this domain. This view, inherited by all of the processes defined below, allocates a number of
Encapsulated History RayleighSubsonic(g.1.2.b)

Individuals:

IdealGas(g); the ideal gas view
State(g.1); the inlet state
State(g.2); the outlet state
State(g.b); the exhaust region conditions
State(g.t); throat conditions
State(g.*); the critical state

Preconditions:

∅

Quantity Conditions:

\[ Q \approx \hat{+} \]
\[ M_1 \approx \hat{L} \]
\[ \frac{p_b}{p^*} \approx \hat{G} \]
\[ \frac{A_2}{A_1} \approx \hat{I} \]
\[ \frac{A_1}{A_t} \approx \hat{I} \]

Relations:

AnyFlow(g.1.2); the general flow process
ReversibleFlow(g.1.2); heat is added reversibly
\[ M_2 \approx \hat{L}; \text{subsonic exit} \]
\[ \frac{M_2}{M_1} \approx \hat{G} \]
\[ \frac{p_2}{p_1} \approx \hat{L} \]
\[ \frac{p_2}{p_b} \approx \hat{I} \]
\[ \frac{p_{c2}}{p_{c1}} \approx \hat{L} \]
\[ \frac{s_2}{s_1} \approx \hat{G} \]
\[ \frac{\dot{m}}{\dot{m}_c} \approx \hat{L} \text{ if } M_2 < 1/\sqrt{k} \text{ then } [T_2/T_1] \approx \hat{G} \]
if \( M_1 > 1/\sqrt{k} \) then \([T_2/T_1] \approx \hat{L} \]

Figure 13.39: A QPT encapsulated history specification for the frictionless flow of an ideal gas in a constant-area duct with heat addition.
; constant-area supersonic flow with heat addition

Encapsulated History RayleighSupersonic(g.1.2.b)

Individuals:
- IdealGas(g); the ideal gas view
- State(g.1); the inlet state
- State(g.2); the outlet state
- State(g.b); the exhaust region conditions
- State(g.t); throat conditions
- State(g.*); the critical state

Preconditions:
- \(0\)

Quantity Conditions:
- \(|Q| \approx \hat{+}\)
- \(|M_1| \approx \hat{G}\)
- \(|M_2| \approx \hat{L}\)
- \(|p_0/p^*| \approx \hat{G}\)
- \(|A_2/A_1| \approx \hat{l}\)
- \(|A_1/A_2| \approx \hat{l}\)

Relations:
- NormalShock(g.1.2); the normal shock process
- ChokedFlow(g); the maximum mass flow density process
- \(|p_b/p_s| \approx \hat{1}\)
- \(|T_2/T_1| \approx \hat{L}\)

Figure 13.40: A QPT encapsulated history specification for supersonic flow of an ideal gas in a constant-area duct with heat addition \(|Q| \approx \hat{+}\).
Encapsulated History RayleighOverExpansion(g.1.2.b)

Individuals:
- IdealGas(g) ; the ideal gas view
- State(g.1) ; the inlet state
- State(g.2) ; the outlet state
- State(g.b) ; the exhaust region conditions
- State(g.t) ; throat conditions

Preconditions:
\(\emptyset\)

Quantity Conditions:
- \(Q \approx \hat{T}\)
- \(A_2/A_1 \approx \hat{1}\)
- \(A_1/A_2 \approx \hat{1}\)
- \(M_1 \approx \hat{C}\)

Relations:
- NormalShock(g.1.2) ; a normal shock occurs in duct
- OverExpansion(g.2.b) ; exhaust requires external adjustment
- \(M_2 \approx 1\) ; sonic exit
- \(s_2/s_1 \approx \hat{G}\)
- \(T_2/T_1 \approx \hat{L}\)
- \(T_{02}/T_{01} \approx \hat{G}\)

Figure 13.41: A QPT encapsulated history specification for an over-expanded flow of an ideal gas in a constant-area duct with heat addition. This history is valid for supersonic upstream flows.
Encapsulated History RayleighPerfectExpansion(g.1.2.b)

Individuals:
IdealGas(g) ; the ideal gas view
State(g.1) ; the inlet state
State(g.2) ; the outlet state
State(g.b) ; the exhaust region conditions
State(g.t) ; throat conditions

Preconditions:
\( \emptyset \)

Quantity Conditions:
\( |Q| \approx \hat{v} \)
\( \left[ A_2/A_1 \right] \approx \hat{i} \)
\( \left[ A_1/A_t \right] \approx \hat{i} \)

Relations:
PerfectExpansion(g.2.b) ; exhaust flow requires no adjustment
\( M_2 \approx \hat{1} ; \) sonic exit
\( s_2/s_1 \approx \hat{G} \)
\( T_{02}/T_{01} \approx \hat{G} \)

Figure 13.42: A QPT encapsulated history specification for the perfectly expanded flow of an ideal gas in a constant-area duct with heat addition. This history is valid for either subsonic or supersonic upstream flows.
Encapsulated History RayleighUnderExpansion(g.1.2.b)

Individuals:
- IdealGas(g); the ideal gas view
- State(g.1); the inlet state
- State(g.2); the outlet state
- State(g.b); the exhaust region conditions
- State(g.t); throat conditions
- State(g.*); the critical state

Preconditions:
∅

Quantity Conditions:
- \(|Q| \approx \dagger\)
- \(|A_2/A_1| \approx \hat{1}\)
- \(|A_1/A_4| \approx \hat{1}\)

Relations:
- UnderExpansion(g.2.b); exhaust requires external adjustment
- If \([M_1] \approx \hat{G}\) then NormalShock(g.1.2) if \([M_1] \approx \hat{L}\) then \([M_2] \approx \hat{1}\)
- If \([M_1] \approx \hat{G} \wedge [T_{02}/T_{01} > T_{01}^*/T_{01}]\) then \([M_2] \approx \hat{G}\)
- If \([M_1] \approx \hat{G} \wedge [T_{02}/T_{01} < T_{01}^*/T_{01}]\) then \([M_2] \approx \hat{1}\)
- \([s_2/s_1] \approx \hat{G}\)

Figure 13.43: A QPT encapsulated history specification for an under-expanded flow of an ideal gas in a constant-area duct with heat addition. This history is valid for either subsonic or supersonic upstream flows, testing the initial Mach number and maximum heat addition to determine the appropriate exit plane Mach number.
Figure 13.44: The individual and elementary process view object hierarchy for interior flows. Relation inheritance proceeds along the arrow paths.
the thermodynamic properties of this substance for inheritance during instantiation.

IdealGas A special instance of the Gas view that defines the characteristics of a thermodynamically ideal gas, including existence of stagnation and critical states for that gas.

State This view enumerates certain quantities of a Gas necessary to define a flow state. A model may instantiate the State view in order to identify important quantities for a dynamic, critical, or stagnation state.

13.5.2 Elementary Process Views

The following list summarizes each of the elementary processes for one-dimensional gas flows. One may apply these processes individually or collectively to solve many significant gas dynamics problems. The list is in alphabetical order by process view name.

AdiabaticFlow One-dimensional adiabatic flow of an ideal gas through a duct. This process view incorporates fundamental elements of the conservation laws simplified for adiabatic flow cases. The AnyFlow relations are inherited.

AnyFlow The general case of one-dimensional flow of an ideal gas through a duct. This process view incorporates the most fundamental elements of the conservation laws applicable to all flows. The IdealGas and State relations are
inherited.

**ChokedFlow** Specification of a one-dimensional flow of an ideal gas that achieves the maximum mass flow rate for the given operating conditions. The *IdealGas* relations are inherited.

**IsentropicChokedFlow** Specification of a one-dimensional flow of an ideal gas that achieves the maximum mass flow rate isentropically for the given operating conditions. The *IsentropicFlow* and *ChokedFlow* relations are inherited.

**IsentropicCompression** Isentropic compression of an ideal gas. The *IsentropicFlow* relations are inherited.

**IsentropicExpansion** Isentropic expansion of an ideal gas. The *IsentropicFlow* relations are inherited.

**IsentropicFlow** One-dimensional isentropic flow of an ideal gas through a duct. This process view incorporates fundamental elements of the conservation laws simplified for isentropic flow cases. The *AdiabaticFlow* and *ReversibleFlow* relations are inherited.

**NormalShock** A normal shock wave process for an ideal gas. The *AdiabaticFlow* relations are inherited. The *ReversibleFlow* process is prohibited.

**OverExpansion** A specification of the one-dimensional duct flow exhaust conditions requiring an external compression adjustment. The *AnyFlow*, *Choked-
Flow and State relations are inherited.

**PerfectExpansion** A specification of the one-dimensional duct flow exhaust conditions requiring no external adjustment. The AnyFlow, ChokedFlow and State relations are inherited.

**ReversibleFlow** An elementary process activated solely to identify flow conditions that are reversible (*e.g.* without friction). The AnyFlow relations are inherited, while activation of the irreversible NormalShock process deactivates the ReversibleFlow process.

**SubsonicCompression** The general case of a one-dimensional subsonic isentropic flow of an ideal gas through a divergent passage. The IsentropicFlow relations are inherited.

**SubsonicExpansion** The general case of a one-dimensional subsonic isentropic flow of an ideal gas through a convergent passage. The IsentropicFlow relations are inherited.

**SupersonicCompression** The general case of a one-dimensional supersonic isentropic flow of an ideal gas through a convergent passage. The IsentropicFlow relations are inherited.

**SupersonicExpansion** The general case of a one-dimensional supersonic isentropic flow of an ideal gas through a divergent passage.
Table 13.1: A cross-reference matrix showing the requisite elemental processes for each of the encapsulated histories. The history from the column headers is defined in the interior flow summary.

UnderExpansion A specification of the one-dimensional duct flow exhaust conditions requiring an external expansion adjustment. The AnyFlow, ChokedFlow and State relations are inherited.

13.5.3 Encapsulated History Views

Several of the interior flow process views apply only to particular situations, but application of these processes conveniently facilitates modeling of these special problems. These special process histories, which in some cases represent the total behavior of a physical device, consolidate many of the elementary process views into one structure. This consolidation eliminates some of the ambiguity that may accrue when applying the elementary processes individually. The list entries are numbered according to the process reference matrix in Table 13.1.

1. CDSubsonic Subsonic isentropic flow of an ideal gas through a convergent-divergent nozzle, with less than the maximum mass flow rate. The IsentropicFlow and State relations are inherited.
2. **CDSubsonicChoked** Subsonic isentropic flow of an ideal gas through a convergent-divergent nozzle with choked flow. The IsentropicChokedFlow and State relations are inherited.


4. **CDPerfectExpansion** Supersonic flow of an ideal gas through a convergent-divergent nozzle with choked flow and no external pressure adjustment. The IsentropicChokedFlow, PerfectExpansion, and State relations are inherited.

5. **CDOverExpansion** Supersonic flow of an ideal gas through a convergent-divergent nozzle with choked flow and an external compression adjustment. The IsentropicChokedFlow, OverExpansion, and State relations are inherited.

6. **CDUnderExpansion** Supersonic flow of an ideal gas through a convergent-divergent nozzle with choked flow and an external expansion adjustment. The IsentropicChokedFlow, UnderExpansion, and State relations are inherited.

