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Nonnormality in the seismic response of primary-secondary systems

Chen, Chen-Kang David, Ph.D.

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NONNORMALITY IN THE SEISMIC RESPONSE
OF PRIMARY-SECONDARY SYSTEMS

by

CHEN-KANG DAVID CHEN

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

John E. Merwin
Professor in the Department of
Civil Engineering, Chairman

Loren D. Lutes
Professor of Civil Engineering
Texas A&M University, Director

John E. Akin
Professor of Mechanical Engineering

Pol D. Spanos
Professor of Civil and Mechanical
Engineering

Houston, Texas
January, 1990
ABSTRACT

Nonnormality in the Seismic Response
of Primary-Secondary Systems

by

Chen-Kang David Chen

Response nonnormality is investigated for a yielding primary structure and a linear secondary system (P-S) subjected to a normally distributed ground acceleration. The nonlinearity considered is bilinear hysteretic (BLH) yielding in the primary structure. The coefficient of excess (COE), which is a normalized fourth cumulant function, is used as a measure of the nonnormality in the current study.

An initial effort focuses on the nonnormality of primary absolute acceleration, since this is the base excitation of a light secondary system. Analytical and numerical results for a nonlinear but nonhysteretic substitute structure are shown to be in good agreement with those from simulation for both mean squared levels and COE of response. It is shown that the acceleration of the primary system can be significantly nonnormal in some situations.

Linear substitute methods are used for analytically evaluating the nonnormality of secondary response. The basic concept is to use a linear model with nonnormal excitation to replace the nonlinear primary element with normal excitation, with the goal of matching the trispectrum for the acceleration of these two systems. The trispectrum is the frequency decomposition of the fourth cumulant function. Periodogram analysis (a special FFT technique for obtaining
polyspectra) is developed for evaluating the trispectral function of BLH primary acceleration. A two filters model (with a more narrowband fourth cumulant filter) gives good approximations for the COE values of secondary response in most cases including both cascade and noncascade analysis.

The probability of failure of secondary response affected by nonnormality due to nonlinearity in the primary is investigated. A nonnormality correction factor (NCF) which is equal to the ratio of the expected life for a Gaussian process to the expected life for the non-Gaussian process is used as an index of the nonnormality effect. Analytical approaches based on knowledge of the first four response cumulants are developed to approximate the NCF values. It is shown that the NCF for first-passage failure generally is more significant than for fatigue failure based on the cases in this study, and both failure modes can be significantly affected by the nonnormality in some situations.
To Lord Jesus Christ
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CHAPTER I
INTRODUCTION

I-1 Introduction:

The term secondary systems is often used to describe various nonstructural elements, such as piping in industrial structures, computer systems in buildings, drilling and exploration equipment on offshore platforms, communication and control devices on space vehicles, etc. Such subsystems often play critical roles in maintaining the operation or safety of the primary subsystem to which they are attached, particularly in the event of extreme loads. Hence, some secondary systems must be designed to survive being subjected to the vibratory effects of an earthquake induced ground motion which is transmitted to such a subsystem through its supporting primary structure.

The theory of the stationary response of linear primary-secondary (P-S) systems to Gaussian (normal) excitation is quite well developed. A review of dynamic analysis of P-S systems can be found in the next chapter. Unfortunately, structural systems under dynamic loading often exhibit nonlinear behavior before serious damage occurs, and the response of a nonlinear system, even under normal excitation, is not normal. However, few studies of nonnormality have been done in the past due to the analytical complexity and difficulty. Some recent studies of fatigue damage accumulation [Hu 1982; Lutes et al. 1984; Winterstein 1985] and of first-passage failure [Grigoriu 1984] have shown that these two important quantities can be significantly affected by nonnormality of the random process studied. This is not surprising since the normal models may significantly misrepresent the frequency of high response levels. Such nonnormality
is particularly likely to occur in a situation involving significant nonlinearity, like the yielding effect in a hysteretic system.

A simple and natural way to include nonnormality in the analysis of a random variable is through consideration of moments higher than the second. In particular, the fourth moments are important for characterizing nonnormality (especially if the random variable is symmetric about its mean value so that the third moment gives no new information). In this study, the kurtosis or the coefficient of excess (COE) (i.e. kurtosis minus 3) will serve as the index to represent the degree of nonnormality of a random process.

This study investigates a simple nonlinear primary-secondary situation in which a very light secondary system is attached to a yielding single-degree-of-freedom (SDOF) bilinear hysteretic (BLH) primary structure. The nonnormality results only from the nonlinearity of the primary structure. The reasons for using such a simple model are that not only is it easy to analyze but also it may fairly accurately represent practical design situations. The use of a SDOF primary system may be justified by the fact that the first mode of the primary system is often of the most interest [US Nuclear Regulatory Guide 1975]. Since the secondary system is usually much less massive than the primary, it is commonly assumed that it does not affect the response of the primary structure. This implies that they can be analyzed as two independent or decoupled sub-systems. This is called cascade analysis. Cascade analysis is very desirable, when it is justified, since it greatly simplifies the analytical work and also gives better intuitive understanding of the system. Cascade analysis has been widely used in many applications and it will be considered in this study. The excitation of the primary system will be taken to be a normal white ground acceleration.
Then the absolute acceleration of the response of the yielding primary structure becomes a nonnormal base excitation of the secondary system. Therefore, the nonnormality of the primary absolute acceleration will first be investigated; then the response of the secondary system to the nonnormal primary acceleration will be studied. Finally, the results will be verified by comparing with the responses obtained from coupled or noncascade analysis.

Nonnormality can have a significant effect on the probability of failure of engineering structures, and this study is an investigation of this phenomenon specifically for P-S systems. Both fatigue damage accumulation and first-passage failure for nonnormal response of the secondary system will be studied after the degree of response nonnormality has been established.

I-2 Objectives:

The objectives of this research will be:

1. to determine the extent to which nonnormality has a significant effect on the risk of seismic failure of secondary systems, and
2. to provide improved methods for predicting the seismic reliability of non-linear primary-secondary systems.

It should be noted that the approximate analytical methods used for predicting nonnormality must be inherently nonlinear. A nonnormal process may be described as one having some nonzero cumulants higher than second order. The nonnormality effect cannot be studied by simple application of equivalent linearization techniques, or essentially equivalent methods involving
perturbation or Gaussian closure. This is because these techniques only provide estimates of the first two cumulants and this partial description of the process gives no information about nonnormality.

The study will include both theoretical derivations and results from computer simulation. Also a computer algorithm needed for numerical implementation of a useful analytical approach (called the method of state space moment equations), will be developed. The major emphasis of the work will be on investigating the nonnormality of the response of a BLH primary structure and of a linear secondary structure. Finally, the reliability of a secondary system affected by nonnormality will be studied and will be compared with the studies based on the normal assumption.

I-3 Outline:

Chapter II will first give a concise review of some of the important literature on dynamic analysis of P-S system. It will be seen that most of work has been limited to linear systems and use of Gaussian assumptions. Chapter II will also give some relevant mathematical background on nonnormality and the calculation of response cumulants of single-degree-of-freedom (SDOF) linear systems as well as the dynamics of SDOF nonlinear systems. The response cumulants of multiple-degree-of-freedom (MDOF) linear systems will also be discussed briefly. Among the available methods for linear systems, state space moment equations provides one of the most convenient ways to evaluate response cumulants. The state space technique is a generalization of the usual Lyapunov equation for calculating second cumulants of response. A general description of the method for describing any order cumulant of any order linear system
will be briefly mentioned, and an algorithm developed for calculating the fourth response cumulant of any order differential equation will be presented in Appendix A.

In chapter III, the P-S systems to be studied and the excitation to be considered will be described. Following that the general ideas and methodology of digital computer simulation will be presented. The crucial issue of accuracy of simulation results will also be discussed and verified. Note that the primary interest of this research is the nonnormality (COE values) of certain processes, and accurate analysis of nonnormality requires considerably more computational time than does the usual root-mean-square (RMS) study.

In order to analytically investigate the nonnormality of primary absolute acceleration (which is the base excitation of the secondary system), a nonlinear nonhysteretic substitute system will be studied in Chapter IV. This model has been suggested previously, but it will now be substantially improved by adjusting the damping in the substitute system to give a better balance between the energy dissipation rate and power input. Also it will be extended to predict the nonnormality of the absolute acceleration of the primary response. There have apparently not been any prior efforts to describe this crucial step in analyzing secondary system dynamics. Both the relative displacement and the absolute acceleration of response of the substitute nonhysteretic primary system will be evaluated. The analytical results will be shown to provide good approximations of those from simulation. A large portion of the material in chapter IV has previously appeared in report form (Chen and Lutes, 1988).

In order to analytically investigate the nonnormal response of the linear
secondary system it is necessary to know the four-dimensional fourth cumulant function for its nonnormal base excitation, which is the primary absolute acceleration. Another form of this same information is the trispectrum, which is the triple Fourier transform of the fourth cumulant function. However, due to the nonlinearity in the primary system, the trispectral function of the response acceleration cannot be found from exact analytical procedures, and it can only be approximated or evaluated from simulated data. In chapter V, periodogram analysis (a special Fourier transform technique) will be studied for estimating cumulant spectra of a given time history. This work will be focused on the trispectral function, which apparently has not been studied previously. The fact that the trispectrum is a function of three arguments and its values are almost always complex, makes both the description and measurement of trispectra quite difficult and challenging. However, simplified techniques will be presented to capture some important aspects of the trispectral function and provide important information for the following nonnormal analysis of the secondary response.

In chapter VI, a linear substitute method will be proposed for analytically evaluating the COE values of the secondary response. The basic approach will be to use a linear model with nonnormal excitation to replace the BLH primary element with normal excitation. The goal will be the matching of the trispectrum for primary acceleration of the substitute linear model to that of the BLH primary system. The choice of the linear filter will be based on the fitting of the power spectral density, and the nonnormal delta correlated excitation will be chosen to achieve matching of the COE of primary acceleration. The philosophy in developing the analytical models is to be as simple as possible while providing
a good estimate, and also providing intuitive insights regarding the P-S system. The noncascade analysis of P-S system will also be studied at the end of chapter VI. Situations will be investigated with the mass of the secondary system equal to 0.1% and 1% of that of the primary system, to demonstrate the effects of P-S interaction.

In chapter VII, both first-passage failure and fatigue failure of secondary response will be studied. The commonly used analytical methods like the Poisson crossing approximation for first-passage failure and the Rayleigh approximation for fatigue failure will be presented as well as more accurate approximations. Previous reliability predictions have generally been limited to the normal process, however, the nonnormal effects on reliability will be studied in this chapter. A nonnormality correction factor (NCF), which is equal to the ratio of the expected life for a Gaussian process to the expected life for the corresponding non-Gaussian process, will be used as the index of the influence of nonnormality. Results will be presented based on the nonnormal response of secondary systems as evaluated in the previous chapters.

Finally, the summary and conclusions of this study will be presented in chapter VIII.
CHAPTER II

BACKGROUND

II-1 Review of dynamic analysis of P-S systems:

The response of secondary systems for earthquake induced ground motions is an interesting and challenging problem from both the analysis and design standpoints. Theoretically, secondary systems can be incorporated as a part of the structure in a dynamic model of the combined system and analyzed by any conventional method. However, this approach in practice presents some difficulties. In particular, for a very light secondary system (such as light equipment), the mass and stiffness matrices of the combined system will have elements with vastly different magnitudes, resulting in numerical difficulties in the dynamic analysis. This difficulty can be overcome by decoupling the P-S system by simply assuming no feedback from the secondary to the primary, thereby taking advantage of the mass "mismatch". This is called cascade analysis.

Floor response spectra represent one of the earliest methods for using decoupled cascade analysis for P-S systems. They were first developed for design input for piping in nuclear power plants in the early sixties. Floor response spectra (like ground response spectra) give a plot of the maximum response of an oscillator versus the natural frequency of the oscillator. For floor response spectra, though, the oscillator corresponds to the secondary system and the input represents motion of the primary system (structure). To evaluate the floor spectra directly requires expensive time history analysis where an ensemble of ground acceleration time histories are used as the input to the
primary structure and the motion of the structure at the support point of the secondary system is then defined as the input to the oscillator [Newmark et al. 1973; Perumont 1984; Scanlan and Sachs 1974; Spanos 1983; Spanos and Vargas 1985; US Nuclear Regulatory Commission 1975]. Biggs and Roesset (1970) first developed a technique for approximating floor response spectra directly from the ground response spectra by using amplification curves. Later, similar approaches were conducted by several researchers based on cascade analysis [Peter et al. 1977; Scanlan and Sachs 1977; Singh 1975, 1980; Vanmarcke 1976, 1977]. The interaction problem was first studied by Newmark (1972), who used a modal approach on the composite P-S system employing the notion of an effective mass ratio. In an alternative approach, Sackman and Kelly (1978, 1979) first utilized the perturbation method analysis to find the dynamic properties of the composite system. More recently a series of papers have been written by Sackman (1983), Der Kiureghian and Moor-Omid (1983) and Igusa (1985) on this subject. In interpreting the results, four important characteristics of a combined P-S system have been identified as: tuning, interaction, nonclassical damping and spatial coupling. Similar approaches have also been used by Singh and Suarez (1987, 1988) on the composite P-S systems. Their approaches are based on synthesis of modal properties of composite P-S systems in terms of the known properties of the individual subsystems; then evaluation of secondary subsystem response through modal combination. Recently, Asfura and Der Kiureghian (1986) have extended the floor response spectrum method by considering the effects of cross-correlations between modal responses and between support motions at different points, as well as the effect of interaction between the primary and the secondary. Suarez and Singh (1989) have also evaluated the effect on floor response spectra of the interaction between two secondary systems.
All the above studies have focused on linear systems and used a Gaussian assumption, while there have been only a few studies on the seismic response of light secondary systems supported on inelastic structures. Kelly (1978), Lin and Mahin (1985), etc., have used floor response spectra in studying yielding primary structures supporting secondary systems. No definite conclusions have been made based on cascade analysis, and nonnormality has not been specifically investigated. Thus, little is known regarding the nonnormal response of nonlinear P-S systems, and further research is needed in this area.

II-2 Background on Nonnormality:

II-2-1 Random Processes:

A random process is a parameterized family of random variables with the parameter (or parameters), belonging to an indexing set (or sets) [Lin, 1976]. In this study, the indexing parameter is time. Hence, a random process can be described as a family of random variables, \( \{ X(t) : t \in T \} \). Let the indexing set \( T \) be discrete, then the probability structure can be defined by the joint probability density function of \( n \) random variables as \( p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) \) where \( X_j = X(t_j) \). Alternatively, the probability structure also may be described by the joint characteristic function,

\[
\Theta(u_1, t_1; \ldots; u_n, t_n) = E[\exp(iu_1 X_1 + \cdots + iu_n X_n)] \tag{2.1}
\]

in which \( E[\cdot] \) means expected value. Note that any lower order joint probability function or characteristic function can be obtained if a higher order one is known.
In many situations it is impractical and/or impossible to work with a complete description of a random process in terms of probability density functions or characteristic functions. One of the most common ways of giving a useful partial description of a process is in terms of moments. Let the order $r$ moment function be written as

$$m_r(t_1, t_2, \ldots, t_r) = E[X_1X_2 \cdots X_r]$$  \hspace{1cm} (2.2)

This can be written as an integral using the order $r$ probability density, or as a derivative of the characteristic function:

$$m_r(t_1, \ldots, t_r) = \frac{1}{i^r} \left. \frac{\partial^r \Theta(u_1, t_1; \ldots; u_n, t_r)}{\partial u_1 \cdots \partial u_r} \right|_{u_1=\cdots=u_r=0}$$  \hspace{1cm} (2.3)

An alternative way to present the information contained in the first $n$ moment functions is in the form of the first $n$ cumulants functions, where the $r$th cumulant is given by

$$k_r(t_1, \ldots, t_r) = \frac{1}{i^r} \left. \frac{\partial^r \ln \Theta(u_1, t_1; \ldots; u_r, t_r)}{\partial u_1 \cdots \partial u_r} \right|_{u_1=\cdots=u_r=0}$$  \hspace{1cm} (2.4)

where $\ln \Theta(u_1, \ldots, u_r)$ is called the log-characteristic function.

Stratonovich (1963) noted that the cumulant functions involving distinct values of time related to correlations of the process at those particular times. Thus, cumulant functions are also called “correlation funtions”. It may be noted that the lower order cumulant functions are simply related to the moment functions. For a zero mean process, in particular, $k_1 = 0$, $k_2 = m_2$ and $k_3 = m_3$. For $r=4$, however, the relation is not quite so simple, since
\[ k_4(t_1, t_2, t_3, t_4) = m_4(t_1, t_2, t_3, t_4) - m_2(t_1, t_2) m_2(t_3, t_4) - m_2(t_1, t_3) m_2(t_2, t_4) - m_2(t_1, t_4) m_2(t_2, t_3) \]  
(2.5)

### II-2-2 Frequency Decomposition of a Stationary Process:

A random process is said to be stationary if its probability density functions are invariant under a shift of the time scale. For a stationary process \( m_2(t_1, t_2) \) and \( k_2(t_1, t_2) \) are functions only of \( t_2 - t_1 \), and similarly \( m_4(t_1, t_2, t_3, t_4) \) and \( k_4(t_1, t_2, t_3, t_4) \) are functions of three time arguments, which can be chosen as \( t_2 - t_1, t_3 - t_1, \) and \( t_4 - t_1 \). Let

\[ C_r(\tau_1, \tau_2, \ldots, \tau_{r-1}) = k_r(t_1, t_2, \ldots, t_r) \]  
(2.6)

in which \( \tau_j = t_{j+1} - t_j \). For the special cases of \( r=2 \) and \( r=4 \), these can be written as

\[ * \quad R(\tau) = C_2(\tau) = k_2(t_1, t_2) \]  
(2.7a)

and

\[ Q(\tau_1, \tau_2, \tau_3) = C_4(\tau_1, \tau_2, \tau_3) = k_4(t_1, t_2, t_3, t_4) \]  
(2.7b)

The Fourier transform of \( R(\tau) \), and its inverse, are given by

\[ S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) \exp^{-i\omega \tau} \, d\tau \]  
(2.8)

and

\[ R(\tau) = \int_{-\infty}^{\infty} S(\omega) \exp^{i\omega \tau} \, d\omega \]  
(2.9)
where $S(\omega)$ is called the power spectral density (or power spectrum) of the random process. It can be shown that $S(\omega)$ is a non-negative, even function of $\omega$.

Analogous to Eq. 2.9, a frequency decomposition of $Q(\tau_1, \tau_2, \tau_3)$ can be made in the following way

$$Q(\tau_1, \tau_2, \tau_3) = \int \int \int_{-\infty}^{\infty} D(\omega_1, \omega_2, \omega_3) \exp^{i(\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3)} d\omega_1 d\omega_2 d\omega_3 \quad (2.10)$$

where $D(\omega_1, \omega_2, \omega_3)$ is a three-dimensional Fourier transformation of $Q(\tau_1, \tau_2, \tau_3)$ and is called trispectral density:

$$D(\omega_1, \omega_2, \omega_3) = \frac{1}{(2\pi)^3} \int \int \int_{-\infty}^{\infty} Q(\tau_1, \tau_2, \tau_3) \exp^{-i(\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3)} d\tau_1 d\tau_2 d\tau_3 \quad (2.11)$$

II-2-3 Properties of Trispectral Density:

It is clear that exchanging any two arguments in $k_4(t_1, t_2, t_3, t_4)$ will not change the value of $k_4$. Thus, the symmetry of $k_4$ is simply that the cumulant is the same for each of the 24 permutations of the four arguments:

$$k_4(t_1, t_2, t_3, t_4) = k_4(t_2, t_1, t_3, t_4) = k_4(t_3, t_2, t_1, t_4) = k_4(t_4, t_2, t_3, t_1) = \ldots \quad (2.12)$$

Rewriting eq. 2.12, the symmetry of a stationary fourth cumulant function, $Q(\tau_1, \tau_2, \tau_3)$, will be
\[ Q(\tau_1, \tau_2, \tau_3) = Q(-\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_1) \]
\[ = Q(\tau_1 - \tau_2, -\tau_2, \tau_3 - \tau_2) \]
\[ = Q(\tau_1 - \tau_3, \tau_2 - \tau_3, -\tau_3) \]
\[ = Q(\tau_2, \tau_1, \tau_3) \]
\[ = \ldots \] (2.13)

Therefore, there are also 24 symmetrical points in \( Q(\tau_1, \tau_2, \tau_3) \).

On the frequency domain, the symmetry of \( D(\omega_1, \omega_2, \omega_3) \) can be obtained similarly by taking the Fourier transformation of each term in eq. 2.13 which gives

\[ D(\omega_1, \omega_2, \omega_3) = D(-\omega_1 - \omega_2 - \omega_3, \omega_2, \omega_3) \]
\[ = D(\omega_1, -\omega_1 - \omega_2 - \omega_3, \omega_3) \]
\[ = D(\omega_1, \omega_2, -\omega_1 - \omega_2 - \omega_3) \]
\[ = D(\omega_2, \omega_1, \omega_3) \]
\[ = \ldots \] (2.14)

In general, for the stationary \( m \)th cumulant function, which is a function of \( m-1 \) arguments, there are \( m! \) symmetrical points in \( D \) as well as in \( Q \). In particular, \( D \) is the same when its three arguments are any choice of three values, in any order, from the set \( \{ \omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3 \} \).

It can be seen from eq. 2.11 that the \( D \) function is usually complex even though \( Q \) is always real. Also it can be noted that \( D(\omega_1, \omega_2, \omega_3) \) and \( D(-\omega_1, -\omega_2, -\omega_3) \) are always a complex conjugate pair. That is,
\[ D(-\omega_1, -\omega_2, -\omega_3) = D^*(\omega_1, \omega_2, \omega_3) \]

in which the star denotes complex conjugate. Similar relationships which can be found are

\[ D(-\omega_1, -\omega_2, \omega_3) = D^*(\omega_1, \omega_2, -\omega_3) \]
\[ D(-\omega_1, \omega_2, -\omega_3) = D^*(\omega_1, -\omega_2, \omega_3) \]
\[ ... \]

Finally, it should be noted that there are certain planes within the frequency domain on which \( D \) is always real. In particular, \( D(\omega_1, -\omega_1, \omega_3) \) is real for all \((\omega_1, \omega_3)\) values since \( D^*(\omega_1, -\omega_1, \omega_3) = D(\omega_1, -\omega_1, \omega_3) \), from the complex conjugate and symmetry properties given above. Of course, there are another five identical planes within other octants in this three dimensional space.

II-2-4 Degree of Nonnormality:

The physical significance of multiple correlations (i.e., correlations between several different random variables) decreases when the order increases. Hence the first few cumulant functions are most important in describing a random process. Many physical problems have small values of the higher order cumulant functions. In fact, for a normal process they are exactly zero for order greater than two. Thus, a simple and natural way to include nonnormality in the analysis of a random variable is through consideration of moments higher than the second. In particular, the fourth cumulant is important for characterizing nonnormality (especially if the random variable is symmetric about its mean value so that the
third cumulant gives no new information). However, in practice, it is not always easy to see the degree of nonnormality of a process directly from the fourth order cumulant function because of the multidimensional nature of that function.

In this study, the coefficient of excess (COE), a special case of normalized fourth cumulant function, will serve as the index to represent the degree of nonnormality of a random process. The COE is a normalized one-dimensional form of \( k_4(t_1,t_2,t_3,t_4) \) corresponding to \( t_1 = t_2 = t_3 = t_4 \), which is same as \( Q(0,0,0) \):

\[
COE = \frac{Q(0,0,0)}{\sigma^4} = \frac{k_4(t,t,t,t)}{\sigma^4} = Kurtosis - 3
\]  

(2.15)

where \( \sigma \) is the root-mean-square value.

For a normal distribution, the COE is equal to zero (i.e. kurtosis=3). When the COE is greater than zero, it means that more probability mass is in the tails of the distribution than for a corresponding normal distribution. If the COE is less than zero, it shows that there is less probability in the tails, giving what may be called an amplitude-limited type distribution. Note, though, that the COE only relates to each individual \( X_j \) random variable, whereas the more general fourth cumulant relates to the joint distribution of up to four such random variables.

II-3 **Response of SDOF Nonlinear System with Normal Excitation:**

A general class of nonlinear oscillators for which the exact analytical
probability density function of stationary response has been obtained is described by

\[ \ddot{z} + f(H)\dot{z} + g(x) = -\ddot{y}(t) \quad (2.16) \]

where \( g(x) \) is an odd nonhysteretic function and \( g(x) > 0 \) for \( x > 0 \), \( H = G(x) + \dot{z}^2/2 \) (energy in the system),

\[ G(z) = \int_0^z g(u)du \]

\( f(H) \) is a positive function, and \( \ddot{y}(t) \) is a Gaussian, white process with mean zero, and autocorrelation function

\[ R(\tau) = E[\ddot{y}(t)\ddot{y}(t + \tau)] = 2\pi S_0 \delta(\tau) \]

in which \( S_0 \) is the power spectral density of the white noise excitation and \( \delta(\tau) \) is the Dirac delta function.

Caughey (1965) solved the Fokker-Planck (F-P) equation for the system of eq. 2.16 to obtain the joint, stationary, probability density function as

\[ p_{\dot{z}z}(q, \dot{q}) = C \exp\left\{-F[H(q, \dot{q})]/\pi S_0\right\} \quad (2.17) \]

where

\[ F(H) = \int_0^H f(h)dh \]
Note that $p_{x\dot{x}}$ depends on the energy $H$ associated with particular combinations of $x$ and $\dot{x}$, rather than depending on $x$ and $\dot{x}$ independently.

It is usually more convenient to write the results in a slightly different form. Let a variable $A \geq 0$ be defined by $H = G(A)$, then one can compute the probability density function of this variable from eq. 2.17 as

$$p_A(a) = CT(a)g(a)\exp\{-F[G(a)]/\pi S_0\}$$  \hspace{1cm} (2.18)

where

$$T(A) = 4 \int_0^A [2G(A) - 2G(x)]^{-\frac{3}{2}} \, dx$$

One can show that $T(A)$ is exactly the period of free vibration of amplitude $A$ of an undamped system, and it will be approximated by

$$T(A) \simeq \frac{2\pi}{\omega_r}$$

where $\omega_r$ is the resonant frequency of the nonlinear system.

