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Equilibrium, efficient-markets, and liquidity in the cash-in-advance model

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Rice University, 1991
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EQUILIBRIUM, EFFICIENT-MARKETS, AND LIQUIDITY
IN THE CASH-IN-ADVANCE MODEL

by

CRISTINO R. ARROYO, III

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
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November, 1990
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ABSTRACT

The existence of equilibrium is a test of the internal consistency of an economic model. In any model with domestic moneys and dividend-yielding assets, the first question is: will money be dominated by assets in equilibrium? In cash-in-advance models domestic moneys always have positive liquidity value as instruments for domestic commodity transactions.

If a non-dominated liquidity role for domestic currencies is posited existence of equilibrium is not usually problematic. For the case where information flows lead to binding cash-in-advance constraints an equilibrium exists in which domestic moneys have positive liquidity value. This equilibrium possesses the unit velocity property, but leads to a sharper characterization of equilibrium market and shadow prices in relation to fundamentals.

That fiat money should be the unique provider of liquidity services is not necessary for equilibrium. It is possible to construct models with well-defined equilibria in which financial assets provide liquidity services. In these models the pricing equations for liquidity-providing assets contain premia for these services over and above risk premia and returns for delaying consumption. Such models can also generate new relationships between the velocity of money or the spot exchange rate and asset returns.
That markets be information-efficient is, however, necessary for equilibrium. Consequently, any rejection of efficient-markets is evidence against the assumptions of the equilibrium theory. Consider, for instance, the case of the efficiency of forward exchange rates vis-a-vis spot rates. Depending on whether forward speculation is consummated through arbitrage of currencies or of assets (e.g., covered and uncovered bonds) the forward efficiency condition will or will not involve liquidity premia. In testing forward efficiency in both models we find, however, there is no material change in the results. Forward efficiency appears to be robust as well to specifications of the utility function. However there is evidence that forward efficiency is not robust to either the measurement of consumption risk, or the choice of covariance estimator of the forecast error.
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\[\text{Cristino R. Arroyo III}\]

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CHAPTER 1
EQUILIBRIUM, EFFICIENT-MARKETS, AND LIQUIDITY
IN THE CASH-IN-ADVANCE MODEL: INTRODUCTION

Recursive dynamic models are the workhorses of contemporary neoclassical macroeconomic theory. This legacy of the rational expectations revolution does not consist merely of a set of useful analytical skeletons and assorted mathematical tools; the models themselves are a fundamental revision of the way we explain causation behind movements in economic aggregates.

Applications of recursive dynamics, however, need not be exclusively concerned with statements regarding aggregates. One interesting observation on the evolution of macroeconomic theory is that, in a way, we have come full circle back to microeconomics again. The preponderance of microfundamental models in aggregative analysis has, in fact, permitted macroeconomists to innovate in some of the traditional preserves of microeconomists. Witness, for example, the research in the area of labor economics that has evolved out of real business cycle models, or the research on coordination problems in games with strategic uncertainty.

Still another example is the subject of this thesis--the application of recursive models to price-theoretic issues in exchange rates and finance. In particular the integration of money and general equilibrium into consumption models of asset prices has yielded an approach to exchange rate, interest rate, and aggregate price level determination that is symmetric to the approach one takes in determining the price of any capital asset. This synthesis represents a gain to the recursive dynamic approach in the sense that there is more generality in the scope of its application and new hypotheses regarding the interrelationships between risk aversion, risk premia, asset prices and interest rates, and exchange
rates and capital flows have been generated by this research program.

The development of dynamic equilibrium models of asset prices may be traced to Breeden (1979); Brock (1979); Cox, Ingersoll, and Ross (1985); and Leroy (1973); among others. These papers focus on the pricing of financial assets without going too deeply into the extensions to economies with fiat money. Introducing money into a dynamic optimization framework is one of the grand old themes of the neoclassical approach, but integration of money into an asset-pricing model is more contemporaneous, explicit formulation of which may find parentage in the influential papers of Townsend (1980) and Lucas (1978). Within the context of integrated monetary-finance models, extensions to international finance issues were subsequently examined by Lucas (1982), Stockman (1980), Svensson and Stockman (1987), and Grilli and Roubini (1989), to name an important few.

This thesis builds on the work of the aforementioned writers in specific ways. Our interests lie in the specification, application, and extension of a standard dynamic equilibrium model of exchange rates and asset prices. To introduce the objectives of this thesis we begin by discussing several problems that have been faced by these researchers which bear upon the ensuing discourse.

1.1. The Existence Issue

The first problem is the problem of existence of equilibrium in a model with money. Closely related to this is the issue of how money ought to be introduced in an asset-pricing model. (Naturally, for the way money enters the model determines whether it will carry value in an equilibrium or be held in zero amounts.) Hahn (1965) has alerted us to the existence question because of the special and anomalous nature of fiat money as an asset in an Arrow-Debreu
world. Relaxation of the Arrow-Debreu assumptions is thence always necessary for any progress towards an answer to this problem.

Positive answers to the existence question and solution methods have since been put forward by Townsend (1980), Lucas (1978), and Lucas and Stokey (1987) for models in which money has value because of a spatial or temporal separation between the institutions of commodity transactions and portfolio allocation. In such models money is regarded as an asset whose only use is for conversion into consumption goods, but is unique among assets for being the only means of commodity purchase. The idea of such institutional separation or similar transactions costs explanations underlies the approach that has come to be known as the cash-in-advance motivation for money. The operational expression of this approach is the cash-in-advance constraint which says that all commodity transactions must be supported by money accumulated in advance of purchases. This is, historically, a very old idea that has caught on recently because of the development of concrete applications. It had been proposed as far back as Tsiang (1966) who called it the finance constraint and Clower (1967) after whom it had been earlier named. These are clearly not the only positive results on the existence question, as the large literature using Samuelsonian overlapping generations shows.

It is obvious that an equilibrium with valued money is absolutely necessary for any model of the (spot) exchange rate. Consequently we set up a version of the Lucas (1978, 1982) model and prove existence for the special case in which the cash-in-advance constraints are binding. This enterprise is not a novel one--Lucas (1978) demonstrates existence-- but is included mainly for completeness. The proof here, moreover, applies to cases in which the marginal utility of money is zero in some states yet no excess cash balances are held. Chapter 2
carries out the exercise and looks at some features of the basic model that have not been examined before.

In this paper we remain neutral on the issue of whether the cash-in-advance specification is the best way to bring money into the analysis. In some respects it is analytically simpler than using an overlapping-generations approach, particularly if one is only interested in expressing a monetary theory of the exchange rate. In certain cases it is identical to the money-in-the-utility function approach, as Feenstra (1986) shows. (Feenstra, 1986 and Stockman, 1989 note, however, that it is not completely identical.) As the application in Chapters 3-4 will illustrate, however, this approach is particularly useful because of available complementary econometric procedures for direct tests of its implications. Owing to these considerations of convenience in both theoretical manipulation and econometric implementation, we have selected the cash-in-advance specification as the approach adopted throughout.

1.2. The Liquidity Issue

This is an issue which has not received much attention from economists working with the asset-pricing framework of Lucas-Svensson, but has merited significant research in the field of finance proper. In Chapter 5 we take up the issue of the liquidity properties of assets in the context of the equilibrium model described above. As we have noted, money is anomalous in the sense of being an asset which yields no returns but is held in positive quantities in a portfolio. The "explanation" proffered by the cash-in-advance approach is that the large transactions costs of using dividend-yielding assets to finance consumption precludes the use of any asset rather than money as a financing instrument for consumption. One major criticism of the cash-in-advance approach is that this
explanation is really no more than an assumption. Without further structural explanations for why fiat money rather than other negotiable instruments minimizes transactions cost, it is still an \textit{ad hoc} way of assigning money positive value. Equilibrium models possessing deeper structural explanations are desirable, but seem to be difficult to specify in a tractable way. Tractability is less of a problem, however, in cash-in-advance models where money is not immediately assumed to be the only liquid or liquidity-providing asset; these turn out to be straightforward extensions of the cash-in-advance approach yet yield new hypotheses. Consequently we examine them in turn, case by case, in Chapter 5, our "extensions" chapter. Arguably, the principal benefit of analyzing these cases is that by doing so we gain a better idea of how restrictive the assumption of the illiquidity of non-money assets is.

1.3. Efficient-Markets Issues

The third and last set of issues we concern ourselves with in this paper revolves around another long-standing problem, that is, whether or not markets are efficient. This issue is relevant to asset-pricing models not necessarily because the policy implications are of considerable moment but because asset-pricing models are particularly well-suited to addressing this economic question.

The efficient-markets hypothesis in general holds that there are no predictable supernormal returns if markets are information-efficient. Specific enough as this sounds the precise meaning of "predictable" and "supernormal" as well as how one measures returns are very much model-dependent concepts.

Chapter 4, however, reduces the efficient-markets implications to a set of parity conditions and asset-pricing results consistent with the nonavailability of
profitable arbitrage where profits are weighted by final consumption services in terms of marginal utilities. These relations come out of the first-order conditions of a consumer maximizing within the framework of cash-in-advance, and are therefore true in any associated equilibrium, particularly the equilibrium discussed in Chapter 2.

Of special interest to us is the international finance result on the efficiency of the forward exchange rate vis-a-vis the spot exchange rate. Indeed the bulk of the text in Chapters 3 and 4 are concerned with expressing and establishing this single relationship. We argue, for example, that there is more than one way to bring forward rates into a model, and the way one introduces forward rates matters, in principle, for the measurement and interpretation of risk premia on forward speculation. More precisely, different approaches to motivating the forward exchange rate yield efficient-markets relations that may or may not contain liquidity premia in addition to risk premia. This idea, while noted in passing by some, has not been the subject of scrutiny in the literature and in this sense our results on forward market efficiency are new.

In Chapter 4 we estimate an empirical version of the theory as applied to forward exchange markets. We then entertain tests of orthogonality conditions implied by the forward efficiency condition of two models, one with liquidity premia, one without. Sensitivity tests to the information lags, the covariance matrix estimator, the specification of the utility function, and the choice of consumption measure are also presented. Chapter 6 concludes with a brief discussion of the interrelationships between equilibrium, efficient-markets, and liquidity.

We therefore begin with the specification of our basic analytical model.
CHAPTER 2
EQUILIBRIUM IN A MODEL OF EXCHANGE RATES AND ASSET PRICES

In this chapter we develop a basic theoretical model of exchange rates and asset prices akin to that of Lucas (1982), Svensson (1985) and Svensson and Stockman (1987) which will be used later for pricing assets and determining spot and forward exchange rates. The purpose of this chapter is to define and demonstrate the existence of a stationary equilibrium for the basic model. Lucas (1978) proves existence of equilibrium in a general asset-pricing model with cash-in-advance constraints for the closed-economy case. In this model the assumptions on information flows are such that the cash-in-advance constraint always binds. In his 1982 paper, Lucas introduces exchange rates but the proof of existence is given as an exercise. Svensson (1985) and Svensson and Stockman (1987) extend the Lucas model to allow for a nontrivial velocity of money and also discuss properties of an equilibrium. They do not supply an existence proof of equilibrium but cite Lucas (1978) for arguments. The exercise of demonstrating equilibrium is thus carried out here for the Lucas (1982) model and the proof here mimics the ideas in his 1978 paper.

In the model of this paper we do not require that money always have positive value at the margin, although we do require that the cash-in-advance constraints bind in equilibrium. Hence, the equilibrium of the model contemplated in this chapter is closer to that of Lucas (1982), where velocity is always one. In a succeeding chapter we extend the basic model to include the possibility that some financial assets are as liquid as money. This will allow the velocity of money (strictly interpreted) to diverge upwards of unity in an equilibrium with binding cash-in-advance constraints. We provide separate existence results there since a proof for the basic monetary model of this chapter is simpler, and it is more convenient to begin with a simple model and subsequently develop modified arguments for the more complicated one.
In both the basic and extended models there will be two currencies which will have value as means of payments for transactions in country-specific commodities, in addition to having value as stores-of-wealth. Part of the solution of these models involves the determination of the correct "shadow price" of currency that is used for transactions purposes. As part of the analysis we examine the equilibrium behavior of this "liquidity value" for money in the basic model. We begin with a description of agents in the economy.

2.1. Preferences and Commodity Space

There are two countries in the economy, referred to as the domestic and foreign country. Each is perfectly specialized in the production of a single commodity. The world economy is inhabited by infinitely-lived individuals identical in their preferences over consumption of these goods across (discrete) time. Denote by \( x_t \in \mathbb{R}_+ \) the quantity consumed at time \( t \) of the domestic good and \( x_t^* \in \mathbb{R}_+ \) the quantity consumed of the foreign good. (Since these are representative agent consumptions, they are in per capita terms, as are all quantities below, unless specified.) We assume that the preferences of the generic individual may be represented by a utility function which is time-separable:

\[
V((x_t, x_t^*)_{t=0,1,...}) = E_0 \sum_{t=0}^{\infty} \beta^t U(x_t, x_t^*).
\]

The period utility function \( U: \mathbb{R}_+^2 \to \mathbb{R} \) is assumed continuously differentiable with derivatives satisfying the following conditions:

ASSUMPTION 2.1. Let \( U_x \) and \( U_x^* \) denote the derivatives of \( U \) with respect to its arguments. We assume that
i. \( U_x > 0 \) and \( U_{x^*} > 0 \) for any \((x,x^*) \in \mathbb{R}_+^2\).

ii. \( \lim_{x \to 0} U_x = +\infty \) and \( \lim_{x \to 0} U_{x^*} = +\infty \).

iii. \( \lim_{x \to +\infty} U_x = 0 \) and \( \lim_{x \to +\infty} U_{x^*} = 0 \).

iv. The Hessian matrix of \( U \) is negative definite so that \( U_{xx} < 0 \) and \( U_{x^* x} < 0 \).

v. Let \( U^n_x \) be the nth-order derivative of \( U \) with respect to \( x \). For any \( x^* \in \mathbb{R}_+ \), there is an \( n' \) for which \( \lim_{x \to 0} U^n_x < +\infty \). A similar property holds for \( U^n_{x^*} \), given \( x \in \mathbb{R}_+ \).

Assumptions 2.1(i)-(iv) are Inada-like conditions that guarantee that (i) \( U \) is strictly increasing, (ii) positive consumption of either good, however small, is infinitely preferable at the margin, to zero consumption of the good, (iii) \( U \) is bounded, and (iv) \( U \) is strictly concave. Assumption (v), however, is not one of "the usual" assumptions. It will not be needed for the proof of existence but it will be useful later on, in order to establish zero (infinite) values of \( xU_x \) (resp. \( x^*U_{x^*} \)) as \( x \) (resp. \( x^* \)) tends to zero (infinity). This function will play a role in our analysis of the behavior of the shadow prices of liquidity. Roughly speaking, what it suggests is that \( \lim_{x \to 0} U_x x = 0 \). This is because if Assumption 2.1(v) holds, without loss of generality we can take \( n' \) to be the smallest order of differentiation for which \( U^n_x \) is bounded. A sequential application of L'Hopital's rule for the indeterminate form \(+\infty/+\infty\) gives:

\[
\lim_{x \to 0} \frac{U_x}{(1/x)} = \lim_{x \to 0} \frac{U_{xx}}{(1/x)^2} = \lim_{x \to 0} (-1)^{(n'+1)} \cdot (1) \cdot (2) \cdot \ldots \cdot (n'-1) \frac{U^n_x}{(1/x)^{n'}} = 0.
\]
2.2. Endowment Process.

The expectation $E_t$ in the preference function above denotes expectation conditional on a state at $t$, $s_t$. It is necessary to define the probability space over which this conditional expectation is taken. In this model the sole source of uncertainty is due to random processes on the per capita supplies $(d_t,d^*_t) \in \mathbb{R}_+^2$ of the two goods. We define the state at $t$ by $s_t = (d_t,d^*_t)$. First we assume that

ASSUMPTION 2.2. The state $s$ belongs to $K \subset \mathbb{R}_+^2$, with $K$ compact and $0 \in K$.

The state $s$ should be strictly nonzero so that the budget sets will be nonempty. That $K$ be compact is not usually restrictive as long as we ensure that the total wealth of the individual is finite. However we will find the compactness assumption useful in ensuring that additional liquidity will have positive value. Consider now the measurable space consisting of $K$ and its Borel subsets $\mathcal{Y}$. We assume further that

ASSUMPTION 2.3. The generating processes for domestic and foreign goods follows a stationary Markov process on the space $(K,\mathcal{Y})$ with transition function $f:K \times \mathcal{Y} \to [0,1]$, i.e.,

i. For any $s \in \mathbb{R}_+^2$, $f(s,\cdot)$ is a probability density on $(K,\mathcal{Y})$.

ii. For any $N \subset \mathcal{Y}$, $f(\cdot,N)$ is $\mathcal{Y}$-measurable.

iii. Let $g(s',\cdot)$ be any continuous function of $s'$. Then $\mathbb{E}g(s) = \int g(s',\cdot)f(s,ds')$ is well-defined if $g$ is bounded and is a continuous function of $s$. The conditional expectation operator $E$ is the Lebesgue integral of $g$ over the measure space $(K,\mathcal{Y},f(s,\cdot))$.

The key property that the transition function $f$ should satisfy is property 2.3(iii)
which says that the operator $E$ maps continuous functions into continuous functions. Note also that $f$ is not a function of time $t$ and that $g(s', \cdot)$ is not a function of $s$. This completes our specification of the stochastic side of the model.

2.3. Markets, Money, and Assets

In this model we introduce national moneys by departing from the centralized trading structure of the standard Arrow-Debreu economy. In its place is imposed a trading structure in which the markets for commodities and for securities are temporally separated. This leads to cash-in-advance constraints on transactions which in turn guarantee that unbacked money will be held in positive amounts in equilibrium together with dividend-yielding securities.\footnote{In the basic model, we posit, specifically, the following trading structure (see Lucas, 1982 or Sargent, 1987 for interpretations). Individuals are assumed to trade goods, as well as moneys and claims on the stochastic production processes (i.e., assets). A unit of the domestic asset is a claim to the random per capita output $d_t$ and similarly for a unit of the foreign asset. There are domestic and foreign moneys, issued monopolistically by respective national governments in fixed per capita supplies $(M, \bar{M}^*) \in \mathbb{R}^{\mathbb{N}^+}$ in every period. A representative individual enters the period $t$ carrying $(M_t, \bar{M}_t^*) \in \mathbb{R}^2$ worth of domestic and foreign currency and $(z_t^*, \bar{z}_t^*) \in Z \subset \mathbb{R}^2$ worth of claims. To rule out solutions in which the individual makes himself infinitely wealthy by unlimited shortselling of assets we assume}

ASSUMPTION 2.4. $Z$ is a compact subset of $\mathbb{R}^2$ containing 0.

Before asset markets open in period $t-1$, information about the next period's real dividends-- i.e., the state $s_t = (d_t, \bar{d}_t^*)$-- become known to the individual, although he
does not collect the actual dividends until period t. Because real dividends are not delivered at the time of goods trading direct barter at t is ruled out and the individual must accumulate cash in advance of future goods purchases--the individual must now choose \((M_t, M_t^*)\) at period t-1 asset markets. This choice is made subject to a wealth constraint described in more detail below. The goods market in period t now opens. Because credit contracts are costly to enforce, the basic model assumes that the two moneys are the sole media of exchange in their respective country's goods market. Therefore in period t goods markets the following cash-in-advance constraints apply:

\[
(2.2a) \quad p_t x_t \leq M_t
\]

\[
(2.2b) \quad p_t^* x_t^* \leq M_t^*
\]

where \((p_t, p_t^*) \in \mathbb{R}_+^2\) are the prices of domestic and foreign goods in terms of their respective currencies. Goods markets subsequently close and real dividends are collected. Period t asset markets then open in which individuals can reallocate their portfolios of claims and currencies according to their total wealth at t, \(Y_t\):

\[
(2.3) \quad Y_t = (M_t - p_t x_t) + e_t (M_t^* - p_t^* x_t^*) + (q_t + p_t d_t) z_t + e_t (q_t^* + p_t^* d_t^*) z_t^*
\]

where \((q_t, q_t^*) \in \mathbb{R}_+^2\) are the prices of claims at t and \(e_t \in \mathbb{R}_+\) is the spot exchange rate of a unit of foreign currency in terms of units of domestic currency. Wealth, measured in domestic currency terms, consists therefore of currency left over after net commodity trades, the currency value of real dividends (i.e., \(p_t d_t z_t\) and \(e_t p_t^* d_t^* z_t^*\)) and capital gains (\(q_t z_t\) and \(e_t q_t^* z_t^*\)). The operative wealth constraint on portfolio reallocations in period t is thus:
\[(2.4) \quad M_{t+1} + e_t M^*_t + q_t z_{t+1} + e_t q_t z^*_t \leq Y_t.\]

Refer to Figure 1 for a summary of the timing conventions.

2.4. The Consumption Problem

We are now in a position to define the problem of the representative consumer in a way that allows us to exploit the theorems of dynamic programming in order to arrive at both a solution to the model and a stationary equilibrium. The individual is initially endowed with a unit apiece of the two kinds of claims and \((M,M^*)\) worth of currencies. To study a stationary equilibrium it is necessary to assume that the individual takes prices \(w\) as given (fixed) functions of the state \(s\) only: \(w(s) = \{ p(s), p^*(s), q(s), q^*(s), e(s) \}\) for any \(t\). The individual also knows the random process \(f(s,N)\). His problem is to choose an optimal policy \(\{ x_t, x^*_t, M_{t+1}, M^*_t, z_{t+1}, z^*_t \}_{t=0,1,..}\) to maximize (2.1) subject to the cash-in-advance constraints (2.2) and the wealth constraint (2.4).

Inasmuch as preferences are time-separable the consumer's problem can be handled recursively by redefining the problem in an equivalent form. Using primes (') for next-period variables (as opposed to unprimed current-period variables), the solution to the consumer's problem above can be expressed equivalently as a sequence of policy functions \(\{ x, x^*, v' \} = \{ x, x^*, M', M^*, z', z^* \}\) together with a value function \(W^*\) that satisfies the equation

\[(2.5) \quad W^*(v,s) = \max_{\{x,x^*,v'\}} U(x,x^*) + \beta \int W^*(v',s')f(s,ds')\]

subject to
(2.6) \((x, x^*, v') \in \Lambda(s, M, M^*, z, z^*)\), where \(\Lambda: \mathbb{R}^6 \rightarrow \mathcal{P}(\mathbb{R}^6)\) is the correspondence defined by

\[
\Lambda(v, s) = \{ (x, x^*, v') \in \mathbb{R}^6 \mid p(s)x \leq M; \text{ and } p(s)x^* \leq M^*; \text{ and } \\
M' + q(s)z' + e(s)[M^* + q(s)z^*] \leq M - p(s)x \\
+ [q(s) + p(s)d]z + e(s)[M^* - p(s)x^* + (q(s) + p(s)d^*)z^*] \}.
\]

We now demonstrate that \(\Lambda\) is a continuous correspondence. It is clear that \(\Lambda\) as defined is nonempty-valued. If the sequence of prices \(w(s)\) are all continuous functions of \(s\) then the graph of \(\Lambda\) is closed in \(\mathbb{R}^6 \times \mathbb{R}^6\). Moreover, if the prices \(w(s)\) are all strictly positive functions of \(s\), then for any bounded subset \(B\) of \(\mathbb{R}^6\), we will have \(\Lambda(B)\) bounded. Therefore by Theorem 3.4 of Stokey, Lucas, and Prescott (1989) \(\Lambda\) is compact-valued and upper hemi-continuous. To establish that \(\Lambda\) is lower hemi-continuous when prices are continuous in \(s\) and strictly positive, we employ the following familiar argument. Consider the correspondence

\[
\Lambda^0(v, s) = \{ (x, x^*, v') \in \mathbb{R}^6 \mid p(s)x < M; \text{ and } p(s)x^* < M^*; \text{ and } \\
M' + q(s)z' \\
+ e(s)[M^* + q(s)z^*] < M - p(s)x + [q(s) + p(s)d]z \\
+ e(s)[M^* - p(s)x^* + (q(s) + p(s)d^*)z^*] \}.
\]

For strictly positive prices and \((M, M^*, d, d^*)\) strictly greater than zero this correspondence is nonempty-valued as \((x, x^*, v') = 0\) always belongs to the image sets. Take any sequence \((v^n, s^n) \rightarrow (v^0, s^0)\) and a point \((x^0, x^{*0}, v^0) \in \Lambda(v^0, s^0)\). Then there is a number \(N\) such that for \(n > N\), \((x^0, x^{*0}, v^0) \in \Lambda(v^n, s^n)\). Hence one can easily construct a sequence \((x^i, x^{*i}, v^i)\) belonging to each image set \(\Lambda(v^n, s^n)\) with \((x^i, x^{*i}, v^i) \rightarrow (x^0, x^{*0}, v^0)\) by choosing any element in the image set.
\( \Lambda(v^n, s^n) \) for \( i = n \leq N \) and then selecting \( (x^i, x^*_i, v^i) = (x^0, x^*_0, v^0) \) for \( i > N \). Therefore \( \Lambda^0 \) must be lower hemi-continuous. But \( \Lambda \) is the closure correspondence of \( \Lambda^0 \). Since the closure correspondence of a lower hemi-continuous correspondence is lower hemi-continuous, we have proven the following result:

**PROPOSITION 2.1.** Let \( w(s) = \{ p(s), p^*(s), q(s), q^*(s), e(s) \} \) be continuous and strictly positive functions of \( s \). Then the correspondence \( \Lambda(v, s) \) is nonempty-valued, convex-valued, compact-valued, and continuous.

We now tackle the question of existence of a bounded, continuous value function \( W^* \) that solves (2.5). Let \( \mathcal{C}(X) \) be the space of bounded continuous functions with domain \( X \) and the norm \( \|f\| = \sup_{x \in X} |f(x)| \).

**PROPOSITION 2.2.** For \( w(s) \) continuous and strictly positive, \( W^* \in \mathcal{C}(\mathbb{R}^6) \) that satisfies (2.5) exists and is unique. Moreover the correspondence

\[ \Pi(v, s) = \{ (x, x^*, v') \in \Lambda(v, s); (x, x^*, v') \text{ satisfies (2.5) for } W^* \} \]

is upper hemi-continuous.

**Proof:** Define the operator \( T \) on \( \mathcal{C}(\mathbb{R}^6) \) by

\[ TW(v, s) = \max_{(x, x^*, v') \in \Lambda(v, s)} U(x, x^*) + \beta \int W(v', s') f(s, ds') \]

where \( W \in \mathcal{C}(\mathbb{R}^6) \). By Assumption 2.1 and Assumption 2.3(iii) \( TW \) is bounded and the expression \( U(x, x^*) + \beta \int W(v', s') f(s, ds') \) is continuous in \( (x, x^*, s) \). Since, by Proposition 2.1, \( \Lambda \) is compact-valued and continuous, Berge's Maximum Theorem
applies and $TW(v,s)$ is a continuous function of $(v,s)$ and the correspondence $\Pi$ is upper hemi-continuous.

