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Axially symmetric harmonic maps and relaxed energy

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Axially Symmetric Harmonic Maps and Relaxed Energy

by

Chi-Cheung Poon

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Axially Symmetric Harmonic Maps and Relaxed Energy
by Chi-Cheung Poon

Abstract.

Here we investigate some new phenomena in harmonic maps that result by imposing a symmetry condition. A map \( u : B^3 \to S^2 \) is called axially symmetric if, in cylindrical coordinates,

\[
u(r, \theta, z) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)\]

for some real-valued function \( \phi(r, z) \), called an angle function for \( u \).

The important notion of the L energy of a map from \( B^3 \) to \( S^2 \) was first studied by H.Brezis, F.Bethuel, and J.M.Coron. In [BBC], the weak \( H^1 \) lower semicontinuity of \( E+8\pi \lambda L \) is proven. Thus, the minimizers of \( E+8\pi \lambda L \) exist. For minimizers of \( E+8\pi \lambda L \), \( 0 < \lambda < 1 \), Bethuel and Brezis [BB] prove that the singularities are only isolated points. Note that such minimizers are still weak solutions of the harmonic map equation.

In this thesis, we treat these problems in the axially symmetric context. By studying a elliptic equation, we show that there is at most one smooth axially symmetric harmonic map corresponding to any given smooth axially symmetric boundary data. We also show that any minimizer in the axially symmetric class of \( E+8\pi \lambda L \), where \( 0 < \lambda \leq 1 \), has only isolate Singularities in minimizers may occur even for \( \lambda = 1 \). These provide the first examples of isolated singularities of degree 0.
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CHAPTER 1
INTRODUCTION

1.1 Background Materials
Let \((M,g)\) be a compact smooth \(m\)-dimensional Riemannian manifold, possibly with nonempty boundary \(\partial M\). Let \((N,h)\) be a compact smooth \(n\)-dimensional Riemannian manifold. Given a map \(u \in C^1(M,N)\), we define the energy density of \(u\) at \(x\) by
\[
e(u(x)) = \frac{1}{2} g^{\alpha \beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij},
\]
and the energy of \(u\) by
\[
E(u) = \int_M e(u) \, dV_M,
\]
where \(dV_M\) is the volume element of \(M\). In terms of local coordinates \(x = (x^1, \ldots, x^m)\) on \(M\) and \(u = (u^1, \ldots, u^n)\) on \(N\), we can write the Euler equations in the form:
\[
\Delta_M u^i + \Gamma^i_{jk} g^{\alpha \beta} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} h_{jk} = 0, \quad i = 1, \ldots, n;
\]
where \(\Gamma^i_{jk}\) is the Christoffel symbols and
\[
\Delta_M = g^{-\frac{1}{2}} \frac{\partial}{\partial x^\alpha} \left( g^{\frac{1}{2}} g^{\alpha \beta} \frac{\partial}{\partial x^\beta} \right)
\]
is the Laplace-Beltrami operator on \(M\).

For \(n \geq 3\), Eells and Sampson [ES], in 1964, proved that if \(M, N\) are compact and without boundary, and if the sectional curvature of \(N\) is nonpositive, then any smooth map from \(M\) to \(N\) can be deformed into a harmonic map. Then Hamilton [Ha], in 1975, gave the existence proof for the Dirichlet problem in case \(n \geq 3\), also assuming that the sectional curvature of \(N\) is nonpositive. The heat flow method was employed in both papers. In 1977, Hildebrandt, Kaul and Widman [HKW] obtained the existence of "small solution" of the Dirichlet problem without the curvature assumption on \(N\).

The partial regularity theory of minimizing harmonic maps, without curvature assumption on \(N\), was established by Schoen and Uhlenbeck [SU1], [SU2] in 1982. They showed that any map which minimizes the energy functional \(E\) is smooth in \(M\).
except at a set of Hausdorff dimension not bigger that \( m - 3 \). When \( m = 3 \), a energy minimizer can have isolated singularities in the interior only.

Harmonic maps are used as models to understand physical problems. We mention one example, that of nematic liquid crystals. In this case, we are interested in maps from domains in \( \mathbb{R}^3 \) into \( S^2 \), the 2-sphere. Let \( \Omega \) be a domain in \( \mathbb{R}^3 \). A map \( u : \Omega \to S^2 \) can be described as a map \( u = (u^1, u^2, u^3) : \Omega \to \mathbb{R}^3 \), so that \( \sum_{i=1,2,3} (u^i)^2 = 1 \). Then the energy functional can be simplified as

\[
E(u) = \int_{\Omega} |\nabla u|^2 \, dx,
\]

and the Euler equation would be

\[
\Delta u^i + |\nabla u|^2 u^i = 0, \quad i = 1, 2, 3.
\]

\[(1.1)\]

In this thesis, we study harmonics maps from \( B^3 \), the unit ball in \( \mathbb{R}^3 \), to \( S^2 \), the unit sphere. We utilize the axial symmetry to simplify the situation sufficiently to allow us to deduce some existence and regularity results. A question we want to solve is:

Given any \( v : \partial B^3 \to S^2 \), is there a smooth harmonic map \( u, u|_{\partial B^3} = v \)?

A method is suggested by Brezis, Coron and Bethuel in [BBC]. They defined the term \( L \)-energy. If a map \( u \) is smooth except at finitely many points, say

\[\{p_1, p_2, \ldots, p_m, n_1, n_2, \ldots, n_m\},\]

\( p_i \)'s has degree +1 and \( n_i \)'s have degree -1. Then

\[
L(u) = \inf_{|\nabla\xi| \leq 1} \sum_i (\xi(p_i) - \xi(n_i)).
\]

They showed that for any \( v : \partial B^3 \to S^2 \), if the degree of \( v \) is 0,

\[
\inf_{u \in H^1(B^3, S^2), u = v \text{ on } \partial B^3} E(u) + 8\pi L(u) = \inf_{u \in C^0(B^3, S^2), u = v \text{ on } \partial B^3} E(u).
\]

This suggests that one may find a smooth harmonic map with prescribed degree zero boundary data by minimizing the functional \( E + 8\pi L \) in \( H^1(B^3, S^2) \). Furthermore, they showed that for any \( 0 < \lambda < 1 \), there is a map \( u_\lambda \) minimizing the functional \( E + 8\pi \lambda L \). Each \( u_\lambda \) is a harmonic map. Also there is a sequence \( \{\lambda_i\} \), so that \( u_{\lambda_i} \) are distinct.
Later, Bethuel and Brezis proved that for \( \lambda < 1 \), the minimizers of \( E + 8\pi\lambda L \) are smooth except at finitely many points in the interior. However, they are unable to obtain the regularity of \( E + 8\pi L \) minimizers.

Here, we try to solve this problem with the help of an axially symmetry. A map \( u : B^3 \to S^2 \) is called axially symmetric if in cylindrical coordinates

\[
u(r, \theta, z) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi),
\]

for some real valued function \( \varphi \). In this case, harmonic map equations can be reduced to a single elliptic equation:

\[
\frac{\partial}{\partial r} r \frac{\partial \varphi}{\partial r} + \frac{\partial}{\partial z} r \frac{\partial \varphi}{\partial z} - \frac{\sin 2\varphi}{2r} = 0.
\]

In the rest of Chapter 1, we will establish the notions of axially symmetric harmonic maps and the \( L \) energy. In Chapter 2, we study the above equation and prove some maximum principle and uniqueness theorems for axially symmetric harmonic maps. The main result in this chapter is: for any axially symmetric boundary, there is at most one smooth axially symmetric harmonic map. We will discuss the regularity of minimizers of \( E + 8\pi\lambda L \) among axially symmetric class. In Chapter 3, we consider the case \( \lambda < 1 \) and in Chapter 4, we consider the case \( \lambda = 1 \). We will show that minimizers of \( E + 8\pi\lambda L, 0 < \lambda < 1 \), among axially symmetric class are smooth except at finitely points. The possible gap between the infimum of energy among all maps and the infimum of energy among smooth maps was pointed out in [HL]. Here we discuss another gap that arises when treating axially symmetric maps. The gap phenomenon provides examples that some minimizers of \( E + 8\pi L \) among axially symmetric class are not smooth, due to the axially symmetry. In Chapter 5, we present some related results which includes a prescribed singularity problem, a heat flow problem and a result concerning liquid crystal droplet. We also prove that for any \( a \) in \( \bar{B}^3 \), there is a harmonic map \( u_a \in C^\infty(B^3 \sim \{a\}, S^2) \), so that \( u_a(x) = x \) on \( \partial B^3 \).

\[ \text{1.2 Axially Symmetric Harmonic Maps} \]
Let $\Omega$ be domain in $\mathbb{R}^3$, $u : \Omega \rightarrow \mathbb{R}^3$. We say $u \in H^1(\Omega, S^2)$ if $u = (u^1, u^2, u^3)$, $u(x) \in S^2$ for almost all $x \in \Omega$ and $u^i \in H^1(\Omega, \mathbb{R})$, i.e.,

$$\int_\Omega |\nabla u|^2 \, dx < \infty, \quad i = 1, 2, 3.$$

$u$ is called weakly harmonic if $u$ satisfies the elliptic system (1.1) weakly, i.e.,

$$\int_\Omega < \nabla u, \nabla \phi > - |\nabla u|^2 < u, \phi > \, dx = 0$$

for all $\phi \in C_0^\infty(\Omega, \mathbb{R}^3)$.

Let $r, \alpha, z$ be cylindrical coordinates in $\mathbb{R}^3$, i.e., $x = r \cos \alpha$, $y = r \sin \alpha$. A map $u : B^3 \rightarrow S^2$ is called, as in [Z], axially symmetric if in $r, \alpha, z$

(1.2) $u(r, \alpha, z) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$

for some real valued function $\varphi(r, z)$. This implies that $u \circ R_\theta \equiv R_\theta \circ u$ for any $\theta \in [0, 2\pi]$, where $R_\theta$ denotes rotation about the z-axis through an angle $\theta$.

Using (1.2), we can simplify the formula for the energy functional of an axially symmetric map $u$,

$$\int_{B^3} |\nabla u|^2 \, dx = 2\pi \int_D r \left( \frac{\partial \varphi}{\partial r} \right)^2 + r \left( \frac{\partial \varphi}{\partial z} \right)^2 + \frac{\sin^2 \varphi}{r} \, drdz.$$

where $D = \{(r, z) : r^2 + z^2 < 1, \quad r > 0\}$. For any function $\varphi : D \rightarrow \mathbb{R}$, define the energy of $\varphi$, also denoted by $E$,

$$E(\varphi) = \int_D r \left( \frac{\partial \varphi}{\partial r} \right)^2 + r \left( \frac{\partial \varphi}{\partial z} \right)^2 + \frac{\sin^2 \varphi}{r} \, drdz.$$

Suppose $\varphi : D \rightarrow \mathbb{R}$, and $E(\varphi) < \infty$. Because of the last term in the integrand, $\sin \varphi(0, z) = 0$ for almost $z \in (-1, 1)$. In the other words, $\varphi$ can assume values $k\pi$, $k \in \mathbb{Z}$ only on the z-axis. If $u$ is the axially symmetric map corresponding to $\varphi$ via (1.2), then the value of $u$ at each point of continuity on the z-axis is necessarily either (0,0,1) or (0,0,-1), i.e., the north pole or the south pole of the sphere.

Any critical point $\varphi$ of $E$ satisfies in $D$ a scalar partial differential equation

(1.3) $\frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial \varphi}{\partial z} \right) - \frac{\sin 2\varphi}{2r} = 0.$

The equation is degenerate on the boundary of $D$ corresponding to the z-axis.
Suppose $\varphi$ is a continuous solution to the equation (1.3) in $D$. By the above discussion, we may assume $\varphi(0,z) = 0$ for all $z \in [-1,1]$. In [Z], Zhang proved

**Proposition 1.1.** Let $\varphi$ be a regular solution to (1.3) in $D$, $\varphi(0,z) = 0$ for all $z \in [-1,1]$. Suppose $u$ is the axially symmetric map corresponding to $\varphi$ via (1.2). Then $u$ is a smooth axially symmetric harmonic map from $B^3$ to $S^2$.

**Proof** By the elliptic theory, $\varphi \in C^0(D) \cap C^\infty(D)$. Therefore $u$ is continuous in $B^3$ and is harmonic except on the $z$-axis. Let $\phi$ be a $C^\infty_0(B^3,\mathbb{R}^3)$ map. Consider the following cut-off function $\eta(t)$, $\eta(t) = 0$ if $0 \leq t \leq 1$, $\eta(t) = 1$ when $2 \leq t$, and $|\eta'(t)| \leq 2$. Let $\eta_k(x,y,z) = \eta(kr)$, where $r^2 = x^2 + y^2$. Then,

$$\int_\Omega \langle \nabla u, \nabla (\eta_k \phi) \rangle - |\nabla u|^2 \langle u, \eta_k \phi \rangle \, dx = 0.$$

By taking $k \to \infty$, we see that

$$\int_\Omega \langle \nabla u, \nabla \phi \rangle - |\nabla u|^2 \langle u, \phi \rangle \, dx = 0.$$

Therefore $u$ is a continuous weakly harmonic map. By the regularity theory for elliptic systems (see [Hi] or [C]), $u$ is a smooth harmonic map in $D$. \qed

Suppose $\varphi$ is continuous in the interior of $D$, but is discontinuous at finitely many points on the $z$-axis. We can apply Proposition 1.1 to subdomains at which $\varphi$ is continuous up to the boundaries and conclude that the corresponding map $u$ is weakly harmonic in $D$. In the case where $\varphi$ is continuous in the interior of $D$, but is discontinuous at a set of $H^1$ measure 0 on the $z$-axis, if $\varphi$ has finite energy, we can repeat the proof of Proposition 1.1 to show that $u$ is also weakly harmonic. Thus Proposition 1.1 can be improved as

**Proposition 1.1'** Let $\varphi$ be a weak solution to (1.3) in $D$. If $\varphi$ is discontinuous only at a set of $H^1$ measure 0 on the $z$-axis and if $\varphi$ has finite energy, then $u$ is also a weakly harmonic map from $B^3$ to $S^2$.

It is easy to check that

$$\Lambda(x) = \frac{(x_1,x_2,x_3)}{|x|} \quad \text{and} \quad \Psi(x) = \frac{(x_1,x_2,-x_3)}{|x|}$$

define homogeneous axially symmetric harmonic maps with $\deg(\pm \Lambda|_{S^2}) = \pm 1$ and $\deg(\pm \Psi|_{S^2}) = \mp 1$. Let $\mathcal{R}_{AS}$ denote the family of axially symmetric harmonic maps $u \in H^1(B^3,S^2)$ such that $u \in C^\infty(B^3 \sim A)$ for some finite subset $A$ of the $z$-axis and such that $u$ coincides near each point $a \in A$ with either $\pm \Lambda(x-a)$ or $\pm \Psi(x-a)$. 
We need the analogue of [BZ, Th.4] which, because of axial symmetry requires a different proof.

**Theorem 1.1** The set $\mathcal{R}_{AS}$ is strongly $H^1$ dense in the family of axially symmetric $H^1$ maps from $B^3$ to $S^2$.

**Proof** Suppose $u \in H^1(B^3, S^2)$ which is axially symmetric and $\epsilon > 0$. Choose $\delta > 0$ so that

$$\int_{\{r < 2\delta\}} |\nabla u|^2 \, dx < \frac{1}{4} \epsilon.$$

On the region $B^3 \cap \{r < 2\delta\}$, $u$ may be represented as above by an angle function $\varphi$ defined on a planar domain. By smoothing $\varphi$, one easily finds an axially symmetric mapping $v \in H^1(B^3 \cap \{r > \delta\}, S^2)$ so that

$$\int_{\{r > \delta\}} |\nabla u - \nabla v|^2 \, dx < \frac{1}{4} \epsilon.$$

Here we may also insist that $v$ have the same trace as $u$ on $\partial B^3 \cap \{r > \delta\}$. It is similarly not difficult to find a finite energy axially symmetric map $w : B^3 \cap \{\delta > r > 2\delta\} \rightarrow S^2$ so that

$$w = v \quad \text{on} \quad B^3 \cap \{r = 2\delta\},$$

$$w = u \quad \text{on} \quad (B^3 \cap \{r = \delta\}) \cup (\partial B^3 \cap \{2\delta > r > \delta\}),$$

and

$$\int_{\{2\delta > r > \delta\}} |\nabla w|^2 \, dx < \frac{1}{4} \epsilon.$$

Finally we define $u_\epsilon : B^3 \rightarrow S^2$ to $v$ on $B^3 \cap \{r > 2\delta\}$ and, on the cylinder on $B^3 \cap \{r < 2\delta\}$, to be the least energy axially symmetric map have trace $u$ on $\partial B^3 \cap \{r < 2\delta\}$ and trace $v$ on $B^3 \cap \{r = 2\delta\}$. So, by energy minimality,

$$\int_{\{2\delta > r\}} |\nabla u_\epsilon|^2 \, dx$$

$$\leq \int_{\{\delta > r\}} |\nabla u|^2 \, dx + \int_{\{2\delta > r > \delta\}} |\nabla w|^2 \, dx$$

$$< \frac{1}{2} \epsilon,$$

hence,

$$\int_{B^3} |\nabla u - \nabla u_\epsilon|^2 \, dx$$

$$\leq \int_{\{2\delta > r\}} |\nabla u|^2 \, dx + \int_{\{2\delta < r\}} |\nabla u_\epsilon|^2 \, dx +$$

$$\int_{\{2\delta < r\}} |\nabla u - \nabla v|^2 \, dx$$

$$< \epsilon.$$
By [HKL2, section 4], \( u_\varepsilon \) is asymptotic to one of the four maps, \( \pm \Lambda(x - a) \) or \( \pm \Psi(x - a) \), at each singular point \( a \). It is now easy to keep the above strict inequality after modifying \( u_\varepsilon \) to make it actually coincide with its tangent map in some neighborhood of each singularity. Thus \( u_\varepsilon \) is the desired nearby function in \( \mathcal{R}_{AS} \). \( \Box \)

**Proposition 1.2** Suppose \( u \) belongs to the family \( \mathcal{R}_{AS} \).

1. \( \deg(u|_{\partial B}) \in \{-1, 0, 1\} \) for every ball \( B \subset B^3 \) centered on the \( z \)-axis. In particular, no two consecutive singularities on the axis can both have degree 1 or both have degree -1.

2. Let \( m^\pm \) respectively, \( n^\mp \) denote the number of singularities at which \( u \) has \( \pm \Lambda \) (respectively, \( \pm \Psi \)) as a tangent map. Then

\[
\deg(u|_{\partial B}) = m^+ + n^- - n^+ - n^-.
\]

**Proof** Suppose the ball \( B \) has radius \( r \) and is centered at \((0,0,a)\). By axial symmetric, \( u|_{\partial B} \) induces a map from the meridian \( \partial B \cap \{x_1 = 0, x_2 \geq 0\} \) to the meridian \( S^2 \cap \{x_1 = 0, x_2 \geq 0\} \) which also sends the poles \( \{(0,0,a-r),(0,0,a+r)\} \) to the poles \( \{(0,0,-1),(0,0,1)\} \). We easily check that \( u|_{\partial B} \) has degree +1 if \( u \) interchanges the poles, and has degree 0 if \( u \) maps both poles to the pole.

To prove (2), we use degree theory for the smooth spatial vectorfield \( u|_{\partial B \sim \text{Sing}(u)} \), and note that \( \deg(\pm \Lambda|_{S^2}) = \pm 1 \) and \( \deg(\pm \Psi|_{S^2}) = \mp 1 \). \( \Box \)

Suppose \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^3 \) and \( u \in H^1(\Omega,S^2) \). Consider the \( L^1 \) vector field

\[
D(u) = (u \cdot u_y \wedge u_x, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y).
\]

Let \( g = u|_{\partial \Omega} \in H^1(\partial \Omega,S^2) \). Following [N], the degree of \( g \) is defined as

\[
\deg(g) = \frac{1}{4\pi} \int_{\partial \Omega} \text{Jac}(g) \, d\mathcal{H}^2
\]

where \( \text{Jac}(g) = D(u) \cdot n \) on \( \partial \Omega \) and \( n \) is the exterior unit normal of \( \Omega \). If \( u, v \in H^1(\Omega,S^2), u|_{\partial \Omega} = v|_{\partial \Omega} \in H^1(\partial \Omega,S^2) \), then \( \deg(u|_{\partial \Omega}) = \deg(v|_{\partial \Omega}) \).

From Theorem 1.1 and Proposition 1.2, we infer that if \( u \) is axially symmetric on an open ball \( B \) centered on the \( z \)-axis and \( u|_{\partial B} \in H^1 \), then the degree of \( u|_{\partial B} \) is either -1, 0, or 1. \( \Box \)

1.3 \textbf{L Energy of an Axially Symmetric Map}
The key notion developed by Brezis, Bethuel and Coron in [BBC] is the $L$-energy

$$L(u) = \frac{1}{4\pi} \sup_{\xi: \Omega \to \mathbb{R}, \|\nabla \xi\|_\infty \leq 1} \left\{ \int_\Omega D(u) \cdot \nabla \xi \, dx - \int_{\partial \Omega} (\text{Jac}(g)) \xi \, d\sigma \right\}$$

which is defined whenever $u \in H^1(\Omega, S^2)$, and $g = u|_{\partial \Omega} \in H^1(\partial \Omega, S^2)$. Here, as in 1.2, $\text{Jac}(g) = D(u) \cdot n$ on $\partial \Omega$ where $D(u) = (u \cdot u_y \wedge u_z, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y)$, and $n$ is the exterior unit normal of $\Omega$. In case $\text{deg}(u|_{\partial \Omega}) = 0$ and $u$ is smooth except at a finite number of singularities $p_i$ of degree $+1$ and $n_i$ of degree $-1$, one has, by [BCL], the elementary formulas

$$\text{div} D(u) = 4\pi \left( \sum_i \delta_{p_i} - \sum_i \delta_{n_i} \right) \text{ in } \mathcal{D}'(\Omega),$$

$$L(u) = \sup_{\xi: \Omega \to \mathbb{R}, \|\nabla \xi\|_\infty \leq 1} \left\{ \sum_i \xi(p_i) - \sum_i \xi(n_i) \right\},$$

the latter expression being the length of the minimal connection or least mass 1-current with boundary $\sum_i \delta_{p_i} - \sum_i \delta_{n_i}$.