Given the probability density of amplitude as in eq. 2.18, the statistics of response of this nonlinear system can be readily evaluated by integrals of eq. 2.18, if $x$ can be written as

$$x(t) = A(t) \cos[\omega_r t + \theta(t)]$$  \hspace{1cm} (2.19)

where $\theta$ is uniformly distributed from zero to $2\pi$ and is statistically independent of $A$. 

II-4 Response of SDOF Linear System with Nonnormal Excitation:

The second order differential equation governing the motion of a typical mass-spring-dashpot system will be

\[ m\ddot{x} + c\dot{x} + kx = f = mp \]  \hspace{1cm} (2.20)

in which \( m, c \) and \( k \) are mass, damping and stiffness of the system respectively, and \( p \) is the negative of the base acceleration. Equation 2.20 can be rewritten as the so-called standard form

\[ \ddot{x} + 2\beta_0\omega_0\dot{x} + \omega_0^2x = p \]  \hspace{1cm} (2.21)

where \( \omega_0^2 = k/m \), is the undamped natural frequency, and \( \beta_0 = c/2\omega_0 m \), is the ratio of the actual damping to the critical damping. It is assumed that \( p \) is a zero-mean random process so that the response is also random with zero-mean. Other terms are taken to be deterministic.

We shall assume that the random excitation begins at \( t=0 \), so that the response of the system can be expressed as a Duhamel integral as

\[ x(t) = \int_{0}^{\infty} p(\tau)h(t-\tau)d\tau \]  \hspace{1cm} (2.22)

or in the frequency domain as

\[ x(t) = \int_{-\infty}^{\infty} \tilde{p}(\omega)H(\omega)\exp(i\omega t)d\omega \]  \hspace{1cm} (2.23)

where
\[ \bar{p}(\omega) = \frac{1}{2\pi} \int_0^\infty p(t) \exp(-i\omega t) dt \]

The function \( h(t) \) is called the impulse response function of the system, and can be written as

\[ h(t) = \begin{cases} \frac{1}{\omega_d} \exp(-\beta_0 \omega_0 t) \sin(\omega_d t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad (2.24) \]

in which \( \omega_d = \omega_0 \sqrt{1 - \beta_0^2} \), the damped natural frequency of the system. The corresponding frequency response function is the Fourier transformation of the impulse response function:

\[ H(\omega) = \int_{-\infty}^\infty h(t) \exp(-i\omega t) dt = \frac{1}{(\omega_0^2 - \omega^2) + i2\beta_0 \omega_0 \omega} \quad (2.25) \]

The cumulant function of the response can be expressed in terms of the cumulant of the excitation and the impulse response function as

\[ k_z(t_1, t_2, \cdots, t_n) = \int_0^{t_1} \cdots \int_0^{t_n} k_p(\tau_1, \tau_2, \cdots, \tau_n) h(t_1 - \tau_1) h(t_2 - \tau_2) \cdots h(t_n - \tau_n) d\tau_1 d\tau_2 \cdots d\tau_n \quad (2.26) \]

where \( k_z \) and \( k_p \) are order \( n \) cumulant functions of response and excitation, respectively; and use has been made of the fact that \( h(t - \tau) = 0 \) for \( t < \tau \). Equation 2.26 shows how knowledge of cumulant functions of excitation can be used to obtain corresponding cumulant functions of response by linear operations.
II-4-1 Stationary Processes:

For a stationary excitation, since the probability distributions are invariant under a shift of the time scale, the nth order cumulant function of response will be a function of only n-1 time arguments. Using $\tau_j = t_{j+1} - t_j$ gives

$$C_x(\tau_1, \tau_2, \cdots, \tau_{n-1}) = \int_0^\infty \cdots \int_0^\infty C_p(\nu_1, \nu_2, \cdots, \nu_{n-1})$$

$$h(\nu) h(\nu + \tau_1 - \nu_1) \cdots h(\nu + \tau_{n-1} - \nu_{n-1}) d\nu d\nu_2 \cdots d\nu_{n-1}$$

(2.27a)

Similarly in the frequency domain, the n-1 dimensional spectral density of response has the following relationship with the n-1 dimensional spectral density of excitation:

$$D_x(\omega_1, \omega_2, \cdots, \omega_{n-1}) = H(\omega_1) H(\omega_2) \cdots H(\omega_{n-1}) H(-\sum_{j=1}^{n-1} \omega_j)$$

(2.27b)

$$D_p(\omega_1, \omega_2, \cdots, \omega_{n-1})$$

in which $D(\omega_1, \omega_2, \cdots, \omega_{n-1})$ is the n-1 dimensional generalization of the trispectral density $D(\omega_1, \omega_2, \omega_3)$ in eqs. 2.10 and 2.11. (Recall that the n-1 dimensional spectral density function is the Fourier transformation of the corresponding stationary nth order cumulant function.)

Note that if $p(t)$ is Gaussian, then $x(t)$ is also Gaussian. Then only the first two cumulants are non-zero and all other higher cumulants vanish. When $n=2$, the second cumulant function, $R_x(\tau)$ of response, can be expressed in terms of the cumulant of the excitation, $R_p(\tau)$ and the impulse function as
\[ R_x(\tau) = \int_0^\infty \int_0^\infty R_p(\tau + \nu_1 - \nu_2) h(\nu_1) h(\nu_2) d\nu_1 d\nu_2 \]  

(2.28)

The spectral density of response is given by

\[ S_x(\omega) = S_p(\omega)|H(\omega)|^2 \]  

(2.29)

If \( p(t) \) is not Gaussian, higher cumulants exist. The third cumulant won't give any new information if the probability distribution is symmetric. The fourth cumulant function of response, \( n=4 \), can be found as

\[ Q_x(\tau_1, \tau_2, \tau_3) = \int \int \int \int_0^\infty Q_p(\tau_1 + \nu_1 - \nu_4, \tau_2 + \nu_2 - \nu_4, \tau_3 + \nu_3 - \nu_4) \]
\[ h(\nu_1) h(\nu_2) h(\nu_3) h(\nu_4) d\nu_1 d\nu_2 d\nu_3 d\nu_4 \]

(2.30)

and

\[ D_x(\omega_1, \omega_2, \omega_3) = H(\omega_1) H(\omega_2) H(\omega_3) H(-\omega_1 - \omega_2 - \omega_3) D_p(\omega_1, \omega_2, \omega_3) \]  

(2.31)

II-4-2 Nonnormal Delta Correlated Excitation:

Let the excitation of eq. 2.20 be a non-Gaussian delta correlated (white noise) process for which

\[ R_p(\tau) = 2\pi S_p \delta(\tau) \]  

(2.32)

and
\[ Q_p(\tau_1, \tau_2, \tau_3) = (2\pi)^3 D_p \delta(\tau_1) \delta(\tau_2) \delta(\tau_3) \]  \hspace{1cm} (2.33)\]

where \( \delta(\tau) \) is the Dirac delta function. Note that the constants \( S_p \) and \( D_p \) are the uniform power spectrum and uniform trispectrum of the excitation, respectively. Without loss of generality, let the mean of the excitation and the response be zero. The second cumulant or moment of response is

\[ R_z(\tau) = 2\pi S_p \int_0^\infty h(\nu) h(\nu + \tau) d\nu \]  \hspace{1cm} (2.34)\]

\[ k_z(t, t) = R_z(0) = 2\pi S_p \int_0^\infty h^2(\nu) d\nu \]
\[ = S_p \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \]  \hspace{1cm} (2.35)\]

\[ = 2S_p \int_0^\infty |H(\omega)|^2 d\omega \]

and the fourth cumulant of response is

\[ Q_z(\tau_1, \tau_2, \tau_3) = (2\pi)^3 D_p \int_0^\infty h(\nu) h(\nu + \tau_1) h(\nu + \tau_2) h(\nu + \tau_3) d\nu \]  \hspace{1cm} (2.36)\]
\[ k_z(t, t, t, t) = Q_z(0, 0, 0) \]
\[ = (2\pi)^3 D_p \int_0^\infty h^4(\nu) d\nu \]
\[ = \int \int \int_{-\infty}^\infty D_z(\omega_1, \omega_2, \omega_3) d\omega_1 d\omega_2 d\omega_3 \]
\[ = D_p \int \int \int_{-\infty}^\infty H(\omega_1)H(\omega_2)H(\omega_3)H\left(-\sum_{j=1}^3 \omega_j\right) d\omega_1 d\omega_2 d\omega_3 \]
\[ = 2D_p \int_0^\infty \int_0^\infty \int_0^\infty H(\omega_1)H(\omega_2)H(\omega_3)H\left(-\sum_{j=1}^3 \omega_j\right) d\omega_1 d\omega_2 d\omega_3 \]
\[ + 6D_p \int_0^\infty \int_0^\infty \int_{-\infty}^0 H(\omega_1)H(\omega_2)H(\omega_3)H\left(-\sum_{j=1}^3 \omega_j\right) d\omega_1 d\omega_2 d\omega_3 \]
\[ (2.37) \]

The COE of response will be equation 2.37 divided by equation 2.35 squared.

If the linear system in the above equations is lightly damped, then eq. 2.25 shows that \( H(\omega) \) has peaks with height of \( O(\beta_0^{-1}) \) near \( \omega = \pm \omega_0 \). From eq. 2.31, it can be observed that the trispectrum of this narrowband process then has a peak near \( (\omega_0, -\omega_0, \omega_0) \) and this peak has a height of \( O(\beta_0^{-4}) \). As mentioned in Sec II-2-3, \( D \) is real on the plane \( (\omega_1, -\omega_1, \omega_3) \) and on the other five “symmetric” planes in the trispectrum. Thus, \( D(\omega_0, -\omega_0, \omega_0) \) is very large and real, as are the other five peaks with equal \( D \) values in other octants. These six high peaks dominate the \( D \) function so that all other points in this three dimensional space are relatively insignificant.
II-5 Response of MDOF Linear System with Nonnormal Excitation:

Let the equation of motion be a set of \( n \) ordinary differential equations of the second order:

\[
M\dddot{\bar{X}} + C\ddot{\bar{X}} + K\bar{X} = \bar{F}(t)
\]  
(2.38)

where \( \bar{X} \) denotes the generalized displacement vector of a linear system with \( n \) degrees of freedom acted upon by a set of generalized forces \( \bar{F}(t) \). \( M, C, K \) are the mass, damping, and stiffness matrices, respectively. This is a simple generalization of eq. 2.20.

In principle, one can evaluate the response cumulants by using the matrix of frequency response functions \( \mathbf{H} \) or the matrix of impulse response functions \( \mathbf{h} \) for the system [Lin, 1976]. By considering the response to harmonic excitations one can obtain the matrix of frequency response functions as

\[
\mathbf{H}(\omega) = \left[ -\omega^2 \mathbf{M} + i\omega \mathbf{C} + \mathbf{K} \right]^{-1}
\]  
(2.39)

the matrix \( \mathbf{H}(\omega) \) is of order \( n \times n \), of course, and is complex and symmetrical in general. The matrix of impulse response functions may be obtained from \( \mathbf{H}(\omega) \) by a Fourier transformation as

\[
\mathbf{h}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{H}(\omega) \exp(i\omega t) d\omega
\]  
(2.40a)

or
\[ h(t) = \begin{cases} [0, I] \exp(-gt)[M^{-1}, 0]^T & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad (2.40b) \]

where

\[ g = \begin{bmatrix} M^{-1}C & M^{-1}K \\ -I & 0 \end{bmatrix} \quad (2.41) \]

Note that \( g \) is an order 2n \times 2n matrix, and \( I \) is the n \times n unit matrix. The element \( H_{jk}(\omega) \), in the matrix \( H(\omega) \), represents the response at point \( j \) due to a complex exponential harmonic excitation acting at point \( k \) with frequency \( \omega \). Similarly, a typical element \( h_{jk}(t) \) in the matrix \( h(t) \) describes the motion at \( j \) due to a unit impulse excitation at \( k \) applied at time \( t=0 \).

The statistical properties of response can be derived from

\[ X_j(t) = \sum_k \int_0^t h_{jk}(t-\tau)F_k(\tau)d\tau \quad (2.42) \]

giving the first two response cumulants as

\[ k_{\bar{X}}(t) = E[\bar{X}(t)] = \int_0^\infty h(t-\tau)E[\bar{F}(\tau)]d\tau \quad (2.43) \]

\[ k_{\bar{X}}(t_1, t_2) = \int \int_0^\infty h(t_1-\tau_1)k_F(\tau_1, \tau_2)h^{-1}(t_2-\tau_2)d\tau_1d\tau_2 \quad (2.44) \]

The fourth order cumulant of response cannot be written as a matrix multiplication, since the fourth cumulant is actually a fourth order (rather than second order) tensor. The summation form for the result is
\[ k_{X_i X_j X_k X_l} (t_1, t_2, t_3, t_4) = \int \int \int_0^\infty \sum_{k_1} \cdots \sum_{k_4} \]
\[ h_{j k_1} (t_1 - \tau_1) \cdots h_{j k_4} (t_4 - \tau_4) \]
\[ k_{F_{k_1} F_{k_2} F_{k_3} F_{k_4}} (\tau_1, \tau_2, \tau_3, \tau_4) d\tau_1 \cdots d\tau_4 \]  \hspace{1cm} (2.45)

where \( j = 1, \ldots, n \). For a stationary process, the mean response vector will be a constant vector, and the \( m \)th order response cumulant can be expressed as a function of \( m-1 \) arguments. If the excitation is delta correlated at each point of application, then the cumulant of response can be further simplified.

The method of normal modes is usually used to simplify \( H \) and \( h \) as well as the cumulant formulations for the MDOF system. In this method a linear transformation is employed to change eq. 2.38 to an uncoupled system. An uncoupled system of equations of motion is, in essence, a set of \( n \) independent equations, each one of which resembles the equation of motion of a SDOF system. The advantage of such a transformation is obvious. However, this method requires the restriction that

\[ CM^{-1}K = KM^{-1}C \]  \hspace{1cm} (2.46)

It can be seen that classical Rayleigh damping, in which \( C \) is a linear combination of the matrices \( M \) and \( K \), is a special case of this.

In P-S system analysis, in general, \( M, C \) and \( K \) are large matrices containing terms of vastly different magnitudes and are difficult to utilize for analysis. For this reason, modal analysis of the composite P-S system has been avoided in most cases.
As can be seen from the above discussion, the computation of cumulant functions involves numerical integration which is vast and complicated, either through the matrix of impulse response functions or the stochastic modal method. The state space method is an efficient alternative approach, which avoids the impulse response function and numerical integration, and which can also be used for dynamic analysis of MDOF system. The state space method has been proved to be an efficient approach to calculate cumulants of response in general MDOF systems [Lutes and Hu, 1986] as well as in P-S systems. This technique will be employed in the current investigation for linear P-S systems under non-normal excitation.

II-6 State Space Method:

For a linear system excited by a delta correlated (white noise) process, state space analysis provides a convenient way to evaluate response moments. Sometimes a non-white excitation process can be approximated by a filtered process which is the output of a linear filter with a delta correlated input process. Hence, the use of the delta correlated excitation process is not necessarily a significant limitation of this approach. Moreover, the state space technique has many merits which are attractive to engineers and researchers. First, as mentioned before, a major obstacle in analysis of MDOF systems often results from a non-classical damping matrix, but classical damping is not required when implementing the state space method. Secondly, the state space method is fairly easy to use, especially for stationary processes. Further, the method can be applied to various situations, even including nonlinear problems if linearization or some closure technique has been employed. Finally, comparing state space
analysis with numerical integration of the equations in the preceding section shows that the state space method provides reliable answers and is significantly more computationally efficient.

II-6-1 Second Response Cumulant:

First it is of interest to calculate response covariances of a second order n-DOF system under stochastic excitation. Multiplying equation 2.32 by the inverse of the mass matrix gives

\[ \ddot{\bar{X}} + M^{-1}C\dot{\bar{X}} + M^{-1}K\bar{X} = M^{-1}\bar{F}(t) = \bar{P}(t) \]  

(2.47)

Let \( \bar{Y} \) be the state vector of responses, \( \bar{X} \) and \( \dot{\bar{X}} \). Rewrite eq. 2.47 to a set of first order differential equations of 2n vectors.

\[ \dot{\bar{Y}} = A\bar{Y} + \bar{G}(t) \]  

(2.48)

where \( A \) is a 2n x 2n matrix, \( \bar{G} \) is an order of 2n vector. These can be expressed as

\[ A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \]  

(2.49a)

\[ \bar{G}(t) = \begin{Bmatrix} \bar{0} \\ \bar{P} \end{Bmatrix} \]  

(2.49b)

If the mean vector of \( \bar{Y} \) is not zero it will be governed by
\[
\frac{d}{dt} E[\bar{Y}] = \left\{ \begin{array}{c}
E[\dot{\bar{X}}] \\
E[\bar{\bar{X}}]
\end{array} \right\} = A \left\{ \begin{array}{c}
E[\dot{\bar{X}}] \\
E[\bar{\bar{X}}]
\end{array} \right\} + \left\{ \begin{array}{c}
\bar{\Delta} \\
E[\bar{P}]
\end{array} \right\} 
\] (2.50)

For the present discussion, presume that the mean vector is zero, so that the covariance matrix of $\bar{Y}$ will be

\[
E[\bar{Y}\bar{Y}^T] = \begin{bmatrix}
E[\bar{X}\bar{X}^T] & E[\bar{X}\dot{\bar{X}}]^T \\
sym. & E[\dot{\bar{X}}\dot{\bar{X}}]^T
\end{bmatrix} 
\] (2.51a)

and will be governed by

\[
\frac{d}{dt} E[\bar{Y}\bar{Y}^T] = AE[\bar{Y}\bar{Y}^T] + E[\bar{Y}\bar{Y}^T]A^T + E[\bar{G}\bar{Y}^T + \bar{Y}\bar{G}^T] 
\] (2.51b)

If the excitation is a zero-mean delta correlated process, the second moment/cumulant will be,

\[
E[P_j(t)P_k(s)] = 2\pi \Phi_{jk} \delta(s - t) 
\] (2.52)

then equation 2.51b can be rewritten as,

\[
\frac{d}{dt} E[\bar{Y}\bar{Y}^T] = AE[\bar{Y}\bar{Y}^T] + E[\bar{Y}\bar{Y}^T]A^T + B 
\] (2.53)

where

\[
B = 2\pi \begin{bmatrix}
0 & 0 \\
0 & M^{-1}\Phi_{pp}M^{-1}
\end{bmatrix}
\]

Eq. 2.53 is the well known Lyapunov equation. Of course it is formulated only for the second moment of response. If the process is stationary, then
\[ \frac{d}{dt} E[\bar{Y}\bar{Y}^T] = 0, \] so that eq. 2.53 becomes an algebraic equation. If the response
is non-stationary then \[ \frac{d}{dt} E[\bar{Y}\bar{Y}^T] \neq 0, \] and eq. 2.53 must be solved as a set of
simultaneous differential equations.

II-6-2  Mth Response Cumulant:

Lutes (1986) has extended the state space analysis formulation to provide
simultaneous linear equations governing the response cumulants of any order. No
attempt will be made to duplicate Lutes' work here; however several important
concepts and conclusions will be discussed, and an algorithm for calculation of
the fourth response cumulant will be presented in Appendix A.

In order to calculate an order \( m \) response cumulant of a system governed by
\( n \) first order linear differential equations, one must evaluate a tensor including
all \( m \)th order response cumulants. Define \( L \) to be a tensor with components
given by

\[ L_{k_1,k_2,\ldots,k_m} = k_{Y_{k_1},\ldots,Y_{k_m}}(t_1,\ldots,t_m) \quad (2.54) \]

where \( Y_j \) denotes the \( j \)-th component of the vector \( \bar{Y} \) and the right-hand-side of
eq 2.54 denotes the joint cumulant of the following set of random variables :

\[ \{Y_{k_1}(t_1),Y_{k_2}(t_2),\ldots,Y_{k_m}(t_m)\} \quad (2.55) \]

The full \( L \) tensor includes \( n^m \) terms, however symmetry reduces the number
of unique terms to the number \( S_{nm} \) of unordered selections of \( m \) terms from \( n \)
possibilities, allowing repetition. This is the binomial coefficient
\[ S_{nm} = \binom{n+m-1}{m} \equiv \frac{(n+m-1)!}{m!(n-1)!} \]  

(2.56)

Now consider \( n \) first order differential equations under stochastic excitation

\[ \dot{\bar{Y}} = A\bar{Y} + \bar{f}(t) \]  

(2.57)

where \( A \) is a matrix of order \( n \times n \). Let the excitation vector be a delta correlated process such that

\[ k_{f_{k_1}, \ldots, f_{k_m}}(t_1, \ldots, t_m) = I_{k_1 \ldots k_m}(t_1) \delta(t_2 - t_1) \ldots \delta(t_m - t_1) \]  

(2.58)

in which \( I \) is the intensity tensor. Then the state space differential equation for \( m \)-th cumulants (the \( S_{nm} \) elements of \( L \)) is

\[ \frac{d}{dt} \bar{L} + Z\bar{L} = \bar{I} \]  

(2.59)

where \( Z \) is a square matrix of order \( S_{nm} \times S_{nm} \), which depends only on \( A \) in eq. 2.57. The elements of \( L \) have been rearranged into the response cumulant vector \( \bar{L} \) of order \( S_{nm} \). Similarly \( I \) has been rearranged into the vector \( \bar{I} \). Solution of such a set of first order differential equations is generally not difficult. In the special case when the response \( X(t) \) is stationary (which requires that \( I \) be stationary), eq. 2.59 is algebraic so the problem is even simpler:

\[ \bar{L} = Z^{-1} \bar{I} \]  

(2.60)
Appendix A contains an algorithm for evaluation of the matrix $Z$ as well as the excitation vector $\vec{F}$. The vector of response cumulants, $\vec{L}$ can be obtained from eq. 2.60 for a stationary process, or by solving the simultaneous equations in eq. 2.59 for a non-stationary process.
CHAPTER III

SYSTEM CONSIDERED AND COMPUTER SIMULATION

III-1 Description of P-S Systems:

The P-S system is modeled as a BLH primary system and a linear secondary system mounted in series (see figure 3.1). The equation of motion can be written as:

\[ m_p (\dddot{x} + \dddot{y}) + c_p \dddot{x} + k_p \phi(x) = c_s \dddot{u} + k_s \dddot{u} \]

or

\[ (\dddot{x} + \dddot{y}) + 2\beta_p \omega_p \dddot{x} + \omega_p^2 \phi(x) = 2\beta_s \omega_s \eta_s \dddot{u} + \omega_s^2 \eta_s \dddot{u} \]  

(3.1)

and

\[ m_s (\dddot{u} + \dddot{x} + \dddot{y}) + c_s \dddot{u} + k_s \dddot{u} = 0 \]

or

\[ (\dddot{u} + \dddot{x} + \dddot{y}) + 2\beta_s \omega_s \dddot{u} + \omega_s^2 \dddot{u} = 0 \]  

(3.2)

where:

- \( x \) denotes the displacement of the primary mass relative to the base,
- \( u \) denotes the displacement of the secondary mass relative to the primary mass,
- \( \dddot{y} \) is the ground acceleration excitation,
- \( m_p \) is the primary mass, \( m_s \) is the secondary mass,
- \( \omega_p = \sqrt{\frac{k_p}{m_p}} \), primary unyielded, undamped natural circular frequency,
- \( \omega_s = \sqrt{\frac{k_s}{m_s}} \), secondary undamped natural circular frequency,
\( \beta_p = \frac{c_p}{2 \omega_p m_p} \), primary small amplitude fraction of critical damping,

\( \beta_s = \frac{c_s}{2 \omega_s m_s} \), secondary fraction of critical damping,

\( \eta = \frac{m_s}{m_p} \), mass ratio, and

\( \phi(x) \) is the bilinear hysteretic restoring force as shown in fig. 3.2.

Note that \( \phi(x) \) is chosen to have a unit slope for small amplitudes and a second slope of \( \alpha \). In general, \( \phi(x) \) depends on previous values of \( z(t) \) but with the limitation that if \( z(t) \) is periodic, then \( \phi(x) \) is also periodic. Note that the right-hand side of eq. 3.1 is the coupling term in the P-S system. This term will be eliminated if cascade analysis is used.

The SDF BLH system has probably been more widely studied than any other class of nonlinear hysteretic oscillator [Caughey 1960; Iwan and Lutes 1968; Lutes 1970]. Two particular values of the slope ratio were chosen to illustrate important situations. These are \( \alpha = 1/2 \), a moderately nonlinear system, and \( \alpha = 1/21 \), a nearly elastoplastic system.

No exact solutions for the statistics of the response of such a hysteretic system to random excitation have yet been obtained by an analytical technique. Thus, a computer simulation program has been used to obtain empirical data for BLH primary and linear secondary systems.

III-2 Excitation:

For the present investigation, the excitation \( \ddot{y}(t) \) represents a ground acceleration. It is taken to be a mean-zero stationary, white, random process with a normal probability distribution, and a uniform power spectral density equal to \( S_0 \) (per radian) for all frequency. That is, the auto-correlation function
is given by

\[ E[\ddot{y}(t_1)\ddot{y}(t_2)] = 2\pi S_0 \delta(t_1 - t_2) \]  \hspace{1cm} (3.3)

in which \( \delta(t) \) is the Dirac delta function.

Various methods exist for simulating stochastic processes [Shinozuka 1977]. In this study, the white noise excitation was simulated using a pulse method [Brinkmann 1980]. The acceleration at the base of the structure was taken to be a sequence of uniformly spaced Dirac delta functions, with each acceleration pulse giving an instantaneous change in the relative velocity \( \dot{z} \). The pulse magnitude, in this study, is a standard normal random number, obtained from subroutine RNNOA in the IMSL-Library (1987), scaled by a constant \( R \), which is given by

\[ R = \sqrt{2\pi S_0 \Delta t} \]  \hspace{1cm} (3.4)

where \( \Delta t \) is the time interval between two adjacent pulses. The interval \( \Delta t \) was chosen to give \( \omega_p \Delta t = 0.1 \) radian, giving approximately 63 pulses per cycle of the unyielded system.

It is convenient to characterize the excitation level by a measure with dimension length, so that the ratios of yield levels to excitation level (Y/N) and root-mean-square response level to excitation level can be plotted as dimensionless quantities. Such a length measure of the excitation level is

\[ N = \frac{(2S_0\omega_p)^{\frac{1}{2}}}{\omega_p^2} \]  \hspace{1cm} (3.5)

For a SDOF linear system as an example, the standard deviation (or RMS value) of displacement response may be expressed in the normalized form as
\[ \frac{\sigma_2}{N} = \sqrt{\frac{\pi}{4\beta_0}} \]

or

\[ \frac{\sigma_2}{Y} = \sqrt{\frac{\pi}{4\beta_0}} \left( \frac{N}{Y} \right) \]

Similarly, the normalized velocity of responses can be written as

\[ \frac{\sigma_2}{\omega_0 N} = \sqrt{\frac{\pi}{4\beta_0}} \]

in which \( \omega_0 \) and \( \beta_0 \) are the natural frequency and damping ratio of this system.