Now since $T$ is a mapping from a complete metric space $\mathcal{C}(\mathbb{R}^6)$ into itself it suffices to show that it is a contraction. It is clear that $T$ satisfies the monotonicity and discounting requirements of Blackwell's Theorem for Contraction Mappings, hence $T$ is a contraction mapping with modulus $\beta$. Therefore a unique fixed point $W^*$ exists that satisfies (2.5), Q.E.D.

We have so far exploited only the continuity and boundedness of $U$ to characterize $W^*$ and the correspondence $\Pi$. We can use the other assumed properties of $U$ to further sharpen our characterization.

**PROPOSITION 2.3.** For strictly positive prices, $W^*$ is strictly increasing in $v$ and strictly concave in $v$. The upper hemi-continuous correspondence $\Pi$ is single-valued, i.e., a continuous function.

Proof: Most of the proof below follows Stokey, Lucas, and Prescott (1989, pp.81-82) but is specialized here for completeness. The proof that $W^*$ is strictly increasing, while straightforward, is our own.

To show $W^*$ is strictly increasing and concave it is useful to remember that if $\mathcal{C}'$ is closed in $\mathcal{C}$ with $\mathcal{C}' \subset \mathcal{C}$ and $T(\mathcal{C}') \subset \mathcal{C}'' \subset \mathcal{C}'$ then $W^* = TW^*$ belongs to $\mathcal{C}''$ (Stokey, Lucas, and Prescott, 1989, p.52). It suffices then to show that under our assumptions $\mathcal{C}' = \{ c \in \mathcal{C}(\mathbb{R}^6); c$ is increasing and concave in $v \}$ and $\mathcal{C}'' = \{ c \in \mathcal{C}; c$ is strictly increasing and strictly concave in $v \}$ meet the specified requirements. Clearly $\mathcal{C}'' \subset \mathcal{C}' \subset \mathcal{C}$ and $\mathcal{C}'$ is closed in $\mathcal{C}$. It remains to show that $T(\mathcal{C}) \subset \mathcal{C}''$. The fact that $\Lambda$ is convex$^3$ and, by Assumption 2.1, that $U$ is strictly increasing and concave guarantees this.
First, it is obvious that when $U$ is strictly increasing in $(x, x^*)$ $TW$ is strictly increasing in $v$ for any $W \in \mathcal{C}$. This is because for any $(x, x^*) \in \Pi(v,s)$, if $\tilde{v} > v$ there is always $(\tilde{x}, \tilde{x}^*) \in \Lambda(\tilde{v},s)$ such that $(\tilde{x}, \tilde{x}^*) > (x, x^*)$ whenever prices are strictly positive. (The inequalities here are used in the "vector" sense.) Therefore the maximum value $TW$ over $\Lambda(v,s)$ should be strictly larger than $TW(v,s)$.

To show that $TW$ is concave: Let $W$ be any element of $\mathcal{C}'$ and let $(x, x^*, v') \in \Pi(v,s)$ and $(\tilde{x}, \tilde{x}^*, \tilde{v}') \in \Pi(\tilde{v},s)$ and write $(x_{\theta}, x^*_{\theta}, v'_{\theta})$ for the convex combination of these two points given $\theta \in (0,1)$. Convexity of $\Lambda$ implies $(x^*_{\theta}, x^*_{\theta}, v'_{\theta}) \in \Lambda(\theta v + (1-\theta)\tilde{v},s)$. Thus,

$$TW(v_{\theta},s) \geq U(x_{\theta}, x^*_{\theta}) + \beta \int W(v'_{\theta},s')f(s,ds')$$

$$> \theta[U(x,x^*) + \beta \int W(v,s')f(s,ds')] + (1-\theta)[U(\tilde{x},\tilde{x}^*) + \beta \int W(\tilde{v},s')f(s,ds')]$$

$$= \theta TW(v,s) + (1-\theta)TW(\tilde{v},s)$$

where the strict inequality comes from the fact that $U$ is strictly concave and $W^* \in \mathcal{C}'$. The last equality comes from $(x, x^*, v') \in \Pi(v,s)$ and $(\tilde{x}, \tilde{x}^*, \tilde{v}') \in \Pi(\tilde{v},s)$.

We have thus shown that if $W$ is any element then of $\mathcal{C}'$ then $TW$ is an element of $\mathcal{C}''$. Hence $T(\mathcal{C}') \subset \mathcal{C}''$, and thus $W^* = TW^*$ is strictly increasing and strictly concave.

Finally, since $U(x,x^*) + \beta \int W^*(v',s')f(s,ds')$ is a strictly concave function and $\Lambda(v,s)$ is convex-valued, $\Pi(v,s)$ must be a singleton, i.e., $\Pi$ must be a (continuous) function, Q.E.D.

That $W^*$ be differentiable is not a trivial assertion. The next proposition shows that $W^*$ is differentiable in the interior of $\mathbb{R}_+ \times Z$. 

PROPOSITION 2.4. If $v_0$ is an interior point of $\mathbb{R}_+ \times Z$ and $(x_0, x_0^*, v_0^*) \in \Pi(v_0, s)$ is such that $(x_0, x_0^*, v_0^*) \in \text{int} \Lambda(v_0, s)$ then $W^*$ is differentiable at $v_0$.

Proof: To show $W^*$ is differentiable one must adapt the usual proof (Stokey, Lucas, and Prescott (1989), p.85) to the case when the return function $U$ does not depend on the state $v$ before invoking the result of Benveniste-Scheinkman (1979). This is done as follows: Let $v_0$ and $(x_0, x_0^*, v_0^*)$ satisfy the conditions of the Proposition. Since $\Lambda$ is continuous there is a neighborhood $N$ of $v_0$ for which $(x_0, x_0^*, v_0^*) \in \Lambda(v, s)$ if $v \in N$. Now define the function $L(v, s) = U(x_0, x_0^*) + \beta \int W^*(v_0^*, s)f(s, ds') - (v-v_0)'I(v-v_0)$ where $I$ is an identity matrix conformable to $v$. It is clear that $L(v, s)$ is concave in $v$ and differentiable in $v$. Because $(x_0, x_0^*, v') \in \Pi(v_0, s)$ we have $L(v_0, s) = W^*(v_0, s)$.

Finally, for any $v$ in the neighborhood $N$,

$$L(v) = U(x_0, x_0^*) + \beta \int W^*(v_0^*, s)f(s, ds') - (v-v_0)'I(v-v_0)$$

$$\leq U(x_0, x_0^*) + \beta \int W^*(v_0^*, s)f(s, ds')$$

$$\leq \max_{(x, x^*, v) \in \Lambda(v, s)} U(x, x^*) + \beta \int W^*(v', s)f(s, ds') \ [\text{since } (x_0, x_0^*, v_0^*) \in \Lambda(v, s)]$$

$$= W^*(v).$$

Hence there is a neighborhood $N$ of $v_0$ and a concave, differentiable function $L$ with domain $N$ such that $L(v_0) = W^*(v_0)$ and $L(v) \leq W^*(v)$ for $v \in N$. Thus the result of Benveniste and Scheinkman applies and $W^*(v)$ is differentiable at $v_0$. Q.E.D.

If $W^*$ is differentiable, then we can analyze solutions to the individual agent’s problem by looking at the first-order (Kuhn-Tucker) conditions to the maximization on the right-hand side of (2.5). By the strict concavity of $W^*$ and the fact that the
constraint set \( \Lambda(v,s) \) is convex-valued and has an interior point, then the Kuhn-Tucker conditions for the (unique) solution to the consumption problem are both necessary and sufficient (Intriligator (1971), p.57). Let \( \theta(s), \theta^*(s), \) and \( \lambda(s) \) be the current-period Kuhn-Tucker multipliers on (2.2a), (2.2b), and (2.3)-(2.4) respectively. The first-order conditions at an interior solution, \((x, x^*, M', M^*') \gg 0\) and \((z', z^*') \in \text{int} \ Z \) are:

\[
\begin{align*}
(2.7a) & & \quad U_x(x,x^*) = [\lambda(s) + \theta(s)]p(s) \quad x > 0 \\
(2.7b) & & \quad U_x^*(x,x^*) = [\epsilon(s)\lambda(s) + \theta^*(s)]p^*(s) \quad x^* > 0 \\
(2.7c) & & \quad \beta \int [\lambda(s') + \theta'(s')]f(s,ds') = \lambda(s) \quad M' > 0 \\
(2.7d) & & \quad \beta \int [\epsilon(s')\lambda(s') + \theta^*(s')]f(s,ds') = \epsilon(s)\lambda(s) \quad M^* > 0 \\
(2.7e) & & \quad [p(s)x - M]\theta(s) = 0 \\
(2.7f) & & \quad [p^*(s)x^* - M^*]\theta^*(s) = 0 \\
(2.7g) & & \quad \beta \int [q(s') + p(s')d']\lambda(s')f(s,ds') = q(s)\lambda(s) \quad (z', z^*') \in Z \\
(2.7h) & & \quad \beta \int [q^*(s') + p^*(s')d^*']e(s')\lambda(s')f(s,ds') = e(s)q^*(s)\lambda(s) \quad (z', z^*') \in Z
\end{align*}
\]

Together with complementary slackness condition between \( \lambda(s) \) and the wealth constraint.
\[ M_{t+1} + e_t M_{t+1}^* + q_t z_{t+1} + e_t q_t z_{t+1}^* \leq (M_t - p_t x_t) + e_t (M_t^* - p_t x_t^*) + (q_t + p_t d_t) z_t + e_t (q_t^* + p_t d_t^*) z_t^* \]

2.5 Asset-Pricing

Regard the above as a system of equations in the vector of prices and shadow prices \{ p(s), p(s)^*, q(s), q(s)^*, e(s), \theta(s), \theta(s)^*, \lambda(s) \}. The system is complete in the sense that the number of equations is the same as there are prices. It is also block-recursive in the following sense: it is not necessary to know what q(s) and q(s)^* are in order to determine p(s), p(s)^*, e(s), \theta(s), \theta(s)^*, and \lambda(s). The first six equations can be used together to solve for the two commodity prices, the exchange rate, and the Kuhn-Tucker multipliers. Suppose these are found to be continuous, strictly positive functions of s. Then the last two conditions amount to a set of pricing equations for assets. Take equation (2.7h). If there exists a fixed function q^* \in \mathcal{C}(\mathbb{R}_+) that will satisfy (2.7h) then such a function determines the price of any asset returning p^*d^* worth of foreign currency given the current state s. Such a function exists-- given equilibrium pricing functions \hat{e} and \lambda apply the Contraction Mapping Theorem to the operator T: \mathcal{C}(\mathbb{R}_+) \longrightarrow \mathcal{C}(\mathbb{R}_+):

\[
Tr^*(s) = \int \{ \beta r(s') + \beta p(s') d^* [\hat{e}(s') \lambda(s')] \} f(s, ds')
\]

where r^*(s) = q^*(s)\hat{e}(s)\lambda(s). Since T is a mapping from a complete metric space into itself, satisfies discounting and is clearly monotonic in r^*, there is a \hat{r}^* \in \mathcal{C} satisfying \hat{r}^* = Tr^*. Therefore q^* = r^*(s)/[\hat{e}(s)\lambda(s)]. Moreover, since the term

\[
\int \beta p(s') d^* [\hat{e}(s') \lambda(s')] f(s, ds')
\]
is strictly positive, then \( r^* \) and \( q^* \) must also be strictly positive. A similar argument can be made about the domestic asset price \( q \).

2.6. Equilibrium

The above discussion on asset prices depended on the existence of bounded, continuous, strictly positive prices \( \{ p(s), p^*(s), e(s), \theta(s), \theta^*(s), \lambda(s) \} \). The existence of such prices is, of course, a question of existence of equilibrium. We begin exploration of this question by defining the usual equilibrium concept applied to models like ours:

**DEFINITION.** A stationary equilibrium is an allocation \( (x, x^*, M', M^*, z', z^*) \); a vector of prices \( w(s) = \{ p(s), p^*(s), q(s), q^*(s), e(s) \} \) bounded, continuous in \( s \), and strictly positive; and a value function \( W^* \) that together solve (2.5) - (2.6) and satisfy the market-clearing conditions (i) \( (x, x^*) = (d, d^*) \), (ii) \( (M', M^*) = (M, M^*) \), and (iii) \( (z', z^*) = (z, z^*) = (1, 1) \)

Digression: By Walras' Law any one of the six market-clearing conditions can be omitted since the sixth must hold if the other five do. This applies in particular to condition (iii). In a different model we could endow individuals with units of claims and allow them to issue or purchase such claims within the shortselling constraint \( (z, z^*) \in Z, Z \) compact. Then Walras' Law would imply that aggregate asset holdings must lie on a hyperplane through 0 in a stationary equilibrium, that is, \( qz + eq^* z^* = 0 \). In such a model the fact that \( (z, z^*) = 0 \) is not a direct consequence of Walras' Law by itself. However, taken together with the assumption that individuals are identical in their preferences, Walras' Law does imply that \( (z, z^*) = 0 \). Thus the assumption
that \( 0 \in Z \), while not really needed in the present context, is absolutely necessary for models with only "inside" assets.

Having defined our concept of equilibrium, we proceed to find one. We will exploit the sufficiency of the individual's first-order conditions in solving for the equilibrium prices. We will omit equations (2.7g) and (2.7h) since by the discussion of the previous section these can be solved block-recursively. Now the equilibrium allocations are already tied down by the market-clearing conditions and the assumption of representative agents. So set \( (x, x^*, M', M^*, z', z^*) \) to their market-clearing values \( (d, d^*, M, M^*, 1, 1) \) in equations (2.7a)-(2.7f). This yields:

\[
(2.8a) \quad U_x(d, d^*) = [\lambda(s) + \theta(s)]p(s)
\]

\[
(2.8b) \quad \int \beta[\lambda(s') + \theta(s')]f(s, ds') = \lambda(s)
\]

\[
(2.8c) \quad [p(s)x - \bar{M}]\theta(s) = 0
\]

\[
(2.8d) \quad U_x^{*}(d, d^*) = [e(s)\lambda(s) + \theta^{*}(s)]p^*(s)
\]

\[
(2.8e) \quad \int \beta[e(s')\lambda(s') + \theta^{*}(s')]f(s, ds') = e(s)\lambda(s)
\]

\[
(2.8f) \quad [p(s)x^* - \bar{M}^*]\theta^*(s) = 0
\]

These are six equations in the state \( s = (d, d^*) \) and the six unknown price functions \( p(s), p^*(s), \) and \( e(s) \) together with the shadow prices \( \theta(s), \theta^*(s), \) and \( \lambda(s) \). Observe that the above system also exhibits block-recursivity: if we can first solve equations (2.8a)-(2.8c) for \( \lambda(s) \) (and \( p(s) \) and \( \theta(s) \)). This function can then be used as
an input in the solution of (2.8d) - (2.8f) for \( e(s), p^*(s), \) and \( \theta^*(s). \) So begin with (2.8a)-(2.8c).

The main problem here is that eq. (2.8c) does not really provide enough information about \( \theta(s) \) in relation to \( p(s) \) to obtain a closed-form equilibrium. However, because we assumed that the information flows are such that the individual knows the future supplies of \( (x, x^*) \) in advance of currency purchases, we must have

\[
(2.9) \quad \hat{p}(s) = \bar{M}/d.
\]

The domestic price level \( \hat{p} \) is strictly positive, bounded, and continuous. Then take

\[
(2.10) \quad \theta(s) = U_x(s)d/M - \beta \int U_x(s')d'/\bar{M}f(s,ds'),
\]

if \( U_x(s)d/M \geq \beta \int U_x(s')d'/\bar{M} f(s,ds') \)

\[ = 0, \quad \text{otherwise}. \]

By Assumption 2.1, \( \theta(s) \) is bounded and it is also continuous, though not necessarily differentiable (see Figure 2). \( \theta \) is obviously nonnegative. Finally, from (2.8b) we find that given \( \theta \) is bounded and continuous, we can once again employ the Contraction Mapping Theorem to find a fixed point \( \lambda \) that solves (2.8b). More directly, this is just the bounded, continuous, strictly positive function

\[
(2.11) \quad \lambda(s) = \beta \int U_x(s')d'/\bar{M} f(s,ds').
\]

Handling the second block of equations in analogous manner, take the bounded, continuous, and strictly positive solution
\[ p^*(s) = \frac{M^*}{d^*} \]

We must now solve jointly the equations

\[ U_x^*(s) = [e(s)\lambda(s) + \theta^*(s)]M^*/d^* \]

\[ \int \beta[e(s')\lambda(s') + \theta^*(s')]f(s,ds') = e(s)\lambda(s) \]

for \( e(s) \) and \( \theta^*(s) \). Substituting the first into the second equation to eliminate the integrand and also \( \lambda(s) \) suggests that we should take

\[ \theta^*(s) = U_x^*(s)d^*/M^* - \beta \int U_x^*(s')d^*/M^* f(s,ds'), \]

if \( U_x^*(s)d^*/M^* \geq \beta \int U_x^*(s')d^*/M^* f(s,ds') \)

\[ = 0, \text{ otherwise.} \]

Given that \( \theta^* \), and \( \lambda \) are bounded, continuous, and nonnegative functions of \( s \), we can once again apply the Contraction Mapping Theorem to the functional equation

\[ Te(s)\lambda(s) = \int [\beta e(s')\lambda(s') + \beta \theta^*(s')]f(s,ds'). \]

For fixed \( \lambda(s) \), \( T \) is a contraction map of \( e(s)\lambda(s) \) implying a fixed-point \( \tilde{e} = T\tilde{e} \) which will solve (2.8e), given \( \theta^* \) and \( \lambda \) described above.

Interestingly enough, the solution \( \tilde{e} \), while clearly bounded and continuous, cannot be guaranteed to be strictly positive without further assumptions, as \( \theta^* \) has not been shown to be strictly positive over some event of nonzero probability, i.e., the integral \( \int \beta \theta^*(s')f(s,ds') \) could be zero.
Define the function \( \varphi(s) = \sum_x (s) d^* \). Then \( \theta^*(s) = M^{*-1}[\varphi(s) - \beta \int \varphi(s') f(s, ds')] \).

It is easy enough to see that

**PROPOSITION 2.5.** If the transition function \( f \) is serially uncorrelated, i.e., \( f(s, ds') = f(ds') \), then \( \theta^*(\tilde{s}) > 0 \) for some \( \tilde{s} \).

**Proof:** If \( f \) is serially uncorrelated then

\[
\theta(s) = M^{*-1}[\varphi(s) - \beta E \varphi^*] = M^{*-1}[\varphi(s) - \beta E \varphi^*].
\]

Since \( \varphi(s) \neq E \varphi^* \) for all \( s \), there is some \( \tilde{s} \) for which \( \varphi(\tilde{s}) > E \varphi^* \), implying \( \theta^*(\tilde{s}) > 0 \), Q.E.D.

To ensure that \( \theta^*(s) \) is strictly positive in the general case one might impose restrictions on preferences. Alternatively, you might entertain restrictions on the transition function \( f \). We therefore make the following assumption:

**ASSUMPTION 2.5.** Let \( \tilde{\varphi} \) be the maximum value of \( \varphi^*(s) \) and let \( \tilde{s} \) attain \( \tilde{\varphi} \). For any \( s \) there is a \( \varphi^0 \) with \( 0 < \varphi^0 < \tilde{\varphi} \) such that the set \( K^0 = \{ s' \in K; \varphi^*(s') < \varphi^0 \} \) has positive measure under \( f(s', \cdot) \).

In the above assumption, the compactness of \( K \) becomes important, because it ensures us that maximum \( \tilde{\varphi} \) exists and the state \( \tilde{s} \) which generates this maximum value is itself a member of \( K \). The following lemma indicates how this assumption helps us:

**LEMMA 2.6.** If Assumption 2.5 holds then there exists an \( \tilde{s} \) such that \( \theta^*(\tilde{s}) > 0 \).

**Proof:** Consider the integral \( \int \varphi^*(s') f(\tilde{s}, ds') \). We know that this integral must be less than or equal to the upper bound \( \tilde{\varphi} \). By Assumption 2.5 however, it must be strictly
less than \( \hat{\varphi} \) since the integration includes states in the set \( K^0 \). Since \( \beta < 1 \) we have

\[
\varphi^*(\tilde{s}) > \beta \int \varphi^*(s')f(\tilde{s},K')
\]

Thus \( \theta^*(\tilde{s}) > 0 \), Q.E.D.

It seems that the compactness of \( K \) assumed in Assumption 2.5 is too restrictive. An weaker assumption that is sufficient for the same result and does not require compactness of \( K \) is available:

ASSUMPTION 2.5'. Let \( \bar{\varphi} \) be the least upper bound for \( \varphi^*(s) \). For any \( s \) there exists a \( \varphi^0 \) with \( 0 < \varphi^0 < \bar{\varphi} \) such that the set \( K^0 = \{ s' \in K; \varphi^*(s') < \varphi^0 \} \) has positive measure under \( f(s,\cdot) \) and its relative complement \( K \setminus K^0 \) is nonempty and has measure less than or equal to \( [(1-\beta)/\beta]\varphi^0 \).

LEMMA 2.6'. If Assumption 2.5' holds then there exists an \( \tilde{s} \) such that \( \theta^*(\tilde{s}) > 0 \).

Proof: Select any \( \tilde{s} \) in \( K \setminus K^0 \). Then,

\[
\beta \int \varphi^*(s')f(\tilde{s},ds') = \beta \int \varphi^*(s)f(s,K^0) + \beta \int \varphi^*(s)f(s,K \setminus K^0)
\]

\[
< \beta \varphi^0 + \beta [(1-\beta)/\beta] \varphi^0
\]

\[
= \varphi^0
\]

\[
\leq \varphi^*(\tilde{s}).
\]
Hence $\theta^*(s) > 0$, Q.E.D.

Note: If, in particular, the relative complement $K \setminus K^0$ has measure zero (as it does in Assumption 2.5') then Lemma 2.6' follows.

Now to determine conditions under which $\tilde{e}(s)$ is strictly positive, we ask-- can Lemma 2.6 (or Lemma 2.6') be extended from a point $\hat{s} \in K$ to a set $N$ in $K$ with positive measure? The answer is yes, if the following assumption holds:

ASSUMPTION 2.6. If $N$ is an open interval in $K$, $\int f(s, N) > 0$, for any $s$.

LEMMA 2.7. If Assumption 2.6 also holds, then $\int \hat{\theta}(s')f(s, N) > 0$ for some $N$.

Proof: From Lemma 2.6 we know that there is some $\hat{s}'$ at which $\hat{\theta}(s') > 0$. Since $\hat{\theta}(s')$ is a continuous function of $s'$ there must be an open interval $N$ of $s'$ over which $\theta^*(s') > 0$ if $\hat{s}'$ is in $N$. By Assumption 2.6, such an interval carries positive measure, hence the result follows, Q.E.D.

Direct application of Lemma 2.7 gives:

COROLLARY 2.8. $\int \beta \theta^*(s')\lambda(s)f(s, ds')$ is strictly positive.

COROLLARY 2.9. Let $\tilde{e}$ be the fixed point of the functional equation (2.14). $\tilde{e}$ is strictly positive, bounded, and continuous in $s$.

More importantly, now that the strict positivity of $\tilde{e}$ is established, we can now legitimately state that the following theorem is true:
THEOREM 2.10. Under Assumptions 2.1(i)-2.1(iv) and (2.2)-(2.6) there exists a stationary equilibrium to the economy described above. In this equilibrium,

\[ p(s)x(s) = \bar{M} \]

and

\[ p^*(s)x^*(s) = \bar{M}^* \]

for any \( s \). This result is also true if Assumption 2.5 is replaced by Assumption 2.5'.

Proof: The requisite allocations are given by \((x, x^*) = (d, d^*); (M', M^*) = (M, M^*);\) and \((z', z^*) = (1, 1)\). The required prices are given by equations (2.9), (2.12), and the fixed points of the contraction mappings (2.14) and

\[
T_1 q(s)\lambda(s) = \int \beta[q(s') + \bar{p}(s')d']\lambda(s') f(s, ds')
\]

\[
T_2 q^*(s)\bar{e}(s)\lambda(s) = \int \beta[q^*(s') + \bar{p}^*(s')d^*][\bar{e}(s')\lambda(s')] f(s, ds'),
\]

given the fixed solutions \( \bar{p}(s), \bar{e}(s), \) and \( \lambda(s) \)

Q.E.D.

2.7. Equilibrium with Nonbinding Cash Constraints

Given the fact that monetary velocity is trivially one in this model, a legitimate question is why not demonstrate equilibria for a model where nonbinding cash-in-advance constraints occur. Such a model, in fact, is available from Svensson
(1985) and Svensson and Stockman (1987). The answer is that it is much more difficult to construct a candidate difference equation in the prices of interest so that an appropriate fixed-point argument can be used to show existence of a bounded continuous solution.

To clarify this point, suppose we allowed for the cash-in-advance constraints to be both binding or not binding by allowing for residual uncertainty discovered only after asset markets have closed. In states when the cash-in-advance constraints are not binding $\theta$ and $\theta^*$ would both be zero. For simplicity consider what happens to the first block of our system of Kuhn-Tucker conditions, (2.8a)-(2.8c). When $\theta = 0$ we can, as before, set $\lambda(s) = \beta \int U_x(s')/p(s') f(s, ds')$, if we can find an equilibrium function $p(s)$, bounded, continuous in $s$, and strictly positive. In addition to these restrictions on $p$ we would also like to select $p$ so that the cash-in-advance constraint does not bind in some of the states in which $\theta$ is zero. These considerations lead us to suggest an equilibrium pricing function $p$ of the form

$$
p(s) = \min \left\{ \bar{M}/d + \left( U_x/p(s) - \beta \int U_x(s')/p(s') f(s, ds') \right); \bar{M}/d \right\},
$$

if $\bar{M}/d + \left( U_x/p(s) - \beta \int U_x(s')/p(s') f(s, ds') \right) > 0$

$$= \varepsilon, \text{ otherwise}
$$

where $\varepsilon > 0$ is an appropriately chosen lower bound to ensure $p$ is nonnegative. When $\theta > 0$ this function chooses $\bar{M}/d$, since the term in brackets $\cdots$ is strictly positive. When $\theta = 0$ then $p < \bar{M}/d$ in states where the term in brackets is negative and is always strictly positive by choice of $\varepsilon$. The problem here is to ensure that there exists a pricing function $p$ that solves

$$p(s) - [U_x(s)/p(s)] = \bar{M}/d - \beta \int U_x(s')/p(s') f(s, ds').$$
This appears not to be a form to which the Contraction Mapping Theorem or related fixed-point theorems can be readily applied. In consideration of this difficulty, we choose to focus below on the results for the equilibrium sketched in Section 2.7.

2.8. The Spot Exchange Rate

The equilibrium spot exchange rate \( \bar{e} \), defined as the fixed-point of (2.14) is given by

\[
\bar{e}(s) = \frac{\bar{M}}{\bar{M}} \int \frac{U_x^*(s')d^*f(s,ds')}{\bar{M}^*} \int U_x(s')d^'f(s,ds')
\]

This equation, derived by Svensson and Stockman (1987) in a slightly different form, tells us that a kind of monetary theory to the exchange rate holds: the exchange rate is partly a function of the relative money stocks in two countries. Thus, if the money supply in the foreign country should experience a permanent expansion, then there is a permanent depreciation of the spot exchange rate.

