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Robust methods tailored for non-Gaussian narrowband array processing

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Robust Methods Tailored for Non-Gaussian Narrowband Array Processing

by

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Abstract

Array processing algorithms generally assume that the received signal, composed of both narrowband signals and noise, is Gaussian, which is not true in general. In the context of the narrowband array processing problem, we develop robust methods to accurately estimate the spatial correlation matrix which also utilize \textit{a priori} information about the matrix structure. For Gaussian processes, structured estimates have been developed which find the maximum likelihood covariance matrix estimate subject to structural constraints on the covariance matrix [8]. However, further problems arise when the noise is non-Gaussian and the estimators for Gaussian processes may lead to grossly inaccurate estimates [17]. By minimizing the worst asymptotic estimate variance, we obtain the robust structured maximum likelihood type estimates (M-estimates) of the spatial correlation matrix in the presence of noises with probability density functions (p.d.f.) in the \(\epsilon\)-contamination and Kolmogorov classes. These estimates are robust against variations in the amplitude distribution of the noise and take into account sensor placement. Given these estimates, existing array processing algorithms designed for Gaussian circumstances can be used on non-Gaussian problems. We also demonstrate a parametric structured estimate of the spatial correlation matrix which allows estimation of the arrival angles directly. A method of \textit{exactly} determining the class of p.d.f.s is developed which only depends on the time domain noise amplitude distributions being second order processes. Knowledge of this p.d.f. class allows development of
algorithms which can be used in the presence of any type of second-order noise process and which perform nearly as well as existing ones do with Gaussian noise.

We examine the maximum number of signals whose parameters can be estimated with a linear array of $M$ equally spaced sensors. Conventionally, when the signals are mutually uncorrelated, this number has been taken to be one less than the number of sensors. We show how to estimate the signals' directions and amplitudes with the number of signals equal to one less than twice the number of sensors. This increase in the number of signals is accomplished by using length $2M$ real signal vectors rather than the usual length $M$ complex vectors. We show that $2M$ of these real vectors are linearly independent with probability one, and, thus, in the presence of additive white noise, the parameters of $2M - 1$ signals can be estimated. An algorithm for determining directions and amplitudes is presented. However, computational complexity limits this algorithm to small $M$ and low time-bandwidth products.
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Chapter 1

Introduction

The objective of narrowband passive array processing is to detect the presence of narrowband signals in the field of the array and to estimate their angles-of-arrival relative to the array. In the general case, it is not known how many, if any, of these signals are present. The problem is further complicated by the signal power being of the same order as the noise power so that signal-to-noise ratios are frequently less than 0 dB.

The illustration in figure 1.1 is a diagram of the processing done by a typical narrowband array algorithm. While we have shown a linear array of equally spaced sensors, the work presented in this thesis is not limited, except in a few specific cases, to this geometry. The narrowband signals impinging on the array are assumed to have a common, known center frequency and to be distant enough to arrive as plane waves. As the signals are narrowband, there is some initial processing done at each sensor in order to select only the frequency of interest. The data from each sensor is sectioned into $N$ equal length blocks and the discrete Fourier transform (DFT) of each block is determined. One value from each DFT corresponding to the center frequency of the signals is extracted and these values form $N$ complex vectors $x_i$, $i = 1, \ldots, N$ where the $j^{\text{th}}$ element of $x_i$ is the DFT value extracted from the $i^{\text{th}}$ block of the $j^{\text{th}}$ sensor. These complex vectors are composed of two parts, the frequency domain representation of the $P$ signals and a noise term, so that

\[ x_i = n_i + \sum_{p=1}^{P} s_{i,p} \]
Figure 1.1: The structure of a typical narrowband array processor
The second moments of these vectors $E[xx']$ is known as the spatial correlation matrix and has the structure

$$
R = Q + SCS'
$$

where $Q$ is the covariance of the noise, the columns of $S$ are the signal vectors, $S'$ denotes the conjugate transpose of $S$, and $C$ is the matrix of correlations between the signals. If the signal vectors are normalized so that for $M$ sensors $s_p's_p = M$, then for a signal propagating in the $-k$ direction at speed $c$ the elements of $s_p$ are

$$
s_{p,m} = \exp \left\{ -j2\pi \frac{f}{c}(z_m \cdot k) \right\}
$$

where $f$ is the center frequency of the signal and the $z_m$ are the sensor positions. For all of the examples in this thesis we will be considering linear arrays of equally spaced sensors (uniform linear arrays). Again, although very little of the work presented here is limited to this particular array geometry, it is the most commonly studied geometry and a large percentage of the array processing literature has concentrated on this particular type of array. For uniform linear arrays and white noise, the form of the spatial correlation matrix becomes

$$
R = \sigma_n^2 I + SCS'
$$

where $\sigma_n^2$ is the noise power and

$$
s_p = \begin{bmatrix} 1 & e^{-j\alpha} & \cdots & e^{-j(M-1)\alpha} \end{bmatrix}^T
$$

with $\alpha = 2\pi \frac{d}{\lambda} \sin \theta_p$ a function of the signal wavelength $\lambda$, the sensor spacing $d$, and the angle-of-arrival of the signal $\theta$. The angles-of-arrival are measured with respect to broadside which is the direction perpendicular to the axis of the array.

The form of the received signal after the DFTs are calculated is that of a sum of complex exponentials of unknown frequency in the presence of additive noise. Consequently, if spectral estimates are used to find the frequencies of these exponentials, then we have succeeded in estimating the angles-of-arrival [18]. As is
shown in figure 1.1, the spectral estimation techniques generally require a covariance matrix estimate for $E[xx']$. Much of this thesis is concerned with obtaining good estimates of the covariance matrix in order to achieve better performance with the spectral estimators. We will be concerned with two spectral estimates: Capon’s method [18] and the eigenvector method [19]. Capon’s method is also known as the minimum energy or maximum likelihood method and is the solution to the optimization problem of finding the steering vector $a$ which minimizes the beam energy $a'R_a$ subject to $a'e = 1$, where $e$ represents a signal arriving in the current direction-of-look. This optimization problem corresponds to minimizing the total beam energy while keeping unity gain in the direction-of-look and has the solution $a = \frac{R^{-1}e}{e'R^{-1}e}$ so that $a'R_a = (e'R^{-1}e)^{-1}$. The eigenvector method is one of a number of subspace methods which employ an eigen decomposition of $R$ to achieve better resolution of the signals. If there are $P$ signals and $P < M$, then the space spanned by these signals is the same as the space spanned by the $P$ largest eigenvalues and their corresponding eigenvectors. Thus, the space spanned by the $M - P$ smallest eigenvalues and corresponding eigenvectors is orthogonal to the signals and, if $R = \sum_{i=1}^{M} \lambda_i v_i v_i'$ is the decomposition of $R$ with the $\lambda_i$s in ascending order, then $\sum_{i=1}^{M-P} \lambda_i v_i v_i'$ is orthogonal to the signals. The eigenvector method is similar to Capon’s method except $\sum_{i=1}^{M-P} \frac{1}{\lambda_i} v_i v_i'$ is used in place of $R^{-1}$. The eigenvector method would result in infinite peaks at the spatial frequencies corresponding to the angles-of-arrival if $R$ were available. However, only estimates of $R$ obtained from the vectors $x_i$ are available and the peaks are not infinite, but, in general, are much greater than for Capon’s method. Because the eigenvector method depends more upon the structure of $R$ than does Capon’s method, it is more sensitive to the quality of the spatial correlation matrix estimate. In fact, if the usual estimate of $R$, $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i'$, is used then there is no further improvement in Capon’s method after $N$ becomes much larger than the number of sensors $M$, but there continues to be much improvement in the eigenvector method.
as $N$ becomes larger and the estimate of $\mathbf{R}$ becomes better [10]. This sensitivity will be evident when we discuss the various methods of estimating $\mathbf{R}$ presented in this thesis.

If the noise amplitude density is Gaussian, then the $\mathbf{x}_i$ are also Gaussian and the spatial correlation matrix estimate $\widehat{\mathbf{R}}$ is the maximum likelihood estimate. However, if the noise is non-Gaussian, then the $\mathbf{x}_i$ will not be Gaussian and $\widehat{\mathbf{R}}$ may be a very poor estimate of $\mathbf{R}$. Very little work has been done in the area of non-Gaussian array processing even though traditional algorithms degrade significantly in the presence of non-Gaussian noise. In addition, there is actually information known about the structure of $\mathbf{R}$ which is not taken into account when finding the estimate $\widehat{\mathbf{R}}$. In this estimation problem as in others, the more a priori information which is used, the better the estimate becomes. We develop estimates which are robust against variations in the noise amplitude density and use the available structural information. A shortcoming of most robust techniques is that they assume the noise amplitude density to belong to a class of densities but do not necessarily know what this class should be. We are able to derive the density class for the frequency domain noise exactly with the only assumption being that the time domain noise amplitude is a second order process. Thus, narrowband array processing is possible when all that is known about the noise amplitude is that is a second order process. Additionally, the performance of these robust algorithms is demonstrated to be good even for the "worst case" noises.