**III-3 Integration Scheme and Statistical Accuracy:**

The equation of motion for this P-S systems can be written in a matrix form as

\[
\begin{bmatrix}
    m_p & 0 \\
    0 & m_s
\end{bmatrix}
\begin{Bmatrix}
    \ddot{x} \\
    \ddot{u}
\end{Bmatrix} +
\begin{bmatrix}
    c_p & 0 \\
    0 & c_s
\end{bmatrix}
\begin{Bmatrix}
    \dot{x} \\
    \dot{u}
\end{Bmatrix} +
\begin{bmatrix}
    (k_1 + k_2) & 0 \\
    0 & k_s
\end{bmatrix}
\begin{Bmatrix}
    \phi(x) \\
    u
\end{Bmatrix} = \ddot{f} \tag{3.6}
\]

where

\[
\ddot{f} = \begin{Bmatrix}
    -m_p \ddot{y} + c_s \dot{u} + k_s u \\
    -m_s (\ddot{x} + \ddot{y})
\end{Bmatrix}
\]

The excitation \( \ddot{y} \) is the stationary, white noise, Gaussian acceleration. Because of the nonlinearity of the restoring force \( \phi(x) \), no exact solution to equation 3.6 has been obtained. However an exact stepwise calculation is possible due to the piecewise linear characteristic of the resistance deformation...
relationship. The computational effort of this approach is greatly reduced by using cascade analysis, for which equation 3.6 will describe two uncoupled systems. Note that the nonlinear function, \( \phi(x) \), can always be described by one of three linear functions with the choice of the proper function depending on the position and velocity of the primary mass, \( m_p \).

For noncascade analysis, equation 3.6 must be solved as simultaneous equations, which can be rewritten as four first order differential equations using the four state variables: \( z, \dot{z}, u \) and \( \dot{u} \). Any of several numerical integration schemes can be used to solve the first order differential equations. In this study, a sixth-order Runge-Kutta-Verner method was used from subroutine IVPRK in the IMSL-Library (1987).

In this simulation, both time averages and ensemble averages have been used in order to obtain better statistical accuracy. Each sample of simulated response was long enough to contain approximately 2000 cycles of response of the unyielded system (\( \omega_p t = 4000 \pi \)). The first 100 cycles of each sample were omitted from calculations, though, on the basis of possible nonstationarity due to initial conditions. Statistical accuracy was improved by using an ensemble of 100 such samples for each process investigated. The reproducibility of the results was verified by comparing numbers obtained from different ensembles and from ensembles of different lengths. The scatter of simulation statistics is discussed in Appendix B. Overall, the ensemble size of 100 and length of 2000 cycles seemed to give an acceptable sample.

The simulation results have also been verified for two limiting situations. First, by decoupling the two subsystems the results for the displacement \( z \) of the
BLH primary system were found to agree very well with analog computer results in an early study [Lutes 1970]. Second, by letting the primary system be linear ($\alpha = 1$), the exact solutions of both primary and secondary response could be found. These results did coincide with the simulation results. Therefore, the overall accuracy and consistency of this simulation has been studied and considered acceptable.
Fig. 3.1 Primary–Secondary System
Fig. 3.2 Restoring force of a BLH system
CHAPTER IV
RESPONSES OF BLH PRIMARY SYSTEM

IV-1 Nonhysteretic Nonlinear Substitute:

As mentioned in section II-3, a general class of nonlinear oscillators for which the exact analytical probability density function of stationary response (eq. 2.14) has been found by Caughey (1965) is described by

\[ \ddot{z} + f(H)\dot{z} + g(z) = -\ddot{y}(t) \]  
\[ (4.1) \]

where \( g(z) \) is an odd nonhysteretic function and \( g(z) > 0 \) for \( z > 0 \), \( H = G(z) + \dot{z}^2/2 \) (energy in the system),

\[ G(z) = \int_{0}^{z} g(y)dy \]

\( f(H) \) is a positive function, and \( \ddot{y}(t) \) is a Gaussian, white process with mean zero.

The nonhysteretic nonlinear substitute method used in this study involves choosing \( f(H) \) and \( g(z) \) in eq. 4.1 in such a way as to give an approximate equivalence between the systems described by eq. 4.1 and eq. 3.1. Approximate response statistics for the hysteretic system are then predicted from the analytical or numerical solution for eq. 4.1.

The actual choice of a nonhysteretic system to approximate a given hysteretic system could be based on any of several methods of comparison of the
two systems. Two rather mathematical approaches have been applied to specific problems by Caughey and Ma (1982) and by Cai and Lin (1987). In the former work, the choice is based on a minimization of the mean squared difference between the two equations of motion, and in the latter the comparison is in terms of the stationary probability distribution solution of the appropriate Fokker-Planck equations. The method of comparison chosen here is less mathematical and somewhat intuitive. It is a modification of the procedure previously used by Lutes (1970). The nonlinear stiffness is taken to be the same as in that earlier work, but the damping function is modified based on more recent results regarding the energy dissipation of hysteretic structures. Specifically, two different nonlinear oscillators will be considered to be approximately equivalent herein if the following two statements hold true:

1. The two oscillators have the same functional relationship between resonant frequency and amplitude of vibration, and
2. The two oscillators have equal energy dissipation per unit time.

To make equivalence statement 1 be precise, the resonant frequency \( \omega_r \) for eq. 3.1 can be taken as that obtained from the energy balance method [Jordan and Smith 1977]:

\[
\omega_r^2 = \frac{1}{\pi A} \int_0^{2\pi} \cos u \phi(A\cos u) \, du
\]  

(4.2)

where \( A \) is amplitude of vibration. For eq. 4.1, one merely replaces \( \phi \) by \( g \) in eq. 4.2. Eq. 4.2 can be shown to be closely related to other common definitions of resonance. For example, eq. 4.2 approximates both the frequency of free vibration of an undamped nonlinear system, and the frequency of maximum response of a hysteretic system with harmonic excitation [Lutes 1970].
Equivalence statement 1 requires that $g(z)$ in eq. 4.1 be chosen such that eq. 4.2 gives the same $\omega_r$ as for eq. 3.1 with some particular functional $\phi$. For the $\phi$ of the bilinear hysteretic system it has been shown [Lutes 1970] that an appropriate $g(z)$ function is

$$\frac{g(z)}{\omega_0^2} = \begin{cases} x & \text{for } |z| \leq Y \\ (1 - \alpha)Y^\frac{2}{3} z^{-1}|z|^\frac{1}{3} + \alpha x & \text{for } |z| > Y \end{cases}$$

(4.3)

Equivalence statement 2 involves the rate of energy dissipation. For convenience, this will be taken as energy per unit mass. The energy dissipation of eq. 4.1 will depend on $f$, which is given to be a function of the energy $H$. If one postulates that $H$ varies slowly as compared to displacement and velocity variations, then the energy for a cycle of amplitude $A$ can be approximated as

$$H = G(A)$$

(4.4)

Hence, the damping function $f$ is only a function of the amplitude of vibration. Neglecting higher-harmonic contributions, the energy dissipated per cycle for eq. 4.1 is

$$ED = \pi f \omega_r A^2$$

(4.5)

The energy dissipated per unit time is then found by dividing by the period of the cycle ($2\pi/\omega_r$) and averaging over all possible amplitudes:

$$E(EDT(A)) = \frac{1}{2} E[f[G(A)]\omega_r^2(A)A^2]$$

(4.6)
The energy dissipated per unit time for the hysteretic system of eq. 3.1 is slightly more complicated. The viscous damping term can be handled as above, with \( f[G(A)] \) in eq. 4.6 replaced by \( 2\beta_0\omega_0 \). The hysteretic energy dissipation, though, will be treated somewhat differently. Previous studies of the power balance in hysteretic oscillators [Lutes and Takemiya 1974; Takemiya and Lutes 1977] have shown that this rate of energy dissipation can be approximated by using the area of the hysteresis loop (EDH), which is the energy dissipated per hysteretic cycle, and a period based on the tangent stiffness of \( \phi \). For the bilinear hysteretic system this tangent stiffness is \( k \) for \( A \leq Y \) and \( k_2 = \alpha k \) for \( A > Y \). Combining these results gives the energy dissipation per unit time in eq. 3.1 as

\[
E(EDT|A) = \beta_0\omega_0 E[\omega_r^2(A)A^2] + \frac{\sqrt{\alpha}\omega_0}{2\pi} E(EDH|A) \tag{4.7}
\]

Comparing eq. 4.6 and eq. 4.7 shows that equal energy dissipation per unit time is achieved if \( f \) is defined by

\[
f = 2\beta_0\omega_0 + \frac{\sqrt{\alpha}}{\pi} \frac{\omega_0 (EDH)}{\omega_r^2 A^2} \tag{4.8}
\]

where

\[
(EDH) = 0 \quad \text{for } A \leq Y
\]

\[
= 4(1 - \alpha)\omega_0^2 Y(A - Y) \quad \text{for } A > Y \tag{4.9}
\]

for the bilinear hysteretic restoring force.

The \( \omega_r \) is given by eq. 4.2, which can be written for the bilinear hysteretic system as
\[
\frac{\omega^2 - \alpha \omega_0^2}{\omega_0^2 (1 - \alpha)} = 1 \quad \text{for } A \leq Y \\
= \frac{1}{\pi} \cos^{-1}(1 - \frac{2Y}{A}) - \frac{2}{\pi} (1 - \frac{2Y}{A})[\frac{Y}{A} (1 - \frac{Y}{A})^{\frac{1}{2}}] \quad \text{for } A \geq Y
\] (4.10)

IV-2 Response Moments of Substitute System:

Recall that Caughey (1965) solved the Fokker-Planck (F-P) equation for the system of eq. 4.1 to obtain the joint, stationary, probability density function as

\[
p_{zz}(q, \dot{q}) = C \exp\{-F[H(q, \dot{q})]/\pi S_0\} \quad (4.11)
\]
or

\[
p_A(a) = CT(a) g(a) \exp\{-F[G(a)]/\pi S_0\} \quad (4.12)
\]
in which the variable \( A \geq 0 \) is defined by \( H = G(A) \). The \( g(A) \) in eq. 4.12 will be evaluated from eq. 4.3, for the corresponding nonhysteretic system, and \( G(A) \) is obtained by integration as

\[
G(A) = \frac{\omega_0^2 A^2}{2} \quad \text{for } A \leq Y \\
= \frac{(1 - \alpha)\omega_0^2 Y^2}{2} [4(\frac{A}{Y})^{1/2} - 3] + \frac{\alpha \omega_0^2 A^2}{2} \quad \text{for } A > Y
\] (4.13)

Using the \( EDH \) of eq. 4.9 for a bilinear hysteretic system in eq. 4.8 to obtain the damping function \( f \) for eq. 4.1 yields
\[ f[G(A)] = 2\beta_0\omega_0 \quad \text{for } A \leq Y \]
\[ = 2\beta_0\omega_0 + \frac{4}{\pi}\sqrt{\alpha(1-\alpha)}\omega_0 \left( \frac{\omega_0}{\omega_r} \right)^2 \frac{Y}{A} (1 - \frac{Y}{A}) \quad \text{for } A > Y \] (4.14)

Since both \( f \) and \( H = G(A) \) can now be easily evaluated as functions of the amplitude \( A \), it is straightforward to use numerical integration to evaluate \( F \) (for \( H > \omega_0^2Y^2/2 \)) for use in eq. 4.12. This gives all the terms in eq. 4.12 except \( C \), which can be obtained from the necessary normalization that the integral of \( p_A(a) \) from zero to infinity must be unity.

In order to evaluate the desired response moments, let the response be written as
\[ x(t) = A(t)\cos[\omega_r t + \theta(t)] \] (4.15)

where \( \omega_r \) is a function of \( A \), and assume that the random phase angle \( \theta \) is uniformly distributed from zero to \( 2\pi \) and is statistically independent of \( A \). Then, the mean-squared value of \( x \) can be written as
\[ E(x^2) = \sigma_x^2 = \frac{E(A^2)}{2} = \frac{1}{2} \int_0^\infty a^2p_A(a)da \] (4.16)

the fourth moment of \( x \) is
\[ E(x^4) = \frac{3}{8}E(A^4) = \frac{3}{8} \int_0^\infty a^4p_A(a)da \] (4.17)

and these two integrals can be easily evaluated by numerical integration.

Let \( \ddot{z} \) denote the absolute acceleration (\( \ddot{x} + \ddot{y} \)) of eq. 4.1. Then
\[ \ddot{z} = -[f(H)\ddot{x} + g(x)] \] (4.18)

Unfortunately, some of the terms in \( E(\ddot{z}^2) \) and \( E(\ddot{z}^4) \) are not quite so easily converted into integrals over \( A \) as were the expectations for displacements.
Looking at the conditional expectations given $A$, the moments of $\bar{z}$ can be shown to be

$$E(\bar{z}^2|A) = f^2[G(A)] \frac{\omega_r^2 A^2}{2} + E[g^2(z)|A]$$  \hspace{1cm} (4.19)$$

and

$$E(\bar{z}^4|A) = f^4[G(A)] \frac{3\omega_r^4 A^4}{8} + 6f^2[G(A)]E[\bar{z}^2 g^2(z)|A] + E[g^4(x)|A]$$  \hspace{1cm} (4.20)$$

After some simplification the conditional expectations involving $g$ (from eq. 4.3) can be shown to be

$$E[g^2(z)|A]/\omega_0^4 = \frac{A^2}{2} - \frac{(1 - \alpha^2)}{\pi} A^2 (\theta^* + \frac{Y}{A} \sin \theta^*)$$
$$+ \frac{4\alpha(1 - \alpha)}{\pi} Y_3/2 A^{1/2} \int_0^{\theta^*} (\cos \theta)^{1/2} d\theta$$
$$+ \frac{(1 - \alpha)^2 Y^3}{\pi} \frac{\log(1 + \sin \theta^*)}{A}$$  \hspace{1cm} (4.21)$$

$$E[\bar{z}^2 g^2(z)|A]/(\omega_0^4 \omega_r^2) = \frac{A^4}{8} - \frac{(1 - \alpha^2)}{4\pi} A^4 [\theta^* + \frac{Y}{A} (1 - 2 \frac{Y^2}{A^2}) \sin \theta^*]$$
$$+ \frac{8\alpha(1 - \alpha)}{5\pi} Y^{3/2} A^{5/2} \int_0^{\theta^*} (\cos \theta)^{1/2} d\theta - \frac{Y}{A} (\frac{Y^3}{2}) \sin \theta^*$$
$$+ \frac{(1 - \alpha)^2 Y^3}{\pi} \frac{A \log(1 + \sin \theta^*)}{1 - \sin \theta^*}$$  \hspace{1cm} (4.22)$$
\[ E[g^4(x)|A]/\omega_0^8 = \frac{3A^4}{8} - \frac{(1 - \alpha^4)}{2\pi} A^{4/2} \left[ \frac{3}{2} (\theta^* + \frac{Y}{A} \sin\theta^*) + \left( \frac{Y}{A} \right)^3 \sin\theta^* \right] \\
+ \frac{8\alpha^3(1 - \alpha)}{5\pi} Y^{3/2} A^{5/2} [3 \int_0^{\theta^*} (\cos\theta)^{1/2} d\theta + 2\left( \frac{Y}{A} \right)^{3/2} \sin\theta^*] \\
+ \frac{12\alpha^2(1 - \alpha)^2}{\pi} Y^3 A \sin\theta^* + \frac{2(1 - \alpha)^4}{\pi} \frac{Y^5}{A} \sin\theta^* \\
+ \frac{8\alpha(1 - \alpha)^3}{\pi} Y^{8/2} A^{1/2} [3 \int_0^{\theta^*} (\cos\theta)^{3/2} d\theta - 2\left( \frac{Y}{A} \right)^{1/2} \sin\theta^*] \]

(4.23)

in which \( \cos\theta^* = Y/A \).

The two integrals of non-integer powers of \( \cos\theta \) in eqs. 4.21 to 4.23 have been approximated analytically (see Appendix C) as

\[ \int_0^{\theta^*} (\cos\theta)^{1/2} d\theta \simeq \theta^* \left[ 1 - \frac{1}{3} \left( \frac{\theta^*}{2} \right)^2 - \frac{1}{11} (0.5784 \ \theta^*)^{10} \right] \]

(4.24)

\[ \int_0^{\theta^*} (\cos\theta)^{3/2} d\theta \simeq \theta^* \left[ 1 - \left( \frac{\theta^*}{2} \right)^2 + \frac{9}{20} \left( \frac{\theta^*}{2} \right)^4 \\
- \frac{2}{11} - \frac{3}{13} \left( \frac{\theta^*}{2} \right)^2 (0.4912 \ \theta^*)^{10} + \frac{1}{21} (0.4912 \ \theta^*)^{20} \right] \]

(4.25)

With appropriate substitutions it is now possible to obtain the desired moments of \( x \) and \( \bar{x} \) in one pass of numerical integration incrementing \( A \) from \( Y \) to some very large value. Specifically, integration of eq. 4.14 gives \( F \) in eq. 4.12, integration of \( (g/\omega_r) \exp(-F/\pi S_0) \) gives \( (2\pi C)^{-1} \) for the same equation, integration of eqs. 4.16 and 4.17 gives the moments of \( x \), and integration of eq. 4.12 times 4.19 and 4.20, respectively, gives the moments of \( \bar{x} \).
IV-3 Analytical Solutions for Limiting Approximations:

If the excitation is either very small or very large, then one can use simpler approximations for $f(H)$ and $g(x)$ than those proposed above and thereby avoid numerical integration. That is, one can obtain analytic expressions for both mean-squared response and fourth moment response.

Lutes (1970) presented simplified approximations for $g(x)$ and $f(H)$ for both very large and very small Y/N values. These approximations were both shown to give analytical approximations for $E(x^2)$, requiring no numerical integration. The following paragraphs present slight modification of these approximations and show that the other desired response moments, $E(x^4), E(z^2)$ and $E(z^4)$, can all be evaluated analytically as well.

IV-3-1 Large Y/N:

In order to model accurately the abrupt increase in damping at the level $x = A$, the $f(H)$ function is taken as

$$f(H) = 2\beta_0\omega_0 \quad for \quad H \leq \omega_0^2 Y^2 / 2$$

$$= 2\beta_0\omega_0 + (2\omega_0 / \pi)\sqrt{\alpha}(1 - \alpha)(2H/\omega_0^2 Y^2 - 1) \quad for \quad H > \omega_0^2 Y^2 / 2$$

(4.26)

For all values of $\alpha$, the function $f(H)$ in eq. 4.17 tends to eq. 4.26 as $H$ tends to $\omega_0^2 Y^2 / 2$. The approximation of $g(x)$ for this limiting case is simply taken as the expression for an unyielding system;

$$g(x) = \omega_0^2 x$$

(4.27)
For this system with nonlinear damping and linear spring the probability distribution of $A$ can be written as

$$p_A(a) = C_1 a \exp(-r_1 a^2) \quad \text{for } a \leq Y$$

$$= C_1 a \exp[-r_2 - r_3(a^2 + r_4)^2] \quad \text{for } a > Y$$

with

$$r_1 = \frac{2\beta_0}{\pi N^2}$$

$$r_2 = \frac{2\beta_0 Y^2}{\pi N^2} - \sqrt{\alpha(1 - \alpha)} N^2$$

$$r_3 = \frac{\sqrt{\alpha}(1 - \alpha)}{\pi^2 N^2 Y^2}$$

$$r_4 = \left(\frac{\pi \beta_0}{\sqrt{\alpha}(1 - \alpha)} - 1\right) Y^2$$

in which $N$ is the length measure of the excitation level as in eq. 3.5.

In order to evaluate $C_1$ and the desired moments of $x$ and $\bar{x}$ (as in section IV-2) it is necessary to obtain the integrals of $a^k$ times $p_A(a)$ for various even $k$ values. To simplify the notation let

$$Q_k = \int_0^Y a^k \exp(-r_1 a^2) da$$

and

$$R_k = \int_Y^\infty a^k \exp[-r_2 - r_3(a^2 + r_4)^2] da$$

The $Q_k$ terms can be evaluated from

$$Q_1 = \frac{1}{2r_1}[1 - \exp(-r_1 Y^2)]$$

and the recursive relationship
\[ Q_k = -\frac{1}{2r_1} Y^{k-1} \exp(-r_1 Y^2) + \frac{k-1}{2r_1} Q_{k-2} \quad \text{for } k \geq 2 \]

The evaluation of \( R_k \) is simplified by using a change of variables \( u = a^2 + r_4 \). Then

\[
R_k = \frac{1}{2} \exp(-r_2) \int_{Y^2 + r_4}^{\infty} (u - r_4)^{(k-1)/2} \exp(-r_3 u^2) du
\]

\[
= \frac{1}{2} \exp(-r_2) \sum_{j=0}^{(k-1)/2} \binom{(k-1)/2}{j} (-r_4)^{-j+(k-1)/2} S_j
\]

in which

\[
S_j = \int_{Y^2 + r_4}^{\infty} u^j \exp(-r_3 u^2) du
\]

can be found from

\[
S_0 = \frac{1}{2} \left( \frac{\pi}{r_3} \right)^{1/2} \text{erfc}[\sqrt{r_3}(Y^2 + r_4)]
\]

\[
S_1 = \frac{1}{2r_3} \exp[-r_3(Y^2 + r_4)^2]
\]

and

\[
S_j = \frac{1}{2r_3} (Y^2 + r_4)^{j-1} \exp[-r_3(Y^2 + r_4)^2] + \frac{j-1}{2r_3} S_{j-2} \quad \text{for } j \geq 2
\]

where \( \text{erfc} \) denotes the complementary error function. From the previous equations, one can find:

\[
C_1^{-1} = Q_1 + R_1
\]

\[
E(x^2) = \frac{C_1}{2}(Q_3 + R_3)
\]
\[ E(z^4) = \frac{3C_1}{8}(Q_5 + R_S) \]

\[ E(z^2) = \frac{C_1 \omega_0^4}{2} [(1 + 4\beta_0^2)(Q_3 + R_3) + \frac{8\beta_0}{\pi Y^2 \sqrt{\alpha}}(1 - \alpha)(R_5 - Y^2 R_S) \]
\[ + \frac{4}{\pi^2 Y^4 \alpha}(1 - \alpha)^2(R_7 - 2Y^2 R_5 + Y^4 R_S)] \]

\[ E(z^4) = \frac{3C_1 \omega_0^8}{8} \{(1 + 4\beta_0^2)^2(Q_5 + R_S) \]
\[ + 8\frac{2\beta_0}{\pi Y^2 \sqrt{\alpha}}(1 - \alpha)(R_7 - Y^2 R_5) + \frac{\alpha(1 - \alpha)^2}{\pi^2 Y^4}(R_9 - 2Y^2 R_7 + Y^4 R_5) \]
\[ + 16\frac{4\beta_0}{\pi Y^2 \sqrt{\alpha}}(1 - \alpha)^2(R_7 - 3Y^2 R_9 + 3Y^4 R_7 - Y^6 R_5) \]
\[ + \frac{4\beta_0}{\pi^3 Y^6 \alpha^3/2}(1 - \alpha)^3(R_{11} - 4Y^2 R_9 + 6Y^4 R_9 - 4Y^6 R_7 + Y^8 R_5)] \}

IV-3-2 Small \( Y/N \):

For \( Y \) sufficiently small the stiffness can be modelled by a linear spring with
the post yielding stiffness;

\[ g(x) = \alpha \omega_0^2 x \]  (4.28)

The appropriate nonlinear damping in this situation can be approximated from
eq. 4.14 for \( A \gg Y \) as

\[ f[G(A)] = f[\alpha \omega_0^2 A^2/2] = 2\beta_0 \omega_0 + \frac{4}{\pi} \omega_0 \frac{(1 - \alpha) Y}{\sqrt{\alpha} A} \]  (4.29)

The probability density can then be written as
\[ p_A(a) = C_2 a \exp[-s_1(a + s_2)^2] \]  

(4.30)

with

\[ s_1 = \frac{2\alpha \beta_0}{\pi N^2} \]
\[ s_2 = \frac{2(1 - \alpha)Y}{\sqrt{\alpha \beta_0 \pi}} \]

In this situation it is not necessary to divide the integrals from \( a = 0 \) to \( a = \infty \) into two separate parts, so it is convenient to write the desired expressions directly in terms of moments of \( A \). Thus, \( C_2 \) is found from \( E(A^0) = 1 \) and the response moments are

\[ E(x^2) = \frac{1}{2} E(A^2) \]

\[ E(x^4) = \frac{3}{8} E(A^4) \]

\[ E(z^2) = \frac{\alpha \omega_0^4}{2} [4\beta_0^2 E(A^2) + \frac{16\beta_0}{\pi} \frac{1 - \alpha}{\sqrt{\alpha}} Y E(A) + \frac{16}{\pi^2} \frac{(1 - \alpha)^2}{\alpha} Y^2] + \frac{\alpha^2 \omega_0^4}{2} E(A^2) \]

\[ E(z^4) = \frac{3\alpha^2 \omega_0^8}{8} [16\beta_0^4 E(A^4) + \frac{128\beta_0^3}{\pi} \frac{1 - \alpha}{\sqrt{\alpha}} Y E(A^3)] + \frac{384\beta_0^2}{\pi^2} \frac{(1 - \alpha)^2}{\alpha} Y^2 E(A^2) + \frac{512\beta_0}{\pi^3} \frac{(1 - \alpha)^3}{\alpha^{3/2}} Y^3 E(A) + \frac{256}{\pi^4} \frac{(1 - \alpha)^4}{\alpha^2} Y^4]
\]

\[ + \frac{6\alpha^3 \omega_0^8}{8} [4\beta_0^2 E(A^4) + \frac{16\beta_0}{\pi} \frac{(1 - \alpha)}{\sqrt{\alpha}} Y E(A^3) + \frac{16}{\pi^2} \frac{(1 - \alpha)^2}{\alpha} Y^2 E(A^2)]
\]

\[ + \frac{3\alpha^4 \omega_0^8}{8} E(A^4) \]
The moments of A can be written as

$$E(A^k) = C_2 \sum_{j=0}^{k+1} \binom{k+1}{j} (-s_2)^{k+1-j} T_j$$

with

$$T_j = \int_{s_2}^{\infty} u^j \exp(-s_1 u^2) du$$

for which

$$T_0 = \frac{1}{2} \left( \frac{\pi}{s_1} \right)^{1/2} \text{erfc} \left( \frac{s_1^{1/2}}{s_2} \right)$$

$$T_1 = \frac{1}{2s_1} \exp(-s_1 s_2^2)$$

and

$$T_j = \frac{(s_2)^{j-1}}{2s_1} \exp(-s_1 s_2^2) + \frac{j-1}{2s_1} T_{j-2} \quad \text{for } j \geq 2$$

IV-4 Numerical Results:

Recall that the Coefficient of Excess (COE) is a measure of how much the distribution of a random variable departs from a normal distribution, and it can be written as:

$$COE(x) = \frac{E(x^4)}{E(x^2)^2} - 3 \quad (4.31)$$

$$COE(\bar{z}) = \frac{E(\bar{z}^4)}{E(\bar{z}^2)^2} - 3 \quad (4.32)$$

This measure is zero for a normal distribution, negative for an amplitude-limited type of distribution, and positive for a distribution with greater than normal
probability of large amplitudes. It will be used here to present the fourth moment results.