7. **CDSupersonicCompression** Supersonic flow of an ideal gas through a convergent-divergent diffuser. The AdiabaticFlow and State relations are inherited.
8. **ConvergentSubsonic** One-dimensional subsonic isentropic flow through a convergent passage, where the throat area is larger than the critical area for the given operating conditions. The IsentropicFlow and State relations are inherited.

9. **ConvergentSonic** One-dimensional subsonic isentropic flow through a convergent passage, where choked flow is achieved. The IsentropicChokedFlow and State relations are inherited.

10. **FannoSubsonic** Subsonic flow of an ideal gas through a constant-area duct with friction. The AdiabaticFlow and State relations are inherited.

11. **FannoSupersonic** Supersonic flow of an ideal gas through a constant-area duct with friction, prior to establishing choked flow. The NormalShock and State relations are inherited.

12. **FannoPerfectExpansion** Flow of an ideal gas through a constant-area duct with friction. Choked flow is established and the exhaust flow requires no external adjustment. The AdiabaticFlow, PerfectExpansion, and State relations are inherited.


15. RayleighSubsonic Subsonic flow through a constant-area duct with heat addition. The AnyFlow, ReversibleFlow, and State relations are inherited.


17. RayleighPerfectExpansion Supersonic flow of an ideal gas through a constant-area duct with heat addition. Choked flow is established and the exhaust flow requires no external adjustment. The PerfectExpansion and State relations are inherited.

18. RayleighUnderExpansion Supersonic flow of an ideal gas through a constant-area duct with heat addition. Choked flow is established and the exhaust flow requires an external expansion adjustment. The UnderExpansion, NormalShock, and State relations are inherited.

19. RayleighOverExpansion Supersonic flow of an ideal gas through a constant-area duct with heat addition. Choked flow is established and the exhaust flow requires an external compression adjustment. The OverExpansion, NormalShock, and State relations are inherited.
Chapter 14

Exterior Flows

The qualitative analysis of the dynamic behavior of gases in what we have called "interior" flows was simplified by the one-dimensional approach generally taken in the quantitative analysis of these same flows. This simplification, along with the concept of an ideal gas, enables solution of the governing equations for many kinds of engineering problems, and indirectly introduces a qualitative perspective that eliminates some of the problems faced in the endeavor to automate the process of qualitative reasoning. This unique perspective allowed us to solve one-dimensional problems without much regard to the concepts of geometry or kinematics. The interior flows ignored, for the most part, the exact size and shape of the containing walls, the relative lengths of combined sections, the relative motions of the gas within those walls, and the interaction of the gas with other matter.

In solving problems in the class of "exterior" (or multi-dimensional) flows, this unique non-geometric and non-kinematic perspective is not available, and therefore we can no longer disregard these important concepts. Unfortunately, as we saw in Chapter 7, developing techniques for qualitative reasoning about these particular concepts is a very difficult (yet heavily investigated) task. Until a strong qualita-
tive implementation of these concepts is available, it will be extremely difficult to implement qualitative models of multi-dimensional gas dynamics.

Nevertheless, the Qualitative Process Theory notions of *individuals* and *processes* retain their usefulness in describing multi-dimensional gas dynamics. This Chapter briefly considers some of the distinct concepts of these complex flows, tangible or otherwise, and proposes that a QPT representation of these concepts is attainable. Section 14.1 restates the fundamental equations of fluid flow from Chapter 12, this time presenting the governing equations for certain classes of multi-dimensional flows. Rather than translating these equations into qualitative relations, as was done in Chapter 13, Section 14.2 simply identifies many of the important concepts that must be accommodated in a qualitative representation. The complex problem of coordinating a multi-discipline approach toward realizing the computational constructs for these concepts is left for future work.

### 14.1 Fundamental Equations

Most of the important concepts of multi-dimensional gas flows can be identified by considering steady, two-dimensional, adiabatic flows. This Section presents the forms of the governing equations for these flows without detailed derivation; proper derivations may be found in [12].

These two-dimensional forms make use of a few convenient definitions. A *streamline* \( l \) in a fluid field is a line that is everywhere tangent to the velocity vector \( \vec{v} \); if
the position vector is \( \vec{r} \), then the equation for a streamline is

\[
d\vec{r} \times \vec{v} = 0.
\]

If a quantity \( q \) is constant along a streamline, then

\[
\vec{v} \cdot \nabla q = 0.
\]

The fluid field \textit{circulation} is the line integral of the velocity vector over a closed curve \( C \),

\[
\Gamma = \oint_C \vec{v} \cdot d\vec{r} = \oint_C d\Gamma = \int_S \vec{N} \cdot (\nabla \times \vec{v}) \, dA,
\]

where \( S \) represents the surface of \( C \) with area \( A \). In terms of the \textit{vorticity} \( \zeta \) of the fluid velocity field, the circulation is given as

\[
\Gamma = \oint_C \vec{v} \cdot d\vec{r} = \int_S \vec{N} \cdot \zeta \, dA.
\]

Considering the rotational velocity \( \omega \) of the fluid at a point \( P \), the vorticity of point \( P \) is

\[
\zeta_P = \nabla \times \vec{v}_P = 2\vec{\omega}_P.
\]

Returning the general vector forms of the governing equations from Chapter 12, we can restate the three conservation laws and the Second Law of Thermodynamics to describe the steady, two-dimensional, adiabatic flow:

**Conservation of Mass** The continuity equation,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]
reduces to the vector equation

$$\nabla \cdot (\rho \vec{v}) = 0$$

for steady flow of a compressible fluid. In two dimensions, this becomes

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} = 0.$$

**Conservation of Energy** The energy equation for unsteady flow involving heat and work is

$$\dot{Q} = \frac{\partial}{\partial t} \int_Y (u + \frac{v^2}{2}) \rho \, d\mathcal{V} + \int_S (u + \frac{v^2}{2}) \rho \vec{v} \cdot \vec{N} \, dA + \int_S p \vec{N} \cdot \vec{v} \, dA + \dot{W}.$$  

For steady adiabatic flow without work addition, this reduces to

$$\nabla \cdot \left[ \left( u + \frac{v^2}{2} + \frac{p}{\rho} \right) \rho \vec{v} \right] = 0,$$

and with the previous result for conservation of mass, along with the definition of enthalpy $h = u + p/\rho$, the vector equation is

$$\vec{v} \cdot \nabla h_0 = 0.$$  

With $v^2 = v_x^2 + v_y^2$, this equation becomes

$$h_0 = h + \frac{v^2}{2},$$

which may be constant either throughout the flow field or along a streamline. If $h_0$ is constant throughout the flow field, the flow is said to be *homogeneous*; if $h_0$ is constant along a streamline, the flow is said to be *adiabatic*. 
Conservation of Momentum For inviscid flow without body forces the conservation of linear momentum is given by

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} + \frac{1}{\rho} \nabla p = 0. \]

For steady flow this reduces to the Euler equation

\[ -\frac{1}{\rho} \nabla p = (\vec{v} \cdot \nabla) \vec{v}, \]

which may be written as

\[ \frac{-1}{\rho} \frac{\partial \rho}{\partial x} = v_z \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} \]
\[ \frac{-1}{\rho} \frac{\partial \rho}{\partial y} = v_z \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \]

For an incompressible rotational flow the quantity

\[ p + \frac{\rho v_z^2}{2} \]

is constant along a streamline; for an incompressible irrotational flow the same quantity is constant throughout the flow field.

Second Law of Thermodynamics For unsteady flow with heat transfer, the Second Law of Thermodynamics is

\[ \int_S \frac{\vec{q} \cdot \vec{N}}{T} dA \leq \frac{\partial}{\partial t} \int_V \rho V dV + \int_S \rho \vec{v} \cdot \vec{N} dA. \]

Reduced for steady flow without heat transfer this equation becomes

\[ \int_S \rho \vec{v} \cdot \vec{N} dA \geq 0, \]
or restated as
\[ \int_V \nabla \cdot s \rho \vec{v} dV \geq 0. \]

Similar to the energy equation, this becomes
\[ \vec{v} \cdot \nabla s = 0, \]
providing the results that either \( s \) may be constant throughout the flow field or \( s \) may be constant along a streamline. If \( s \) is constant throughout the flow field, the flow is said to be *homentropic*; if \( s \) is constant along a streamline, the flow is *isentropic*. Alternatively, combining these thermodynamic relations with the Euler equation, one may use the Crocco equation
\[ \zeta \times \vec{v} = T \nabla s - \nabla h_0 \]

to correlate the fluid rotation with its velocity and thermodynamic properties.

These non-linear governing equations defy exact analytic solution except for special cases. More commonly, one obtains reasonable solutions to the more complex cases by using approximation techniques, particularly numerical methods such as the *method of characteristics* or *finite-element analysis*.

Obviously, proper accounting of the geometric concepts of *streamline* and *flow field* must be made before even the simplest versions of the governing equations can be translated into qualitative form. This may also introduce a solution to the problem of representing a vector in our qualitative mathematics; perhaps by decomposing the vector into multiple scalar quantities, or perhaps by providing another
geometric notion of spatial direction. Whereas in the one-dimensional applications the “direction” was “downstream,” the complex concepts of field and rotation invalidate this simplification, allowing fluid particles to travel in one than one “direction” simultaneously. So along with the qualitative definitions of “field” and “direction,” there must also be a corresponding kinematic representation of geometric shapes, such as surfaces and volumes, and a scheme for describing the motion of these shapes through space. Once this foundation has been established, we can build a qualitative theory of exterior gas flows upon it.

14.2 Qualitative Concepts

Many of the important qualitative descriptions of multi-dimensional gas flows will arise from work in qualitative models of space and kinematics. The special cases for which analytic solutions of the governing equations are available rely heavily on concepts from rigid-body kinematics and from the geometries of polar and cylindrical coordinate systems. Although the principles of qualitative physics do not, in general, require an intrinsic coordinate system, we require some frame of reference in order to describe motion and dynamics.

Even without a mechanical or heuristic translation of the governing equations into qualitative relations, we may still identify some of the concepts that will likely be included in qualitative models for engineering applications.¹ Furthermore, rather

¹Most of these do not seem to qualify as “common sense” concepts, though one may argue that
than disassembling the vector quantities into their spatial components and writing
thousands of qualitative proportionality expressions to describe physical indirect
influences, we choose not to depart from the earlier speculations about elementary
interior flows being describable entirely by direct relations. Maintaining this spirit,
while lacking the proper qualitative foundations for geometric reasoning, we simply
identify by name some of the individual views, process views, and encapsulated
histories that seem to be logical continuations of the corresponding interior flow
components.