There is a fundamental limit to the number of signals whose angles-of-arrival can be unambiguously estimated with an $M$ sensor uniform linear array. If we consider the signal vector of equation 1.1, we see that it has the Vandermonde form, and, as a result, any $M$ different vectors of this form are linearly independent; any more than $M$ are linearly dependent. Therefore, if the number of signals is less than or equal to $M - 1$, the signal vectors form a unique set whose linear combination cannot be duplicated by a linear combination of any other set of signal vectors
containing fewer than \( M \) vectors. Thus, using this representation, at most \( M - 1 \) angles-of-arrival can be estimated unambiguously. However, we are able to use a more general signal representation which involves estimating the relative phase offset of each signal. This phase offset is non-zero because over a short observation period, such as those corresponding to the blocks of data for which each DFT is calculated, the narrowband signal appears to be a sinusoid which may have any phase relative to the array. Although the phase offset cancels out when estimating \( \mathbf{R} = E[\mathbf{x}\mathbf{x}'] \) for the signal vectors of equation 1.1, for the real signal vectors formed by concatenating the real and imaginary parts of the complex signal vectors the phase offset remains. This different signal representation allows us to estimate up to \( 2M - 1 \) angles-of-arrival unambiguously but at the cost of having to estimate all of the phase offsets.
Chapter 2

Robust Estimation of Structured Covariance Matrices

Most narrowband array processing algorithms use an estimate of the spatial correlation matrix either implicitly or explicitly. Consequently, a good estimate of the spatial correlation matrix is essential for reliable performance, and the better the estimate is, the better performance we can expect. We will examine methods of using all the information known about the structure of the spatial correlation matrix to obtain good estimates of it in both Gaussian and non-Gaussian noise.

Assuming that the signals are narrowband about a common known frequency and distant enough to be planar when they arrive at the array, for an $M$ sensor array the elements of the received signal $s$ propagating in the direction $-k$ at speed $c$ are

$$s_m = \exp\left\{-j2\pi \frac{f}{c} (z_m \cdot k)\right\}, \ m = 1, \ldots, M$$

where $f$ is the frequency of the signal and $z_m$, $m = 1, \ldots, M$, are the sensor positions. If the noise is uncorrelated with the signals, then the form of the spatial correlation matrix is

$$R = \sigma_n^2 Q + SC^\prime$$

(2.1)

where $Q$ is the noise correlation matrix normalized so that $\sigma_n^2$ is the noise power, $S$ is an $M \times P$ matrix whose columns are the signal vectors, $C$ is the matrix of correlations between the signals, and $S^\prime$ is the complex conjugate transpose of $S$. 
For our simulations we used a linear array of equally spaced sensors, also known as a uniform linear array. For this array geometry the signal vectors have the form

\[ s = \begin{bmatrix} 1 & e^{-j\alpha} & e^{-j2\alpha} & \cdots & e^{-j(M-1)\alpha} \end{bmatrix}^T \]

where \( \alpha = 2\pi \frac{d}{\lambda} \sin \theta \) is a function of the signal wavelength \( \lambda \), the angle-of-arrival of the signal with respect to broadside \( \theta \), and the sensor spacing \( d \). If only white noise is present and the signals are completely incoherent with each other, then

\[ \mathbf{R} = \sigma_n^2 \mathbf{I} + \sum_{p=1}^{P} \sigma_p^2 \mathbf{s}_p \mathbf{s}_p^T \]

where \( P \) is the number of signals and \( \sigma_p^2 \) is the power of the \( p \)th signal. In this case the spatial correlation matrix is complex Toeplitz, and we may want to find the best complex Toeplitz estimate of \( \mathbf{R} \).

The usual procedure for estimating the covariance matrix \( \mathbf{R} \) when given vector samples \( \mathbf{x}_i \) of a Gaussian process is the maximum likelihood estimate

\[ \widehat{\mathbf{R}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i^T \]

where the number of samples \( N \) is also known as the time-bandwidth product. However, in general \( \widehat{\mathbf{R}} \) will not have the same structure as \( \mathbf{R} \). For example, if \( \mathbf{R} \) is known to be Toeplitz, \( \widehat{\mathbf{R}} \) is rarely Toeplitz. In such cases, one would suppose that a better estimate of \( \mathbf{R} \) would be an estimate which is always Toeplitz. The easiest way to construct a Toeplitz estimate of \( \mathbf{R} \) is to average the diagonals of \( \widehat{\mathbf{R}} \). While this estimate has been shown [21] to be closer to \( \mathbf{R} \) in Frobenius norm than \( \widehat{\mathbf{R}} \), there is no reason to suppose it to be the best Toeplitz estimate. Several methods of estimating Toeplitz matrices have been proposed [8,9,32]. We discuss a direct Toeplitz estimate of \( \mathbf{R} \), the maximum likelihood Toeplitz estimate.

Suppose we define the spatial correlation matrix in terms of the signal and noise parameters

\[ \mathbf{R}(\sigma_n^2, \sigma_1^2, \cdots, \sigma_p^2, \theta_1, \cdots, \theta_P) = \sigma_n^2 \mathbf{I} + \sum_{p=1}^{P} \sigma_p^2 \mathbf{s}_p \mathbf{s}_p^T \]

(2.2)
where the number of signals \( P \) is assumed to be known. These parameters impose a structure on \( \mathbf{R} \) which can be used to define a class of possible estimates just as has been done with the Toeplitz constraint. Thus, we can maximize over the allowable values of the signal and noise parameters to find the maximum likelihood estimate of \( \mathbf{R} \) subject to the structure of equation 2.2. This method provides a parametric estimate of \( \mathbf{R} \) which yields the desired angle-of-arrival estimates directly.

In general, the non-Gaussian problem is poorly suited for covariance matrix estimation. The obvious approach is

\[
\max_{\mathbf{R}^* \in \mathcal{R}} f_X(x|\mathbf{R}^*)
\]

where \( f_X(\cdot) \) is a multivariate non-Gaussian probability density function (p.d.f.). However, a non-Gaussian p.d.f. is often not easily parameterized in terms of the covariance matrix; this simple formulation may be unreasonable. In addition, when dealing with non-Gaussian problems, estimating the angles-of-arrival by estimation of the covariance matrix is not necessarily even approximately optimum, and optimal angle-of-arrival estimation may involve the estimation of some other statistic. So, for any given p.d.f. the optimum solution may not bear any relation to the Gaussian solution at all.

We will see that the robust estimation problem provides a solution to most of the above mentioned non-Gaussian difficulties. Covariance matrix estimation will be feasible and, even for non-Gaussian noise, the covariance matrix still contains the same information about the signals and noise as in the Gaussian case. Thus, the robust covariance matrix estimate can be used to find the angles-of-arrival in the same manner as the usual estimate.

Figure 2.1 illustrates the advantages of using a robust estimate of the spatial correlation matrix when the noise is non-Gaussian. Both beampatterns were generated with a Toeplitz estimate of \( \mathbf{R} \). However, for the black curve it was assumed that the noise was Gaussian. For the gray curve, the robust estimate for the \( \epsilon-\)
Figure 2.1: The beampatterns in this figure differ only in the spatial correlation matrix estimate which was used. The black curve was generated with the maximum likelihood Toeplitz estimate of the spatial correlation matrix assuming the noise to be Gaussian. The gray curve used the robust Toeplitz estimate for the $\epsilon$-contamination noise class. In both cases, the eigenvector method was used assuming two signals to be present. There are two signals with unit power placed at 0 degrees (broadside) and +7.5 degrees. The noise has unit power and is the worst case noise in the $\epsilon$-contamination class for $\epsilon = 0.25$ for a five sensor uniform linear array. The time-bandwidth product is 25.
contamination class was used and it resulted in a better beampattern. As can be seen, Gaussian estimates can degrade significantly in non-Gaussian noise, but robust estimates can be designed to yield good performance for a variety of noise amplitude distributions.