The numerical results of root-mean-square response from eqs. 4.16 and 4.19 are plotted in figures 4.1 to 4.4, and of coefficient of excess from eqs. 4.31 and 4.32 are plotted in figures 4.5 to 4.8, for the cases of $\alpha = 1/2$ and 1/21, of damping $\beta = 1\%$ and 5%. The simulation results for the bilinear hysteretic system are also included in figures 4.1 to 4.8, as are the analytical approximations from IV-3 for very large and very small $Y/N$ values.

IV-4-1 RMS Values:

Figures 4.1 to 4.4 show that for both the moderately nonlinear system ($\alpha = 1/2$) and the nearly elastoplastic system ($\alpha = 1/21$), the response levels of rms displacement and absolute acceleration obtained from numerical integration for the nonhysteretic system agree quite well with the simulation results for the corresponding hysteretic system.

The rms displacement levels have been investigated in various previous studies. The values obtained here for the substitute nonhysteretic structure are similar to those previously obtained by a power balance method [Lutes and Takemiya 1974], and are substantially better than those previously obtained by Lutes (1970) for a slightly different nonlinear nonhysteretic model. The largest error is for the nearly elasto-plastic structure with intermediate values of the yield level and 5% viscous damping. Lutes and Takemiya (1974), concluded that similar errors were primarily due to errors in the calculation of the average frequency of the system. If the same conclusion applies in the present study,
then modification of eq. 4.2 or 4.3 could be expected to improve these rms displacement predictions. The rms levels of absolute acceleration predicted for the nonhysteretic system are seen to be in very good agreement with those for the bilinear hysteretic system. The errors are sometimes significantly smaller for acceleration than for displacement. The writer is not aware of prior studies of the acceleration levels.

Recall that the "small Y/N" and "large Y/N" results in the figures have been obtained from purely analytical solution of simplified equations (as in section IV-3). These results are always good for the appropriate Y/N ranges. For intermediate Y/N values no such simplified approximation has been obtained, but the figures show that the results for large and small Y/N do also give considerable information regarding the intermediate range.

One may also note that because the stiffness term dominates the acceleration response when Y/N is either very large or very small, \( \sigma_z/\omega_0^2 N \) tends to \( \alpha \sigma_z/N \) for small values of Y/N, and to \( \sigma_z/N \) for large values of Y/N. Thus, contrary to displacement response, \( \alpha = 1/21 \) generally gives lower mean-squared acceleration response than does \( \alpha = 1/2 \).

IV-4-2 COE Values:

Analytical approximation of the fourth moment of \( z \) for the bilinear hysteretic system with \( \alpha = 1/2 \) was presented by Lutes and Takemiya (1974), but the corresponding simulation values were not obtained. The COE(\( \bar{z} \)) is the primary focus of this study, and it had apparently not been previously studied by either analytical or empirical methods.
From figs. 4.5 and 4.6, one can see that the COE of the displacement response of the nonhysteretic system agrees well with simulation results for \( \alpha = 1/2 \) but not for \( \alpha = 1/21 \). The excessively large values of \( COE(x) \) predicted when \( \alpha \) is small may reflect an inadequate choice of \( g(x) \) according to eq 4.3.

Figures 4.7 and 4.8 show that the COE of the absolute acceleration response of the nonhysteretic system agrees quite well with the simulation results for the corresponding hysteretic system for both \( \alpha = 1/2 \) and \( \alpha = 1/21 \). The accuracy of this approximation is somewhat surprising in light of the large errors in \( COE(\tilde{z}) \) for \( \alpha = 1/21 \) and intermediate Y/N values. Apparently \( \tilde{z} \) is much more affected by \( z \) than by \( x \) in this situation for a nearly elastoplastic structure. Figures 4.7 and 4.8 show that \( \tilde{z} \) can be quite substantially nonnormal, and that this is often in the direction of a negative COE value. The fact that \( \tilde{z} \) is nonnormal implies that the response of a secondary system attached to this primary system will also be nonnormal. If the mass of the secondary system is much smaller than that of the primary system, then \( \tilde{z} \) can be considered as the secondary excitation. If the mass ratio is not very small, then interaction between the two systems may change the \( COE(\tilde{z}) \) value.

The fact that the \( COE(\tilde{z}) \) is often negative is good news in that it implies that very large values of \( \tilde{z} \) are less likely than for a normal process with the same RMS value. This should somewhat reduce the probability of damage in the secondary system. The opposite and undesirable situation of \( COE(\tilde{z}) > 0 \) does occur in some circumstances, though. Specifically, moderately small Y/N values lead to \( COE(\tilde{z}) > 0 \) unless the system is nearly elastoplastic and \( \beta_0 \) is not very small.
IV-5 Summary:

In this chapter, response nonnormality has been investigated only for a yielding primary structure subjected to a normally distributed ground acceleration. This is viewed as a preliminary step in the nonlinear investigation of primary-secondary systems, since this primary acceleration would be the base excitation of a light secondary system. Results have been presented both from simulation and from analysis of a simplified nonhysteretic nonlinear substitute structure. Obtaining response moments (RMS and COE) for the substitute structure generally requires simple numerical integration, although closed-form solutions have been obtained for simplifications appropriate to either large or small values of the yield level.

From the numerical results the following observations can be made:

1. nonnormality of the response of a secondary system should definitely be investigated, since the absolute acceleration of the primary structure is sometimes decidedly nonnormal.

2. The most nonnormal response acceleration found was in the direction of amplitude limiting (COE ≈ -1.5). Nonnormality in the opposite sense (COE ≈ 1.0) was also observed, though, for smaller values of the yield level.

3. The nonhysteretic substitute system used here gave quite good predictions of both the RMS and COE of the acceleration response.

4. The substitute system also gave very good predictions of the RMS values of displacement response, but the COE values for displacement were significantly in error for a nearly elastoplastic structure.
Fig. 4.1 RMS of primary displacement for $\alpha=0.5$
Fig. 4.2 RMS of primary displacement for $\alpha=1/21$
Fig. 4.3 RMS of primary absolute acceleration for $\alpha=0.5$
Fig. 4.4 RMS of primary absolute acceleration for $\alpha=1/21$
Fig. 4.5 COE of primary displacement for $\alpha=0.5$
Fig. 4.6 COE of primary displacement for $\alpha=1/21$
Fig. 4.7 COE of primary absolute acceleration for $\alpha=0.5$
Fig. 4.8 COE of primary absolute acceleration for $\alpha=1/21$
CHAPTER V

TRISPECTRAL ANALYSIS

V-1 Introduction:

In the preceding chapter, a method has been demonstrated (use of the nonlinear nonhysteretic substitute system) which can provide satisfactory estimates of the coefficient of excess of the primary system response, without using computer simulation. However, knowledge of the coefficient of excess of the primary acceleration is not enough to allow evaluation of the coefficient of excess for the response of the linear secondary system. This is similar to the problem of evaluating the RMS value (or variance) of the secondary response, which requires, not only the variance but the autocorrelation function or the power spectral density of the primary system acceleration. From eq. 2.27, it can be seen that in order to obtain the coefficient of excess of the secondary system, one must know not only the coefficient of excess of the primary acceleration but the whole fourth cumulant function of primary acceleration, \( Q(\tau_1, \tau_2, \tau_3) \), or the trispectrum, \( D(\omega_1, \omega_2, \omega_3) \). Thus, in order to evaluate the COE value of the secondary response, both the power spectral density and the trispectral density of the primary acceleration should be investigated. The trispectrum of the primary acceleration has not been studied previously and will present a challenging task in the current study.

Since the primary is nonlinear, no analytical form of the trispectrum of primary response can be obtained. However, the trispectrum of primary absolute acceleration can be evaluated numerically from simulated data, and this will provide an "experimental" \( D_z \) function for the nonnormal base excitation of the
secondary system. The empirical $D_3$ function will also be used for comparisons with the $D_3$ from analytical models in the following chapter. It turns out that finding a simulated $D_3(\omega_1, \omega_2, \omega_3)$ function is not a trivial problem. A direct method of finding $D(\omega_1, \omega_2, \omega_3)$ for a given time history consists of first numerically evaluating necessary moments and cumulants, then implementing eq. 2.11 by a numerical triple Fourier integral of the fourth order cumulant. This is theoretically feasible but involves very considerable computation. Another approach is through the "Periodogram", which was first introduced by Schuster (1898). The basic idea of Periodogram analysis is to estimate the kth order ($k \geq 2$) spectral function by using a finite Fourier transform of a single time series. This has been practically applied up to the third order spectrum, which is called the bispectrum [Brillinger and Rosenblatt 1967a,b; Hasselman et al. 1963; Subba Rao and Gabr 1984]. In this study, the method is extended to investigate the fourth order spectrum, which is called the trispectrum. Prior to considering periodogram analysis, though, it is useful to investigate some general characteristics of trispectral functions.

V-2 Methods for Comparing Trispectral Functions:

Finding an adequate approximation of the $D(\omega_1, \omega_2, \omega_3)$ function is considerably more difficult than the more common frequency domain problem of approximating the power spectral function, $S(\omega)$. In the latter situation one can plot $S(\omega)$ versus $\omega$ and use this plot in making judgements regarding the adequacy of an approximation. This is made easier by the facts that $S(\omega)$ is real, and it is an even function so that only $\omega \geq 0$ need to be considered. The $D(\omega_1, \omega_2, \omega_3)$, however, not only is defined on a three dimensional space


but also is generally complex. Thus, fitting $D$ amounts to fitting two functions (the real and imaginary parts), each being a function of three arguments. The symmetry of $D(\omega_1, \omega_2, \omega_3)$ (see Section II-2-3) helps somewhat, but it also is rather complicated inasmuch as it is basically a four-dimensional symmetry in a three-dimensional space. Thus, it is very difficult to conceive of any simple plotting scheme that would reveal all aspects of the $D(\omega_1, \omega_2, \omega_3)$ function for all points within a domain having a given finite range for each frequency argument. Because of these difficulties, the approximations presented here are compared with simulation data only in certain limited regions of the three-dimensional space of $\omega$ values. The following paragraph explains why one particular region is considered more important than most other regions.

Recall from Section II-4-2 the nature of $D_z(\omega_1, \omega_2, \omega_3)$ when $z$ is the response of a lightly damped linear oscillator having a delta correlated excitation. Then $H(\omega)$ has peaks with height of $O(\beta_0^{-1})$ near $\omega = \pm \omega_0$. This, in turn, gives $D_z(\omega_1, \omega_2, \omega_3)$ as having peaks of $O(\beta_0^{-4})$ near $(\omega_0, -\omega_0, \omega_0)$ and each of the other five points "symmetric" to this point. Furthermore, these six high peaks dominate the $D_z$ function, so that all other points are relatively insignificant. In the present situation, the primary system is nonlinear so the behavior of $D_z(\omega_1, \omega_2, \omega_3)$ will surely not be this simple, but some similarity may still be expected. Thus, it is anticipated that $D_z(\omega_1, \omega_2, \omega_3)$ may be dominated by major peaks near points like $(\omega_r, -\omega_r, \omega_r)$, where $\omega_r$ denotes a type of "resonant" frequency of the bilinear hysteretic primary system. The value of $\omega_r$ is unknown, but the existence of such peaks can be investigated by studying $D_z$ in the vicinity of the line $(\omega, -\omega, \omega)$. This line through one octant of the three-dimensional $\omega$ space must include the point $(\omega_r, -\omega_r, \omega_r)$ if it exists, and the existence of a high
peak of $D_\omega$ along this line will at least partially confirm the assumed similarity between the present problem and the one involving a linear primary system. Note, also, that $D$ is a real function at every point on the $(\omega_1, -\omega_1, \omega_3)$ plane, simplifying the study of $D$ in this vicinity.

**V-3 Periodogram Analysis:**

**V-3-1 Polyspectra:**

In order to implement the Fourier transform in a digital computer, the discrete Fourier transform (DFT) has to be used. In 1965, Cooley and Tukey developed an efficient DFT algorithm, called the Fast Fourier transform (FFT), which tremendously increased the computational speed. Therefore, the FFT has become a universal standard algorithm for the DFT and also enhanced the feasibility of using periodogram analysis.

Suppose that an order $k$ stationary process $\{X_n | n = 1, \ldots, N\}$ is known on the set $\{\Delta t, \ldots, j\Delta t, \ldots, N\Delta t\}$ in which $\Delta t$ is the sample interval and $T = N\Delta t$ is the total length of the time sample. The finite Fourier transform of the process $\{X_n\}$ is defined by

$$d_x(\tau \Delta \omega) = \sum_{n=1}^{N} (X_n - \bar{X}) \exp(-i \frac{2\pi n \tau}{N})$$  \hspace{1cm} (5.1)

in which $\bar{X}$ denotes the sample mean and $\Delta \omega = 2\pi / T$ is the frequency increment. Note that $d_x((\tau + N) \Delta \omega) = d_x(\tau \Delta \omega)$ so that $d_x$ is a periodic function. Similarly the inverse Fourier transform of $d_x(\omega)$:
\[ X_n = \bar{X} + \frac{1}{N} \sum_{r_m=1}^{N} d_e(r \Delta \omega) \exp(i \frac{2\pi n r}{N}) \]

gives \(X_{n+N} = X_n\). Thus, we will consider \(\{X_n\}\) to have this periodicity in all calculations involving the Fourier transform. This simplifies certain calculations.

For example, we can write the second cumulant as

\[ C_2(\tau_n) = k_2(t_j, t_j + \tau_n) = \sum_{j=1}^{N} (X_j - \bar{X})(X_{j+n} - \bar{X}) \tag{5.2} \]

in which \(\tau_n = n \Delta t\). Even though the \(j+n\) subscript on the final term goes outside the original range of 1, \ldots, \(N\), the term is unambiguously defined by the periodicity property. Note from eq. 5.2 that \(C_2\) also has the periodicity property

\[ C_2(\tau_{n+N}) = C_2(\tau_n) \]

this periodicity also extends to higher order cumulants such as \(C_k(\tau_1, \ldots, \tau_{k-1})\) which is periodic in each of its \(k-1\) time arguments.

The kth-order polyspectrum (or kth-order cumulant spectrum) is defined by an order \((k-1)\) Fourier transform of the kth-order cumulant function. This is the same idea as in Sec. II-2-2, but the discrete form can be written as

\[ f_k(\omega_1, \ldots, \omega_k) = \left( \frac{\Delta t}{2\pi} \right)^{k-1} \sum_{n_1=1}^{N} \cdots \sum_{n_{k-1}=1}^{N} C_k(\tau_1, \ldots, \tau_{k-1}) \exp \left[ -i \sum_{j=1}^{k-1} \omega_j n_j \Delta t \right] \tag{5.3} \]

where \(\tau_j = n_j \Delta t, 1 \leq n_j \leq N\) and \(\omega_j = r_j \Delta \omega\). 
A sufficient condition to assure that the above Fourier transform exists is,

$$\sum_{n_1} \cdots \sum_{n_{k-1}} |C_k(\tau_1, \ldots, \tau_{k-1})| < \infty$$

In general, $f_k(\omega_1, \ldots, \omega_k)$ is complex-valued and bounded. The final argument of $f_k$ in eq. 5.3 is determined from the condition that the sum of the $k$ variables satisfy

$$\sum_{j=1}^{k} \omega_j = 0$$

Points in the general $k$ dimensional frequency space which satisfy this condition are said to belong to the principal manifold, which is actually of dimension $k - 1$. The function $f_k$ is only defined on this manifold. Since the second-order cumulant $C_2(\tau)$ is just the covariance function, it follows that the second-order polyspectrum is exactly the same as the conventional power spectrum, i.e. $f_2(\omega, -\omega) \equiv S(\omega)$. The third-order polyspectrum, $f_3(\omega_1, \omega_2, -\omega_1 - \omega_2)$ has been called the bispectrum, and the fourth-order spectrum, $f_4(\omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3)$ has been called the trispectrum. Since all polyspectra of higher than second order vanish if $\{X_n\}$ is Gaussian, the power spectrum is the only necessary information for a Gaussian process. On the other hand, the bispectrum, trispectrum and all higher-order polyspectra can be regarded as measures of the departure of the process from Gaussianity. In this study, the bispectrum vanishes due to the symmetric distribution of $\{X_n\}$, so that the trispectrum becomes the most important representation of the non-Gaussian process.
V-3-2 The Estimation of Polyspectra:

The basic idea of the periodogram analysis involves using the finite Fourier transform of eq. 5.1 on a sampled time series. The relevant products of these finite Fourier transforms are then "smoothed" by averaging over neighboring sets of frequencies to produce estimates of the required polyspectrum.

Let $I_N(\omega_1, \ldots, \omega_k)$ be called the kth-order periodogram, or briefly periodogram, and be defined as

$$I_N(\omega_1, \ldots, \omega_k) = \frac{(\Delta t)^{k-1}}{N(2\pi)^{k-1}} \prod_{j=1}^{k} d_x(\omega_j)$$  \hspace{1cm} (5.4)

It can be shown that the expected value of a kth-order periodogram is an asymptotically unbiased estimate of the kth-order polyspectrum (cumulant spectrum) as

$$\lim_{N \to \infty} E[I_N(\omega_1, \ldots, \omega_k)] = f_k(\omega_1, \ldots, \omega_k)$$  \hspace{1cm} (5.5)

provided that the $\omega_1, \ldots, \omega_k$ do not lie in any proper submanifold of the principal manifold, with the submanifolds defined as

$$\sum_{j \in J} \omega_j = 0$$

in which $J$ is a nonvacuous proper subset of $1, \ldots, k$. The expected value in eq. 5.5 typically diverges as $N \to \infty$ if the $\omega$'s do lie in a proper submanifold [Brillinger and Rosenblatt 1967a, 1967b].
It can, however, be proved that the periodogram is not a consistent estimate in the sense of mean square convergence. That is, the variance of $I_N(\omega_1, \ldots, \omega_k)$ does not tend to 0 when $N \to \infty$. To construct a consistent estimate one must "smooth" the function $I_N(\omega_1, \ldots, \omega_k)$ by using a weight function which becomes increasingly more concentrated as the sample size $N$ goes to $\infty$ [Priestley 1988]. There are many possible choices for a specific form of the weight function (or window). Two of the commonly used forms are those of Hanning and Bartlett, but a simpler form is used here.

To estimate $f_k(\omega_1, \ldots, \omega_k)$ at any point that is not in a submanifold, one can simply "smooth" or average the periodogram in the neighborhood of the point. For an estimate at a point in a submanifold, one must average the periodogram for $\omega$'s in a neighborhood of the point, but not actually in the submanifold [Brillinger and Rosenblatt 1967a, 1967b].

The estimation of bispectra has been investigated quite extensively [Hasselman et al. 1963; Subba Rao and Gabr 1984; Choi et al. 1985], but is not within the scope of this study. It appears that very little has been done on the estimation of trispectra. The principal manifold for the trispectrum is $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$, and the possible submanifolds of interest have any $\omega_i + \omega_j = 0$, for $i, j \in \{1, 2, 3, 4\}$. Unfortunately, a region of particular interest is along the line $(\omega, -\omega, \omega, -\omega)$ and all points on this line do lie in these submanifolds. This somewhat complicates the estimation of trispectra in the periodogram analysis. In order to obtain information about the trispectrum on submanifolds, as mentioned early, one needs to take the average of values fairly near the submanifold. This can be accomplished with any simple weighted average over some range of the k-th order periodogram. However, the crucial issue
is how large a range to use for the average? To investigate this problem, a rectangular window (an “on-off” average) has been employed for simplification. The periodogram, $\tilde{I}_N$, smoothed by the rectangular window with width equal to $(2n + 1)\Delta \omega$, can be defined as

$$
\tilde{I}_N(\omega_1, \ldots, \omega_4) = \frac{1}{M} \sum_{b_1 = -n\Delta \omega}^{n\Delta \omega} \cdots \sum_{b_4 = -n\Delta \omega}^{n\Delta \omega} I_N(\omega_1 + b_1, \ldots, \omega_4 + b_4) W(\omega_1 + b_1, \ldots, \omega_4 + b_4)
$$

In which $W(\eta_1, \eta_2, \eta_3, \eta_4) = 1$ if the four frequencies do lie in the manifold $\eta_1 + \eta_2 + \eta_3 + \eta_4 = 0$ but off the submanifolds, $\eta_i + \eta_j \neq 0$ for $i \neq j$ and $W(\eta_1, \eta_2, \eta_3, \eta_4) = 0$ otherwise. The normalization term $M$ in eq. 5.6 is the total number of $I_N$ within the range of four dimensional smoothing,

$$
M = \sum_{b_1 = -n\Delta \omega}^{n\Delta \omega} \cdots \sum_{b_4 = -n\Delta \omega}^{n\Delta \omega} W(\omega_1 + b_1, \ldots, \omega_4 + b_4)
$$

When a proper average width $(2n + 1)\Delta \omega$ is chosen, eq. 5.6 will give a consistent estimate of $D(\omega_1, \omega_2, \omega_3)$ on the submanifolds.

A simple first-order linear system with damping has been employed for obtaining an appropriate value for $n$ in eq. 5.6. The equation of motion for the linear system subjected to a delta correlated excitation can be written as

$$
\dot{z} + bz = y(t) = \sum A_j \delta(t - j\Delta t)
$$

in which the $A_j$'s are independent random variables. The impulse response function, $h(t)$, and the transfer function, $H(\omega)$, are given by
\[ h(t) = \exp(-bt) \]
\[ H(\omega) = \frac{1}{b + i\omega} \]

Therefore, both the response power spectral density, \( S_z(\omega) \) and the trispectrum \( D_z(\omega, \omega, -\omega) \) (on a submanifold) can be obtained analytically as

\[ S_z(\omega) = \frac{E(A^2)}{2\pi\Delta t} \frac{1}{\omega^2 + b^2} \]  \hspace{1cm} (5.8)

and

\[ D_z(\omega, \omega, -\omega) = \frac{E(A^4) - 3E(A^2)^2}{(2\pi)^3\Delta t} \left(\frac{1}{\omega^2 + b^2}\right)^2 \]  \hspace{1cm} (5.9)

Note that \( \Delta t \) is the time increment of the process. The fourth moment, \( E(A^4) \) and second moment, \( E(A^2) \), of the excitation can be chosen so as to completely define the response trispectrum. The ratio of \( D(\omega, \omega, -\omega)/S(\omega)^2 \) can be used as an index of the normalized tripectrum which is somewhat similar to the COE value. It is clear from equations 5.8 and 5.9 that \( D/S^2 \) for this process \( X(t) \) is a constant and is equal to the \( COE(A) \) times \( \frac{\Delta t}{2\pi} \). Thus, the extent to which \( X(t) \) is non-Gaussian is directly related to the \( COE(A) \), and if \( A \) is Gaussian then \( X(t) \) is also Gaussian. For the numerical simulation the parameters have been chosen as \( \Delta t = 0.1(\text{sec}) \), and \( b = 0.5(\text{sec}^{-1}) \). The \( COE(A) \) values have been chosen to be 0 (for a Gaussian process) and 22.2 for a non-Gaussian process which gives \( D/S^2 = 0.353 \). Figs. 5.1 and 5.2 illustrate the implementation of eq. 5.6 to find \( D(\omega, \omega, -\omega)/S^2(\omega) \) values at three different frequencies. It can be seen that when \( n \) is between 14 and 17, the smoothed periodogram gives a quite good estimate for the trispectrum. It should also be noted that when
n is larger than 18, the estimate becomes chaotic, but these values have not been plotted in the figures. Overall, one can conclude that the estimation of the trispectrum on the submanifolds is feasible, however, it requires averaging over a fairly large range of frequencies. One disadvantage of this approach is that the frequency averaging causes the estimations of the trispectrum from the periodogram analysis to appear quite broadband, even if the process of interest is narrowband.

V-3-3 Trispectrum of Primary Acceleration:

This section presents the results of using eq. 5.6 to estimate the trispectrum for the primary absolute acceleration of the nonlinear primary system for the case: $\alpha = 0.5$, $\beta_p = 1\%$, $Y/N = 1$. In order to obtain accurate simulated results, both ensemble averaging and block averaging have been used. The term block averaging refers to a procedure of generating a very long time history then dividing it into a number of blocks covering different time intervals. The finite Fourier transform of each block is then calculated and these transforms are averaged over the different blocks [Priestley 1981]. Thus, block averaging is essentially the same as ensemble averaging except that the time samples (blocks) in the former approach are related to each other, rather than being independent. In this study, a block contains 4096 (or $2^{12}$) time increments. The normalized time increment, $\omega_p \Delta t$ has been chosen to be 0.1 radian giving approximately 63 excitation pulses per cycle of the unyielded system (the same as in the other simulations for the P-S system). The ensemble consists of four long time histories, each of which is divided into 10 blocks. The resulting block and ensemble averaging seems to give satisfactory simulation results for trispectral
analysis.

To avoid the difficulty of describing the full four dimensional behavior of $D(\omega_1, \omega_2, \omega_3)$, one possible approach is to restrict attention to some particular plane within the frequency space. In particular, it seems desirable to study a plane containing a line like $(\omega, -\omega, \omega, -\omega)$, since it is anticipated that $D$ may have major peaks on such lines. Obviously, there are infinitely many planes containing the line, $(\omega, -\omega, \omega, -\omega)$ in the $(\omega_1, \omega_2, \omega_3, -\omega_1 - \omega_2 - \omega_3)$ domain. One simple choice is the plane described by $\omega_1$ and $\omega_3$, and given by $(\omega_1, -\omega_1, \omega_3, -\omega_3)$. The line $(\omega, -\omega, \omega, -\omega)$ is clearly the diagonal of this plane.