14.2.1 Individual Views

Unlike an interior flow, an exterior flow is not necessarily confined to a container.
The Hayes contained-liquid concepts (described in Section 7.2.2) might now play
a bigger role in describing the material of interest. For instance, a gas might be
a ContainedGas or FreeGas, StationaryGas or MobileGas, ExcitedGas or RestingGas.
The space occupied by the gas can be a ZoneOfSilence or a ZoneOfAction, perhaps
subdivided by GasFields. The limits of the fields might be a SolidBoundary, a
LiquidBoundary, or a JetBoundary, or contain a Vortex. The fields in turn may also
possess qualities such as HomentropicField or HomenergicField or RotatingField, and
the field may possess a Streamline of constant properties or quantity. The exterior
gas may possess new quantities such as Vorticity, AngularVelocity, and Circulation.

the astute (mechanical) observer of nature will require some means to describe anything it sees.
14.2.2 Process Views

Introducing multiple dimensions into the models requires complex process views to describe the dynamic situation. Now we have the possibilities of ObliqueShock and ConicalShock to complement the existing NormalShock process. We must introduce the processes of ExpansionTurn and CompressionTurn to describe flow around corners, as well as MachWave, WaveReflection and WaveCancellation to describe interactions with gases and other bodies. Along with the turning processes we introduce new landmark values for the geometric quantity spaces such as MaximumTurnAngle and MinimumDeflectionAngle. These concepts can be further abstracted to the notions of Lift and Drag, along with Turbulence, Combustion, and Mixing.

14.2.3 Encapsulated Histories

The most exciting application potential for these multi-dimensional individuals and processes is in their combination into encapsulated histories. In elementary cases these might be applied to AcuteTurningFlow, ObtuseTurningFlow, PrandtlMeyerFlow, or even ShockDiamond. In more general cases these might be applied to RadialFlowFan, AxialFlowFan or Propeller, and then combined with Combustion and CDNozzle processes to create a history abstraction of GasTurbine performance. The WaveReflection and WaveCancellation processes may be combined into SupersonicDiffuser and SupersonicNozzle histories, and then along with Combustion and ShockDiamond into complex RocketEngine and RamJet abstractions. The Lift, Drag, and
TurningFlow processes and histories may combine into AirfoilFlow and WingFlow histories. Ultimately, these lead to the combination of WingFlow and GasTurbine or Propeller into an extremely complex AircraftFlow history that qualitatively describes the gas flows over an entire aircraft; an impressive accomplishment indeed.
Chapter 15

Gas Dynamics Envisionments

This Chapter presents some examples that demonstrate the process and capabilities of qualitative reasoning, and of the gas dynamics theory in particular. Section 15.2 exercises most of the elements of the theory in several example problems. These problems address a variety of qualitative reasoning issues, exercise many of the theory elements, and present the results in varying ways. Before pursuing these problems, however, Section 15.1 briefly describes some of the knowledge representation and reasoning assumptions required to establish a starting point.

15.1 Model Construction

There are a wide variety of ways to set up models of gas dynamics problems. The activity of defining a model that maximizes its simulation potential is an arduous task, but it is an important contributor to the success of computational reasoning systems. Automating this process of model creation has received considerable attention in the qualitative reasoning community [21,87,139]. In this Section we review some of the considerations for constructing and model, and specifically address many of the assumptions made for the gas dynamics examples to follow.
The discretization of the problem space is one important consideration. Rather than solving for a continuous behavior, we assume that we can solve the problem by approximation, isolating and consolidating behaviors occurring between select points. In quantitative models, the identification of states along a path or points in time generally coincides with what is already known about about that point, such as observed measurements or initial conditions. For performance problems we might also select points that represent the “terminals” of that various device models that we have available for solving a multi-device problem: this is straightforward only if one knows what the devices are. For design problems we try to match our available device models to observed or desired flow characteristics. For qualitative models we perform the same sort of selections, striving to identify the unique states that conform to the requirements necessary to employ the special processes or encapsulated histories of the available qualitative theories. In the examples of Section 15.2 the identification of important states is already complete.

Identifying the quantity spaces for the physical parameters is another modeling issue. For describing change and for landmark comparison purposes, the qualitative theory uses ratios with values in the space \( Q_R \). Using the Mach number to define flow conditions almost totally defines the qualitative values of the other physical parameters, such that there is not usually a need to perform limit analysis for these parameters in their quantity spaces. Nevertheless, most of the parameters have similar quantity spaces delimited by landmark values corresponding to their values
in the *stagnation* and *critical* states, such as

\[
[T] \in \{(0, T^*), T^*, (T^*, T_0), T_0\},
\]

\[
[\rho] \in \{(0, \rho^*), \rho^*, (\rho^*, \rho_0), \rho_0\},
\]

\[
[l] \in \{0, (0, l^*), l^*, (l^*, +\infty)\}
\]

where the \([\cdot]\) operator refers here to a mapping from the real number space to the appropriate quantity space for that parameter (not to \(Q_R\)). Within the individual and process views of the qualitative theory used in the examples below, however, all of the parameter values refer to qualitative values in \(Q_R\), creating ratios from the landmark values for that parameter if necessary before performing the mapping.

For example,

\[
[T_0/T] \in Q_R,
\]

\[
[\rho^*/\rho] \in Q_R,
\]

\[
[M] \in Q_R.
\]

These landmark values also provide common reference states for establishing convenient correspondences within the models, using the QPT Correspondence feature.

For example,

\[
\text{Correspondence}([M] \approx \hat{1}. [T] \approx [T^*])
\]

\[
\text{Correspondence}([M] \approx \hat{1}. [l] \approx [l^*])
\]

\[
\text{Correspondence}([M] \approx \hat{0}. [T] \approx [T_0])
\]

\[
\text{Correspondence}(s, s_0)
\]
To simplify the presentation of the examples we make a few assumptions about an automated reasoning system that might perform these analyses:

- We assume that the reasoning system is able to carry out the composition of ratios, and that it is able to perform constraint satisfaction using the RRI algebra.

- We assume that some mechanism is available for instantiating and traversing the discretized topology of the domain, restricting state pair matches to connected and adjacent states.

- We assume that the direction of fluid travel is downstream, and that the aggregate behavior of the fluid is consistent throughout the model. This should permit localized behaviors such as eddy currents, dead regions, and turbulence.

- We assume that a complete library of modeling components includes individual and process views for both design and performance problems.

- We assume that the reasoning system is capable of attaching suitable explanations to special process conditions such as ChokedFlow.

To conserve space, we list only the interesting relations made available by the composite individual and process structures; trivial or unexciting results are not pre-

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1Technically, the components of the theory should themselves enforce this requirement. The initial assumption about all quantities being positive implicitly defines a reference that prohibits negative conditions such as a reverse flow. A velocity check in the view preconditions might serve to enforce this standard.
sented. Unless otherwise specified, we assume also that there is no work done on
the fluid \((W_2/W_1) \approx \hat{1}\), and that the gas involved behaves like an ideal gas. When
given the ideal gas condition, we assume that the requirements for the \text{IdealGas}
individual view are satisfied and that the preconditions and quantity conditions
hold throughout the problem; \textit{e.g.}, the gas does not disappear unless the model
specifically terminates its existence.

15.2 Example Problems

Example 6 Consider the flow of an ideal gas through the one-dimensional passage
with an abrupt change in cross-sectional area shown in Figure 15.1. Both portions of
the duct, where \(A_2/A_1 = 1\) and \(A_3/A_2 = 1\) (and \(A_2\) is double-valued), are frictionless
and without heat transfer. Describe the total environment for this situation, and
compare the results with a continuously-varying cross-sectional area approximation
for the passage, where \([A_3/A_1] \approx \hat{G}\).

To generate the total environment we will assume all of the values for \([M_1] \in Q_R\)
and propagate possible solutions downstream to stations 2 and 3, without prespec-
ifying the pressures or temperatures at the inlet or outlet. The passage areas are
fixed at \([A_2/A_1] \approx \hat{1}\) and \([A_3/A_2] \approx \hat{1}\). Since the gas \(g\) is ideal, we know the \text{IdealGas}(g)
and \text{Gas}(g) views are part of the individual structure at each point. We
also use the \text{State} individual view to define flow states at each of the three points:
\text{State}(1.g), \text{State}(2.g) and \text{State}(3.g). First, we will investigate each case of \(M_1\)
Figure 15.1: The abrupt change in cross-sectional area diagram for example 6. The passage is assumed to be one-dimensional with no requirements placed on intake or exhaust conditions.

individually:

1. \([M_1] \approx 0\). Using the individuals \(\text{IdealGas(g)}\), \(\text{State(1.g)}\) and \(\text{State(2.g)}\), we activate the AnyFlow process. This process imposes no additional quantity
conditions, while providing the relations

\[
\begin{bmatrix}
\frac{m_2}{m_1}
\end{bmatrix} = \hat{i}
\]

\[
\begin{bmatrix}
p_2
\end{bmatrix}_{p_1} + \begin{bmatrix}
a_2
\end{bmatrix}_{A_1} + \begin{bmatrix}
v_2
\end{bmatrix}_{v_1} \approx 3 \quad \Rightarrow \quad \begin{bmatrix}
p_{2/rh0_1}
\end{bmatrix} + \begin{bmatrix}
v_2
\end{bmatrix}_{v_1} \approx 2
\]

\[
\begin{bmatrix}
v_2
\end{bmatrix}_{v_1} + \begin{bmatrix}
a_2
\end{bmatrix}_{h_1} + \begin{bmatrix}
w_2
\end{bmatrix}_{Q_1} + \begin{bmatrix}
w_2
\end{bmatrix}_{W_1} \approx 2 \quad \Rightarrow \quad \begin{bmatrix}
v_2
\end{bmatrix}_{v_1} + \begin{bmatrix}
a_2
\end{bmatrix}_{h_1} \approx 2
\]

\[
\begin{bmatrix}
a_2
\end{bmatrix}_{A_1} + \begin{bmatrix}
p_2
\end{bmatrix}_{p_1} + \begin{bmatrix}
p_2
\end{bmatrix}_{p_1} + 2 \otimes \begin{bmatrix}
v_2
\end{bmatrix}_{v_1} + \begin{bmatrix}
p_2
\end{bmatrix}_{F_1} \approx 4 \quad \Rightarrow \quad \begin{bmatrix}
p_2
\end{bmatrix}_{p_1} + \begin{bmatrix}
p_2
\end{bmatrix}_{p_1} + 2 \otimes \begin{bmatrix}
v_2
\end{bmatrix}_{v_1} \approx 4
\]

\[
\begin{bmatrix}
p_2
\end{bmatrix}_{p_1} \approx \begin{bmatrix}
p_2
\end{bmatrix}_{p_1} + \begin{bmatrix}
p_2
\end{bmatrix}_{F_1} \otimes \hat{i}
\]

\[
\begin{bmatrix}
a_2
\end{bmatrix}_{h_1} \approx \begin{bmatrix}
p_2
\end{bmatrix}_{F_1}
\]

\[
\begin{bmatrix}
p_2
\end{bmatrix}_{c_1} \approx \begin{bmatrix}
p_2
\end{bmatrix}_{F_1}
\]

if \([M_1] < [M_2]\) then \[\begin{bmatrix}
p_2/T_{02}
\end{bmatrix}_{T_2/T_{02}} \approx \hat{L}\]

if \([M_1] = [M_2]\) then \[\begin{bmatrix}
p_2/T_{02}
\end{bmatrix}_{T_2/T_{02}} \approx \hat{i}\]

if \([M_1] < [M_2]\) then \[\begin{bmatrix}
p_2/p_{02}
\end{bmatrix}_{p_2/p_{02}} \approx \hat{L}\]

if \([M_1] = [M_2]\) then \[\begin{bmatrix}
p_2/p_{02}
\end{bmatrix}_{p_2/p_{02}} \approx \hat{i}\]