Even in the general nonsmooth case $u \in H^1(\Omega, S^2)$ with $\text{deg}(u|_{\partial \Omega}) = 0$, Giaquinta, Modica and Soucek [GMS1], [GMS2] have shown that

$$L(u) = M(I_u)$$

for some 1-dimensional integer multiplicity rectifiable current $I_u$ in $\hat{\Omega}$ (for some bounded $\hat{\Omega} \subset \subset \Omega$) which minimizes mass among 1-dimensional rectifiable currents $I$ in $\hat{\Omega}$ that have spt$(I)$ in $\hat{\Omega}$ and that satisfy the equation

$$-\partial I \times [S^2] = \partial [\text{graph}(u)] \perp (\Omega \times S^2).$$

Here we will discuss some important restrictions on $I_u$ that are imposed by the axially symmetric condition.

**Proposition 1.3** Suppose $u \in H^1(B^3, S^2)$ is axially symmetric with $\text{deg}(u|_{\partial B^3}) = 0$. Then the associated current $I_u$ is unique and equals $\pm [J_u]$, for some Lebesgue measurable subset $J_u$ of $B^3 \cap (z-\text{axis})$. Thus the 1-density of the measure associated with $I_u$ is either one or zero at Lebesgue almost all points on the $z$-axis, and the orienting vectorfield is $\pm e_3 \cdot \chi_A$, where $\chi_A$ is the characteristic function of a set $A$.

**Proof** For the uniqueness, note that the minimizing property implies that any such $I_u$ must have support in the $z$-axis. The difference of two such rectifiable currents
would have boundary 0 and so, by the constancy theorem, [F, 4.1.7] be a multiple of \(\mathbb{R} \times \{(0,0)\}\). Since it also has finite mass, this multiple must zero.

To see the structure of \(I_u\) consider first the case \(u \in \mathcal{R}_{AS}\). Here \(u\) has, by 1.2, an even number of singularities whose degrees alternate between -1 and +1 as one ascends the \(z\)-axis. The associated current \(I_u\) is determined by simply choosing alternate intervals and orienting them all with \(+e_3\) or \(-e_3\) according to whether the top singularity has degree +1 or -1.

In the general case we may choose by Theorem 1.1 a sequence \(u_i \in \mathcal{R}_{AS}\) that is strongly convergent to \(u\). We may assume that all members of the sequence of currents \(I_{u_i}\) are orineted in the same direction, say \(+e_3\). Here weak convergence of the currents \(I_{u_i}\) corresponds to convergence of the characteristic functions \(\chi_{I_{u_i}}\) in \(\mathcal{D}'\). After passing the subsequence, these functions convergence weakly in \(L^* = L^\infty\) to a function \(f\) on the \(z\)-axis with \(0 \leq f \leq 1\). However, by [GMS1], the currents \(I_{u_i}\) converge weakly to \(I_u\), and \(I_u\) is an integer multiplicity rectifiable current. Thus \(f\) has values 0 or 1 almost everywhere and \(I_u\) corresponds to the Lebesque measurable set \(J_u = \{(0,0,z) : f(0,0,z) = 1\}\) with orientation \(+e_3\). \qed
CHAPTER 2
MAXIMUM PRINCIPLE AND UNIQUENESS

2.1 Introduction

As in Chapter 1, a map $u : B^3 \rightarrow S^2$ is called axially symmetric if in cylindrical coordinates

$$u(r, \theta, z) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

for some real-valued function $\varphi(r, z)$. Let $D = \{(r, z) : r \geq 0, r^2 + z^2 \leq 1\}$. If $u$ is a smooth harmonic map, then

$$\frac{\partial}{\partial r}(r \frac{\partial \varphi}{\partial r}) + \frac{\partial}{\partial z}(r \frac{\partial \varphi}{\partial z}) - \frac{\sin 2\varphi}{2r} = 0$$

in $D$, $\varphi$ is continuous and $\sin \varphi(0, z) = 0$ for all $z$. We may assume that $\varphi(0, z) = 0$ for all $z$.

In [Z], D. Zhang proved the existence of smooth axially symmetric harmonic maps, under the assumption that $\max_{\partial D} |\varphi| < \pi$. Then R. Hardt, D. Kinderlehrer and F.H. Lin [HKL] extended Zhang's result to the case that $\max_{\partial D} |\varphi| \leq \pi$. Here, we give an answer to the uniqueness problem for smooth axially symmetric harmonic maps. Our main results are:

**Theorem 2.1** Let $\varphi$ be a continuous solution of (2.1) in $D$. If $|\varphi| \leq k\pi$ on $\partial D$, then $|\varphi| \leq k\pi$ in $D$.

**Theorem 2.2** Let $\varphi, \phi$ be continuous solutions of 2.1 in $D$. If $\varphi = \phi$ on $\partial D$, then $\varphi = \phi$ in $D$.

In fact, the proof of Theorem 2.2 only requires that $\varphi - \phi$ is continuous in $D$. Thus, we can draw the same conclusion for axially symmetric harmonic maps with isolated singularities. For example,

**Theorem 2.3** Let $\varphi, \phi \in C^0(D \sim \{(0, 0)\})$ be solutions of 2.1. Suppose $\varphi = \phi$ on $\partial D$. If $\varphi$ and $\phi$ have the same asymptotic behavior at the origin, i.e., $\varphi - \phi$ is continuous in $D$, then $\varphi = \phi$ in $D$.

In section 2.6, we will give a slightly improved version of Theorem 2.1. Also we will prove that if $\varphi$ and it's first derivatives vanish on a segment of the $z$-axis, then $\varphi \equiv 0$. 
2.2 A Maximum Principle

Maximum Principle [GT, p35] Suppose \( u \geq 0 \) and is a super-solution of a linear elliptic equation, i.e.,

\[
a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b^i \frac{\partial u}{\partial x_i} + cu \leq 0.
\]

If \( u = 0 \) at an interior minimum, then \( u \) is identically equal to 0, irrespective of the sign of the function \( c \).

Corollary Let \( \varphi \) and \( \phi \) be solutions of (2.1) in \( D \) and \( \varphi \geq \phi \). If at some point \( (r_0, z_0) \), \( r_0 > 0 \), \( \varphi(r_0, z_0) = \phi(r_0, z_0) \), then \( \varphi = \phi \) in \( D \).

Proof Since \( \varphi - \phi \geq 0 \) and \( \varphi - \phi \) satisfies the equation

\[
\frac{\partial}{\partial r} \left( r \frac{\partial (\varphi - \phi)}{\partial r} \right) + \frac{\partial}{\partial z} \left( z \frac{\partial (\varphi - \phi)}{\partial z} \right) - \frac{\cos(\varphi + \phi) \sin(\varphi - \phi)}{r} (\varphi - \phi) = 0,
\]

by the Maximum Principle, \( \varphi - \phi \equiv 0 \) in \( D \) if \( \varphi - \phi \) vanishes at some point \( (r_0, z_0) \) with \( r_0 > 0 \).

2.3 Proof of Theorem 2.1

Let \( \tau \) be a \( z \)-independent solution of (2.1). Then as a function of \( r \), \( \tau \) satisfies the ordinary differential equation:

\[
rr'' + r' - \frac{\sin 2\tau}{2r} = 0.
\]

By multiplying the equation by \( 2rr' \), we obtain \( (r^2(r')^2)' = (\sin^2 \tau)' \). Thus

\[
r^2(r')^2 = \sin^2 \tau - c.
\]

In [Z], D. Zhang studied this ordinary differential equation in the case \( c = 0 \) and obtained a family of solutions \( \{\tau_t, 0 < t < 1\} \) so that

a) \( \tau_t(0) = 0 \);
b) for any \( r > 0 \), \( \tau_t(r) \to 0 \) as \( t \to 0 \);
c) \( \tau_{t_1} < \tau_{t_2} \) if \( t_1 < t_2 \);
d) for any \( r > 0 \), \( \tau_t(r) \to \pi \) as \( t \to 1 \);
e) for any point \( (r, z) \), \( r > 0 \), \( 0 < z < \pi \), there is a solution \( \tau_t \) so that \( \tau_t(r) = z \);
f) for any \( 1 > t > 0 \), \( \tau_t(r) \to \pi \) as \( r \to \infty \).
Also, one can easily check that for all positive integer \( k, \{ \tau_t + k\pi, 0 < t < 1 \} \) forms another family of solutions, so that \( k\pi \leq \tau_t + k\pi \leq (k + 1)\pi \).

Let \( \varphi \) be a continuous solution of (2.1), \(|\varphi| \leq k\pi \) on \( \partial D \) and \( \varphi(0, z) = 0 \) for all \( z \).

Let \( M = \max_D \varphi = \varphi(r_0, z_0) \). By assumption, \( r_0 > 0 \).

Case 1 \( \quad M = m\pi \) for some integer \( m > k \).

The constant function \( \psi = m\pi \) is a solution of 2.1 and \( \psi \geq \varphi \). However, \( \psi(r_0, z_0) = m\pi = \varphi(r_0, z_0) \). By the maximum principle, it is impossible.

Case 2 \( \quad m\pi < M < (m + 1)\pi \) for some integer \( m > k \).

Let \( \{ \tau_t, 0 < t < 1 \} \) be the family of solutions of (2.2) described in the above. Define \( \psi_t = \tau_t + m\pi \). Then \( \psi_t \)'s are also solutions of (2.1). Since the set \( \{ (r, z) : \varphi(r, z) > m\pi \} \) is contained in the interior of \( D \), \( t = \min \{ s : \psi_s \geq \varphi \} \) exist. Moreover \( \psi_t \geq \varphi \) in \( D \) and there is a point \((r_1, z_1)\), \( r_1 > 0 \) such that \( \psi(r_1, z_1) = \varphi(r_1, z_1) \). Again, by the maximum principle, we obtain a contradiction. \( \square \)

### 2.4 More \( z \)-independent Solutions

Here we consider the following initial value problem:

\[
(2.3) \quad r\varphi'' + \varphi' - \frac{\sin2\varphi}{2r}, \quad \varphi(r_0) = \frac{\pi}{2}
\]

for some \( r_0 > 0 \).

As in section 2.3, we can reduce the equation to the form:

\[
(2.4) \quad r^2(\varphi')^2 = \sin^2 \varphi - c, \quad \varphi(r_0) = \frac{\pi}{2}.
\]

From section 2.3, there is a solution \( \varphi_0 \) such that \( r^2(\varphi_0')^2 = \sin^2 \varphi_0 \) and \( \varphi_0(r_0) = \frac{\pi}{2} \), \( \varphi_0(0) = 0 \). By minimizing the functional

\[
\mathcal{F}(\mu) = \int_{r_0}^{s} r(\mu')^2 - \frac{\sin^2 \mu}{r} \ d r
\]

among functions \( \mu(s) = z \) and \( \frac{\pi}{2} \leq \mu(r) \leq \varphi_0(r) \) for \( r_0 \leq r \leq s \), we can see that there is a solution \( \mu \) such that \( \mu(r_0) = \frac{\pi}{2} \) and \( \mu(s) = z \), for any \( \frac{\pi}{2} < z < \varphi_0(s) \). In fact, this function is a solution to (2.4) for some \( c > 0 \). Therefore we obtain a family of solutions

\[ F = \{ \mu_c(r), r_0 < r < \infty; 0 < c < 1 \} \]
such that $\frac{\pi}{2} < \mu_c(r) < \varphi_0(r)$, for $r > r_0$. Moreover one can check that $\lim_{c \to 0} \mu_c(r) = \varphi_0(r)$ and $\lim_{c \to 0} \mu_c(r) = \frac{\pi}{2}$ for any $r > r_0$.

Next, choose any $x > 0$. We minimize the functional $F$ among functions $\mu(x) = x$ and $\varphi_0(r) < \mu(r) < \frac{\pi}{2}$, for $x < r < r_0$. Then we obtain a function $\mu$ which is a solution of (2.3), and thus a solution of (2.4) for some $c$. By the uniqueness of solution of ordinary differential equation, this $\mu$ must be an extension of a member in the family $F$. Since $x$ is arbitrary, the function $\mu_c$ lives on the whole positive real line, $\{ r : r > 0 \}$. Also, the function is monotone increasing and $\lim_{r \to 0} \mu_c(r)$ exist. From equation (2.4), $\sin^2 \mu_c(0) = c$. Recall that we obtain equation (2.4) by multiplying $2r\varphi'$ to equation (2.3):

$$2r^2 \mu_c'' + 2r(\mu_c')^2 = \mu_c' \sin 2\mu_c.$$

When $c \neq 0$, $\sin 2\mu_c(0) \neq 0$. Therefore, $\mu_c'(0) = 0$.

Now we can rewrite the family $F$ as:

$$F(r_0) = \{ \mu_c(r) : 0 \leq r < \infty, \mu_c(r_0) = \frac{\pi}{2}; 0 < c < 1 \}.$$

When $r > r_0$, $\frac{\pi}{2} < \mu_c(r) < \varphi_0(r)$. When $0 \leq r < r_0$, $\varphi_0(r) < \mu_c(r) < \frac{\pi}{2}$. Note that since $\mu_c(0) \neq 0$, it cannot generate a smooth axially symmetric harmonic map, in contrary to those solutions obtained by D. Zhang.

**Remark** Here we basically solve equation (2.4) with $1 > c > 0$. It is clear that there is no solution for $c > 1$. When $c = 1$, the constant function $\frac{\pi}{2}$ is the only solution. When $c = 0$, the solution is the one obtained by D. Zhang. Of course, one can also solve the equation (2.4) with $c < 0$. However, one can also see that the solutions go to $-\infty$ as $r \to 0$.

### 2.5 Proof of Theorem 2.2

Suppose $\varphi, \phi$ be two solutions to (2.1), $\varphi \geq \phi$ on $\partial D$. Let $\psi = \varphi - \phi$, and let $M = \max_D \psi$.

**Case 1** $(2m - 1)\pi < M \leq 2m\pi$, for some $m > 0$.

In the region $\Omega_1 = \{ (r, z) : \psi(r, z) > (2m - 1)\pi \}$, $\sin \psi \leq 0$, so

$$\frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) + \frac{\partial}{\partial z}(r \frac{\partial \psi}{\partial z}) = \cos(\varphi + \phi) \sin \psi \geq \frac{\sin \psi}{r}.$$

...
If $M = 2m\pi$, the constant function $\nu = M$ satisfies

$$\frac{\partial}{\partial r} \left( r \frac{\partial \nu}{\partial r} \right) + \frac{\partial}{\partial r} \left( r \frac{\partial \nu}{\partial z} \right) - \sin \frac{\nu}{r} = 0.$$ 

Therefore in $\Omega_1$, $\nu - \psi \geq 0$ and

$$\frac{\partial}{\partial r} \left( r \frac{\partial (\nu - \psi)}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial (\nu - \psi)}{\partial z} \right) \leq \frac{\cos \frac{(\nu + \psi)}{2}}{r} \sin \frac{\nu - \psi}{2} (\nu - \psi).$$

Hence, by the maximum principle, $M \neq 2m\pi$.

If $(2m - 1)\pi < M < 2m\pi$, first choose a function $\varphi_0$ which is a solution to equation 2.4 with $c = 0$ and $2\varphi_0 + 2(m - 1)\pi > \psi$ in $\Omega_1$. Let $\varphi_0(r_0) = \frac{\pi}{2}$ at some point $r_0 > 0$ and $F(r_0)$ be the family of solutions to equation (2.4) obtained in section 2.4. Define $\nu_c = 2(m - 1)\pi + 2\mu_c$ for each $1 > c > 0$. Then

$$\frac{\partial}{\partial r} \left( r \frac{\partial \nu_c}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial \nu_c}{\partial z} \right) - \frac{\sin \nu_c}{r} = 0.$$

Let $\alpha = \max \{ c : \nu_c \geq \psi \}$. $\nu_c - \psi \geq 0$ in $\Omega_1$ and vanishes at some point in $\Omega_1$. Also, from (2.5),

$$\frac{\partial}{\partial r} \left( r \frac{\partial (\nu_\alpha - \psi)}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial (\nu_\alpha - \psi)}{\partial z} \right) - \frac{\cos \frac{(\nu_\alpha + \psi)}{2}}{r} \sin \frac{(\nu_\alpha - \psi)}{2} (\nu_\alpha - \psi) \leq 0.$$

By the maximum principle, we obtain a contradiction.

**Case 2** \hspace{1em} $2m\pi < M \leq (2m + 1)\pi$, for some $m \geq 0$.

In the region $\Omega_2 = \{(r, z) : \psi(r, z) > 2m\pi\}$, $\sin \psi \geq 0$. So

$$\frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial \psi}{\partial z} \right) = \cos (\varphi + \phi) \frac{\sin \psi}{r} \geq \frac{\sin \psi}{r}.$$  \hspace{1em} (2.6)

If $M = (2m + 1)\pi$, the constant function $\gamma = M$ satisfies

$$\frac{\partial}{\partial r} \left( r \frac{\partial \gamma}{\partial r} \right) + \frac{\partial}{\partial r} \left( r \frac{\partial \gamma}{\partial z} \right) + \frac{\sin \gamma}{r} = 0.$$

Therefore in $\Omega_2$, $\gamma - \psi \geq 0$ and

$$\frac{\partial}{\partial r} \left( r \frac{\partial (\gamma - \psi)}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial (\gamma - \psi)}{\partial z} \right) \leq - \frac{\cos \frac{(\gamma + \psi)}{2}}{r} \sin \frac{(\gamma - \psi)}{2} (\gamma - \psi).$$

Hence, by the maximum principle, $M \neq (2m + 1)\pi$.

If $2m\pi < M < (2m + 1)\pi$, first choose a function $\phi_0$ which is a solution to equation (2.4) with $c = 0$ and $(2m + 1)\pi - 2\phi_0 > \psi$ in $\Omega_2$. Let $\phi_0(r_0) = \frac{\pi}{2}$ at some point $r_0 > 0$.
and \( F(r_0) \) be the family of solutions to equation (2.4) obtained in section 2.4. Define \( \gamma_c = (2m + 1)\pi - 2\mu_c \) for each \( 1 > c > 0 \). Then
\[
\frac{\partial}{\partial r}(r \frac{\partial \gamma_c}{\partial r}) + \frac{\partial}{\partial z}(r \frac{\partial \gamma_c}{\partial z}) + \frac{\sin \gamma_c}{r} = 0.
\]

Let \( \beta = \max \{ c : \gamma_c \geq \psi \} \). \( \gamma_\beta - \psi \geq 0 \) in \( \Omega_2 \) and vanishes at some point in \( \Omega_2 \). Also, from (2.6),
\[
\frac{\partial}{\partial r}(r \frac{\partial (\gamma_\beta - \psi)}{\partial r}) + \frac{\partial}{\partial z}(r \frac{\partial (\gamma_\beta - \psi)}{\partial z}) + \frac{\cos \frac{(\gamma_\beta + \psi)}{2} \sin \frac{(\gamma_\beta - \psi)}{2}}{r} (\gamma_\beta - \psi) \leq 0.
\]

By the maximum principle, we obtain a contradiction. \( \square \)

The proof is completed.

2.6. Other Results
The following result is a slight improvement of Theorem 2.1.

**Theorem 2.4** Let \( \varphi \) be a solution of (2.1) in \( D \). \( \varphi(0, z) = 0 \) for all \( z \), and \( m\pi \leq \max_D \varphi \leq (m + \frac{1}{2})\pi \) for some integer \( m \), then \( \max_D \varphi \leq \max_D \varphi \).

**Proof** From Theorem 2.1, we only have to show that \( M \) does not lie in the interval \( (\max_D \varphi, (m + 1)\pi) \), where \( M = \max_D \varphi \).

Suppose \( M > (m + \frac{1}{2})\pi \). As before, we can find a function \( \nu \) from a family \( F(r_0) \), with appropriate choice of \( r_0 \), so that \( m\pi + \nu \geq \varphi \) in the region \( \{(r, z) : \varphi(r, z) > (m + \frac{1}{2})\pi\} \) and they have a point in common there. Since \( m\pi + \nu \) is also a solution to (2.1), it is impossible.