A contour map plotted for the $(\omega_1, -\omega_1, \omega_3, -\omega_3)$ plane is shown in figure 5.3 (recall that $D$ has no imaginary part on this plane). The plot has been split into two parts, with figure 5.3a giving more detail on negative $D_3$ values and figure 5.3b concentrating on positive $D_3$. Note that the trispetrum has been “smoothed” over a fairly wide range ($n=15$), so that each number on the map does not represent the “real” trispectrum value but an averaged value. It can be seen that a very high positive peak occurs at normalized frequency $(\omega/\omega_p)$ coordinate $(0.72, -0.72, 0.72, -0.72)$ and two negative troughs appear symmetrically to the line $(\omega, -\omega, \omega, -\omega)$ at coordinates of about $(1.0, -1.0, 0.8, -0.8)$ and $(0.8, -0.8, 1.0, -1.0)$. It should be noted that the frequency $(0.72)$ giving a peak of this function is exactly the same as the frequency giving the maximum power spectral density of this yielding BLH system.

In order to investigate further the negative troughs, another plane which is orthogonal to the line $(\omega_1, -\omega_1, \omega_3, -\omega_3)$ and which contains these two troughs has been investigated. This plane can be defined as $(\omega_1, \omega_2, 2\omega - \omega_1, -2\omega - \omega_2)$.
where $a$ is the distance from the origin to the plane and it has been chosen as $a = 0.85\omega_p$. A contour map of the real part of $D_z$ for this plane has also been plotted in fig. 5.4, with major emphasis being placed on the negative $D_z$ values. It is interesting to note that the negative part is basically shaped like a ring (or donut). The center of the ring seems to be located at about $(0.9, -0.9, 0.9, -0.9)$ which is a little higher than the positive peak on $(\omega, -\omega, \omega, -\omega)$. Figs. 5.3 and 5.4, provide some valuable qualitative as well as quantitative information. Fig. 5.3 can also be used to compare these simulation results with the "smoothed" $D$ function from substitute linear models in the following chapter. This will allow assessment of the acceptability of schemes for "matching" the trispectrum for a BLH system with that for a substitute linear system.
Fig. 5.1 Trispectrum of normal process
Fig. 5.2 Trispectrum of nonnormal process
Fig. 5.4 Contour Map (II) $D_z(\omega_1, \omega_2, 2a - \omega_1), a = 0.85\omega_p$
CHAPTER VI

RESPONSE OF LINEAR SECONDARY SYSTEM

VI-1 Introduction:

The central interest of this study is to develop an analytical model, which can approximate the trispectrum, $D(\omega_1, \omega_2, \omega_3)$ for the absolute acceleration of the nonlinear primary system, as simulated in the preceding chapter. This will give a description of the nonnormal base excitation of the linear secondary system, so that the COE value of the secondary response can then be evaluated analytically. Note that in order to evaluate the COE value of the secondary response, the analytical model also must adequately approximate the power spectral density of the primary acceleration. However, the estimation of a power spectral density is much simpler than the approximation of a trispectrum.

The basic approach used here consists of simultaneously replacing the nonlinear primary system with a substitute linear primary and replacing the Gaussian excitation of the original primary with a non-Gaussian excitation. Obviously, the non-Gaussian excitation is required in the substitute system, since a Gaussian excitation of a linear primary would give a Gaussian primary response (and a Gaussian secondary response). A major advantage of using a linear substitute system is that it allows the use of linear methods (such as state space moment equations) to find the secondary response. The major question is whether it is possible to find a substitute primary system and a substitute excitation such that the $D(\omega_1, \omega_2, \omega_3)$ function for the primary response acceleration is adequately approximated. It should be noted that the substitute excitation and the original excitation are both delta-correlated, so
that they differ only in probability distribution.

The term "equivalent linearization technique" as generally used in analysis of nonlinear vibrations refers to a somewhat different method from the "linear substitute method" used here. The former term is often used to refer to some version of the Krylov and Bogoliubov method (1943), in which the parameters of the equivalent linear system are obtained by minimizing some measure of the difference between the original and the linearized system. The RMS value of the response of the "equivalent linear system" can then be found and the power spectral density can also be approximated [Caughey 1959; Spanos and Iwan 1978]. In this study, the linear substitute system also has a substitute excitation. This makes it infeasible to evaluate parameters by a strict minimization technique, so more intuitive and approximate methods are used. Also, the linearization has been extended for approximating the absolute acceleration in the mean square sense, whereas linearization has usually concentrated on displacement and velocity response.

In some situations a much simpler concept is used in lieu of matching the \( D_2(\omega_1, \omega_2, \omega_3) \) function at any particular point. Recall that the COE is a normalized fourth order cumulant for the special case when all time arguments are the same, \( k_4(t, t, t, t) \), and is the triple integral of the \( D(\omega_1, \omega_2, \omega_3) \) function over the entire frequency space. Thus, a good approximation of \( D_2(\omega_1, \omega_2, \omega_3) \) would necessarily give a good approximation of the COE of \( \tilde{z} \) (although the inverse is not necessarily true). In some situations one can determine some parameters in an approximation of \( D_2(\omega_1, \omega_2, \omega_3) \) on the basis of matching the approximate COE to a simulated value. Matching of the COE value is a reasonable condition to impose on any good approximation for \( D(\omega_1, \omega_2, \omega_3) \),
and COE matching is generally much simpler than D matching, even along a
prescribed line in the $\omega$ space. The idea involved here is exactly equivalent to
noting that matching a target variance value is a reasonable condition for any
good approximation of a target $S(\omega)$ function.

**VI-1-1 Simulation Results for Secondary Response:**

Before proceeding to the linear substitute method for the P-S system, it is
appropriate to summarize the results from simulation and seek to understand
the physical phenomenon of secondary response. The simulation results for the
coefficient of excess of secondary response, $COE(u)$, versus the frequency ratio
($\omega_s/\omega_p$) can be found in figs. 6.1 to 6.6 for the BLH system with $\alpha = 0.5$
and 1/21 and with the excitation level varied to give the $Y/N$ and $\sigma_x/Y$ values
shown.

It can be seen from the figures that the $COE(u)$ is nearly 0 (Gaussian)
at a low frequency ratio, goes to an asymptotic value when the frequency ratio
becomes large (usually about 5 or 6), and generally has a peak (local extremum)
at some intermediate frequency. At low $\omega_s/\omega_p$ values, the secondary response
($u$) is proportional to the absolute displacement of primary response ($z + y$). The
low frequency $COE(u)$ values show that $z + y$ is essentially normal. At first this
may seem surprising, but it can be explained by considering the magnitudes of
the component terms. In particular, consider the variances. Since the variance
of the normal delta correlated excitation, $y$, is infinity compared to a finite
variance of the nonnormal displacement, $z$, the sum should be dominated by
$y$, so should be nearly normal. When the secondary becomes very stiff, $u$
becomes proportional to the primary absolute acceleration ($\ddot{x} + \ddot{y}$ or $\ddot{z}$), and
the $COE(\tilde{z})$ is usually nonnormal because of the nonlinearity in the primary system. The COE of primary absolute acceleration has been studied in chapter IV, and the values there agree with the asymptotic values in figs. 4.7 and 4.8. It is presumed that the local peak of $COE(u)$ at an intermediate frequency is due to an effective "tuning" between the secondary system and a "resonant" frequency of the nonlinear primary system. This resonant frequency, which will be denoted by $\omega_r$, is smaller than $\omega_p$, particularly for small $Y/N$ values. Note that the tuning peak value of $COE(u)$ has the same sign as the $COE(u)$ for $\omega_s >> \omega_p$.

Figures 6.7 and 6.8 illustrate the $COE(u)$ values at tuning for $\alpha = 0.5$ and $\alpha = 1/21$, respectively. The tuning $COE(u)$ values are plotted versus $\sigma_z/Y$ for the primary system. In general, each tuning value occurs for a different frequency ratio, and these $\omega_s/\omega_p$ values are given in parentheses adjacent to data points on the figures. It may be noted that $COE(u)$ varies from negative values for small $\sigma_z/Y$ to positive values for large $\sigma_z/Y$. This is similar to the trends for $COE(\tilde{z})$ (see figs. 4.7 and 4.8) but the magnitudes and zero crossing frequencies are different. For $COE(u)$, the zero crossings occur at $\sigma_z/Y \simeq 2$ for $\alpha = 0.5$, and $\sigma_z/Y \simeq 25$ for $\alpha = 1/21$. The most significant nonnormality of secondary response at tuning can be up to $COE(u) = 1$, which occurs at $\sigma_z/Y$ values of 5 to 15 for $\alpha = 0.5$, and down to $COE(u) = -1$ at $\sigma_z/Y$ about 0.5 for both $\alpha$ values. The tuning frequencies, $\omega_s/\omega_p$, are usually smaller than unity since the resonant frequency of a yielding BLH system is less than $\omega_p$. Normally, $\omega_s/\omega_p$ decreases as the excitation level is increased and varies from 1 to 0.71 for $\alpha = 0.5$ and 1 to 0.22 for $\alpha = 1/21$.

It is interesting to note that the $COE(u)$ values for $\alpha = 1/21$ are generally
smaller than for $\alpha = 0.5$, indicating that an increase of the second slope, $\alpha$, has increased the nonnormality of secondary response in this case. This is consistent with the result in chapter IV that the nonnormality of primary absolute acceleration (input to the secondary system) is more significant for the $\alpha = 0.5$ case.

VI-2 Single Linear Filter Model:

Fig. 6.9 illustrates the principle of using a linear substitute method for the analysis of a P-S system. The basic idea is to use a linear filter having a nonnormal excitation to replace the BLH primary element having a normal excitation; with the hope that both the second and the fourth order cumulants of the primary absolute acceleration for the substitute system will match those of the original system.

The choice of the linear substitute element will be primarily based on matching the power spectral density (or its area which is the mean square value) of the primary absolute acceleration. A third order linear substitute system has previously been shown to be useful for predicting the power spectral density for the response displacement of a BLH system [Hsieh 1979; Lutes and Jan 1983]. This model has been employed in other studies and is now extended for predicting the power spectral density of response acceleration by matching the RMS values of response displacement and velocity (Appendix D). Alternatively, a second order linear substitute system has also been considered, with parameters chosen to achieve approximate matching of the RMS values of velocity and absolute acceleration of responses (Appendix D). The comparisons of power spectral density of $\tilde{z}$ for these two linear models can be found in figs.
6.10 and 6.11. It can be seen that the response of the second order linear system better approximates that of the BLH system, especially when $\alpha$ is small (like $1/21$). Therefore, the second order linear system will be employed in the current study as the linear substitute system.

A delta correlated excitation with parameter value $D_0$ (see eq. 2.33) was chosen in this study. The value of $D_0$ was chosen such that the model matched the COE of the primary response acceleration. Let $k_n(\cdot)$ denote the stationary $n$th cumulant function. Since $COE(\ddot{z})$ or $k_4(\ddot{z})$ is known, the constant $D_0$ can be obtained from the equation:

$$k_4(\ddot{z}) = (2\pi)^3 D_0 \int_0^\infty h_p^4(t) dt$$

(6.1)

in which $h_p(t)$ is the impulse response function for the primary acceleration. The $h_p$ for the second order linear substitute system can be found in Appendix D.

After the nonlinear primary of the P-S system has been replaced by a linear substitute system, the fourth and second response cumulants of the secondary response ($k_4(u)$ and $k_2(u)$) can be calculated from the state space method or other linear methods. The calculation of the cumulant functions for a fourth order linear system subjected to delta correlated excitation can be found in Appendix A. Figures 6.12 and 6.13 show representative COE values of the secondary response from this single linear filter substitute method and from computer simulation. It can be observed that the linear model works well for the two limiting ranges of $\omega_s << \omega_p$ and $\omega_s >> \omega_p$.

When $\omega_s << \omega_p$, the displacement response of the secondary system is
proportional to the absolute displacement of the primary response, which is nearly Gaussian in general. Thus, \( COE(u) \) approaches zero for \( \omega_s << \omega_p \), and this is true for either a linear or nonlinear primary. At the other extreme of \( \omega_s >> \omega_p \), the displacement of the secondary response is proportional to the absolute acceleration of the primary response. Recall, though, that the linear substitute primary has been chosen to match \( COE(\ddot{z}) \) to that of the nonlinear primary. Thus, the substitute system must match \( COE(u) \) for \( \omega_s >> \omega_p \). For intermediate values of \( \omega_s/\omega_p \), though, the linearized model completely misses the tuning peak of \( COE(u) \) which appears in the simulation data.

If the linear substitute model had adequately matched the general D function (or Q function) of the BLH primary acceleration, then it would also have have matched the nonnormality of the secondary response of the original system. In fact, though, the \( COE(u) \) for the simulated secondary response was not matched by the response of this particular linear model. Thus, the D function for the primary acceleration must not have been adequately matched even though \( k_4(\ddot{z}) \) was matched. A case of \( \alpha = 0.5 \), \( \beta_p = 1\% \) and \( Y/N = 1 \) has been studied in order to experimentally investigate the matching of the primary acceleration D functions between the linear model and the BLH system. Fig. 6.14 illustrates the “smoothed” \( D_\ddot{z} \) function from this single linear filter model for comparison with the \( D_\ddot{z} \) function from the BLH system (periodogram analysis) as shown in fig. 5.3. From the contour maps, it can be observed that the BLH \( D_\ddot{z} \) function is not accurately matched by the linear system especially, near the peak, even though the total volume under the \( D_\ddot{z} \) (i.e., the \( COE(\ddot{z}) \) value) has been matched.

The failure of the single linear filter model to predict a tuning peak
of $COE(u)$ can perhaps best be seen by considering the response of the secondary system to an “equivalent” delta correlated excitation. The idea of an equivalent delta correlated excitation is to seek to accurately model $S_u(\omega)$ and $D_u(\omega_1, \omega_2, \omega_3)$ only in the neighborhood of the major peaks of these two functions. If almost all the significant contributions to $k_2(u)$ and $k_4(u)$ come from these neighborhoods, then this technique will give accurate estimates of the cumulants. In general one can expect the approximation to be acceptable when the secondary system is lightly damped, since $S_u(\omega)$ will then be very large for $\omega \approx \pm \omega_s$ and $D_u(\omega_1, \omega_2, \omega_3)$ will be very large near $(\omega_s, \omega_s, -\omega_s)$ and its five symmetric points (see Sec II-4-2). The constant values for $S_\xi$ and $D_\xi$ for this delta correlated excitation of the secondary should then be $S_\xi(\omega_s)$ and $D_\xi(\omega_s, \omega_s, -\omega_s)$ in order to properly model $S_u$ and $D_u$ in the neighborhoods of these points. It should be noted that this delta correlated excitation approach may not work well if $S_\xi(\omega_s)$ and $D_\xi(\omega_s, \omega_s, -\omega_s)$ are much smaller than $S_\xi$ and $D_\xi$ at some other points in the frequency space. For example, if $S_\xi(\omega_r) \gg S_\xi(\omega_s)$ then $S_u(\omega)$ can be expected to have major peaks both near $\omega = \pm \omega_r$ and near $\omega = \pm \omega_s$, and the delta correlated excitation approach would ignore the contribution of the former of these peaks.

One expects $S_\xi(\omega)$ and $D_\xi(\omega_1, \omega_2, \omega_3)$ to achieve their largest values near points $\omega_r$ and $(\omega_r, \omega_r, -\omega_r)$, respectively. So long as $S_\xi(\omega_s)$ and $D_\xi(\omega_s, \omega_s, -\omega_s)$ are not much much smaller than $S_\xi(\omega_r)$ and $D_\xi(\omega_r, \omega_r, -\omega_r)$, the equivalent delta correlated excitation approximation gives

$$S_u(\omega) \approx S_\xi(\omega_s)|H_s(\omega)|^2 \quad (6.2a)$$

and
\[ D_u(\omega_1, \omega_2, \omega_3) \simeq D_z(\omega_s, \omega_s, -\omega_s)H_s(\omega_1)H_s(\omega_2)H_s(\omega_3)H_s(- \sum_{j=1}^{3} \omega_j) \] (6.2b)

For the single linear filter model for the primary these can be rewritten as

\[ S_u(\omega) \simeq S_0 |H_p(\omega_s)|^2 |H_s(\omega)|^2 \] (6.3a)

and

\[ D_u(\omega_1, \omega_2, \omega_3) \simeq D_0 |H_p(\omega_s)|^4 H_s(\omega_1)H_s(\omega_2)H_s(\omega_3)H_s(- \sum_{j=1}^{3} \omega_j) \] (6.3b)

Integrating eq. 6.3a with respect to \( \omega \), and eq. 6.3b with respect to \( \omega_1, \omega_2 \) and \( \omega_3 \) then gives the approximations for \( k_2(u) \) and \( k_4(u) \). Dividing the latter of these by the former squared gives the approximation of COE(u). Note, though, that the characteristic \( H_p \) of the linear primary enters eqs. 6.3a and 6.3b only as the constant value \( H_p(\omega_s) \) and it completely cancels out of the COE(u) approximation. Thus, using the equivalent delta correlated excitation of the secondary system causes COE(u) to be completely independent of the characteristics of the single linear filter substitute primary system. In particular, COE(u) is independent of \( \omega_r \) and this precludes the possibility of obtaining a COE(u) peak at tuning (\( \omega_s \simeq \omega_r \)).

It should be noted that \( \omega_s << \omega_p \) and \( \omega_s > > \omega_p \) are situations in which the equivalent delta correlated excitation approximation may not be justified, since they may give \( S_z(\omega_s) << S_z(\omega_r) \) and \( D_z(\omega_s, \omega_s, -\omega_s) << D_z(\omega_r, \omega_r, -\omega_r) \). As noted above, the single linear filter model does work well in these extreme cases.
For $\omega_s \simeq \omega_r$, though, the equivalent delta correlated result should be reasonably accurate, and the absence of a tuning peak in $COE(u)$ appears to confirm this conclusion.

The tuning peaks of $COE(u)$ in figs. 6.1 to 6.6 show that $D_\delta$ must be more sharply peaked in the vicinity of $(\omega_r, \omega_r, -\omega_r)$ than was predicted by the single linear filter model. This is confirmed by the contour map comparison of $D_\delta$ in figs. 5.3 and 6.14 (even though the smoothing in these latter plots hides much of the details). The following section presents an alternate model chosen to give this more peaked $D_\delta$ by using a more narrowband filter for the fourth cumulant.

VI-3 Two Filters Model:

The fact that P-S frequency tuning causes a peak (local extremum) of the COE value of the secondary response provides evidence that the fourth cumulant of the primary response is more narrowbanded than is the second cumulant. In order to approximate this tuning peak, another model, called the two filters model, is proposed as shown in fig. 6.15. The only change from the single filter model is in the addition of a separate filter, $H_{p4}(\omega)$ for finding the fourth cumulant of $\tilde{E}$ in the analytical model. The parameters of this fourth cumulant filter are chosen on the basis of matching the peak COE value of the secondary response at tuning.

The maximum values of $S(\omega)$ and $D(\omega, -\omega, \omega)$ appear to occur at essentially the same frequency, in general. Thus, the linear fourth cumulant filter will be taken to have the same resonant frequency as the second cumulant filter (as investigated in the previous section). This leaves the damping ratio (or
bandwidth) as the only parameter to be determined for the fourth cumulant filter, and this can be chosen to match the height of the peak in $COE(u)$ at P-S tuning. A convenient way to present the results of this parameter choice will be as a bandwidth ratio, $B_r$, which is defined as the ratio of the damping of the second cumulant filter to the damping of the fourth cumulant filter. A preliminary study of the bandwidth ratio based on matching the tuning peak $COE(u)$ values can be found in fig. 6.16. It is observed that the $B_r$ values generally scatter around 2 for $\alpha = 1/21$, and also for $\alpha = 0.5$ if $\sigma_x/Y \leq 2$. For $\sigma_x/Y \geq 2$ and $\alpha = 0.5$, the $B_r$ value is usually closer to 3.

Figs. 6.17 and 6.18 illustrate the estimates of $COE(u)$ at tuning by using several $B_r$ values in the two filters model (The frequency ratios, $\omega_s/\omega_p$ corresponding to these peak COE values are given in parentheses). It can be seen that the $COE(u)$ predictions obtained by using a single bandwidth ratio such as $B_r = 2.5$ may be acceptable for many purposes, but sometimes have significant errors. To obtain better estimates, one can use a larger $B_r$ value (like 3) for $\alpha = 0.5$ when $\sigma_x/Y > 2$ and a smaller $B_r$ value (like 2) for all the other situations shown.

Using the $B_r$ as 2 and 3 (according to the preceding observations), $COE(u)$ values have been obtained for several different $Y/N$ values for 1% damping and $\alpha = 0.5$ and 1/21. These results from the two filters model are shown in figs. 6.19 to 6.24 for comparison with those from simulation. Note that figs. 6.19 and 6.22 are the same situations as in figs. 6.12 and 6.13, respectively. One can see that the $COE(u)$ at tuning has been significantly improved by using the two filters model with a narrower fourth cumulant filter. It should be noted that theoretically $B_r$ should go to unity in the two limiting cases: $Y \to \infty$ or 0, since
the nonlinear primary tends to a linear system (a single linear filter primary) in these situations. However, the $COE(u)$ values also approach zero in these two extreme situations. Using $B_r = 2$ or $3$ even in these situations seems to give acceptable errors in $COE(u)$, since the absolute values are so small. Table 6.1 shows all the parameter values used in the two filters model for the situations studied here.

Even though the tuning peaks of $COE(u)$ have been quite accurately matched by using the two filters model, the error in the $COE(u)$ for an intermediate frequency range (say $\omega_s \approx 2\omega_p$) still has not been significantly improved. In order to allow more detailed investigation of this remaining discrepancy, the smoothed $D_2$ function for the two filters model has been evaluated for comparison with the simulation results for the BLH system. Fig. 6.25 shows the “smoothed” $D_2$ function contour plots for the linear model for the case: $\alpha = 0.5$, $\beta_p = 1\%$ and $Y/N = 1$. When this plot is compared with figs. 5.3 and 6.14, it can be observed that the peak in the contour map has been significantly improved by using the more narrowband fourth cumulant filter. However, figs. 5.3 and 5.4 also show negative $D_2$ values on a “ring” area centered at $(\omega, \omega, -\omega)$ for a frequency $\omega$ higher than $\omega_r$ at the positive peak. These negative values do not appear in fig. 6.25. In fact, they could not possibly appear for any linear substitute primary since the trispectrum of the output of any linear model cannot be negative if the input constant $D_0$ is positive. This reveals an inherent shortcoming of any linear model for approximating the trispectrum of a BLH system. Namely, the BLH system sometimes has frequency regions giving a trispectrum of the “opposite” sign, and a linear model with delta correlated input never gives this behavior.
One can again consider the idea of an equivalent delta correlated excitation to seek to explain the discrepancy of the two filters model for intermediate frequencies above tuning. Consider $\alpha = 0.5$, $\beta_p = 1\%$ and $Y/N = 1$, since that is the situation for which the smoothed trispectrum from simulation has been presented (figs. 5.3 and 5.4). The negative values of $D_z(\omega, -\omega, \omega)$ for $\omega/\omega_p \approx 1$ could be expected, based on the equivalent delta correlated excitation model, to give negative $COE(u)$ values for $\omega_s \approx \omega_p$. Fig. 6.19 shows that this does, in fact occur. The results are not exactly in agreement with the delta correlated excitation predictions since the bandwidth of the secondary system is finite. Nonetheless, the negative $D_z$ trispectrum values in the vicinity of $(\omega, -\omega, \omega)$ for $\omega \approx \omega_p$ should be expected to reduce the $COE(u)$ values for $\omega_s$ anywhere in this vicinity. Since the linear model never gives these negative $D_z$ values, it should overpredict the $COE(u)$ in this area.

Thus, one can conclude that the discrepancy of $COE(u)$ in the intermediate frequency range is inevitable when using the linear models. Fortunately, the linear system estimate is always on the conservative side for positive $COE(u)$ values (which are of most interest), and the largest COE values usually occur either at tuning or in the asymptotic region. Thus, the discrepancy for intermediate frequencies may not be a problem for practical applications.

V-6 Noncascade Study for P-S system:

The basic assumption of the P-S systems is that the mass ratio $m_s/m_p$ is relatively small such that the interaction between primary and secondary can usually be neglected. For RMS values, it has been shown that ignoring the interaction effects would be acceptable so long as the frequencies of the
two systems are not close, but a significant error on the conservative side may occur when tuning exists [Crandall and Mark, 1963; Kelly and Sackman, 1978]. For COE values, however, the influence of the secondary system has not been studied previously, therefore, the effects of interaction on the nonnormality of the secondary system is of interest and is investigated in this section.

For the noncascade study of P-S systems, the Rugge-Kutta method has been employed for solving the coupled BLH primary and linear secondary in the simulation, as has been mentioned in chapter III. The two filters model can be extended with little effort to provide an analytical approach, once the parameters of cascade analysis have been established. The model is illustrated in fig. 6.26 in which the parameters of the two linear substitute primary systems and of the nonnormal excitation can be obtained from the previous discussion. In particular \( k_p = k_{p2} \), with the value chosen according to eq. D.22, and \( c_{p4} = c_{p2}/B_r \), with \( c_{p2} \) chosen according to eq. D.21. The influence of the secondary system can be studied by considering the mass ratio \( \eta = m_s/m_p \) in the state space equations (eq. 3.1), in which \( \eta = 0 \) was used for cascade analysis.

Case studies have been done by first investigating the mass ratio \( \eta = 1\% \). Comparisons between the analytical approach and simulation results for COE(u) can be found in fig. 6.27 for \( \alpha = 0.5 \) and 1/21. It can be seen that the effects of interaction are significant at tuning, for which the change in the COE value can be 50\% for the 1\% mass ratio. However, the influences of the secondary system are relatively small at other frequencies. Note that there is almost no effect in the asymptotic frequency range due to the existence of the secondary system. Therefore, the study of interaction effects on nonnormality can be focused on the tuning situation only. Since the 1\% mass ratio is usually an upper bound for P-S
systems and cascade analysis ($\eta = 0$) is the lower bound, another intermediate mass ratio of $\eta = 0.1\%$ has also been investigated. For different excitation levels, the tuning peak values of the COE of secondary response have been studied from both simulation and the analytical approach. Figures 6.28 and 6.29 illustrate the results from simulation and from the two filters model using $\eta = 0, 0.1\%$, and 1\% for different excitation levels.