where \([W_2/W_1] \approx \hat{i}\), \([Q_2/Q_1] \approx \hat{i}\) and \([F_2/F_1] \approx \hat{i}\). The conditional cases for \([M_1] > [M_2]\) have been eliminated since \([M_1] \approx \hat{0}\) and therefore cannot be greater than \([M_2]\). This reduces the possible solutions to those for

\[
\begin{bmatrix}
T_2/T_{02}
\end{bmatrix}_{T_2/T_{02}} \in \{\hat{L}, \hat{i}\}
\]

and

\[
\begin{bmatrix}
p_2/p_{02}
\end{bmatrix}_{p_2/p_{02}} \in \{\hat{L}, \hat{i}\}.
\]
Since \( v_1 = 0 \), \( T_1 \approx T_{01} \) and \( p_1 \approx p_{01} \), and we require

\[
\begin{bmatrix}
T_2 \\
T_{02}
\end{bmatrix} \in \{ \hat{L}, \hat{i} \}
\]

and

\[
\begin{bmatrix}
p_2 \\
p_{02}
\end{bmatrix} \in \{ \hat{L}, \hat{i} \},
\]

which is consistent with the IdealGas relations. For \( |M_1| \approx \hat{0} \), these relations provide

\[
\begin{bmatrix}
T_{01} \\
T_1
\end{bmatrix} \geq \hat{i}
\]
\[
\begin{bmatrix}
p_{01} \\
p_1
\end{bmatrix} \geq \hat{i}
\]
\[
\begin{bmatrix}
T_1^* \\
T_1
\end{bmatrix} \approx \hat{L}
\]
\[
\begin{bmatrix}
p_1^* \\
p_1
\end{bmatrix} \approx \hat{L}
\]

Since the flow experiences no heat transfer it is adiabatic, so that \( |Q_2/Q_1| \approx \hat{i} \) and we can activate the AdiabaticFlow process so long as \( |s_2/s_1| \in \{ \hat{i}, \hat{G} \} \). This process adds

\[
\begin{bmatrix}
h_2 \\
h_1
\end{bmatrix} \oplus \begin{bmatrix}
v_2 \\
v_1
\end{bmatrix} \approx 2
\]
\[
\begin{bmatrix}
\rho_2 \\
\rho_1
\end{bmatrix} \oplus \begin{bmatrix}
v_2 \\
v_1
\end{bmatrix} \approx 2
\]
\[
\begin{bmatrix}
T_2^* \\
T_1^*
\end{bmatrix} \approx \hat{i}
\]
\[
\begin{bmatrix}
T_{02} \\
T_{01}
\end{bmatrix} \approx \hat{i}
\]

if \( |M_1| < |M_2| \) then \( \begin{bmatrix}
T_2^* \\
T_1^*
\end{bmatrix} \approx \hat{L} \)

if \( |M_1| = |M_2| \) then \( \begin{bmatrix}
T_2^* \\
T_1^*
\end{bmatrix} \approx \hat{i} \)
to the list of available relations. If $s_2/s_1 \approx \hat{G}$, the second law of thermodynamics requires

$$
\hat{G} \approx \left[ \frac{T_2}{T_1} \right] \otimes \left[ \frac{p_2}{p_1} \right] \oplus \hat{1}.
$$

From the AdiabaticFlow relations and the inequality constraint on $[M_2]$, we reduce the number of possible solutions to the second law to two: if $[T_2/T_1] \approx \hat{L}$ then $[p_2/p_1] \approx \hat{L}$, or if $[T_2/T_1] \approx \hat{1}$ then $[p_2/p_1] \approx \hat{1}$. Both of these solutions are consistent with the ideal gas relationships $[T_{01}/T_1] \geq \hat{1}$ and $[T_{02}/T_2] \geq \hat{1}$, along with the adiabatic flow relation for stagnation temperature change $[T_{02}/T_{01}] \approx \hat{1}$ since the combination equation provides an ambiguous result if $[T_0/T] \approx \hat{G}$,

$$
\left[ \frac{T_2}{T_1} \right] \approx \left[ \frac{T_{02}}{T_{01}} \right] \otimes \left[ \frac{T_{01}}{T_1} \right] \otimes \left[ \frac{T_{02}}{T_2} \right],
$$

$$
\left[ \frac{T_2}{T_1} \right] \approx \hat{1} \otimes \hat{G} \otimes \hat{G},
$$

$$
\left[ \frac{T_2}{T_1} \right] \approx \hat{?}
$$

which includes $\hat{L}$, and if $[T_0/T] \approx \hat{1}$ the equation provides the expected $[T_2/T_1] \approx \hat{1}$. However, pursuing the case where $[T_2/T_1] \approx \hat{1}$, we have

$$
\left[ \frac{T_2}{T_1} \right] \approx \hat{1} \Rightarrow \left[ \frac{h_2}{h_1} \right] \approx \hat{1} \Rightarrow \left[ \frac{v_2}{v_1} \right] \Rightarrow \left[ \frac{\rho_2}{\rho_1} \right] \approx \hat{1} \Rightarrow \left[ \frac{p_2}{p_1} \right] \approx \hat{1},
$$

which is inconsistent with the second law requirement that $[p_2/p_1] \approx \hat{L}$. Therefore, for an increasing entropy between 1 and 2, the only adiabatic solution is $M_2 > M_1$, or $[M_2] \in \{ \hat{L}, \hat{1}, \hat{G} \}$. 
Because the flow is frictionless, \( [f] = 0 \), the ReversibleFlow process is active, though it provides no additional problem-solving relations. It does, however, allow us to activate the IsentropicFlow process for the \( [s_2/s_1] \approx \hat{1} \) case. This additional process provides the convenient relations

\[
[p_2/p_1] \approx [T_2/T_1]
\]

if \( [M_1] < [M_2] \) then \( [p_2/p_1] \approx \hat{L} \)

if \( [M_1] = [M_2] \) then \( [p_2/p_1] \approx \hat{i} \)

\[
[rho_2/rho_1] \approx [T_2/T_1]
\]

\[
[p_{02}/p_{01}] \approx \hat{i}
\]

\[
[p_2^*/p_1^*] \approx \hat{i}
\]

\[
[rho_2^*/rho_1^*] \approx \hat{i}
\]

which again restrict the values for pressure change \( [p_2/p_1] \in \{\hat{L}, \hat{i}\} \). If we first choose the \( [p_2/p_1] \approx \hat{L} \) case, we have

\[
\left[ \frac{p_2}{p_1} \right] \approx \hat{L} \Rightarrow \left[ \frac{rho_2}{rho_1} \right] \approx \hat{L} \Rightarrow \left[ \frac{T_2}{T_1} \right] \approx \hat{L},
\]

which is consistent with the earlier requirements, providing the same result as the non-isentropic case, \( [M_2] \in \{\hat{L}, \hat{i}, \hat{G}\} \). However, either of the values \( [M_2] \approx \hat{i} \) or \( [M_2] \approx \hat{G} \) cause the ChokedFlow and IsentropicChokedFlow processes to activate in addition to the processes already activated. These processes restrict the mass flow rate \( \dot{m} \) to the mass flow rate for choked flow \( \dot{m}_c \), but provide no other relations to sort out the possibilities.
Similarly, for the $[p_2/p_1] \approx \hat{1}$ case, we find $[T_2/T_1] \approx \hat{1}$ and $[M_2] \approx [M_1]$. Even with the additional ideal gas relations
\[
\begin{align*}
\frac{T_2}{T_2} &= [M_2] \\
\frac{P_2}{P_2} &= [M_2]
\end{align*}
\]
and the isentropic relation $[p_2^* / p_1^*] \approx \hat{1}$, the ambiguity of the qualitative multiplication and division operators do not provide constraint-checking assistance, so we assume that all three values for $[M_2]$ are possible.

Since no more processes activate under the above conditions, we close the world and the final process structure for the latter case becomes

$$PS = \{ \text{AnyFlow, ReversibleFlow, AdiabaticFlow,}$$
$$\text{IsentropicFlow, ChokedFlow,}$$
$$\text{IsentropicChokedFlow} \}.$$  

Summarizing the results for $[M_1] \approx \hat{1}$, we have $[M_2] \approx \hat{1}$ if the "flow" is isentropic (a trivial result), or $[M_2] \in \{ \hat{L}, \hat{1}, \hat{G} \}$. Table 15.1 provides the list of possible states.$^2$

2. $[M_1] \approx \hat{L}$. This case is identical to the case $[M_1] \approx \hat{1}$ except that it permits the conditional $[M_2] < [M_1]$, which in turn permits $[T_2/T_1] \approx \hat{G}$ for the adiabatic

---

$^2$The $[M] \approx 0$ cases arise because we have not defined a "flow" requirement such as $\dot{m} > 0$ in the AnyFlow process. Two interesting issues arise when one permits $\dot{m} = 0$: (1) the ratio $\dot{m}_2/\dot{m}_1$ is undefined, suggesting that the deduction system should declare that no "flow" solution is possible with the available views, and (2) allowing $\dot{m} = 0$ requires that one or more of the quantity spaces for $A$, $\rho$ and $\nu$ contain the element 0.
\[ [M_1] \approx \hat{0} \]

<table>
<thead>
<tr>
<th>$M_2$</th>
<th>$T_2$</th>
<th>$T_1$</th>
<th>$P_2$</th>
<th>$P_1$</th>
<th>$u_2$</th>
<th>$u_1$</th>
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<tbody>
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</table>

Table 15.1: Example 6 solutions for $[M_1] \approx \hat{0}$.

case and $[p_2/p_1] \approx \hat{G}$ for the isentropic case. For either case we have the solution $[M_2] \in \{\hat{0}, \hat{L}\}$, which is consistent with constraints in both process structures. Therefore, we have the list of possible states shown in Table 15.2.

3. $[M_1] \approx \hat{i}$. The results for this case are the same as the results for the $[M_1] \approx \hat{L}$ case except for the special case where $[M_1] \approx [M_2]$, in which all flow parameters remain steady. The isentropic portion of this case immediately activates the ChokedFlow and lsentropicChokedFlow processes because of the condition $\hat{m}_1/\hat{m}_{1c} = 1$ set by the lsentropicFlow relations for $[M_1] \geq \hat{i}$. These results are summarized in Table 15.3.