Thus \( m\pi < M < (m + \frac{1}{2})\pi \). In the region \( \{(r, z) : \varphi(r, z) > m\pi\} \), \( \cos \varphi > 0 \). Since
\[
\frac{\partial}{\partial r}(r \frac{\partial \varphi}{\partial r}) + \frac{\partial}{\partial z}(r \frac{\partial \varphi}{\partial z}) - \frac{\cos \varphi}{r} \sin \varphi = 0,
\]

\( \varphi \) cannot have an interior maximum. Hence we have the conclusion. \( \square \)

**Theorem 2.5** Let \( \varphi \geq 0 \) be a solution of (2.2) in \( D \). If
\[
\varphi(0, z) = 0 = \frac{\partial}{\partial r} \varphi(0, z),
\]
for all \( z \), then \( \varphi \) is identically equal to 0 in \( D \).
**Proof** By the assumptions, \( \varphi \) will generate an axially symmetric harmonic map. Thus \( \varphi \) is analytic in \( D \sim \{(r, z) \in \partial D : r \geq 0\} \). Let \( \varphi = r^2 \phi \) for some positive smooth function \( \phi \). Note that

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{\sin 2\varphi}{2r^2} - \frac{1}{r} \frac{\partial \varphi}{\partial r},
\]

and

\[
\lim_{r \to 0} \frac{1}{r} \frac{\partial \varphi}{\partial r}(r, z) = 2\phi(0, z), \quad \lim_{r \to 0} \frac{\sin 2\varphi(r, z)}{2r^2} = \phi(0, z).
\]

Therefore, when \( r \) is small enough,

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial^2 \varphi}{\partial z^2} \leq 0.
\]

By the boundary point lemma [GT, Lemma 3.4], \( \varphi \equiv 0. \) \( \square \)
CHAPTER 3
PARTIAL REGULARITY FOR RELAXED ENERGY
MINIMIZERS, PART I

3.1 Minimization of Relaxed Energies Among Axially Symmetric Maps

For \( u \in H^1(B^3, S^2) \), as in Chapter 1, let \( E(u) \) denote the ordinary energy of \( u \)

\[
E(u) = \int_{B^3} |\nabla u|^2 \, dx,
\]

and \( L(u) \) be as in section 1.3. For each \( \lambda \in [0, 1] \), we will consider the functional

\[
E_\lambda = E + 8\pi \lambda L.
\]

When \( \lambda = 0 \), then \( E_0 \) is the usual energy functional \( E \).

The weak \( H^1 \) lower semicontinuity of \( E + 8\pi \lambda L \) is proved in [BBC] and leads to the existence of \( E + 8\pi \lambda L \) minimizers among all \( H^1 \) maps. Such minimizers are still weak solutions of the harmonic map equation. These problems are formulated in the context of cartesian currents by M. Giaquinta, G. Modica, J. Soucek [GMS1], [GMS2] who interpret \( L(u) \) as the mass of a 1-dimensional integer multiplicity rectifiable current \( I_u \) and who prove the finiteness of the 1-dimensional measure of the singularities of an \( E + 8\pi \lambda L \) minimizer. This interesting result may not be the optimal estimate. In fact, for \( 0 < \lambda < 1 \), Brezis and Bethuel [BB] proved that the singularities are only isolated points, just as in the case of \( E \)-minimizers [SU1], [SU2].

We will treat these problems in the axially symmetric context. In this chapter, we consider the case \( 0 < \lambda < 1 \). In the next chapter, we consider the case \( \lambda = 1 \).

Lemma 3.1 For any \( \lambda \in (0, 1] \) and any degree 0, axially symmetric map \( g \in H^1(\partial B^3, S^2) \), there exists a map \( u_\lambda \in H^1(B^3, S^2) \) that minimizes \( E_\lambda \) among all axially symmetric maps having trace \( g \) on \( \partial B^3 \). The map \( u_\lambda|_{B^3} \) is real analytic away from the \( z \)-axis.

Proof Recalling the proof of [BBC, Th. 3], we see that, for every \( \lambda \in (0, 1] \), the functional \( E_\lambda \) is lower semicontinuous for the weak \( H^1 \) topology. Since \( g(x/|x|) \) provides at least one map with \( u|_{\partial B^3} = g \) and \( E_\lambda(g(x/|x|)) < \infty \), the existence readily follows.
To see the regularity off the z-axis, we assume $0 < r_0 < 1$, $-1 < z_0 < 1$, and
\[ \sigma = (r_0^2 + z_0^2)^{1/2} < 1. \]
Then the ring shaped domain
\[ \Omega = \{(x, y, z) : ((x^2 + y^2)^{1/2} - r_0)^2 + (z - w_0)^2 < \sigma^2 \} \]
does not intersect the z-axis and hence misses the support of $I_u$. It follows that $u|_\Omega$ minimizes energy among axially symmetric maps and is thus, by [2] or [HKL2], real analytic on $\Omega$.

**Proposition 3.1** Let $g \in H^1(\partial B^3, S^2)$, $\deg(g) = 0$. Let $u_\lambda$ be $E_\lambda$ minimizers, $0 < \lambda \leq 1$, among all maps in $H^1(B^3, S^2)$ having trace $g$ on $\partial B^3$.

1. If $0 < \lambda_1 < \lambda_2 \leq 1$, $u_i = u_{\lambda_i}$ for $i = 1, 2$, then
   \[ \int_{B^3} |\nabla u_1|^2 \, dx \leq \int_{B^3} |\nabla u_2|^2 \, dx. \]

2. There is a subsequence $\lambda_i$ such that $u_i = u_{\lambda_i}$ converges strongly in $H^1(B^3, S^2)$ to a map $u_1$ which is a $E_1$ minimizer.

**Proof** Suppose (1) is not true,
\[ \int_{B^3} |\nabla u_1|^2 \, dx > \int_{B^3} |\nabla u_2|^2 \, dx. \]

By the minimality,
\[ \int_{B^3} |\nabla u_1|^2 \, dx + \lambda_1 8 \pi L(u_1) \leq \int_{B^3} |\nabla u_2|^2 \, dx + \lambda_1 8 \pi L(u_2). \]

Therefore $L(u_1) < L(u_2)$. Also,
\[ \int_{B^3} |\nabla u_2|^2 \, dx + \lambda_1 8 \pi L(u_2) \]
\[ = \int_{B^3} |\nabla u_2|^2 \, dx + \lambda_2 8 \pi L(u_2) - (\lambda_2 - \lambda_1) 8 \pi L(u_2) \]
\[ < \int_{B^3} |\nabla u_1|^2 \, dx + \lambda_2 8 \pi L(u_1) - (\lambda_2 - \lambda_1) 8 \pi L(u_1) \]
\[ = \int_{B^3} |\nabla u_1|^2 \, dx + \lambda_1 8 \pi L(u_1). \]

It contradicts the minimality of $u_1$. This proves (1).

From (1) $\int_{B^3} |\nabla u_\lambda|^2 \, dx$ forms an increasing sequence. Let $v$ be a smooth map from $B^3$ to $S^2$. Since $L(v) = 0$,
\[ \int_{B^3} |\nabla u_\lambda|^2 \, dx + \lambda 8 \pi L(u_\lambda) \leq \int_{B^3} |\nabla v|^2 \, dx. \]
So \( \int_{B^3} |\nabla u_\lambda|^2 \, dx \) is bounded and there is a sequence \( \lambda_i \) such that \( u_i = u_{\lambda_i} \) converges weakly to a map \( u_1 \). For any smooth map \( w \) from \( B^3 \) to \( S^2 \),

\[
\int_{B^3} |\nabla u_i|^2 \, dx + \lambda_i 8\pi L(u_i) \leq \int_{B^3} |\nabla w|^2 \, dx.
\]

By the lower semicontinuity of the functional \( E_1 \),

\[
\int_{B^3} |\nabla u_1|^2 \, dx + 8\pi L(u_1) \leq \int_{B^3} |\nabla w|^2 \, dx.
\]

From [BBC],

\[
\inf \{ E_1(u) : u \in H^1(B^3, S^2) \} = \inf \{ E(u) : u \in C^\infty(B^3, S^2) \}.
\]

Therefore our \( u_1 \) minimizes \( E_1 \). By (1), \( E(u_i) \leq E(u_1) \) for all \( i \). The lower semicontinuity of \( E \) implies that

\[
E(u_1) \leq \lim_{i \to \infty} E(u_i) \leq E(u_1).
\]

Therefore

\[
E(u_1) = \lim_{i \to \infty} E(u_i),
\]

and \( u_i \) converges to \( u_1 \) strongly in \( H^1(B^3, S^2) \).

\[\square\]

3.2 Cartesian Currents

To study regularity of \( E_\lambda \) minimizers along the axis, it is useful to be able to localize the problem as well as study limits of minimizers. Note that one cannot simply localize by restricting the mapping because the \( L \) term is defined globally.

We now consider the slightly more general notion of an axially symmetric cartesian current. These arise as the cartesian current [GMS1], [GMS2] limits of graphs of axially symmetric maps. Modifying [GMS1], [GMS2] slightly, we define, for any axially symmetric open set \( \Omega \) in \( \mathbb{R}^3 \), \( \text{cart}_{\text{AS}}(\Omega, S^2) \) as the class of 3-dimensional currents \( T \) in \( \mathbb{R}^3 \times S^2 \) such that

\[
\mathrm{spt}(\partial T) \subset \partial \Omega \times S^2 \quad \text{and}
\]

\[
T = \llbracket \text{graph}(u_T) \rrbracket + \llbracket S^2 \rrbracket
\]
where \( u_T \in H^1(\Omega, S^2) \) is axially symmetric and \( I_T = \pm[J_T] \) for some Legesque measurable subset \( J_T \) of \( \Omega \cap (z-\text{axis}) \). Here \([A]\) denotes the current corresponding to integration over \( A \) with the obvious suitable orientation.

Here the current \( I_T \) is actually almost determined by the map \( u_T \). More precisely, since \( T \) has no boundary in \( \partial \Omega \times S^2 \), there are, for a fixed axially symmetric \( u_T \in H^1(\Omega, S^2) \) only 2 or 3 possibilities for the corresponding current \( I_T \). In fact, if \( \text{div} \ D(u) = 0 \) in \( \Omega \), i.e.,

\[
\int_\Omega D(u) \cdot \nabla \xi \, dx = 0 \quad \text{for all} \quad \xi \in C^1_0(\Omega, S^2),
\]

then one must have \( I_T = 0 \) or \( \pm[\Omega \cap (z-\text{axis})] \). If \( \text{div} D(u_T) \neq 0 \) in \( \Omega \), then

\[
\partial[\text{graph}(u_T)] = -\partial[J_T] \times [S^2] = \partial[\Omega \cap (z-\text{axis}) \sim J_T] \times [S^2]
\]

in \( \Omega \) for a unique \([J_T]\).

For any \( T \in \text{cart}_{AS}(\Omega, S^2) \) and \( 0 < \lambda \leq 1 \), we define

\[
\mathcal{D}_\lambda(T, \Omega) = \int_\Omega |\nabla u_T|^2 \, dx + 8\pi \lambda M(I_T).
\]

As in [GMS1], [GMS2] it follows that, for any 2-dimensional current \( Q \) occurring as the boundary of some axially symmetric cartesian current, the family

\[
\{ T \in \text{cart}_{AS}(\Omega, S^2) : \partial T = Q \}
\]

contains an element minimizing \( \mathcal{D}_\lambda \).

Next we consider the relation with \( E_\lambda \) minimizing mappings. For any axially symmetric \( u \in H^1(B^3, S^2) \) with \( \deg(u|_{\partial \Omega}) = 0 \) and any open ball \( \Omega \subset B^3 \) centered on the \( z \)-axis, we find that the current

\[
T_{u,\Omega} = [\text{graph}(u|_\Omega)] + (I_u \sqcap \Omega) \times [S^2]
\]

belongs to \( \text{cart}_{AS}(\Omega, S^2) \) and that

\[
I_{T_{u,\Omega}} = I_u.
\]

If \( u \) minimizes \( E_\lambda \), then \( T_{u,\Omega} \) minimizes \( \mathcal{D}_\lambda(\cdot, \Omega) \).

In fact, for any \( T \in \text{cart}_{AS}(\Omega, S^2) \) with \( \partial T = \partial T_{u,\Omega} \), we may let

\[
v = \begin{cases} 
  u & \text{on } B^3 \sim \Omega, \\
  u_T & \text{on } \Omega,
\end{cases}
\]
and see that, in \( B^3 \),

\[
\partial[\text{graph}(v)] = \partial[\text{graph}(u|_{B^3 - \Omega})] + \partial[\text{graph}(u_T|\Omega)]
\]
\[
= \partial(I_u \subseteq (B^3 - \Omega) \times \{S^2\}) + \partial((I_T \subseteq \Omega) \times \{S^2\})
\]
\[
= \partial((I_u \subseteq (B^3 - \Omega) + (I_T \subseteq \Omega)) \times \{S^2\}).
\]

Since \( \deg(v|\partial B^3) = 0 \), Proposition 1.3 implies that

\[
I_u \subseteq (B^3 - \Omega) + I_T \subseteq \Omega = I_v = \pm[I_v]
\]

for some measurable subset \( J_v \) of \( \Omega \cap (x\text{-axis}) \). Moreover, since \( u \) is \( E_\lambda \) minimizing and since \( v|_{\partial B^3} \equiv u|_{\partial B^3} \),

\[
\int_{B^3} |\nabla u|^2 \, dx + 8\pi\lambda \mathcal{M}(I_u) = E_\lambda(u)
\]
\[
\leq E_\lambda(v)
\]
\[
= \int_{B^3} |\nabla v|^2 \, dx + 8\pi\lambda \mathcal{M}(I_v).
\]

Subtracting \( \int_{B^3 - \Omega} |\nabla u|^2 dx + 8\pi\lambda \mathcal{M}(I_u \subseteq (B^3 - \Omega)) \) from both sides gives the desired minimizing property

\[
\mathcal{D}_\lambda(T_u, \Omega, \Omega) = \int_\Omega |\nabla u|^2 dx + 8\pi\lambda \mathcal{M}(I_u \subseteq \Omega)
\]
\[
\leq \int_\Omega |\nabla u_T|^2 dx + 8\pi\lambda \mathcal{M}(I_T \subseteq \Omega)
\]
\[
= \mathcal{D}_\lambda(T, \Omega).
\]

### 3.3 \( \epsilon \)-regularity

**Lemma 3.2** Let \( u \) be a minimizer of \( E + 8\pi\lambda L \), \( 0 \leq \lambda < 1 \), in the axially symmetric class. Suppose \( \varphi \) is the angle function \( u \). There exists \( \delta > 0 \), \( \delta < \pi/2 \), depending on \( \lambda \), so that if \( |\varphi| \leq \delta \) on \( \partial D \), then \( |\varphi| \leq \delta \) in \( D \).

In particular, if \( \lambda = 0 \), then we can take \( \delta = \pi/2 \).

**Proof** Suppose not. Let \( \Omega = \{(r, z) : \varphi(r, z) > \delta\} \). Define \( \phi : D \to [0, \pi] : \phi = \varphi \) in \( D \sim \Omega \) and \( \phi = \delta \frac{\pi - \phi}{\pi - \delta} \) in \( \Omega \). When \( \delta \) is small enough,

\[
\int_\Omega r\frac{\partial \phi}{\partial r} + r\frac{\partial \phi}{\partial z} + \frac{\sin^2 \phi}{r} \, drdz
\]
\[ \geq 2 \left( \frac{\delta}{\pi - \delta} \right)^2 \int_{\Omega} r \left( \frac{\partial \varphi}{\partial r} \right) + r \left( \frac{\partial \varphi}{\partial z} \right) + \frac{\sin^2 \varphi}{r} \ drdz. \]

However, by applying the proof of [BB, Lemma 1] to the axially symmetric case,

\[ \int_{\Omega} r \left( \frac{\partial \varphi}{\partial r} \right) + r \left( \frac{\partial \varphi}{\partial z} \right) + \frac{\sin^2 \varphi}{r} \ drdz \leq \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \int_{\Omega} r \left( \frac{\partial \phi}{\partial r} \right) + r \left( \frac{\partial \phi}{\partial z} \right) + \frac{\sin^2 \phi}{r} \ drdz. \]

We have a contradiction when \( \delta \) is small enough.

The last statement is a direct application of the result of [HKW]. \( \square \)

**Corollary 3.1** Let \( u \) be a \( E \)-minimizer of in the axially symmetric class. Suppose \( \varphi \) is the angle function of \( u \). Then \( \text{card}\{\text{singular points of } u\} \leq \text{card}\{(r, z) \in \partial D : \varphi(r, z) = \pi/2\} \).

**Proof** Let \( A = \{(r, z) \in \tilde{D} : \varphi(r, z) = \pi/2\} \). By lemma 3.2, \( A \) cannot bound a subdomain in \( D \). Therefore each connected component of \( A \) can hit the \( z \)-axis at most once. This proves the Corollary. \( \square \)

**Lemma 3.3** For each \( \lambda \in (0, 1] \) there is a positive \( \varepsilon_0 = \varepsilon_0(\lambda) \) so that if \( u : B^3 \to S^2 \) minimizes \( E_\lambda \) among axially symmetric maps and if \( E_\lambda(u) < \varepsilon_0 \), then \( u|_{B^3_{1/2}} \) is smooth.

**Proof** We only have to repeat the proof in [HKL2, 4.1] and apply Lemma 3.2. \( \square \)

### 3.4 Monotonicity Inequality and Tangent Maps

**Lemma 3.4** Suppose \( \lambda \in (0, 1] \) and \( u : B^3 \to S^2 \) minimizes \( E_\lambda \) among axially symmetric maps. For any point \( a \) on the \( z \)-axis with \( |a| < 1 \) and numbers \( 0 \leq r < s < 1 - |a| \),

\[ r^{-1} \int_{B_r(a)} |\nabla u|^2 dx + 8\pi \lambda M [I_u \mathcal{L} B_r(a)] \leq s^{-1} \int_{B_s(a)} |\nabla u|^2 dx + 8\pi \lambda M [I_u \mathcal{L} B_s^3(a)], \]

with equality if and only if

\[ u(x) = u \left( a + \frac{sx}{|x-a|} \right) \quad \text{for } r \leq |x-a| \leq s, \]
and \( I_u \) equals 0 or \( \pm [J] \) where \( J \) is either \( B^3 \cap (x\text{-axis}) \) or one of the two components of \( B^3 \cap (x\text{-axis}) \sim \{a\} \).

The proof is to repeat some of the arguments of [GMS1], since the constructions of [GMS1, Th. 4] preserve axial symmetry.

**Theorem 3.1** Suppose \( \lambda \in (0,1] \) and \( u_0 \) is a weak limit in \( H^1 \) of a sequence of maps \( u_i : B^3 \to S^2 \) that minimize \( E_\lambda \) among axially symmetric maps. Then

\[
\lim_{i \to \infty} E_\lambda(u_i) = E_\lambda(u_0),
\]

and \( u_0 \) minimizes \( E_\lambda \) among axially symmetric maps. If \( \lambda < 1 \), then

\[
\lim_{i \to \infty} E(u_i) = E(u_0), \quad \lim_{i \to \infty} L(u_i) = L(u_0),
\]

and \( u_i \) converges strongly in \( H^1 \) to \( u \).

We will see in the next Chapter that the second conclusion may not be true for minimizers of \( E_1 \).

**Proof** Following the proof of [BBC, Th. 3], we have that

\[
E_\lambda(u_0) \leq \liminf_{i \to \infty} E_\lambda(u_i).
\]

To show that \( E_\lambda(u_0) \geq \limsup_{i \to \infty} E_\lambda(u_i) \), we will construct a comparison map \( v_i \) precisely as in the proof of [HKL2, 4.2]. That is, we again note that \( u_i \to u_0 \) in \( C^2_{\text{loc}}(B^3 \sim \{r = 0\}) \), and let, for a small positive \( \delta \), let \( C^\pm_\delta \) denote the solid polar regions

\[
C^\pm_\delta = \{x : 1 - \delta < |x| < 1, |(x_1, x_2)| < \delta|x|\}.
\]

On the shell \( \{1 - \delta \leq |x| \leq 1\} \), we use the linear interpolation between \( u_i \) and \( u_0 \),

\[
l_i(x) = \delta^{-1}(|x| + \delta - 1)u_i + \delta^{-1}(1 - |x|)u_0,
\]

to define a comparison map

\[
v_i(x) = u_0(x) \quad \text{for } |x| \leq 1 - \delta,
\]

\[
v_i(x) = \frac{l_i(x)}{|l_i(x)|} \quad \text{for } 1 - \delta \leq |x| \leq 1 \text{ and } |(x_1, x_2)| \geq \delta|x|,
\]

\[
v_i(x) = \text{the homogeneous \(-\) degree \(-\) 0 extension of } v_i|_{B^3_\delta} \text{ with center } a^\pm_\delta = (0,0, \pm (1 - \frac{1}{2}\delta)) \quad \text{for } x \in C^\pm_\delta.
\]
Clearly \( v_i \) is axially symmetric with \( v_i|_{\partial B^3} = u_i, \ v_i|_{B_1-\delta} = u_0|_{B_1-\delta} \), and, for \( i \) sufficiently large,

\[
\int_B |\nabla v_i|^2 dx \leq \int_{B_1-\delta} |\nabla u_0|^2 dx + c\Lambda \delta
\]

where \( \Lambda = \sup_{1-\delta \leq \rho \leq 1} \int_{\partial B_\rho} |\nabla u_i|^2 dx \).