2.1 Estimating Structured Covariance Matrices

Burg et al. [8] have shown how to find structured maximum likelihood estimates of real covariance matrices of Gaussian processes. Although the matrices we are interested in estimating are complex, we will concentrate on real matrix estimators for the greater generality which they allow. Length $M$ complex Gaussian random vectors are actually circularly symmetric length $2M$ real Gaussian random vectors.\(^1\) For circularly symmetric Gaussian random vectors, the estimators can be designed for complex covariance matrices without any loss of generality. However, in the non-Gaussian case we will need the generality afforded by using $2M \times 2M$ real covariance matrices rather than $M \times M$ complex covariance matrices as, in general, even if a non-Gaussian random vector is circularly symmetric there is not necessarily any equivalent complex random vector. Therefore, although for Gaussian processes the structured covariance matrix estimation problem is easily derived for the complex case, we will stay with real covariance matrices to avoid confusion later on when dealing with non-Gaussian processes. Burg et al. showed that if we choose the structured estimate, $R^*$, of a real matrix $R$ so that it maximizes the p.d.f. of the samples $x_i$ assuming that the true covariance matrix is $R^*$, then we arrive at the equation

$$\begin{align*}
\text{maximize } & \quad g(x_i, R^*) = -\ln[\det(R^*)] - \frac{1}{N} \sum_{i=1}^{N} x_i^T R^{-1} x_i \\
R^* \in \mathcal{R} & \quad \end{align*}$$

\(^1\) A complex Gaussian random vector $z = x + jy$ is circularly symmetric if $E\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x^T & y^T \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \text{ so that } E\{zz^*\} = A + jB.$
where \( \mathcal{R} \) is the set of allowable \( R^* \), e.g., all positive definite Toeplitz matrices. If we let \( S = (1/N) \sum_{i=1}^{N} x_i x_i^T \), then the objective function becomes

\[
\text{maximize} \quad \left\{ g(S, R^*) = -\ln[\det(R^*)] - \text{tr}(R^{-1}S) \right\}
\]

(2.3)

In addition, Burg et al. showed that the variation of \( g(S, R^*) \) is

\[
\delta g(S, R^*) = \text{tr}[(R^{-1}SR^{-1} - R^{-1})\delta R^*]
\]

where \( \delta R^* \) is the variation of \( R^* \), i.e., the matrix of variations of the elements of \( R^* \). Burg et al. have shown that, if \( \mathcal{R} \) is a closed set of nonnegative definite symmetric matrices, then \( \mathcal{R} \) contains a maximum value of \( g(S, R^*) \). Thus, for the sets we will be considering, the maximum exists in the interior of \( \mathcal{R} \) and a necessary condition for the maximization of \( g(S, R^*) \) is that \( \delta g(S, R^*) = 0 \) for any feasible variation of \( R^* \). Therefore,

\[
\text{tr}[(R^{-1}SR^{-1} - R^{-1})\delta R^*] = 0
\]

(2.4)

Often the matrix \( \delta R^* \) has the same constraints on its structure as the matrix \( R^* \) does, i.e., any \( R^* \in \mathcal{R} \) is a valid \( \delta R^* \), such as when \( R^* \) is required to be a real Toeplitz matrix. In such a situation, we can substitute \( R^* \) for \( \delta R^* \) in equation 2.4, it simplifies to \( \text{tr}[R^{-1}S] = 2M \), and we have the following optimization problem:

\[
\text{maximize} \quad \left\{ g(S, R^*) = -\ln[\det(R^*)] \right\} \quad \text{subject to} \quad \text{tr}(R^{-1}S) = 2M
\]

(2.5)

This constrained optimization problem will be valid for all of the structures which we will consider. For our simulations we found the unconstrained optimization problem to be better suited to the readily available nonlinear optimization packages. However, the constraint \( \text{tr} \left( R^{-1}S \right) = 2M \) is useful for scaling the initial estimate which is passed to the optimization routine, and use of the constraint greatly decreased the amount of time needed to find the maximum. Routines written this way were also easily modified for the robust estimates to follow.
2.1.1 Structures for Narrowband Array Processing

We have shown how to estimate real structured covariance matrices. However, the particular matrices we are interested in are spatial correlation matrices which are complex Toeplitz matrices. Complex Toeplitz matrices can be estimated using the theory for real matrices if the appropriate structures for the real matrices can be determined. In particular, we will estimate a $2M \times 2M$ real matrix whose terms can be combined to give an $M \times M$ complex Toeplitz matrix. Let $\mathbf{R}_C$ denote an $M \times M$ complex matrix and $\mathbf{R}_R$ the corresponding $2M \times 2M$ real matrix. $\mathbf{R}_R$ can be broken down into four submatrices; namely $\mathbf{R}_R = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{bmatrix}$. To see the relationship between $\mathbf{R}_R$ and $\mathbf{R}_C$, suppose $\mathbf{R}_C$ is found by averaging the outer products of length $M$ complex vectors, then we can picture $\mathbf{R}_R$ as being formed by averaging the outer products of the length $2M$ real vectors formed by concatenating the real and imaginary parts of the complex vectors, i.e., $\mathbf{R}_R = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i^T$ implies that

$$\mathbf{R}_R = \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} \text{Re}(\mathbf{x}_i) \\ \text{Im}(\mathbf{x}_i) \end{bmatrix} \begin{bmatrix} \text{Re}(\mathbf{x}_i^T) & \text{Im}(\mathbf{x}_i^T) \end{bmatrix}$$

Thus, $\mathbf{A} = \text{Re}(\mathbf{x}_i)\text{Re}(\mathbf{x}_i^T)$, $\mathbf{B} = \text{Im}(\mathbf{x}_i)\text{Im}(\mathbf{x}_i^T)$, and $\mathbf{C} = \text{Re}(\mathbf{x}_i)\text{Im}(\mathbf{x}_i^T)$. So we see that to form $r_c(m, n)$, the $(m, n)^{th}$ element of $\mathbf{R}_C$, we simply calculate

$$r_c(m, n) = a(m, n) + b(m, n) + j(c(m, n) + c(n, m))$$

Therefore, the constraints on $\mathbf{R}_R$ so that $\mathbf{R}_C$ is complex Toeplitz are:

$$a(m, n) + b(m, n) = a(k, l) + b(k, l) \quad \text{if } |n - m| = |l - k|,$$
$$-c(m, n) + c(n, m) = -c(k, l) + c(l, k) \quad \text{if } n - m = l - k, \quad \text{(2.6)}$$
$$-c(m, n) + c(n, m) = c(k, l) - c(l, k) \quad \text{if } n - m = k - l$$

We can see that the constraints on the variation of $\mathbf{R}_R$ are satisfied by $\mathbf{R}_R$ itself, and, thus, this estimation problem can be solved by the optimization problem of either equation 2.3 or equation 2.5.
We can also estimate the signal and noise parameters directly by imposing the structure of equation 2.1 on our estimate. As this structure is defined by the signal and noise parameters, estimates for these parameters are determined automatically when the maximum likelihood structured estimate of $R$ is found. Although this structure easily allows for the estimation of the signal correlations, for our actual simulations we assumed the structure of equation 2.2. Thus, estimates were found for the same data as for the Toeplitz case and comparisons could be made between the two methods. A similar estimator has been proposed by Bresler [6] and Böhme [5] for estimating the noise power and the correlations between the signals assuming that the angles-of-arrival are known. Most previous parametric estimators such as those of Kenefic [20], Ziskind and Wax [35], and Linebarger and Johnson [22] have solved other optimization problems.

### 2.1.2 Simulations for Gaussian Noise

Determining which matrix estimate is closest to the actual matrix is difficult as there is no clear choice for the distance measure. Consequently, we choose to compare the estimates by seeing how they perform in narrowband array processing problems. Thus, the quality of the estimates is determined by the quality of the beampatterns associated with each estimate of the spatial correlation matrix.

Figure 2.2 displays the beampattern resulting from the different spatial correlation matrix estimates being compared. The same simulation data was used for each beampattern in this figure. There are two unit power signals at 0 degrees and +10 degrees impinging on a five sensor uniform linear array. The noise present also has unit power and is white and Gaussian. The time-bandwidth product is 100. Four matrix estimates were used in figure 2.2:

1. $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i^T$

2. $R_2^T$, the maximum likelihood Toeplitz estimate.
Figure 2.2: The same data was used for each plot in this figure. The black curves were generated by using Capon’s method on the appropriate spatial correlation matrix estimate, and the gray curves were generated by the eigenvector method. The matrix estimates used are: (a) $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i^*$, (b) the maximum likelihood Toeplitz estimate $\hat{R}_T$, and (c) the maximum likelihood parametric estimates $\hat{R}_2$ and $\hat{R}_4$. There are two signals with unit power placed at 0 degrees (broadside) and +10 degrees. The noise is white, Gaussian noise with unit power. The time-bandwidth product is 100.
3. $R_2^*$, the maximum likelihood parametric estimate assuming that two signals are present.

4. $R_4^*$, the maximum likelihood parametric estimate assuming four signals to be present.

When finding the parametric estimate, you can either assume that the number of signals is known or you can estimate the maximum number of signals possible. Hence, the two parametric estimates, $R_2^*$ and $R_4^*$.

For a five sensor uniform linear array, to find $R_7^*$ we must maximize a nonlinear function in 39 variables. The submatrices $A$, $B$, and $C$ of $R_R$ each have 25 elements, but $A$ and $B$ are symmetric which means that they have only 15 degrees of freedom. Also, by using the constraints in equation 2.6, several of the elements in $R_R$ can be written in terms of the other elements. So we actually only have 15 independent elements in $A$, 5 elements in $B$, and 19 elements in $C$ which cannot be written in terms of each other. To find $R_2^*$ and $R_4^*$, the optimization problems only involved the signal and noise parameters; 5 parameters for $R_2^*$ and 9 parameters for $R_4^*$.