What about a temporary monetary expansion? To examine this question assume that \( f \) is serially uncorrelated, i.e., \( f(s,ds') = f(ds') \). Let \( \bar{e}' \) be next period's exchange rate. The change in \( \bar{e} \) is

\[
\bar{e}' - \bar{e} = \left[ \frac{\bar{M}'}{\bar{M}^*} \right] \int \frac{U_x^*(s'')d^*f(ds'')}{U(x)ds''} - \left[ \frac{\bar{M}}{\bar{M}^*} \right] \int \frac{U_x^*(s')d^*f(ds')}{U(x)s'd'f(ds')}
\]
\[
\begin{bmatrix}
\bar{M}' & \bar{M} \\
\bar{M}^* & M^*
\end{bmatrix} \cdot \frac{\int U_x^*(s)d^* f(ds)}{\int U_x(s)d f(ds)}
\]

Between two periods, the rate of appreciation of the spot exchange rate, \( \pi_d = [\bar{e}(s') - e(s)]/\bar{e}(s) \), is just the rate of relative monetary expansion

\[
\pi_d = \left[ \frac{(\bar{M}/M^*)}{(\bar{M}'/M^*)} \right] - 1
\]

\[
= \left[ \frac{\bar{M}'}{\bar{M}} \cdot \frac{M^*}{M^*} \right] - 1.
\]

Since there are no lagged effects, a one-period increase in the money supply of the foreign country leads to a current depreciation and the exchange rate remains stable until the old money supply levels are restored. When the foreign money supply decreases to pre-expansion levels the exchange rate then appreciates (in the same period as the money supply contraction) to pre-expansion levels. As far as the rate of appreciation is concerned a relatively higher rate of money growth abroad (at home) implies a higher rate of depreciation (appreciation) of the exchange rate.

The equilibrium exchange rate also depends on the relative expected marginal utility per percentage point of GNP growth:

\[
\frac{\int U_x^*(s)d^* f(ds)}{\int U_x(s)d f(ds)}
\]

This interpretation is due to the fact that one way to look at the term \( U_x^*x^* \) is to
decompose it as \((\Delta U/\Delta x^*) \cdot x^* = \Delta U \cdot (\Delta x^*/x^*)^{-1}\). The last expression has the interpretation of being the change in \(U\) over the growth rate of foreign output. This suggests that a relative increase (decrease) in the expected growth rate of output in the foreign (domestic) country will cause the exchange rate to depreciate (appreciate). Intuitively the depreciation results because expectations of productivity growth abroad in the next period generate an increase in the current period demand for foreign real balances that must be accumulated in advance of future consumption. The increase in the demand for foreign currency drives the relative price of the domestic currency to the foreign currency down. Unlike the usual model in which exchange rates depend on current productivity growth the requirement of cash-in-advance makes today's exchange rate conditional on tomorrow's productivity instead.

2.9. The Multipliers \(\theta\) and \(\theta^*\)

The Kuhn-Tucker multipliers \(\theta(s)\) and \(\theta^*(s)\) carry the interpretation of shadow prices or marginal utilities of domestic and foreign liquidity (the other multiplier \(\lambda(s)\) is the marginal utility of wealth.) To conclude this chapter we discuss some of the properties of these liquidity prices as they have been constructed in equations (2.10) and (2.13). We know that, in an equilibrium with binding cash-in-advance constraints the mappings \(\theta\) and \(\theta^*\) are continuous in \(s\). We have also seen that, with a few assumptions, one can guarantee that \(\theta^*\) is strictly positive over events of nonzero probability measure. A version of Assumption 2.5 (or Assumption 2.5') can also be made to ensure that \(\theta\) is also strictly positive over some sets of nonzero measure. In this section we would like to examine if more can be said about how these functions behave as the state \(s = (\text{d}, \text{d}^*)\) varies. The point is that we might wish to rank states in the sense of a state \(s^0\) being "no worse" than another state \(s^1\), if \(\text{d}^0, \text{d}^{*0} \geq (\text{d}^1, \text{d}^{*1})\) and "better" than state \(s^1\) if, in addition, \(\text{d}^0 > \text{d}^1\) or \(\text{d}^{*0} > \text{d}^{*1}\). What we
would then like to establish are sufficient conditions for the mappings \( \vartheta \) and \( \vartheta^* \) to be monotonic functions of \( s \). That they should be monotonic is partly intuitive— if we are in a "good" state and consuming more than in an "average" state, we are required to support higher consumption levels with the same stock of moneys. With no excess money balances this requires price levels to be smaller when \( \vartheta \) and \( \vartheta^* \) are positive and for \( \vartheta \) and \( \vartheta^* \) not to decrease. If there are excess balances, then \( \vartheta \) and \( \vartheta^* \) should go from zero to positive levels and not decrease thereafter. Unfortunately the conditions we need to impose in order to prove the main proposition of this section are somewhat strong, hence the results below should be regarded as first steps in the development of more general results.

Recalling that \( \vartheta \) and \( \vartheta^* \) are given by (2.10) and (2.13), let us continue to make the assumption of no serial correlation in \( f \). Moreover we will eventually need to make the assumption that \( U \) is separable in \( x \) and \( x^* \). Then variations in \( \vartheta \) and \( \vartheta^* \) due to the state \( s \) are governed only by the terms \( U_x(d)d \) and \( U_x^*(d^*)d^* \). We examine below the behavior of \( \vartheta^* \)— similar considerations apply to \( \vartheta \). The following lemma provides the desired monotonicity result:

**LEMMA 2.11.** Let Assumption 2.1(i) hold. Define the coefficient of relative risk aversion by

\[
\rho(x^*) = \left| \frac{x^* U''(x^*)}{U'(x^*)} \right|.
\]

A necessary and sufficient condition for the function \( \varphi(x^*) = x^* U'(x^*) \) to be strictly increasing in \( x^* \) is that \( \rho < 1 \) for any \( x^* \).
Proof: The first derivative of $\varphi$ with respect to $x^*$ is

$$
\varphi'(x^*) = U'(x^*) \cdot [1 - \rho(x^*)]
$$

which is strictly positive under Assumption 2.1(i) if and only if $\rho < 1$ for any $x^*$, Q.E.D.

COROLLARY 2.12. If $0 < \rho < 1$ the $x^*$-elasticity of $\varphi^*$ is less than or equal to 1.

Proof: The $x^*$-elasticity of $\varphi^*$ is given by

$$
\mu_{x^*}(x^*, \varphi^*) = \left| \frac{x^* \varphi'(x^*)}{\varphi^*(x^*)} \right|
$$

$$
= 1 - \rho(x^*),
$$

Q.E.D.

With the help of Lemma 2.11 and Corollary 2.12 we can establish the following result:

PROPOSITION 2.13. Let Assumptions (2.1)-(2.6) hold. In addition to (i) $0 < \rho < 1$ for all $x^*$, assume that (ii) there is no serial correlation in $f$, i.e., $f(s, ds') = f(ds')$; and (iii) $U$ is separable in $x$ and $x^*$. Then there exists a unique critical value $d^{**}$ such that

$$
\theta(d^*) > 0 \text{ if } d^* > d^{**}
$$

and

$$
\theta(d^*) = 0 \text{ if } d^* \leq d^{**}.
$$
Moreover, \( \theta^* \) is a bounded, continuous function, with \( d^* \)-elasticity less than 1, and is strictly increasing in the interval \( d^* > d^{**} \).

Proof: First, \( \theta^*(d^*) \) is given by

\[
\theta^*(d^*) = M^{-1}(\phi^*(d^*) - \beta E \phi^*).
\]

Clearly the elasticity of \( \theta^* \) with respect to \( d^* \) is less than 1 because of Corollary 2.12. Now \( \phi^*(0) = 0 \) by virtue of Assumption 2.1(v). Moreover, \( \phi^* \) is a strictly increasing function of \( d^* \) by Lemma 2.11. We thus only need to establish that \( \phi^*(d^*) > \beta E \phi^* \) for some \( d^* \) in order to claim that a critical value \( d^{**} \) exists. (Refer to Figure 3.) This must be true as \( \phi^*(d^*) \neq E \phi^* \) for all \( d^* \), hence there must be some \( d^* \) for which \( \phi^*(d^*) > E \phi^* \). Therefore \( d^{**} \) exists such that \( \theta^*(d^*) > 0 \) when \( d^* > d^{**} \) and \( \theta^*(d^*) = 0 \) when \( d^* \leq d^{**} \). Since \( \phi^*(d^*) \) is a strictly increasing function of \( d^* \) when \( \rho(d^*) < 1 \), \( \theta^* \) is also strictly increasing in the range \( d^* > d^{**} \), Q.E.D.

NOTES TO CHAPTER 2

1. We explore this property further in the succeeding chapter on equilibrium with liquid assets.

2. \( \mathcal{P}(S) \) denotes the power set of \( S \).

3. The correspondence \( \Lambda(\nu) \) is convex in \( \nu \) if for any \( l \in \Lambda(\nu, s) \) and \( I \in \Lambda(\tilde{\nu}, s) \) we have \( \theta l + (1-\theta)I \in \Lambda(\theta \nu + (1-\theta)\tilde{\nu}, s) \).
CHAPTER 3
EFFICIENT FORWARD CURRENCY MARKETS IN
THE CASH-IN-ADVANCE MODEL

We have seen that the basic model of Chapter 2 generates equilibrium prices for currency and financial assets. These pricing relations embody the efficiency hypothesis of asset markets, which states that equilibrium asset prices imply zero predictable speculative profits after adjusting for risk. In this chapter we examine the efficiency of forward foreign exchange markets in the context of the equilibrium model with cash-in-advance constraints. The purpose of this chapter is to provide the theoretical expression of the efficient-markets hypothesis for forward currencies in a form that will be amenable to empirical rejection. In the succeeding chapter we carry out the econometric testing.

3.1. An Overview of Forward-Spot Efficiency

To motivate the problem this chapter addresses, it is useful to view the discussion in the context of the research on efficient forward exchange markets.

At the qualitative level, the efficiency hypothesis applied to foreign exchange markets states that risk-adjusted forward speculative profits on future values of the spot rate are not predictable. Despite the voluminous research on this issue, only recently (Korajczyk, 1985; Mark, 1985) have researchers found support for forward-spot exchange rate efficiency, and only after discarding simple versions of the efficient-markets hypothesis in which agents are presumed risk-neutral\(^1\).

In the literature, forward market efficiency was initially taken to mean that the forward rate is a (conditionally) unbiased predictor of the spot rate. This, it has been
argued, is a straightforward consequence of covered and uncovered interest parity conditions holding simultaneously. As the unbiasedness property is not consistent with risk-aversion on the part of agents, when research uniformly rejected the unbiasedness hypothesis\(^2\) it became more reasonable to suppose that risk premia exist which could explain deviations from unbiasedness and uncovered interest parity. Initially simple linear regression models for the risk premium were posited; a time-invariant risk premium and a white noise error term would be sufficient to explain the deviations of forward rates from spot exchange rates. As further research (notably Hansen and Hodrick, 1984 and Korajczyk, 1985; there were others that followed) failed to support the simple linear regression specification, many researchers tried alternative specifications of the linear model that would stand a better chance of fitting the data. The specifications that have developed out of this research program posit the existence of time-varying risk premia in order to admit of the more complex covariance structure of prediction errors that have been observed, such as that of conditional heteroskedasticity and serial correlation. However the success of these empirically-oriented models in supporting the hypothesis of efficient-markets has been, thus far, mixed.\(^3\)

An alternative approach to the problem of reconciling forward-spot efficiency with the evidence exploits the fact that many of the dynamic asset pricing models generate efficiency conditions in the form of estimable orthogonality conditions. Another reasonable line of attack to the problem of explaining risk premia in the forward market is to respecify the theory which generates the estimable model. The appeal of this approach, besides providing a source of alternative possible sources of risk premia, is that it can also suggest reasons for why simple forms of the efficient-markets hypothesis may fail to obtain in the forward market. By integrating forward and spot markets into a larger, equilibrium model of international finance, the issue of efficiency could be understood structurally and any deviations from efficiency
or interest parity might be shown to come from the presence of structures in the model economy that are not necessarily inconsistent with the behavior of rational agents in markets characterized by rapid information diffusion. An empirically successful version of this kind of model would not only reconcile the data with the efficiency hypothesis but could also explain rigorously why deviations were observed in the first place.

The development of the cash-in-advance model of international trade in Lucas (1982) provided a starting point. A standard neoclassical equilibrium model was proposed integrating exchange in monetary and financial assets. When uncertainty was introduced into the model with risk-averse agents, risk premia were implied by optimizing behavior which depended principally on the covariation of asset returns with the marginal utility of consumption or the growth rate thereof. There was some criticism attendant to the Lucas cash-in-advance model--many were and are still uncomfortable about what, precisely, are the underlying rigidities in an economy that give rise to the cash-in-advance constraints, so specified. Additionally, the original Lucas model adopted a timing assumption on information flows that left no residual uncertainty in between the respective trading sessions for goods and assets. This specification implied that the velocities of both domestic and foreign moneys are unity, a property which was viewed as somewhat restrictive. Nevertheless, when Mark (1985) tested the moment restrictions imposed by the Lucas model on forward speculative profits he did not obtain strong rejections, lending credence to the theoretical approach to the risk premium. Further evidence gathered by Hodrick (1989) seemed to be in line with Mark's initial findings. The approach using orthogonality conditions has thus experienced moderate success in reconciling market efficiency with exchange rate data.

Focusing on the theoretical underpinnings of the risk premium has had important consequences, however, on the interpretation of empirical results. Adopting
a more complex specification of the underlying uncertainty and the characteristics of preferences over uncertain outcomes has led to formulations of the efficiency hypothesis that are not robust to changes in assumptions made about these structures. Consequently, empirical tests based on theoretical models are necessarily joint tests of the hypothesis and key assumptions of the underlying theory (such as the assumption of the way expectations are formed, the form of the objective function, the timing and form of uncertainty present in the model, the assumptions on the structure of asset markets and trading, etc.). It has become difficult, but not altogether inappropriate, to separate tests of the empirical validity of the efficient-markets hypothesis from structural features of the generating model. As efficient-market asset pricing rules become more model-dependent, tests of efficient markets are correspondingly more model-specific. The important acceptance criterion to keep in mind, therefore, is the robustness of the hypothesis across a broad range of plausible alternative specifications of the theory.

As is well-known, cash-in-advance constraints drive a "wedge" between the marginal utility of wealth and the marginal utility of consumption in states when the consumer does not hold enough currency to support desired consumption. Intuitively this is because money provides nontrivial liquidity services, in addition to having value as a component of wealth. Technically, this is a result of nonzero Kuhn-Tucker multipliers on binding cash-in-advance constraints. Recalling equation (2.7a) we have, typically,

$$U_x = (\lambda + \theta)p$$

and not just $U_x = \lambda p$. Whenever this is the case it has been observed (Townsend, 1987; Svensson, 1985a) that the relevant risk factor in the asset-pricing equations is no
longer the marginal utility of consumption $U_x/p$ but the marginal utility of wealth $\lambda$. It would seem only natural that related pricing conditions such as forward market efficiency should be likewise affected. This aspect, however, has received less attention in the literature. We will argue, in this chapter, that the presence of this "wedge" between the marginal utility of consumption and the marginal utility of wealth makes the avenue through which forward trades are carried out material to the measurement of the risk premium. Specifically we will compare the efficiency conditions implied by an asset-arbitrage approach to forward speculation proposed in the next section and an earlier approach suggested in Svensson and Stockman (1987) which relies on currency-arbitrage. Chapter 4 then checks the findings of Mark (1985) for the asset-arbitrage model of forward exchange rates.

3.2. An Asset-Arbitrage Approach to Forward Exchange

The model of this section is called an asset-arbitrage model of forward exchange rates, in contradistinction to an earlier approach to forward rates suggested in Svensson and Stockman (1987) (henceforth abbreviated as SS). In the SS model, forward exchange rates are motivated by the device of currency trade at both asset and goods markets. Since they end up defining the forward exchange rate as a rate of exchange of currency, the SS model may be called a currency-arbitrage model of forward exchange rates. The model developed here will not provide for currency exchange in both goods and asset trading sessions. We allow individuals to adjust their portfolios of assets and currencies only in the asset market, as in the original cash-in-advance setups. Instead, forward rates are introduced implicitly through interrelationships between the returns on risky and riskless financial assets denominated in the two currencies. Thus described, we may call the approach herein considered an asset-arbitrage approach to forward exchange. We describe our model
first, examine some of its implications, and then compare it with the SS goods-market model.

3.2.1. Model and Tradeable Assets

The model we use is the model of Chapter 2 with a few modifications. We introduce several other assets that can be components of the portfolios of individuals in addition to the claims on the random production processes \( d_t \) and \( d_t^* \). These additional assets will be inside assets, therefore net aggregate holdings of these assets are zero. Moreover, with identical individuals this also implies zero equilibrium asset holdings at the individual level. For convenience in the ensuing discussion let us refer to the domestic and foreign currency units described in Chapter 2 as dollars and pounds, respectively. We continue to assume that currencies are made available from the outside in fixed supplies every period. We also maintain the assumption that the claims on the dividend processes are likewise outside assets that individuals are endowed with. As before the problem of the individual is to maximize the expected present value of his lifetime utility:

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t u(x_t, x_t^*)
\]

where and \( E_t z_{t+j} \) is a shorthand notation for the conditional statistical expectation described previously as \( \int z(s_{t+j}) f(w_t) ds_{t+j} \). We reinterpret the state at \( t+j \), \( s_{t+j} \), more broadly to include, in addition to the realizations (\( d_t, d_t^* \)), signals that make up an information set at \( t, \Omega_t \). We will also explicitly use notation for time periods \( t, t+1, \ldots \) to make it convenient for the empirical development of Chapter 4. The timing conventions are the same: We assume that the individual arrives in period \( t \) with predetermined holdings (\( M_t, M^*_t \)) worth of domestic and foreign money (in dollars...
and pounds, \((z_t, z_t^*)\) shares of domestic and foreign securities which are claims to a random endowment process \((d_t, d_t^*)\) governing the supply of consumption goods, and, in addition, \((B_t, B_{1t}^*, B_{2t}^*)\) units of domestic and foreign bonds (with units measured in dollars and pounds respectively).

Goods markets open during which domestic and foreign cash-in-advance constraints on goods trades are in effect. Our cash-in-advance constraints are, as before:

\[
(3.1a) \quad p_t^* x_t \leq M_t
\]

\[
(3.1b) \quad p_t x_t \leq M^*_t
\]

After goods markets are closed but before asset markets open, individuals receive payments of dividends and returns on the asset and bond holdings \((z_t, z_t^*, B_t, B_{1t}^*, B_{2t}^*)\) brought into period \(t\). As before the returns on the claims \((z_t, z_t^*)\) consist of the realizations of the stochastic dividend processes \((d_t, d_t^*)\) with transition function \(f(s_t, d_{t-1})\). The information on the realizations of \((d_t, d_t^*)\) are known before the individual chooses \((M_t, M_t^*)\) during period \(t-1\) asset markets.

The important modification is that there are other assets in the model. The first new asset is a covered domestic bond. This asset returns \((1+R_{t+1})B_{t+1}\) dollars at time \(t+1\) for every \(B_{t+1}\) dollars invested at time \(t\). We assume the return \(R_{t+1}\) is known as of time \(t\), i.e., this is a dollar bond with a sure dollar return one period ahead. The second asset is a covered foreign bond which, for every \(B_{1t+1}^*\) pounds of covered foreign bonds the individual chooses at time \(t\) returns \(f_t^{t+1}(1+R_{t+1}^*)B_{1t+1}^*\) dollars, where the rate \(f_t^{t+1}\), what we presently call the one-period forward exchange rate, is a number known at \(t\). We also assume that the pound rate of return \((1+R_{t+1}^*)\) is known at time \(t\). Thus, the covered foreign bond yields a sure dollar return on a
pound-denominated investment. Finally, the third asset is an uncovered foreign bond which returns \( e^*_{t+1}(1+R^*_{t+1})B^*_{2t+1} \) dollars. The quantity \((1+R^*_{t+1})\) is the same pound rate of return as that of the covered bond, \((1+R^*_{t+1})\) but the dollar return \( e^*_{t+1}(1+R^*_{t+1})B^*_{2t+1} \) depends on the spot rate of exchange \( e^*_{t+1} \) of dollars for pounds in \( t+1 \) asset markets. Since the future spot exchange rate \( e^*_{t+1} \) is unknown as of time \( t \) when the individual selects bond holdings \((B^*_t, B^*_1, B^*_2)\), the return to the uncovered bond is uncertain as of \( t \).\(^5\)

With these markets in mind, let \( q_t \) and \( q^*_t \) be, respectively, the dollar and pound prices of domestic endowment claims and foreign endowment claims at period \( t \). After the dividends on the claims and the currency returns on the bonds are distributed asset markets open. In the asset trading session the individual may reallocate his portfolio of currency, claims, and bonds within bounds of the overall wealth constraint

\[
\begin{align*}
(3.2) \quad M^*_{t+1} + e^*_{t} M^*_{t+1} + B^*_{t+1} + e^*_{t}(B^*_{1t+1} + B^*_{2t+1}) + q^*_t z^*_{t+1} + e^*_{t} q^*_t z^*_{t+1} \\
\leq (M_t - p_t x_t) + e^*_{t}(M^*_{t} - p^*_t x^*_t) + (1 + R_t)B^*_t + f^*_t (1 + R^*_t)B^*_1 \\
+ e^*_{t}(1 + R^*_t)B^*_2 + (q^*_t + p^*_t d^*_t)z^*_t + e^*_{t}(q^*_t + p^*_t d^*_t)z^*_t
\end{align*}
\]

in which the dollar has been chosen as the numeraire.

Returning to the model of the paper, write the vector of all period \( t \) prices and returns in the model \((p^*_t, p^*_t, q^*_t, q^*_t, e^*_{t}, f^*_{t+1}, R^*_t, R^*_t)\) compactly as \( \pi^*_t \) and the vector of period \( t \) state variables \((M^*_t, M^*_t, B^*_t, B^*_1, B^*_2, z^*_t, z^*_t)\) as \( v^*_t \). Recursivity in the individual's problem allows us to restate the solution of the maximization problem as the solution to the following dynamic programming problem. Define the value function \( W \) implicitly as the fixed point of the functional equation:
(3.3) \[
TW(v_t, \pi_t) = \max_{(x_t^*, x_t^*, v_{t+1}^*)} U(x_t^*, x_t) + \beta E_t W(v_{t+1}; \pi_{t+1})
\]

subject to (3.1)-(3.2), nonnegativity of \((x_t^*, x_t^*, M_t, M_t^*)\), and

(3.4) \[
(z_{t+1}^*, z_{t+1}^*, B_{t+1}^*, B_{1t+1}^*, B_{2t+1}^*) \in Z,
\]

\(Z\) is a compact set in \(\mathbb{R}^g\) containing 0 in its interior.

The state at \(t+1\), \(s_{t+1}\), over which the individual forms the conditional expectation \(E_t\) is \((d_{t+1}, d_{t+1}^*)\). To form a time \(t\) prediction of \(s_{t+1}\) the individual uses the information set \(\Omega_t\). We assume that \(\Omega_t\) consists of all prices dated \(t\) or earlier, all variables dated \(t\) or earlier, the bond return components \((1+R_{t+1}^*), (1+R_{t+1}^*)\), and the forward rate \(f_{t+1}^t\). By carrying out the constrained maximization on the right-hand side of (3.3) we solve the representative consumer's problem.

In this model it is not required that the individual hold nonnegative amounts of any of the bonds, which are the inside assets. The assumption that individuals are identical, however, implies that the equilibrium quantities of the bonds be zero because there can be no shortselling in the aggregate. Given that 0 is an interior point of \(Z\) this will mean that the first-order relationships between the bond returns will hold with equality.

Compactness of \(Z\) means that short sales of bonds are limited. This is convenient for showing that equilibrium exists, as illustrated in the model of Chapter 2, however it is not crucial for establishing that the first-order relationships between the bond returns (from which we will derive the forward efficiency condition) must hold with strict equality. It is possible to relax the assumption of limited shortselling to obtain the efficient-markets relationships we derive below by using arbitrage-pricing arguments. The existence of our first-order equality relationships
between the bond returns is necessary to ensure the nonexistence of profitable arbitragers, which must, in turn, hold if a solution is to exist for the maximization problem. This derivation is developed in Appendix A. In the balance of this chapter we maintain the assumption that $Z$ is compact and short-selling is limited.

3.2.2. Solution of the Consumer's Problem

Letting $\theta_t$ and $\theta_t^*$ be Kuhn-Tucker multipliers corresponding to (3.1a) and (3.1b) respectively and $\lambda_t$ the Kuhn-Tucker multiplier for (3.2), we have it that for an interior solution (i.e., one where $x_t > 0$ and $x_t^* > 0$) to the domestic consumer's problem the following are necessary:

\begin{align}
(3.5a) & \quad U_{x_t} - (\theta_t + \lambda_t)p_t = 0 \\
(3.5b) & \quad U_{x_t^*} - (\theta_t^* + e_t\lambda_t)p_t^* = 0 \\
(3.5c) & \quad \beta(1 + R_{t+1})E_t^t \lambda_{t+1} = \lambda_t \\
(3.5d) & \quad \beta(1 + R_{t+1})E_t^t e_{t+1} \lambda_{t+1} = e_t \lambda_t \\
(3.5e) & \quad \beta(1 + R_{t+1})E_t e_{t+1} \lambda_{t+1} = e_t \lambda_t \\
(3.5f) & \quad \beta E_t(\theta_{t+1} + \lambda_{t+1}) = \lambda_t \\
(3.5g) & \quad \beta E_t(\theta_{t+1}^* + e_{t+1} \lambda_{t+1}) = e_t \lambda_t
\end{align}
\[ (3.5h) \quad \beta E_t(q_{t+1} + p_{t+1}d_{t+1})\lambda_{t+1} = q_t\lambda_t \]

\[ (3.5i) \quad \beta E_t(q_{t+1}^* + p_{t+1}^*d_{t+1}^*)e_{t+1}\lambda_{t+1} = e_tq_t^*\lambda_t \]

\[ (3.5j) \quad \theta_t(M_t - p_tx_t) = 0 \quad \text{and} \quad \theta_t^*(M_t^* - p_t^*x_t) = 0 \]

\[ (3.5k) \quad (x_tx_t^*M_tM_t^*\theta_t\theta_t^*\lambda_t) \geq 0. \]

The remaining first-order conditions are the cash-in-advance constraints (3.1), the wealth constraint (3.2), and a set of transversality conditions.\(^6\)

At an optimum the representative domestic individual allocates his consumption so that the marginal utility of (domestic) consumption \( U_{x_t} \) is equal to the sum of the marginal utility of dollars as a liquidity service \( (p_t\theta_t) \) and the marginal utility of dollars as a component of wealth \( (p_t\lambda_t) \). (In this model \( \theta_t \) and \( \lambda_t \) are measured in terms of utility per dollar and \( \theta_t^* \) in terms of utility per pound.) The condition on the allocation of foreign goods consumption is similar. Hence we may occasionally refer to money's total marginal utility as equivalent to the marginal utility of consumption without any impropriety. In this model, as in many of its predecessors, money has value not merely because it represents additional general purchasing power but also because it is particularly useful in financing consumption, which no other asset in the model does. The cash-in-advance constraints drive a "wedge" (Svensson, 1985b) between the utility of money and the utility of wealth.