4. $[M_1] \approx \hat{G}$. Finally, the case for $M_1 > 1$ provides an interesting process structure. The AnyFlow, AdiabaticFlow and ChokedFlow processes remain active, per the problem specification, but now the condition $[M_1] \approx \hat{G}$ satisfies the
\[ [M_1] \approx \hat{L} \]

<table>
<thead>
<tr>
<th>[M_2]</th>
<th>[e_2] [e_1]</th>
<th>[c_2] [c_1]</th>
<th>[t_2] [t_1]</th>
<th>[\varepsilon_2] [\varepsilon_1]</th>
<th>[u_2] [u_1]</th>
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<td>[\hat{G}]</td>
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<td>[\hat{G}]</td>
<td>[\hat{L}]</td>
</tr>
</tbody>
</table>

Table 15.2: Example 6 solutions for \([M_1] \approx \hat{L}\).

\[ [M_1] \approx i \]

<table>
<thead>
<tr>
<th>[M_2]</th>
<th>[e_2] [e_1]</th>
<th>[c_2] [c_1]</th>
<th>[t_2] [t_1]</th>
<th>[\varepsilon_2] [\varepsilon_1]</th>
<th>[u_2] [u_1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\hat{G}]</td>
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<td>[\hat{i}]</td>
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</tbody>
</table>

Table 15.3: Example 6 solutions for \([M_1] \approx \hat{i}\).
\[ [M_1] \approx \hat{G} \]

<table>
<thead>
<tr>
<th>$M_2$</th>
<th>$\frac{p_2}{p_1}$</th>
<th>$\frac{T_2}{T_1}$</th>
<th>$\frac{\rho_2}{\rho_1}$</th>
<th>$\frac{s_2}{s_1}$</th>
<th>$L$</th>
<th>$\hat{G}$</th>
<th>$\hat{G}$</th>
<th>$\hat{G}$</th>
<th>$\hat{G}$</th>
<th>$\hat{L}$</th>
</tr>
</thead>
</table>

Table 15.4: Example 6 solutions for $[M_1] \approx \hat{G}$.

quantity condition of the NormalShock process. This process clearly specifies the downstream conditions,

\[ [M_2] \approx \hat{L} \]
\[ \frac{p_2}{p_1} \approx \hat{G} \]
\[ \frac{T_2}{T_1} \approx \hat{G} \]
\[ \frac{\rho_2}{\rho_1} \approx \hat{G} \]
\[ \frac{s_2}{s_1} \approx \hat{G} \]

removing all ambiguity in the qualitative relations; there is only one solution to the flow characteristics at state 2. The NormalShock process deactivates the ReversibleFlow and IsentropicFlow processes because of the irreversible shock and the increase in entropy. The process structure for this case is

\[ PS = \{ \text{AnyFlow, AdiabaticFlow, ChokedFlow, NormalShock} \} \]

This result is summarized in Table 15.4.

Now, given the predicted values at state 2 for each of these $[M_1]$ cases, we must similarly propagate $M_2$ downstream to state 3, determining $M_3$ and the possible
exit plane conditions. In fact, since the first section of the duct (from 1 to 2) did not eliminate any of the possible values of \([M_2]\), the solution process for the second section of duct (from 2 to 3) is exactly the same as for the first. Again, we determine the various combinations of processes that lead to the results \([M_3] \in Q_R\), albeit along different process combination paths from the original \([M_1]\). In each case, the individual structure is the same: \(IS = \{\text{Gas}(g), \text{IdealGas}(g), \text{State}(1.g), \text{State}(2.g), \text{State}(3.g)\}\), while the active processes change. If one were to construct the tree of Mach number possibilities for the total envisionment, the first layer (for \([M_1]\)) would have four branches, the second layer (for \([M_2]\)) would have thirteen branches (four each for \([M_1] \in \{\hat{0}, \hat{L}, \hat{r}\}\) and one for \([M_1] \approx \hat{G}\)), and the third layer would have 43 branches (thirteen for each case of \([M_1] \in \{\hat{0}, \hat{L}, \hat{r}\}\) and four for \([M_1] \approx \hat{G}\)). Instead of drawing this tree, Table 15.5 summarizes the process structures and possible Mach number combinations for either portion of the duct.

If we ignore the middle station along the duct and consider the gross behavior of the gas between 1 and 3, which in fact may happen automatically if the deduction mechanism is able to select the \(\text{State}(1.g)\) and \(\text{State}(3.g)\) (i.e. there is no requirement that the states be adjacent), we may treat the duct as a divergent section with \(A_3 > A_1\). Now, using the same total envisionment process as before with \([M_1] \in \{\hat{0}, \hat{L}, \hat{r}, \hat{G}\}\), we determine the results from this new perspective. Just as before, the \(\text{Gas}(g), \text{IdealGas}(g), \text{State}(1.g), \text{State}(3.g)\) individual views are valid, along with the \(\text{AnyFlow}(g.1.3)\) and \(\text{AdiabaticFlow}(g.1.3)\) process views, but in this
Table 15.5: Example 6 process structure summary. This table holds for either portion of the duct, where $(i,j) \rightarrow (1,2)$ or $(2,3)$. The "." notation indicates that the corresponding process is active under the imposed conditions.
case the constant-area simplifications of the continuity and momentum equations cannot be made, so for example

\[
\begin{bmatrix}
\rho_3 \\
\rho_1
\end{bmatrix} \oplus \hat{G} \oplus \begin{bmatrix}
v_3 \\
v_1
\end{bmatrix} \approx 3.
\]

Pursuing each of the initial values individually, we may compare these results with the earlier ones.

1. \([M_1] \approx \hat{0}\). If we first assume that there is an entropy increase, admitted by the AdiabaticFlow process, then we have the requirement \([T_3/T_1] \in \{\hat{1}, \hat{L}\}\) as before. If we let \([T_3/T_1] \approx \hat{1}\), then

\[
\begin{bmatrix}
T_3 \\
T_1
\end{bmatrix} \approx \hat{1} \Rightarrow \begin{bmatrix} h_3 \\
h_1 \end{bmatrix} \approx \hat{1} \Rightarrow \begin{bmatrix} v_3 \\
v_1 \end{bmatrix} \approx \hat{1} \Rightarrow \begin{bmatrix} \rho_3 \\
\rho_1 \end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} p_3 \\
p_1 \end{bmatrix} \approx \hat{G}
\]

which is not consistent with the second law of thermodynamics. If we let \([T_3/T_1] \approx \hat{L}\), then

\[
\begin{bmatrix}
T_3 \\
T_1
\end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} h_3 \\
h_1 \end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} v_3 \\
v_1 \end{bmatrix} \approx \hat{G} \Rightarrow \begin{bmatrix} \rho_3 \\
\rho_1 \end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} p_3 \\
p_1 \end{bmatrix} \approx \hat{?},
\]

where the only value of \([p_3/p_1]\) satisfying the second law is \(\hat{L}\). This is the case of \(M_3 > M_1\), as before, where \([M_3] \in \{\hat{L}, \hat{1}, \hat{G}\}\).

If we instead assume that there is no entropy increase, then the IsentropicFlow relations become available, along with the constraint \([p_3/p_1] \in \{\hat{L}, \hat{1}\}\) since \([M_3] \neq [M_1]\). If we let \([p_3/p_1] \approx \hat{L}\), then

\[
\begin{bmatrix}
p_3 \\
p_1
\end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} T_3 \\
T_1
\end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} \rho_3 \\
\rho_1 \end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} h_3 \\
h_1 \end{bmatrix} \approx \hat{L} \Rightarrow \begin{bmatrix} v_3 \\
v_1 \end{bmatrix} \approx \hat{G}
\]
\[ [M_1] \approx \hat{0}, [A_3/A_1] \approx \hat{G} \]

<table>
<thead>
<tr>
<th>$\hat{M}_3$</th>
<th>$e_3$</th>
<th>$e_1$</th>
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<td>$\hat{G}$</td>
<td>$\hat{L}$</td>
</tr>
<tr>
<td>$\hat{L}$</td>
<td>$\hat{i}$</td>
<td>$\hat{G}$</td>
<td>$\hat{G}$</td>
<td>$\hat{G}$</td>
<td>$\hat{L}$</td>
</tr>
</tbody>
</table>

Table 15.6: Example 6 solutions for $[M_1] \approx \hat{0}$.

, satisfying all of the constraints so that again $[M_3] \in \{\hat{L}, \hat{i}, \hat{G}\}$. If we let $[p_3/p_1] \approx \hat{i}$, then

\[
\begin{align*}
\begin{bmatrix} p_3 \\ p_1 \end{bmatrix} & \approx \hat{i} \Rightarrow \begin{bmatrix} T_3 \\ T_1 \end{bmatrix} \approx \hat{i} \Rightarrow \begin{bmatrix} h_3 \\ h_1 \end{bmatrix} \approx \hat{i} \Rightarrow \begin{bmatrix} u_3 \\ u_1 \end{bmatrix} \approx \hat{i} \Rightarrow \begin{bmatrix} \rho_3 \\ \rho_1 \end{bmatrix} \approx \hat{L},
\end{align*}
\]

which is inconsistent with the isentropic flow relations. Combining the two cases we see that $[M_3] > [M_1]$, and therefore $[M_3] \neq \hat{0}$. This eliminates one of the solutions for each portion of the duct in the previous model, and four possibilities overall. Table 15.6 summarizes the results.

2. $[M_1] \approx \hat{L}$. For this situation we have a special heuristic process to eliminate some of the possibilities encountered earlier. The IdealGas(g), IsentropicFlow(g,1,3) and $[A_3/A_1] \approx \hat{G}$ constraints satisfy the quantity conditions for the SubsonicCompression(g,1,3) process, providing the specific results

\[
\begin{bmatrix} p_3 \\ p_1 \end{bmatrix} \approx \hat{G},
\]
\[ [M_1] \approx \hat{L}, [A_3/A_1] \approx \hat{G} \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
[M_3] & e_3 & e_1 & T_3 & T_1 & v_3 \\
\hline
\hat{G} & \hat{G} & \hat{L} & \hat{L} & \hat{L} & \hat{G} \\
\hat{L} & \hat{G} & \hat{L} & \hat{L} & \hat{G} & \hat{L} \\
\hline
\end{array}
\]

Table 15.7: Example 6 solutions for \([M_1] \approx \hat{L}\).

\[
\begin{bmatrix} v_3 \\ v_1 \end{bmatrix} \approx \hat{L},
\begin{bmatrix} \rho_3 \\ \rho_1 \end{bmatrix} \approx \hat{G},
\]

which enable us to calculate

\[
\begin{bmatrix} T_3 \\ T_1 \end{bmatrix} \approx \hat{G},
\]

\([M_3] < [M_1],\)

so that \([M_3] \in \{\hat{0}, \hat{L}\}\). These results reduce the number of possible isentropic solutions encountered in the previous model. However, the adiabatic results with an entropy increase remain. For comparison, Table 15.7 summarizes the new results.

3. \([M_1] \approx \hat{I}\). Unfortunately, this case does not have a special process that simplifies the isentropic solution. The adiabatic solutions remain the same as the previous model. The isentropic solutions are same except that we eliminate
\[ [M_1] \approx \hat{1}, [A_3/A_1] \approx \hat{G} \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
G & G & \hat{L} & \hat{L} & \hat{L} & \hat{G} \\
\hline
\hat{G} & \hat{1} & \hat{L} & \hat{L} & \hat{L} & \hat{G} \\
\hline
\hat{L} & \hat{G} & \hat{G} & \hat{G} & \hat{G} & \hat{L} \\
\hline
\hat{L} & \hat{1} & \hat{G} & \hat{G} & \hat{G} & \hat{L} \\
\hline
\hat{0} & \hat{G} & \hat{G} & \hat{G} & \hat{G} & \hat{L} \\
\hline
\end{array}
\]

Table 15.8: Example 6 solutions for \([M_1] \approx \hat{1}\).

the case \([M_3] \approx \hat{1}\) because the increasing area and constant density and velocity conditions do not satisfy the continuity equation. Table 15.8 repeats the results, eliminating the inconsistent case.