To estimate \( L(u_i) \), we observe that for almost all \( \delta \) the slices of the cartesian currents

\[
\partial(T_{u_i,B^3} \cap B_1-\delta) - (\partial T_{u_i,B^3}) \cap B_1-\delta
\]

converge weakly to the slice

\[
\partial(T_{u_0,B^3} \cap B_1-\delta) - (\partial T_{u_0,B^3}) \cap B_1-\delta
\]

of \( u_0 \). In particular, the degree of the slices (see [GMS2, section 3])

\[
\partial(T_{v_i,B^3} \cap C^\pm_\delta) - (\partial T_{v_i,B^3}) \cap C^\pm_\delta
\]

coincide, for \( i \) sufficiently large with the degree of the slices

\[
\partial(T_{u_0,B^3} \cap C^\pm_\delta) - (\partial T_{u_0,B^3}) \cap C^\pm_\delta.
\]

It follows that \( L(v_i) \leq L(u_0) + 2\delta \). Thus the minimality of \( u_i \) implies

\[
E_\lambda(u_i) \leq E_\lambda(v_i) \leq \int_{B_1-\delta} |\nabla u_0|^2 dx + 8\pi L(u_0) + (c\Lambda + 16\pi \lambda)\delta
\]

Again letting \( i \to \infty \) and \( \delta \to 0 \) we conclude that

\[
\limsup_{i \to \infty} E_\lambda(u_i) \leq E_\lambda(u_0).
\]

This, combined with the weak \( H^1 \) lower semi-continuity of \( E_\lambda \), proves that

\[
\lim_{i \to \infty} E_\lambda(u_i) = E_\lambda(u_0).
\]

To see that \( u_0 \) minimizes \( E_\lambda \) among axially symmetric maps, we may use a similar argument by combining the argument of [HL2,6.4] with the above construction. If \( u_0 \) were not \( E_\lambda \) minimizing, then there would exist a positive number \( \alpha \) and an axially symmetric mapping \( v \) with

\[
v|_{\partial B^3} = u_0|_{\partial B^3} \quad \text{and} \quad E_\lambda(v) + \alpha \leq E_\lambda(u_0).
\]
For any $r < s < 1$ and $r = r(\alpha)$ sufficiently close to 1, the mapping
\[
\begin{align*}
  w(x) &= v\left(\frac{x}{r}\right) \quad \text{for } |x| < r, \\
  w(x) &= u_0\left(\frac{((1-s)|x| + sr - s)}{(r-s)}x\right) \quad \text{for } r < |x| < s, \\
  w(x) &= u_0(x) \quad \text{for } s < |x| < 1,
\end{align*}
\]
satisfies not only $E_\lambda(w) + \frac{1}{2} \alpha < E_\lambda(u_0)$ and $w|_{\partial B^3} = u_0|_{\partial B^3}$ but also
\[
w|_{(B^3 \sim B_s)} = u_0|(B^3 \sim B_s).
\]

We will now repeat the construction of [HL2,6.4]. For a small positive $\delta$, we interpolate in the annular region $s < |x| < s + \delta$ between $w$ and $u_i$ using the axially symmetric construction of $v_i$ from [HKL,4.2] as above. We obtain as in [HL2,6.4], for $i$ sufficiently large, an axially symmetric mapping $w_i$ such that
\[
w_i|_{\partial B^3} = u_i|_{\partial B^3} \quad \text{and} \quad E(w_i) + \frac{1}{2} \alpha < E(u_i).
\]

By considering the degree of the slices of the corresponding cartesian currents, we again find that the $L$ term is bounded by $2\delta$. By choosing $\delta$ sufficiently small we obtain that
\[
E_\lambda(w_i) < E_\lambda(u_i),
\]
a contradiction.

Finally, we infer from the lower semicontinuity of $E_1 = E + 8\pi L$ under weak $H^1$ convergence that
\[
E_1(u_0) \leq \liminf_{i \to \infty} E_1(u_i).
\]

This, along with the convergence of $E_\lambda(u_i)$ to $E_\lambda(u_0)$ implies, in case $\lambda < 1$, that
\[
\limsup_{i \to \infty} E(u_i) = \limsup_{i \to \infty} (1 - \lambda)^{-1}[E_\lambda(u_i) - \lambda E_1(u_i)]
\]
\[
= (1 - \lambda)^{-1} \limsup_{i \to \infty} E_\lambda(u_i) - \lambda(1 - \lambda)^{-1} \liminf_{i \to \infty} E_1(u_i)
\]
\[
\geq (1 - \lambda)^{-1}[E_\lambda(u_0) - E_1(u_0)]
\]
\[
= E(u_0).
\]

Since $E$ is clearly also weakly $H^1$ lower semicontinuous, we conclude that
\[
E(u_0) = \lim_{i \to \infty} E(u_i).
\]
This implies that
\[
L(u_0) = \lambda^{-1} E_\lambda(u_0) - E(u_0) = \lim_{i \to \infty} \lambda^{-1} E_\lambda(u_i) - \lim_{i \to \infty} E(u_i) = \lim_{i \to \infty} L(u_i).
\]
It also implies, along with the weak $H^1$ convergence, the strong $H^1$ convergence
\[
\lim_{i \to \infty} \int_{B^3} |\nabla u_i - \nabla u_0|^2 dx = \lim_{i \to \infty} \left[ \int_{B^3} |\nabla u_i|^2 dx - \int_{B^3} |\nabla u_0|^2 dx \right] - 2 \lim_{i \to \infty} \int_{B^3} (\nabla u_i - \nabla u_0) \cdot \nabla u_0 dx = 0 - 0.
\]

**Theorem 3.2** Suppose $0 < \lambda \leq 1$, $u : B^3 \to S^2$ minimizes $E_\lambda$ among axially symmetric maps, and $a \in B^3 \cap z$-axis. Any sequence of positive numbers approaching $0$ contains a subsequence $r_i$ such that the cartesian currents $[\text{graph}(u|_{B_{r_i}(a)})] + [I_u \mathbb{L} B_{r_i}(a)] \times [S^2]$ converge weakly as $i \to \infty$ to an axially symmetric cartesian current
\[T = [\text{graph}(u_T)] + I_T \times [S^2]\]
that minimizes $D_\lambda$. The mapping $u_T$ is a homogeneous harmonic mapping. If $u_T$ is constant, then $I_T = \emptyset$ or $\pm [B^3 \cap z$-axis]. If $u_T$ is nonconstant, then $\text{Sing} u_T = \{0\}$, and $I_T = \pm [B^3 \cap \text{neg.} \ z$-axis] or $\pm [B^3 \cap \text{pos.} \ z$-axis].

**Proof** These conclusions follow readily from Lemma 3.4 and Theorem 3.1 as in the corresponding arguments for energy minimizing maps [SU1]. The classification of the tangent maps follows readily from section 1.3 and the fact that nonconstant axially symmetric harmonic maps from $S^2$ to itself are smooth and of degree $\pm 1$. □

5. Partial Regularity of Minimizers in Case $\lambda < 1$

**Lemma 3.5** (Compare [HKL1, 2.3]) There is a positive number $\epsilon$ and, for $0 < \lambda < 1$, a positive number $C_\lambda$ so that if $u$ minimizes $E_\lambda$ as above, $a \in B^3 \cap z$-axis, $B_r(a) \subset B^3$, and $r^{-1} \int_{B_r(a)} |\nabla u|^2 dx < \epsilon$, then
\[
(r/2)^{-1} \int_{B_{r/2}(a)} |\nabla u|^2 dx < C_\lambda [r^{-1} \int_{B_r(a)} |\nabla u|^2 dx]^{1/2} [r^{-3} \int_{B_r(a)} |u - u_{n,r}|^2 dx]^{1/2}
\]

where $u_{n,r} = u_{n,r}(x)$ is the projection of $u$ onto $B_{r/2}(a)$.
where \( u_{a,r} = r^{-3} \int_{B_r(a)} u \, dx \).

Proof Let \( \varphi \) be the angle function corresponding to \( u \), and recall from [HKL2,4.1] the general estimate that

\[
\int_{B_r(a)} |\nabla \cos \varphi| \, dx \leq \frac{1}{2} \int_{B_r(a)} |\nabla u|^2 \, dx < \frac{1}{2} \varepsilon.
\]

As in the proof of [HKL1,2.3], we may choose a number \( s \in [\frac{1}{2} r, r] \) so that we have the 3 simultaneous inequalities

\[
\begin{align*}
\frac{r}{8} \int_{\partial B_s(a)} |\nabla \cos \varphi| d\mathcal{H}^2 & \leq \frac{8}{16} \int_{B_r(a)} |\nabla u|^2 \, dx, \\
\frac{r}{16} \int_{\partial B_s(a)} |\nabla \tan u| d\mathcal{H}^2 & \leq \frac{16}{16} \int_{B_r(a)} |\nabla u|^2 \, dx, \\
\frac{r}{16} \int_{\partial B_s(a)} |u - u_{a,r}|^2 d\mathcal{H}^2 & \leq \frac{16}{16} \int_{B_r(a)} |u - u_{a,r}|^2 \, dx.
\end{align*}
\]

From the first inequality we conclude as in [HKL2,4.2] that, for \( \varepsilon \) sufficiently small, \( u \) maps \( \partial B_r(a) \) continuously into a small neighborhood of the North or South Pole. Let \( v \) be the \( H^1 \) extension of \( u|_{\partial B_r(a)} \) that minimizes energy among all maps, and \( w \) be the axially symmetric extension of \( u|_{\partial B_r(a)} \) that minimizes energy among axially symmetric maps. By minimality as in [SU3], both \( v \) and \( w \) must have images in the corresponding Northern or Southern Hemisphere. The uniqueness theorem of [JK] and [HKW] then implies that \( v \equiv w \). In particular, \( v \) is axially symmetric, and its absolute energy minimality provides, as in [HKL1] and [HL2], the estimate

\[
\int_{B_s(a)} |\nabla v|^2 \, dx
\]

\[
\leq C \left( \int_{\partial B_s(a)} |\nabla \tan v|^2 d\mathcal{H}^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_s(a)} |v - \xi|^2 d\mathcal{H}^2 \right)^{\frac{1}{2}}.
\]

for all \( \xi \in \mathbb{R}^3 \). The axial symmetry of \( v \) allows us to repeat the proof of [BB, Lemma 1] to infer that

\[
\int_{B_s(a)} |\nabla u|^2 \, dx \leq (1 + \lambda)(1 - \lambda)^{-1} \int_{B_s(a)} |\nabla v|^2 \, dx.
\]

Noting that \( v = u \) on \( \partial B_s(a) \) and taking \( \xi = u_{a,r} \), the lemma now follows by putting together the above 4 inequalities. \( \square \)

**Corollary 3.2** There is an arbitrarily positive number \( \theta = \theta(\lambda) < \frac{1}{4} \) so that if \( u \) is as above with \( \varepsilon \) (sufficiently small depending on \( \lambda \) and \( \theta \)), \( a \in B^3 \cap (z\text{-axis}) \), and \( B_r(a) \subset B^3 \), then

\[
(\theta r)^{-1} \int_{B_r(a)} |\nabla u|^2 \, dx \leq \theta r^{-1} \int_{B_r(a)} |\nabla u|^2 \, dx.
\]
Proof Using Lemma 3.5, we may essentially repeat the argument of [HKL1,2.4].\hfill\Box

Theorem 3.3 There is a positive number $\epsilon$ and, for $0 < \lambda < 1$, a positive number $C_\lambda$ so that if $u$ minimizes $E_\lambda$ as above, $a \in B^3 \cap \text{z-axis}$, $B_r(a) \subset B^3$, and $r^{-1}\int_{B_r(a)} |\nabla u|^2 dx < \epsilon$, then $u$ is real analytic in a neighborhood of $a$.

Proof By the higher regularity theory [S], [M,6.7], it suffices to show that $u$ is Hölder continuous in a neighborhood. For this we wish to apply Morrey’s Lemma [M,3.5.2] near $a$. It suffices, as in [HKL1,4.7], to find a positive $\Lambda < 1$ so that

$$
(\theta^{3/2} \frac{T^r}{4})^{-1} \int_{B_{\theta^{3/2}T^r/4}(b)} |\nabla u|^2 dx \leq \Lambda r^{-1} \int_{B_r(b)} |\nabla u|^2 dx
$$

for all sufficiently small $r$ and all points $b$ near $a$.

To prove this estimate, we consider the 2 cases:

Case 1, $b_2^2 + b_3^2 \leq \theta^2 r^2/16$.

Here

$$
B_{\theta^{3/2}T^r/4}(b) \subset B_\theta(b) \subset B_{\theta r/2}((b_1,b_2,0)) \subset B_{r/2}((b_1,b_2,0)) \subset B_r(b).
$$

Using these inclusions and the Corollary 3.2, we deduce that

$$
(\theta^{7/4} r/4)^{-1} \int_{B_{\theta^{7/4} r/4}(b)} |\nabla u|^2 dx
\leq 2\theta^{-3/4} (\frac{r\theta}{2})^{-1} \int_{B_{\theta r/2}((b_1,b_2,0))} |\nabla u|^2 dx
\leq 2\theta^{-3/4} \theta (r/2)^{-1} \int_{B_{r/2}((b_1,b_2,0))} |\nabla u|^2 dx
\leq 4\theta^{1/4} r^{-1} \int_{B_r(b)} |\nabla u|^2 dx
$$

and it suffices to have $\theta < 4^{-4}$.

Case 2, $b_2^2 + b_3^2 > \theta^2 r^2/4$.

Here $u|_{B_{\theta r/4}(b)}$ is smooth because $B_{\theta r/4}(b)$ does not intersect the $z-$axis. So, for $\epsilon$ sufficiently small we may apply the interior estimate of [S, 2.2], which, after integration, gives

$$
\int_{B_{\theta^{7/4} r/4}(b)} |\nabla u|^2 dx
\leq (4\pi/3)(\theta^{7/4} r/4)^{-3} C(\theta r/4)^{-3} \int_{B_{\theta r/4}(b)} |\nabla u|^2 dx
= (\theta^{7/4} r/4) \theta^{3/2} (4\pi/3) C(\theta r/4)^{-1} \int_{B_{\theta r/4}(b)} |\nabla u|^2 dx
$$
where \( C \) is a universal constant. Since trivially

\[
(\theta r/4)^{-1} \int_{B_{\theta r/4}(b)} |\nabla u|^2 dx \leq (\theta/4)^{-1} \int_{B_r(b)} |\nabla u|^2 dx,
\]

\[
(\theta^7/4r^4/4)^{-1} \int_{B_{\theta^7/4r/4}(b)} |\nabla u|^2 dx \leq (\theta^3/4)^{-1} (16\pi/3) C (\theta/4)^{-1} \int_{B_r(b)} |\nabla u|^2 dx,
\]

and it suffices to have \( \theta < (3/16\pi C)^2 \).

\[ \square \]

**Theorem 3.4** Suppose \( 0 < \lambda < 1 \), \( u \in H^1(B^3, S^2) \) is axially symmetric, \( u|_{\partial B^3} \in H^1(\partial B^3, S^2) \), \( \deg(u|_{\partial B^3}) = 0 \), and \( u \) minimizes \( E_\lambda \) among axially symmetric maps. Then

1. \( u \) is real analytic away from a discrete subset \( \text{Sing}(u) \) of \( B^3 \cap (z-\text{axis}) \).

2. \( u \) is \( C^{k,\alpha} \) smooth near \( \partial B^3 \) in case \( u|_{\partial B^3} \) is \( C^{k,\alpha} \) smooth where \( k \in \{1, 2, \ldots, \infty, \omega\} \) and \( 0 < \alpha < 1 \).

**Proof** Suppose for contradiction that there existed a sequence of distinct singular points \( a_i \) converging to a singular point \( a \) in \( B^3 \). Then \( a \) and each \( a_i \) belong to the \( z-\text{axis} \) by section 3.1. Let \( r_i = 2|a_i| \). By Theorem 3.1 and 3.2 we find, after passing to a subsequence, that the maps \( u(r_i(\cdot) + a) \) converge strongly in \( H^1 \) a map \( u_0 \) with \( \text{Sing}(u_0) = \{0\} \). Passing to another subsequence we assume that \( a_i/2|a_i| \) converges to a point \( b \in \partial B_{1/2} \). By the regularity of \( u_0 \) at \( b \), we may choose a sufficiently small positive \( \rho < 1/4 \) so that

\[
\rho^{-1} \int_{B_{\rho}(b)} |\nabla u_0|^2 dx < \epsilon
\]

with \( \epsilon \) as in the Theorem. The strong convergence gives the estimate

\[
(r_i;\rho)^{-1} \int_{B_{r_i}(a_i)} |\nabla u|^2 dx < \epsilon
\]

for \( i \) sufficiently large. But then the Theorem would imply that \( u \) is smooth at \( a_i \), a contradiction. Thus \( u \) is real analytic away from a discrete subset of \( B^3 \cap (z-\text{axis}) \).

Assuming next that \( u|_{\partial B^3} \) is \( C^{1,\alpha} \) it suffices to show that \( u \) is Hölder continuous near \( \partial B^3 \) because we may then use the higher boundary regularity theory as in [HKL1, 5.6] to get the \( C^{k,\alpha} \rightarrow C^{k,\alpha} \) result. Away from the \( z-\text{axis} \), \( u \) is \( E \) minimizing among axially symmetric maps. So by [HKL2, 4.2] it only remains to establish this Hölder
continuity near the poles \((0, 0, \pm 1)\). For this we follow the regularity argument of [HKL2,4.2]. First note that, an argument similar to Section 3.2 and Theorem 3.2 as in [HKL2,4.2] shows that there is a homogeneous tangent map of \(u_0\) at \((0, 0, \pm 1)\). Since \(u_0|_{S^2 \cap \{x_3 < 0\}}\) is harmonic and \(u_0|_{S^2 \cap \{x_3 = 0\}}\) is constant, namely \(u((0, 0, \pm 1))\), we find that \(u_0\) is constant. One can modify the proof of Theorem 3.1 to show that the convergence of \(u(r_1(\cdot) + (0, 0, \pm 1))\) to \(u_0\) is strong in \(H^1\). Thus,

\[
    r^{-1} \int_{B_r((0,0,\pm 1)) \cap \mathbb{B}^3} |\nabla u|^2 \, dx \to 0 \text{ as } r \to 0,
\]

and it remains to establish a boundary version of the small energy regularity theorem 3.3. Here we proceed as in [HKL1, section 5] finding the boundary versions of Lemma 3.5, Corollary 3.2 and [BB, Lemma].
4.1 Energy Gaps

In [HK1] an example was given of smooth degree 0 boundary data $g : \partial B^3 \to S^2$ for which the energy minimizing map necessarily had singularities. Here there was a definite gap

$$\inf \left\{ \int_{B^3} |\nabla u|^2 dx : u \in H^1(B^3, S^2), u|_{\partial B^3} = g \right\}$$

$$< \inf \left\{ \int_{B^3} |\nabla u|^2 dx : u \in H^1(\tilde{B}^3, S^2), u|_{\partial B^3} = g \right\}.$$

In [HKL2,6.2] and [Z] it was observed that this example could be done in the axially symmetric context with boundary data whose corresponding angle function had values in $[0, \pi)$. Here we describe another example of axially symmetric boundary data which exhibits another gap.