The cash-in-advance multipliers \( \theta_t, \theta_t^* \), which we have interpreted as the marginal utility of liquidity, have been called "liquidity premia." To make sense as a model with money, we should have \( \theta_t > 0 \) over some states with positive measure. This requirement, it turns out, is related to the necessity of a positive interest rate in
some states. With a riskless rate of return $R_{t+1}$ in the model, the following is true:

PROPOSITION 3.1. If, for any $t$, the marginal utility of domestic money, $U_{x_t}/p_t$, is positive over some states of nonzero measure, and the domestic interest rate $R_{t+1}$ is also positive over some states of nonzero measure, then at a solution to the consumer's problem, the domestic cash-in-advance multiplier, $\theta_t$, must also be positive over some states of nonzero measure, for any $t$. The converse is also true.

Notes: (a) The states over which $U_{x_t}/p_t$, $R_{t+1}$, and $\theta_t$ are positive need not be the same states for all of these. (b) We recognize that a statement similar to the necessity part of Proposition 3.1 has appeared before in another version of the cash-in-advance model (Svensson, 1985b).

Proof: We present a contrapositive proof of the first part of the proposition--that is, if $\theta_t = 0$ almost everywhere for some $t'$, then $R_{t+1} = 0$ almost everywhere for some $t''$ if all other conditions of the proposition are true. The proof uses two relations from the first-order conditions that will have some usefulness elsewhere in this paper. These are:

\begin{equation}
(3.6a) \quad \lambda_t = \beta E_t U_{x_{t+1}}/p_{t+1}
\end{equation}

which can be obtained from substitution of (3.6a) into (3.6f) and

\begin{equation}
(3.6b) \quad \theta_t = U_{x_t}/p_t - \beta E_t U_{x_{t+1}}/p_{t+1}
\end{equation}

which can be obtained from substituting (3.6a) back into (3.5a).
Now, at an optimum, we know that (3.5c) above also holds. This and the expression for $\lambda_t$ in (3.6a) imply that, at any $t$,

\[(3.7) \quad (1 + R_{t+1}) E_t \beta E_{t+1} U_{x_{t+2}} / p_{t+2} = E_t U_{x_{t+1}} / p_{t+1}\]

Suppose now that $\theta_t = 0$ almost everywhere for some $t'$, say at $t+1$. Then $E_t \theta_{t+1} = 0$. Then by (3.6b), the preceding equation becomes

\[(1 + R_{t+1}) E_t U_{x_{t+1}} / p_{t+1} = E_t U_{x_{t+1}} / p_{t+1}\]

Since, by assumption, $U_{x_{t+1}} / p_{t+1}$ is positive over some states of nonzero measure, and is never negative (negative prices can be precluded by monotonicity of $U$ and regularity conditions on the consumption space) its expectation $E_t U_{x_{t+1}} / p_{t+1}$ is positive. But then

\[(1 + R_{t+1}) = 1\]

almost everywhere, implying $R_{t+1} = 0$ almost everywhere for $t'' = t$.

To prove the converse: If $R_{t+1} = 0$ almost everywhere at some $t''$, say $t$, substitution into (3.7) above gives

$$E_t \beta E_{t+1} U_{x_{t+2}} / p_{t+2} = E_t U_{x_{t+1}} / p_{t+1}$$

Looking at this relation and using (3.6b), we have
\[ E_t \theta_{t+1} = 0. \]

As \( \theta_t \geq 0 \) almost everywhere and for any \( t \), the last equation can only mean \( \theta_t = 0 \) almost everywhere, at \( t' = t+1 \), Q.E.D.

As we have shown in the discussion of Chapter 2, in an equilibrium \( p_t \) is strictly positive in every state, hence \( U_{x_t}/p_t \) is positive in at least some states. This can also be done for the modified model of this chapter. Moreover adding assumptions like Assumptions 2.5 and 2.6 will ensure that in this equilibrium \( \theta_{t+1} \) will be positive over some states of nonzero probability measure. The second part of Proposition 3.1 (i.e., \( \theta_{t+1} > 0 \) in some states implies \( R_{t+1} > 0 \) in some states) then means that in such an equilibrium the domestic interest rate \( R_{t+1} \) will be positive for some states. Now the first part of the proposition (\( R_{t+1} > 0 \) in some states implies \( \theta_{t+1} > 0 \) in some states) has the following interpretation: imagine an equilibrium existed in which \( \theta_t = 0 \) almost everywhere for all \( t \). This can happen only in a situation in which the level of per capita domestic money held by individuals into \( t \) is always high enough to accommodate all planned purchases of domestic goods \( x_t \) the consumer makes (in positive probability states). The first half of the proposition says that such a scenario is inconsistent with the perfectly certain rate of return on a dollar invested at \( t \), \( R_{t+1} \), being positive at all. The intuition behind this is suggested by the first-order conditions on domestic money and domestic bonds. If \( R_{t+1} \) were anything but zero, since it only costs \( \lambda_t \) if one were to release an increment of money for investment in one-period domestic bonds, and the expected return on such a riskless activity is not \( \beta E_t \lambda_{t+1} \), the implicit return from hoarding the money into the next period when \( \theta_{t+1} \) is zero--this must equal \( \lambda_t \) if moneyholdings are efficiently allocated across periods) but the strictly greater amount \( (1+R_{t+1})\beta E_t \lambda_{t+1} \), the original allocation of \( M_t \)
and \( B_t \) could not have been optimal to begin with. In the spirit of Hartley (1988) the liquidity return on money, \( \theta_t \), has to match the explicit return on the alternative asset:

\[
E_t \theta_{t+1}/E_t \lambda_{t+1} = R_{t+1}.
\]

Summing up, Proposition 3.1 says that in a stationary equilibrium with binding cash-in-advance constraints the interest rate \( R_{t+1} \) will be positive with nonzero probability. If there is an equilibrium in which \( p_t x_t < M_t \) almost surely, then \( R_{t+1} = 0 \) almost surely. Note, however, that Proposition 3.1 above does not show \( \theta_t > 0 \) for all \( t \) almost everywhere whenever \( R_{t+1} > 0 \) almost everywhere for all \( t \). This is, in fact, not generally true.\(^8\)

Analogous propositions hold for the multiplier on the foreign cash-in-advance constraint, \( \theta_t^* \), and the foreign interest rate \( R_t^* \), only now we must also include as an assumption the requirement that \( e_t \) be positive over some state of positive measure. This additional requirement is also guaranteed by an equilibrium like that of Chapter 2 (see Corollary 2.9. in Chapter 2.)

Henceforth assume that \( R_{t+1} > 0 \) over some states of positive measure, so that the cash-in-advance model has "nontrivially binding" cash-in-advance constraints. We can derive several efficient-markets pricing relationships from conditions conditions (3.5c)-(3.5i). Elementary substitutions and rearrangements allow us to highlight the asset-pricing and currency arbitrage conditions in the following

**PROPOSITION 3.2.** The following relations hold at a consumer's optimum:

\[
\frac{f_{t+1}^p}{e_t} = \frac{(1 + R_{t+1})}{(1 + R_{t+1}^*)}
\]
(Uncovered Interest Parity with Risk Premium)
\[
\frac{E_t e_{t+1} \lambda_{t+1}}{e_t E_t \lambda_{t+1}} = \frac{(1 + R_{t+1})}{(1 + R_{t+1}^*)}
\]

(Forward-Spot Efficiency)
\[
E_t \left[ \frac{(f_{t+1} - e_{t+1})}{e_t} \lambda_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \right] = 0
\]

(Efficient-Markets Asset-Pricing)
\[
\beta E_t [(1 + R_{t+1})(\lambda_{t+1}/\lambda_t)] = 1
\]
\[
\beta E_t [(1 + R_{t+1}^*)(e_{t+1}/e_t)(\lambda_{t+1}/\lambda_t)] = 1
\]
\[
\beta E_t \left[ \frac{(q_{t+1} + p_{t+1} d_{t+1})}{q_t} \lambda_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \right] = 1
\]
\[
\beta E_t \left[ \frac{(q_{t+1}^* + p_{t+1}^* d_{t+1}^*)}{q_{t}^*} \frac{e_{t+1}}{e_t} \lambda_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \right] = 1
\]

(Currency-Valuation Equations)
\[
\beta E_t [(\theta_{t+1} + \lambda_{t+1})/\lambda_t] = 1
\]
\[
\beta E_t [(\theta_{t+1}^* + e_{t+1} \lambda_{t+1})/e_t \lambda_t] = 1.
\]
3.3. Risk Premia and Forward Efficiency: Comparisons

With Svensson and Stockman (1987)

In their paper, Svensson and Stockman (1987) introduce spot and forward currency markets by assuming that agents can trade currencies during goods markets at what they call the spot exchange rate and can also trade currencies during asset markets at what they call the forward exchange rate. Specifically, let $\bar{M}_t$ and $\bar{M}_t^*$ be the quantities of domestic and foreign currency held by the agent from period $t-1$ asset markets into period $t$ goods markets. Now they adopt the timing convention that information regarding the process $(d_t, d_t^*)$ becomes known only after individuals have selected $\bar{M}_t$ and $\bar{M}_t^*$. At period $t$ goods markets then, they choose new holdings of currencies $M_t$ and $M_t^*$ at the exchange rate of $e_t$ pounds for a dollar. Hence at goods markets there is the additional constraint

$$M_t + e_t M_t^* = \bar{M}_t + e_t \bar{M}_t^*$$

and then at time $t$ asset markets, their wealth constraint is

$$\bar{M}_{t+1}^* +  \hat{\gamma}_t^{t+1} M_{t+1}^* + q_t z_{t+1} +  \hat{\gamma}_t^{t+1} q_t^* z_{t+1}^* \leq M_t - p_t x_t + \hat{\gamma}_t^{t+1} (M_t^* - p_t x_t^*) + (q_t + p_t d_t) z_t +  \hat{\gamma}_t^{t+1} (q_t^* + p_t d_t^*) z_t^*.$$ 

Svensson and Stockman regard $\hat{\gamma}_t^{t+1}$ as the forward exchange rate prevailing at $t$. This, they argue, is due to the fact that when ones solves the model $\hat{\gamma}_t^{t+1}$ can be expressed as the sum of the expected spot exchange rate and a term that has an interpretation as a risk premium. In support of this interpretation they allude to a similarity with a risk premium obtained by Hodrick and Srivastava (1984)9. An equivalence, however, was not explicitly established. It would have been reassuring if
they had done this, on account of the subtle but important assumption they implicitly make on the liquidity of forward speculative returns. In their model individuals speculating on the spot rate $e_{t+1}$ buy currency at period $t$ asset markets at the forward rate $r_{t+1}^{t+1}$. Hence we might call this mechanism of forward transaction a currency-arbitrage approach.

The first improvement on this currency-arbitrage model made by the asset-arbitrage setup suggested in this chapter is that the asset-arbitrage approach yields covered interest parity out of a straightforward no-profit argument between trades in covered and uncovered bonds. While covered parity cannot be shown inconsistent with the model of SS, it is not implied by the SS model's first-order conditions.

Moreover, when there is some residual uncertainty that only gets resolved in between goods and asset trading sessions then covered interest parity will generally not obtain. Intuitively this is because the forward rate and the spot rate are not determined at the same point of time within the period $t$. If considerable new information arrives between these two decision points then the current forward rate can have very little to do with the current spot rate. Their model may predict deviations from covered interest parity that are wider than observed empirically, as the size of such potential deviations depend crucially on the kind of new information that arrives. More importantly, such disparities may be predictable on the basis of news components occurring between markets.

It is well-known that current spot rates are highly correlated with current forward rates, more so than the current forward rate correlates with the spot rate a period ahead. Although empirical research has shown that there have been deviations from covered interest parity, these have been small and may not have much to do with information flow as with simple transactions costs of making covered arbitrages. The desirability of preserving covered parity in a model is premised on these observations.
The proposition that it is additional information driving observed deviations from covered parity in a currency-arbitrage model of forward exchange might be tested if one could identify the information that occurs between trading sessions. Observability, however, is precisely the problem since we never really observe the kind of sequential trading dichotomy assumed in cash-in-advance models so as to be able to say with little arbitrariness what information arrives before goods markets and what arrives after goods markets. In order to get an empirically testable proposition one would have to specify what additional information individuals collect between markets as part of the assumptions of the theoretical model. Further, one would need some justification why such information should be unknown when commodity trades are made but known when asset trading takes place.

Next, we observe that with the cash-in-advance constraints specified above, allowing individuals to trade currency at goods markets in effect means that the total stock of currency at t, $\bar{M}_t + e_t\bar{M}^*_t$, is what matters for transactions. If the individual knew precisely what $e_t$ was at the time of choosing $\bar{M}_t$ and $\bar{M}^*_t$, his choice of the separate quantities $\bar{M}_t$ and $\bar{M}^*_t$ would be of no particular relevance to the consumption choice since he will reallocate these anyway to suit his consumption plan in period $t$. In their model (given the timing assumption on information flows) the only reason for caring about the composition of the total money stock is because individuals will speculate on deviations between the rates $e_{t+1}$ and $T_{t+1}^t$. That is to say, in their model, forward speculation is the reason why national currencies exist instead of one world currency (measured as an index with fixed weights). In the absence of this speculative motive an individual could choose to hold nothing but dollars (or pounds) into period $t$, and then merely adjust the composition of his currency portfolio at goods markets to finance purchases of domestic and foreign goods. The spot exchange rate $e_t$ in the model would be indeterminate. Ultimately, then, the
motivation behind having distinct national moneys in the SS formulation is that national moneys become the channel through which investors speculate on the exchange rates $r_{t+1}^t$ and $e_t$. Whether this is a reasonable explanation for having distinct national currencies is debatable.

In the currency-arbitrage setup the returns to holding currency into period $t+1$ accrue not at $t+1$ asset markets, but at $t+1$ goods markets instead. This is precisely because forward speculation is carried out in terms of holding currency into $t+1$. At $t+1$ goods markets, individuals can use said currency to finance their consumption, thereby making forward speculation valuable not only because it is a way of transferring wealth across time periods, but also because it provides liquidity returns at $t+1$ goods markets. While one could certainly make such an assumption about the liquidity of the returns to forward trading, there is no a priori reason why the returns to a forward contract should be modelled as being any more liquid than the returns to speculation in all other financial assets. In contrast the asset-arbitrage approach to forward exchange introduces forward rates through covered foreign bond returns accruing at $t+1$ asset markets--after $t+1$ goods markets have closed. This makes forward speculative returns no more liquid than, say, the dividends on the claims $Z_t Z_{t*}$. This will also enable us to isolate the "premium" to forward speculation which is due to uncertainty about the future value of wealth in complete symmetry to the premium on risky assets traded at asset markets. We will show momentarily that the two approaches imply different risk premia.

Were one to maximize (3.1) subject to the cash-in-advance constraints (3.2), the currency constraint (3.8) and the SS wealth constraint (3.9) one could show that the efficient-markets condition is (replacing their notation for the forward exchange rate $f_{t+1}^t$ with $f_{t+1}^t$)
\[
E_t \left[ \frac{(f_{t+1}^t - e_{t+1})}{e_t} \cdot \Delta^{SS}_{(t+1,t)} \right] = 0
\]

where \( \Delta^{SS}_{(t+1,t)} = \frac{\lambda_{t+1}^t + \theta_{t+1}^t}{\lambda_{t+1}^t + \theta_t^t} \).

This condition is what was expected: the "discount factor" \( \Delta^{SS}_{(t+1,t)} \) on futures speculation includes liquidity components \( \theta_{t+1} \) and \( \theta_t \) in the forward efficiency condition. This is precisely because of the liquidity services provided by maturing forward contracts in the currency-arbitrage formulation and will not appear in an approach that models the forward exchange mechanism through arbitrage in assets (bonds). The premium for currency speculation using a currency-arbitrage approach will therefore also contain liquidity components, and can be obtained by employing the conditional covariance decomposition

\[
E_t g_{t+1}^t h_{t+1} = E_t g_{t+1}^t E_t h_{t+1} + \text{cov}_t(g_{t+1}^t, h_{t+1}^t).
\]

Multiply the SS forward efficiency condition by \( e_t^t \epsilon \Omega_t^t \) and use the decomposition to get the risk-premium form of forward currency prices:

\[
f_{t+1}^t = E_t e_{t+1}^t + [E_t \Delta^{SS}_{(t+1,t)}]^{-1} \cdot \text{cov}_t(e_{t+1}^t, \Delta^{SS}_{(t+1,t)})
\]

where the second term in the sum is the forward risk premium.

One can now ask the question: when does the asset-arbitrage approach to forward exchange predict different risk/liquidity premia from the currency-arbitrage approach? If we hold \( \lambda_{t+1}^t \) and \( \lambda_t^t \) fixed and expand the risk-premium equation of SS around \( \theta_{t+1}^t = 0 \) and \( \theta_t^t = 0 \) we get\(^{10} \)
$$\Delta^{SS}(t+1,t) = \frac{\lambda_{t+1}}{\lambda_t} + L(t+1,t)$$

$$L(t+1,t) = \left[ \frac{1}{\lambda_t} \left[ \theta_{t+1} - \theta_t \frac{\lambda_{t+1}}{\lambda_t} \right] \right] + \text{higher-order terms}$$

So that the monetary approach has

$$f_t^{t+1} = E_t e_{t+1} + \{E_t \left[ \left( \frac{\lambda_{t+1}}{\lambda_t} \right) + L(t+1,t) \right] \}^{-1} \cdot (\text{cov}_t(e_{t+1}, \lambda_{t+1}/\lambda_t) + \text{cov}_t(e_{t+1}L(t+1,t)))$$

whereas the asset-arbitrage approach does not involve $L(t+1,t)$ in either the expectation term for $\Delta^{SS}(t+1,t)$ or in the second conditional covariance term.

A case may be made perhaps for why $E_t L(t+1,t) = 0$. It is more difficult, however to make a case for the conditional covariance term being zero. Disregarding the higher-order terms in $L(t+1,t)$, it may be said that the currency-arbitrage model yields the same premium as the asset-arbitrage model only in the case when the conditional covariance of $e_{t+1}$ and the relative growth rate of the two components of the marginal utility of money, $\left[ \frac{\theta_{t+1}}{\theta_t} - \frac{\lambda_{t+1}}{\lambda_t} \right]$ is zero. Otherwise, the forward risk premia are quite different.

This finding is related to what happens under both models when the liquidity premium at $t+1$, $\theta_{t+1}$ is nonzero. Consider first the case when $\theta_t = 0$ in all states and all $t$. The requirement then that one transact with money in goods markets is not essential to the solution for $x_t$. Now in both the currency-arbitrage model and the
asset-arbitrage model of this paper, \( \lambda_t + \theta_t = u_x_t/p_t \). If the null of \( \theta_{t+1} = 0 \) a.e. is true then the risk premium for forward speculation is the same under all three models since \( L(t+1,1) = 0 \) always. Under this case the risk premium is, in fact, the same one used in some seminal studies of forward market efficiency with risk premia (Cox, Ingersoll, and Ross, 1981; Hansen and Hodrick, 1983; Hodrick and Srivastava, 1984; Mark, 1985; and various others). That is, for the case of \( \theta_t = 0 \) a.e.,

\[
E_t \left[ \frac{f_t + 1 - e_t + 1}{e_t} \cdot \frac{u_x_{t+1}}{p_{t+1}} \cdot \frac{1}{u_x_t/p_t} \right] = 0.
\]

which is the formulation in Hansen and Hodrick (1984), for instance.

Under the more realistic supposition that \( \theta_{t+1} > 0 \) in some states, the forward market efficiency condition of SS is still

\[
E_t \left[ \frac{f_t + 1 - e_t + 1}{e_t} \cdot \frac{u_x_{t+1}}{p_{t+1}} \cdot \frac{1}{u_x_t/p_t} \right] = 0.
\]

This is identical to what would obtain in the barter model of Lucas (1982) if one were to use the asset-arbitrage approach there. Using a currency-arbitrage model of forward rates is equivalent to using an asset-arbitrage approach in a nonmonetary (barter) model as far as forward market efficiency is concerned. Theirs is a model in
which the presence of a transactions demand for money does not directly impact forward-spot efficiency but affects this only indirectly through the marginal utility of consumption at t+1.

In a cash-in-advance model where $\theta_{t+1} > 0$ for some states, the asset-arbitrage approach does not yield identical efficient-markets equations to either the currency-arbitrage model or a Lucas barter model with asset-arbitrage. The discount factor in the model of Section 3.2 is not the ratio $\frac{(U_{x,t+1}/p_{t+1})}{(U_{x,t}/p_t)}$ obtained in a barter model but is $\frac{(U_{x,t+1}/p_{t+1}) - \theta_{t+1}}{(U_{x,t}/p_t) - \theta_t}$ instead. If one were to extend the asset-arbitrage model to allow for shocks to $M_{t+1}$ from monetary policy, such policy can directly affect forward efficiency at t through impacts on time t expectations of $\theta_{t+1}$ or the covariance of $\theta_{t+1}$ with the future spot rate $e_{t+1}$.

On this point it is worth noting that because of the assumption of the currency-arbitrage model of SS that moneys can be traded at goods as well as asset markets, the effect on consumption choice of domestic and foreign money supply shocks is symmetric. Since all that matters at goods trading is the total stock of currency the individual brings into the market and not the composition of his currency portfolio, a shock that doubles $\bar{M}_t$, for instance, is equivalent in effect to a shock that doubles $\bar{M}_t^*$ for fixed $e_t$. By contrast, in the asset-arbitrage approach of this paper, the use of bonds to motivate forward exchange rates allows supply shocks of different currencies to have differential effects on the consumption choice whatever be $e_t$.

The issue that neither the asset-arbitrage approach nor the currency-arbitrage model of SS treats adequately has to do with the difference between a forward price and a futures price. Conceptually speaking, the distinction is that under forward prices, one need not sacrifice any resources at t in order to speculate--agents simply agree to rate of exchange at time t for a trade that takes place at t+1. A futures contract,
however, involves a delivery at $t$ of some resource in exchange for delivery of a contracted amount of something else at $t+1$. If so then the appropriate interpretation of $T^t_{t+1}$ in the SS model is that of a futures price of currency. In order for individuals to speculate on the "forward rate" they must deliver a dollar worth of bonds, say, at time $t$ asset markets for the number of pounds determined by the "forward rate" at $t$ and then hold these pounds over into the next period in which the $t+1$ spot rate prevails, and then convert it back into dollars at $t+1$.

In the asset-arbitrage approach of this paper agents also need to give up wealth in the form of security returns or leftover money at time $t$ asset markets. They can also issue bonds or securities and in doing so will give up resources at time $t$ that can be used to buy other assets or money. Hence, $f^t_{t+1}$ in the model of Section 3.2 is, strictly speaking, also a futures rate. Perhaps the best way to model forward rates should be via pure gambling contracts on deviations of the realized spot rate $e^t_{t+1}$ from the expectation $f^t_{t+1}$ in which delivery of returns accrues at $t+1$ but no wealth need be surrendered at $t$ in order to gamble. In this respect forward speculation should resemble a 100% margin loan on stock speculation. The precise timing of when such contracts mature will then determine whether or not a liquidity premium will appear in addition to a risk premium.

3.4. Arbitrage-Pricing for Assets

The forward-spot efficiency condition expressed in Proposition 3.2 is an arbitrage condition on the relative price of forward currency. It is necessitated by the fact that the representative individual with monotonic preferences can always make himself wealthier by undertaking an arbitrage in covered and uncovered bonds if the price of forward cover relative to the spot exchange rate is not in line with the efficient-markets condition. The sense in which the individual is wealthier in this
stochastic context is that his expected returns from arbitrage are larger than required to compensate for the risk associated with the covariance of his returns with $\lambda_{t+1}$, the value of next-period wealth. There are other arbitrage pricing conditions in this model associated with the claims which also allow the individual to transfer wealth across periods. These were the asset-pricing equations given in Proposition 3.2. The formal demonstration of how the forward efficiency condition, the interest parity conditions and the asset-pricing equations are recovered using arbitrage arguments is provided separately in Appendix A.

It turns out that the pricing conditions on the claims $z_t^*$, $z_t^*$ are identical for a SS version of the model except that in the SS formulation one will get $f_{t+1}$ in place of $e_t$. This statement suggests that one way of testing the two models would be to assess the empirical performance of a specification that uses $f_{t+1}$ in lieu of $e_t$. However, it is not likely that data on forward and spot exchange rates can reject one specification in favor of the other would because, as mentioned above, a cursory examination of the time series of $f_{t+1}$ and $e_t$ show that these track each other very closely.

3.5. Conclusions

Buried in a footnote in Hansen and Hodrick (1983) is a comment on the forward exchange rate in a monetary model of exchange rates:

"Considerable controversy exists in the literature regarding how the use of fiat money in a model should be motivated. The implications for the determination of spot exchange rates of various alternative strategies, such as placing real balances in the utility function, cash-in-advance constraints, or physical and intertemporal barriers to trade, do differ. We conjecture that motivating and introducing forward markets into the various models
may result in additional differences in the joint spot and forward exchange rate processes." [Footnote 4, pp. 116-117, Hansen and Hodrick (1983).]

In conclusion we wish to express support for this point. The discussion of the asset-arbitrage approach to forward exchange vis-a-vis the currency-arbitrage approach of Svensson-Stockman shows that alternative modelling choices matter for the theoretical expression of forward market efficiency. While the conjecture of Hansen and Hodrick has not been the focus of attention in the research on efficient forward currency markets, it is not a patently false conjecture. It is correct and material to the way we explain and measure any deviations from efficient markets that have been observed.

NOTES TO CHAPTER 3

1. Ito and Quah (1989) is one exception, however there has been some debate about the stationarity of the data and the size of their sample (Hodrick, 1987, p.39.)

2. Again, Ito and Quah is an exception. It is noteworthy that researchers have singled out the uncovered interest parity condition as the channel by which deviations from forward market efficiency manifest themselves. The empirical support for covered interest arbitrage is more considerable.

3. The papers of Domowitz and Hakkio (1985) and Korajczyk (1985) illustrate this observation quite well. They are discussed in the foregoing text.