It can be seen that the analytical predictions normally agree well with simulation results for different mass ratios, and the effects of interaction in P-S systems at tuning can be significant for the most nonnormal situations. Therefore, the influence of the secondary system should be taken into account if the nonnormality of secondary response is significant.
(a). BLH : $\alpha = 0.5$, $\beta_p = 1\%$ :

<table>
<thead>
<tr>
<th>Y/N</th>
<th>$\omega/\omega_p$</th>
<th>damping of $k_2$</th>
<th>damping of $k_4$</th>
<th>$D_0\omega_p/S_0^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30.</td>
<td>1.0</td>
<td>1.0 %</td>
<td>0.5 %</td>
<td>-30.04</td>
</tr>
<tr>
<td>15.</td>
<td>1.0</td>
<td>1.53 %</td>
<td>0.76 %</td>
<td>-139.84</td>
</tr>
<tr>
<td>9.</td>
<td>0.87</td>
<td>2.87 %</td>
<td>1.44 %</td>
<td>-101.64</td>
</tr>
<tr>
<td>1.5</td>
<td>0.72</td>
<td>8.61 %</td>
<td>2.87 %</td>
<td>10.0</td>
</tr>
<tr>
<td>1.</td>
<td>0.72</td>
<td>7.8 %</td>
<td>2.6 %</td>
<td>35.74</td>
</tr>
<tr>
<td>0.2</td>
<td>0.71</td>
<td>2.15 %</td>
<td>0.72 %</td>
<td>41.69</td>
</tr>
</tbody>
</table>

(b). BLH : $\alpha = 1/21$, $\beta_p = 1\%$ :

<table>
<thead>
<tr>
<th>Y/N</th>
<th>$\omega/\omega_p$</th>
<th>damping of $k_2$</th>
<th>damping of $k_4$</th>
<th>$D_0\omega_p/S_0^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30.</td>
<td>1.0</td>
<td>1.0 %</td>
<td>0.5 %</td>
<td>-32.55</td>
</tr>
<tr>
<td>7.</td>
<td>0.99</td>
<td>4.75 %</td>
<td>2.4 %</td>
<td>-72.48</td>
</tr>
<tr>
<td>3.</td>
<td>0.91</td>
<td>17.25 %</td>
<td>8.6 %</td>
<td>-33.0</td>
</tr>
<tr>
<td>2.</td>
<td>0.85</td>
<td>50. %</td>
<td>25. %</td>
<td>-31.71</td>
</tr>
<tr>
<td>0.6</td>
<td>0.22</td>
<td>22.4 %</td>
<td>8.9 %</td>
<td>7.01</td>
</tr>
<tr>
<td>0.4</td>
<td>0.22</td>
<td>16.6 %</td>
<td>6.6 %</td>
<td>53.08</td>
</tr>
</tbody>
</table>

Table 6.1 Parameters in Two Filters Model
BLH: alpha=0.5, damping=1%
Linear secondary's damping=1%

Fig. 6.19 Two filters model for secondary response, α=0.5
Primary: BLH, $\alpha=0.5$, damping=1%
Secondary: Linear, damping=1%

Fig. 6.2 Simulation for secondary response, $\alpha=0.5$
Primary : BLH, alpha=0.5, damping=1%
Secondary : Linear, damping=1%

\[ \frac{\omega_s}{\omega_p} \]

\[ \text{COE (u)} \]

\[ Y/N=15 \]
\[ \text{RMS(s)/Y}=0.5 \]

Fig. 6.3 Simulation for secondary response, \( \alpha=0.5 \)
Primary: BLH, $\alpha=1/21$, damping=1%
Secondary: Linear, damping=1%

Fig. 6.4 Simulation for secondary response, $\alpha=1/21$
Primary: BLH, $\alpha=1/21$, damping=1%
Secondary: Linear, damping=1%

Fig. 6.5 Simulation for secondary response, $\alpha=1/21$
Primary: BLH, alpha=1/21, damping=1%
Secondary: Linear, damping=1%

Fig. 6.6 Simulation for secondary response, \( \alpha=1/21 \)
Primary: BLH, slope=0.5, damping=1%
Secondary: Linear, damping=1%

Fig. 6.7 Simulation for secondary response at tuning, α=0.5

Primary: BLH, slope=1/21, damping=1%
Secondary: Linear, damping=1%

Fig. 6.8 Simulation for secondary response at tuning, α=1/21
- Simulation:

\[ S_0 \rightarrow \text{BLH} \rightarrow \ddot{Z} \rightarrow \text{Linear} \rightarrow u \]

- Single Linear Filter:

\[ S_0 \rightarrow \text{Linear} \rightarrow \ddot{Z}_{\text{app}} \rightarrow \text{Linear} \rightarrow u_{\text{app}} \]

\[ D_0 \]

Nonnormal

\[ S_0 \rightarrow \text{Linear P-S} \rightarrow u_{\text{app}} \rightarrow \text{COE}(u) \]

\[ D_0 \]

Nonnormal

\[ h(t) = h_p(t) * h_s(t) \]

\[ k_s(u), k_s(u) \]

Fig. 6.9 Linear substitute method of P-S analysis
Fig. 6.10 PSD of primary absolute acceleration for $\alpha=0.5$
Fig. 6.11 PSD of primary absolute acceleration for $\alpha=1/21$
Fig. 6.12 Single filter model for secondary response, $\alpha=0.5$
BLH: $\alpha=1/21$, damping=1%

Linear secondary's damping=1%

$\omega_s/\omega_p$

![Graph 1](image1)

$\text{COE (u)}$

$Y/N=0.4$

$\text{RMS}(x)/Y=49$

Fig. 6.13 Single linear filter for secondary response, $\alpha=1/21$
Fig. 6.14 Trispectrum of single filter model $D_{3}(\omega_1, -\omega_1, \omega_3)$
Two Filters Model:

\[ S_0 \rightarrow H_{p2}(\omega) \rightarrow H_s(\omega) \rightarrow k_2(u) \]

\[ D_0 \rightarrow H_{p4}(\omega) \rightarrow H_s(\omega) \rightarrow k_4(u) \]

Fig 6.15 Two linear filters model
Primary: BLH, damping=1%
Secondary: Linear, damping=1%

Fig. 6.16 Preliminary study for bandwidth ratios
Fig. 6.17 Bandwidth ratios of two filters model, $\alpha=0.5$
Primary: BLH, slope=1/21, damping=1%
Secondary: Linear, damping=1%

Fig. 6.18 Bandwidth ratios of two filters model, $\alpha=1/21$
Primary: BLH, alpha=0.5, damping=1%
Secondary: Linear, damping=1%

Fig. 6.1 Simulation for secondary response, $\alpha=0.5$
Fig. 6.20 Two filters model for secondary response, $\alpha=0.5$
BLH: alpha=0.5, damping=1%
Linear secondary's damping=1%

Fig. 6.21 Two filters model for secondary response, α=0.5
BLH: $\alpha = 1/21$, damping = 1%
Linear secondary's damping = 1%

Fig. 6.22 Two filters model for secondary response, $\alpha = 1/21$
BLH: alpha=1/21, damping=1%
Linear secondary's damping=1%

Fig. 6.23 Two filters model for secondary response, α=1/21
Fig. 6.24 Two filters model for secondary response, α=1/21
Fig. 6.25 Trispectrum of two filters model $D_3(\omega_1, -\omega_1, \omega_2)$
Fig. 6.26 Linear substitute method (two filters model)
Primary: BLH, $\alpha=0.5$, damping=1%
Secondary: Linear, damping=1%

$\omega_s / \omega_p$

$\text{COE (u)}$

$\Box: \eta=0 \quad \Box: \eta=1\%$ (simulation)
$---$: Two filters model ($\eta=1\%$)

$Y/N=1$
$\text{RMS}(x)/Y=5.4$

---

Primary: BLH, $\alpha=1/21$, damping=1%
Secondary: Linear, damping=1%

$\omega_s / \omega_p$

$\text{COE (u)}$

$\Box: \eta=0 \quad \Box: \eta=1\%$ (simulation)
$---$: Two filters model ($\eta=1\%$)

$Y/N=0.4$
$\text{RMS}(x)/Y=49$

---

Fig. 6.27 Two filters model for secondary response, $\eta=1\%$
Primary: BLH, $\alpha=0.5$, damping=1%
Secondary: Linear, damping=1%

Fig. 6.28 Noncascade COE of secondary response, $\alpha=0.5$
Primary: BLH, $\alpha=1/21$, damping=1%
Secondary: Linear, damping=1%

Simulation: $\eta=0$: □, $\eta=0.1%$: ◦, $\eta=1%$: ■
Two filters model: $\eta=0$: ⋯, $\eta=0.1%$: ——, $\eta=1%$: ——

$\omega_s/\omega_p$

$\text{RMS}(x)/Y$

Fig. 6.29 Noncascade COE of secondary response, $\alpha=1/21$
CHAPTER VII

RELIABILITIES AFFECTED BY NONNORMALITIES

VII-1 Introduction:

Often when a dynamic system is subjected to random excitation, it is important to determine the probability that the system will not malfunction during a specific period of time. The reliability function \( R(T) \), which is the probability of nonfailure during the time interval \( 0 \leq t \leq T \), is therefore of central interest in most applications of random vibration theory. The two most common failure modes in civil engineering are (i) first-passage failure, and (ii) fatigue failure. The former mode could also be called first crossing failure, and it represents the situation in which failure occurs the very first time that some response quantity crosses a specified threshold. An example would be if fracture of a brittle structural member occurs the first time that the stress in the member reaches a critical level. In the fatigue failure mode, failure is due to an accumulation of many small increments of damage inflicted throughout the life of the system. In this chapter, general reliability estimates will be presented based only on RMS response and a normal distribution assumption, as well as those based on methods considering nonnormality, in which higher order response cumulants are included. First-passage failure will be studied first, and fatigue damage will be investigated in the following section. Finally the methods will be applied to obtain reliability values for a secondary system affected by nonnormality due to nonlinearity in the primary system. This will be in section VII-4 and will be based on data from the previous chapter.
VII-2 First-Passage Failure:

Let $R(b,T)$ be the probability that the absolute value of the random process $\{X(t)\}$ remains below the level $b$ at all times in the interval $[0,T]$:

$$R(b,T) = P[|X(t)| \leq b : 0 \leq t \leq T]$$

$$\simeq P[|X(0)| \leq b] P[t(b) \geq T] \quad (7.1a)$$

in which $t(b)$ is the time when $|X(t)|$ first reaches the level $b$. This can also be written as

$$R(b,T) \simeq R_0 \exp(- \int_0^T \alpha_b(t) dt) \quad (7.1b)$$

in which $R_0 = P[|X(0)| \leq b]$. Note that $b$ is a failure level (symmetric two-sided barrier), and $\alpha_b(t)$ is called the decay rate. This decay rate is the conditional probability density of first-passage at time $t$ given no passage prior to $t$, which is the conditional expected rate of upcrossings at time $t$, given no upcrossings prior to $t$. When $t$ becomes large, $\alpha_b(t)$ tends to a constant $\alpha_b$ if $X(t)$ is stationary. This constant limiting decay rate is of interest in this study. The probability $R_0$ of starting below the threshold is generally affected by the initial conditions on $\{X(t)\}$ as well as by the level $b$. However, if $b$ is relatively large, one can approximate $R_0$ by unity. In this case, the reliability for stationary response can be approximated by

$$R(b,T) \simeq \exp(-\alpha_b T) \quad (7.1c)$$

The decay rate, $\alpha_b$, becomes the only necessary information to predict first-passage failures in stationary processes with large $b$ values.
If the conditional joint distribution of $X(t)$ and $\dot{X}(t)$ given no prior upcrossings were known, then $\alpha_b$ could be simply evaluated from an integral of that distribution. In most cases, however, the distribution is not known, so some other approach must be used. The "Poisson Crossings" approximation is often used to simplify the procedure by using an integral of the unconditional density of $X(t)$ and $\dot{X}(t)$. This yields $\alpha_b = 2\nu_b$, where $\nu_b$ is the stationary unconditional expected rate of upcrossing of the level $b$. The value of $\nu_b$ can be found from a classical result of S.O. Rice (1954) as

$$\nu_b = \int_0^\infty \dot{x} p_X(b, \dot{x}) d\dot{x}$$

(7.2)

VII-2-1 Normal Processes:

If $X(t)$ and $\dot{X}(t)$ are jointly normal with zero mean, then equation 7.2 can be simplified to:

$$\nu_b = \frac{1}{2\pi} \frac{\sigma_{\dot{x}}}{\sigma_x} \exp(-\frac{b^2}{2\sigma_x^2})$$

(7.3)

$$= \nu_0 \exp(-\frac{b^2}{2\sigma_x^2})$$

in which $\sigma_{\dot{x}}$ and $\sigma_x$ are the standard deviation of $X$ and $\dot{X}$ and $\nu_0$ is $\nu_b$ for $b = 0$. If the crossings of the level $b$ are assumed to constitute a Poisson process then the probability density, $p_T(T)$ can be taken as an exponential distribution:

$$p_T(T) = (2\nu_b) \exp[-(2\nu_b)T] \quad T \geq 0$$

Then the mean time to failure can be written as
\[ E[T] = \frac{1}{2\nu_b} \quad (7.4a) \]

and the reliability function in eq. 6.1c can be written as

\[ R(b, T) = \exp(-2\nu_b T) \quad (7.4b) \]

Equation 7.4 is the well-known Poisson first-passage prediction which has been widely used in applications.

**VII-2-2 Poisson Correction Factor:**

It has been shown that any tendency of the crossings to cluster in clumps will usually act to decrease \( \alpha_b \). Although the Possion crossings approximation provides a very simple formula, it will introduce a great amount of error on the conservative side if employed for a narrowband process. Comparisons of results from the Poisson crossings approximation and from simulations have been plotted as \( \alpha_b/2\nu_b \) versus \( b/\sigma_X \) for various bandwidth values [Crandall 1970; Roberts 1976; Lutes et al. 1980]. Note that the ratio \( \alpha_b/2\nu_b \) may be considered as a Poisson correction factor, since it gives a measure of the effect of the conditioning event of no prior upcrossings in \([0, T]\). If the conditioning has no effect then the upcrossings truly are Poisson distributed and \( \alpha_b/2\nu_b = 1 \). It has been shown that the greatest discrepancy generally occurs at \( b/\sigma_X \approx 2 \).

For the response of a 1% damped linear oscillator, the value of \( 2\nu_b \) from the Poisson approximation may be up to 500% of the true \( \alpha_b \) value [Lutes et al. 1980]. Therefore, the clustering effect can be quite significant and it has been studied by several authors through analytical approaches as well as empirical
work [Mark 1966, Vanmarcke 1975, Roberts 1976, Lutes et al. 1980, etc.]. The Poisson correction factor is generally related to the bandwidth of the process for any given \( b/\sigma_X \) value. Mark (1966) and Vanmarcke (1975), gave closed form analytical approximations of the Poisson correction factor as a function of bandwidth and Roberts (1976) gave numerical results from another analytical approximation. Lutes et al. (1980) presented empirical formulas for better fitting of empirical data. Among all these, Vanmarcke's analytical formula (1975) is probably the best in terms of having an analytical basis and generally giving adequate approximations.

In formulating his analytical approximation, Vanmarcke used a bandwidth parameter \( q \) defined by

\[
q = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}} \tag{7.5}
\]

in which

\[
\lambda_b = \int_0^\infty \omega^b G(\omega) d\omega \tag{7.6}
\]

where \( G(\omega) \) represents the one-sided power spectral density of a zero-mean process \( \{X(t)\} \). For the stationary response of a simple oscillator, \( q \) can be approximated as

\[
q = \left[ \frac{4}{\pi} \beta (1 - 1.1 \beta) \right]^{1/2} \tag{7.7}
\]

provided that the damping ratio \( \beta \) is small. The Poisson correction factor of Vanmarcke's formula (1975) can then be written as
\[
\frac{c_b}{2\nu_b} = \frac{1 - \exp \left[ -\sqrt{\frac{\pi}{2}} g^{1.2} r \right]}{1 - \exp \left[ -r^2 / 2 \right]}
\] (7.8)

in which \( r = b/\sigma_z \).

VII-2-3 Nonnormal processes:

If \( X(t) \) is not normal, then equation 7.2 cannot be integrated easily and the procedure of obtaining \( \nu_b \) may become very complicated. However, two alternative simplifying approximations can be made:

\[
\frac{\nu_2}{\nu_0} = \frac{p_X(x)}{p_X(0)}
\] (7.9a)

\[
\frac{\nu_2}{\nu_0} = \exp \left( -\frac{1}{2}[g^{-1}(x)]^2 \right)
\] (7.9b)

Equation (7.9a) is precise for the situation in which \( X(t) \) and \( \dot{X}(t) \) are independent as well as uncorrelated, and equation (7.9b) is correct if \( X(t) = g[U(t)] \), a monotone functional transformation of the standard normal process \( U(t) \) [Winterstein, 1988]. In the general situation neither of these particular cases strictly applies.

Since the probability density function (PDF) of \( X(t) \), \( p_X(x) \), is usually unknown in practice, approximation of \( p_X(x) \), in equation 7.9a, based on a limited number of moments/cumulants has been widely used. For example the Charlier and Edgeworth series [Crandall, 1980] give:
\[ p_{X_0}(x_0) = \varphi(x_0) \left[ 1 + \sum_{n=1}^{N} h_n H_n(x_0) \right] \] (7.10)

In terms of the standard normal PDF, \( \varphi(x_0) = (2\pi)^{-1/2} \exp(-x_0^2/2) \), and the Hermite polynomials \( H_n(x_0) \). Note that \( X_0 = [X(t) - m_X]/\sigma_X \) in eq. 7.10. The Hermite polynomial of degree \( n \), \( H_n(x_0) \), is defined as a function which satisfies the relationship given by,

\[ \frac{d^n}{dx_0^n} \exp(-x_0^2/2) = (-1)^n H_n(x_0) \exp(-x_0^2/2) \quad n = 0, 1, 2, \ldots \] (7.11)

This yields lower order polynomials of:

\[ H_0(x_0) = 1 \]

\[ H_1(x_0) = x_0 \]

\[ H_2(x_0) = x_0^2 - 1 \] (7.12)

\[ H_3(x_0) = x_0^3 - 3x_0 \]

\[ H_4(x_0) = x_0^4 - 6x_0^2 + 3, \text{ etc.} \]

and a derivative relationship of:

\[ \frac{d}{dx_0} H_n(x_0) = n H_{n-1}(x_0) \] (7.13)

The coefficients \( h_n \) in eq. 7.10 can be determined from

\[ h_n = \frac{1}{n!} E[H_n(X_0)] \] (7.14)

so that
\[ h_0 = 1 \]
\[ h_1 = E[X_0] = 0 \]
\[ h_2 = \frac{1}{2!} E[X_0^2 - 1] = 0 \] \hspace{1cm} (7.15)
\[ h_3 = \frac{1}{3!} E[X_0^3] \]
\[ h_4 = \frac{1}{4!} E[X_0^4 - 3] = \frac{1}{4!} COE[X_0], \text{ etc.} \]

Let \( \chi_n = E[X_0^n(t)] \), central moments of \( X_0(t) \), which can be related to \( h_n \) as

\[ \chi_n = n! \left( h_n + \frac{h_{n-2}}{1!2} + \frac{h_{n-4}}{2!2^2} + \cdots \right) \] \hspace{1cm} (7.16)

For example, \( h_1 = h_2 = 0 \), \( h_3 = \chi_3/6 \) and \( h_4 = (\chi_4 - 3)/24 \). Note that \( \chi_3 \) and \( \chi_4 \) are called the skewness and kurtosis, respectively. The coefficient of excess, in particular, is equal to \( \chi_4 \) minus 3.

It has been shown that the first four cumulants are sufficient to capture a great deal of the nonnormal characteristics of stochastic response [Winterstein 1988]. Equation 7.10 is therefore typically truncated at \( N=4 \). Combining equation 7.9a and the truncated equation 7.10 yields

\[ \frac{\nu_x}{\nu_0} = \exp\left(-\frac{x_0^2}{2}\right) \left[ \frac{1 + h_3(x_0^3 - 3x_0) + h_4(x_0^4 - 6x_0^2 + 3)}{1 + 3h_4} \right] \] \hspace{1cm} (7.17)

In a recent work by Winterstein (1988), comparisons of \( \nu_x/\nu_0 \) have been made between Charlier and Edgeworth series for \( p_X(z) \) (eq. 7.17) and approximations obtained from equations 7.9a and b with the \( g \) function in equation 7.9b based on a Hermite series. It has been shown that the latter model is more
stable and gives better approximations. It is convenient, at this point, to replace \( g^{-1}(x) \) in eq. 7.9b by \( u(x) \) giving

\[
\frac{\nu_x}{\nu_0} = \exp \left( -\frac{u^2(x)}{2} \right) \tag{7.18}
\]

The Hermite series may also be used for approximating \( p_X(x) \) in equation 7.9a. In particularly, \( X = g(U) \) gives

\[
F_X(x) = P[X(t) \leq x] = \Phi[u(x)] \tag{7.19a}
\]

so that

\[
p_X(x) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} u^2(x) \right] \frac{du(x)}{dx} \tag{7.19b}
\]

and substituting this into the approximation of eq. 7.9a gives

\[
\frac{\nu_x}{\nu_0} = \exp \left[ -\frac{u^2(x)}{2} \right] \frac{du(x)}{dx} \tag{7.20}
\]

For most cases, eq. 7.20 seems to give better predictions for large \( x/\sigma_x \) values than does eq. 7.18. Presumably this means that the independent \( X \) and \( \dot{X} \) assumption of eq. 7.9a is in better agreement with real processes than is the monotone transform assumption of eq. 7.9b. Since it is generally more accurate, eq. 7.20 will be used in the current study of nonnormality.

For choosing appropriate \( u(x) \) functions to use in eq. 7.20 (or eq. 7.18), it is necessary to consider separately two different nonnormal situations. In Winterstein’s (1988) terminology a system is called “hardening” if the form of \( u \) versus \( x \) is concave upward. The opposite situation of \( u \) versus \( x \) being convex
upward is called "softening". For an $x$ process with nonzero skewness ($\chi_3 \neq 0$), hardening refers to a negative skewness and softening to a positive skewness. For the situation considered here the skewness is zero ($\chi_3 = 0$), therefore, hardening refers to a negative COE and softening to a positive COE.

(i) Hardening Response ($\chi_3 < 0$ or $\chi_3 = 0$ and $COE < 0$) :

In this situation, let $u(x)$ be in the form of a Hermite polynomial:

$$u(x) = g^{-1}(x) = x_0 - \sum_{n=3}^{N} h_n H e_{n-1}(x_0)$$  \hspace{1cm} (7.21)

Note that $h_n$ for the leading term of the summation in eq. 7.21 will be negative so that the $u$ versus $x$ is concave upward (a hardening response).

(ii) Softening Response ($\chi_3 > 0$ or $\chi_3 = 0$ and $COE > 0$) :

In this situation, let $g(U)$ be in the form of a Hermite polynomial:

$$\frac{X - mx}{\sigma_X} = X_0 = g(U) = k \left[ U + \sum_{n=3}^{N} \hat{h}_n H e_{n-1}(U) \right]$$  \hspace{1cm} (7.22)

$$= k \left[ U + \hat{h}_3 (U^2 - 1) + \hat{h}_4 (U^3 - 3U) + \ldots \right]$$

in which $k$ is a scaling factor ensuring that $X_0(t)$ has unit variance and its value can be obtained from the "second-order" approximation [Winterstein, 1987] :

$$k = \left[ 1 + \sum_{n=3}^{N} (n - 1)! \hat{h}_n^2 \right]^{-1/2} = (1 + 2\hat{h}_3^2 + 6\hat{h}_4^2)^{-1/2}$$  \hspace{1cm} (7.23a)
The coefficient \( \hat{h}_n \) can be expressed in terms of the corresponding Hermite moment \( h_n \) of eq. 7.14 as

\[
\hat{h}_4 = \frac{\sqrt{1 + 36h_4} - 1}{18} = \frac{\sqrt{1 + 1.5(\chi_4 - 3)} - 1}{18} \tag{7.23b}
\]

\[
\hat{h}_3 = \frac{h_3}{1 + 6h_4} = \frac{\lambda_3}{4 + 2\sqrt{1 + 1.5(\chi_4 - 3)}} \tag{7.23c}
\]

By inverting eq. 7.22, \( u(x) \) can be found as

\[
u(x) = \left[ \sqrt{\xi^2(x) + c + \xi(x)} \right]^{1/3} - \left[ \sqrt{\xi^2(x) + c - \xi(x)} \right]^{1/3} - a \tag{7.24}
u(x)
\]

where

\[
\xi(x) = 1.5b \left( a + \frac{x - \mu \chi}{k\sigma \chi} \right) - a^3
\]

in terms of the constants \( a = \frac{\hat{h}_3}{3h_4} \), \( b = \frac{1}{3h_4} \), and \( c = (b - 1 - a^2)^3 \).

Figure 7.1 shows the normalized crossing rate \( \nu_1/\nu_0 \) obtained from eq. 7.20 along with either eq. 7.21 or 7.24 for several different COE values. It can be seen that the nonnormal effects on crossing rate are especially significant when \( x/\sigma_x \) becomes large (\( \geq 4 \)).

Let the nonnormality correction factor (NCF) of first-passage failure, \( Q \), be defined as the ratio of the mean time to failure for a normal process to the mean time to failure for a nonnormal process. Employing eq. 7.4a, the \( Q \) value can also be expressed in term of crossing rates as
\[ Q = \frac{\text{E(T) for normal process}}{\text{E(T) for nonnormal process}} = \frac{\nu_z}{\nu_u} \]  

(7.25a)

Note that the term \( \nu_z/\nu_u \) can be obtained from eq. 7.20 as simply

\[ \frac{\nu_z}{\nu_u} = \frac{du(z)}{dx} \]  

(7.25b)

Fig. 7.2 illustrates the values of \( Q \) for several different COE values and \( x/\sigma_z \) values, which are relevant to the current study of nonnormality in secondary response.