To solve the optimization problems for $R_T^*$, $R_2^*$, and $R_4^*$, we used an unconstrained quasi-Newton routine from IMSL to minimize the objective function $g(S, R^*)$ in equation 2.3. This routine requires an initial value for the matrix estimate. For the Toeplitz estimate we started with the Toeplitz matrix found by averaging the diagonals of $\frac{1}{N} \sum_{i=1}^{N} x_i x_i^*$. For the parametric estimates, the initial noise and signal powers were set to one and the angles-of-arrival were uniformly spaced between $-90$ and $+90$ degrees. The constraint in equation 2.5 was also useful in establishing initial values to pass to the optimization routine. The constraint controls the scale of $R^*$ and can be satisfied easily by simply scaling the initial parameters — all of the variables for $R_T^*$ and just the noise and signal powers.
for $R_2^*$ and $R_4^*$. Much time was saved in the optimization process as satisfying the constraint at the outset made the initial estimate closer to the final answer.

Figure 2.2 shows the performance of two beamforming algorithms using the various estimates of $R$. Each figure shows both Capon’s method (the black curve), also known as the minimum energy or maximum likelihood method [18], and the eigenvector method (the gray curve) proposed by Johnson and DeGraaf [19]. For the eigenvector method we assumed that the number of signals was known. As can be seen, Capon’s method cannot resolve the two sources with any of the five matrix estimates. Capon’s method is a relatively poor measure of how good the matrix estimate is as it is insensitive to the quality of estimate after it gets within a certain distance of $R$ [10]. Consequently, other figures in this paper will concentrate on using the eigenvector method. The other beampatterns in this figure clearly show the advantages of using all of the information you have and finding a structured estimate when using the eigenvector method. When the estimate $\hat{R}$ is used, the sources are not resolved. The sources are resolved for the Toeplitz estimate $R_T^*$, and also for $R_2^*$ and $R_4^*$. For this data, the estimates $R_2^*$ and $R_4^*$ yield virtually identical beampatterns, so only one figure is shown for both of these estimates. These estimates use more information about the structure of $R$ than the other estimates and are able to deliver correspondingly better estimates of $R$. The eigenvector method is clearly more sensitive to the estimate of $R$ and is a much better indicator of estimate quality.

2.2 Robust Estimation

Robust techniques can be used to estimate covariance matrices of non-Gaussian processes [13,17,24,25,34]. These techniques find the estimator that minimizes the maximum asymptotic estimate variance over a class of processes. The process class is usually specified by the class of possible probability density functions (p.d.f.) of
the process samples. Two typical classes are the \( \epsilon \)-contamination class
\[
f(x) = (1 - \epsilon)\phi(x) + \epsilon h(x)
\]  
(2.7)
where \( \phi(x) \) is the zero mean, unit variance Gaussian density function, \( 0 \leq \epsilon < 1 \) is some constant describing the uncertainty of the nominal Gaussian model, and \( h(x) \) is any valid p.d.f., and the Kolmogorov class
\[
\sup_x |F(x) - \Phi(x)| < \epsilon
\]  
(2.8)
where \( \Phi(x) \) is the Gaussian distribution function, \( 0 \leq \epsilon < 1 \) is some constant, and \( F(x) = \int_{-\infty}^{x} f(\alpha) d\alpha \) is the distribution function corresponding to \( f(x) \). The \( \epsilon \)-contamination class described by equation 2.7 is by far the most common and the easiest to use, but the Kolmogorov class of equation 2.8 is known to be more important in array processing problems and is in some ways the more natural class to use [33]. Note that for either class the constant \( \epsilon \) must be known to define the class unambiguously. For the covariance matrix estimation problem we are interested in these classes in their multivariate form
\[
f(x) = (1 - \epsilon)\phi(x) + \epsilon h(x)
\]  
(2.9)
and
\[
\sup_{C \in \mathbb{C}} |F(C) - \Phi(C)| < \epsilon
\]  
(2.10)
where \( x \in \mathbb{R}^{2M} \) and \( C \) contains all the Borel-measurable convex sets in \( \mathbb{R}^{2M} \). In this case, \( F(C) \) and \( \Phi(C) \) are the integrals of their respective density functions over the set \( C \).

Let's consider the robust estimation problem. We want to find the structured maximum likelihood type estimates (M-estimates) which minimize the maximum asymptotic estimate variance for each of these classes of p.d.f.'s. Huber [17] has several results for robust M-estimates where an M-estimate is any estimate found by
\[
\min_{\mathbf{R}^*} \sum_{i=1}^{N} \rho(x_i; \mathbf{R}^*)
\]
for some arbitrary function \( \rho \). The maximum likelihood structured estimate fits into this class if we consider

\[
\rho (x; R^*) = \begin{cases} 
-\ln f_X(x|R^*) , & R^* \in \mathcal{R} \\
\infty , & R^* \notin \mathcal{R}
\end{cases}
\]

From results obtained in Huber [17, sections 4.4 and 4.6], we know that, if we want an estimator which not only minimizes the maximum asymptotic variance but is also asymptotically efficient, then the estimator should be designed for the "worst case" density in the class. Huber shows that the asymptotic variance of our estimator is \( \frac{1}{I(f)} \) where \( I(f) \) is the Fisher information with respect to the variable being estimated. Also, each of the two p.d.f. classes of equations 2.9 and 2.10 contain a density which will minimize the Fisher information \( I(f) \) for its class and these densities are unique for each class. The asymptotic variance of the estimator designed for the worst case noise is maximized in the presence of the worst case density and is less for any other density in the class. Consequently, the worst asymptotic variance for the robust estimator is \( \frac{1}{I(f)} \). Thus, if we find the density for each class which minimizes Fisher information with respect to the covariance matrix, we have found our robust estimates.

There are several ways we can simplify our search for the worst case densities. As we are estimating the covariance matrix, we will assume that all of the densities in our classes have the correct covariance matrix. Let \( f(x | R) \) denote any p.d.f. in the class and, when \( R \) is equal to the identity matrix \( I \), let \( f(X | I) = f_X(x) \), then

\[
f(x | R) = (\det(R))^{-1/2} f_X \left( R^{-1/2} x \right)
\]

If we assume that \( R \) depends differentiably on some parameter \( \theta \) (we will see later that the choice of \( \theta \) does not matter), then the Fisher information is

\[
I(f(\cdot | R)) = E \left\{ \left[ \frac{\partial}{\partial \theta} \ln f(x | R) \right]^2 \right\}
\]
\[
E \left\{ \left[ \text{tr} \left( \frac{\partial}{\partial \theta} R^{-1/2} \right) + \frac{\partial f_X(x)}{f_X(x)} \right]^2 \right\}
\]

The only part of the Fisher information which is dependent on our choice of \( f(\cdot) \) is independent of \( R \). Therefore, the same p.d.f. is always worst case irrespective of the value of \( R \), and, for the purpose of finding the worst case density, we can assume that \( R = I \). In addition, we can simplify our problem to a univariate problem once again by use of the following theorem.

**Theorem 2.1** If we assume that \( E[xx^T] = I \) and \( E[x] = 0 \), then the density minimizing Fisher information over the set of densities in either equation 2.9 or equation 2.10 is always spherically symmetric.

**Proof** Jensen's inequality states that \( E[g(x)] \geq g[E(x)] \) for any \( g \) which is a convex function. The Fisher information of \( f \) with respect to the parameter \( \theta \)

\[
I(f) = E \left[ \left( \frac{\partial}{\partial \theta} \ln[f] \right)^2 \right]
\]

is a convex function [17]. Consider the \( 2M \)-dimensional p.d.f. \( f_X \) belonging to either of the above two classes. Clearly the p.d.f.

\[
\tilde{f}(x) = \frac{1}{B} \int_{\|y\|=\|x\|} f_X(y) dy,
\]

where \( B \) is the area of the \( 2M \)-dimensional sphere of radius \( \|x\| = \sqrt{x^T x} \), is a member of the same class as \( f_X \) and is spherically symmetric. We will show that \( I(\tilde{f}) \leq I(f_X) \). \( \tilde{f} \) is a spherically symmetric p.d.f. such that \( \tilde{f}(x) \) is the average of all values of \( f_X(y) \) such that \( \|y\| = \|x\| \).

We can write \( I(\tilde{f}) = I(E[\tilde{f}]) \) where \( \tilde{f} \) is chosen from the class \( \mathcal{F} \) of rotations of \( f_X \).

\[
\mathcal{F} = \{ \tilde{f} \mid \exists \text{ an orthogonal matrix } H \in \mathcal{O} \text{ s.t. } \tilde{f}(x) = f_X(Hx), \forall x \in \mathbb{R}^{2M} \}
\]
Assign a distribution to $\mathcal{F}$ such that the distribution of $\tilde{f}$ over $\mathcal{F}$ is uniform, i.e., all are equally likely. Therefore, $I(E(\tilde{f})) = I(\tilde{f})$.

Consider $E[I(f)]$ for any $\tilde{f} \in \mathcal{F}$. Note that the Fisher information with respect to the covariance matrix is the same for any $\tilde{f}$ in this class since it is invariant to rotations when $R = I$. This invariance is obvious from the definition of $I(f)$ if a change of variables is made from $x$ to $Hx$.