4. Hence the generic name "consumption-beta" capital asset pricing model applies.
5. The use of an explicit pound rate of return $R_{t+1}$ is an expositational convenience. The idea we have in mind is equivalent to the following alternative scenario: assume there is a number $C_{1t+1}$ which is a sure return in time $t+1$ dollars on a pound-denominated contract agreed to at $t$. Let $C_{2t+1}(e_{t+1})$ be the time $t+1$ dollar return on another pound-denominated contract at $t$ which is made contingent only on the realization of $e_{t+1}$ and information at time $t$. If the form of the latter return obeys $C_{2t+1}(e_{t+1}) = k(\Omega_t)e_{t+1}$, where $k$ is a number depending on time $t$ information alone, then the forward exchange rate is given implicitly according to $t^{t+1} = e_{t+1}C_{1t+1}/C_{2t+1}(e_{t+1}) = C_{1t+1}/k$. Also, $(1+R^*_{t+1}) = C_{1t+1}/e_{t+1} = C_{2t+1}/e_{t+1} = k$ by an arbitrage argument on $k$ and a sure pound return $R^*_{t+1}$. The idea here is that the uncertainty about the dollar returns on the uncovered contract come only from uncertainty about the future spot exchange rate.

6. Because of the infinite horizon there are a set of transversality conditions that require the expected discounted value of each asset to go to zero as the discount horizon becomes infinite. With $\nu_t$ being the vector of state variables and $\pi^Y_t$ their prices, these transversality conditions are \[ \lim_{j \to -\infty} E_t \beta^j \pi^Y_{t+j} \circ \nu_{t+j} = 0, \] where $\circ$ is the direct product operator.

7. In the model this can happen only if the money supply created by the governments is always large enough to support all planned purchases except possibly in zero measure states.

8. The most that can be shown is that $E_t \theta_{t+1} > 0$ whenever $R_{t+1} > 0$ a.e.
9. It is also similar to that obtained in Stockman (1983) up to a term that results from Jensen's inequality.

10. The choice of $\theta_{t+1} = 0$ and $\theta_t = 0$ is for convenience only. We could have chosen any nonnegative level of $\theta_{t+1}$ and $\theta_t$ around which to expand without changing the qualitative result that the forward risk premia of the asset-arbitrage model of forward rates does not include liquidity services of forward profits, while the monetary approach does.

11. Very loosely speaking, this might happen in the asset-arbitrage model since $E_t \theta_{t+1} = R_{t+1} E_t \lambda_{t+1}$, so that, on average, the growth rates of $\theta_t$ and $\lambda_t$ will be the same when domestic interest rates are stable and people predict accurately.
CHAPTER 4
AN ECONOMETRIC ANALYSIS OF FORWARD MARKET EFFICIENCY

4.1. On the Robustness of Forward Market Efficiency

We have noted in the preceding chapter that the empirical evidence on forward market efficiency has been mixed. The emerging consensus at this point, however, is that if forward market efficiency holds in some form, it is not likely to be a simple one. For instance, consider the earlier experience with what is known as the unbiasedness-uncorrelatedness hypothesis. Let $e_{t+j}$ be the spot exchange rate $j$ periods away (or its logarithm or log-difference), $f_{t}^{t+j}$ be the $j$-period forward rate (or its log or log-difference), and let $\Omega_{t}$ be information in period $t$. The unbiasedness-uncorrelatedness hypothesis maintains that in the linear regression

$$e_{t+j} = \alpha + \beta f_{t}^{t+j} + \epsilon_{t+j}$$

where $e_{t+j}$ is the forecast error in $t+j$, the null should be that $\alpha = 0$ and $\beta = 1$ (unbiasedness) and $E(\epsilon_{t+j} \cdot \Omega_{s}) = 0$ (uncorrelatedness). A voluminous amount of research effort has gone into testing precisely this specification of market efficiency and the thorough investigator is referred to Hodrick (1987) for a good review of the empirical evidence on this specification. In summary, it has been found that $\alpha$ is typically nonzero, suggesting the presence of risk premia (Hansen and Hodrick, 1984; Korajczyk, 1983; Hsieh, 1982; more recently Hakkio and Rush, 1989) though some earlier studies dissent (Cornell, 1980; Frankel, 1982.) Moreover, the intercept $\alpha$ has been observed to be unstable over time indicating the potential for models with time-varying risk premia. The parameter
β has been found to be less than one and (somewhat mysteriously) negative (Bilson, 1981; Cumby and Obstfeld, 1984). Equally damaging to the unbiasedness-uncorrelatedness hypothesis are results that have found the forecast errors $\varepsilon_{t+j}$ not to be white noise; $\varepsilon_{t+j}$ in the above specification has been found to be serially correlated, correlated with components of $\Omega_t$ such as forward premia and past forward profits (Hansen and Hodrick, 1984) and not conditionally homoskedastic (Cumby and Obstfeld, 1984).

The poor results of tests of the unbiasedness-uncorrelatedness hypothesis that were generated by simple linear regression models encouraged the development of models with risk premia. Two approaches along this line have been noteworthy. One approach is to maintain the linear hypothesis above but allow for time-variation in the intercept $\alpha$ (Domowitz and Hakkio, 1983) or for ARCH (Domowitz and Hakkio, 1983) or serial correlation (Korajczyk, 1985) in the forecast error process. The results here are somewhat mixed— the Domowitz-Hakkio model finds support for $\alpha = 0$ but rejects $\alpha = 0$ and $\beta = 1$. Korajczyk’s model, however, provides more promising results in that unbiasedness is supported and serial correlation properties of the forecast errors are tied to time-variation in real interest rate differentials.

A second approach which has had some success in reconciling the data with market efficiency bases the estimating equations on orthogonality conditions that typically come out of dynamic asset-pricing models. The stochastic components of the model are introduced as part of the theory and the corresponding (usually nonlinear) Euler equations are derived. The advent of orthogonality condition estimation methods such as the Generalized Method of Moments (GMM) technique (Hansen, 1982 and Hansen and Singleton, 1982) provided the necessary statistical tools.
Though research that has used the second framework is not extensive relative to studies that have used linear regression forms of the unbiasedness-uncorrelatedness hypothesis, the results have thus far been favorable to forward market efficiency. Outstanding in this respect were the results obtained by Mark (1985) who estimated a Lucas model of forward rates and found that the moment restrictions implied by the Lucas model were not strongly rejected. It was difficult to say, however, that risk-neutrality was rejected by Mark's results since the reported standard errors of the risk-aversion parameters were high. More recently, Hodrick (1989) has been able to replicate aspects of Mark's results and find some support for a risk-averse model.

The research in this section builds on the work of Mark and Hodrick. We adopt essentially the same approach that these two authors have used in their respective papers but we look at forward-spot efficiency in some detail on a country-by-country basis. As a check for robustness we also generate estimates for various alternative structures such as the form of the utility function and the measure of the consumption risk premium, the composition of the information set, the estimator of the forecast error covariances, and the assumptions on the liquidity of forward speculative returns. Without preempting ourselves we suggest that the results of our econometric analysis make us less optimistic about Euler-equation models of the risk-premium than the earlier results of Mark and Hodrick would warrant. Our findings are that forward efficiency is not robust to some of the perturbations mentioned above.

The rest of this section is organized as follows: Section 4.2. develops the estimating model, Section 4.3. introduces notation and the GMM technique together with some discussion of the pros and cons of the procedure. In Section 4.4 we present and discuss our results.
4.2. Econometric Implementation

The model of Chapter 3 gives us a foundation for the development of a tractable empirical model of forward efficiency. To arrive at such a specification we now exploit equations (3.6a) to eliminate the unobservable multipliers and expression for forward market efficiency condition in terms of observables and parameters.

Take the forward efficiency condition of Proposition 3.2 and multiply through by $\lambda_t p_t / U_{x_t}$ inside the expectations operator ($\lambda_t p_t / U_{x_t}$ is a known quantity as of time $t$ so this is legitimate). By (3.6a) and application of the law of iterated expectations, we have

$$E_t \left[ \frac{f_t^{t+1} - e_t^{t+1}}{e_t} \cdot \frac{U_{x_t}(b)}{U_x(b)} \cdot \frac{p_t}{p_{t+2}} \right] = 0$$

Where $b$ is a vector of parameters that proceed from a parametric specification of $u$. One approach to estimating this equation is to take the expression inside the expectation and to set it equal to an error term that becomes known at $t+2$. We can generalize the above for $j$-period forward rates for $j \geq 2$ according to:

$$h_{t+j}(b) \equiv \left[ \frac{f_t^{t+j} - e_t^{t+j}}{e_t} \cdot \frac{U_{x_t}(b)}{U_x(b)} \cdot \frac{p_t}{p_{t+j+1}} \right] = \epsilon_{t+j}$$

Forward market efficiency then implies that $E_t \epsilon_{t+j} = 0$. It is also follows from efficient-markets that for any $\Gamma_t \subset \Omega_t$, $E(\epsilon_{t+j} \cdot \Gamma_t) = 0$. However, it will be noted in examining how $\epsilon_{t+j}$ covaries with elements of the $(t+j)$th information set $\Omega_{t+j}$.
we find that, for any \( k, j \),

\[
E(\varepsilon_{t+j}^\Gamma \Gamma_{t+k}) \neq 0 \text{ in general for } 0 < k \leq j.
\]

And in particular, that

\[
E(\varepsilon_{t+j}^t \varepsilon_{t+k}^t) \neq 0 \text{ in general for } 0 < k \leq j
\]

*although* \( E(\varepsilon_{t+j}^t \varepsilon_{t+k}^t) = 0 \) for \( k \leq 0 \). This is a well-known property of models in which the sampling interval is finer than the prediction interval over which individuals make their forecasts. These facts suggest that one possible way to proceed in estimating (4.1) would be to estimate a regression of the form

\[
(4.2) \quad h_{t+j}(b) = c\Gamma_t + \varepsilon_{t+j}.
\]

This regression approach amounts to the estimation of an equation that is nonlinear in the parameters \( b \) of \( h_{t+j} \). The estimation is complicated by two facts: (i) \( \varepsilon_{t+j} \) follows what might be called (Brown and Ligeralde, 1989) an \( m \)-order band-diagonal (denoted \( \text{BD}(m) \) here) process with \( m = j - 1 \), as implied by the across-period covariances derived above and possible conditional heteroskedasticity. (Note that the order of the BD process corresponds to the number of *intervening* observation periods between highest time index appearing in \( h \) and the reference period \( t \).) *Simultaneously*, (ii) Some structural variables appearing in \( h_{t+j} \) may not be taken to be econometrically exogenous, thereby posing problems for the consistency of direct GLS estimation on (2.2) if the order of the BD process exceeds 0. (The information set may contain past
period forward profits $\frac{t^{d}_{t-k} - c_{t}}{c_{t-k}}$, which are not exogenous but lagged endogenous variables.) If one is willing to give a specific model to the conditionally heteroskedastic BD process underlying $\varepsilon_{t+j}$ then under some stationarity assumptions on the structural variables in $h_{t+j}$ and $\Gamma_{t}$ one can estimate (4.2) by nonlinear generalized least squares using a covariance matrix for $\varepsilon_{t+j}$ that takes into account facts (i) and (ii), or by maximum likelihood, given a distributional assumption for $\varepsilon_{t+j}$. The test of the validity of the model would be a test of the null hypothesis that the coefficient vector $\mathbf{c}$ be zero. This was the approach adopted in earlier research on forward efficiency and closely related issues. (Hansen and Hodrick, 1980; and Brown and Maital, 1981 both derive the appropriate covariance matrix for general versions of (4.2).)

The alternative estimation procedure follows the generalized method of moments (GMM) technique as developed in Hansen (1982) and Hansen and Singleton (1982) with modifications for small samples suggested in Newey and West (1987) and Eichenbaum, Hansen, and Singleton (1988). This approach yields consistent estimates as does the approaches of Hansen and Hodrick (1980) and Brown and Maital (1981) under the assumption that covariance matrix is BD above and is robust to certain kinds of misspecification in the covariance structure. Before discussing the caveats attendant to our use of this technique, we outline the basic GMM procedure as applied to equation (4.1) so that its usefulness and limitations can be made clearer. For the purposes of this paper, we will be interested primarily in empirical implementation of relationship (4.1), although a more complete treatment would involve estimation of empirical counterparts of the set of first-order conditions to the consumption problem of Chapter 3.
4.3. A Review of the Generalized Method of Moments Procedure

Beginning from (4.1), suppose the sample is of size \( T+j \). Assume a sequence of information set vectors, \( \{\Gamma_t\}_{t=1}^{\ldots,T} \) each \( \Gamma_t \) being of dimension \((i \times 1)\). Define now the sample orthogonality conditions by

\[
\xi_{t+j}(b) = h_{t+j}(b) \otimes \Gamma_t.
\]

\( \xi_{t+j} \) may be considered the vector of \( j \)-periods away forecast errors at \( t \). The theory says that \( \mathbb{E}(\xi_{t+j}) = 0 \) and the principle of generalized method of moments estimation is to find the parameters \( b \) that will minimize the distance between the sample counterpart of \( \mathbb{E}(\xi_{t+j}) \) and 0. The sample counterpart of the true expectation in the orthogonality conditions is the sample mean

\[
g_{T}(b) = T^{-1} \sum_{t=1}^{T} \xi_{t+j}(b)
\]

\( g_T(b) \) being, then, an \( i \times 1 \) vector. A GMM estimator for \( b \) is given by the estimator which minimizes

\[
I_T(\beta) = g_T(\beta)'W g_T(\beta)
\]

where \( W \) is a \((i \times i)\) positive definite weighting matrix. Hansen (1982) shows that the optimal choice of \( W \) is to set it equal to the inverse of

\[
H = \sum_{k=-m}^{m} \mathbb{E}(\xi_{t+j}(b)\xi_{t+j-k}(b)')
\]
where \( m = j - 1 \), for \( j \geq 2 \), is the number of nonzero off-diagonals in the true covariance matrix of \( \xi_{t+j}(b) \). As \( b \) is not known beforehand, neither is \( H \) nor \( W \). Hence we use the consistent estimate

\[
H_T = T^{-1} \sum_{t=1}^{T} \{ \xi_{t+j}(b_{NLS})' \xi_{t+j}(b_{NLS}) + \sum_{k=1}^{m} [ \xi_{t+j}(b_{NLS})' \xi_{t+j-k}(b_{NLS}) + \xi_{t+j-k}(b_{NLS})' \xi_{t+j}(b_{NLS})] \}
\]

where \( b_{NLS} \) is the estimate that minimizes

\[
G_T(b) = g_T(b)'g_T(b).
\]

Letting \( W_T \) be the positive definite inverse of \( H_T \), a feasible GMM estimator \( b_{GMM} \) is that which minimizes

\[
\hat{G}_T(b) = g_T(b)'W_Tg_T(b).
\]

It has also been shown by Hansen (1982) that, under some standard stationarity assumptions on the structural variables in \( g_T(b) \), \( b_{GMM} \) is consistent and that

\[
T^{1/2}(b_{GMM} - b) \xrightarrow{d} N(0, (\frac{\partial g}{\partial b}'W\frac{\partial g}{\partial b})^{-1})
\]

where \( \frac{\partial g}{\partial b} \) is evaluated at the true parameter \( b \). A consistent estimate of the asymptotic covariance matrix above is:

\[
V_T = (D_T'W_TD_T)^{-1}
\]
where $D_T$ is the $(i \times p)$ matrix ( $p$ is the length of the parameter vector $b$) given by

$$F_T = T^{-1} \sum_{t=1}^{T} \left\{ \frac{\partial h_{t+j}}{\partial b} \otimes \Gamma_t \right\}$$

with $\frac{\partial h_{t+j}}{\partial b}$ evaluated at $b_{NLS}$.

Hansen (1982) establishes that the statistic $J = (T+j) \cdot \hat{\Gamma}_T - \chi^2(i-p)$. $i$ is the number of variables in the information set at $t$ and $p$ is the number of parameters to be estimated, so $i - p$ can be interpreted as the number of overidentifying restrictions, or excess orthogonality conditions the model must also satisfy. This fact amounts to a statistical test of the validity of the model. If $(T+j) \cdot \hat{\Gamma}_T$ is "too large" statistically at a chosen level of significance, the model does not satisfy all other orthogonality conditions it is required by theory to satisfy. We refer below to this statistic as "Hansen's J-statistic."

When $T$ is small, $H_T$ may not be positive semidefinite. In the application below we, in fact, rarely find $H_T$ to be positive definite in the estimations. To handle this case one can use the Newey-West covariance matrix in place of $H_T$:

$$N_T = T^{-1} \sum_{t=1}^{T} \left\{ \xi_{t+j}(b_{NLS}) \xi_{t+j}(b_{NLS})' \right\}$$

$$+ \sum_{k=1}^{m(T)} \left\{ \left[ \frac{m(T) + 1}{m(T) + 1} - 1 \right] (\xi_{t+j}(b_{NLS}) \xi_{t+j-k}(b_{NLS})' + \xi_{t+j-k}(b_{NLS}) \xi_{t+j}(b_{NLS})') \right\}$$

The matrix $N_T$ is guaranteed to be positive semidefinite (Newey and West, 1984) although it can be singular in finite samples. In eq. (4.3) $m(T)$ is no longer the order of the BD error process but is now some function of the sample
size $T+j$ that grows more slowly than $(T+j)^{1/4}$. We consider various settings of $m(T)$ in the estimations that follow.

One problem with the use of the Newey-West weighting matrix is that it may not be appropriate for the type of error process specified above. The Newey-West matrix downweights the off-diagonal elements of the weighting matrix and does not necessarily impose the condition that all but a finite number of off-diagonals in the true covariance matrix are zero. The theory above, however, says that $E(\xi_{t+j} \xi_{t+k}') = 0$ for $k < 0$, that is, the error process is an $MA(m)$ process or is band-diagonal $BD(m)$.

There is an alternative covariance estimator, the "modified Durbin" weighting matrix (Eichenbaum, Hansen, and Singleton, 1988) that is consistent and positive semidefinite like the Newey-West matrix, but does not downweight the off-diagonals and imposes the zero serial correlation restriction on the higher-order lags. The Durbin weighting matrix is based on the Wold decomposition of the process

$$\xi_t = e_t + B_1 e_{t-1} + \cdots + B_m e_{t-m}$$

where $m = j - 1$ as before. The matrix $W$ in the minimand $\hat{G}_T$ would then be the inverse of the matrix

$$D = (I + B_1 + \cdots + B_m)\Omega(I + B_1' + \cdots + B_m')$$

where $\Omega = e'e/(T+j)$. The Durbin method involves estimating the $B_i$ matrices in two steps: first, using the finite order autoregression

$$\xi_t = A_1 \xi_{t-1} + \cdots + A_{L(T)} \xi_{t-L(T)} + e_t$$
estimate the $A_i$ matrices. (The lag order $L(T)$ should be increasing with sample size.) Then using the estimates of the $A_i$ matrices, use the residuals $\hat{\xi}_t$ (which are predicted forecast errors) in the regression

$$\hat{\xi}_t = B_1 \hat{e}_{t-1} + \cdots + B_m \hat{e}_{t-m} + \nu.$$

This second step yields estimates $\hat{B}_1, \ldots, \hat{B}_m$ that can be used to calculate a consistent estimate of $\Omega$:

$$\hat{\Omega} = [T-L(T)-m]^{-1} \Sigma_t^{T} \hat{v}_t \hat{v}_t'$$

where $\hat{v}_t$ are the residuals of the second regression above. The Durbin weighting matrix which is a positive semidefinite, consistent estimate of the matrix $D$ above is given by

$$D_T = (I + \hat{B}_1 + \cdots + \hat{B}_m) \hat{\Omega} (I + \hat{B}_1' + \cdots + \hat{B}_m').$$

Once again, we note that in calculating the estimate $D_T$ we imposed the restriction that $E(\xi_{t+j}^T \xi_{t+k}) = 0$ for $k < 0$.

In using either of the Newey-West or Durbin matrices, the appropriate asymptotic test for $b_{GMM}$ is now based on

$$T^{1/2}(b_{GMM} - b) \xrightarrow{d} N(0, \frac{\partial g}{\partial b} W \frac{\partial g}{\partial b}^{-1} \frac{\partial g}{\partial b} W' HW \frac{\partial g}{\partial b}^{-1} \frac{\partial g}{\partial b} W \frac{\partial g}{\partial b}^{-1})$$
where \( H = \sum_{k=-m}^{m} E( \xi_{t+j}^{(b)} \xi_{t+j-k}^{(b)'} ) \), as above and \( W \) is the probability limit (i.e., plim) of either \( D_T^{-1} \) or \( N_T^{-1} \).

One reason for the appeal of GMM as a procedure for estimating Euler equations is that it does not impose too much structure on the error process in order to get consistent parameter and covariance estimates. In this sense it is robust to many possible covariance structures because the estimated covariance matrix it uses is a sample equivalent of the true covariance matrix (however, it is not robust to the number of off-diagonals set to zero or whether or not the nonzero off-diagonal terms ought to be downweighted.) In addition it provides a very handy statistic for testing the statistical validity of the model, which is one of the objectives of empirical study.

There are a few disadvantages to the use of GMM, however. The most obvious drawback is that because the estimator depends on one's choice of the elements of the information set \( \Gamma_t \), it is not generally efficient nor asymptotically efficient unless one knows what the optimal set of instruments \( \Gamma_t \) are. This problem does not have a practical solution, as Hansen and Singleton (1982) note, for it requires the researcher to completely specify the economic environment, in which case maximum-likelihood methods would yield efficient estimators, though often these are intractable. So different information sets/instruments can change the estimates and the power of tests.

Another drawback has to do with the presence of unit roots in the data, which the stationarity assumption rules out. Specifically if \( \xi_{t+j} \) has a unit root then GMM, as any other time-series procedure, will not provide consistent estimates of \( b \). This can be handled by transforming the data so that \( \xi_{t+j} \) becomes stationary (i.e., by taking logs or differencing or using growth rates), but
in doing so one may end up testing a model that is quite different from that suggested by the theory. Techniques have been proposed for handling the case where there are time trends (Hansen, Heaton, and Ogaki, 1988; and Ogaki, 1988) and the tradeoff in using these is that one typically needs to make assumptions about the trend equations for the variables concerned. In the empirical section we test that the ratios \[ \frac{f_{t+j}^t - e_{t+j}}{e_t}, \frac{p_t}{p_{t+j+1}}, \frac{U_x(b)}{U_{x_t}(b)} \] are all stationary, as are forward speculative profits \[ \frac{f_{t+j}^t - e_{t+j}}{e_t} \], whose lags form the instrument set.

A second limitation is that although GMM is robust to many specifications of the stochastic side of the model, it is not robust to specifications of the nonstochastic side of the model. In particular, the choice of the utility function \( U \) in the model may lead to trivial minimization problems. In situations in which one or more parameters \( b' \) in \( b \) enter \( g_T(b) \) in such a way that \( g_T(b) \rightarrow 0 \) or \( g_T(b)'g_T(b) \rightarrow 0 \) as \( b' \rightarrow 0 \) or \( b' \rightarrow \pm \infty \), GMM simply lets \( b \) go to a corner or to infinity, thereby trivially minimizing the quadratic form \( \hat{A}_T \). The implication of this is that one's choice of specification for \( g_T(b) \) is constrained to those specifications that can "identify" the parameters. This situation may also arise if the data in the sample takes on particular values. For example, consider what happens if \( x'_{t+j+1}/x_t > 1 \) for all \( t \) and utility is of the constant relative risk aversion form in \( x_t \):

\[
(4.4a) \quad U(x_t) = (1-b)^{-1} \cdot x_t^{1-b}.
\]
Marginal utility is \( U_{x_t} = x_t^b \) so that

\[
h_{t+j}^{(b)} = \left[ \frac{f_t^{t+j} e_{t+1}}{e_t} \left( \frac{x_{t+j+1}}{x_t} \right)^b \cdot \frac{p_t}{p_{t+j+1}} \right]
\]

and

\[
g_T^{(b)} = T^{-1} \sum_{t=1}^{T} \{ h_{t+j}^{(b)} \otimes \Gamma_t^{(b)} \}
\]

To minimize \( \hat{\Delta}_T = g_T^{(b)} W_T g_T^{(b)} \), given that \( x_{t+j+1}/x_t > 1 \) always, GMM lets \( b \rightarrow \infty \) whatever be \( W_T \) or the sample values of the forecast error \( (f_t^{t+j} e_{t+j})/e_t \).

If \( x_{t+j+1}/x_t \) is not always greater than 1, however, GMM is possible, although one should expect that estimates of \( b \) will tend to be large the more frequently one observes \( x_{t+j+1}/x_t > 1 \) and large values of the forecast error in the same periods.

As a first attempt to determine the pervasiveness of this potential problem we employ two specifications in addition to the usual constant relative risk-aversion utility function described above. These are the separable quadratic utility form

\[
(4.4b) \quad U(x, x^*) = -(b-x)^2 - (c-x^*^2), \quad x \leq b, \quad x^* \leq c
\]

and a separable form of a logarithmic utility function which figures prominently in studies involving linear expenditure systems (LES):

\[
(4.4c) \quad U(x, x^*) = \ln(x-b) + \ln(x^*-c), \quad x \geq b, \quad x^* \geq c
\]

In the empirical analysis the information sets \( \Gamma_t \) considered usually consist of a constant term and lags of the forecast error for both the currency being
examined and the five other currencies as well:

\[
\Gamma_t = \{ 1; \left[ \frac{f_{t-n}^i - e_t^i}{e_{t-n}^i} \right]_{i = 1, \ldots, 6} \}_{n = 1, \ldots, N}
\]

where \( i \) indexes the countries/currencies and \( N \) is the total number of lags of the forecast errors. Finally to compare across specifications of forward trading we present analogous estimates from the currency-arbitrage model of forward rates suggested in Svensson-Stockman (1987). Appendix B summarizes the specifications we entertain in the next section.

4.4. Data and Estimation

4.4.1. Data Set

The data set used in this paper is a monthly series from June 1973 to July 1983 consisting of 122 data points. Spot and 3-month forward exchange rates together with stock index levels and T-bill rates for six OECD countries, Canada, France, Germany, Netherlands, Switzerland, and the U.K., were obtained from OECD Main Economic Indicators Historical Statistics, 1983. This period conforms more or less with the time period used in prior studies that have used the GMM framework (Mark, 1985; Hodrick, 1987b; 1989). Also in conformity with Mark's research we select the U.S. individual for the representative agent so that consumption, population, and prices were in U.S. levels and all exchange rates were relative to the dollar. Consumption expenditure, imports, price, and population data were obtained from the U.S. Commerce Department's Business

We employ two measures of consumption. The first is the sum of personal consumption expenditures on nondurables and services, which is among the usual measures that have been used. The second measure is equal to nondurables plus services less U.S. imports of merchandise. Since monthly series on U.S. services imports were not available and neither were separate price deflators for expenditures on nondurables and on services we simply added together seasonally adjusted annualized personal consumption expenditures on nondurables and on services and then divided by 12. We then removed merchandise imports before deflating everything by the implicit price deflator for personal consumption expenditures and U.S. noninstitutional population. This second measure of consumption is closer to the interpretation given $x_t$ in the two-country theoretical models discussed in chapter 3.

4.4.2. Estimation Results

Before presenting the estimation results a couple of preliminary explorations are in order. First we describe the data to be used and test for its stationarity, since the GMM procedure requires stationarity. We then show that the models and the sample used are capable of replicating the results of Mark (1985) and Hodrick (1989), if not exactly, at least essentially.