4. \([M_1] \approx \hat{G}\). For this situation we again have a special heuristic process to eliminate some of the possibilities. TheIdealGas(g), IsentropicFlow(g.1.3) and \([A_3/A_1] \approx \hat{G}\) constraints satisfy the quantity conditions for the SupersonicExpansion(g.1.3) process, providing the specific results

\[
\begin{align*}
[p_3] & \approx \hat{L}, \\
[p_1] & \\
[v_3] & \approx \hat{G}, \\
[v_1] & \\
[\rho_3] & \approx \hat{L}, \\
[\rho_1] & \\
\end{align*}
\]

which enable us to calculate

\[
\begin{align*}
[T_3] & \approx \hat{L}, \\
[T_1] & \\
\end{align*}
\]
\[ [M_1] \approx \hat{G}, [A_3/A_1] \approx \hat{G} \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\hat{M}_3 & \frac{v_3}{v_1} & \frac{\rho_3}{\rho_1} & \frac{T_3}{T_1} & \frac{p_3}{p_1} & \frac{u_3}{u_1} \\
\hline
\hat{L} & \hat{G} & \hat{G} & \hat{G} & \hat{L} \\
\hat{G} & \hat{1} & \hat{L} & \hat{L} & \hat{G} \\
\hline
\end{array}
\]

Table 15.9: Example 6 solutions for \([M_1] \approx \hat{G}\).

\[ [M_3] > [M_1], \]

so that \([M_3] \approx \hat{G}\). The ChokedFlow and IsentropicChokedFlow processes are also activated. In this case we have *added* an isentropic solution where there was not one before. The previous model eliminated the isentropic possibilities with an entropy-increasing shock wave. This adiabatic solution involving the NormalShock process remains valid for this model. Table 15.9 provides the new results.

Summarizing this example, we see that the decisions regarding design of a model and the assumptions used as the initial conditions may lead to diverse results. Obviously, the number of spurious results decreases with increasing assumptions; but the knowledge of what modeling tools are available (such as the SupersonicExpansion process) may also affect the model design decisions. The choice to model this abrupt-area-change passage as a continuous-area-change divergent passage obviously affected the quality of the results. Since the length of the passage does not enter into either model, this could in fact be a very reasonable assumption. We also
Figure 15.2: The passage diagram for example 7, a three-section constant-area duct experiencing friction in one section and heat addition in another. The length of the second and third sections are independently variable across $l^*$. 

see that while the first model provided 43 possible paths with a seemingly more accurate specification (and representation) of two stages, the second model provides better results with only 12 paths and one stage.

Example 7 Consider the flow of an ideal gas through the constant-area passage shown in Figure 15.2. Between stations 1 and 2 the duct is frictionless and insulated from heat transfer. Between stations 2 and 3 the flow experiences friction with the duct walls, but is insulated from heat transfer. Between stations 3 and 4 the walls are again frictionless, but there is heat $Q$ added to the flow. Describe the process structures of the attainable environment for the case where the incident Mach number $M_1$ is supersonic.

Determining the attainable environment simply requires that we propagate the effects of the initial value $[M_1] \approx \hat{G}$ downstream through state 4. We are given the IdealGas(g) view and State(1.g), State(2.g), State(3.g) and State(4.g), so that the
individual view structure is

\[ IS = \{ \text{Gas}(g), \text{IdealGas}(g), \text{State}(1.g), \text{State}(2.g), \text{State}(3.g), \text{State}(4.g) \} \]

and constant. The duct is constant-area so that \([A_2/A_1] \approx \hat{1}, [A_3/A_2] \approx \hat{1},\) and \([A_4/A_3] \approx \hat{1}.\)

With \([f] \approx \hat{0}\) and \([Q] \approx \hat{0}\) between states 1 and 2, such that \([Q_2/Q_1] \approx \hat{1}\) and \([F_2/F_1] \approx q_{one},\) the AnyFlow(g.1.2), ReversibleFlow(g.1.2), AdiabaticFlow(g.1.2), and IsentropicFlow(g.1.2) processes are active without any initial condition. Adding the given condition \([M_1] \approx \hat{G},\) however, causes the NormalShock(g.1.2) and ChokedFlow(g) processes to activate. The entropy change caused by the shock wave causes \([s_2/s_1] \approx \hat{G},\) so that the IsentropicFlow(g.1.2) and ReversibleFlow(g.1.2) processes can no longer be active. With the resulting process structure,

\[ PS = \{ \text{AnyFlow}(g.1.2), \text{AdiabaticFlow}(g.1.2), \text{ChokedFlow}(g), \text{NormalShock}(g.1.2) \} \]
we have the constraint relations

\[
\begin{align*}
[M_2] & \approx \hat{L} \\
[p_2/p_1] & \approx \hat{G} \\
[T_2/T_1] & \approx \hat{G} \\
[\rho_2/\rho_1] & \approx \hat{G} \\
[s_2/s_1] & \approx \hat{G} \\
[\dot{m}/\dot{m}_e] & \approx \hat{I} \\
[h_2/h_1] & \approx \hat{G} \\
[v_2/v_1] & \approx \hat{L} \\
[c_2/c_1] & \approx \hat{G}
\end{align*}
\]

Now the state at 2 is known with respect to the state at 1. We now propagate these results through to state 3. We assume for the moment that the third section represents the exhaust region for the second section, providing State(3B,g) and \(p_{3b}\). This regions introduces a requirement for a limit analysis on \([p_b/p^*]\) for \(\hat{G}, \hat{I}\) and \(\hat{L}\). When required, we will also be performing a limit analysis on two different values for the variable length \(l_2\) of this section: \([l_2/l_2^*] \approx \hat{G}\) and \([l_2/l_2^*] \approx \hat{L}\).

1. \([p_b/p^*] \approx \hat{G}\). With this limit hypothesis the FannoSubsonic(g.2.3.3B) encapsulated history activates (which we consider part of the process structure PS).
This provides the results

\[ [M_2] \approx \hat{L} \]
\[ [M_2/M_1] \approx \hat{G} \]
\[ [p_2/p_1] \approx \hat{L} \]
\[ [s_2/s_1] \approx \hat{G} \]
\[ [F_2/F_1] \approx \hat{L} \]
\[ [\dot{m}/\dot{m}_c] \approx \hat{L} \]

along with the relations for the AdiabaticFlow and AnyFlow processes. This introduces an inconsistency in the choked flow conditions: in the first portion of the duct the ChokedFlow process was active, specifying \([\dot{m}/\dot{m}_c] \approx 1\), whereas the proposed solution for second portion requires \([\dot{m}/\dot{m}_c] \approx \hat{L}\). This inconsistency rules out this hypothesis. A limit analysis on \(l_2\) is not required for this situation. To satisfy curiosity, if we were to pose the limit hypothesis of \([p_3/p_5] \approx \hat{1}\) we could activate the OverExpansion(g.3.3B) process with the ChokedFlow(g) condition, but we cannot activate the FannoOverExpansion(g.2.3.3B) process because \([M_2]\) does not satisfy the quantity condition \([M_2] \approx \hat{G}\).

2. \([p_b/p^*] \approx \hat{1}\). When the back pressure equals the critical pressure and \([p_3/p_5^*] \approx \hat{1}\) with the ChokedFlow(b) process active, the requirements for the PerfectExpansion(g.3.3B) process are satisfied, thereby activating this process. In combination with the AdiabaticFlow(g.2.3) process, this activates the FannoP-
erfectExpansion(g.2.3,3B) history, setting \([M_3] \approx \hat{1}\) and \([s_3/s_1] \approx \hat{C}\) as a possible solution. The process structure for this situation is

\[
PS = \{\text{AnyFlow(g.2.3)}, \text{AdiabaticFlow(g.2.3)}, \text{ChokedFlow(g)}, \text{PerfectExpansion(g.3.3B)}, \text{FannoPerfectExpansion(g.2.3,3B)}\}.
\]

If on the other hand \([p_3/p_5^*] \not\approx \hat{1}\), then we have no new processes to add to the AnyFlow and ChokedFlow structure, providing the unsatisfying result \([M_3] \in Q_R\).

3. \([p_6/p^*] \approx \hat{L}\). When the back pressure is less than the critical pressure and \([p_3/p_5^*] \approx \hat{1}\) with the ChokedFlow(g) process active, the requirements for the UnderExpansion(g.3.3B) process are satisfied, which activates this process. Together with the AdiabaticFlow(g.2.3) process, these activate the FannoUnderExpansion(g.2.3,3B) history. This sets \([M_3] \approx \hat{1}\) under the relational condition that \([M_2] \approx \hat{L}\). We do not need to perform a limit analysis on the length \(l_2\) in this subsonic case. The process structure for this situation is

\[
PS = \{\text{AnyFlow(g.2.3)}, \text{AdiabaticFlow(g.2.3)}, \text{ChokedFlow(g)}, \text{UnderExpansion(g.3.3B)}, \text{FannoUnderExpansion(g.2.3,3B)}\}.
\]

Again, if the limit hypothesis for the exit plane were \([p_3/p_5^*] \not\approx \hat{1}\), then we have no new processes to add to the AnyFlow and ChokedFlow structure, providing the result \([M_3] \in Q_R\).
Now that the limit analysis for the second section is complete, we propagate the results through the third section, where there is heat addition but no friction, to determine the values at state 4. Again, we will hypothesize an region 4B in which we perform a limit analysis to impose various exhaust conditions on the flow.

To propagate the solution we select the results for the second section that set \([M_3] \approx \hat{1}\), supported by several process structures. The ChokedFlow\((g)\) and AnyFlow\((g,3,4)\) processes remain active, though now the AdiabaticFlow process cannot be activated due to the condition \([Q_4/Q_3] \approx \hat{G}\). The heat is added reversibly, so that ReversibleFlow\((g,3,4)\) is valid, but since the AdiabaticFlow\((g,3,4)\) process is not active, we cannot activate the companion IsentropicFlow\((g,3,4)\) process. With \([M_3] \approx \hat{1}\), the only encapsulated history that can be used to describe the flow through the third section is the RayleighPerfectExpansion\((g,3,4,4B)\) history, which requires the ChokedFlow\((g)\) and PerfectExpansion\((g,4,4B)\) processes along with the constraint that \([p_4/p_4^*] \approx \hat{1}\) and \([p_0/p^*] \approx \hat{1}\). This leaves \([M_4] \approx \hat{1}\) without consideration of \([M_3]\) or the length of the passage. The process structure for this combination is

\[
\text{PS} = \{\text{AnyFlow}(g,3,4), \text{ChokedFlow}(g), \text{PerfectExpansion}(g,4,4B), \\
\text{RayleighPerfectExpansion}(g,3,4,4B)\}.
\]

The other limit hypotheses for \([p_0/p^*]\) will activate the UnderExpansion\((g,4,4B)\) and OverExpansion\((g,4,4B)\) processes, though neither of the RayleighUnderExpansion\((g,3,4,4B)\) or RayleighOverExpansion\((g,3,4,4B)\) processes sheds light on \([M_4]\) be-
cause they do not consider initially sonic flow. With only the AnyFlow(g.3.4), ChokedFlow(g), and ReversibleFlow(g.3.4) processes active, the resultant exit plane Mach number can be \([M_4] \in Q_R\) due to the ambiguity of the qualitative arithmetic.