**Theorem 4.1** There exists axially symmetric boundary data $g : \partial B^3 \to S^2$ for which

$$\inf \left\{ \int_{B^3} |\nabla u|^2 dx : u \in H^1 \cap C^0(\tilde{B}^3, S^2), u|_{\partial B^3} = g \right\}$$

$$< \inf \left\{ \int_{B^3} |\nabla u|^2 dx : u \in H^1 \cap C^0(\tilde{B}^3, S^2),
\quad u \text{ is axially symmetric, } u|_{\tilde{B}} = g \right\}.$$

**Proof** One may make a construction very similar to [HL1]. In a spherical cap of radius $\epsilon$ centered at the North Pole, one chooses the angle function corresponding to $g$ to be strictly increasing from 0 to $2\pi$. In the large central region it should be identically $2\pi$. In a small spherical cap of radius $\epsilon$ centered at the South Pole, it should be monotonically decreasing from $2\pi$ to 0. As in the proof of Proposition 1.2 one sees that the two dimensional degree of $g|_{\partial B^3 \cap \{0\} \times \mathbb{R}^2}$ (as a map into the unit circle in the $yz$-plane) is also zero and that $g$ admits some continuous axially symmetric extension $v$ to all of $B^3$. To obtain a lower bound for the energy of any such $v$ we may assume that $v|_{\tilde{B}}$ is smooth and axially symmetric and use the co-area
\[ \int_{B^3} |\nabla v|^2 \, dx \geq 2 \int_{B^3} |J_2 Dv|^2 \, dx = 2 \int_{S^2} \mathcal{H}^1(v^{-1}\{y\}) \, d\mathcal{H}^2 y. \]

For any regular value \( y \) of \( v \) different from 0 or \( \pi \), the level set \( v^{-1}\{y\} \) is a smooth one manifold that lies entirely in the intersection of \( B^3 \) with the vertical plane \( P \) that passes through \( y \) and the \( z \)-axis. The boundary of \( v^{-1}\{y\} \) consists of 4 points, 2 near each of the poles. We claim that 2 of the components of \( v^{-1}\{y\} \) must join points that lie near opposite poles. In fact, otherwise there would be a component \( \Gamma \) of \( v^{-1}\{y\} \) joining 2 points near the same pole. Let \( A \) be the short arc of \( P \cap S^2 \). By the construction of \( g \), the two dimensional degree of \( v \) on \( \Gamma \cup A \) is 2. But this contradicts that \( v \) is continuous on the region in \( P \) bounded by \( \Gamma \cup A \). Thus, \[ \mathcal{H}^1(v^{-1}\{y\}) \geq 2 - 2\epsilon \]

for \( \mathcal{H}^2 \) almost all \( y \), and \[ \int_{B^3} |\nabla v|^2 \, dx \geq 16\pi(1 - \epsilon). \]

On the other hand, to construct a small energy continuous (but not axially symmetric) extension \( u \) of \( g \), we first examine a related problem. Consider, on the lower half ball \( B^- = B^3 \cap \{x_3 \leq 0\} \), smooth axially symmetric boundary data \( G \) where \( G \equiv (0,0,1) \) on the lower hemisphere \( \partial B^- \cap \{x_3 < 0\} \) and the angle function of \( G \) is strictly increasing from 0 to \( 2\pi \) on the upper disk \( \partial B^- \cap \{x_3 = 0\} \). Since \( G \) has (3-dimensional) degree 0 it admits a continuous finite energy extension \( w \). The mapping \( w \) is not axially symmetric because the 2-dimensional degree of \( G|_{\partial B^- \cap \{(0) \times \mathbb{R}^2\}} \) (as a map into the unit circle in the \( yz \)-plane) is 2. Scaling this problem down, we see that on the small hemispherical region \( B^- \cap \{x_3 \leq 0\} \) the boundary data \( G(\epsilon) \) admits the smooth continuous extension \( w(\epsilon) \) whose energy is \( c \epsilon \) with \( c = \int_{B^-} |\nabla w|^2 \, dx. \)

Returning to the function \( g : \partial B^3 \to S^2 \) described above, we may define \( u \) on \( \partial B(0,0,1) \cap B^3 \) to be identically \((0,0,1)\). For \( \epsilon \) small, the region \( B(0,0,1) \cap B^3 \) may be deformed to \( B^- \) by a map close to the identity in Lipschitz norm. For sufficiently small \( \epsilon \), it is now clear that one may obtain \( g \) as above along with a continuous extension \( u : B^3 \to S^2 \) so that \( u \equiv (0,0,1) \) on \( B^3 \sim (B(0,0,1) \cup B(0,0,-1)) \), and \[ \int_{B^3} |\nabla u|^2 \, dx < 4\epsilon \epsilon. \]

The energy estimates for \( u \) and \( v \) now establish the theorem. \( \square \)
Theorem 4.2 There exists an axially symmetric region $\Omega \subset \mathbb{R}^3$ and axially symmetric boundary data $h : \partial \Omega \to S^2$ for which there are 2 gaps

$$\inf \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H^1(\Omega, S^2), \right.$$ 

$$u \text{ is axially symmetric, } u|_{\partial \Omega} = h \left\} \right.$$ 

$$< \inf \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H^1 \cap C^0(\overline{\Omega}, S^2), u|_{\partial \Omega} = h \right\} \right.$$ 

$$< \inf \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H^1 \cap C^0(\overline{\Omega}, S^2), \right.$$ 

$$u \text{ is axially symmetric, } u|_{\partial \Omega} = h \right\}.$$

Proof Let $\Omega$ be the dumbbell-shaped domain

$$B_1^3 \left( 0, 0, \frac{1}{2} N \right) \cup B_1^3 \left( 0, 0, \frac{1}{2} N \right) \cup B_2^2(0) \times [\frac{-1}{2} N, \frac{1}{2} N]$$

where $N$ is large and $\epsilon$ is small. On the top ball $B_1 \left( 0, 0, \frac{1}{2} N \right)$, we choose $h$ to be similar to that of [HL1]. That is, in the small polar region $\partial B_1 \left( 0, 0, \frac{1}{2} N \right) \cap B_{2\epsilon} \left( 0, 0, \frac{1}{2} N + 1 \right)$, $h$ should have an angle function that is strictly increasing from 0 to $\pi$. For the central region

$$B_1 \left( 0, 0, \frac{1}{2} N \right) \sim \left( B_{2\epsilon} \left( 0, 0, \frac{1}{2} N + 1 \right) \cup B_{2\epsilon} \left( 0, 0, \frac{1}{2} N - 1 \right) \right)$$

it should also be identically $\pi$. For the lower polar sector

$$\partial B_1 \left( 0, 0, \frac{1}{2} N \right) \cap \left( B_{2\epsilon} \left( 0, 0, \frac{1}{2} N - 1 \right) \sim B_1 \left( 0, 0, \frac{1}{2} N - 1 \right) \right)$$

it should increase from $\pi$ to $2\pi$. Along the long thin neck it should remain identically $2\pi$. For the boundary data on the bottom half of $\partial \Omega$, we simply let $h(x, y, -z) = h(x, y, z)$ so that the resulting angle function decreases from $2\pi$ back to 0.

We may find a fairly small energy (discontinuous) axially symmetric extension $u$ of $h$ by letting $u$ be identically $(0, 0, 1)$ in the long cylindrical neck of $\Omega$ and be identically $(0, 0, -1)$ in the large middle regions of the top and bottom balls. To complete the construction of $u$ we use, as in the proof of [HL1], homogeneous extension centered at the four points $(0, 0, \frac{1}{2} N + 1 - \epsilon)$, $(0, 0, \frac{1}{2} N - 1 + \epsilon)$, $(0, 0, -\frac{1}{2} N + 1 - \epsilon)$, $(0, 0, -\frac{1}{2} N - 1 + \epsilon)$. The resulting function has energy

$$\int_\Omega |\nabla u|^2 \, dx \leq 33\pi \epsilon$$

for $\epsilon$ sufficiently small.
Next we observe, as in [HL1], that for any smooth \( v \in C^1(\Omega, S^2) \) with \( v|_{\partial \Omega} = h \) and, for \( H^2 \) almost all \( y \in S^2 \), the fiber \( v^{-1}\{y\} \) must contain at least two curves, one traversing the length of the top ball and one traversing the length of the bottom ball. Using the co-area formula as in Theorem 4.1 we see that

\[
\int_{\Omega} |\nabla v|^2 dx \geq 16\pi(1 - 2\varepsilon).
\]

On the other hand, we may find such a smooth (but not axially symmetric) map \( v \) with energy

\[
\int_{B} |\nabla v|^2 dx \leq c(\varepsilon)
\]

where \( c(\varepsilon) \) is independent of \( N \). One simply chooses \( v \) to be identically \((0, 0, 1)\) in \( \Omega \sim (B_1(0, 0, 1/2) \cup B_2(0, 0, -1/2)) \). Then we have degree-0 boundary data on the two balls and we may extend to a smooth map with energy independent of \( N \).

Finally we reason again as in Theorem 4.1 to see that for any axially symmetric function \( w \in C^1(\Omega, S^2) \) with \( w|_{\partial \Omega} = h \) and for \( H^2 \) almost all \( y \in S^2 \), the fiber \( w^{-1}\{y\} \) must contain two curves traversing the entire length of the middle cylinder. Again by the co-area formula

\[
\int_{\Omega} |\nabla w|^2 dx \geq 16\pi(N - 2\varepsilon).
\]

Putting these inequalities all together we have, for \( \varepsilon \) sufficiently small and \( N > (c(\varepsilon) + 32\pi\varepsilon)/16\pi \), that

\[
\inf \left\{ \int_{\Omega} |\nabla u|^2 dx : u \in H^1(\Omega, S^2), \text{uis ax. sym.}, u|_{\partial \Omega} = h \right\}
\]

\[
\leq 33\pi\varepsilon
\]

\[
< 16\pi(1 - 2\varepsilon)
\]

\[
\leq \inf \left\{ \int_{B}^3 |\nabla u|^2 dx : u \in H^1 \cap C^0(B^3, S^2), u|_{\partial B^3} = h \right\}
\]

\[
\leq c(\varepsilon)
\]

\[
< 16\pi(N - 2\varepsilon)
\]

\[
\leq \inf \left\{ \int_{B}^3 |\nabla u|^2 dx : u \in H^1 \cap C^0(B^3, S^2), \text{u is ax. sym.}, u|_{\partial B^3} = h \right\}
\]

which completes the proof. \( \Box \)

4.2 An Energy Comparison Lemma
The next lemma was suggested by the work [BBC]. While it is clear from [HL1], [BCL], [ABL], [B], and [BBC] that the energy increase necessary to add a dipole of separation $d$ is approximately $8\pi d$, the lemma shows that, in some circumstances, one can do this with an energy increase strictly less than $8\pi d$.

**Lemma 4.1** Suppose $u$ is a $C^2$ axially symmetric map of $B^3$ to $S^2$ and $\nabla u(0) \neq 0$. Then, for all sufficiently small positive $\delta$, there exists an axially symmetric map $v_\delta : B^3 \to S^2$ such that $v_\delta = u$ outside $B_{2\delta}$, $v_\delta$ is Lipschitz except for the 2 singularities $(0,0, \pm \delta)$ where $v_\delta$ has degree $\pm 1$ or $\mp 1$, and

$$
\int_{B^3} |\nabla v_\delta|^2 \, dx < \int_{B^3} |\nabla u|^2 \, dx + 8\pi (2\delta).
$$

**Proof** Let $\varepsilon$ be a positive number whose dependence on $\delta$ will be determined later, and let $a_{\pm \delta} = (0,0, \pm \delta)$. By smoothness and axial symmetry, the restriction of $u$ to the $z$–axis is either identically $(0,0,1)$ or identically $(0,0, -1)$. To fix notations, suppose it is $(0,0,1)$. The arguments below can be easily modified for the $(0,0, -1)$ case.

Let $\varphi = \varphi(r,z)$ be the angle function corresponding to $u$ as in the introduction. We may choose $\varphi$ to be smooth with $\varphi(0, \cdot) \equiv 0$. Since $\nabla u(0) \neq 0$, $\left(\frac{\partial \varphi}{\partial r}\right)(0,0) \neq 0$, and, by Taylor’s theorem,

$$
\varphi(r,0) = r \left(\frac{\partial \varphi}{\partial r}\right)(0,0) + O(r^2).
$$

Consider the 2 solid cones

$$
C^\pm = \{(r,z) : 0 < \pm z < \delta - \varepsilon^{-1} r\},
$$

their complementary regions in corresponding cylinders

$$
G^\pm = \{(r,z) : \delta - \varepsilon^{-1} r \leq \pm z < \delta\},
$$

and the hemispherical caps

$$
H^\pm = \{(r,z) : r^2 + (z \mp \delta)^2 < \varepsilon^2 \delta^2, \pm z \geq \delta\}.
$$

We now define $v_\delta$. First, on

$$
B^3 \sim (C^+ \cup G^+ \cup H^+) \cup (C^- \cup G^- \cup H^-),
$$

let $v_\delta = u$. Second on the horizontal disk

$$
B^2_{\varepsilon \delta}(0) \times \{0\},
$$
let \( v_\delta \) coincide with the unique conformal axially symmetric homeomorphism that maps onto the large spherical region

\[
\{ x \in S^2 : |x - (0,0,1)| > \tan \varphi(\epsilon \delta, 0) \}.
\]

Finally, on the two regions

\[
\Omega^\pm = C^\pm \cup G^\pm \cup H^\pm,
\]

let \( v_\delta \) be homogeneous extension with center \( a_{\pm \delta} \) of the boundary map \( v_\delta|_{\partial \Omega^\pm} \), which has been defined above.

We now find the total energy of \( v_\delta \) by computing its energy on each of the regions \( H^\pm, C^\pm, G^\pm \).

First we note that

\[
\int_{H^\pm} |\nabla v_\delta|^2 dx = \epsilon \delta \int_{\partial H^\pm \cap \{ x : x_3 \geq \delta \}} |\nabla u|^2 dx = O(\epsilon^3 \delta^3)
\]

because \( |\nabla u| \) is bounded.

Second we write

\[
\int_{C^\pm} |\nabla v_\delta|^2 dx = \int_{D^\pm} |\nabla v_\delta|^2 dx + \int_{E^\pm} |\nabla v_\delta|^2 dx
\]

where \( D^\pm = \{(r,z) : r^2 + (z \mp \delta)^2 \leq \delta^2, \pm z \leq \delta - \epsilon r \} \) and \( E^\pm = C^\pm \sim D^\pm \). Since radially projection from \( a_{\pm \delta} \) maps the spherical cap

\[
\sum^\pm = \partial B^3_{\epsilon \delta}(a_{\pm \delta}) \cap \{(r,z) : \pm z < \delta - \delta (1 + \epsilon^2)^{-1/2}\}
\]

conformally onto \( B^2_{\epsilon \delta}(0) \times \{0\} \) and \( v_\delta|_{B^2_{\epsilon \delta}(0) \times \{0\}} \) is conformal, \( v_\delta|\Sigma^\pm \) is also conformal, and so

\[
\int_{\Sigma^\pm} |\nabla \tan v_\delta|^2 d\mathcal{H}^2
\]

\[
= 2\mathcal{H}^2 \left( v_\delta(B^2_{\epsilon \delta}(0) \times \{0\}) \right)
\]

\[
= 2\mathcal{H}^2 \left( v_\delta(B^2_{\epsilon \delta}(0) \times \{0\}) \right)
\]

\[
= 2(4\pi - [1 - \cos \varphi(\epsilon \delta, 0)])
\]

\[
= 8\pi - \varphi(\epsilon \delta, 0)^2 + O(\epsilon^4 \delta^4)
\]

\[
= 8\pi - \left( \left( \frac{\partial \varphi}{\partial r} \right)(0,0) \right)^2 \epsilon^2 \delta^2 + O(\epsilon^3 \delta^3).
\]
So, by homogeneity,

\[
\int_{D_{\ell}} |\nabla v_\ell|^2 \, dx \\
\leq \delta \sum_{\pm} |\nabla_{\tan} v_\ell|^2 \, d\mathcal{H}^2 \\
= \delta \left( 8\pi - \left( \left( \frac{\partial \varphi}{\partial r} \right) (0,0) \right)^2 \cdot \varepsilon^2 \delta^2 + O(\varepsilon^3 \delta^3) \right) \\
= 8\pi \delta - \left( \left( \frac{\partial \varphi}{\partial r} \right) (0,0) \right)^2 \cdot \varepsilon^2 \delta^3 + O(\varepsilon^3 \delta^4).
\]

To estimate \( \int_{E_{\ell}} |\nabla v_\ell^\pm|^2 \, dx \), we observe by [BC, p207] (exactly as in [BBC]) that

\[
|\nabla v_\ell(x,y)|^2 \leq \left\{ \begin{array}{ll}
C \varepsilon^4 \delta^4(\varepsilon^4 \delta^4 + r^2)^{-2} & \text{for } 0 \leq r \leq \frac{1}{2} \varepsilon \delta, \\
C & \text{for } \frac{1}{2} \varepsilon \delta \leq r \leq \varepsilon \delta.
\end{array} \right.
\]

Thus by Fubini's theorem,

\[
\int_{E_{\ell}} |\nabla v_\ell^\pm|^2 \, dx \\
\leq C \left( \int_0^{\varepsilon^{\delta/2}} \varepsilon^4 \delta^4(\varepsilon^4 \delta^4 + r^2)^{-2}(\delta^{-1} r^2) r \, dr + \int_{\varepsilon^{\delta/2}}^{\varepsilon \delta} (\delta^{-1} r^2) r \, dr \right) \\
= O \left( \varepsilon^4 \delta^3 \ln \left( 1/\varepsilon \delta \right) \right).
\]

(A factor \( \delta^{-1} \) was missing in [BBC].)

Finally we compute \( \int_{G_{+}} |\nabla v_\ell|^2 \, dx = 2 \int_{G_{+}} |\nabla v_\ell|^2 \, dx \) by representing \( v_\ell \) by an angle function \( \Psi \). Thus, on \( G_{+} \),

\[
\Psi(r,z) = \varphi \left( \varepsilon \delta, \delta - \varepsilon \delta \left( \frac{\delta - z}{r} \right) \right),
\]

and the partial derivatives are

\[
\Psi_r(r,z) = \left( \frac{\varepsilon \delta (\delta - z)}{r^2} \right) \cdot \varphi_z \left( \varepsilon \delta, \delta - \varepsilon \delta \left( \frac{\delta - z}{r} \right) \right),
\]

\[
\Psi_z(r,z) = \left( \frac{\varepsilon \delta}{r} \right) \cdot \varphi_z \left( \varepsilon \delta, \delta - \varepsilon \delta \left( \frac{\delta - z}{r} \right) \right).
\]

So the energy is

\[
\int_{G_{+}} |\nabla v_\ell|^2 \, dx
\]
\[ \begin{align*}
&= 2\pi \int_{0}^{t} \int_{\delta - r/\epsilon}^{t/\epsilon} (r \Psi_r^2 + r \Psi_z^2 + r^{-1} \sin^2 \Psi) dz dr \\
&= 2\pi \int_{0}^{t/\epsilon} \int_{\delta - r/\epsilon}^{t/\epsilon} \left( \epsilon^2 \delta^2 |\varphi_z|^2 \left( r^{-2} + (\delta - z)r^{-4} \right) + r^{-1} \sin^2 \Psi \right) dz dr \\
&= 2\pi \int_{0}^{t/\epsilon} \int_{\delta - r/\epsilon}^{t/\epsilon} \left( \epsilon^2 \delta^2 |\varphi_z|^2 \left( r^{-1} + z^2 r^{-3} \right) + r^{-1} \sin^2 \Psi \right) dz dr,
\end{align*} \]

the latter equality resulting by the change of variable \( z \to \delta - z \). Next we note that \( \varphi_z \) vanishes on the \( z \)-axis, and that, near the \( z \)-axis, \( u(\cdot, \cdot, z) \) is approximately conformal. More precisely, by Taylor's theorem,

\[ |\varphi_z| = O(\epsilon), \quad \sin^2 \Psi = 1/2|\nabla u(0)|^2 \epsilon^2 \delta^2 + O(\epsilon^3 \delta^3). \]

Thus,

\[ \int_{G^+} |\nabla v_\delta|^2 dx = O(\epsilon^2 \delta^5) + O(\delta^5) + \pi |\nabla u(0)|^2 \epsilon^2 \delta^3 + O(\epsilon^3 \delta^4) \]

Combining our estimates, we now have that

\[ \begin{align*}
\int_{\Omega^+ \cup \Omega^-} |\nabla v_\delta|^2 dx &= 16\pi \delta - \left( \left( \frac{\partial \varphi}{\partial r} \right)(0,0) \right)^2 \cdot \epsilon^2 \delta^3 + 2\pi |\nabla u(0)|^2 \epsilon^2 \delta^3 + \\
&\quad + O(\epsilon^3 \delta^3) + O \left( \epsilon^4 \delta^3 \ln \left(1/\epsilon \delta\right) \right) + O(\delta^5).
\end{align*} \]

On the other hand it is also very easy to estimate the energy of \( u \) on \( \Omega^+ \cup \Omega^- \) using Taylor's theorem,

\[ \begin{align*}
\int_{\Omega^+ \cup \Omega^-} |\nabla u|^2 dx
&= \int_{C^{++} \cup C^{--} \cup C_{G^+} \cup C_{G^-}} |\nabla u|^2 dx + \int_{H^+ \cup H^-} |\nabla u|^2 dx \\
&= (|\nabla u(0)|^2 + O(\delta)) \operatorname{vol}(C^+ \cup C^- \cup G^+ \cup G^-) + O(\epsilon^3 \delta^3) \\
&= 2\pi |\nabla u(0)|^2 \epsilon^2 \delta^3 + O(\epsilon^2 \delta^4) + O(\epsilon^3 \delta^3).
\end{align*} \]

Putting these two together, we cancel the common term \( 2\pi |\nabla u(0)|^2 \epsilon^2 \delta^3 \), and obtain the relation

\[ \begin{align*}
\int_{B^3} |\nabla v_\delta|^2 dx &= \int_{B^3} |\nabla u|^2 dx + 8\pi (2\delta) - \left( \left( \frac{\partial \varphi}{\partial r} \right)(0,0) \right)^2 \cdot \epsilon^2 \delta^3 \\
&\quad + O(\epsilon^3 \delta^3) + O \left( \epsilon^4 \delta^3 \ln \left(1/\epsilon \delta\right) \right) + O(\delta^5) O(\epsilon^2 \delta^4).
\end{align*} \]
It now suffices to take $\epsilon = \delta^{1/2}$ and choose $\delta$ sufficiently small to obtain the desired strict inequality. \hfill \Box

**Lemma 4.2** If $u \in H^1(B^3, S^2)$ minimizes $E_1$ among axially symmetric maps, $u|_{\partial B^3} \in H^1(\partial B^3, S^2)$, and $\deg(u|_{\partial B^3}) = 0$, then the complement of $\text{spt}(I_u) = \tilde{J}_u$ (see section 1.3) is dense in the $z$-axis.

**Proof** If $u$ is constant, then $I_u = 0$ and the Lemma is trivially true.

Suppose now that $u$ is nonconstant and, for contradiction, that $\text{spt} (I_u) = \tilde{J}_u$ contains an open interval $J \subset B^3 \cap (z$-axis). Then since $I_u = \pm [J_u]$, the constancy theorem [F, 4.1.7] implies that $I_u \subseteq J = \pm [J_u]$, hence, $\text{div}D(u) = 0$ in $J$.

Since $\int_{B^3} |\nabla u|^2 \, dx < \infty$, we may, by [SU1,2.7], choose a point $a \in J$ and, for any $\epsilon > 0$, a positive $r$ so that $B_r(a) \cap (z - \text{axis}) \subset J$ and

$$r^{-1} \int_{B_r(a)} |\nabla u|^2 \, dx < \epsilon.$$ 

Then, by choosing $\epsilon$ sufficiently small and using Fubini’s theorem, we may, as in [HKL2,4.1], choose a positive $s < r$ so that $u|_{\partial B_s(a)}$ is continuous and so that the angle function for $u$ has small enough oscillation to cause the image of $u$ to lie entirely in either the Northern or Southern Hemisphere. As before, an energy minimizing map $v$ with $v|_{\partial B_s(a)} = u|_{\partial B_s(a)}$ is axially symmetric, has values entirely in the same hemisphere, and has no singularities in $B_s(a)$; in particular,

$$\partial [\text{graph}(v)] = [\text{graph} (v|_{\partial B_s(a)})].$$

Letting $w = v$ on $B_s(a)$ and $w = u$ on $B^3 \sim B_s(a)$, we conclude that $L(w) = L(u)$. Moreover, since

$$E(w) = \int_{B_s(a)} |\nabla v|^2 \, dx + \int_{B^3 \sim B_s(a)} |\nabla u|^2 \, dx$$

$$\leq \int_{B_s(a)} |\nabla u|^2 \, dx + \int_{B^3 \sim B_s(a)} |\nabla u|^2 \, dx$$

$$= E(u),$$

and $w|_{\partial B^3} = u|_{\partial B^3}$, $w$ is also $E_1$ minimizing. Replacing $u$ by $w$, we may now assume that $u$ is smooth on $B_s(a)$.