Table 1 supplies descriptive information on the sample. Realized nominal forward profits for the six currencies considered were, on average, positive but were typically smaller than 1 per cent of the current spot exchange rate. Their standard errors, on the other hand, were typically of the order of six or seven percent. This, in fact, bears out the typical observation that forward profits are quite volatile. That forward profits are stationary appears to be borne out by the
fact that the calculated autocorrelations at various lags die out after two or three months. On the other hand the price ratio $p_t/p_{t+3}$ either exhibits some nonstationarity over the period or follows a high-order AR process, as can be seen from the slow decline of the sample autocorrelations. (This turns out not to be as serious a problem as one would think, since in the estimating model the price ratio and nominal forward profits are multiplied by each other and it turns out that the product of these two factors is stationary. This result is discussed shortly.) Real consumption growth, however, appears to be stationary and, on average, exhibits little if no growth whether measured in terms of nondurables plus services (NDS) or nondurables plus services less imports (NDS-Imports). Quite importantly, though, we observe that the standard errors of real consumption growth are generally about six times smaller than the standard errors of forward profits, indicating that consumption growth is a much smoother series than the series it is intended to explain.

More formal tests of the presence of unit roots in the data are reported in Table 2. (See Fuller, 1976; or Phillips, 1987 for more details on the testing procedure carried out below.) In this table we present Dickey-Fuller (DF) and augmented Dickey-Fuller (ADF) statistics for the joint null that $\rho = 1$ and $\alpha = 0$ in the regression

$$v_t = \rho v_{t-1} + \alpha + \nu_t,$$

against the alternative hypothesis that $\rho < 1$ and $\alpha \neq 0$. $v_t$ represents the variable of interest (e.g. consumption growth or forward profits.) The relevant test statistics are:

$$DF^T = T(\hat{\rho} - 1)$$
and

\[ \text{DF}^t = T^{-1/2} \cdot \text{DF}^T \cdot \frac{[\Sigma T_{v_t}^2]^{1/2}}{[\Sigma T_{(v_t - \hat{\rho} v_{t-1})^2}]^{1/2}} \]

The asymptotic distributions of these statistics are quite complex so we compare the sample statistics with their critical values as tabulated by Fuller (1976). If the null hypothesis is rejected in favor of the above alternative then the series \( \{v_t\}_{t=1,\ldots,T} \) will be covariance stationary albeit with nonzero mean due to a nonzero \( \alpha \). In conducting the unit root tests the variables of interest are realized real forward profits \( \frac{f_{t+3}^t - e_{t+3}}{e_t} \) and consumption growth \( \frac{x_t}{x_{t+j}} \), which constitute equation \( h_{t+j}(b) \), and nominal forward profits \( \frac{f_{t+3}^t - e_{t+3}}{e_t} \), which are the instrumental variables. The results in Table 2 indicate that the presence of a unit root in the data can be rejected, for the most part, in favor of stationarity at a significance level of 1 per cent. This is true even if we correct for potential serial correlation in \( v_t \) up to order 50, as can be seen in the significance of the augmented Dickey-Fuller statistics ADF(10) and ADF(50). Stationarity of the data is not compromised either by using a currency-arbitrage specification \( (j = 3) \) instead of an asset-arbitrage specification \( (j = 4) \).

Table 3 presents joint GMM estimates on the full system of six currencies using the Newey-West matrix to weight the sample orthogonality conditions. The numbers there indicate that in the essential respects the data replicates the earlier findings of Mark (1985) and Hodrick (1989). The estimates of the risk-aversion parameter are uniformly larger than one and positive. The fact that the
risk-aversion estimates are significantly large and positive was reported by Hodrick, and indicates that the data used are consistent with risk-aversion on the part of agents. Moreover, like these two writers we find here no evidence that the orthogonality conditions are substantially violated; the J-statistics fall far below the level necessary for model rejection. We can now proceed to analyze the robustness of this forward efficiency result to the specification changes in the model suggested at various points in the above text.

Tables 4 shows the results of GMM on a country-by-country basis on the six currencies for what can be regarded as the benchmark model specification in this study. In this table we use a currency-arbitrage formulation of the forward trading mechanism, CRRA utility, a consumption measure consisting of nondurables plus services expenditures, and the Newey-West weighting matrix, which are the most commonly used settings for this kind of model. For instruments in the information set we use 1 - 3 lags of forward profits and a constant term. This, too, is standard, however an important fact to remember in assessing the strength of the results that follow is that forward profits have not been regarded as particularly good instruments for rejecting forward efficiency relative to other instruments, say, forward premia. Others before (Mark, 1985; and Hodrick, 1989) have found that using forward premia tends to increase the number of observed rejections.

The main result in Table 4 is that the results for the system of six currencies holds at the disaggregated individual currency level as well. We find that the US dollar/Canadian dollar exchange rates are not orthogonal to information, but for the most part we do not find strong and regular rejections of forward efficiency for other currencies. We again observe large and positive relative risk aversion parameters that are significantly different from one, indicating a rejection of a null of risk-neutrality in favor of risk-aversion.
The large risk-aversion estimates are, to some degree, stylized results in the GMM estimation tradition. At this point our belief is that this is mainly an artifact of the use of a relatively smooth series (consumption growth) to explain or track a highly volatile one (real forward profits.) Consequently, one needs a lot of curvature in the nonlinear transformation of the smooth series to minimize the distance between the volatile series and zero.

In Tables 5 and 6 we entertain the possibility that the utility function used may not be the correct one. In place of constant relative risk-aversion, we consider two simple alternative utility specifications: quadratic and LES utility, but do not change the other settings of the model. We do find that the US dollar/Swiss franc exchange rates may not exhibit forward efficiency in addition to the US dollar/Canadian dollar exchange rates. However the additional evidence produced is by no means a convincing rejection of forward efficiency. The estimated parameters, too, are well within two standard errors of the region in which the utility function is concave. Comparison with Table 4 shows little difference; we therefore conclude that forward efficiency is generally robust to the choice of utility function used in measuring aversion to risk.

Table 7 now presents results pertinent to the asset-arbitrage formulation of forward trades proposed in Chapter 3. Using an asset-arbitrage formulation is tantamount to an assumption that the returns on forward speculation are not any more liquid than returns on other assets. If we should find that forward market efficiency is compromised by this change in assumption, then one interpretation would be that that there might be liquidity premia as well as risk premia built into the excess returns from forward speculation. (Another interpretation, of course, would be that forward efficiency in these dynamic asset-pricing models is not very robust.) However it should be kept in mind that any such rejection (non-rejection) should be taken as informal evidence of the existence
(nonexistence) of liquidity premia, since the model in use does not provide a nested test of the hypotheses of no liquidity premia.

We, in fact, find little evidence in Table 7 that rejects the asset-arbitrage mechanism for forward speculation. If anything, there is a very slight improvement in the model's performance in the sense that, except for the equation for the Swiss franc with 1 information lag, the chi-square statistics that measure rejection of orthogonality are uniformly smaller in value compared to their counterparts in Table 4. The risk-aversion parameter estimates, relative to their measured standard errors, increase a bit but not by very much. Thus, it appears that forward market efficiency is robust to the specification of the way forward trading is carried out. This is informal evidence against the need to use liquidity premia to explain movements in forward speculative returns.

We turn now to checking the robustness of forward efficiency to changes in the stochastic assumptions of the model. The way one estimates the covariance matrix of the model parameters makes a difference not just because different information is provided about the likely range in which the parameters lie. It is also material to the test of forward market efficiency relied upon in this study, i.e. the estimator used for the matrix \( \mathbf{W} \) in section 4.3 above is also used in the calculation of Hansen's J-statistic. Given that the usual covariance estimator, the Newey-West matrix, may not be appropriate for the band-diagonal forecast error process predicted by the theory, we need to see if using a more appropriate alternative estimator substantially upsets the nice results that we have so far observed. In fact, it has been suggested (Eichenbaum, Hansen, and Singleton, 1988) that taking into account the restrictions imposed by the theory on the error process may increase the power of the J-test.

In Table 8 we hold the same all the settings used in generating Table 4 except that now we employ the modified Durbin estimate instead of that of
Newey and West's. Here we obtain some important findings for the theory. The risk-aversion parameter estimates have become smaller, which is desirable, given the fact that highly risk-averse preferences are considered by many a form of model rejection. More importantly, though, the results in the table show strong and consistent rejections of orthogonality in terms of the values of Hansen's J for nearly all currencies in the sample. We interpret this result as evidence against forward market efficiency; tests of forward efficiency may not be robust to the choice of covariance estimator.

Thus far we have considered only one measure of the risk factor--growth in personal consumption expenditures on nondurables and services. Theory, however, would have us consider measuring consumption differently, to distinguish between domestic output and foreign output. In Table 9 we report results from estimating the benchmark model using spending on nondurables plus services less spending on imports as the measure of risk. The results here are almost as dramatic as the results in Table 8. We find strong rejections of forward efficiency, but no longer in terms of the values of the J-statistics, rather, the model now generates negative and significant relative risk aversion parameters. Further, an inspection at Table 10 in which we use the import-free measure of consumption together with the Durbin covariance estimator suggests that negative CRRA parameter estimates and large J-statistics are not mutually exclusive in the data. This reinforces the finding that forward efficiency is also sensitive to the measurement of the consumption risk premium.

4.5. Conclusions

The picture from the results of this chapter contrasts starkly with the favorable results of earlier studies using the GMM framework. Relative to what
theory would have one believe it is even more unsettling: risk-aversion estimates are large, and sometimes large and negative; instead of orthogonality, readily available information is correlated with risk-adjusted speculative profits; results are sensitive to small changes in instruments, consumption measures, and the covariance matrix chosen. What is to be believed?

Though this study extends the debate on whether forward exchange rates exhibit efficient-market properties, there are still some unanswered questions. For instance, is it the small sample size or true model misspecification that is behind the failure of the benchmark model? Little is known about the small-sample properties of the Durbin covariance matrix vis-a-vis the Newey-West estimator. Nonetheless one implication of the results above is clear: more work is necessary before either a satisfactory model of forward profits or a general rejection of forward efficiency can be claimed.

It has been noted that if enough tests are conducted, market efficiency will be rejected along some dimension. This is an almost certainly true statement and thus it is emphasized that while the results call into question the robustness of forward efficiency, there is no certainty at this point that these results themselves are robust. It is possible that forward efficiency has generally more empirical support than the above case study suggests. To establish this possibility as a likelihood, however, requires more evidence on Euler-equation models of forward efficiency. If anything, the earlier experience with the linear regression framework gives us additional motivation for such an undertaking.

If ultimately we should find that the results of this paper are robust, and forward efficiency is not, we are left with the uncomfortable question of what is being rejected. Market efficiency is a joint hypothesis of unbounded rationality, organized markets, auxiliary assumptions of functional form and forcing processes, etc. Which of the usual suspects is guilty? To seek a precise answer to this
question may be asking too much of the simple models of this chapter. Still, there are a few clues to what is needed. A different measure of risk may have to be developed. Consumption growth is generally too smooth to be expected to track the volatility of forward speculative profits, and may, in fact have less to do with speculative motives than other measures, such as investor incomes. To the extent that speculators do not adjust consumption in response to changes in income over the speculation period may overstate or understate the profits required for arbitrage to take place. With regard to employing new measures of risk, formal tests of the existence of additional explanatory factors, like liquidity premia, will be helpful in eliminating some of the candidate risk measures. Secondly, more evidence on the power and small-sample properties of alternative covariance estimators like the Durbin and Newey-West matrices is needed in order to be sure that this auxiliary assumption is not the principal culprit. A larger study would also be well-advised to increase sample sizes in order to eliminate any small-sample biases. Lastly, new models of rational agents or the speculative process may provide more powerful alternatives to test the standard asset-pricing model against.

Finally, if it were true, in fact, that forward exchange markets are not efficient, could one actually demonstrate that this can be exploited to generate nonzero, risk-adjusted arbitrage profits on a consistent basis? This is not clear but for many the answer to this question may be the only acceptable proof.
NOTES TO CHAPTER 4

1. More generally, \( j \) should be the length of the prediction interval, i.e., the difference between the highest time subscript appearing in the expression whose time \( t \) expectation is being taken, and \( t \).

2. For the asset-arbitrage model of Chapter 3, \( j = 3 \) since we use 3-month forward rates sampled monthly. So \( m = 3 \) whenever the Hansen matrix can be inverted because of the presence of \( U_{x_{t+j+1}} \) in the forward efficiency condition. For the currency-arbitrage model of Svensson-Stockman discussed in Chapter 3, \( j = 3 \) and \( m = 2 \) because there we use \( U_{x_{t+j}} \).
CHAPTER 5
EQUILIBRIUM WITH LIQUID ASSETS

In this chapter we consider an extension of the basic model introduced in Chapter 2 which will allow for the presence of assets that are close substitutes to money in terms of the liquidity services provided. In typical asset-pricing models with cash-in-advance constraints we normally think of currency as being the sole means by which all transactions in commodities are supported. This, of course, is a violation of reality--casual empiricism suggests that certain goods may be purchased by interest-bearing checking accounts or charge cards, or secured with negotiable papers. This violation has proven to be a useful one for addressing certain issues like the valuation of money in a general equilibrium context, the monetary approach to exchange rates, general equilibrium responses to inflationary financing of government deficits, and so forth. However it is worth examining the implications of relaxing the assumption that money is the unique means of payment in real transactions if only to understand how restrictive (or unrestrictive) such an assumption really is. This exercise is particularly worthwhile if it is possible to do so without also having to explicitly say why money is a medium of exchange in an economy. In this respect, our basic intuition should be a familiar doctrine: as long as the net returns on other liquid assets do not dominate the net returns to money in the probability sense, money will still be held in nonzero amounts for its transactions services. Hence the assumption of a non-interest bearing asset (flat money) as the unique medium of exchange in a cash-in-advance economy is restrictive only to the extent that nonnegative asset returns in the economy are "an almost sure thing."

Earlier extensions of the cash-in-advance framework in this direction are those of Lucas (1984) and Lucas and Stokey (1983, 1987). In their models money is still unique in its liquidity services over a class of goods termed cash goods, but there is
another class of goods, called credit goods which can be bought with either money or
promises to pay that are settled in the period asset market. Asset prices in their model
incorporate the marginal utility of credit goods, while the valuation of money arises
from the marginal utilities of both cash and credit goods. Hartley (1988) and Englund
and Svensson (1988) modify the approach of Lucas and Stokey by differentiating
between cash, checking accounts, and less liquid financial assets. Hartley examines
the liquidity services provided by money and checking accounts and concludes that
the presence of interest-bearing checking accounts does not preclude the use of money
even in the purchase of "check goods" because of foregone returns on the exercise of
check balances. Englund and Svensson examine issues of financial innovation and
interventions in intermediation services in the form of reserve requirements.

A modelling alternative to expanding the menu of goods that can be purchased
in the economy is to expand the menu of assets that are traded. In the model of this
paper an asset will be characterized as being more or less liquid according to: (1)
when dividend returns accrue, whether in goods markets or asset markets; (2) when
capital gains accrue (i.e., when the asset is sold), whether in goods or asset markets;
and (3) when the asset must be purchased-- in goods or in asset markets. Grilli and
Roubini (1989), for example, consider the case when financial assets must be
purchased at goods markets with cash-in-advance, just as goods and services are. The
returns of financial assets of their model, however, provide liquidity only in the asset
trading session. This allows them to prove several assertions regarding the effect of
controls on international capital flows on exchange rates and asset prices.

In the model of this chapter we shall not distinguish between assets according to
(3). Rather, we consider variation of assets according to (1) and (2) only. Instead
of examining the most general model possible, we focus, thus, on cases that are the
reverse of that in Grilli and Roubini (1989)-- there are no restrictions on financial
asset trades, and assets yield some kind of liquidity services in the purchase of goods
and services. One motivation for examining this case is relevant in the context of international exchange—the possibility exists that controls and imperfections are larger in international money markets rather than in asset markets. It has been observed, for instance, that exchange rates, the relative prices of one currency for another, are notoriously unstable, perhaps more so than returns on most assets. Even when exchange rates are controlled—directly, by various types of fixed exchange rate regimes, or indirectly by central bank intervention in the foreign exchange markets—stabilizing the exchange rate is often a policy that can be pursued only when internal balance is not seriously compromised. Hence exchange rate stabilization policy itself is often unstable. Part of the reason behind an investor's participation in international asset markets might then be to provide liquidity in the financing of international transactions when the exchange rate is volatile. By investing in foreign money market funds, for example, a domestic investor can shield himself from unfavorable exchange rate movements which erode the relative price of domestic currency for foreign currency, thereby guaranteeing liquidity in foreign goods markets and at the same time securing, in some states, positive foreign currency dividends.

In the sequel the models considered take on the following general form. Assume that Assumptions (2.1)-(2.3) of Chapter 2 hold, suitably reinterpretated. There will now be two kinds of assets in each country, quantities denoted by \( z_1 \) and \( z_2 \) (resp., \( z_1^* \) and \( z_2^* \)). Asset 2 will be the nonliquid asset— it is an asset purchased in asset markets and any returns from holding it accrue as wealth during asset markets. Asset 1, on the other hand will be the liquid asset, purchased also at asset markets, but possessing \textit{ex ante} returns \( r \) (resp., \( r^* \)) that accrue at goods markets. The crucial assumption we make in this model is that we can treat the liquid assets as outside rather than inside assets in order for money not to be rate-of-return dominated.\(^1\)

The ensuing discussion in this chapter is organized according to the various specifications of the return function \( r \) that we wish to examine. That is, for each
specification of asset returns we will, in turn, look at the problem of existence of a stationary equilibrium with binding liquidity constraints. Letting \( v = (M, M^*, z_1, z_2, z_1^*, z_2^*) \) and now letting the state \( s \) be equal to total consumption of domestic and foreign output \((\bar{x}, \bar{x}^*)\) we now, in the manner of Chapter 2, write the general form of the maximization problem as

\[
(5.1) \quad \text{TW}(v,s) = \max \{x,x^*,v'\} \ U(x,x^*) + \beta \int W(v',s') f(s,ds')
\]

subject to \( x, x^*, M, M^* \) all strictly positive and

\[
(5.2a) \quad p(s)x \leq M + rz_1
\]

\[
(5.2b) \quad p(s)x^* \leq M^* + r^*z_1^*
\]

\[
(5.3) \quad M' + e(s)M^* + q_1(s)z_1' + q_2(s)z_2' + e(s)[q_1^*(s)z_1^* + q_2^*(s)z_2^*] \\
\leq [M + rz_1 - p(s)x] + e(s)[M^* + rz_1^* - p(s)x^*] + [q_1(s) + r]z_1 \\
+ e(s)[q_1^*(s) + r^*]z_1^* + [q_2(s) + p(s)d_2]z_2 + e(s)[q_2^*(s) + p(s)d_2^*]z_2^*.
\]

\[
(5.4) \quad (z_1^*, z_2^*, z_1^*, z_2^*) \in Z \subset \mathbb{R}^4, \ Z \text{ is compact and contains } 0.
\]

Evidently, versions of Propositions 2.1-2.4 hold for this model. Rather than repeat ourselves we therefore begin our analysis by describing the general solution to the consumption problem. Let \( \theta(s), \theta^*(s), \) and \( \lambda(s) \) be the Kuhn-Tucker multipliers on (5.2a), (5.2b) and (5.3) respectively. Among the first-order conditions for an interior solution are:

\[
(5.5a) \quad U_x(s) = [\theta(s) + \lambda(s)] p(s) \ ; \ x > 0
\]
\[(5.5b) \quad \beta \int [ \theta(s') + \lambda(s')] f(s, ds') = \lambda(s) ; \quad M > 0 \]

\[(5.5c) \quad \beta \int [ q_2(s') + p(s')d_2^*] \lambda(s') f(s, ds') = q_2(s)\lambda(s) \]

\[(5.5d) \quad U_x^*(s) = [ \theta^*(s) + e(s)\lambda(s)] p^*(s) ; \quad x^* > 0 \]

\[(5.5e) \quad \beta \int [ \theta^*(s') + e(s)\lambda(s')] f(s, ds') = e(s)\lambda(s) ; \quad M^* > 0 \]

\[(5.5f) \quad \beta \int [ q_2^*(s') + p^*(s')d_2^*] e(s')\lambda(s') f(s, ds') = q_2^*(s)e(s)\lambda(s) \]

Equations (5.5a)-(5.5f) do not vary with our specification of \( r \) and \( r^* \), hence implications derived from these will not change in the subsections below. Specifically, what these equations suggest is that we should seek equilibria in which

\[(5.6) \quad \lambda(s) = \beta \int U_x(s')/p(s') f(s, ds') \]

\[(5.7) \quad \theta(s) = U_x(s)/p(s) - \beta \int U_x(s')/p(s') f(s, ds') \]

\[(5.8) \quad \theta^*(s) = U_x^*(s)/p^*(s) - \beta \int U_x^*(s')/p^*(s') f(s, ds') \]

\[(5.9) \quad e(s) = \frac{\int U_x^*(s')/p^*(s') f(s, ds')}{\int U_x(s')/p(s') f(s, ds')} \]

and \( (q_2(s), q_2^*(s)) \) are fixed-point functions for
(5.10) \[ T q_2(s) \lambda(s) = \beta \int \left[ q_2(s') + p(s') d_2 \right] \lambda(s') f(s, ds') \]

(5.11) \[ T^* q_2(s) e(s) \lambda(s) = \beta \int \left[ q_2^*(s') + p^*(s') d_2^* \right] e(s) \lambda(s') f(s, ds') \]

all conditional on the existence of strictly positive pricing functions (p(s), p^*(s)) satisfying the remaining conditions:

(5.12) \[ p(s)x = M + rz_1 \]

(5.13) \[ p^*(s)x^* = M^* + rz_1^* \]

(5.14) \[ \beta \int \left[ q_1^*(s') \lambda(s') + r[\Theta(s') + \lambda(s')] \right] f(s, ds') = q_1(s) \lambda(s) \]

(5.15) \[ \beta \int \left[ q_1^*(s') e(s') \lambda(s') + r[\Theta^*(s') + e(s') \lambda(s')] \right] f(s, ds') = q_1^*(s) e(s) \lambda(s) \]

where (r, r^*) are functions to be specified below.

More concretely, in the next sections r will be allowed to depend on p, q_1, and s in specific ways. Since an analogous treatment can be made for r^*, as is apparent from the symmetry of the solution structure, it suffices to look at the solution for the domestic equations (5.6), (5.7), (5.10), (5.12), and (5.14) to get at a solution. Thus we will omit analysis of the behavior of the foreign goods, asset, and shadow prices when doing so would be unduly repetitive.

5.1. Model 1: Fixed Financial Dividends

The simplest model of liquid returns is one in which the return function r in (5.12) is a fixed positive function of the state s: r = r(s). This is a model in which the
domestic asset guarantees r units of domestic currency per share as liquid dividends, depending, possibly, on the state s. The other component of the returns to this kind of asset—capital gains—is not, however, readily realized and cannot support consumption until the asset is liquidated in asset markets. Therefore the kind of asset we are contemplating here is much like a Treasury note when r is independent of s, and much like a money market fund when r depends on s.

Let us set the initial endowment of moneys that the individual holds to \((\tilde{M}, \tilde{M}^*)\) and assets \((z_1^*, z_2^*; z_1^*, z_2^*)\) to \((1, 1, 1, 1)\) respectively. In the manner of Chapter 2, it is natural then to define a stationary equilibrium as a sequence of bounded, continuous, and strictly positive prices \((\tilde{p}(s), \tilde{q}_1(s), \tilde{q}_2(s), \tilde{p}_1^*(s), \tilde{q}_1^*(s), \tilde{q}_2^*(s), \tilde{e}(s))\) that solve equations (5.6)-(5.15); an allocation of goods, moneys, and assetholdings \((x, x^*, M, M^*, z_1, z_2, z_1^*, z_2^*)\) equal to the aggregate supplies \((d_2, d_2^*, M, M^*, 1, 1, 1, 1)\); and lastly a value function \(W^*\) that satisfies \(TW^* = W^*\) in (5.1).

Restricting attention only to the domestic side of the model, it can be checked that finding an equilibrium comes down to finding bounded, strictly positive pricing functions p and q1 that satisfy (5.12) and (5.14).

5.1.1. Existence

That a solution exists for this case is established directly: first, when \(r = r(s) > 0\) for all s, bounded, and continuous, we have from (5.12)

\[
\tilde{p}(s) = \frac{\tilde{M} + r(s)}{\tilde{x}}
\]

which is bounded, continuous, and strictly positive. Therefore the equilibrium Kuhn-Tucker multipliers \(\lambda(s)\) and \(\theta(s)\) given in (5.6) and (5.7) are well-defined,
bounded, continuous, and nonnegative (strictly positive, in the case of \( \lambda \)). Given the fixed function \( \lambda \), define \( v(s) = q_1(s)\lambda(s) \). The mapping

\[
Tv(s) = \beta \int v(s') f(s,d's') + \beta \left[ \frac{U_X(s')\bar{\lambda}}{M + r(s')} \right] f(s,d's')
\]

is easily seen to be a contraction map of \( v \), yielding a fixed point \( \hat{v} = Tv = \tilde{q}_1 \lambda \). The solution \( \tilde{q}_1 \) will be strictly positive since \( \lambda \) is strictly positive and the second integral of the right-hand side is also strictly positive. As \( \tilde{q}_1 \) is a bounded and continuous function of \( s \), we have found a solution to the block of domestic equations (5.6), (5.7), (5.10), (5.12), and (5.14). Following this same approach we can find a counterpart pricing function \( \tilde{p}^* \) for the foreign goods sector and use this to establish strictly positive asset prices \( \tilde{q}_1^*, \tilde{q}_2^* \), a nonnegative multiplier \( \tilde{\delta}^*(s) \), and a strictly positive exchange rate \( \tilde{e}(s) \) given by (5.9). Hence we claim that

**PROPOSITION 5.1.** There exists a stationary equilibrium to the model described by (5.1)-(5.4) with \( r = r(s) \). In this equilibrium, the cash-in-advance constraints bind.

The above model, though obviously a simple one, is nonetheless capable of generating interesting and testable hypotheses. We consider three issues in turn: exchange-rate determination, the velocity of money, and asset pricing for liquid assets.