Selecting the second section limit hypotheses that do not provide \([M_3] \approx \hat{1}\), and thus do not trigger the Fanno histories, we cannot obtain anything but spurious results owing to the ambiguity of the AnyFlow process. The choking inconsistency encountered in the FannoSubsonic(g.2.3.3B process raises an issue regarding the validity of the model, primarily regarding the occurrence of the normal shock in the first portion of the duct. Although the NormalShock process was activated, it did not have to be activated, and in fact the FannoSupersonic history could have activated a NormalShock(g.1.3) process if we ignored state 2 and applied these processes between 1 and 3. With a passage length less than the critical length, the choked flow at 3 would be subsonic, allowing the RayleighPerfectExpansion(g.3.4.4B) or RayleighUnderExpansion(g.3.4.4B) processes to accelerate the flow to the sonic exit at 4. Allowing the upstream conditions to adapt to the downstream conditions is the best representation of the actual physical behavior.

**Example 8** Consider the flow of an ideal gas through the divergent passage shown in Figure 15.3. The walls of the passage are frictional, \([\dot{f}] \neq \hat{0}\), and heat is added to the gas along the passage walls between 1 and 2. Describe the inter-parameter influences determined by the constraint equations for the active individual and process view structures.
Figure 15.3: The problem diagram for example 8, a divergent passage with heat addition $Q$ and friction $f$. The passage is assumed to be one-dimensional with no requirements placed on intake or exhaust conditions.

From the given information we know $[A_2/A_1] \approx \hat{G}$, $[Q_2/Q_1] \approx \hat{G}$, and $[F_2/F_1] \approx \hat{G}$, while the individual view structure is

$$\text{IS} = \{\text{Gas(g)}, \text{IdealGas(g)}, \text{State(1.g)}, \text{State(2.g)}\}.$$  

With these individuals we may activate the AnyFlow(1.2.g) process without further
preconditions or quantity conditions. This provides the familiar relations

\[
\begin{align*}
\left[ \frac{m_2}{m_1} \right] &= \hat{1} \\
\left[ \frac{\rho_2}{\rho_1} \right] \oplus \left[ \frac{A_2}{A_1} \right] \oplus \left[ \frac{v_2}{v_1} \right] &\approx 3 \\
\left[ \frac{v_2}{v_1} \right] \oplus \left[ \frac{h_2}{h_1} \right] \oplus \left[ \frac{Q_2}{Q_1} \right] \oplus \left[ \frac{W_2}{W_1} \right] &\approx 2 \\
\left[ \frac{A_2}{A_1} \right] \oplus \left[ \frac{p_2}{p_1} \right] \oplus \left[ \frac{\rho_2}{\rho_1} \right] \oplus 2 \otimes \left[ \frac{v_2}{v_1} \right] \oplus \left[ \frac{F_2}{F_1} \right] &\approx 4 \\
\left[ \frac{p_2}{p_1} \right] &\approx \left[ \frac{\rho_2}{\rho_1} \right] \oplus \left[ \frac{T_2}{T_1} \right] \oplus \hat{1} \\
\left[ \frac{h_2}{h_1} \right] &\approx \left[ \frac{T_2}{T_1} \right] \\
\left[ \frac{e_2}{e_1} \right] &\approx \left[ \frac{T_2}{T_1} \right] \\
\text{if } [M_1] > [M_2] &\text{ then } \left[ \frac{T_2/T_{02}}{T_1/T_{01}} \right] \approx \hat{G} \\
\text{if } [M_1] < [M_2] &\text{ then } \left[ \frac{T_2/T_{02}}{T_1/T_{01}} \right] \approx \hat{L} \\
\text{if } [M_1] = [M_2] &\text{ then } \left[ \frac{T_2/T_{02}}{T_1/T_{01}} \right] \approx \hat{1} \\
\text{if } [M_1] > [M_2] &\text{ then } \left[ \frac{p_2/p_{02}}{p_1/p_{01}} \right] \approx \hat{G} \\
\text{if } [M_1] < [M_2] &\text{ then } \left[ \frac{p_2/p_{02}}{p_1/p_{01}} \right] \approx \hat{L} \\
\text{if } [M_1] = [M_2] &\text{ then } \left[ \frac{p_2/p_{02}}{p_1/p_{01}} \right] \approx \hat{1}
\end{align*}
\]

Since the problem conditions do not activate any of the other processes, we close the world so that these relations become the constraints. Since this gives rise to many different solutions, it is interesting to resort back to the QPT representation scheme and write the contraints in terms of qualitative proportionalities. In so doing, we decompose the original relations (direct influences) into a set of interparameter relationships (indirect influences) that describe the physical behavior in terms of action-reaction pairs. Of course, this decomposition loses much of the original
filtering power of the full constraint relationships, introducing the *all else being equal* condition. Focusing first on the momentum equation

\[ \frac{A_2}{A_1} \oplus \frac{p_2}{p_1} \oplus \frac{\rho_2}{\rho_1} \oplus 2 \otimes \frac{v_2}{v_1} \oplus \frac{F_2}{F_1} \approx 4, \]

we may construct the influence diagram shown in Figure 15.4.

Since we have not identified dependent and independent variables, the interparameter dependencies are bidirectional. Each dependency pair may be written as an indirect influence using the \( \alpha_Q \) operators, requiring two expressions to note the bidirectional influence of each pair. Since there are five unique parameters, there are four pairs of influence expressions for each parameter. The four pairs involving pressure \( p \), for example, are

\[
\begin{align*}
p & \alpha_{Q-} v \\
p & \alpha_{Q+} F \\
p & \alpha_{Q-} A \\
p & \alpha_{Q-} \rho
\end{align*}
\]

\[
\begin{align*}
v & \alpha_{Q-} p \\
F & \alpha_{Q+} p \\
A & \alpha_{Q-} p \\
\rho & \alpha_{Q-} p
\end{align*}
\]
Carrying out a similar decomposition of the continuity equation,

\[
\left[ \frac{\rho_2}{\rho_1} \right] \oplus \left[ \frac{A_2}{A_1} \right] \oplus \left[ \frac{v_2}{v_1} \right] \approx 3,
\]

we obtain the influence pairs

\[
\begin{align*}
p \propto_{Q_-} v & \quad & v \propto_{Q_-} p \\
p \propto_{Q_-} A & \quad & A \propto_{Q_-} p \\
p \propto_{Q_-} \rho & \quad & \rho \propto_{Q_-} p
\end{align*}
\]

which are already included in the influence pair set for the momentum equation; therefore, they are redundant and can be eliminated. However, the state equation for the ideal gas

\[
\begin{bmatrix} p_2 \\ p_1 \end{bmatrix} \approx \begin{bmatrix} \rho_2 \\ \rho_1 \end{bmatrix} \oplus \begin{bmatrix} T_1 \\ T_1 \end{bmatrix} \oplus \mathbf{i}
\]

provides the influence pairs

\[
\begin{align*}
p \propto_{Q_+} \rho & \quad & \rho \propto_{Q_+} p \\
p \propto_{Q_+} T & \quad & T \propto_{Q_+} p \\
\rho \propto_{Q_-} T & \quad & T \propto_{Q_-} \rho
\end{align*}
\]

which result in a contradiction for \( p \propto_{Q} \rho \) with the influences from the momentum equation.\(^3\) Nevertheless, both cases may be explored. After performing this decomposition process on the remaining relationships, we obtain the set of qualitative proportionalities shown in Table 15.10.

---

\(^3\)This is a good example of how the process of decomposing physical relationships into qualitative proportionalities introduces ambiguity into the model.
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Table 15.10: The set of qualitative proportionality (indirect influence) pairs for the AnyFlow and IdealGas views of example 8. Each constraint requires expressions to enforce the bidirectional influences.
Employing these indirect influence pairs obviously increases the complexity of the model. For this small flow problem involving eleven physical parameters, decomposing the direct influence relations creates a larger problem of managing these 50 constraint pairs and their respective quantity spaces. After selecting the dependent variables and the corresponding set of direct influences, we may reduce this complexity by eliminating some of the indirect influences.

**Example 9** Consider the flow of an ideal gas through the varying-area channel shown in Figure 15.5. Between stations 1 and 2 the channel is divergent only; between stations 2 and 3 the channel is convergent only; and between stations 3 and 4 the channel is again divergent-only, exhausting into a region of back pressure $p_b$. The cross-sectional area of throat $A_3 = A_1$ is less than that of the channel entrance $[A_1]$. The entire channel is frictionless and experiences no heat transfer. Describe the parameter history and quantity space for the dynamic pressure $p$ throughout this passage, assuming $M_1 \approx 0$ and $p_b$ variable.

Drawing from our experience from example 7, we can make educated decisions about how to model this passage before setting out to perform the simulation and history generation. If we were influenced by the prespecification of stations along the duct, it would seem that the passage model requires the serial combination of three components: two divergent passages and one convergent passage. This combination

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4 Admittedly, this is a difficult quantitative problem to solve even under ideal one-dimensional conditions.
Figure 15.5: The problem diagram for the combination divergent-convergent-divergent nozzle of example 9.

would likely lead to more spurious qualitative results than is necessary, as seen in example 7.

Recognizing the shape of the passage between stations 2 and 4 as a convergent-divergent nozzle, we can make a better modeling decision by combining only two components in series: a divergent nozzle connected to a convergent-divergent nozzle. The encapsulated histories we have available to describe the performance of the convergent-divergent nozzle provide convenient abstractions for this modeling task. The component combination with the divergent nozzle, however, introduces spurious solutions that we prefer to avoid.

The best modeling decision requires abstraction of the entire passage as one convergent-divergent nozzle. The assumptions made in the one-dimensional analysis of gas flows refer only to the minimum cross-sectional area of the duct, not to the shape of the passage before or after this throat. The CDNozzle-series of encapsulated histories encourage this abstraction by considering only the intake, throat
and exhaust areas. The only quantity condition for area enforced by these histories is that the throat be smaller than both the intake and exhaust. Proper application of these conditions requires that we identify the throat location; in this problem the throat location is prespecified.

For the qualitative model of Figure 15.5, then, we assign station 1 as the intake, station 3 as the throat and station 4 as the exhaust. The \( \text{Gas}(g) \), \( \text{IdealGas}(g) \), \( \text{State}(1.g) \), \( \text{State}(t.g) \), \( \text{State}(4.g) \), and \( \text{State}(4B,g) \) individuals are active, where \( 4B \) symbolically represents the exhaust (back pressure) region. Since \( [M_1] \approx 0 \), we exploit the Correspondence relations that set the parameter values at \( \text{State}(1.g) \) to those of the stagnation state \( \text{State}(01.g) \). This requires that \( p_1 = p_{01} \), which becomes the root of the parameter history for \( p \). Since the available convergent-divergent nozzle histories require some specification of the operating pressure ratio \( [p_b/p_{01}] \), we perform a limit analysis on each element of the operating pressure ratio quantity space

\[
\frac{p_b}{p_{01}} \in \{[0, r_{p,c3}], [r_{p,c3}, r_{p,c3}], (r_{p,c3}, r_{p,c2}), [r_{p,c2}, r_{p,c1}], [r_{p,c1}, r_{p,c1}], (r_{p,c1}, 1), [1, 1] \}.
\]

The pressure quantity spaces resulting from each operating pressure ratio hypothesis are shown in the partial ordering format; that is, arrows connect the dynamic pressure and landmark values in a graph that shows a relative magnitude ordering, with the head of the arrow pointing to the larger value. Pressure quantities for which we are unable to provide relative magnitude estimates, as will be the case for \( p_2 \), remain unconnected.
Figure 15.6: Example 9: the dynamic pressure quantity space for $[p_{4B}/p_{01}] < [r_{p_{03}}]$.