If the Poisson assumption can be employed for a nonnormal processes, then the limiting decay rate \( \alpha_z \) will be:

\[ \alpha_z = 2\nu_z \]

Whenever the bandwidth/clustering effect must be considered, the Poisson correction factor \( \alpha_z/2\nu_z \) can be obtained from Vanmarcke's equation (eq. 7.8) or some other approximation, and used along with the nonnormality correction factor.

VII-3 Stochastic Fatigue Failure:

Fatigue failure results from the gradual accumulation of damage. This damage accumulation is readily observable in some crack propagation problems. However, in most crack initiation problems, evidence of damage is not visible
prior to crack formation. The term fatigue is normally used to describe both of the mentioned situations, and the concept of accumulated damage is used to denote progress toward failure whether or not that progress is observable or measurable. The fatigue damage is presumed to start at (or very near) zero for a new specimen, and to grow monotonically, with failure occurring when the total damage reaches unity.

In order to investigate stochastic fatigue damage, it is first necessary to know the S-N curve, or “fatigue curve” from constant amplitude periodic tests. Commonly, the S-N curve is approximated by a straight line on the log-log scale and the equation can be written as

\[ N(S) = KS^{-m} \]  

(7.26)

in which \( S \) represents the stress range (=twice the amplitude), ie., valley to peak excursion, of the cycle; \( K \) and \( m \) are constants which depend on the material; and \( N(S) \) is the number of cycles at which failure occurs for constant range \( S \).

The central issue in stochastic fatigue is to estimate the fatigue life (T) of a structure under random excitation. Since T is a random variable, \( \mu_T \) or \( E[T] \) is then usually used for characterization of T. Typically \( \mu_T \gg \sigma_T \), so that one can concentrate on estimating the expected fatigue life, \( \mu_T \), by letting \( E[D(\mu_T)] \approx 1 \). The term \( D(t) \) denotes the damage function giving the accumulated fatigue damage at time \( t \). \( D(t) \) is a nondecreasing function of time, defined to be zero initially and to be unity when failure occurs at the failure time \( T \).

According to the commonly used Palmgren-Miner (P-M) hypothesis, the
damage in any cycle is a function only of the loading in that cycle, i.e., any cycle of a given range S always causes the same amount of damage in any situation.

Applying the P-M hypothesis with the S-N curve of eq. 7.26 yields the expected fatigue damage per cycle as

\[ E[\Delta D] = E[N(S)]^{-1} = \frac{1}{K} E[S^m] = \frac{1}{K} \int_0^\infty s^m p_S(s) ds \quad (7.27) \]

The expected fatigue damage per unit time will be equal to

\[ E[\dot{D}(t)] = E[\Delta D] \nu \quad (7.28) \]

in which \( \nu \) is the rate of occurrence of cycles. Thus, the expected fatigue life can be written as

\[ \mu_T = \frac{1}{E[\dot{D}(t)]} = \frac{1}{E[\Delta D] \nu} \quad (7.29) \]

In order to evaluate the fatigue life from eq. 7.29, it is necessary to know the probability distribution of stress ranges and the rate of occurrence of cycles. This is not an easy task except in the special case in which the stress response is Gaussian and narrowband.

**VII-3-1 Rainflow Analysis (Time Series Method):**

A time series approach may be considered to be a “quasi-experimental” method for fatigue calculations, since a time history must be simulated or generated from a known power spectral density (psd). Such a time series or
simulation method is quite general, and does not require that the stress response be narrowband or Gaussian. For a given time history, the fatigue damage can be fairly easily evaluated if the P-M rule is employed. One must identify the cycles, count their number and determine the stress ranges. Quite a few cycle identification algorithms have been proposed, and rainfall analysis is generally recognized as being among the methods which produce the best results [Dowling 1972; Fuchs and Stephens 1980; Wirsching and Light 1980]. An asset of rainfall analysis is that not only does it count all the cycles, but also it does not lose large cycles which happen to be interrupted by small cycles. Computer algorithms for rainfall analysis can be found from Zimmerman (1983) or Downing and Socie (1982).

VII-3-2 Rayleigh Method (Spectral Method):

Although the rainfall method provides a very rational procedure for dividing a time history into cycles, its procedure is somewhat complicated and significant computation is involved in generating the time history, identifying extrema, then identifying cycles. A spectral method based only on a given psd and use of the Gaussian assumption, is generally significantly easier to implement. If the process \{X(t)\} is Gaussian, then its amplitude (or envelope) will be a Rayleigh distribution. For a narrowband process the stress range \( S \) is clearly twice the amplitude, so it also has the Rayleigh probability distribution. In this case \( E[\Delta D] \) can be evaluated from eq. 7.27 as

\[
E[\Delta D] = \frac{1}{K} [2\sqrt{2} \sigma_X]^{m} \Gamma(1 + \frac{m}{2})
\]

\[
= \frac{1}{K} [2\sqrt{2} \lambda_0^{1/2}]^{m} \Gamma(1 + \frac{m}{2}) \tag{7.30}
\]
in which $\Gamma(\cdot)$ denotes the gamma function; and $\sigma_X = \lambda_0^{1/2} =$ RMS value of process \(\{X(t)\}\). $\lambda_0$ denotes the spectral moment of order 0 which can be obtained from eq. 7.5.

For this narrowband case one can take the rate of occurrence of cycles to be the zero crossing rate $\nu_0$, which can be found as

$$\nu_0 = \frac{1}{2\pi} \frac{\sigma_X}{\sigma_X} = \frac{1}{2\pi} \frac{\lambda_2^{1/2}}{\lambda_0^{1/2}}$$  \hspace{1cm} (7.31)

Combining eq. 7.29, 7.30 and 7.31, the fatigue life can be readily obtained, and this is the well-known Rayleigh method. The Rayleigh method employs narrowband and Gaussian assumptions to provide a simple formula for predicting the fatigue life.

For an oscillator subjected to a broadband excitation, the response will be narrowband if the damping is sufficiently small. For practical situations the narrowband approximation may not be strictly justified, but the Rayleigh approximation appears to be conservative in those cases. That is, if the process is Gaussian, but not very narrowband, then the Rayleigh approximation predicts a shorter fatigue life than does rainflow analysis. Because of its simplicity and apparent conservatism, the Rayleigh approximation has become the most widely used stochastic method in engineering applications.

**VII-3-3 Bandwidth Correction Factor:**

If the stress process is not narrowband, then one may choose to correct the Rayleigh method to account for the bandwidth. Lutes et al. (1984), Corazao
(1981) and Wirsching and Light (1980) have used rainflow analysis to investigate the error from the Rayleigh approximation and presented a bandwidth correction factor $\kappa$ defined as

$$\kappa = \frac{E[T] \text{ by Rayleigh approximation}}{E[T] \text{ by rainflow analysis}} \quad (7.32)$$

The factor $\kappa$ approaches unity for a narrowband process, and generally decreases for more broadband processes. For many practical situations one finds $\kappa$ in the range of 0.7 to 1.0, however other situations have been investigated in which $\kappa$ is as small as 0.5 or 0.6.

VII-3-4 Nonnormal processes:

If the fatigue stress is a nonnormal process, then the Rayleigh method may not give a conservative result if the process has a higher probability of large extrema than does the Gaussian process. Lutes et al. (1984) first investigated the effects of nonnormality on fatigue calculations made by rainflow analysis. They introduced a nonnormality correction factor (NCF) defined by:

$$L = \frac{E(T) \text{ for normal process}}{E(T) \text{ for nonnormal process}} \quad (7.33)$$

in which the normal and nonnormal processes have the same time of occurrence of extrema and zero-crossings and the same RMS values, but differ in probability distribution. When the S-N curve is as given in eq. 7.26, this gives

$$L = \frac{E(S^n) \text{ for nonnormal process}}{E(S^n) \text{ for normal process}} \quad (7.34)$$
Winterstein (1985) recently employed the Hermite series to predict nonnormal effects on fatigue damage. As in section VII-2, let $g(U)$ be a monotonic function of a standard normal process. Then if the normal process $U(t)$ has a peak at level $Y$, the nonnormal process $X(t) = g[U(t)]$ has a corresponding peak at $g(Y)$. Thus, if $U(t)$ is sufficiently narrowband, then $E[S^m]$ in eq. 7.34 can be expressed as

$$E[S^m] = E[(g(Y) - g(-Y))^m]$$  \hspace{1cm} (7.35a)

and if $g$ is an odd function this reduces to

$$E[S^m] = (2)^m E[(g(Y))^m]$$  \hspace{1cm} (7.35b)

Note that $Y$ has a Rayleigh distribution since $U(t)$ is narrowband. For the COE $>0$, substituting eq. 7.22 into eqs. 7.34 and 7.35, gives the first-order estimate of the nonnormality correction factor (NCF) as

$$L = 1 + m(m - 1)h_4$$  \hspace{1cm} (7.36)

A second-order estimate [Winterstein 1988], which will give a better approximation if $m$ is large (larger than 5), or if the nonnormality is greater, is

$$L = \left(\frac{\sqrt{\pi k}}{2W!}\right)^m \frac{(mW)!}{(m/2)!}$$  \hspace{1cm} (7.37)

where

$$W^2 = \frac{4}{\pi}(1 + h_4 + \hat{h}_4) - 1$$
The comparison between these two estimates and simulation results from Lutes et al. (1984) can be found in figures 7.3(a) and 7.3(b) for the different kurtosis values. In the current study, the kurtosis is less than 4.0 (COE < 1.0), so the first-order (linear) approximation is accurate enough from an engineering point of view. Hence, eq. 7.36 will be used for the NCF instead of eq. 7.35.

Nonnormality with the COE < 0 generally results in a reduced rate of fatigue damage (for which the NCF is less than unity) compared to a normal process. However, the above calculation technique does not work so well in this situation. Recall that the monotone Hermite polynomial for COE<0 (eq. 7.21) is for \( u(x) = g^{-1}(x) \), whereas, eq. 7.35 requires moments of \( g(Y) \). The truncated Hermite expansion can be inverted (similar to eq. 7.24) to give an expression for \( g(u) \), but evaluation of the expectation in eq. 7.35 is still a problem. No acceptable analytical approximation of this calculation has been found so an alternate approximation (not using Hermite expansions) will be used.

Let \( A = g(Y) \) denote the amplitude or peak value of the narrowband nonnormal process \( X(t) = g[U(t)] \), where \( Y \) is the Rayleigh amplitude of the normal \( U(t) \) process. Rather than using a general series expansion let

\[
A = g(Y) = c_1 Y + c_2 G(Y)
\]  

(7.38)

in which \( G(Y) \) is a specified nonlinear function, and \( c_1 \) and \( c_2 \) are two constants. Choosing an appropriate \( G(Y) \) function and appropriate constants \( c_1 \) and \( c_2 \), allows a variety of situations to be approximated. The COE value of the narrowband process, \( X \) can be obtained from the moment functions of \( A \), since
\[ E[X^2] = \frac{1}{2} E[A^2] \]

and

\[ E[X^4] = \frac{3}{8} E[A^4] \]

Thus

\[ COE(X) = \frac{3}{2} \frac{E[A^4]}{E[A^2]^2} - 3 \quad (7.39) \]

If \( m \) is chosen as an integer then the \( E[A^m] \) moments can be obtained from a binomial expansion of eq. 7.38 as

\[ E[A^m] = c_1^m \binom{m}{0} E[Y^m] + c_1^{m-1} c_2 \binom{m}{1} E[Y^{m-1} G(Y)] + \cdots + c_2^m \binom{m}{m} E[G^m(Y)] \]

In the current study, the \( G(Y) \) has been chosen to be \( Y^{1/5} \) in order to offer \( COE(X) \) between -1.42 (for \( c_1 = 0 \)) and 0 (for \( c_2 = 0 \)), which contains all the negative values of COE in the simulations of secondary response. Note that different \( c_1 \) and \( c_2 \) values will produce different RMS values of \( X \) as well as different COE values. Eq. 7.34 gives the NCF on the condition that the normal and nonnormal process have the same RMS values. Rather than explicitly solving for \( c_1 \) and \( c_2 \) to give this RMS condition, one can normalize eq. 7.34 by the appropriate RMS values, giving

\[ L = \frac{(E[A^m]/\sigma_X^m)}{(E[Y^m]/\sigma_Y^m)} \quad (7.40) \]

One can then vary the ratio \( c_2/c_1 \) and plot \( L \) versus \( COE(X) \).
The comparison of the estimates from the nonlinear transformation, \( A = c_1 Y + c_2 Y^{1/5} \) with simulation results from Hu (1982) can be found in fig. 7.4. It can be observed that the nonlinear transformation method gives reasonable approximations of the NCF for hardening response. Therefore, eq. 7.40 will be used in predicting the NCF for negative \( COE(X) \) values.

If the process is non-narrowband as well as nonnormal, then both the bandwidth and nonnormal corrections could be applied to the Rayleigh method. However, it has been shown that non-Gaussian effects often are more important than bandwidth [Lutes et al. 1984], and also the effects of nonnormality on the fatigue failure are of more interest in this study. Therefore, only the nonnormality correction factor will be evaluated in the following section.

**VII-4 Numerical Results:**

As can be seen from chapter VI, the most significant nonnormality of secondary response occurs which the secondary frequency \( (\omega_s) \) is either tuned to a resonance of the primary system or is much larger than the frequency of the primary system. These two secondary frequency situations will be referred to as tuning and the asymptotic region, respectively. The numerical results for the nonnormality correction factors (NCF) are presented here only for these two critical situations. The NCF has also been evaluated for both cascade analysis \((m_s/m_p = 0)\) and noncascade analysis \((m_s/m_p = 0.1\%, 1\%)\) in situations for which the COE values can be found in Sec. VI-4.

The NCF for first-passage failure \((Q)\) of secondary response can be evaluated from eq. 7.25. The two \( u/\sigma_u \) values which have been considered are 3 and 4.
Note that $u$ is the displacement of the secondary system and $\sigma_u$ is the RMS value. For tuning, the NCF values can be found in figs. 7.5 and 7.6 for $\alpha = 0.5$ and $1/21$, respectively. The $Q$ values are plotted versus the RMS ductility $(\sigma_x/Y)$ of the nonlinear primary system. Each $Q$ value shown corresponds to the local extreme value of $COE(u)$ achieved at P-S tuning. In general the value of $\omega_s/\omega_p$ giving this tuning is different for each $\sigma_x/Y$ value. These tuning values of $\omega_s/\omega_p$ are given in parentheses adjacent to selected data points on the figures. Fig. 7.7 illustrates the NCF of the first-passage failure in the asymptotic region for which the secondary displacement response becomes proportional to the primary absolute acceleration. Only cascade analysis is shown in fig. 7.7 since the mass ratio has no practical significance in this asymptotic region.

As noted earlier, and illustrated in figs. 7.1 and 7.2, nonnormality has a much greater effect on first-passage when the barrier level is higher. This is supported by figs. 7.5 to 7.7 in which the $Q$ values for $u/\sigma_u = 4$ diverge from unity much more than those for $u/\sigma_u = 3$. For $u/\sigma_u = 4$ it can be seen that the NCF can be much greater than unity, indicating that neglecting nonnormality may significantly underestimate the probability of first-passage failure. In particular, $Q$ is approximately 6 at tuning for $\alpha = 0.5$ and $\sigma_x/Y$ in the range of 2 to 10. Similarly large values occur in the asymptotic frequency region for this same system. Neglecting nonnormality in these situations would clearly be unacceptable.

When $COE(u) < 0$, the NCF is less than unity. Figs. 7.5 to 7.7 show that these deviations are sometimes huge. For example, neglecting nonnormality for the asymptotic frequency region would underestimate the time to first passage by over 6 orders of magnitude for the system with $\alpha = 1/21$ and $\sigma_x/Y \approx 4$. While
this discrepancy is very large, it does not have as much practical significance as the \( Q > 1 \) situations. For \( Q < 1 \) neglecting nonnormality may sometimes cause large errors, but it is a conservative procedure in that it overestimates the probability of failure.

From figs. 7.5 and 7.6 one notes that noncascade analysis brings the NCF values at the tuning frequency closer to unity. This, of course, is because the nonzero mass ratio reduces the nonnormality of the response of the tuned secondary, as shown in chapter VI.

The NCF of fatigue failure (\( L \)) for the secondary response can be calculated from eq. 7.36 for positive COE values and eq. 7.40 for negative COE. The fatigue constant, \( m \), has been chosen to have values of 3 and 5, in order to present results appropriate to usual welded structures. The results are presented in figs. 7.8 and 7.9 for tuning and in fig. 7.10 for the asymptotic region. The form of the plots is the same as in the preceding figures for \( Q \). For \( m = 3 \), the \( L \) values are generally less than 1.25, indicating that it may be acceptable to neglect the nonnormality effects in this situation. However, when \( m \) becomes as large as 5, the NCF can be up to 1.75, so that the effects probably should not be ignored. It is also interesting to note that for the same degree of nonnormality, the NCF of fatigue failure (\( L \)) values seem much smaller than the NCF of first-passage failure (\( Q \)) values for the \( m \) and \( u/\sigma_u \) values considered. This is an indication that first-passage failure is more sensitive than fatigue failure to the probability distribution of the extreme values. Thus, consideration of nonnormality effects is more critical for first-passage failure than for fatigue failure.

Overall, it can be seen that the NCF can be significant in some situations,
indicating that reliability predictions for secondary response can be greatly in error if nonnormality is ignored. It also can be observed that if nonnormality is neglected, then the probability of failure of secondary response will generally be overestimated for small $\sigma_z/Y$ values and underestimated when $\sigma_z/Y$ becomes large. Fortunately the former situation will more commonly occur when the yielding takes place in the seismic response of a primary system. It may also be noted that for the same value of $\sigma_z/Y$, $\alpha = 0.5$ usually gives a larger nonnormality correction than $\alpha = 1/21$, and the difference is quite significant. In addition, it has been shown that the reliability effects of nonnormality are as significant in the asymptotic region as at tuning. Furthermore, secondary system are commonly designed to operate in the asymptotic frequency region, and nonnormality effects have usually been neglected in the past.
Fig. 7.1 Normalized crossing rates

Normalized crossing rates

First-Passage failure

\( x/RMS(x) \)
Fig. 7.3(a) NCF of fatigue failure (large kurtosis)
Nonnormality Correction Factor

- □ simulation
- first-order estimate
- second-order estimate

Fig 7.3(b) NCF of fatigue failure (small kurtosis)
NCF of fatigue failure for negative COE values

Simulation (Hu, 1982): m=3 □, m=4 ○, m=5 ▼

\[ A = C_1 Y + C_2 Y^{1/\alpha} \]

Fig. 7.4 NCF of fatigue failure for negative COE values
Primary: BLH, \( \alpha=0.5 \), damping=1%
Secondary: Linear, damping=1%

\[ \eta=0: \quad \eta=0.1\%: \quad \eta=1\%: \]

\[ u/RMS(u)=3 \]

\( \omega_s/\omega_p \)

\( (0.72) \quad (0.71) \)

\( (0.87) \quad (1) \)

RMS(\( x/Y \))

\[ \text{(First-passage failure)} \]

---

Primary: BLH, \( \alpha=0.5 \), damping=1%
Secondary: Linear, damping=1%

\[ \eta=0: \quad \eta=0.1\%: \quad \eta=1\%: \]

\[ u/RMS(u)=4 \]

\( \omega_s/\omega_p \)

\( (0.72) \quad (0.71) \)

\( (0.87) \quad (1) \)

RMS(\( x/Y \))

\[ \text{(First-passage failure)} \]

---

Fig. 7.5 NCF of first-passage failure at tuning for \( \alpha=0.5 \)
Primary: BLH, $\alpha = 1/21$, damping = 1%
Secondary: Linear, damping = 1%

$\eta = 0$: — — $\eta = 0.1%$: ▽ $\eta = 1%$: — —
$u/RMS(u) = 3$

$\omega_s/\omega_p$

(0.22)
(0.85)
(0.97)
(0.99)

(First-passage failure)

Fig. 7.6 NCF of first-passage failure at tuning for $\alpha = 1/21$
Primary: BLH, alpha=0.5, damping=1%
Secondary: Linear, damping=1%

Fig. 7.7 NCF of first-passage failure at asymptotic region
Fig. 7.8 NCF of fatigue failure at tuning for $\alpha=0.5$
Primary: BLH, $\alpha = 1/21$, damping = 1%
Secondary: Linear, damping = 1%

Fig. 7.9 NCF of fatigue failure at tuning for $\alpha = 1/21$
Fig. 7.10 NCF of fatigue failure at asymptotic region
CHAPTER VIII

SUMMARY AND CONCLUSIONS

Some recent studies have shown that the reliability of a structure can be significantly affected by nonnormality of the stochastic structural response. This is not surprising since use of a normally distributed model may significantly misrepresent the frequency of the high response levels, which are likely to contribute to failure. Such nonnormality is particularly likely to occur in a situation involving significant nonlinearity, like the yielding effect in a hysteretic system. In this study, response nonnormality has been investigated in a system composed of a bilinear hysteretic (BLH) yielding primary structure and a linear secondary system subjected to a normally distributed ground acceleration. The secondary system is much less massive than the primary structure and it would usually represent some nonstructural element. The behavior of secondary systems is very important since they often play critical roles in maintaining the operation or safety of the primary structure in the event of extreme loads. This study has focused on nonnormality due to structural yielding in the primary system, and has considered the effects of nonnormality on the probability of failure of the secondary system.

The fourth cumulant function and the simplified, normalized form called the coefficient of excess (COE) have been used to characterize nonnormality in this study. This nonnormality has first been studied for the absolute acceleration of the response of the primary system, since this acceleration is the base excitation of the secondary system, then it has been studied for the relative displacement of the secondary system.
Numerical simulation has been used to obtain COE values for comparison with the results of various analytical methods. In order to obtain results with small statistical variation a combination of ensemble averaging and time averaging has been used. Each ensemble has contained 100 samples and each sample has contained approximately 2000 cycles of response of the primary structure.

The trispectrum, which is the Fourier transform of the fourth cumulant function, has been investigated in a few situations in order to gain better understanding into the nonnormal behavior. Attention on the trispectrum has focused on the vicinity of a single line within the three dimensional frequency space, since that line has been shown to contain the dominant frequency components in at least some important situations. Furthermore, the trispectrum is real along this particular line whereas it is complex over most of the frequency space. Periodogram analysis has been used to obtain smoothed trispectra from discrete Fourier transforms of simulated time histories. This has required empirical determination of appropriate averaging schemes and development of plotting schemes to reveal the most important features of the complex and complicated trispectrum.

One analytical method investigated for predicting the response acceleration of the BLH primary system has used a simplified nonhysteretic nonlinear substitute structure. The method is similar in principle to an existing technique, but significant improvements have been made to the substitute model. Obtaining response moments for the substitute structure generally requires simple numerical integration, although closed-form solutions have also been obtained for simplifications appropriate to either large or small values of the yield level.
The analytical approaches used for calculating the nonnormal secondary response have been based on the concept of using a linear model with nonnormal excitation to replace the BLH primary element with normal excitation. The goal has been the matching of the trispectrum for primary accelerations of the substitute linear model to that of the BLH primary system. The choice of the linear filter has been based on the fitting of the power spectral density, and the nonnormal delta correlated excitation has been chosen to achieve matching of the COE of the primary acceleration. This approach called the single filter model, was eventually extended to allow use of two different substitute primary systems. In this "two filters model" the only difference was that a more narrow band filter was used to predict the fourth cumulant of the response.

Most of this study has considered cascade analysis, in which the response of the primary structure is assumed to be unaffected by the presence of the secondary system. Some study has also been given to noncascade analysis of P-S systems, using both analytical and simulation approaches. In these noncascade analyses the mass of the secondary structure has been taken as 0.1% and 1% of the primary mass ($\eta = 0.1\%$ and 1%).

Finally, the effects of nonnormality on the probability of failure of secondary systems have been studied for both first-passage failure and stochastic fatigue failure. A nonnormality correction factor (NCF), has been defined as the ratio of mean life to failure for a normal process to the mean life to failure for the nonnormal process. Analytical approaches have been used to approximate the NCF values. In most situations a Hermite moment series, based only on the first four cumulant functions, has been employed for representation of a non-Gaussian process. However, evaluating the fatigue failure for a COE value less than zero,
required a different approach, so the non-Gaussian process was represented by a cruder nonlinear transformation of a Gaussian process.

Several observations and conclusions can be drawn based on the results of the above studies:

1. The nonhysteretic nonlinear substitute system gives quite good predictions of the RMS and COE values of the response acceleration of the primary system. The most nonnormal response acceleration found was in the direction of amplitude limiting (COE \( \simeq -1.5 \)). Nonnormality in the opposite sense (COE \( \simeq 1.0 \)) was also observed, though, for smaller values of the yield level. The trends of COE values versus yield level are similar for response absolute acceleration and response displacement. Both have positive COE values at low yield levels and negative COE values at large yield levels. Also increasing the damping in the BLH system, will generally reduce the nonnormality of response displacement and acceleration.

2. The response of the secondary system was nearly normal when the secondary frequency was much less than the primary frequency (\( \omega_s/\omega_p << 1 \)). The secondary COE was the same as that of the primary response acceleration when \( \omega_s/\omega_p >> 1 \) (called the asymptotic frequency region). In addition the secondary COE had a "tuning peak" when \( \omega_s \) approximated the resonant frequency of the primary structure. The single filter model accurately approximated the COE in the low frequency and asymptotic frequency region, but completely failed to predict the tuning peak of the COE.

3. The two filters model gives quite good estimates for the COE of secondary response in most frequency regions. In particular, the empirical tuning
peaks of the COE can be adequately approximated by proper choice of the bandwidth ratio. The optimal bandwidth ratio varies from 2 to 3 for the cases studies here. Using a single bandwidth ratio of $B_r = 2.5$ may be acceptable for many purposes, but sometimes gives significant errors.

4. The trispectrum of the primary acceleration is somewhat different from that of a linear system with delta correlated excitation. Based on the periodogram results, the trispectrum has a dominant peak at the expected location $(\omega_r, -\omega_r, \omega_r)$ but also has a nearby "donut" shaped region having a trispectrum of the opposite sign. This unexpected region of the opposite sign precludes the possibility of accurately fitting the entire secondary COE curve by any linear substitute model of the type used here. The two filters model gives reasonably good matching of the dominant peak of the trispectrum.

5. The two filters model somewhat mispredicts the COE of secondary response in an intermediate frequency range between the tuning and the asymptotic regions. This discrepancy is due to the "opposite sign" portion of the trispectrum, and is inevitable for any linear system. Fortunately, the linear system estimate is always on the conservative side for positive COE values of secondary response, and also the largest COE values usually occur either at tuning or in the asymptotic region. Thus, the discrepancy for the intermediate frequencies may not be a problem for practical applications.