### 5.1.2. The Spot Exchange Rate

In the model with liquid assets, we can easily see that the equilibrium spot exchange rate need not be correlated with \( \bar{M} \) but is strongly correlated with broadly defined money, here taken to be \( \bar{M} \) plus the currency returns \( r \). Hence a form of the
monetary theory of the exchange rate holds: from (5.9) and (5.16) we have

\[ \tilde{c}(s) = \frac{\int \left\{ U_x^*(s') \tilde{x}^*(s') / [\tilde{M}^* + r(s')] \right\} f(s, ds')}{\int \left\{ U_x(s') \tilde{x}(s') / [\tilde{M} + r(s)] \right\} f(s, ds')} \]

if \( r(s) \) does not depend on \( s \), in particular, the spot exchange rate is now equal to

\[ \tilde{c}(s) = \frac{(\tilde{M} + r)}{(\tilde{M}^* + r^*)} \frac{\int U_x^*(s') \tilde{x}^*(s') f(s, ds')}{\int U_x(s') \tilde{x}(s') f(s, ds')} \]

The implication of the theory here is that if one were to study exchange rates by positing a monetary model of the exchange rate, to the extent that there are other liquidity-providing assets that are bilaterally traded one would need to consider measuring money more broadly by using asset return data as well as M1 or M2 data. Evidence that declines in the returns of domestic M2-like assets or assets like domestic T-Bills lead to exchange rate depreciation would be evidence potentially in support of the kind of theory this section suggests. This result is similar in spirit to that of Grilli and Roubini (1989) who argue that when there are capital controls the equilibrium exchange rate is affected by the volume of bilateral capital flows. The difference is that the flow of returns on holdings of domestic liquid assets tends to make the exchange rate appreciate in this model whereas, because of capital controls, increased trading in domestic assets leads to the opposite effect in their model. This is, of course, due to the fact that asset returns are money-augmenting in our model but asset transactions are money-utilizing in the model of Grilli-Roubini.
5.1.3. The Velocity of Money

Svensson (1985) presents a model of money in which the cash-in-advance constraints need not be binding because the information flow may be such that at the time individuals transact in goods markets there is residual uncertainty about the state thus ensuring a precautionary motive for money balances over and above the amount required for transactions. For a fixed length of the transactions period, Svensson's specification can potentially explain departures of monetary velocity \( v \equiv px/M \) from 1 in the downward direction.

One problem with using Svensson's model to explain velocity is that because his explanation hinges on the timing of information availability we are forced to specify the kind of residual uncertainty that appears between the goods markets and asset markets in order to falsify the theory. This is a difficulty because we do not have a clear idea of how, in the real world, separation of asset and goods trading is achieved in markets, let alone what information flows in the interim. Hence, the specification of what constitutes the residual uncertainty causing individuals to hold precautionary balances between trading sessions is made even more arbitrary.

The model of this section suggests an alternative explanation which is based on observable variables that may be easier to define. Additionally, in the equilibrium of the model specified in Section 5.1 we have \( px = M + rz \). If we continue to measure monetary velocity by \( vl \equiv px/M \) we find that \( vl = 1 + (rz/M) \), so that the model of this section predicts deviations of \( vl \) on the upside of unity whenever returns to liquid assets are strictly positive and aggregate supplies of these assets are positive. In this model \( vl \) turns out to be positively correlated with nominal returns on liquid assets and negatively correlated to the nominal money stock, another hypothesis which is testable.

That Svensson's model predicts monetary velocities on one side of unity and the
model of this chapter predicts monetary velocity on the other side of unity need not imply that the two models are inconsistent with each other. That depends on the length of the transactions period, as \( M \) is a stock whereas \( px \) is a periodic flow. If both models, however, were to measure the length of the period in the same way, however, then indeed the two will be incompatible and can be the subject of empirical tests.

The fact that measured velocity will be in excess of one, however, may limit the usefulness of the model of this chapter to transaction periods of about four months or more. For example, US data sampled at frequencies shorter than four months or so will have calculated velocity numbers less than one. Only to the extent that semestral consumption is subject to the liquidity constraint proposed can the model above be of some use.

There is a further prediction for the velocity of broadly defined money. Suppose that \( z_1 \), in fact, consisted of M2 assets and a portfolio of other assets such as money market mutual funds and T-Bills. Say, for argument's sake, that \( rz_1 = \delta z_{11} + (1-\delta)z_{12} \) where \( z_{11} \) are financial assets like the kind just mentioned and \( z_{12} \) are M2-like assets. Suppose we were to define velocity by \( v_2 = px/M2 = px/[M + (1-\delta)z_{12}] \). Then the fact that the liquidity constraints bind suggests that

\[
v_2 = 1 + \frac{\delta z_{11}}{[\bar{M} + (1-\delta)z_{12}]}\]

which says potential explanation for the deviation of \( v_2 \) from unity. Again, it may not be fruitful to use this theory to study velocity over periods shorter than a few months. However an additional challenge to this explanation of velocity movements can be raised based on even annual data. If one looks at annual data one finds that historically \( v_2 \) has been (mysteriously) constant over the period 1960-1976 (and to
some extent, even earlier) with v2 lying between 2.32 and 2.44. The above theory would be falsified if we found, for example that \( r_1^2z_{11}/(M + (1-\delta)r_2^2z_{12}) \) was highly unstable over this same period or if the data indicates \( r_1^2z_{11}/(M + (1-\delta)r_2^2z_{12}) \) was too small or too large relative to v2. Also the theory predicts a negative correlation of v2 with the stock of money M2 and a positive correlation with returns on the liquid non-M2 assets. This is another implication that may be falsified by the data.

Despite the potential for its rejection, the above theory is attractive as an alternative to Svensson's explanation and may be more useful because in implementing this theory of velocity we need not specify the information variables and the timing of information flows to test it. We do need to specify the kinds of assets which provide liquidity returns, however we have a clearer idea of what these are likely to be compared to what we know about what constitutes residual uncertainty in a cash-in-advance model.

5.1.4. Pricing of Liquid Assets

Let us write \( E_t y_{t+1} \) for \( \int y(s_{t+1}) f(s_t, ds_{t+1}) \). Equation 5.14 tells us that, unlike nonliquid assets, the price of liquid assets includes the expected value of future liquidity services over and above the expected value of future asset returns as an addition to the consumer's wealth. That is,

\[
q_{1t} = \beta E_t [(q_{1t+1} + r_{t+1})(\lambda_{t+1}/\lambda_t)] + \beta E_t [r_{t+1}(\theta_{t+1}/\lambda_t)]
\]

If the returns r were not available at goods markets the second term on the right-hand side would necessarily be zero. Thus the implication of the theory for asset pricing is that financial assets which deliver returns more frequently than other assets with the same effective rate of return\(^3\) should have a higher valuation in the market simply
because the returns represent liquidity for current transactions. This, of course, should not be the case in a model with complete Arrow-Debreu markets because one should always be able to borrow against future wealth in order to obtain liquidity today. The fact is, however, that the model is not one of complete markets because only certain assets and money can buy goods today-- lifetime wealth cannot.

In the sequel, theories of spot exchange rates, monetary velocity, and asset-pricing analogous to those outlined in Sections 5.1.2 - 5.1.4 will come out of the model depending on exactly how \( r \) is specified. Where these are interesting we shall discuss them, but where they are straightforward versions of the ones we have developed above they will merely be noted in passing, or not at all.

5.2. Model 2: Liquid Real Dividends

Consider now the model where \( r = p(s)d_1 \) (and \( r^* = p^*(s)d_1^* \)), i.e., the domestic liquid asset generates a domestic consumption good dividend priced at \( p \). In this case \( r \) depends on \( s \), just as in Model 1 above but we allow it to depend, additionally, on the price level \( p \), albeit in a simple multiplicatively separable way. For this case the price level \( p \) is determined according to:

\[
\hat{p}(s) = \frac{M}{\bar{x} - d_1}
\]

Therefore \( \lambda \) and \( \theta \) can be calculated from (5.6) and (5.7) respectively. Equation (5.14) implies that the mapping

\[
Tv(s) = \int \beta v(s') f(s,ds') + \int \beta U_x(s') d_1 f(s,ds').
\]
is a contraction of \( v(s) = q_1(s)\lambda(s) \). Therefore a fixed point \( T\tilde{v} = \tilde{v} = \tilde{q}_1\lambda \) exists, implying a function \( q_1 \), continuous, bounded, and strictly positive for any \( s \), that solves (5.14). Applying a similar argument to the block of equations involving the foreign country prices yields \( p^* = M^*/(\tilde{x}^* - d_1^*) \) which then determines \( \theta^* \) from (5.8) and then the exchange rate \( \tilde{e}(s) \) from (5.9). The foreign liquid asset price is then the fixed point of a contraction mapping like \( T \) above with \( v(s) = q_1(s)\tilde{e}(s)\lambda(s) \) and \( U_x^*(s)d_1^* \) replacing \( U_x(s)d_1 \). As all of these prices and the exchange rate are strictly positive we conclude that

**PROPOSITION 5.2.** There exists a stationary equilibrium to the model with \( r = pd_1 \) and \( r^* = p^*d_1^* \). In this equilibrium, the cash-in-advance constraints bind.

The velocity of money (measured using the M1 concept) in this version of the model turns out to be positively related to output of the sector with liquidity returns and negatively related to aggregate real money balances:

\[
v_1 = 1 + \frac{d_1}{M/p}
\]

Also, this model implies that the spot exchange rate is a function of the relative money stocks in both countries. It also turns out to be a function of relative expected change in utility levels, like before, but changes in utilities are now weighted by the outputs of the sectors with no liquidity returns, \( (d_2, d_2^*) \):

\[
\tilde{e}(s) = \frac{M}{M^*} \frac{\int U_x*(s')d_2^*(s') f(s, ds')}{\int U_x(s')d_2(s') f(s, ds')},
\]
whereas before the marginal utilities $U_x$, $U_x^*$ were weighted by $\bar{x}$ and $\bar{x}^*$.

5.3. Model 3: Liquid Capital Gains

The previous section considered the possibility of equilibrium when the returns $r$, $r^*$ are functions of the consumption prices $p$, $p^*$ in their respective economies. Now we consider equilibrium for a model in which the return is a function of the asset price $q_1$, specifically, we set $r = q_1$ and $d_1 = 0$ in the general model of Section 5.1. This is a model, then, of asset prices when assets are valued only for their potential worth at liquidation in goods markets. For this case the relevant equations are

$$\beta \int [\theta(s') + \lambda(s')]q_1(s') f(s, ds') = \lambda(s)q_1(s)$$

and

$$p(s) = \frac{\bar{M} + q_1(s)}{\bar{x}}$$

to get the equilibrium values of $p(s)$ we solve for the asset price $q_1$ first. Substituting in $U_x(s)/p(s)$ for $\theta(s) + \lambda(s)$ and $\beta \int U_x(s')/p(s')f(s, ds')$ for $\lambda(s)$, we derive

$$\int \frac{U_x(s')}{p(s')} q_1(s') = q_1(s) \int \frac{U_x(s)}{p(s)}$$

Replacing $p(s)$ with the right-hand side of the second equation above and rearrangement yields a functional mapping $Tq_1(s)$:
\[ Tq_1(s) = \frac{\int \frac{U_x(s')\tilde{x}}{\tilde{M} + q_1(s')} q_1(s')f(s,ds')}{\int \frac{U_x(s')\tilde{x}}{\tilde{M} + q_1(s')} f(s,ds')} \]

Now a formal solution of the above equation requires us to use the Schauder Theorem\(^5\) to establish existence of a price \( q_1 \) that solves this functional equation. However direct inspection suggests immediately that letting \( q_1(s) = q \), i.e., a constant function, solves the functional equation. Letting \( q \) be a positive constant yields a bounded, continuous, strictly positive function \( \tilde{q}_1 \) which then allows us to solve for \( \tilde{p} \) and the shadow prices \( \lambda \) and \( \theta \). Performing a symmetric procedure to derive \( \tilde{q}_1^* \), then \( \tilde{p}^* \) and \( \theta^* \), and finally \( \tilde{e} \) yields an equilibrium for this type of model.

Two things should be noted: first, the equilibrium so constructed may appear somewhat unusual because it is one in which the equilibrium liquid asset price \( \tilde{q}_1 \) is independent of the state \( s \). In this sense, \( \tilde{q}_1 \) is arbitrary. All other prices, however, will be functions of \( s \) through the consumptions (\( \tilde{x}, \tilde{x}^* \)). What matters, ultimately, is the real return of the asset, \( \tilde{q}_1/\tilde{p} \), since its only use is in obtaining consumption at goods markets.

Secondly, we did not rule out the possibility that the solution to the functional equation \( T, \tilde{q}_1 \), is zero. To rule out a zero solution requires assumptions that guarantee \( \theta \) will be strictly positive at \( \tilde{q}_1 = 0 \). These conditions were provided in Assumptions (2.7) and (2.7') of Chapter 2.

In the model of this section the equilibrium exchange rate is again a function of relative capital flows through the liquid asset prices \( q_1 = q \) and \( q_1^* = q^* \):
\[
\hat{\sigma}(s) = \frac{(\hat{\bar{M}} + q \hat{\bar{\bar{q}}})}{(\hat{\bar{\bar{M}}} + q \hat{\bar{\bar{q}}})} \int U_x(s') \hat{x}(s') f(s, ds') \int U_x(s') \tilde{x}(s') f(s, ds')
\]

Once again, the effect of greater flows in the domestic liquid assets is a depreciation of the nominal exchange rate-- the opposite case to that suggested in Grilli and Roubini (1989). And once again, this is for the reason that liquid assets are money substitutes in the model of this section.

Finally, in the model with liquid capital gains velocity is a function not just of the price level of goods and services, but also the price of money substitutes: taking velocity $v1$ to mean $px/\bar{M}$, we have

\[
v1 = 1 + q1/\bar{M}.
\]

This is, again, a testable proposition.

5.4. Model 4: Interest-Bearing Checking Accounts

Let us modify the model introduced in Section 5.1 to make things more interesting. Suppose that the liquid asset took the form of an exercise option of the following kind: the asset can be held into asset markets or used to support transactions in the goods markets. Each unit of the asset held into asset markets will yield dividends $pd_1$ and may be retraded at the price $q_1$. Thus total returns to trading a unit of this asset in asset markets are $(q_1 + pd_1)$. Now if the asset is, instead, liquidated at goods markets to finance consumption each unit liquidated will return $r$ units of currency but assets used up in goods markets do not entitle the asset holder to returns in the asset market. Hence the liquidity returns are $r$ per unit at goods markets. Now
the consumer, given that he enters the period with \( z_1, z_1^* \) units of the liquid assets, must decide what proportions \( \Phi \) and \( \Phi^* \) of these to hold into asset markets, the balance being the proportion used to finance consumption in the period. The proportions \( \Phi \) and \( \Phi^* \) then carry the interpretation of being "exercise option policies," since the consumer in exercising the option of using \((1-\Phi)\) of liquid assets for consumption foregoes the returns (capital gains plus dividends) from not exercising his option. One kind of asset that falls in this class of exercise options is an interest-bearing checking account or negotiable order of withdrawal; in the language of the model just described, \( r \) would be equal to 1 and \((q_1 + pd_1) = (1 + k)\) where \( k \) is the interest rate paid on the balance of the checking account.

The new maximization program that must be solved is given by (5.1) subject to

\[
(5.20) \quad p(s)x \leq M + r(1-\Phi)z_1
\]

\[
(5.21) \quad p(s)x^* \leq M^* + r^*(1-\Phi^*)z_1^*
\]

\[
(5.22) \quad M' + e(s)M^* + q_1(s)z_1' + q_2(s)z_2' + e(s)[q_1(s)z_1^* + q_2(s)z_2^*] \\
\leq [M + r(1-\Phi)z_1 - p(s)x] + e(s)[M^* + r^*(1-\Phi^*)z_1^* - p(s)x^*] \\
+ [q_1(s) + p(s)d_1]\Phi z_1 + e(s)[q_1(s) + p(s)d_1]\Phi z_1^* \\
+ [q_2(s) + p(s)d_2]z_2 + e(s)[q_2(s) + p(s)d_2]z_2^*,
\]

together with (5.4). As before we look at the necessary conditions to the domestic side of the model. These are (5.5a)-(5.5c), as before, together with

\[
(5.23) \quad \beta \int \{ [1-\Phi(s')]r[\theta(s') + \lambda(s')] + \Phi(s')[q_1(s') + p(s')d_1]\lambda(s') \} f(s,ds') \\
= q_1(s)\lambda(s); \quad z_1 > 0,
\]
\[(5.24) \quad [(q_1(s) + p(s)d_1)\lambda(s)z_1 - r[\theta(s) + \lambda(s)]z_1 - \gamma(s)] \leq 0; \quad \Phi \geq 0; \text{ with complementary slackness,}\]

\[(5.25) \quad \gamma(s) \geq 0; \quad [1 - \Phi(s)] \geq 0; \text{ with complementary slackness,}\]

\[(5.26) \quad p(s)x \leq M + r[(1 - \Phi(s)]z_1.\]

where \(\gamma(s)\) is a Kuhn-Tucker multiplier. We can set \(z_1 = 1, z_2 = 1, x = \tilde{x} = d_1 + d_2\), in equilibrium, as before, and conjecture that (5.26) holds with equality. Now, however, the optimal option policy \(\Phi\) must be solved for using the first-order conditions mentioned together with the wealth constraint (5.22). Since solving for \(\Phi\) is somewhat cumbersome, we show below that we can construct equilibrium prices for any choice of \(\Phi \in [0, 1]\).

Before we show such an equilibrium, however, we note that there is an interesting result that comes out of the model which is reminiscent of Modigliani-Miller-type theorems. It is most easily explained for the case when the optimal policy \(\Phi\) is a function whose values lie in the interior of \([0, 1]\). In this case \(\gamma(s) = 0\) and the first inequality in (5.22) holds with equality:

\[(5.26) \quad [(q_1(s) + p(s)d_1)\lambda(s) - r[\theta(s) + \lambda(s)] = 0\]

where we have set \(z_1 = 1\). Looking now at the liquid asset-pricing equation (5.23), we find that (5.26) implies

\[(5.27) \quad \beta \int [q_1(s') + p(s')d_1]\lambda(s') f(s, ds') = q_1(s)\lambda(s),\]
the standard sort of asset-pricing equation for an asset with dividends \( p d_1 \) and price \( q_1 \). Therefore we are led to conclude that

PROPOSITION 5.4. Let \( \Phi \) be any option policy with values in \((0,1)\). The equilibrium price \( q_1 \) of the liquid asset is the same regardless of \( \Phi \).

which says that the proportioning, \( \Phi \), is irrelevant for the asset price. The reasoning behind this is clear: in an interior solution the returns from not exercising the option are equated at the margin of choice to the returns from exercising the option (i.e., the liquidity returns). Since prices, market and shadow, are determined at the marginal \( \Phi \), the net value of the entire stock of \( x_1 \) must be the same whether we measure returns in terms of market prices or in terms of shadow prices.

Now let us continue with the case where \( \Phi \) is a fixed function taking values in \((0,1)\) and establish that there is an equilibrium for this case. Conjecture once again that equilibrium values of \( \lambda \) and \( \theta \) will be given by equations (5.6) and (5.7) once we find an equilibrium value for \( p \) which is strictly positive. As \( \Phi \) is a fixed function, we then can set

\[
(5.28) \quad p(s) = \frac{\bar{M} + r(1-\Phi)}{\bar{x}} > 0, \quad \text{for any } \bar{x} > 0, \bar{M} > 0
\]

Therefore \( \lambda \) and \( \theta \) are determined. From (5.27) we can define a mapping \( T^1: \mathcal{E}(K) \rightarrow \mathcal{E}(K) \) by

\[
T^1 v(s) = \beta \int v(s')f(s,ds') + K(s)
\]
where \( v(s) = q_1(s)\lambda(s) \) and \( K(s) = \beta \int \tilde{p}(s)d_1\lambda(s) f(s,ds') \). This mapping is again a contraction of \( v \) implying the existence of a fixed-point function \( \tilde{v} = T^1\tilde{v} \), in turn implying a function \( \tilde{q}_1 = \tilde{v}/\lambda \) which will solve (5.27). Since \( K(s) \) is strictly positive, so should be \( \tilde{q}_1 \). Similarly, we can construct a mapping \( T^2 \) for \( w = q_2\lambda \) which will give us a positive function \( \tilde{q}_2 \) that will solve the asset-pricing equation for the nonliquid asset, (5.5c). Having solved for the domestic prices, we can employ the same kind of steps we have just followed to solve the first-order conditions involving the foreign variables. First we set \( \Phi^\ast = \Phi^\ast \) and let the foreign liquidity constraint bind to get \( \tilde{p}^\ast \), whereupon \( \tilde{\theta}^\ast \) can be calculated from (5.8). We then use equation (5.9) to obtain the exchange rate \( \tilde{e} \), and proceed to construct contraction mappings that will give \( \tilde{q}_1^\ast \) and \( \tilde{q}_2^\ast \) in the manner illustrated above. As all of \( \{ p, p^\ast, e, q_1, q_2, q_1^\ast, q_2^\ast \} \) will be bounded, continuous, and strictly positive functions of \( s \), we arrive at the following

**PROPOSITION 5.5.** For any choice of \( \Phi, \Phi^\ast \) with values in (0,1) there is a stationary equilibrium for the model described in this section. In this equilibrium the liquidity constraints bind.

To show then that a stationary equilibrium will exist, for any \( \Phi \) in the interval [0,1] it suffices to examine the corners \( \Phi = 0 \) and \( \Phi = 1 \). The case where \( \Phi = 0 \) clearly will have an equilibrium, because the implied first-order conditions are identical to that of the model of Section 5.1 except for the condition that

\[
\beta \int [q_1(s') + p(s')d_1] f(s,ds') \leq q_1(s)\lambda(s)
\]

(with a similar inequality for \( q_1^\ast \)). But we can always choose the price \( \tilde{q}_1 \) that
satisfies the above with equality, since, by arguments in Section 5.1, there will be such a \( \tilde{q}_1 \).

Finally, in the case where \( \Phi = 1 \), we have identical first-order conditions to those of the model of this section, except that now \( \gamma(s) \geq 0 \), and in place of (5.26) we have

\[
(5.29) \quad [(q_1(s) + p(s)d_1)\lambda(s) - r[\theta(s) + \lambda(s)]] - \gamma(s) = 0.
\]

Also, equation (5.27) holds. One solution, then, is to choose

\[
(5.30) \quad \gamma(s) = [(q_1(s) + p(s)d_1)\lambda(s) - r[\theta(s) + \lambda(s)]],
\]

if \( [(q_1(s) + p(s)d_1)\lambda(s) - r[\theta(s) + \lambda(s)] > 0 \)

\[= 0 \]; otherwise.

The corresponding solution for \( p \) is \( \tilde{p} = \bar{M}/x \). So \( \lambda, \theta, \tilde{q}_2, \) and \( \tilde{q}_1 \) can be determined as described for the case when \( \Phi \in (0,1) \). It is then a straightforward exercise to show that we can find bounded, continuous, and strictly positive prices \( p^*, q_1^*, q_2^* \), and \( e \) that will solve the foreign side of the model's first-order conditions, given that \( \Phi^* = 0 \) or \( \Phi^* = 1 \). Hence, we can strengthen the statements of Propositions (5.4) and (5.5) to

THEOREM 5.6. Let \( \Phi \) be any option policy with values in \([0,1] \). The equilibrium price \( q_1 \) of the liquid asset is determined regardless of \( \Phi \).

THEOREM 5.7. For any choice of \( \Phi, \Phi^* \) with values in \([0,1] \) there is a stationary equilibrium for the model described in this section. In this equilibrium, the liquidity constraints bind.
5.5. A Nonexistence Example

Thus far we have entertained models of liquid assets or liquidity-providing assets in which a stationary equilibrium with binding liquidity constraints could be shown to exist. For each of the models described it was necessary to verify that such an equilibrium could be constructed. One might criticize this approach on the grounds that one should be establishing general results instead of checking every imaginable model that can be developed. Our response to this is to note that the models discussed above are not that highly specialized to begin with (the framework of Section 5.4, for example, admits several types of liquid assets, not just checking accounts.) Secondly, general results come at a high price in terms of the restrictiveness of the assumptions necessary to derive them. We illustrate this point with an example showing how, with a small variation in the structure of the basic model existence of a solution (equilibrium) implies a restrictive condition must be satisfied by preferences or the stochastic process \( f. \)

The example is constructed as follows: take the basic model and posit that the returns on the liquid asset consist of both real dividends and capital gains, i.e., let \( r = q_1 + pd_1, \) where \( p \) is the price of the consumption good. This seems plausible enough since we have looked at models where one or the other of real dividends or capital gains provided current-period liquidity.

Assume that a solution to the consumer’s problem defined in equations (5.1)-(5.4) exists in which assets and money are demanded in positive amounts. Then it follows, necessarily that (5.5a)-(5.5f) should hold together with

\[
\beta \int [q_1(s') + p(s')d_1'] [\lambda(s') + \theta(s')] f(s,ds') = q_1(s)\lambda(s)
\]
together with its counterpart equation for the foreign liquid asset. Suppose now that
the timing of information flows is such that the liquidity constraints bind. Then we
also have, necessarily,

\begin{equation}
\frac{M + q_1(s)}{\bar{x} - \bar{d}_1} = \frac{\dot{p}(s)}{\dot{x}}
\end{equation}

Using the fact that \( U_x(s) = [\lambda(s) + \theta(s)]p(s) \), and that \( \lambda(s) = \beta \int U_x(s')/p(s') f(s,ds') \)
from (5.5a) and (5.5b), respectively, we can substitute the Kuhn-Tucker multipliers \( \lambda \)
and \( \theta \) out of (5.31) to get

\[ \int [q_1(s') + p(s')d_{11}] U_x(s')/p(s') f(s,ds') = q_1(s) \int U_x(s')/p(s') f(s,ds') \]

Using (5.32) to eliminate \( q_1(s) \) from this equation gives

\[ \int \dot{x} U_x(s') f(s,ds') = p(s) [\dot{x} - d_{11}] \int U_x(s')/p(s'). \]

This last equation is an implicit functional equation in the unknown function \( p \). Write
\( v(s) = [p(s)]^{-1} \). The functional equation implied is

\begin{equation}
Tv(s) = [\dot{x} - d_{11}] \int \frac{U_x(s')v(s')f(s,ds')}{U_x(s')\dot{x} f(s,ds')} \int U_x(s')\dot{x} f(s,ds').
\end{equation}

Consider now the problem that this functional equation presents us with: we must find
a solution function \( v^* = Tv^* \) which is bounded away from zero and bounded from
above as well. (If \( v \) were not bounded in both directions, then either \( p \) is infinity or
zero, neither of which is an admissible solution for p.) Let \([y, \tilde{v}]\) be an interval containing the values \(v\) takes on; \(y\) is not zero and \(\tilde{v}\) is not plus infinity. It can easily be shown that

**PROPOSITION 5.8.** For any \(v(s) \in [y, \tilde{v}]\), \(Tv(s) \in [y, \tilde{v}]\) only if

\[
\frac{[\bar{x} - \bar{d}]}{U_x(s') f(s, ds')} = 1.
\]

To prove this result one shows the contrapositive, i.e., that whenever the left-hand side \(\frac{[\bar{x} - \bar{d}]}{U_x(s') f(s, ds')}\) is less than (resp., greater than) one then the mapping \(T\) applied to the function \(v\) at its lower bound \(y\) (resp., upper bound \(\tilde{v}\)) will map outside of \([y, \tilde{v}]\).