1. $[p_{4B}/p_{01}] \approx [0, r_{p_{03}}]$. With the operating pressure ratio below the third critical ratio, choked isentropic flow occurs with a supersonic exhaust that undergoes an external expansion adjustment. The encapsulated history CDUnderExpansion(g.1.4.4B) provides the IsentropicChokedFlow(g.1.4) and UnderExpansion(g.4.4B) processes, which set $[M_4] \approx \hat{G}$, $[p_4] > [p_{4B}]$, $[p_{4B}] < [p^*]$, $[p_4] < [p^*]$, and $[p_{04}] \approx [p_{01}]$. The sonic throat provides $[p_3] \approx [p^*]$. Figure 15.6 shows the pressure quantity space for this process structure.

2. $[p_{4B}/p_{01}] \approx [r_{p_{03}}]$. When the operating pressure ratio is exactly equal to the third critical ratio, choked isentropic flow occurs with a supersonic exhaust that requires no external adjustment. The encapsulated history CDPerfectExpansion(g.1.4.4B) provides the IsentropicChokedFlow(g.1.4) and PerfectExpansion(g.4.4B) processes, which provide $[M_4] \approx \hat{G}$, $[p_4] \approx [p_{4B}]$, $[p_4] < [p^*]$, and $[p_{04}] \approx [p_{01}]$. The sonic throat provides $[p_3] \approx [p^*]$. Figure 15.7 shows the pressure quantity space for these conditions.
Figure 15.7: Example 9: the dynamic pressure quantity space for $[\tau_{p,3}] \approx [p_{4B}/p_{01}]$.

Figure 15.8: Example 9: the dynamic pressure quantity space for $[\tau_{p,3}] < [p_{4B}/p_{01}] < [\tau_{p,3}]$.

3. $[p_{4B}/p_{01}] \approx (\tau_{p,3}, \tau_{p,2})$. With the operating pressure ratio between the second and third critical ratios, choked conditions occur and the isentropic flow must undergo an external compression process in order to match the imposed back pressure. The encapsulated history CDOverExpansion(g.1.4.4B) activates the processes IsentropicChokedFlow(g.1.4) and OverExpansion(g.4.4B), which fix $[M_4] \approx \hat{G}$, $[p_4] < [p_{4B}]$, $[p_4] < [p^*]$, $[p_{4B}] < [p^*]$, and $[p_{01}] < [p_{04}]$. The sonic throat provides $[p_3] \approx [p^*]$. Figure 15.8 shows the corresponding pressure quantity space.

4. $[p_{4B}/p_{01}] \approx (\tau_{p,3}, \tau_{p,1})$. With the operating pressure ratio between the first
and second critical ratios, choked flow occurs and a normal shock stands in the divergent section or exit plane of the nozzle. This shock positions itself so that the exhaust pressure matches the imposed back pressure. The encapsulated history CDNormalShock(g.1.4.4B) realizes this situation, providing the NormalShock(g.3.4) and ChokedFlow(g), but eliminating from possibility the IsentropicFlow(g.1.4) process. The sonic throat provides \([p_3] \approx [p^*]\), and the subsonic exhaust provides \([p_4] \approx [p_{4B}]\). Furthermore, the normal shock introduces a stagnation pressure increase \([p_{04}/p_{01}] \approx \hat{C}\), which we include in the pressure quantity space (Figure 15.9).

5. \([p_{4B}/p_{01}] \approx [r_{p_{e1}}]\). At the first critical pressure ratio the flow is choked, reaching the maximum mass flow density, but subsonic downstream of the throat. This condition activates the CDSubsonicChoked(g.1.4.4B) history, which specifies that the IsentropicChokedFlow(g.1.4) process is active, the exhaust pressure is equal to the imposed back pressure \([p_4/p_{04}] \approx [p_{4B}/p_{04B}]\), sonic conditions exist at the throat \([M_3] \approx \hat{1}\), and subsonic conditions exist at the exit.
plane $|M_4| \approx \hat{L}$. Since the flow is isentropic, we also deduce $|p_4| \approx |p_{4B}|$, and along with the ideal gas relationships we conclude $|p_3/p^*| \approx \hat{1}$ and $|p_3| < |p_4|$. This quantity space is shown in Figure 15.10.

6. $|p_{4B}/p_{01}| \approx (r_{p_{c1}}, 1)$. With the operating pressure ratio $p_{4B}/p_{01}$ in the interval between the first critical ratio $r_{p_{c1}}$ and 1, this satisfies the quantity conditions for the encapsulated history CDSubsonic(g.1.4.B). This history specifies that the flow through the nozzle is isentropic (IsentropicFlow(g.1.4)), subsonic ($|M_4| \approx \hat{L}$, $|M_3| \approx \hat{L}$), and that the exhaust pressure equals the imposed back pressure $|p_4/p_{04}| \approx |p_{4B}/p_{04B}|$. Since the flow is isentropic, we also deduce $|p_4| \approx |p_{4B}|$, $|p_3/p^*| \approx \hat{G}$, and $|p_3| < |p_4|$. The quantity space for $p$ (Figure 15.11) is similar to the previous case, but now depicts the inequality $|p^*| < |p_3|$.

7. $|p_{4B}/p_{01}| \approx \hat{1}$. This condition does not activate any of the encapsulated histories for nozzle performance, but it is sufficient to activate the Isentrop-
Figure 15.11: Example 9: the dynamic pressure quantity space for $|r_{p_{41}}| < |p_{4B}/p_{01}| < \hat{1}$.

Figure 15.12: Example 9: one possible dynamic pressure quantity space for $|p_{4B}/p_{01}| \approx \hat{1}$.

icFlow(g,1,4) process, providing the correct results $|M_4| \approx \hat{0}$ and $|p_4| \approx |p_{01}|$, but also the spurious results $|M_4| \in \{\hat{L}, \hat{1}, \hat{G}\}$ and $|p_4| < |p_{01}|$ encountered in other problems. The quantity space for $p$ assuming the no-flow situation is shown in Figure 15.12.
Chapter 16

Conclusions

Qualitative reasoning is an important topic in artificial intelligence research. The discipline of qualitative physics specializes this topic to one of determining how to describe the natural behaviors of the physical world, and of how to represent the important concepts through computer programs, so that we may then provide machines with the ability to reason, in a "common sense" manner, about their environment.

This dissertation contributes a collection of axioms for building models of the physical domain of gas dynamics, identifying the important concepts as elements of a theory. These qualitative elements coincide with the fundamental laws of nature commonly applied in science and engineering problems, while enabling more complex structures to be built from them. In particular, these models may be deployed in intelligent computer-aided design systems for application to the difficult problems associated with conceptual and preliminary design tasks. The theory satisfies the requirements for qualitative physics models described in Section 2.2, and furthermore, since the components of the theory are constructed from the fundamental laws of nature, they overcome many of the limitations of "shallow"
expert system models.

The theory is presented within the framework of an existing qualitative reasoning ontology called *Qualitative Process Theory*. This ontology presumes that the fundamental mechanism for change in a dynamic system is a compilation of active processes. The current theory identifies and describes these processes for the gas dynamics domain, and shows how these processes activate and deactivate based on conditions existing in the problem environment. It creates a new algebraic scheme called *qualitative ratios* to assist in the qualitative description of the change, drawing parallels with the problem-solving techniques engineers already use. Combined with a logical inference engine, models built from these processes provide a means for performing qualitative reasoning and simulation for real-world problems.

The theory covers the phenomena associated with dynamic behavior of gases from an academic standpoint. It considers the flow of coherent fluid structures at the macroscopic level, translating an introductory academic discussion of gas dynamics into a straightforward, tutorial, qualitative discussion. Although the general form of many of the quantitative equations is presented, the theory accommodates only the simplified forms resulting primarily from the one-dimensional steady flow and ideal gas assumptions. These simplifications are consistent with the endeavor and purpose of qualitative reasoning. The result is a set of process descriptions that may be composed into models for simulating the dynamic behavior of gases.

Development of this theory raises many issues and extensions, associated pri-
arily with the engineering applications of qualitative reasoning technology. The foremost issue on this front is determining how far the qualitative models should be applied to a particular problem before turning to quantitative techniques. Applying these qualitative reasoning techniques to more elaborate problems, or even requiring the qualitative results to be more accurate, requires more detail in the models and the theory. For the gas dynamics theory these extensions might include:

- Completion of the set of process descriptions for multi-dimensional flow;
- Identification of constructs for chemically reacting, radiation-induced, and multiple-phase flows;
- Consideration of flows in the presence of applied electric and magnetic fields;
- Consideration of chaotic flows such as turbulence and combustion;
- Consideration of plasmas as a form of matter and comparison of their qualitative behavior with the qualitative behavior of gases;
- Investigation of process descriptions that accommodate both design and performance problems simultaneously;
- Analysis of the behavior of still gases, possibly from the microscopic standpoint.

All of these extensions will be useful in the intelligent computer-aided design arena. However, one may argue that many of these extensions seem to depart from the stated
goal of providing "common sense" reasoning capabilities to machines, particularly as to whether issues such as radiation-induced flows or microscopic trajectories qualify as "common sense" observations.

There are also certain issues associated with the computational implementation of this theory of gas dynamics, although most of these issues actually pertain to the implementation of qualitative theories in general. For example, how does a program generate a topology network to discretize the problem into interesting states? How are complex networks traversed, considering the possibility of multiple arcs at the nodes? How might the complexity of generating an interpretation or prediction be reduced by focusing on the determination of the value of a particular parameter? Given the results, what actions should the program take to change the current (or predicted) situation? How might the model automatically adapt itself optimize its representation of a changing situation? These (and many other) issues hold the exciting challenges for developers of inferencing schemes and qualitative reasoning ontologies.

Obviously, there are many interesting qualitative theories of the physical world yet to be developed. Combined with the current theory of gas dynamics, a companion theory of liquid dynamics would complete a consolidated theory of general fluid dynamics, able to reason about unknown flows and liquid-gas interactions. A theory of heat transfer might then be added to permit reasoning about the effects of conduction, convection and radiation to fluid dynamics. Combined with a theory
of solid matter that included distinctions between white, black and gray bodies, this suite of qualitative theories would address a vast problem space and provide a machine with tremendous "common sense" reasoning capabilities.

"The holy grail of qualitative physics is a complete set of models, spanning the space of all the physical domains people know, able to characterize human models from the person on the street up to the best experts, capable of supporting efficient application programs, and so forth. Like traditional physics, we will probably never get there. But we will certainly learn interesting things on the way."

—Ken Forbus [58]
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