6. The interaction forces in noncascade analysis can significantly reduce the nonnormality of secondary response, especially at tuning. At other frequencies, the interaction effects are relatively small and can be neglected. The reduction of the COE of secondary response can be up to 100% for a 1% mass ratio at some tuning frequencies. The two filters model adequately
approximates this effect in general.

7. The NCF for first-passage of a level four times the RMS value can be much greater than unity, indicating that neglecting nonnormality may significantly underestimate the probability of first-passage failure. In particular, the NCF is approximately 6 both at tuning and in the asymptotic frequency region for certain parameter values. Neglecting nonnormality in these situations would be unacceptable. If the exponent \( m \) in the fatigue law is as small as 3, then the NCF of fatigue failure of the secondary response is generally less than 1.25, indicating that it may be acceptable to neglect the nonnormality effects in this situation. However, when \( m \) becomes as large as 5, the NCF can be up to 1.75, so that the effects probably should not be ignored. The influence of nonnormality in first-passage failure generally is more significant than in fatigue failure based on the cases in this study. Consideration of nonnormality effects is more critical for first-passage failure than for fatigue failure, since first-passage failure is more sensitive than fatigue failure to the probability of the extreme values.

8. Overall, it can be seen that the NCF for failure can be significant in some situations, indicating that reliability predictions for secondary response can be greatly in error if nonnormality is ignored. The probability of failure of secondary response will generally be overestimated for small \( \sigma_{x}/Y \) values and underestimated when \( \sigma_{x}/Y \) values become large. It may also be noted that for the same yielding level, \( \alpha = 0.5 \) usually gives a larger nonnormality correction than \( \alpha = 1/21 \), and the difference is quite significant. In addition, it has been shown that the reliability effects of nonnormality are as significant in the asymptotic region as at tuning, which is particularly pertinent since secondary systems are commonly designed to operate in the
asymptotic frequency region and the nonnormality effects have usually been
neglected in the past.
APPENDIX A

STATE SPACE METHOD

Consider a $N$ first order linear differential equations under stochastic excitation as

$$\frac{d}{dt} \tilde{Y} = A\tilde{Y} + \tilde{f}(t) \quad (A.1)$$

where $A$ is a matrix of order $N \times N$, $\tilde{f}$ and $\tilde{Y}$ are the excitation and response vectors respectively. Let the excitation vector be a non-Gaussian delta correlated process up to order $M$ cumulants, such that

$$k_{F_{k_1}, \ldots, F_{k_M}}(t_1, \ldots, t_M) = I_{k_1 \ldots k_M} \delta(t_2 - t_1) \cdots (t_M - t_1)$$

in which $I$ is the $M$th order intensity tensor. As explained in chapter II, the state space differential equation for $M$th response cumulants comes out in a form involving $M$th order tensors. The equation can be rewritten though in the form

$$\frac{d}{dt} \tilde{L} + Z\tilde{L} = \tilde{I} \quad (A.2)$$

where $Z$ is a square matrix of order $S_{NM} \times S_{NM}$, which depends only on $A$ in eq. A.1. The unknown response cumulant vector, $\tilde{L}$, and delta correlated excitation vector, $\tilde{I}$, are of order $S_{NM}$. The term $S_{NM}$ is given by a binomial coefficient :
\[ S_{NM} = \frac{(N + M - 1)!}{M!(N - 1)!} \]

Solution of eq. A.2 is generally not difficult so long as the matrix \( Z \) and excitation vector \( \vec{f} \) can be found. If the response \( Y(t) \) is stationary, eq. A.2 can even be simplified to an algebraic equation as

\[ Z\vec{L} = \vec{f} \] (A.3)

which gives

\[ \vec{L} = Z^{-1}\vec{f} \] (A.4)

The only practical difficulty is in finding the terms of the \( Z \) matrix and the \( \vec{f} \) vector.

**A-1 Algorithm:**

The following algorithm has been developed for assembling the matrix \( Z \) of order \( S_{NM} \times S_{NM} \) directly from matrix \( A \) (given in eq. A.1). It is based on a function \( g \) which relates the order of the elements in vectors \( \vec{f} \) and \( \vec{L} \), to the order of the terms in the original tensor forms. The form used is based on considering only \( 1 \leq I_1 \leq I_2 \leq \cdots \leq IM \leq N \) within the symmetric \( L \) tensor. The terms RI and RO are used to denote the row and column numbers, respectively in \( Z \). That is, RI denotes a row in \( \vec{f} \) and RO denotes a row in \( \vec{L} \). First one initializes the \( Z \) matrix to be all zeros, then the following sequence of steps creates the proper matrix.
1. Do I₁ from 1 to N
2. Do I₂ from I₁ to N
3. Do I₃ from I₂ to N

⋮

M. Do IM from IM-1 to N
M+1. RI=g(I₁,I₂,...,IM)
M+2. Do Q from 1 to M
M+3. RO=g(Q,I₂,I₃,...,IM)
M+4. Z(RI,RO)=Z(RI,RO)+A(I₁,Q)
M+5. RO=g(I₁,Q,I₃,...,IM)
M+6. Z(RI,RO)=Z(RI,RO)+A(I₂,Q)

⋮

2M+3. RO=g(I₁,I₂,I₃,...,Q)
2M+4. Z(RI,RO)=Z(RI,RO)+A(IM,Q)
2M+5. NEXT Q
2M+6. NEXT I₁
2M+7. NEXT I₂
2M+8. NEXT I₃

⋮

3M+5. NEXT IM

In the present study, the fourth cumulant is of the most interest. The form of the \(g\) function which has been chosen is given by

\[
RI = g(I₁, I₂, I₃, I₄)
\]

\[= RI₁ + RI₂ + RI₃ + RI₄ \quad (A.5)\]
where

\[ RI_1 = (I_1 - 1) \left[ \frac{N(N + 1)(N + 2)}{6} - \frac{I_1}{12} (3N^2 + 7N + 3) + \frac{I_1^2}{24} (4N + 5) - \frac{I_1^3}{24} \right] \]
\[ RI_2 = (I_2 - 1) \left[ \frac{N(N + 1)}{2} - \left( \frac{N}{2} + \frac{1}{3} \right) I_2 + \frac{I_2^2}{6} \right] \]
\[ RI_3 = (I_3 - 1) \left[ N - \frac{I_3}{2} \right] \]
\[ RI_4 = I_4 \quad \text{(A.6)} \]

in which \( 1 \leq I_1 \leq I_2 \leq I_3 \leq I_4 \leq N \).

Limiting attention to the case in which the delta correlated components of \( \tilde{f}(t) \) (in eq. A.1) are independent, gives \( I \) as being identically zero except for RI terms which are due to only one nonzero term \( j \) in \( \tilde{f}(t) \) giving

\[ RI = g(j,j,j,j) \quad \text{(A.7)} \]

**A-2 Example:**

In this section, the fourth response cumulant of a second order linear primary and a second order linear secondary system under delta correlated excitation is illustrated. The equation of motion can be written as:

\[ (\ddot{z} + \dot{y}) + 2\beta_p \omega_p \dot{z} + \omega_p^2 z - 2\eta \beta_s \omega_s \dot{u} - \eta \omega_s^2 u = 0 \quad \text{(A.8a)} \]

and

\[ (\ddot{u} + \ddot{z} + \dot{y}) + 2\beta_s \omega_s \dot{u} + \omega_s^2 u = 0 \quad \text{(A.8b)} \]
where $x$, $u$, and $y$ denote the relative primary displacement, the relative secondary displacement and the ground acceleration, respectively. The terms $eta_p$, $\omega_p$ are the damping ratio and natural frequency of the primary, $\beta_s$, $\omega_s$ are the damping ratio and natural frequency of the secondary, and $\eta = m_s/m_p$ is the mass ratio. Let

$$
\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ u \\ \dot{u} \end{pmatrix}
$$

(A.9)

then eq. A.8 can be rewritten into four first order differential equations as

$$
\dot{\vec{Y}} = A\vec{Y} + \vec{f}
$$

(A.10)

where

$$
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\omega_p^2 & -2\beta_p\omega_p & \eta\omega_s^2 & 2\eta\beta_s\omega_s \\
0 & 0 & 0 & 1 \\
\omega_p^2 & 2\beta_p\omega_p & -\omega_s^2(1 + \eta) & -2\beta_s\omega_s(1 + \eta)
\end{pmatrix}
$$

(A.11)

and

$$
\vec{f} = \begin{pmatrix} 0 \\ -\ddot{y} \\ 0 \\ 0 \end{pmatrix}
$$

(A.12)

With $N=4$, the matrix $Z$ will be order 35x35 for calculating the fourth response cumulant ($M=4$) since

$$
S_{44} = \frac{(4 + 4 - 1)!}{4!(4 - 1)!} = 35
$$
Given the matrix $A$ in eq. A.11 and $M=N=4$, the matrix $Z$ is easily obtained by following the algorithm in the preceding section. The vector $\vec{r}$ is a zero vector of order 35 except for one term which is for

$$RI = g(2, 2, 2, 2) = 21$$

Note that $I_{2222}$ is the fourth cumulant of the excitation when all time arguments are zero. The response vector $\vec{L}$ in eq. A.4 can be solved for by any linear method. The fourth cumulant of secondary displacement will be the $g(3, 3, 3, 3)$ term of vector $\vec{L}$, which is $L(31)$.

A similar approach can be used for calculating the second response cumulant. Since it is much simpler and easier than the fourth cumulant, its illustration is not given here.
APPENDIX B
SCATTER IN SIMULATION STATISTICS

It is not uncommon that the statistical scatter increases as the order of moments increases in time-average moment simulation. Since the fourth moment/cumulant function is of the most interest in this research, the statistical variances may cause a significant error. To obtain an estimate with a small statistical variance, it is helpful to use a combination of time average and ensemble average. The object of this work is, therefore, to investigate the necessary length of the time history and size of ensemble average which will give a tolerable statistical deviation.

One simple method for gaining information about the statistical variations in this study is to let the primary system be linear. Then the response cumulants of the primary and secondary systems can be evaluated analytically, and the statistical variations for each different combination of time and ensemble averages can also be obtained. Two different lengths of time history have been used to verified the scattering effects, one is 200 cycles and the other is 2000 cycles.

A single time series was first generated for this linear P-S system. Using 200 cycles gave deviations of COE of about 0.37 in primary response and 0.5 in secondary response. The corresponding deviations were 0.05 and 0.22 for 2000 cycles. It can be seen that the longer time history did improve the variations, but the scatters are still quite large and cannot be accepted. Second, in order to investigate the variations between the time and ensemble average, several samples were simulated for 2000 cycles and ten times more 200 cycles samples
were also considered, so that these two simulations had the same total number of increments. The comparisons can be found in figs B.1 and B.2 for RMS and COE values of primary and secondary response. For the same no of simulated increments, the time average has shown to be superior to the ensemble average in most cases. Overall, the ensemble size of 100 and length of 2000 cycles seemed to give an acceptable sample and these values will be used in the current simulation for the nonlinear P-S system.
Fig. B.1 Statistical error of primary response
Fig. B.2 Statistical error of secondary response
APPENDIX C

APPROXIMATION OF CERTAIN INTEGRALS

For use in eqs. 4.21 to 4.23 it is necessary to approximate the integral of 
\((\cos \theta)^{1/2}\) from zero to some upper limit \(\theta^*\) which is always between zero and \(\frac{\pi}{2}\). 
This is easily accomplished if \((\cos \theta)^{1/2}\) on \(0 \leq \theta \leq \frac{\pi}{2}\) can be approximated by some simple polynomial. It is most important that this approximation be quite good for small \(\theta\) values, since these values will contribute most to the integral. In order to obtain this good approximation for small \(\theta\) it was decided to include the first two terms from the power series for \((\cos \theta)^{1/2}\). Thus 

\[ (\cos \theta)^{1/2} = (1 - \frac{\theta^2}{2} + O(\theta^4))^{1/2} \]
\[ = 1 - \frac{\theta^2}{4} + O(\theta^4) \quad (C.1) \]

Numerical comparison shows that using only these first two terms gives an approximation of \((\cos \theta)^{1/2}\) which is quite good for a surprisingly large range of \(\theta\) values. In particular the error is less than 0.04 for \(\theta \leq 1.2\) rad. Beyond that point the errors become more significant because this parabola only drops to 0.383 at \(\theta = \frac{\pi}{2}\), where the target function is zero. To give a more accurate approximation it was rather arbitrarily decided to add one more term to eq. A.1 which would give the proper value for \(\theta = \frac{\pi}{2}\) without significantly changing the values from eq. C.1 for smaller \(\theta\) values. By trial and error the approximation was chosen as

\[ (\cos \theta)^{1/2} \approx 1 - \left(\frac{\theta}{2}\right)^2 - (0.5784 \theta)^{10} \quad (C.2) \]

The maximum error in this expression is about 0.12 for \(\theta \approx 1.35\), but the error in
the integral of \((\cos\theta)^{1/2}\) must be considerably less than this. One further test of eq. C.2 was made by squaring both sides of the relationship and integrating this expression from zero to \(\theta^*\). The left-hand-side, of course, gives \(\sin\theta^*\) and the right-hand-side approximates this within 1% over the entire range from zero to \(\frac{\pi}{2}\). Thus, eq. C.2 was judged to be an adequate approximation, and its integral is exactly eq. 4.24.

For the integral of \((\cos\theta)^{3/2}\) a slight modification of the above procedure was used. Obviously one could simply take the third power of eq. C.2 and integrate that expression, but a slightly simpler form was considered preferable. It is not very efficient to use the the first two terms in the power series expansion for \((\cos\theta)^{3/2}\) since \(1-\frac{3}{4}\theta^2\) decreases too rapidly, becoming negative for \(\theta > 1.155\). The alternative used here was to approximate \((\cos\theta)^{3/4}\) in much the same way as was done for \((\cos\theta)^{1/2}\). Thus,

\[
(\cos\theta)^{3/4} \simeq 1 - \frac{3}{8}\theta^2 - (0.4912 \theta)^{10}
\]

(C.3)

This is, in fact, a better approximation than eq. C.2, being within 0.015 for all pertinent \(\theta\) values. This expression was then squared to give

\[
(\cos\theta)^{3/2} \simeq 1 - \frac{3}{4}\theta^2 + \frac{9}{64}\theta^4 - 2(0.4912 \theta)^{10}
\]

\[
+ \frac{3}{4}\theta^2(0.4912 \theta)^{10} + (0.4912 \theta)^{20}
\]

(C.4)

Integration then gives eq. 4.25.
APPENDIX D
LINEAR SUBSTITUTE SYSTEMS

D-1 Third Order Linear System:

Consider the third order linear system with governing equations of motion given by:

\[ m\ddot{x} + c_1\dot{x} + k_1x + k_2(x - s) = -m\ddot{y} \]
\[ c_2\dot{s} = k_2(x - s) \quad (D.1) \]

As shown in figure D.1, the branch of the BLH model containing the Coulomb slider has now been replaced by a dashpot and spring \((c_2 \text{ and } k_2)\) in series and the motion across the dashpot has been taken to be \(s(t)\). Let

\[ \frac{k_1}{m} = \omega_1^2 \]
\[ \frac{k_2}{m} = \omega_2^2 \]
\[ \frac{c_1}{m} = 2\beta_1 \omega_1 \]
\[ r = \frac{\omega_0 c_2}{k_2} \quad (D.2) \]

where
\[ \omega_0 \] is the BLH unyielded, undamped frequency, 
\[ \omega_1 \] is the undamped natural frequency when \(c_2 = 0\), 
\[ \omega_2 \] is a measure of the increase in frequency due to \(k_2\), 
\[ \beta_1 \] is the damping ratio when \(c_2 = \infty\), and 
\[ r \] is a dimensionless measure of the \(c_2\) damping.
Then eq. D.1 becomes

\[ \ddot{x} + 2\beta_1 \omega_1 \dot{x} + \omega_2^2 (x - s) = -\ddot{y} \]
\[ x = \frac{r}{\omega_0} \dot{s} + s \]  \hspace{1cm} (D.3)

The transfer function and impulse function for the response displacement can be found in Hseih (1979). However, in this study the response of primary interest is the absolute acceleration:

\[ \ddot{x} + \ddot{y} = -\left[ 2\beta_1 \omega_1 \dot{x} + \omega_2^2 x + \omega_2^2 (x - s) \right] \]  \hspace{1cm} (D.4)

The harmonic transfer function of response acceleration can be obtained from eq. D.3 by letting

\[ \ddot{y} = e^{i\omega t} \]

and

\[ s = H_s(\omega)e^{i\omega t} \]

This gives

\[ H_s(\omega) = -\frac{1}{a_1 + b_1 i} \]  \hspace{1cm} (D.5)

with

\[ a_1 = \omega_2^2 - (1 + 2\beta_1 r)\omega^2 \]
\[ b_1 = 2\beta_1 \omega + \omega_2^2 r \omega + \omega_2^2 r \omega - r \omega^3 \]

Similarly
&amp; 
\[ z = \frac{r}{\omega_0} \cdot \dot{s} + s = \left( \frac{r}{\omega_0} i\omega + 1 \right) H_s(\omega) e^{i\omega t} \]
gives
\[ H_{\ddot{x}}(\omega) = \left( \frac{r}{\omega_0} i\omega + 1 \right) H_s(\omega) \tag{D.6} \]
and
\[ H_{\ddot{z}}(\omega) = (i\omega) H_{\ddot{x}}(\omega) \\
= (i\omega - \frac{r}{\omega_0} \omega^2) H_s(\omega) \tag{D.7} \]

Substituting eqs. D.5 to D.7 to eq. D.4, the transfer function of \( \ddot{x} + \ddot{y} \) can be obtained as
\[ H_{\ddot{x} + \ddot{y}}(\omega) = -(a_2 + b_2 i) H_s(\omega) = \frac{a_2 + b_2 i}{a_1 + b_1 i} \tag{D.8} \]
in which
\[ a_2 = \omega_1^2 - 2r_1 \omega^2 \\
b_2 = 2r_1 \omega + \omega_1^2 r \omega + \omega_2^2 r \omega \]

To obtain directly the impulse response function of absolute acceleration, one can let \( \ddot{y} = \delta(t) \), \( z(t) = h_z(t) \) in eq. D.3, and use the initial conditions of \( z(0) = h_z(0) = 0 \), \( \dot{z}(0) = 1 \) and \( s(0) = 0 \). Presuming an \( \exp(-\lambda t) \) form for the homogenous solution gives the cubic equation to be solved as:
\[ d_1 \lambda^3 + d_2 \lambda^2 + d_3 \lambda + d_4 = 0 \tag{D.9} \]
in which

\[ d_1 = -\frac{r}{\omega_0} \]
\[ d_2 = 1 + 2\beta_1 \omega_1 \frac{r}{\omega_0} \]
\[ d_3 = -(2\beta_1 \omega_1 + \omega_1^2 \frac{r}{\omega_0} + \omega_2^2 \frac{r}{\omega_0}) \]
\[ d_4 = \omega_1^2 \]

The three roots should be of the form \( \lambda_0, \lambda_1 + \omega i, \lambda_1 - \omega i \), so that the responses can be written as

\[ s = A_0 \exp(-\lambda_0 t) + \exp(-\lambda_1 t) \left[ A_1 \sin \omega t + A_2 \cos \omega t \right] \]
\[ x = B_0 \exp(-\lambda_0 t) + \exp(-\lambda_1 t) \left[ B_1 \sin \omega t + B_2 \cos \omega t \right] \quad (D.10) \]
\[ \dot{x} = C_0 \exp(-\lambda_0 t) + \exp(-\lambda_1 t) \left[ C_1 \sin \omega t + C_2 \cos \omega t \right] \]

The \( B \) coefficients were found by Hseih (1979) to be

\[ B_0 = \frac{\omega_0 (1 - \frac{r}{\omega_0} \lambda_0)}{r[(\lambda_0 - \lambda_1)^2 + \omega^2]} \]
\[ B_1 = \frac{1}{\omega} + B_0 \left( \frac{\lambda_0 - \lambda_1}{\omega} \right) \quad (D.11) \]
\[ B_2 = -B_0 \]

The \( A \) and \( C \) coefficients can similarly be found to be given by:

\[ C_0 = -B_0 \lambda_0 \]
\[ C_1 = -(\lambda_1 B_1 + \omega B_2) \quad (D.12) \]
\[ C_2 = -(\lambda_1 B_2 - \omega B_1) \]
\[ A_0 = \frac{1}{1 - \frac{x}{\omega_0} \lambda_0} B_0 \]
\[ A_1 = \frac{(1 - \frac{x}{\omega_0} \lambda_1)B_1 + (\frac{x}{\omega_0} \omega)B_2}{(1 - \frac{x}{\omega_0} \lambda_1)^2 + \omega^2(\frac{x}{\omega_0})^2} \]  \hspace{1cm} (D.13)
\[ A_2 = -A_0 \]

The impulse response function for absolute acceleration can then be evaluated by substituting eqs. D.10 to D.13 into eq. D.4.

For given values of \( \alpha k_p \), \( \beta \) (or \( c_p \)) and given RMS values of displacement and velocity for the BLH primary system, Hseih (1979) considered two types of matching which can be chosen in order to make a third order linear system resemble the BLH system:

1. Choose \( k_1 = \alpha k_p \) and \( k_2 = (1 - \alpha)k_p \) (or \( \omega_1^2 = \alpha \omega_0^2 \) and \( \omega_2^2 = (1 - \alpha)\omega_0^2 \)), then the \( c_1 \) and \( c_2 \) (or \( \beta_1 \) and \( r \)) are determined by matching the RMS responses of displacement and velocity.

2. Choose \( k_1 = \alpha k_p \) and \( c_1 = c_p \) (or \( \omega_1^2 = \alpha \omega_0^2 \) and \( 2\beta_1 \omega_1 = 2\beta_0 \omega_0 \)), then the \( c_2 \) and \( k_2 \) (or \( r \) and \( \omega_2 \)) should be chosen to match the RMS responses of displacement and velocity.

The power spectral density of the absolute acceleration for the third order linear models obtained by these two criteria are compared in figs. D.2 and D.3 with the results from simulation of the BLH system. The results show that option 1 is a better approximation than option 2 in general. Hence, option 1 will be used for the third order linear model in the chapter VI.
D-2 Second Order Linear System:

The equation of motion of a second order linear system subjected to a ground acceleration can be written as:

\[ m\ddot{x} + c\dot{x} + kx = -m\ddot{y} \]  \hspace{1cm} (D.14a)

or

\[ \ddot{x} + 2\beta \omega_1 \dot{x} + \omega_1^2 x = -\ddot{y} \] \hspace{1cm} (D.14b)

The transfer function of response displacement and velocity are

\[ H_x(\omega) = \frac{1}{(\omega_1^2 - \omega^2) + 2\beta \omega_1 \omega i} \] \hspace{1cm} (D.15)

and

\[ H_\dot{x}(\omega) = \frac{i\omega}{(\omega_1^2 - \omega^2) + 2\beta \omega_1 \omega i} \] \hspace{1cm} (D.16)

The transfer function of the absolute acceleration ($\ddot{x} + \ddot{y}$ or $\ddot{z}$) can then be derived from the relationships:

\[-(\ddot{x} + \ddot{y}) = 2\beta \omega_1 \dot{x} + \omega_1^2 x\]

which gives

\[ H_{\ddot{x} + \ddot{y}}(\omega) = H_x(\omega)(2\beta \omega_1 (i\omega) + \omega_1^2) \]

\[ = \frac{\omega_1^2 + 2\beta \omega_1 \omega i}{(\omega_1^2 - \omega^2) + 2\beta \omega_1 \omega i} \] \hspace{1cm} (D.17)
If the excitation is delta correlated and $S_0$ is its constant power spectral density, then the power spectral density of absolute acceleration can be obtained as

$$S_z(\omega) = S_0 |H_z(\omega)|^2$$  \hfill (D.18)

The mean square response for absolute acceleration of this second order linear system has been found by Crandall and Mark (1973) as

$$\sigma_z^2 = \frac{\pi}{2} \frac{S_0 \omega_1}{\beta} (1 + 4\beta^2)$$  \hfill (D.19)

Hence, the damping ratio which will cause matching of a given $\sigma_z$ value can be obtained from a second order algebraic equation:

$$4\beta^2 - b\beta + 1 = 0$$  \hfill (D.20)

where

$$b = \left[ \frac{\sigma_z}{\omega_0^2 Y} \right]^2 \left( \frac{\omega_0}{\omega_1} \right)^2 \left( \frac{4}{N} \right)^2$$

Note that $\omega_0$ is the unyielded, undamped natural frequency of the BLH system, and $Y/N$ is the yielding level. This gives

$$\beta = \frac{b \pm \sqrt{b^2 - 16}}{8}$$  \hfill (D.21)

One choice of the parameters which is valuable in the current study is to simultaneously match mean square velocity and acceleration of the second order linear system to those of the BLH system. A simple solution which gives a good approximation of this matching is to use
\[
\frac{\omega_1}{\omega_0} = \frac{\sigma_z}{\sigma_{\dot{z}}} \tag{D.22}
\]

and determine \( \beta \) from eq. D.21. The power spectral density for response absolute acceleration can be evaluated from eq. D.18 once the parameters of the second order linear system have been determined.
BLH System

Third Order Linear System

Fig. D.1 Third order linear system
BLH: $\alpha = 0.5$, damping = 1%

$S_n(\omega/\omega_0)$ vs $\omega/\omega_0$

- BLH
- 3rd order #1
- 3rd order #2

$Y/N = 1$

$\text{RMS}(x)/Y = 5.4$

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BLH: $\alpha = 0.5$, damping = 1%

$S_n(\omega/\omega_0)$ vs $\omega/\omega_0$

- BLH
- 3rd order #1
- 3rd order #2

$Y/N = 9$

$\text{RMS}(x)/Y = 0.6$

---

Fig. D.2 PSD for the third order linear system, $\alpha = 0.5$
BLH: alpha=1/21, damping=1%

\[ S_n(\omega/\omega_0) \]

\[ Y/N=1 \]
\[ \text{RMS}(x)/Y=10.7 \]

\[ \omega/\omega_0 \]

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BLH: alpha=1/21, damping=1%

\[ S_n^r(\omega/\omega_0) \]

\[ Y/N=7 \]
\[ \text{RMS}(x)/Y=0.9 \]

\[ \omega/\omega_0 \]

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Fig. D.3 PSD for the third order linear system, \( \alpha=1/21 \)
REFERENCES