The meaning of this result is that solution functions to functional equations resembling (5.33) which are continuous, bounded, and strictly positive are not to be expected under general conditions. If \(\frac{[\bar{x} - \bar{d}]}{U_x(s') f(s, ds')} = 1\) does not hold (why should it?) or made to hold in the model, then stationary equilibria of the sort we have been looking at, with binding liquidity constraints, cannot possibly exist. This motivates the model-specific approach which we have adopted in this chapter, and also rounds off the discussion.
NOTES TO CHAPTER 5

1. One way of motivating the existence of liquid assets is to think of the economy as having, besides firms, a government sector and a banking sector. The government sector's role is to supply M1 through issue of currency and controls over demand deposits. The banking sector's role is to provide liquidity in the form of M2 assets and liquid or liquidity-providing securities--interest-bearing checking accounts, money market mutual funds, etc. This separation of roles between the government and the banking sector is also a pedagogical convenience only--it is immaterial for this model to know who supplies the moneys and the assets. All that matters is that these are supplied by someone. In the model of this paper, these assets are treated as "outside" assets supplied in fixed quantities. If we chose to we could have let these be "inside" assets and then, because agents are identical, equilibrium net asset holdings will be zero under the proviso that the returns on the liquid asset are not almost surely greater than one. If this were not the case, then the demand for money balances will be zero implying no equilibrium will exist if money supplies are positive.

2. See, for example, Gordon (1978), pp. 435-439.

3. By effective rate of return we mean to say the rate of return calculated after allowing for the fact that dividends, once they accrue, can be reinvested should they not be used to finance consumption. One way to imagine the comparison made here is to take a case in which the returns cannot be reinvested at all. What the model is saying is that given two securities returning currency dividends with the same total annual returns r, the security that pays the
dividends on a monthly basis should carry a higher price relative to one that pays out dividends at year-end. This is due to the usefulness of the monthly returns in supporting monthly transactions. As asset markets are not complete in this model, the time when the returns become available matters for consumption.

4. The state $s$ has to be reinterpreted here as $s = (d_1^t + d_2^t, d_1^s + d_2^s)$, i.e. the sum of the outputs from the two random production processes in each country.

5. In a form convenient for application to the problem at hand, the Schauder Fixed-Point Theorem says the following. Let $K$ be a bounded subset of $\mathbb{R}^1$ and $\mathcal{C}(K)$ be the space of bounded, continuous functions on $K$, let $I$ be a nonempty, closed, bounded, and convex subset of $\mathcal{C}(K)$. If a mapping $T:I \to I$ is continuous and the family of functions $T(I)$ is an equicontinuous family, then $T$ has a fixed point in $I$. We could select $I$ to be the interval of values which $q_1$ takes, so that the mapping $T$ defined in Section 5.3 will map from $I$ into $I$. Under assumptions on the transition function $f(s,ds')$ akin to that in Stokey, Lucas, and Prescott (1989, p. 520) we can show that $T$ will be continuous and the family $T(I)$ equicontinuous.
CHAPTER 6
CONCLUSIONS

This paper is concerned with three issues—equilibrium, efficient-markets, and liquidity—in the application of recursive dynamic modelling to the analysis of foreign exchange and asset markets. To tie up the discussion and to give perspective to our results, a few words are in order on the interrelationships between the three.

Existence of equilibrium is a test of the consistency of an economic model with maximization, feasibility, and market clearing, i.e., it is a requirement that a model must satisfy for internal consistency. Because of fiat money's special nature, in a model with both money and assets, the critical question is always: does the structure of the model admit of positive value for money at the margin, or will money be dominated by financial assets in any equilibrium? Underlying this is the even deeper philosophical question: what is money?

It is clear that in the absence of assumptions that imbue money with characteristics that are not shared by any other asset, money will be valueless at the margin if assets yield positive real returns in some states. In simple overlapping-generations models, for instance, this particular result is ubiquitous since a riskless interest-bearing asset is an even better vehicle than money for transferring value across separated generations. One must, of necessity, entertain other kinds of market incompleteness in order to arrive at a sensible model of valued money and assets.

The answer provided by cash-in-advance or Clower-constraint models is that money has liquidity services for transactions in temporally or spatially separated asset and commodity markets. The extreme case of this specification posits money as the unique instrument for the retirement of debts in commodity transactions. This is an heroic assumption, but a very useful one for purposes of constructing a tractable
model that admits of an equilibrium. Nonetheless it is an assumption that can be relaxed in the direction of including liquid or liquidity-providing assets as long as care is taken not to dominate money's own liquidity services. This is the direction taken in Chapter 5.

The cash-in-advance model of international exchange is even more stringent in the assumptions required for equilibrium because not only must the returns of money not be dominated by asset returns, the services of a particular national currency should not be completely dominated by another country's currency. Yet it is not obvious that currencies would be imperfect substitutes in the absence of rigidities like legal restrictions, information costs, or transactions costs. We have left this issue an open one, preferring to work within the framework imposed by the imperfect substitutes assumption for different currencies.

Once a non-dominated role for money is established, however, the question of the existence of equilibrium is not generally problematic since the structure of the cash-in-advance model will possess the requisite properties for invoking elementary equilibrium proofs. This is the point of the earlier sections of Chapter 2. We then proceed to demonstrate existence of a special kind of equilibrium: a stationary equilibrium in which money has, with positive probability, positive marginal value for transactions. The cost of this specialization is that the equilibrium we present has strictly binding cash-in-advance constraints a la Lucas (1978) and further restrictions need to be imposed on the underlying stochastic process in the model. The gain is that we are able to characterize the equilibrium prices and shadow prices of the model more exactly in terms of their relationships with each other and their relationships to fundamentals in the model. In particular we are able to recover a version of the monetary theory of the exchange rate in which the exchange rate depends on relative money stocks and marginal utilities; we also obtain a monotonocity result for the marginal liquidity prices $\theta$ and $\theta^*$ as they relate to the underlying dividend processes
These results constitute the latter sections of Chapter 2.

Now the efficiency of markets is often regarded as a necessary condition for equilibrium, hence it is a more primitive concept than equilibrium. It is possible to discuss market efficiency without reference to a specific equilibrium, as arbitrage-pricing theory suggests. However the necessity of efficient-markets for equilibrium is important to keep in mind because any rejection of efficient-markets is, indirectly, evidence against our equilibrium theory. This reason alone probably justifies the long-standing interest in market efficiency, considerations of profit-seeking activity aside.

We study the issue of efficient markets more modestly by focusing on forward-spot exchange rate efficiency rather than examining all the possible efficient-markets conditions one can derive out of the asset-pricing model of Chapter 3. Evidence is subsequently presented in Chapter 4 showing that in a six-currency study the hypothesis of forward efficiency is not always consistent with well-defined or risk-averse preferences. Unfortunately for the strength of these results, rejections of forward efficiency are not overwhelming or robust.

Chapter 3 developed a model of forward exchange that is consistent with covered and uncovered interest parity, but perhaps the most important contribution made there has to do with the relationship between efficient-markets and liquidity. This relationship can be put thusly: in a model where monetary or financial assets possess special liquidity services, efficient-markets pricing requires that liquidity services be counted as part of the returns of an asset or money. Therefore in assigning liquidity services to specific assets in a model, care must be taken in interpreting or measuring asset return premia to include (or exclude) liquidity premia as a potential source of above normal returns.

This doctrine is behind our use of an asset-arbitrage approach to forward exchange in contrast to a currency-arbitrage, or monetary approach suggested in the
literature. The asset-arbitrage approach to forward exchange does not assign liquidity services to forward profits; the monetary approach does. We do find, in the empirical study, little difference between a model with liquidity premia in forward returns and one without. However the evidence presented does not constitute a formal test of the hypothesis of no liquidity premia in forward returns and should be regarded as indicative at this point.

This doctrine also motivates the study of equilibrium with liquid assets in Chapter 5. As a starting point for research into the presence of significant liquidity premia we need to develop several useful equilibrium models in which some (not all) of the return components of financial assets—dividends, capital gains, or exercise value—provide liquidity services. We resolve the existence question for a number of proposed models in Chapter 5. As a consequence we are able to derive appropriate equilibrium asset-pricing equations in each specific case where equilibrium exists. We are also able to use the idea of liquid or liquidity-providing assets to develop alternative theories of monetary velocity and the exchange rate to those that have been proposed in recent research. Empirical tests for the presence and importance of liquidity premia might then be based on these structures.
REFERENCES


FIGURE 1

TIMING CONVENTIONS FOR THE BASIC MODEL

individual trades
in period t-1 asset
markets [ chooses
(M_t, M_t^*, z_t, z_t^*)]

period t dividends are
paid out here

Time t-1

period t dividends and other
information variables are
known

individual trades
in period t goods markets
[ chooses (x_t, x_t^*)]

Time t

individual trades
in period t asset markets
[chooses (M_{t+1}, M_{t+1}^*, z_{t+1}, z_{t+1}^*)]

Time t+1
FIGURE 2
BEHAVIOR OF THE LIQUIDITY MULTIPLIER, GENERAL CASE
FIGURE 3
EXISTENCE OF A SOLUTION WHEN $\theta$ IS INCREASING
<table>
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<th>STD.ERR</th>
<th>AUTOCORRELATIONS</th>
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(2) $\frac{p_t}{p_{t+3}} = 0.976$ 0.008 0.93 0.82 0.70 0.61 0.57 0.52 0.46 0.44 0.40 0.36 0.30 0.23

(3) $\frac{x_t}{x_{t+3}} = 0.999$ 0.009 0.57 0.21 -0.25 -0.25 -0.14 0.01 0.10 0.16 0.16 0.20 0.15 0.09

NDS Imports   1.000 0.0116 0.37 0.19 -0.20 -0.12 -0.04 -0.13 0.16 0.09 0.12 -0.03 -0.09 -0.12
TABLE 2

STATIONARITY TESTS
MODEL: $v_t = \rho v_{t-1} + \alpha + \nu_t$

$H_0$: $\rho = 1$ and $\alpha = 0$

Critical values (1% significance): -20.70
- 3.43

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TABLE 3

CRRA PARAMETER ESTIMATES, JOINT GMM ESTIMATION
(CONSUMPTION = NDS, NEWEY WEST LAGS = 2,3,4)

**KEY:**
- Estimate
- (Std.Err.)
- Hansen's J-statistic
- (Marg. significance of J)

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Notes: Starting value of parameter was 0.9. An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen's weighting matrix was employed in all iterations.
## TABLE 5

QUADRATIC UTILITY PARAMETER ESTIMATES
CURRENCY ARBITRAGE MODEL
(CONSUMPTION = NDS, NEWEY-WEST LAGS = 2)

**KEY:**
- Estimate
- (Std.Err.)
- Hansen's J-statistic
- (Marg. significance of J)

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Notes: Starting value of parameter was the maximum consumption level in the sample. An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen's weighting matrix was employed in all iterations.
### TABLE 6

**LES UTILITY PARAMETER ESTIMATES**  
**CURRENCY ARBITRAGE MODEL**  
(CONSUMPTION = NDS, NEWEY-WEST LAGS = 2)

**KEY:** Estimate  
(Std.Err.)  
Hansen’s J-statistic  
(Marg. significance of J)

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**Notes:** Starting value of parameter was 0.99 times the minimum consumption in the sample.  
An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen’s weighting matrix was employed in all iterations.
TABLE 7
CRRA PARAMETER ESTIMATES, ASSET ARBITRAGE MODEL
(CONSUMPTION = NDS, NEWNEY-WEST LAGS = 3)

**KEY:** Estimate
(Std.Err.)
Hansen's J-statistic
(Marg. significance of J)

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Notes: Starting value of parameter was 0.9. An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen's weighting matrix was employed in all iterations.
### TABLE 8

**CRRA PARAMETER ESTIMATES W/DURBIN COVARIANCE CURRENCY ARBITRAGE MODEL**

(Consumption = NDS, Durbin Lags = 2)

**KEY:**
- Estimate
- (Std.Err.)
- Hansen's J-statistic
- (Marg. significance of J)

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Notes: Starting value of parameter was 0.9. An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen's weighting matrix was employed in all iterations.
### TABLE 9
CRRA PARAMETER ESTIMATES, CURRENCY ARBITRAGE MODEL
(CONSUMPTION = NDS-IMPORTS, NEWEY-WEST LAGS = 2)

**KEY:**
- Estimate
- (Std.Err.)
- Hansen’s J-statistic
- (Marg. significance of J)

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<td>(0.67)</td>
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Notes: Starting value of parameter was 0.9. An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen's weighting matrix was employed in all iterations.
TABLE 10
CRRA PARAMETER ESTIMATES W/DURBIN COVARIANCE
CURRENCY ARBITRAGE MODEL
(Consumption = NDS-Imports, Durbin lags = 2)

**KEY:**
- Estimate
- (Std.Err.)
- Hansen's J-statistic
- (Marg. significance of J)

**INSTRUMENT LAGS**

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Notes: Starting value of parameter was 0.9. An asterisk (*) denotes significance at better than 10%. An "H" denotes that Hansen's weighting matrix was employed in all iterations.
APPENDIX A

ARBITRAGE PRICING DERIVATION OF THE FORWARD EFFICIENCY, INTEREST PARITY, AND ASSET PRICING RELATIONS

Denote the consumer's wealth at \( t+1 \) by \( Y_{t+1} \). In the model of Chapter 3,

\[
Y_{t+1} = M_{t+1}^{*}p_{t+1}^{*}x_{t+1} + e_{t+1}^{*}(M_{t+1}^{*}p_{t+1}^{*}x_{t+1}^{*}) + (1+R_{t+1}^{*})B_{t+1}^{*}
+ t_{t+1}^{*}(1+R_{t+1}^{*})B_{2t+1}^{*} + s_{t+1}(1+R_{t+1}^{*})B_{2t+1}^{*} + (q_{t+1}^{*}p_{t+1}^{*}d_{t+1}^{*})z_{t+1}^{*}
+ e_{t+1}^{*}(q_{t+1}^{*}p_{t+1}^{*}d_{t+1}^{*})z_{t+1}^{*}
\]

The time \( t \) wealth constraint is

\[
M_{t+1} + e_{t}^{*}M_{t+1} + B_{t+1} + e_{t}^{*}(B_{1t+1}^{*} + B_{2t+1}^{*}) + q_{t}z_{t+1} + e_{t}q_{t}^{*}z_{t+1} \leq Y_{t}
\]

Now in can be shown that if there were, in addition, Arrow-Debreu contingent claims that could be traded at asset markets, the price at \( t \) of \( a_{t+j} \) claims to consumption at \( t+j \) contingent on the state \((d_{t+j}^{*}d_{t+j}^{*})\) belonging to a subset of the sample space \( D \) is given by

\[
\Lambda_{t}(D)a_{t+j} = \int_{(d_{t+j}^{*}d_{t+j}^{*}) \epsilon D} \beta \frac{\lambda_{t+j}}{\lambda_{t}} a_{t+j}
\]

where \( \lambda_{t} \) is the Kuhn-Tucker multiplier on the wealth constraint as described previously. When \( D \) is the entire sample space for \((d_{t+j}^{*}d_{t+j}^{*})\) over which the Markov process \( f \) of Chapter 2 is defined,
Applying now the operator $\Lambda_t$ to $Y_{t+1}$ and adding the resulting equation to the time $t$ wealth constraint yields, after rearrangements,

\[
\Lambda_t Y_{t+1} + M_{t+1} + e_t M_{t+1}^* + B_{t+1} + e_t (B_{1t+1} + B_{2t+1}^*) + q_t z_{t+1} + e_t q_t^* z_{t+1} \\
+ \Lambda_t (p_{t+1} x_{t+1} + e_{t+1} p_{t+1} x_{t+1}) - \Lambda_t M_{t+1} + e_{t+1} M_{t+1}^* + (1 + R_{t+1}) B_{t+1} \\
+ f_{t+1} (1 + R_{t+1}) B_{1t+1} + e_{t+1} (1 + R_{t+1}) B_{2t+1} + (q_{t+1} + p_{t+1} d_{t+1}) z_{t+1} \\
+ e_{t+1} (q_{t+1} + p_{t+1} d_{t+1}) z_{t+1}^* \leq Y_t.
\]

Observe that we can bring $B_{t+1}$ out of $\Lambda_t$ since $B_{t+1}$ is chosen at time $t$ asset markets:

\[
\Lambda_t (1 + R_{t+1}) B_{t+1} = E_t \frac{\lambda_{t+1}}{\lambda_t} (1 + R_{t+1}) B_{t+1} = B_{t+1} E_t \frac{\lambda_{t+1}}{\lambda_t} (1 + R_{t+1}) B_{t+1}.
\]

As similar considerations apply to $M_{t+1}$, $M_{t+1}^*$, $B_{1t+1}^*$, $B_{2t+1}^*$, $z_{t+1}$, and $z_{t+1}^*$ we can regroup terms, factoring out these choice variables to get:

\[
\begin{align*}
\Lambda_t Y_{t+1} + \Lambda_t (p_{t+1} x_{t+1} + e_{t+1} p_{t+1} x_{t+1}^*) + (1 - \Lambda_t) M_{t+1} + (e_t - \Lambda_t e_{t+1}) M_{t+1}^* \\
+ [1 - \Lambda_t (1 + R_{t+1})] B_{t+1} + [e_t - \Lambda_t f_{t+1} (1 + R_{t+1})] B_{1t+1} \\
+ [e_t - \Lambda_t (1 + R_{t+1})] B_{2t+1} + [q_t - \Lambda_t q_{t+1} + p_{t+1} d_{t+1}] z_{t+1} \\
+ [e_t q_t - \Lambda_t e_{t+1} (q_{t+1} + p_{t+1} d_{t+1}) z_{t+1}^*] \leq Y_t.
\end{align*}
\]

To apply arbitrage pricing, let us interpret this last equation. $\Lambda_t Y_{t+1}$ is the expected present value of wealth in period $t+1$. $\Lambda_t (p_{t+1} x_{t+1} + e_{t+1} p_{t+1} x_{t+1}^*)$ is
the expected present value of consumption in period \( t+1 \). \( H_t \) is the consumer's present wealth. Now since \( (M_{t+1}, M^*_{t+1}) \) are bounded between zero and \( (\tilde{M}_{t+1}, \tilde{M}^*_{t+1}) \) since the choice of \( B_{t+1} B_{1t+1} B_{2t+1} z_{t+1} \), and \( z^*_{t+1} \) can take on any values, positive or negative, the consumer can drive either or both of \( \Lambda_t Y_{t+1} \) or \( \Lambda_{t+1} (p_{t+1} x_{t+1} + e_{t+1} p_{t+1} x^*_{t+1}) \) to infinity even with finite wealth \( Y_t \) unless the terms in braces [ ] are zero. That is, unless prices and returns on bonds and securities satisfy the no-arbitrage conditions

\[
\begin{align*}
(A.1) & \quad 1 - \Lambda_t (1 + R^*_{t+1}) = 0 \\
(A.2) & \quad e_t - \Lambda_t e_{t+1} (1 + R^*_{t+1}) = 0 \\
(A.3) & \quad e_t - \Lambda_t e_{t+1} (1 + R^*_{t+1}) = 0 \\
(A.4) & \quad q_t - \Lambda_t (q_{t+1} + p_{t+1} d_{t+1}) = 0 \\
(A.5) & \quad e_t q^*_t - \Lambda_t e_{t+1} (q^*_{t+1} + p^*_{t+1} d^*_{t+1}) = 0
\end{align*}
\]

Using the definition of \( \Lambda_t \), (A.4) and (A.5) can be rewritten as

\[
\begin{align*}
E_t \beta \frac{(q_{t+1} + p_{t+1} d_{t+1})}{q_t} \lambda_{t+1} \frac{\lambda_t}{\lambda} = 1 \\
E_t \beta \frac{(q^*_t + p^*_t d^*_t)}{q^*_t} \frac{e_{t+1}}{e_t} \frac{\lambda_{t+1}}{\lambda_t} = 1
\end{align*}
\]

which are the asset-pricing equations in Proposition 3.2.
To recover the covered interest parity, uncovered interest parity, and forward efficiency conditions obtained in the text use the definition of $\Lambda_t$ to rewrite (A.1), (A.2), and (A.3) as

(A.1') $E_t \beta(1+R_{t+1})^{\frac{\lambda_{t+1}}{\lambda_t}} = 1$

(A.2') $E_t \beta^{t+1}(1+R_{t+1}^*)^{\frac{\lambda_{t+1}}{\lambda_t}} = e_t$

(A.3') $E_t \beta e_{t+1}(1+R_{t+1}^*)^{\frac{\lambda_{t+1}}{\lambda_t}} = e_t$

From (A.1') and (A.2') we get covered interest parity:

$e_t E_t \beta(1+R_{t+1})^{\frac{\lambda_{t+1}}{\lambda_t}} = E_t \beta^{t+1}(1+R_{t+1}^*)^{\frac{\lambda_{t+1}}{\lambda_t}}$

$e_t \beta(1+R_{t+1})E_t^{\frac{\lambda_{t+1}}{\lambda_t}} = \beta^{t+1}(1+R_{t+1}^*)E_t^{\frac{\lambda_{t+1}}{\lambda_t}}$

$(1+R_{t+1}) = \frac{f^{t+1}}{e_t} (1+R_{t+1}^*)$

From (A.1') and (A.2') we get uncovered interest parity (with a risk premium)

$e_t E_t \beta(1+R_{t+1})^{\frac{\lambda_{t+1}}{\lambda_t}} = E_t \beta e_{t+1}(1+R_{t+1}^*)^{\frac{\lambda_{t+1}}{\lambda_t}}$
\[ e_t \beta(1+R_{t+1}) \frac{1}{\lambda_t} E_{t+1} \lambda_{t+1} = \beta(1+R_{t+1}^*) \frac{1}{\lambda_t} E_{t+1} e_{t+1} \lambda_{t+1} \]

\[ e_t(1+R_{t+1}) E_t \lambda_{t+1} = (1+R_{t+1}^*) E_t e_{t+1} \lambda_{t+1} \]

\[ \frac{E_t e_{t+1} \lambda_{t+1}}{e_t E_t \lambda_{t+1}} = \frac{(1+R_{t+1})}{(1+R_{t+1}^*)} \]

Finally, forward market efficiency proceeds from (A.2') and (A.3'):

\[ E_t \beta_t^{t+1} (1+R_{t+1}^*) \frac{\lambda_{t+1}}{\lambda_t} = E_t \beta \epsilon_{t+1} (1+R_{t+1}^*) \frac{\lambda_{t+1}}{\lambda_t} \]

\[ \beta(1+R_{t+1}^*) E_t^{t+1} \frac{\lambda_{t+1}}{\lambda_t} = \beta(1+R_{t+1}^*) E_t e_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \]

\[ E_t^{t+1} \frac{\lambda_{t+1}}{\lambda_t} = E_t e_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \]
APPENDIX B
SUMMARY OF EMPIRICAL SPECIFICATIONS

MODEL 1 (Asset-Arbitrage Model):

\[
E_t \left[ \frac{f_{t+k}^j - e_{t+k}^j}{e_t} \cdot \frac{U_{x_{t+4}}/p_{t+4}}{U_{x_t}/p_t} \otimes \Gamma_t \right] = 0. \quad (m = 3)
\]

MODEL 2 (SS Monetary or Currency-Arbitrage Model):

\[
E_t \left[ \frac{f_{t+k}^j - e_{t+k}^j}{e_t} \cdot \frac{U_{x_{t+3}}/p_{t+3}}{U_{x_t}/p_t} \otimes \Gamma_t \right] = 0. \quad (m = 2)
\]

Specification of \( \Gamma_t \):

\[
\Gamma_t = \left( 1, \left[ \frac{f_{t-k}^j - e_{t-k}^j}{e_t} \right] \right) \quad k = 1, 2, \text{ or } 3 \quad \text{(Forward profits)}
\]

Specifications of \( U \):

\[
U(x,x^*) = \frac{x_t^{1-b}}{1-b} + \frac{x_t^{1-c}}{1-c} \quad ; \quad b, c \in [0,1]. \quad \text{(Separable CRRA)}
\]

starting value of \( b = 0.9 \).

\[
U(x,x^*) = - (b-x)^2 - (c-x^*)^2 \quad ; \quad x \leq b, x^* \leq c. \quad \text{(Separable Quadratic)}
\]

starting value of \( b = (\max_t(x_t)) \).
\[ U(x, x^*) = \ln(x-b) + \ln(x^*-c) ; \quad x \geq b, \ x^* \geq c \quad \text{(Separable LES)} \]

starting value of \( b = 0.99 \cdot \min_t (x_t) \).

**Consumption:** Either of seasonally adjusted personal consumption expenditures on nondurables plus services (NDS) or NDS less seasonally adjusted merchandise imports (NDS-Imports)

**Covariance Matrix:** Hansen, Newey-West, or Durbin Weighting matrix. Let

\[ \xi_{t+j}(b) = h_{t+j}(b) \odot \Gamma_t. \]

**Hansen:**

\[ H_T = T^{-1} \sum_{t=1}^T \{ \xi_{t+j}(b_{\text{NLS}})\xi_{t+j}(b_{\text{NLS}})' \]

\[ + \sum_{k=1}^m \left[ \xi_{t+j}(b_{\text{NLS}})\xi_{t+j-k}(b_{\text{NLS}})' + \xi_{t+j-k}(b_{\text{NLS}})\xi_{t+j}(b_{\text{NLS}})' \right] \}

**Newey-West:**

\[ N_T = T^{-1} \sum_{t=1}^T \{ \xi_{t+j}(b_{\text{NLS}})\xi_{t+j}(b_{\text{NLS}}) \]

\[ + \sum_{k=1}^m \left[ \frac{1}{m(T)+1} \left[ \xi_{t+j}(b_{\text{NLS}})\xi_{t+j-k}(b_{\text{NLS}})' + \xi_{t+j-k}(b_{\text{NLS}})\xi_{t+j}(b_{\text{NLS}})' \right] \right] \}

\( b_{\text{NLS}} \) minimizes

\[ G_T(\beta) = g_T(\beta)'g_T(\beta). \]
Now let

$$\xi_t = e_t + B_1 e_{t-1} + \cdots + B_m e_{t-m}$$

$$D = (I + B_1 + \cdots + B_m)\Omega(I + B_1' + \cdots + B_m')$$

where $\Omega = \epsilon'\epsilon/(T+j)$.

**Durbin:**

$$D_T = (I + \hat{B}_1 + \cdots + \hat{B}_m)\hat{\Omega}(I + \hat{B}_1' + \cdots + \hat{B}_m').$$

$$\hat{\Omega} = [T-L(T)-m]^{-1} \sum_{t=L(T)+m+1}^{T} \hat{\nu}_t\hat{\nu}_t'$$

where $\hat{\nu}_t$ are residuals of

$$\hat{\xi}_t = B_1\hat{e}_{t-1} + \cdots + B_m\hat{e}_{t-m} + \nu_t$$

and $\hat{e}_t$ are residuals of

$$\xi_t = A_1\xi_{t-1} + \cdots + A_{L(T)}\xi_{t-L(T)} + e_t$$

$b_{GMM}$ is the parameter which minimizes

$$\hat{G}_T(\beta) = g_T(\beta)'W_Tg_T(\beta).$$

for any of either $W_T = H_T^{-1}, N_T^{-1}$, or $D_T^{-1}$. 