We now apply Theorem 2.5 to $u \left(a + \frac{1}{2}s(\cdot)\right)$ and find a point $b \in B_s/2(a) \cap (z$-axis) with $\nabla u(b) \neq 0$. Then we may apply Lemma 4.1 to $u \left(b + \frac{1}{2}s(\cdot)\right)$ and find a positive $\delta < s/4$ along with an axially symmetric map $v_\delta : B_{s/2}(b) \to S^2$ such that $v_\delta = u$
outside $B_{2\delta}(b)$, $\upsilon_\delta$ is Lipschitz except for the 2 singularities $b \pm (0,0,\delta)$ where $\upsilon_\delta$ has degree $\pm 1$ or $\mp 1$, and

$$\int_{B_{1/2}(b)} |\nabla \upsilon_\delta|^2 dx < \int_{B_{\delta/2}(b)} |\nabla u|^2 dx + 8\pi (2\delta).$$

Letting $u_\delta = \upsilon_\delta$ on $B_{\delta/2}(b)$ and $u_\delta = u$ on $B^3 \sim B_{\delta/2}(b)$, we see that $u_\delta|_{\partial B^3} = u|_{\partial B^3}$ and that

$$E(u_\delta) = \int_{B_{1/2}(b)} |\nabla \upsilon_\delta|^2 dx + \int_{B^3 \sim B_{\delta/2}(b)} |\nabla u|^2 dx < E(u) + 8\pi (2\delta).$$

To compute $L(u_\delta)$, note that $I_{u_\delta}$ is oriented in the same direction as $I_u$ because $u_\delta$ and $u$ coincide on $B^3 \sim B_{\delta/2}(b)$, that is, $I_{u_\delta} = \pm [J_{u_\delta}]$ if $I_u = \pm [J_u]$. From the equalities

$$J_{u_\delta} \cap (B^3 \sim B_{\delta/2}(b)) = J_u \cap (B^3 \sim B_{\delta/2}(b))$$

$$J_u \cap (B_{\delta/2}(b)) = J \cap (B_{\delta/2}(b)), $$

and the nature of the two singularities $b \pm (0,0,\delta)$ of $u_\delta$, it is clear that

$$J_{u_\delta} = J_u \sim (b - (0,0,\delta), b + (0,0,\delta)).$$

Thus $L(u_\delta) = H^1(J_{u_\delta}) = H^1(J_u) - 2\delta = L(u) - 2\delta$, and

$$E_1(u_\delta) = E(u_\delta) + 8\pi L(u_\delta) < E(u) + 8\pi (2\delta) + 8\pi (L(u) - 2\delta) = E_1(u),$$

contradicting the $E_1$ minimality of $u$. \qed


4.3 Maps with an Angle Function in $[-\pi, \pi]$

In [2] D. Zhang proved that for any smooth axially symmetric boundary data $g$ which has degree 0 and whose angle function has values in $(-\pi, \pi)$, there exists a smooth harmonic $u$ with $u|_{\partial B^3} = g$ and with angle function also in $(-\pi, \pi)$.

This was improved slightly in [HKL2,3.1] to allow values in $[-\pi, \pi]$ provided the boundary data also has degree 0. The degree 0 condition is automatic in the $(-\pi, \pi)$ case, and can be guaranteed in the $[-\pi, \pi]$ case by simply requiring that $g(0,0, -1) = g(0,0,1)$. 

Recall from Chapter 1 that the family $\mathcal{R}_{AS}$ consisted of axially symmetric maps from $B^3$ to $S^2$ which are smooth except a finite subset of $B^3 \cap (z-\text{axis})$. Here we will consider the subfamily

$$\mathcal{R}_{AS}^\tau = \{ u \in \mathcal{R}_{AS} : u|_{B^3-\text{Sing}(u)} = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \text{ for some smooth } \varphi = \varphi(r, z) \text{ which takes values in } [-\pi, \pi] \}.$$ 

Being in $\mathcal{R}_{AS}^\tau$ imposes a restriction on the singularities. If a map $u \in \mathcal{R}_{AS}^\tau$ has a singularity with tangent map $\pm \Psi$ (respectively, $\pm \Lambda$) (see section 1.2 for $\Psi$ and $\Lambda$), then any possible singularity adjacent along the $z-$axis must have tangent map $\pm \Lambda$ (respectively, $\mp \Psi$). (Recall from section 1.2 that if $u$ were in the larger class $\mathcal{R}_{AS}$, the adjacent singularity could also have tangent map $\pm \Psi$ (respectively, $\mp \Lambda$).) We will see in Chapter 5 that this is essentially the only condition on the singularities for harmonic maps in $\mathcal{R}_{AS}^\tau$.

Taking the closure of $\mathcal{R}_{AS}^\tau$ in $H^1$, we define

$$A^\tau = \{ u \in \overline{\mathcal{R}_{AS}^\tau} : u|_{\partial B^3} \in H^1(\partial B^3, S^2) \text{ and } \deg(u|_{\partial B^3}) = 0 \}.$$ 

The next lemma concerns singularity cancellation for a map $u$ at the expense of increasing energy approximately $8\pi L(u)$. This property is true in the class of general harmonic maps [BBC, Cor. 2], false in the class of axially symmetric harmonic maps (see section 4.4 below), but true for the subclass $A^\tau$.

**Lemma 4.3** For any $u \in A^\tau$ with $\deg(u|_{\partial B^3}) = 0$ and $\epsilon > 0$, there is a smooth map $v \in A^\tau$ such that $v|_{\partial B^3} = u|_{\partial B^3}$ and

$$E(v) \leq E_1(u) + \epsilon.$$ 

**Proof** Replacing $u(x)$ by $u_{\sigma}(\sigma^{-1}x)$ for some $\sigma > 1$ where

$$u_{\sigma}(x) = \begin{cases} u(x) & \text{for } x \in B_1, \\ u \left( \frac{x}{|x|} \right) & \text{for } x \in B_\sigma \sim B_1, \end{cases}$$

we may assume that $u$ is smooth near $\partial B^3$. The proof of Theorem 1.1 then shows that $A^\tau \cap \mathcal{R}_{AS}^\tau$ is strongly $H^1$ dense in $A^\tau$, and, by the proof of Proposition 1.3, we may also assume that $u \in A^\tau \cap \mathcal{R}_{AS}^\tau$. Then $L(u) = \sum_{i=1}^n |b_i - a_i|$ where $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ are consecutive singularities of $u$ on the $Z-$axis. Finally replacing $\epsilon$ by $\epsilon/n$, and, for each $i$, $u(x)$ by $u \left( \rho_i \left( x - \frac{1}{2}(a_i + b_i) \right) \right)$ for a suitable $\rho_i > \frac{1}{2}|b_i - a_i|$, we may reduce
to the case where \( u \) has only two singularities at points \( a_\pm = (0, 0, \pm \rho) \) for some positive \( \rho < 1 \).

By axial symmetry, the angle function \( \varphi \) corresponding to \( u \) has
\[
\varphi(0,0,1) = 0 \text{ or } \pi.
\]
Suppose for convenience that it equals 0. (Otherwise replace \( \varphi \) by \( \pi - \varphi \).) Then, on the \( z \)-axis, \( \varphi \) is identically 0 below \( a_- \) and above \( a_+ \) and is identically \( \pi \) in between.

By definition of \( \mathcal{R}_{\Delta S} \),
\[
\int_{\partial B_s(a_\pm)} |\nabla_{\tan} u|^2 dH^2 = 8\pi
\]
for all sufficiently small \( s > 0 \). Since \( \frac{\partial u}{\partial x} = 0 \) on the \( z \)-axis between \( a_- \) and \( a_+ \), we can find a positive \( \delta = \delta(\epsilon, s) < s \) so that
\[
|\partial \varphi \partial z| \leq \epsilon \text{ on } T = B^2_s(0) \times [-\rho + s - \delta, \rho - s + \delta].
\]
We define \( v = u \) outside of \( B_s(a_-) \cup T \cup B_s(a_+) \). In \( T \sim (B_s(a_-) \cup B_s(a_+)) \), define
\[
v(r, \theta, z) = (\cos \theta \sin \Psi, \sin \theta \sin \Psi, \cos \Psi)
\]
where \( \Psi(r, z) = \tan^{-1}(r\delta^{-1} \tan \varphi(\delta, z)) \). This implies that \( v|_{B^2_s(0) \times \{z\}} \) is conformal. Finally on the two spheres \( \partial B_s(a_\pm) \), \( v \) is defined and satisfies the conditions in Theorem 3.1 of [HKL2]. Therefore there are smooth harmonic mappings \( v_\pm \in A^r \)
with \( v_\pm|_{\partial B_s(a_\pm)} = v|_{\partial B_s(a_\pm)} \), and we may let \( v = v_\pm \) in \( B_s(a_\pm) \).

For each \( z \in [-\rho + s - \delta, \rho - s + \delta] \),
\[
\int_{B^2_s(0) \times \{z\}} \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \, dx \, dy = 2H^2 \left( v(B^2_s(0) \times \{z\}) \right) \leq 8\pi
\]
by the conformality of \( v|_{B^2_s(0) \times \{z\}} \). Moreover, \( |\frac{\partial v}{\partial z}| = O(\epsilon) \) in \( T \). Thus,
\[
\int_T |\nabla v|^2 \, dx
\]
\[
= \int_{-\rho + s - \delta}^{\rho - s + \delta} \int_{B^2_s(0) \times \{z\}} \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \, dx \, dy \, dz
\]
\[
= 8\pi(2\rho) + O(\epsilon^2).
\]
In \( \partial B_s(a_\pm) \cap T \),
\[
|\nabla v|^2 = 2\pi \left( \left( \frac{\partial \Psi}{\partial r} \right)^2 + \left( \frac{\partial \Psi}{\partial z} \right)^2 + r^{-2} \sin^2 \Psi \right)
\]
\[
\leq C \left( \delta^{-2} + \epsilon^2 + \delta^{-2} \right)
\]
\[
\leq C \delta^{-2}.
\]
Hence,
\[
\begin{align*}
\int_{\partial B_s(a_{\pm})} |\nabla \tan u|^2 d\mathcal{H}^2 & = \int_{\partial B_s(a_{\pm}) \sim T} |\nabla \tan v|^2 d\mathcal{H}^2 + \int_{\partial B_s(a_{\pm}) \cap T} |\nabla \tan v|^2 d\mathcal{H}^2 \\
& \leq 8\pi + C.
\end{align*}
\]

Since \( v \) is harmonic and smooth in the two balls \( B_s(a_{\pm}) \) we deduce from [HL2,4.1] that
\[
\int_{B_s(a_{\pm})} |\nabla v|^2 dx \leq s \int_{\partial B_s(a_{\pm})} |\nabla \tan v|^2 d\mathcal{H}^2 = O(s).
\]

We now conclude that the energy of \( v \) satisfies
\[
\begin{align*}
E(v) & \leq \int_{B^3} |\nabla u|^2 dx + \int_T |\nabla v|^2 dx + \int_{B_s(a_{-})} |\nabla v|^2 dx + \int_{B_s(a_{+})} |\nabla v|^2 dx \\
& = E(u) + 8\pi(2\rho) + O(\varepsilon^2) + O(s),
\end{align*}
\]
and the proof is completed by choosing \( s \) small enough. \( \Box \)

**Theorem 4.3** Suppose \( g : \partial B^3 \to S^2 \) is a continuous \( H^1 \) axially symmetric map which has degree 0 and whose angle function lies in \([\pi, \pi] \). Then
\[
\inf_{v \in A^* \cap C^\infty \setminus \partial B^3} E(v) = \inf_{u \in A^* \cap \partial B^3} E_1(u),
\]
and the infemas are attained by some real analytic axially symmetric harmonic function constructed as in [HKL2, Th.3.1]. Moreover, every minimizer is real analytic on \( B^3 \). A minimizer is also \( C^{k,\alpha} \) smooth at \( \partial B^3 \) provided \( g \) is \( C^{k,\alpha} \) smooth where \( k \in \{1, 2, \cdots, \infty, \omega\} \) and \( 0 < \alpha < 1 \).

**Proof** The inequality \( \leq \) follows from Lemma 4.3, and the opposite inequality is immediate because \( L(v) = 0 \) whenever \( v \in C^\infty \cap H^1(B^3, S^2) \).

We recall now the construction of [HKL2, Th.3.1]. We may assume that \( g(0,0,1) = (0,0,1) \) and \( g(0,0,-1) = (0,0,1) \) because \( g \) has degree 0. Choose a positive \( \eta = \eta(g) < \frac{1}{2} \) such that
\[
(0,0,1) \cdot g > \frac{1}{2} \text{ on } \partial B^3 \cap \{ r < 3\eta \}.
\]

Associated with this \( \eta \) is a positive \( \varepsilon \) and the piecewise barrier \( \Phi^+_{\varepsilon}(r) \) constructed in the proof of [HKL2, Th.3.1]. Let \( \varphi = \varphi(r, z) \) be an energy minimizer subject to the constraints that
\[
0 \leq \varphi(r, z) \leq \Phi^+_{\varepsilon}(r) \text{ and } u|_{\partial B^3} = g.
\]
where
\[ u(r, \theta, z) = (\cos \theta \sin \varphi(r, z), \sin \theta \sin \varphi(r, z), \cos \varphi(r, z)). \]

Here, we can again use [HKL2, Lemma 3.2] to compare with the energy minimizer in an annular cylinder to guarantee that we have strict inequality
\[ \varphi(r, z) < \Phi_\epsilon^+(r) \quad \text{for} \quad 2\eta \leq r \leq 2\eta + \epsilon. \]

We claim that \( u \) minimizes energy among maps in \( A^\pi \cap C^{\infty} \) which coincide with \( g \) on \( \partial B^3 \). If not, there is a \( v \in A^\pi \cap C^{\infty} \) with
\[
E(v) < E(u) \quad \text{and} \quad v|_{\partial B^3} = g.
\]

Then \( v(0,0,\cdot) \equiv (0,0,1) \) by the continuity and axial symmetry of \( v \). We can find a sufficiently small positive \( \epsilon_1 \leq \epsilon \) so that the angle function \( \varphi_v \) corresponding to \( v \) satisfies
\[
0 \leq \varphi_v(r, z) \leq \Phi_\epsilon^+(r) \quad \text{for} \quad 2\eta \leq r \leq 2\eta + \epsilon.
\]

where \( \varphi_\epsilon^+(r) \) is as in [HKL2, Th.3.1]. Let \( \varphi_1 = \varphi_1(r, z) \) be an energy minimizer subject to the new constraints that
\[
0 \leq \varphi_1(r, z) \leq \Phi_\epsilon^+(r) \quad \text{and} \quad u_1|_{\partial B^3} = g
\]
where \( u_1(r, \theta, z) = (\cos \theta \sin \varphi_1, \sin \theta \sin \varphi_1, \cos \varphi_1). \)

By (4.2) and the minimality of \( \varphi_1 \),
\[
E(u_1) \leq E(v).
\]

However since \( u_1|_{\partial B^3} = g \), we see, as in the use of [HKL2,3.2] in the proof of [HKL2,3.1], that \( \varphi_1 \) also satisfies
\[ \varphi_1(r, z) < \Phi_\epsilon^+(r) \quad \text{for} \quad 2\eta \leq r \leq 2\eta + \epsilon. \]

By the strict maximum principle with the barriers \( \Phi_\delta^+ \) for \( 0 < \delta \leq \epsilon \), we deduce that \( \varphi_1 \) must satisfy the original constraints
\[ 0 \leq \varphi_1(r, z) \leq \Phi_\epsilon^+(r). \]
hence,

\[(4.4) \quad E(u) \leq E(u_1).\]

Combining (4.1), (4.3), and (4.4) gives the desired contradiction.

Finally suppose \( w \in A^e \), \( E_1(w) = \inf_{u \in A^e, u|_{\partial B^3} = g} E_1(u) \), and \( a \) is a point on \( B^3 \cap (z-\text{axis}) \). Since, when working with functions in \( A^e \), all constructions in the proofs of Lemma 4.1 and Lemma 4.2 keep the angle function in the interval \([-\pi, \pi]\), we see that we may choose a positive \( \rho < 1 - |a| \) so that \( \partial B^3_\rho(a) \cap \text{spt}(I_w) = \emptyset \) and \( \partial B^3_\rho(a) \cap \text{Sing}(w) = \emptyset \). By the above discussion we may find a smooth \( v \in A^e \) so that \( v|_{\partial B_\rho(a)} = w|_{\partial B_\rho(a)} \) and

\[E(v) \leq E_1(u|_{B_\rho(a)}).\]

Replacing \( w \) by \( v \) on \( B_\rho(a) \) still gives an \( E_1 \) minimizer and so, by analytic continuation, \( w \) coincides with \( v \) on \( B_\rho(a) \). In particular, \( w \) is real analytic at \( a \).

For the boundary regularity we need only show, as in Corollary 3.2, that \( w \) is Hölder continuous near the poles \((0, 0, \pm 1)\). Here we can choose a positive number \( \rho_\pm \) so that

\[\partial B_{\rho_\pm}((0, 0, \pm 1)) \cap \text{spt}(I_w) = \emptyset\]

and

\[\partial B_{\rho_\pm}((0, 0, \pm 1)) \cap \text{Sing}(w) = \emptyset.\]

Since the barriers used in [HKL2, section 3] and above only required that the the boundary data be Lipschitz at the axis, we may repeat the above arguments to obtain a Hölder continuous axially symmetric map \( v_\pm : B^3 \cap B_{\rho_\pm}((0, 0, \pm 1)) \to S^2 \) with angle function in \([-\pi, \pi]\), with

\[v_\pm|_{B^3 \cap B_{\rho_\pm}((0,0, \pm 1))} = w|_{B^3 \cap B_{\rho_\pm}((0,0, \pm 1))}\]

and with \( E(v_\pm) \leq E_1(u|B^3 \cap B_{\rho_\pm}((0,0, \pm 1))) \). Replacement by \( v_\pm \) and analytic continuation shows that \( u \) coincides with \( v_\pm \) near \((0,0, \pm 1)\).

\[\square\]

### 4.4 Partial Regularity of Minimizers in Case \( \lambda = 1 \)
For boundary data whose angle function goes beyond the range \([-\pi, \pi]\) an \(E_1\) minimizer may have singularities. We already have 2 examples. For the boundary data \(g\) constructed in Theorem 4.1 one easily verifies that
\[
\begin{align*}
\inf \{ E_1(u) : u &\in H^1(B^3, S^2), u \text{ is axially symmetric, } \\
&\quad \text{and } u|_{\partial B^3} = g \} \\
\approx &\quad 2(8\pi \varepsilon) + 8\pi(4\varepsilon) \\
<< &\quad 2(8\pi) \\
\approx &\quad \inf \{ E(u) : u \in H^1 \cap C^0(\bar{B}^3, S^2), u \text{ is axially symmetric, } \\
&\quad \text{and } u|_{\partial B^3} = g \}.
\end{align*}
\]
For the boundary data \(h\) constructed in Theorem 4.2 one also finds that
\[
\begin{align*}
\inf \{ E_1(u) : u &\in H^1(B^3, S^2), u \text{ is axially symmetric, } \\
&\quad \text{and } u|_{\partial B^3} = h \} \\
\approx &\quad 4(8\pi \varepsilon) + 8\pi(4) \\
<< &\quad 2N(8\pi) \\
\approx &\quad \inf \{ E(u) : u \in H^1 \cap C^0(\bar{B}^3, S^2), u \text{ is axially symmetric, } \\
&\quad \text{and } u|_{\partial B^3} = h \}.
\end{align*}
\]
Thus minimizers of \(E_1\) in the axially symmetric class for boundary data \(g\) or \(h\) have singularities. Next, we will show that such singularities are isolated and of degree 0.

**Theorem 4.4** Suppose \(u \in H^1(B^3, S^2)\) is axially symmetric,
\[
u|_{\partial B^3} \in H^1(\partial B^3, S^2), \quad \operatorname{deg}(u|_{\partial B^3}) = 0,
\]
and \(u\) minimizes \(E_1\) among axially symmetric maps. Then
1. \(u\) is real analytic away from a discrete subset \(\operatorname{Sing}(u)\) of \(B^3 \cap (z-\text{axis})\).
2. \(\operatorname{Sing}(u)\) is finite in case \(u|_{\partial B^3}\) is continuous.
3. \(L(u) = 0\).
4. \(\operatorname{deg}(u|_{\partial B_r(a)}) = 0\) whenever \(B_r(a) \subset B^3\) and \(\partial B_r(a) \cap \operatorname{Sing}(u) = \emptyset\).

**Proof** We already know from section 3.2 that \(u\) is real analytic away from the \(z-\text{axis}\). By Theorem 4.3 the complement \(U\) of \(\operatorname{spt}(I_u)\) in the \(z-\text{axis}\) is dense. Moreover, by [SU1,2.7] and Lemma 3.3,
\[
U \cap \operatorname{Sing}(u) = \{ a \in U : \liminf_{r \to 0} r^{-1} \int_{B_r(a)} |\nabla u|^2 dx > 0 \}\]
is a relatively closed subset of $U$ of 1-dimensional Hausdorff measure zero. Thus,

$$V = B^3 \cap (z - \text{axis}) \sim \text{spt}(I_u) \sim \text{Sing}(u)$$

is an open and dense in $B^3 \cap (z - \text{axis})$.