Thus, $E[I(\tilde{f})] = I(\tilde{f})$ for any $\tilde{f} \in \mathcal{F}$, so choose $I(f_X)$. Therefore, as $I(E[\tilde{f}]) \leq E[I(\tilde{f})]$, then $I(\tilde{f}) \leq I(f_X)$, and the worst case density is always spherically symmetric as long as all the members of $\mathcal{F}$ are in the class being considered. 

As the worst case density is spherically symmetric, we need only concern ourselves with designing the estimator for spherically symmetric densities. Thus the two classes of densities which we are interested in are

$$f(r) = (1 - \epsilon)\phi(r) + \epsilon h(r), \quad 0 \leq \epsilon \leq 1$$ (2.11)

and

$$\sup_r |F(r) - \bar{F}(r)| < \epsilon, \quad 0 \leq \epsilon \leq 0.5$$ (2.12)

where $r = ||x|| = \sqrt{x^T x}$ for any $x \in \mathbb{R}^{2M}$, $\phi(r) = (2\pi)^{-M} e^{-r^2/2}$, and the distribution corresponding to the density $f(r)$ is denoted as $F(r) = C_{2M} \int_0^r f(\alpha)\alpha^{2M-1}d\alpha$ with $C_{2M} = 2(\pi)^M / \Gamma(M)$ being the volume of the $2M$-dimensional unit sphere. For the Kolmogorov class, $\epsilon$ must be less than 0.5 because of symmetry constraints. This constraint on $\epsilon$ is easiest to see in the univariate case where the distributions of any two symmetric random variables cannot differ by more than 0.5 because they both go through 0.5 at the origin.

We now just have to consider noise densities of the form

$$f(x| r^*) \propto \det(r^*)^{-1/2} f[(x^T r^{-1} x)^{1/2}]$$
where \( f(\cdot) \) is any univariate density. The density \( f_X(x| \mathcal{R}^*) \) describes the class of processes known as elliptically symmetric [26]. In this case, the objective function becomes
\[
g(x_i, \mathcal{R}^*) = -\frac{N}{2} \ln[\det(\mathcal{R}^*)] + \sum_{i=1}^{N} \ln f[(\text{tr}(\mathcal{R}^{*-1}x_ix_i^T))^{1/2}] \tag{2.13}
\]
Taking the variation of this objective function we get
\[
\delta g(x_i, \mathcal{R}^*) = -\frac{N}{2} \text{tr}(\mathcal{R}^{*-1} \delta \mathcal{R}^*) - \frac{1}{2} \sum_{i=1}^{N} \frac{f'(\text{tr}(\mathcal{R}^{*-1}x_ix_i^T))^{1/2})[\text{tr}(\mathcal{R}^{*-1}x_ix_i^T)]^{-1/2}\text{tr}[\mathcal{R}^{*-1}(\delta \mathcal{R}^*)\mathcal{R}^{*-1}x_ix_i^T]}{f[(\text{tr}(\mathcal{R}^{*-1}x_ix_i^T))^{1/2}]}
\]
The techniques used by Burg et al. [8] can be modified to show that the solution will be in the interior of \( \mathcal{R} \) for the structures we are considering. Thus, we can require that \( \delta g(x_i, \mathcal{R}^*) = 0 \) at the maximum for any feasible variation of \( \mathcal{R}^* \). If the structural constraint on \( \delta \mathcal{R}^* \) is satisfied by \( \mathcal{R}^* \), as is the case for the structures we are considering, we can once again replace \( \delta \mathcal{R}^* \) by \( \mathcal{R}^* \) and the equation remains true. The constraint then becomes
\[
-\frac{N}{2} \text{tr}(\mathcal{I}) - \frac{1}{2} \sum_{i=1}^{N} \frac{f'(\text{tr}(\mathcal{R}^{*-1}x_ix_i^T))^{1/2})[\text{tr}(\mathcal{R}^{*-1}x_ix_i^T)]^{1/2}}{f[(\text{tr}(\mathcal{R}^{*-1}x_ix_i^T))^{1/2}]} = 0
\]
or
\[
\frac{1}{N} \sum_{i=1}^{N} u[(\text{tr}(\mathcal{R}^{*-1}x_ix_i^T))^{1/2}] = 2M \tag{2.14}
\]
where \( u(r) = -\frac{f'(r)}{f(r)}r \). We note that for Gaussian processes this simplifies to \( \text{tr}(\mathcal{R}^{*-1} \mathcal{S}) = 2M \). So, for this more general case we have the optimization problem: maximize equation 2.13 subject to equation 2.14. For the robust estimator, the density chosen is the worst case density in whichever of the two classes of densities is being considered.

2.2.1 The \( \epsilon \)-contamination Class

We are interested in finding the robust maximum likelihood structured covariance matrix estimate for the class of densities in equation 2.11 and want to minimize
the worst performance of our estimator over this class of density functions. As has been discussed previously, this minimax problem is equivalent to designing the estimator in equation 2.13 for the p.d.f. which minimizes the Fisher information for that estimator over the entire class of allowable p.d.f.s.

The worst case density for the \( \epsilon \)-contamination class has been given by Huber [16,17] as

\[
    f_0(r) = \begin{cases} 
    (1 - \epsilon)\phi(a) \left( \frac{a}{r} \right)^2, & 0 \leq r \leq a \\
    (1 - \epsilon)\phi(r), & a < r < b \\
    (1 - \epsilon)\phi(b) \left( \frac{b}{r} \right)^2, & r \geq b 
    \end{cases}
\]

where

\[
    a = \begin{cases} 
    0, & p - \kappa \leq 0 \\
    \sqrt{p - \kappa}, & p - \kappa > 0 
    \end{cases}
\]

\[
    b = \sqrt{p + \kappa}
\]

and \( \kappa \) is chosen so that \( C_{2M} \int_0^\infty f_0(r)r^{2M-1}dr = 1 \). In [17, page 236] a table is given which provides values of \( a \) and \( b \) for various values of \( \epsilon \) and \( M \) (in the table \( p = 2M \)). For our examples we chose \( \epsilon = 0.1 \) and \( M = 5 \) yielding values of \( a = 2.235 \) and \( b = 3.874 \).

Figure 2.3 shows how this worst case density compares to the nominal Gaussian density. As both densities are spherically symmetric and only functions of \( r = \|x\| \), the multivariate densities can be described by the densities of \( r \) shown in this figure. The black curve is the worst case p.d.f. for the \( \epsilon \)-contamination class for \( \epsilon = 0.1 \) and \( M = 5 \). The gray curve is the 10-dimensional Gaussian p.d.f and is shown for comparison. As can be seen, the worst case density is heavier in the locations which hurt variance estimation, i.e., in the tail and near the origin.
Figure 2.3: The gray curve is the density of the radius of the 10-dimensional multivariate white Gaussian density. The black curve is the density of the radius of the worst case density for the ε-contamination class for ε = 0.1 and M = 5.

For this \( f_0(r) \), \( u_0(r) \) is

\[
\begin{align*}
  u_0(r) &= -\frac{f_0'(r)}{f_0(r)} = \\
  &= \begin{cases} 
    a^2, & 0 \leq r \leq a \\
    r^2, & a < r < b \\
    b^2, & r \geq b
  \end{cases}
\end{align*}
\]

and substituting this into equations 2.13 and 2.14 yields the robust estimator

\[
\text{maximize} \quad \begin{align*}
  \mathbf{R}^* \in R \\
  -\ln[\det(\mathbf{R}^*)] - \frac{1}{N} \sum_{0 \leq r_i \leq a} a^2 \left[ 2 \ln \left( \frac{r_i}{a} \right) + 1 \right] - \\
  \frac{1}{N} \sum_{a < r_i < b} r_i^2 - \frac{1}{N} \sum_{r_i \geq b} b^2 \left[ 2 \ln \left( \frac{r_i}{b} \right) + 1 \right]
\end{align*}
\]

(2.15)
or also the equivalent constrained optimization problem

\[
\begin{align*}
\text{maximize} & \quad -\ln[\det(R^*)] - \frac{2}{N} \sum_{0 \leq r_i \leq a} a^2 \ln \left( \frac{r_i}{a} \right) - \frac{2}{N} \sum_{r_i \geq b} b^2 \ln \left( \frac{r_i}{b} \right) \\
\text{subject to} & \quad \frac{1}{N} \sum_{i=1}^{N} u_{0}(r_i) = 2M
\end{align*}
\]

(2.16)

where \( r_i = [\text{tr}(R^{-1}_i x_i x_i^T)]^{1/2} \).

### 2.2.2 The Kolmogorov Class

We have derived the worst case density for the Kolmogorov class by using the same variational methods that Huber used to find the worst case density for the \( \epsilon \)-contamination class. Details of this derivation are provided in appendix A. For the particular case of \( \epsilon = 0.1 \) and \( M = 5 \) we have

\[
f_0(r) = \begin{cases} 
\phi(a) \left( \frac{a}{r} \right)^{2M} & , 0 \leq r \leq a \\
\phi(a) \left( \frac{a}{r} \right)^{2M} \cos^2 \left( \frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}} \right) & , a \leq r \leq \sqrt{2M} \\
\phi(b) \left( \frac{b}{r} \right)^{2M} \cos^2 \left( \frac{\omega_2}{2} \ln \frac{r}{\sqrt{2M}} \right) & , \sqrt{2M} \leq r \leq b \\
\phi(b) \left( \frac{b}{r} \right)^{2M} & , r \geq b
\end{cases}
\]

with \( a = 2.267227, b = 4.167853, \omega_1 = 4.071497, \omega_2 = 5.025638, k_1^2 = 6.725245, \) and \( k_2^2 = 14.180138 \).

Figure 2.4 shows how this worst case density for \( \epsilon = 0.1 \) and \( M = 5 \) compares to the nominal Gaussian density. Once again, the actual densities which are shown are the densities of \( r \). This worst case density is also heavier in the tail and near the origin. At \( a \) the difference in the distributions is \( F_0(r) - \Phi(r) = \epsilon \) and at \( b \) the
Figure 2.4: The gray curve is the density of the radius of the 10-dimensional multivariate white Gaussian density. The black curve is the density of the radius of the worst case density for the Kolmogorov class for $\epsilon = 0.1$ and $M = 5$.

difference is also $\epsilon$ but the other way, $\Phi(r) - F_0(r) = \epsilon$. Thus the bound on the difference of the distribution functions is met twice.

This density yields

$$u_0(r) = \begin{cases} 
  k_1^2 & , \quad 0 \leq r < a \\
  2M + \omega_1 \tan \left( \frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}} \right) & , \quad a \leq r \leq \sqrt{2M} \\
  2M + \omega_2 \tan \left( \frac{\omega_2}{2} \ln \frac{r}{\sqrt{2M}} \right) & , \quad \sqrt{2M} \leq r \leq b \\
  k_2^2 & , \quad r \geq b
\end{cases}$$

Once again by substituting this into equations 2.13 and 2.14 we get the robust estimators for this class of densities. As can be seen in appendix A, the function
$w_0(r)$ is important not only for defining the constraint in the robust estimator, but also for determining the worst case density.

2.2.3 Simulations for Non-Gaussian Noise

For our non-Gaussian simulations the setup is identical to those simulations done for Gaussian noise except for the noises which were used. Once again we used a five sensor uniform linear array and had two signals at 0 degrees and $+10$ degrees. For figure 2.5 the noise is the worst case noise for the $\epsilon$-contamination class with $\epsilon = 0.1$. Similarly, for figure 2.6 the noise is the worst case noise for the Kolmogorov class with $\epsilon = 0.1$. The four matrix estimates which are used in figures 2.5 and 2.6 are $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i'$, and the robust versions of $R_7^*$, $R_2^*$, and $R_4^*$. The robust estimates used in each figure are the ones corresponding to the p.d.f. class that the noise was worst case for. We used the same optimization software as for the Gaussian case and just varied the function to be optimized. Again we scaled our initial estimates to satisfy the constraint in the constrained version of the optimization problem. Only the beampatterns for the eigenvector method are shown in these figures as it is more sensitive to the matrix estimate than is Capon’s method.

The initial estimates for $R_2^*$ and $R_4^*$ are the same as in the Gaussian case — unit power for the signals and noise and equally spaced angles-of-arrival between $-90$ and $+90$ degrees. The initial estimate for $R_7^*$ consists of the robust equivalent of averaging the diagonals of the sample covariance matrix. For the main diagonal value, we found the robust estimate of the variance taking all of the elements of all of the sample vectors together as the samples to calculate the variance estimate from. The technique of robust variance estimation is covered by Huber [17, chapter 5]. The values of the other diagonals can be calculated similarly by taking, for the $j^{th}$ diagonal, all pairs of elements in each sample vector whose positions in their
Figure 2.5: The same data was used for each plot in this figure. The curves were generated by the eigenvector method assuming that two signals are present. The matrix estimates used are: (a) \(\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i^T\), (b) the robust Toeplitz M-estimate \(R_T^2\), and the robust parametric M-estimates (c) \(R_2^*\) and (d) \(R_4^*\). There are two signals with unit power placed at 0 degrees (broadside) and +10 degrees. The noise has the worst case density for the \(\epsilon\)-contamination class and is white with unit power. The time-bandwidth product is 100.
Figure 2.6: The same data was used for each plot in this figure. The curves were generated by the eigenvector method assuming that two signals are present. The matrix estimates used are: (a) $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i^T$, (b) the robust Toeplitz M-estimate $R_T$, and the robust parametric M-estimates (c) $R_r^*$ and (d) $R_r^*$. There are two signals with unit power placed at 0 degrees (broadside) and +10 degrees. The noise has the worst case density for the Kolmogorov class and is white with unit power. The time-bandwidth product is 100.
sample vector differ by $j$ and remembering that

$$E[xy] = \frac{1}{4} \left[ E[(x + y)^2] - E[(x - y)^2] \right]$$

Thus, the covariance of $x$ and $y$ can be calculated using variances, and it is trivial to generalize this to the case that $x$ and $y$ are complex.

For figures 2.5 and 2.6, beampatterns should only be compared within the same figure and not between these figures. As $\epsilon$ is allowed less of a range for the Kolmogorov class than the $\epsilon$-contamination class, $\epsilon = 0.1$ allows more corruption in the Kolmogorov class, as is evident in figures 2.3 and 2.4. Thus, the beampatterns are worse in figure 2.6 than in figure 2.5 because the noise contamination is worse. The simple average is seen to yield extremely poor estimates in both of these figures. The robust estimates all perform well, and we see that as more structural information is used, the better the matrix estimate is. It should be noted that, as these are the worst case noises, these estimates will perform better in the presence of any other p.d.f. in their classes. Thus, they will do better if the noise is Gaussian and, in fact, already compare favorably with the Gaussian estimates for figure 2.2. Thus, little performance is lost when using these robust estimators, even if the noise is Gaussian, and much is to be gained when the noise is non-Gaussian.
Chapter 3

On Resolving \(2M - 1\) Signals with an \(M\) Sensor Uniform Linear Array

The problem of determining how many narrowband signals can be resolved with a linear array of equally spaced sensors (a uniform linear array) has been examined in great detail [7,28,31] and has been determined to be at most one less than the number of sensors. A subtle aspect of the usual problem formulation is the representation of narrowband signals as complex-valued signals. An equivalent, and seemingly less convenient, representation of a narrowband signal is as a two-vector consisting of the real and imaginary (in-phase and quadrature) components. While these representations are equivalent, the mathematical operations they permit are quite different. Chief among these is multiplication; the four constituents of complex multiplication — real×real, imag×real, real×imag, and imag×imag — cannot be individually extracted from the product of two complex numbers. In the real representation, these products are easily determined. This richness can result in signal processing gains, here expressed as determining the directions and amplitudes of more signals than sensors in a uniform linear array.

The usual signal model [18] has plane-wave signals propagating toward a uniform linear array in the presence of additive noise; they are assumed to be narrowband about a common known frequency and the noise is assumed to be statistically independent of the signals and spatially white. If there are \(N\) “snapshots”, the output samples from the sensors are of the form

\[
x_i = n_i + \sum_{p=1}^{P} a_{i,p} s_p, \quad i = 1, \ldots, N
\]  

(3.1)
where \( \mathbf{n}_i \) is a vector of the noise samples, \( P \) is the number of signals, \( a_{i,p} \) is a complex number representing the amplitude and relative phase of the \( p^{th} \) signal for the \( i^{th} \) snapshot, and 
\[
\mathbf{s}_p = \begin{bmatrix} 1 & e^{j\theta_p} & \cdots & e^{j(M-1)\theta_p} \end{bmatrix}^T.
\]
The angle-of-arrival for the \( p^{th} \) signal can be determined from 
\[
\theta_p = -2\pi(d/\lambda) \sin \psi_p
\]
where \( d \) is the spacing between the sensors, \( \lambda \) is the wavelength of the signals, and \( \psi_p \) is the angle-of-arrival of the \( p^{th} \) signal relative to broadside. \( N \) is also known as the time-bandwidth product.

The complex spatial correlation matrix of the sensor outputs has the form

\[
\mathbf{R} = \sigma_n^2 \mathbf{I} + \mathbf{S} \mathbf{C} \mathbf{S}^T
\]  
(3.2)

where \( \sigma_n^2 \) is the noise power, \( \mathbf{I} \) is the identity matrix, \( \mathbf{S} \) is an \( M \times P \) matrix whose columns are \( \mathbf{s}_p \), \( \mathbf{S}' \) is the conjugate transpose of \( \mathbf{S} \), and \( \mathbf{C} \) is the matrix of correlations between the signals. We will be assuming that \( \mathbf{C} \) has full rank so as to avoid the complications which result from having completely coherent signals.