Suppose $B$ is an open ball in $B^3$ centered on the $Z$-axis with $\partial B \cap (z - \text{axis}) \subset V$. Then $u|_{\partial B}$ is real analytic and of degree 0 because $\partial B$ does not intersect either $\text{Sing}(u)$ or $\text{spt}(I_u)$. To prove (1), (3), and (4) we may, by replacing $B^3$ by each such $B$ and rescaling, assume that

$$u|_{\partial B^3}$$

is real analytic and of degree 0.

We also assume, to fix notations, that $u(0,0,1) = (0,0,1)$. (The case $u(0,0,1) = (0,0,1)$ is treated by a similar argument.) Then $u(0,0, -1) = (0,0,1)$ also because $u|_{\partial B^3}$ is axially symmetric and has degree 0. Moreover,

$$u(0,0,\alpha) = (0,0,1) \text{ for any point } (0,0,\alpha) \in V$$

because $u|_{\partial B^3(1-\alpha)}((0,0,\frac{1}{2}(1+\alpha)))$ has degree 0.

By Lemma 3.1, the set

$$A = \{(x_1, x_2, x_3) \in B^3 : x_1 = 0, x_2 > 0, u((x_1, x_2, x_3)) = (0,0, -1)\}$$

is a real analytic subset of the open half-disk $B^3 \cap \{x_1 = 0, x_2 > 0\}$. The set $A$ has dimension at most one because otherwise [F,3.4.8(11)] $A$ would contain a relatively open subset of the half-disk, $u^{-1}\{(0,0, -1)\}$ would contain an open subset of $B^3$, and $u$ would be identically $(0,0, -1)$ by analytic continuation. Also $A$ does not contain any isolated points because otherwise such a point would give an isolated extrema for the angle function $\varphi$ and contradict the maximum principle. Thus, for each point $a \in A$, there is a positive $\rho$ so that $A \cap \bar{B}_\rho(a)$ is a finite union of rectifiable paths which have both endpoints in $\partial B\rho(a)$, which pass through $a$, and which intersect only at $a$ (i.e. there are an even number of branches emanating from $a$ [BH,3.7]). Since $u$ is analytic near $\partial B^3$, there is a positive $\sigma < 1$ so that $A \sim B_\sigma$ is a finite union of disjoint arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$ whose closures join a point of $\partial B_\sigma$ to a point of $\partial B^3$. Finally we observe that $A$ does not contain any simple closed curves. Otherwise such a curve $\Gamma$ would enclose an open subregion $R$ of the open half-disk, and we could obtain another $E_1$ minimizer by simply changing $u$ on the corresponding solid region $\{x \in B^3 : (0,|x_2|, x_3) \in R\}$ to be identically $(0,0, -1)$. Analytic continuation would again give a contradiction.
We now consider the behavior of a maximal (i.e., inextendible) path $\Gamma$ in $A$. Since $A$ does not contain any simple closed curves, $\Gamma$ is the image of some embedding $\gamma$ of $\mathbb{R}$ into the open half-disk. We claim that $\lim_{t \to +\infty} \gamma(t)$ exists. This is clear if $\gamma(t)$ exits from $B_\sigma$ sometime as $t$ is increasing. Suppose now that $\gamma(t)$ lies in $B_\sigma$ for all $t$ sufficiently large and $K$ is the set of limit points of sequences $\gamma(t_i)$ corresponding to sequences $t_i$ approaching $\infty$. From the previous paragraph we see that $K$ is a nonempty compact subset of $B_\sigma \cap (z-$axis). If $K$ contained two points $(0,0,\alpha)$ and $(0,0,\beta)$ with $\alpha < \beta$, we could by Lemma 4.2 choose an open interval $J$ in $V \cap \{(x_0,x_3) : x_3 < x < \beta\}$. Since $u$ is analytic on $\{x : (0,0,x_3) \in J\}$ and $A$ is one dimensional, we may find a point $(0,0,z)$ in $J$ so that the slice $A \cap \{x : x_3 = z\}$ is a finite set. However, since $K$ contains both $(0,0,\alpha)$ and $(0,0,\beta)$, the intermediate value theorem readily implies that the subset $\Gamma \cap \{x : x_3 = z\}$ must be infinite. This contradiction shows that $K$ has exactly one point and hence $\lim_{t \to +\infty} \gamma(t)$ exists. Similarly, $\lim_{t \to -\infty} \gamma(t)$ exists.

To complete our qualitative description of $A$ we will show that for each such maximal curve curve $\gamma$ in $A$, at least one of the limit points $\lim_{t \to -\infty} \gamma(t)$ or $\lim_{t \to +\infty} \gamma(t)$ lies in $\partial B^3$. Suppose for contradiction that neither limit point belongs to $\partial B^3$. Then both lie in $B_\sigma \cap (z-$axis). In case these limit points coincide, $\gamma(\mathbb{R})$ would again enclose an open subregion of the open half-disk. Changing $u$ on the corresponding solid region to be identically $(0,0,-1)$ would again increase neither $E$ nor $L$ and again lead to a contradiction by analytic continuation. So now suppose that $J$ is an open interval in $B_\sigma \cap (z-$axis) whose endpoints are the two limit points of $\gamma$, and consider the region $R$ in the open half-disk bounded by $\gamma(\mathbb{R}) \cup J$. Then

$$\Omega = \{x \in B^3 : (0,|x_2|,x_3) \in R\} \cup J$$

is an open region in $\mathbb{R}^3$ bounded by the surface $\{x \in B^3 : (0,|x_2|,x_3) \in \gamma(\overline{R})\}$. Let $v$ be the mapping obtained from $u$ by changing $u$ on $\Omega$ to be identically $(0,0,-1)$. To compare $E_1(v)$ with $E_1(u)$, recall from Lemma 1.3 that

$$I_u \cup (B^3 \sim \Omega) = I_u \cup (B^3 \sim \Omega)$$

so that if $I_u = \pm[J]$, then $I_v = \pm[J_v]$ where $J_v \sim J = J_u \sim J$. Moreover, by the constancy theorem [F,4.1.7],

either $[J_v] \cup \Omega = 0$ or $[J_v] \cup \Omega = [J]$.
In case $\lfloor J_v \rfloor \subseteq \Omega = 0$,

$$E_1(v) = \int_B |\nabla v|^2 dx + 8\pi \mathcal{H}^1(J_v)$$
$$= \int_{B^3 \setminus \Omega} |\nabla u|^2 dx + 8\pi \mathcal{H}^1(J_u \sim J)$$
$$\leq E_1(u),$$

and the $E_1$ minimality of $v$ again gives a contradiction by analytic continuation.

In case $\lfloor J_v \rfloor \subseteq J = \lfloor J \rfloor$, we deduce that

$$\partial (I_u \cup \Omega) = -\partial (I_u \cup (B^3 \sim \Omega))$$
$$= -\partial (I_u \cup (B^3 \sim \Omega))$$
$$= \partial (I_u \cup \Omega)$$
$$= \pm \partial \lfloor J \rfloor$$
$$= \pm (\delta_{(0,0,\beta)} - \delta_{(0,0,\alpha)}).$$

Recalling Lemma 3.4, we see that the cartesian current

$$T_{u,\Omega} = \lfloor \text{graph}(u|_{\Omega}) \rfloor + (I_u \cup \Omega) \times \lfloor S^2 \rfloor$$

has

$$\partial T_{u,\Omega}$$
$$= \left\{ -\partial \lfloor \text{graph}(u|_{B^3 \setminus \Omega}) \rfloor - (I_u \cup (B^3 \sim \Omega)) \times \lfloor S^2 \rfloor \right\} \cup B^3$$
$$= \left\{ -\partial \lfloor \text{graph}(v|_{B^3 \setminus \Omega}) \rfloor - (I_v \cup (B^3 \sim \Omega)) \times \lfloor S^2 \rfloor \right\} \cup B^3$$
$$= \partial T_{v,\Omega}$$
$$= \lfloor \partial \Omega \times \{(0,0,-1)\} \rfloor \pm \left( \delta_{(0,0,\beta)} \times \lfloor S^2 \rfloor \right) \mp \left( \delta_{(0,0,\alpha)} \times \lfloor S^2 \rfloor \right).$$

Thus the cartesian current

$$S = T_{u,\Omega} + \lfloor (B^3 \sim \Omega) \times \{(0,0,-1)\} \rfloor$$

has $\partial S = \lfloor \partial B^3 \times \{(0,0,-1)\} \rfloor \pm \left( \delta_{(0,0,\beta)} \times \lfloor S^2 \rfloor \right) \mp \left( \delta_{(0,0,\alpha)} \times \lfloor S^2 \rfloor \right)$. From [GMS2] and [BCL] we deduce the lower bound

$$\int_{\Omega} |\nabla u|^2 dx + 8\pi \mathcal{H}^1(J_u \cap J) = \mathcal{D}(S, B^3) \geq 8\pi |\beta - \alpha|.$$
Thus,

\[
\begin{align*}
E(v) &= \int_B |\nabla v|^2 \, dx + 8\pi \mathcal{H}^1(J_v) \\
&= \int_{B^3 - \Omega} |\nabla u|^2 \, dx + 8\pi \mathcal{H}^1(J_u \sim J) + 8\pi |\beta - \alpha| \\
&\leq \int_{B^3 - \Omega} |\nabla u|^2 \, dx + 8\pi \mathcal{H}^1(J_u \sim J) + \int_{\Omega} |\nabla u|^2 \, dx + 8\pi \mathcal{H}^1(J_u \cap J) \\
&= E_1(u),
\end{align*}
\]

\(v\) is \(E_1\) minimizing, and a contradiction again follows by unique continuation.

An important consequence of the previous paragraph is that, by following such
maximal curves in \(A\), one can associate, to each limit point of \(A\) on the \(z\)-axis, a
distinct arc \(\Gamma_i\) of \(A \sim B_\varphi\). Thus the set \(\bar{A} \cap (z\text{-axis})\) is a finite set containing no
more than \(j\) points.

Let \(a \in B^3 \cap (z\text{-axis}) \sim \bar{A}\), and choose an open ball \(B\) in \(B^3 \sim \bar{A}\) that is centered
at \(a\) and has

\[
\partial B \cap \text{Sing}(u) = \emptyset \quad \text{and} \quad \partial B \cap \text{spt}(I_u) = \emptyset.
\]

Then, as before, \(u|_{\partial B}\) is real analytic and \(u\) equals \((0,0,1)\) at the two points of
\(\partial B \cap (z\text{-axis})\). Moreover, since \(B \cap \bar{A} = \emptyset\), we may choose on \(\bar{B}\) an angle function
\(\varphi\) for \(u\) so that \(\varphi\) vanishes on \(\partial B \cap (z\text{-axis})\) and \(|\varphi| < \pi\). We may now apply
Theorem 4.3 to find a smooth (real analytic) axially symmetric map \(v : B \to S^2\) with
\(v|_{\partial B} = u|_{\partial B}\) and \(E(v) = E_1(u)\). By analytic continuation, \(u\) must coincide
with \(v\) on \(B\). Thus \(u\) is analytic on all of \(B\),

\[
B \cap \text{Sing}(u) = \emptyset, \quad \text{and} \quad B \cap \text{spt}(I_u) = B \cap \text{spt}(I_v) = \emptyset.
\]

We now conclude that \(\text{Sing}(u)\) is contained in the finite set \(\bar{A} \cap (z\text{-axis})\) and that
\(L(u) = 0\). This completes the proof of (1) and (3).

Suppose

\[
B_r(a) \subset B^3
\]

and

\[
\partial B_r(a) \cap \text{Sing}(u) = \emptyset.
\]

To show that \(\text{deg}(u|_{\partial B_r(a)}) = 0\), it suffices to consider the case when \(a \in \text{Sing}(u)\) and
\(B_r(a) \cap \text{Sing}(u) = \{a\}\). But then if \(\text{deg}(u|_{\partial B_r(a)})\) were nonzero, \(a \in \text{spt}(\text{div} D(u))\) and
\( \partial I_u \) would be nonzero in \( B^3 \), contradicting that \( 8\pi M(I_u) = L(u) = 0 \). This proves (4).

To verify (2) we now assume only that

\[
\text{if } u|_{\partial B^3} \text{ is a continuous } H^1 \text{ map,}
\]

and, as before, that \( u(0,0,1) = (0,0,1) \). Choose a positive \( \rho < 1 \) so that

\[
u^{-1}\{(0,0,-1)\} \cap \partial B^3 \cap B_\rho((0,0,1)) = \emptyset,
\]

and, by Fubini's theorem as in [HKL2,4.1], so that

\[
F = u^{-1}\{(0,0,-1)\} \cap \partial B_\rho((0,0,1)) \cap \{x_1 = 0, x_2 > 0\}
\]

is a finite set. We find that, as before, each point \( a \) of \( \text{Sing}(u) \) other than \((0,0,1)\) or \((0,0,-1)\) is a limit point of a maximal curve in

\[
\{(x_1,x_2,x_3) \in B^3 \cap B_\rho((0,0,1)) : u(x_1,x_2,x_3) = (0,0,-1), x_1 = 0, x_2 > 0\}
\]

whose other limit belongs to \( F \). Since only finitely many such curves approach each point of \( F \), \( B_\rho((0,0,1)) \cap \text{Sing}(u) \) is finite. Similarly, \( B_\rho((0,0,-1)) \cap \text{Sing}(u) \) is finite, and the finiteness of \( \text{Sing}(u) \) now follows from (1). \( \square \)

4.5 Remarks

(1) **Boundary Regularity**

It seems likely that boundary regularity does not hold for \( E_1 \) minimizers. A likely counterexample is \( u|_{B^3_{1/2}}(0,0,-1/2) \) where \( u : B^3 \to S^2 \) is an \( E_1 \) minimizer whose axially symmetric boundary data \( g \) is defined by having its restriction to the circle \( S^2 \cap \{x_1 = 0\} \) being a constant speed 2 map to itself. \( g \) admits no continuous axially symmetric extension, and we conjecture that an \( E_1 \) minimizer for \( g \) has a single singularity at the origin, and that \( u|_{\partial B^3_{1/2}}(0,0,-1/2) \) is smooth.

(2) **Minimizing \( E - 8\pi \lambda L \).**

From Lemma 1.3 we have the universal bound

\[
L(u) = M(I_u) = \mathcal{H}^1(J_u) \leq 2
\]
for any $u \in H^1(B^3, S^2)$ such that $u|_{\partial B^3}$ has degree 0. From the proofs of [BBC, Thm.3] and Lemma 3.1, we readily obtain, for any $\lambda \in (0, 1]$ and any degree 0, axially symmetric boundary data, the existence of an $E - 8\pi \lambda L$ minimizer among axially symmetric maps. To analyse the regularity of such a minimizer, we observe that if $I_u = \pm [J]$, then

\[
E(u) - 8\pi \lambda M(J_u) = E(u) - 8\pi \lambda \mathcal{H}^1(J_u) = E(u) + 8\pi \lambda \mathcal{H}^1(B^3 \cap (z - \text{axis}) \sim J_u) - 16\pi
\]

Thus minimizing $E - 8\pi \lambda L$ among axially symmetric mappings with boundary data $g$ is equivalent to minimizing $D_{\lambda} (\cdot, B^3)$ among axially symmetric cartesian currents with boundary equal to

\[
\{ [\text{graph} g] \pm (\delta_{(0,0,1)} - \delta_{(0,0,-1)}) \times [S^2] \}.
\]

From Lemma 3.4, Theorem 3.4, and Theorem 4.4, we deduce that the singularities of an $E - 8\pi \lambda L$ minimizer form a discrete subset of $B^3 \cap (z - \text{axis})$.

Finally, using some arguments from the proof of Theorem 4.4 we can show that the minimization problem in Theorem 4.3 for functions in $A^\pi$ is equivalent to one over the larger class of all axially symmetric functions.

**Theorem 4.5** If $u$ is an $E_1$ minimizer among axially symmetric maps and $g = u|_{\partial B^3}$ is a continuous $H^1$ axially symmetric map which has degree 0 and has an angle function with values in $[-\pi, \pi]$ then $u$ is real analytic on $B^3$ and admits an angle function with values in $[-\pi, \pi]$. In particular, by Theorem 4.3,

\[
\inf \left\{ \int_\Omega |\nabla w|^2 \, dx : w \in H^1(\Omega, S^2), w \text{ is axially symmetric,} \\
w|_{\partial \Omega} = g \right\}
\]

\[
= \inf \left\{ \int_\Omega |\nabla w|^2 \, dx : w \in A^\pi \cap C^\infty(\Omega, S^2), w \text{ is axially symmetric,} \\
w|_{\partial \Omega} = g \right\}.
\]

**Proof** As in the proof of Theorem 4.4 one examines the set

\[
A = \{(x_1, x_2, x_3) \in B^3 : x_1 = 0, x_2 > 0, u((x_1, x_2, x_3)) = (0, 0, -1)\}.
\]

Suppose that $u$ does not admit an angle function with values in $[-\pi, \pi]$. Then by Theorem 4.3 and the hypothesis there is a component $R$ of the complement of $A$ in
the open half-disk whose closure intersects \( \partial B^3 \) only possibly at the poles. Following the proof of Theorem 4.4, \( \overline{A} \cap (z-\text{axis}) \) is discrete in \( B^3 \), and \( J = \overline{R} \cap (z-\text{axis}) \) is a countable union of disjoint closed intervals accumulating possibly only at the poles. As before, we let \( v \) be the mapping obtained from \( u \) by changing \( u \) on

\[
\Omega = \{ x \in B^3 : (0, |x_2|, x_3) \in R \} \cup J
\]

to be identically \((0,0,-1)\), and verify that

\[
E_1(v) = \int_B |\nabla v|^2 dx + 8\pi \mathcal{H}^1(J_v) \leq E_1(u).
\]

Thus, \( v \) is \( E_1 \) minimizing, and a contradiction again follows by unique continuation. \( \Box \)
CHAPTER 5
OTHER RELATED RESULTS

5.1 Prescribed Singularity

Here we recall from Chapter 4, section 4.3 the class $\mathcal{R}_{\mathcal{A}}^\gamma$ of axially symmetric maps with only finitely many singularities and an angle function in $[-\pi, \pi]$.

**Theorem 5.1** Let $g$ be a smooth axially symmetric Lipschitz map of $\partial B^3$ into $S^2$ with angle function in $[-\pi, \pi]$ and with $g(0,0,-1) = g(0,0,1)$. For any finite subset $a^-, a^+_1, a^-_2, a^+_2, \ldots, a^-_j, a^+_j$ of consecutive points on the $z$-axis, there exists an axially symmetric harmonic map

$$u \in \mathcal{R}_{\mathcal{A}}^\gamma \cap C^\infty \left( \tilde{B}^3 \sim \bigcup_{i=1}^j \{a^-_i\} \cup \{a^+_i\} \right)$$

which satisfies $u|_{\partial B^3} = g$ and which has a singularity at $a^\pm_i$ of degree $\pm 1$, for each $i = 1, 2, \ldots, j$.

**Proof** The conditions on $g$ imply that it has degree 0. By the remarks in the beginning of section 4.3 it is not difficult to construct a (not necessarily harmonic) map $v \in A^\gamma \cap C^\infty \left( \tilde{B}^3 \sim \bigcup_{i=1}^j \{a^-_i\} \cup \{a^+_i\} \right)$ which satisfies $u|_{\partial B^3} = g$ and which has a singularity at $a^\pm_i$ of degree $\pm 1$ for each $i = 1, 2, \ldots, j$. Recall from [BBC] the weak $H^1$ lower-semicontinuity of the functional

$$F_u(u) = E(u) + 8\pi L(u,v)$$

where

$$L(u,v) = (4\pi)^{-1} \sup_{\xi : B^3 \rightarrow \mathbb{R}, \|\xi\|_{\infty} \leq 1} \left\{ \int_{\Omega} (D(u) - D(v)) \cdot \nabla \xi \, dx \right\}.$$ 

Thus there exists a mapping $u \in A^\gamma$ that minimizes $F_u$ among mappings in $A^\gamma$ that have trace $g$ on $\partial B^3$.

As in section 1.3 we readily find that

$$L(u) = M(I_u)$$

for some 1-dimensional multiplicity one rectifiable current $I_u$ with

$$-\partial I_u \times [S^2] = (\partial[\text{graph}(u)] - \partial[\text{graph}(v)]) \ll (\Omega \times S^2).$$
Let $a$ be any point in $B^3 \cap (z-\text{axis}) \sim \bigcup_{i=1}^j \{a_i^-\} \cup \{a_i^+\}$ and $\rho = \text{dist}(a, \partial B^3 \cup \bigcup_{i=1}^j \{a_i^-\} \cup \{a_i^+\})$. Then, as in section 1.3,

$$I_u \perp B_\rho(a) = \pm [J]$$

for some measurable subset $J$ of $B^3_\rho(a) \cap (z-\text{axis})$. As in Lemma 4.2 we find that the complement of $\bar{J}$ is dense in $B_\rho(a) \cap (z-\text{axis})$. So we can find a positive $\sigma < \rho$ so that $\bar{J} \cap \partial B_\sigma(a) = \emptyset$. But then, by the smoothness of $v|_{B_\sigma(a)}$, $u|_{B_\sigma(a)}$ is $E_1$ minimizing with $L(u|_{B_\sigma(a)}) = H^1(J)$. From Theorem 4.3 we conclude that $H^1(J) = 0$ and $u|_{\partial B(a)}$ is smooth.