In our other works [33,34] we have had reason to estimate \( 2M \times 2M \) real correlation matrices rather than the usual \( M \times M \) complex ones. For real correlation matrices the form of the spatial correlation matrix is

\[
\tilde{\mathbf{R}} = \frac{\sigma_n^2}{2} \mathbf{I} + \tilde{\mathbf{S}} \tilde{\mathbf{C}} \tilde{\mathbf{S}}^T
\]  
(3.3)

which is essentially the same as in equation 3.2, where \( \tilde{\mathbf{S}} \) is now the \( 2M \times P \) matrix whose columns are length \( 2M \) real signal vectors

\[
\tilde{\mathbf{s}}_p = \begin{bmatrix} \text{Re}\{\mathbf{s}_p\} \\ \text{Im}\{\mathbf{s}_p\} \end{bmatrix}
\]

formed by concatenating the real and imaginary parts of the complex signal vectors \( \mathbf{s}_p \) defined previously. Again we will assume that the signal correlation matrix \( \tilde{\mathbf{C}} \) has full rank.

Note that the representations in equations 3.2 and 3.3 are equivalent if the data \( \mathbf{x}_i \) are circularly symmetric. A complex random vector \( \mathbf{z} = \mathbf{u} + j\mathbf{v} \) is said to be circularly symmetric if \( \mathbf{u} \) and \( \mathbf{v} \) have the properties that 
\[
\mathbb{E}\{\mathbf{u}\mathbf{u}^T\} = \mathbf{A} = 
\]
\[ \mathbb{E}\{vv^T\} \text{ and } \mathbb{E}\{vu^T\} = B \text{ satisfies } B^T = -B. \] Under these circumstances, the joint covariance matrix of \( u \) and \( v \) has the form [15]

\[
\mathbb{E}\left\{\begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} u^T & v^T \end{bmatrix}\right\} = \begin{bmatrix} \mathbb{E}\{uu^T\} & \mathbb{E}\{uv^T\} \\ \mathbb{E}\{vu^T\} & \mathbb{E}\{vv^T\} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A & -B \\ B & A \end{bmatrix},
\]

and the complex covariance matrix of \( z \) is

\[
\mathbb{E}\{zz'\} = \mathbb{E}\{uu^T + vv^T\} + j\mathbb{E}\{vu^T - uv^T\} = A + jB.
\] (3.5)

Thus, two components — \( A \) and \( B \) — specify both representations and each component can be found from either of them. If we assume that equation 3.4 holds for \( \widetilde{R} \) in equation 3.3, then \( \widetilde{R} \) can always be determined from \( R \), but, if these assumptions do not hold, then \( \widetilde{R} \) cannot be found. However, because of the \( \tilde{s}_p \), \( \widetilde{R} \) does not generally have the form of equation 3.4 and cannot be determined from \( R \). Thus information contained in \( \widetilde{R} \) might be lost when \( R \) is used instead, and there may be gains associated with using \( \widetilde{R} \) in place of \( R \) in order to retain all the available information.

In the usual complex model, the addition of each signal vector increases the rank of the signal part of \( R \) by one and no more than \( M - 1 \) signals can be resolved. This increase occurs because any \( P \) distinct signal vectors are linearly independent for \( P \leq M \). If the same properties hold for the real signal representations, then we may be able to resolve up to \( 2M - 1 \) signals. Thus, a substantial gain in the number of resolvable signals might be realized by using a slightly different signal model than usual, i.e., real signals instead of complex. By the number of "resolvable" signals we mean the number of signals whose directions of propagation can be determined by the array processing algorithm if the signals are sufficiently strong and well-spaced. The "resolution" problem we are addressing in this paper concerns the theoretical maximum of this number. We are not concerned with closeness of the angles-of-arrival of these signals or with their signal-to-noise ratios; we assume that these factors are not preventing us from resolving the signals.
3.1 Determining the Maximum Number of Resolvable Signals

Using Schmidt's [28] terminology, the signal vectors $s_1, \ldots, s_{J+1}$ are rank $J$ ambiguous if and only if they are linearly dependent and no subsets of these vectors are linearly dependent. Note that linear arrays always have a rank 1 ambiguity, as a signal vector appears the same as any other signal vector contained in the same cone of ambiguity. The cone of ambiguity contains all vectors which form the same angle-of-arrival relative to the axis of the linear array. We will only concern ourselves with higher order ambiguities. If an array possesses a rank $J$ ambiguity for a set of vectors $s_1, \ldots, s_{J+1}$, then $s_1, \ldots, s_J$ can be combined to form $s_{J+1}$. Therefore, if there are $J$ or more signals, we cannot reliably estimate their angles-of-arrival; however, we can for $J - 1$ or fewer signals. We will use the rank ambiguity of a uniform linear array to determine how many signals can be extracted with $M$ sensors.

For a uniform linear array, up to $M$ complex signal vectors form a linearly independent set of vectors. The matrix formed by the $M$ vectors is of the Vandermonde form

$$
\begin{bmatrix}
1 & \cdots & 1 \\
e^{i\theta_1} & \cdots & e^{i\theta_M} \\
e^{i2\theta_1} & \cdots & e^{i2\theta_M} \\
\vdots & & \vdots \\
e^{i(M-1)\theta_1} & \cdots & e^{i(M-1)\theta_M}
\end{bmatrix}
$$

and has the determinant $\prod_{m>n}^{M} \left(e^{i\theta_m} - e^{i\theta_n}\right)$ [27] which is non-zero as long as all of the angles-of-arrival are different. Therefore, these signal vectors are rank $M$ ambiguous and up to $M - 1$ signals can be estimated without any ambiguities other than those resulting from the cone of ambiguity.

Suppose we now consider the real signal vectors $\bar{s}_p = [1 \cos(\theta_p) \cdots \cos((M-1)\theta_p) \ 0 \ \sin(\theta_p) \ \cdots \ \sin((M-1)\theta_p)]^T$ which are found by concatenating the
real and imaginary parts of the complex signal vectors $s_p$. For different angles-of-
arrival, any $2M - 1$ of these vectors form a linearly independent set of vectors (see Appendix B), and these signal vectors are rank $2M - 1$ ambiguous. Therefore, if
it is possible to use these vectors, we can reliably estimate up to $2M - 2$ signals — twice the previous number.

The major limitation in using these signal vectors is the assumption that the phase offsets of all the signals are zero. This assumption is unrealistic as the signals will impinge on the array with phase offsets varying between $0$ and $2\pi$, and these signal vectors cannot be used. The phase offsets are irrelevant in the complex case and disappear when the outerproducts of the data samples are taken in order to estimate $R = E\{xx'\}$. However, in the real case, the phase offsets are very important and must be estimated. Real signal vectors which explicitly incorporate phase are of the form $\tilde{s}_p = [\cos(\alpha_p) \cos(\alpha_p + \theta_p) \ldots \cos(\alpha_p + (M-1)\theta_p) \sin(\alpha_p) \sin(\alpha_p + \theta_p) \ldots \sin(\alpha_p + (M-1)\theta_p)]^T$. Assuming that all of the angles-of-arrival are different and that the phase offsets, $\alpha_p$, are independent and uniformly distributed on $[0, 2\pi)$, any $2M$ of these vectors are linearly independent with probability one (see Appendix C). That is, there are sets containing $2M$ of these vectors which are linearly dependent, but, as a subset of all possible sets containing $2M$ of these vectors, they have measure zero. The assumption of independent, uniformly distributed phase offsets is reasonable as we are assuming that no phase offset is more likely than another and the phase offset of one signal is not affected by that of another signal.

### 3.2 An Algorithm to Estimate $2M - 1$ Angles-of-Arrival

Suppose we know the number of signals and wish to find the maximum likelihood estimate of the signals' parameters: their angles-of-arrival, their amplitudes, and their phases. If the noise is assumed to be additive, white, and Gaussian, the
problem simplifies to a least-squares problem

\[
\min_{\Theta, \Lambda, \Omega} \sum_{i=1}^{N} \left( \tilde{x}_i - \sum_{p=1}^{P} \tilde{s}(\theta_p, \sigma_p, \alpha_{i,p}) \right)^T \left( \tilde{x}_i - \sum_{p=1}^{P} \tilde{s}(\theta_p, \sigma_p, \alpha_{i,p}) \right)
\]

There are \( N \) samples denoted by \( \tilde{x}_i, i = 1, \ldots, N \), which are real vectors formed by concatenating the real and imaginary parts of the complex data samples in equation 3.1. The signals are of the form \( \tilde{s}(\theta_p, \sigma_p, \alpha_{i,p}) = \sigma_p \cdot [\cos(\alpha_{i,p}) \cos(\alpha_{i,p} + \theta_p) \cdots \cos(\alpha_{i,p} + (M-1)\theta_p) \sin(\alpha_{i,p}) \sin(\alpha_{i,p} + \theta_p) \cdots \sin(\alpha_{i,p} + (M-1)\theta_p)]^T \). The spaces being minimized over are \( \Theta = \{\theta_1, \cdots, \theta_P | -\pi \leq \theta_p \leq \pi, p = 1, \ldots, P; \theta_i \neq \theta_j \text{ for all } i \neq j \} \), \( \Lambda = \{\alpha_{i,p} | 0 \leq \alpha_{i,p} < 2\pi; i = 1, \ldots, N; p = 1, \ldots, P \} \), and \( \Omega = \{\sigma_1, \cdots, \sigma_P | \sigma_p > 0, p = 1, \ldots, P \} \). Note that because the signals are narrowband and not necessarily pure sinusoids, the phase offsets \( \alpha_{i,p} \) for the signals are allowed to vary from sample to sample. We are assuming that for each signal the amplitude and angle-of-arrival remain constant during the observation period. The optimization problem in equation 3.6 is simply a nonlinear least-squares problem which fits the estimated parameters to the data.