Thus $I_u = 0$ and $u$ is smooth on $B^3 \sim \bigcup_{i=1}^j \{a_i^-\} \cup \{a_i^+\}$. A similar argument using Lemma 4.2 and small balls centered at $(0,0,\pm 1)$ shows that $u$ is smooth near $\partial B^3$. Since

$$\partial \left[\text{graph}(u)\right] \subset (\Omega \times S^2) = \partial \left[\text{graph}(v)\right] \subset (\Omega \times S^2) = \sum_{i=1}^j (\delta_{a_i^+} - \delta_{a_i^-}) \times [S^2],$$

the mapping $u|_{\partial B_\epsilon(a^+_0)}$ has degree $\pm 1$ for all sufficiently small positive $\epsilon$.  

\[\square\]

5.2 An Example

It is well known that $u_0(x) = x/|x|$ is the unique minimizer of the energy functional $\int_{B^3} |\nabla u|^2 dx$ among maps $u \in H^1(B^3, S^2)$ such that $u(x) = x$ for $x \in \partial B^3$ [BCL].

Here we show that

**Theorem 5.2** For any $x_0$ in $\bar{B}^3$, there is a harmonic map $u : B^3 \to S^2$ such that

1. $u(x) = x$ on $\partial B^3$;
2. $u$ is smooth in $\bar{B}^3 \sim \{x_0\}$, i.e., $x_0$ is the only singularity of $u$.

**Proof** First suppose that $x_0$ is in the interior of $B^3$. By rotating $B^3$ and $S^2$, we may assume that $x_0 = (0,0,a)$, $0 < a < 1$. We look for harmonic maps which is axially symmetric in this coordinates system.

For any smooth $\phi : D \to \mathbb{R}$, where $D = \{(r,z) : r^2 + z^2 < 1, \quad r > 0\}$, define

$$E(\phi) = 2\pi \int_D r \left( \frac{\partial \phi}{\partial r} \right)^2 + r \left( \frac{\partial \phi}{\partial z} \right)^2 + \frac{\sin^2 \phi}{r} \quad drdz.$$

Any critical point $\phi$ of $E$ satisfies in $D$ the P.D.E.

$$\frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial \phi}{\partial z} \right) - \frac{\sin 2\phi}{2r} = 0.$$

(5.1)
Let \( \rho, \theta \) be polar coordinates centered at \((0,a) \in D\), i.e., \( r = \rho \sin \theta, \; z = \rho \cos \theta + a \). In coordinates \( \rho, \theta \), (5.1) becomes
\[
\rho \sin \theta \left( \frac{\partial^2 \varphi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \varphi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right) + \sin \theta \frac{\partial \varphi}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial \varphi}{\partial \theta} = \frac{\sin 2\varphi}{2 \rho \sin \theta} = 0.
\]

Suppose \( \varphi \) is independent of \( \rho \), hence \( \frac{\partial^2 \varphi}{\partial \rho^2} = \frac{\partial^2 \varphi}{\partial \theta^2} = 0 \), and
\[
0 = \sin \theta \frac{d^2 \varphi}{d \theta^2} + \cos \theta \frac{d \varphi}{d \theta} - \frac{\sin 2\varphi}{2 \sin \theta}
= \frac{d}{d \theta} \left( \sin \theta \frac{d \varphi}{d \theta} \right) - \frac{\sin 2\varphi}{2 \sin \theta}.
\]
Thus
\[
\frac{d}{d \theta} \left( \left( \sin \theta \frac{d \varphi}{d \theta} \right)^2 \right) = \sin 2\varphi \frac{d \varphi}{d \theta}
= \frac{d}{d \theta} \left( \sin^2 \varphi \right)
\]
and
\[
\left( \sin \theta \frac{d \varphi}{d \theta} \right)^2 = \sin^2 \varphi + C,
\]
for some constant \( C \). If we set \( \varphi(0) = 0 \), then \( C = 0 \) and
\[
(5.2) \quad \left| \sin \theta \frac{d \varphi}{d \theta} \right| = |\sin \varphi|.
\]

The general solution of (5.2) is:
\[
\cos \varphi_c = \frac{\sinh c + \cosh c \cos \theta}{\cosh c + \sinh c \cos \theta}, \quad -\infty < c < \infty.
\]
When \( c = 0 \), \( \varphi_0 = \theta \); when \( c > 0 \), \( \varphi_c \leq \theta \); and when \( c < 0 \), \( \varphi_c \geq \theta \). Also \( \varphi_c \to 0 \) as \( c \to \infty \) and \( \varphi_c \to \pi \) as \( c \to -\infty \).

Define \( A = \{(r,z) : r^2 + z^2 = 1, \; r \geq 0\}, \; g : A \to [0,\pi], \; g(r,z) = \arctan(r/z) \).
If we substitute \( \varphi \) by \( g \) in (1) and restrict \( r, z \) to \( \{(r,z) : r^2 + z^2 = 1\} \), then (1) gives the identity map from \( S^2 \) to \( S^2 \). In coordinates \( \rho, \theta \), \( A = \{(\rho,\theta) : \rho = \rho_0(\theta), \; 0 \leq \theta \leq \pi\} \) for some function \( \rho_0 \). Write \( g(\theta) = g(\rho_0(\theta) \sin \theta, \rho_0(\theta) \cos \theta + a) \). Then \( g \) satisfies
\[
\frac{\sin g}{\cos g - a} = \tan \theta.
\]
Clearly \( g(\theta) \leq \theta \). Also \( g \) is montone increasing, \( g(0) = 0 \), \( g(\pi) = \pi \), and
\[
\frac{d}{d\theta} g(\theta) = \frac{1 - 2a \cos g + a^2}{1 - a \cos g}.
\]
Therefore \( g'(0) = 1 - a \), and \( g'(<\pi) = 1 + a \). We can find some \( c > 0 \) such that
\[
\varphi_c(\theta) \leq g(\theta) \leq \theta = \varphi_0(\theta).
\]

Now we can proceed as in [Z] and consider the following problem:

\[
\text{Minimize } E(\psi) = 2\pi \int_D \left( r \left( \frac{\partial \psi}{\partial r} \right)^2 + r \left( \frac{\partial \psi}{\partial z} \right)^2 + \frac{\sin^2 \psi}{r} \right) \ dr \, dz
\]

among maps \( \psi : D \to [0, \pi] \), \( \psi = g \) on \( A \),

(5.3) \quad \varphi_c(\theta) \leq \psi(\rho, \theta) \leq \varphi_0(\theta).

As in [Z], a maximum principle implies that equality holds only for \( \theta \in \{0, \pi\} \) because the constraints \( \varphi_c, \varphi_0 \) are critical points of \( E \). Thus a minimizer \( \psi_a \) is a critical point of \( E \) and is regular in \( D \). Both \( \varphi_c \) and \( \varphi_0 \) are continuous in \( \bar{D} \sim \{(0, a)\} \)

\[
\varphi_c = \varphi_0 = 0 \quad \text{for } \ r = 0, z > a,
\]

\[
\varphi_c = \varphi_0 = \pi \quad \text{for } \ r = 0, z < a.
\]

By the constraint (5.3), we conclude that \( \psi_a \) is continuous in \( \bar{D} \sim \{(0, a)\} \),

\[
\psi_a = 0 \text{ for } r = 0, z > a \quad \text{and} \quad \psi_a = \pi \text{ for } r = 0, z < a.
\]

Then
\[
u_a(r, \alpha, z) = (\sin \psi_a \cos \alpha, \sin \psi_a \sin \alpha, \cos \psi_a)
\]
is the desired axially symmetric harmonic map.

Now suppose that \( x_0 \) is on the boundary of \( B^3 \). By rotating \( B^3 \) and \( S^2 \), we may assume that \( x_0 = (0, 0, 1) \).

For any \( 0 < a < 1 \), let \( v_a : B^3 \to S^2 \) be obtained by the homogeneous extension, with respect to the point \( (0, 0, a) \), of the identity map from \( S^2 \) to \( S^2 \). One can compute the energies of \( v_a \), \( 0 < a < 1 \), as in [BCL, 7.B] and see that they are uniformly bounded. If \( u_a \) is the axially symmetric harmonic map we obtained in the above, then \( E(u_a) \leq E(v_a) \) and \( \{E(u_a), 0 < a < 1\} \) is uniformly bounded. Thus
there is a subsequence \( \{u_{a_i}\} \), as \( a_i \to 1 \), \( u_{a_i} \) converges weakly in \( H^1(B^3, S^2) \) to a map \( u_1 \) which is also axially symmetric. Also, \( u_1 \) is harmonic, \( u_1(x) = x \) on \( \partial B^3 \) in the sense of traces. Moreover \( u_1 \) is completely regular on \( \tilde{B}^3 \sim \{(0,0,1)\} \). 

\[ \square \]

**Remarks**

(1) According to the definition of R. Schoen [S], the map \( u_1 \) in the above is stationary.

(2) Suppose \( v \) is a harmonic map from \( B^3 \) to \( S^2 \), \( v(x) = x \) on \( \partial B^3 \), and \( v \) is smooth in a neighborhood of \( \partial B^3 \). If \( v \) is stationary, then \( v(x) = x/|x| \). The proof is quite easy. If \( v \) is stationary and is smooth in a neighborhood of \( \partial B^3 \), then it satisfies the monotonic inequality:

\[
\int_{B^3} |\nabla v|^2 dx + \int_{\partial B^3} |r v|^2 d\sigma \leq \int_{\partial B^3} |\nabla_{\tan} v|^2 d\sigma,
\]

where \( d\sigma \) is the volume element on \( \partial B^3 \), \( \nabla_{\tan} \) is the tangential derivatives on \( \partial B^3 \). Therefore

\[
\int_{B^3} |\nabla v|^2 dx \leq \int_{\partial B^3} |\nabla_{\tan} v|^2 d\sigma \leq 8\pi.
\]

However, all maps from \( B^3 \) to \( S^2 \) have energies bigger than \( 8\pi \) [BCL]. The map \( x/|x| \) is the only map has energy \( 8\pi \). Hence, \( v = x/|x| \).

### 5.3 A Heat Flow Problem

The evolution problem for harmonic maps from the 3 dimensional ball to the 2 sphere with prescribed boundary data has been studied by many mathematicians. In [G], the existence of of axially symmetric heat flow from \( B^3 \) to \( S^2 \) is established. Here, we consider the case where the boundary data has degree one and the solution has a singularity at the origin.

As in section 5.2, a map \( u : B^3 \to S^2 \) is axially symmetric if

\[
u(r, \alpha, z) = (\sin \varphi \cos \alpha, \sin \varphi \sin \alpha, \cos \varphi)\]

for some smooth \( \varphi : D \to \mathbb{R} \), where \( D = \{(r, z) : r^2 + z^2 < 1, \quad r > 0\} \). \( v : B^3 \times \mathbb{R} \to S^2 \) is an axially symmetric heat flow solution if \( v \) satisfies the harmonic heat flow equation

\[
v_t = \Delta v + |\nabla v|^2 v
\]
and for each \( t \), the map \( v(\cdot, t) \) is axially symmetric. Let

\[
v(r, \alpha, z, t) = (\sin \varphi(r, z, t) \cos \alpha, \sin \varphi(r, z, t) \sin \alpha, \cos \varphi(r, z, t))
\]

where \( \varphi : D \times \mathbb{R} \to \mathbb{R} \). Then \( v \) is an axially symmetric heat flow if

\[
(5.4) \quad \varphi_t = \Delta \varphi + \frac{1}{r} \frac{\partial \varphi}{\partial r} - \frac{\sin 2\varphi}{2r^2}.
\]

Grotowski showed that given a smooth function \( \varphi_0, \ 0 \leq \varphi_0 \leq \pi \), there is solution \( \varphi(\cdot, t) \) to (5.4) so that \( \varphi(\cdot, 0) = \varphi_0 \) and \( \varphi(\cdot, t) = \varphi_0 \) on \( \partial D \) for all \( t > 0 \). Also he proved that when \( t \to \infty \), \( \varphi(\cdot, t) \) converges to a smooth solution to (5.1).

Here we consider the same heat equation (5.4) with initial data \( \varphi_0 \) which is smooth in \( D \) but is discontinuous at the origin.

**Theorem 5.3** Let \( \varphi_0 \) be a function so that

1. \( 0 \leq \varphi_0 \leq \pi \),
2. \( \varphi_0(0, z) = 0 \) for \( z > 0 \) and \( \varphi_0(0, z) = \pi \) for \( z < 0 \).
3. There are \( \varphi_1 \) and \( \varphi_2 \), solutions to (5.2) obtained in section 5.2 so that \( \varphi_1 \leq \varphi_0 \leq \varphi_1 \).

Then there is a solution \( \varphi(\cdot, t) \) to equation (5.4), \( \varphi(\cdot, 0) = \varphi_0 \), \( \varphi(\cdot, t) = \varphi_0 \) on \( \partial D \) for all \( t > 0 \). Also when \( t \to \infty \), \( \varphi(\cdot, t) \) converges to a solution \( \varphi_\infty \) to (5.1) \( \varphi - \infty = \varphi_0 \) on \( \partial D \).

For any \( \epsilon > 0 \), define \( D^\epsilon = D \cap \{(r, z) : r > \epsilon \} \). Since (5.4) is uniformly parabolic in \( D^\epsilon \), there is a smooth solution to (5.4) in \( D^\epsilon \), call it \( \varphi^\epsilon \), so that \( \varphi^\epsilon(\cdot, 0) = \varphi_0 \) and \( \varphi^\epsilon(\cdot, t) = \varphi_0 \) on \( \partial D^\epsilon \) for all \( t > 0 \).

**Lemma 5.1** Let

\[
E^\epsilon(t) = \int_D r \left( \frac{\partial \varphi^\epsilon}{\partial r} \right)^2 + r \left( \frac{\partial \varphi^\epsilon}{\partial z} \right)^2 + \frac{\sin^2 \varphi^\epsilon}{r} \, drdz.
\]

\( E^\epsilon(t) \) is non-increasing. In fact,

\[
\frac{d}{dt} E^\epsilon(t) \leq - \int_{D^\epsilon} \left( \frac{\partial \varphi^\epsilon}{\partial t} \right)^2 \, drdz.
\]

**Proof** Since \( \varphi^\epsilon \) is a heat flow solution, using integration by parts,

\[
\frac{d}{dt} E^\epsilon(t)
\]
\[
\begin{align*}
&= \frac{d}{dt} \int_{D^t} r \left( \frac{\partial \varphi^t}{\partial r} \right)^2 + r \left( \frac{\partial \varphi^t}{\partial z} \right)^2 + \frac{\sin^2 \varphi^t}{r} \ dr dz \\
&= \int_{D^t} r \left( \frac{\partial \varphi^t}{\partial r} \right) \frac{\partial}{\partial r} \left( \frac{\partial \varphi^t}{\partial t} \right) + r \left( \frac{\partial \varphi^t}{\partial z} \right) \frac{\partial}{\partial z} \left( \frac{\partial \varphi^t}{\partial t} \right) + \frac{\sin 2 \varphi^t}{r} \frac{\partial \varphi^t}{\partial t} \ dr dz \\
&\quad - \int_{\partial D^t} \frac{\partial}{\partial t} \left( r \frac{\partial \varphi^t}{\partial r} + r \frac{\partial \varphi^t}{\partial z} \right) \ ds \\
&\quad + \int_{\partial D^t} \frac{\partial \varphi^t}{\partial t} \ dr dz \\
&= - \int_{D^t} \left( \frac{\partial \varphi^t}{\partial t} \right)^2 \ dr dz.
\end{align*}
\]

When $\epsilon \to 0$, on each $A \subset \subset D$, $\varphi^\epsilon$ converges to a function $\varphi$ uniformly. Therefore $\varphi$ also satisfies (5.4) in $A$. By Lemma 5.1,

\[
E^t(t) \leq E^t(0) = \int_{D^t} r \left( \frac{\partial \varphi_0}{\partial r} \right)^2 + r \left( \frac{\partial \varphi_0}{\partial z} \right)^2 + \frac{\sin^2 \varphi_0}{r} \ dr dz.
\]

Let

\[
E(t) = \int_{D} r \left( \frac{\partial \varphi}{\partial r} \right)^2 + r \left( \frac{\partial \varphi}{\partial z} \right)^2 + \frac{\sin^2 \varphi}{r} \ dr dz.
\]

By letting $\epsilon \to 0$, we have

\[
E(t) \leq \int_{D} r \left( \frac{\partial \varphi_0}{\partial r} \right)^2 + r \left( \frac{\partial \varphi_0}{\partial z} \right)^2 + \frac{\sin^2 \varphi_0}{r} \ dr dz < \infty.
\]

Moreover, we can repeat the proof of Lemma 5.1 to assert that $E(t)$ is non-increasing.

By a maximum principle for parabolic equations,

\[
\varphi_1 \leq \varphi^\epsilon(\cdot, t) \leq \varphi_2
\]

for all $t > 0$ and for each $\epsilon > 0$. Therefore,

\[
\varphi_1 \leq \varphi(\cdot, t) \leq \varphi_2
\]

for all $t > 0$. Hence, $\varphi$ is continuous in $\bar{D} \sim \{(0, 0)\}$. The requirement that $\varphi = \varphi_0$ on $\partial D$ follows immediately.

From Lemma 5.1,

\[
\frac{d}{dt}(t) \leq - \int_{D} \left( \frac{\partial \varphi}{\partial t} \right)^2 \ dr dz.
\]
Integrating both sides for in $[0,T]$, we have

$$\int_0^T \int_D \left( \frac{\partial \varphi}{\partial t} \right)^2 \, dr \, dz \, dt \leq E(0) - E(T) \leq E(0).$$

By letting $T \to \infty$, we see that, $\frac{\partial \varphi}{\partial t} \to 0$ in $L^2$. Therefore $\varphi(\cdot,t) \to \varphi - \infty(\cdot)$ which is a solution to (5.1).

This proves Theorem 5.3.

5.4 A Liquid Crystal Droplet Problem

The map $u(x) = x/|x|$ from $B^3$ to $S^2$ is a very interesting map. In the following, we show a minimizing property of the pair $(B^3, u)$. A problem concerning drops of liquid crystal is to find a domain $\Omega$ and a map $u : \Omega \to S^2$ which minimizes $E(u)$ + the surface tension on $\partial \Omega$ under a volume constraint and a boundary condition. Here we consider a case in which the surface tension is simply a constant times the area of $\partial \Omega$ and the boundary condition is that $u$ coincides with the unit normal of $\partial \Omega$ on the boundary of $\Omega$. For more physical and mathematical details, one can consult [V].

**Theorem 5.4** Let $u(x) = x/|x|$ for $x \in B^3$. Then for any positive constant $c_0$,

$$\int_{B^3} |\nabla u|^2 \, dx + c_0 \text{area}(\partial B^3) \leq \int_{\Omega} |\nabla v|^2 \, dx + c_0 \text{area}(\partial \Omega)$$

for any smooth convex domain $\Omega \subset \mathbb{R}^3$, $\text{vol}(\Omega) = \text{vol}(B^3)$, and map $v : \Omega \to S^2$ such that $v(x) = n(x)$ where $n(x)$ is the unit outward normal of $\Omega$ at $x \in \partial \Omega$.

**Proof** Let $\Omega$ be a smooth convex domain in $\mathbb{R}^3$ and $n(x)$ is the unit outward normal of $\Omega$ at $x \in \partial \Omega$. Suppose $v : \Omega \to S^2$ and $v(x) = n(x)$ for $x \in \partial \Omega$. From Lemma 1 in [L],

$$\int_{\Omega} |\nabla v|^2 \geq \int_{\Omega} (\text{div}(v))^2 - (\text{trace}(\nabla v))^2 \, dx$$

$$= \int_{\Omega} \text{div}(uv) - (\nabla v)u \, dx$$

$$= \int_{\partial \Omega} \text{div}(v)(v \cdot n) - (\nabla u)u \cdot n \, d\sigma$$

where $d\sigma$ is the volume form of $\partial \Omega$. The proofs of the identities can be found in the Lemma 2 of [L].
Since \( v(x) = n(x) \) for \( x \in \partial \Omega \), and \( |v| = 1 \),

\[
(\nabla v) \cdot n = (\nabla v) n \cdot v = v \cdot \frac{\partial v}{\partial n} = \frac{1}{2} \frac{\partial |v|^2}{\partial n} = 0.
\]

Thus,

\[
\int_{\Omega} |\nabla v|^2 \geq \int_{\Omega} \text{div}(n) d\sigma.
\]

Let \( H \) be the mean curvature of \( \partial \Omega \). Using divergence theorem,

\[
\int_{\Omega} |\nabla v|^2 dx + c_0 \text{area}(\partial \Omega) \geq \int_{\partial \Omega} 2H dx + c_0 \text{area}(\partial \Omega)
\]

\[
\geq 2 \text{area}(\partial \Omega) + c_0 \text{area}(\partial \Omega)
\]

\[
\geq 2 \text{area}(\partial B^3) + c_0 \text{area}(\partial B^3)
\]

\[
\geq 8\pi + c_0 \text{area}(\partial B^3)
\]

by the isoperimetric inequality and formula (5.14) in [O]. Since

\[
\int_{B^3} |\nabla u|^2 dx = 8\pi,
\]

\[
\int_{B^3} |\nabla u|^2 dx + c_0 \text{area}(\partial B^3) \leq \int_{\Omega} |\nabla v|^2 dx + c_0 \text{area}(\partial \Omega).
\]

\( \square \)
References


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