Figures 3.1–3.3 contain histograms of the results obtained by using this method on the output of a two-sensor array with 100 independent simulations per figure. All histograms have 90 bins over the range \(-90^\circ\) to \(90^\circ\). The nonlinear least-squares routine NL2SOL [12] was chosen to solve equation 3.6 because it performs well for both large and small residual problems, i.e., for both low and high signal-to-noise ratios (SNR) [11]. The performances of many similar routines suffer for the low SNR case when there is a large difference between the data and the best signal match. For the initial parameter values which were passed to NL2SOL, we set the signal amplitudes to one, the phase offsets all to zero, and chose equally spaced angles-of-arrival between \(-90^\circ\) and \(90^\circ\). The SNR is the ratio of the signal power at one sensor to the noise power at one sensor. For one signal the real spatial correlation matrix can be written as \( \tilde{R} = \frac{\sigma^2}{2} I + \frac{\sigma^2}{2} \tilde{s} \tilde{s}^T \) where \( \tilde{s} = [\cos(\alpha) \cos(\alpha + \theta) \cdots \cos(\alpha + (M-1)\theta) \sin(\alpha) \sin(\alpha + \theta) \cdots \sin(\alpha + (M-1)\theta)]^T \).
Figure 3.1: Histogram of the bearing estimate obtained from equation 3.6 for a 2 sensor array with spacing $\lambda/2$. There is one signal present at $20^\circ$ with 20 dB SNR, and the time-bandwidth product is 25.

and the SNR is $\sigma_y^2/\sigma_n^2$. Because the initial estimates passed to NL2SOL were not necessarily close to their correct values, there were a few trials for which NL2SOL did not converge due to these poor starting points. These trials were not used to generate the histograms, and each histogram represents 100 trials for which NL2SOL determined that convergence had been obtained.

For the histogram in figure 3.1 there was only one signal at $20^\circ$ with a SNR of 20 dB and the time-bandwidth product was 25. Previous methods are able to easily estimate the angle-of-arrival of one signal with 2 sensors. It is seen that our method also has no trouble with one signal. In fact, the difference between the
Figure 3.2: Histograms of the bearing estimates obtained from equation 3.6 for a 2 sensor array with spacing $\lambda/2$. There are two signals present at $5^\circ$ and $20^\circ$ with 20 dB SNR for both of them, and the time-bandwidth product is 25.

The mean of the estimates of the angle-of-arrival and the true value of $20^\circ$ is only $0.05^\circ$ and the standard deviation of the estimates is only $0.426^\circ$.

As seen in figures 3.2 and 3.3, however, our method can estimate the angles-of-arrival of 2 or 3 signals with a 2 sensor array, whereas previous methods are limited to one angle-of-arrival. For figure 3.2 the time-bandwidth product is again 25 and there are two signals at $5^\circ$ and $20^\circ$, both with SNR's of 20 dB. For each simulation two angles-of-arrival were generated. We ordered these and generated two histograms; one for the smallest angle from each simulation and one for the largest. Thus, the histogram in figure 3.2 has two different shadings, i.e., one for each of the two individual histograms which were generated. For the bearings
where the histograms overlapped, the individual histograms are summed to produce a total histogram indicating the probability that any bearing will be chosen by the algorithm as one of the two angles. The means of the estimates are $6.33^\circ$ for the smaller estimates (an error of $1.33^\circ$) and $18.24^\circ$ for the larger estimates (an error of $1.71^\circ$). The standard deviation of the smaller valued estimates is $2.456^\circ$ and for the larger valued estimates is $2.137^\circ$.

Figure 3.3 displays a histogram for 3 signals at $-15^\circ$, $5^\circ$, and $20^\circ$ with $20$ dB SNR and a time-bandwidth product of 25. This histogram was generated in an identical manner to that of figure 3.2 except that 3 histograms were created and summed. As can be seen in figure 3.3, our algorithm has little trouble estimating...
3 angles-of-arrival with a 2 sensor array. The means of the estimates are still very good: \(-12.66^\circ\) for the smallest estimates (an error of \(2.34^\circ\)), \(5.93^\circ\) for the middle estimates (an error of \(0.93^\circ\)), and \(19.10^\circ\) for the largest estimates (an error of \(0.9^\circ\)). The standard deviation of the estimates has increased some, though, resulting in \(8.959^\circ\) for the smallest estimates from each simulation, \(4.409^\circ\) for the middle estimates, and \(6.557^\circ\) for the largest estimates.

In summary, performance with a 2 sensor array is extremely good for numbers of signals ranging from 1 to the maximum 3. There is some increase in the variance of the estimates as the number of signals increases due mainly to the larger number of parameters which must be estimated. When three signals are present, we are operating near the maximum number of parameters which can be estimated given the number of data samples available. Therefore, the noise affects the solution much more and the signals’ angles-of-arrival are not as accurate. The increase in the variances of the angle-of-arrival estimates is due partly to there being so many more parameters to estimate and the algorithm being limited by machine precision as to how close the estimates can be. The more parameters there are per equation, the larger the error due to the limits of machine precision.
Chapter 4

Array Processing in the Presence of Arbitrary Noise Amplitude Distributions

The typical narrowband array processing algorithm computes the Fourier transform of the sensors' outputs twice. For the first transform, the outputs of each sensor are windowed and the transform at the particular frequency of interest is calculated for each block of data. The spatial correlation matrix is then estimated, and the second transform involves estimating the spectrum corresponding to this matrix. Robust techniques have been used to estimate covariance matrices of non-Gaussian processes when the process is known to belong to a certain class of non-Gaussian processes [13,17,24,25,34]. These techniques find the estimator that minimizes the maximum asymptotic estimate variance over a class of processes. The process class is usually specified by the class of possible probability density functions (p.d.f.) of the process samples. Two typical classes are the \( \epsilon \)-contamination class

\[
f(x) = (1 - \epsilon)\phi(x) + \epsilon h(x), \tag{4.1}
\]

where \( \phi(x) \) is the Gaussian density function, \( 0 \leq \epsilon < 1 \) is some constant describing the uncertainty of the nominal Gaussian model, and \( h(x) \) is any valid p.d.f., and the Kolmogorov class

\[
\sup_x |F(x) - \Phi(x)| < \epsilon, \tag{4.2}
\]

where \( \Phi(x) \) is the Gaussian distribution function, \( 0 \leq \epsilon < 1 \) is some constant, and \( F(x) = \int_{-\infty}^{x} f(\alpha)d\alpha \) is the distribution function corresponding to \( f(x) \). The \( \epsilon \)-contamination class described by equation 4.1 is by far the most common and the easiest to use, but the Kolmogorov class of equation 4.2 will be more important in
this work and we shall show it to be the more natural class to use. Note that for either class the constant $\epsilon$ must be known before the estimation can be performed. A major drawback of robust methods is that $\epsilon$ is almost never known \textit{a priori} and is frequently difficult to derive or to estimate accurately. Additionally, some types of classes may suit a particular problem better than others, but the easiest class to work with, typically the $\epsilon$-contamination class, is usually chosen because the best class cannot be determined. We will derive \textit{explicitly} which class to use for the narrowband array processing problem and what the corresponding uncertainty parameter $\epsilon$ is.

The signals impinging on the array of sensors are assumed to be narrowband about a common, known frequency, and the noise is assumed to be statistically independent of the signals and spatially white. If there are $n$ samples, the output sample vectors taken across the array are of the form

$$\mathbf{\xi}_i = \mathbf{\eta}(iT) + \sum_{p=1}^{P} \mathbf{\nu}_p(iT), \quad i = 1, \ldots, n$$

where $\mathbf{\eta}(\cdot)$ is a vector of noise samples, $P$ is the number of signals, $\mathbf{\nu}_p(\cdot)$ are the signal vectors, and $T$ is the sampling period. We will be assuming that it is possible to sample at a rate such that the $\mathbf{\xi}_i$ are statistically independent.

Most narrowband algorithms proceed by evaluating the spectrum of the output at the frequency of the signal. The vectors $\mathbf{\xi}_i$ are windowed by forming $N$ blocks of length $K$ ($K = n/N$). Each block is Fourier transformed and the vector at the frequency $k_0$ of the signals is retained.

$$\mathbf{x}_i = \sum_{l=(i-1)(n/N)}^{i(n/N)-1} \mathbf{\xi}_l e^{-j \frac{2\pi k_0 l}{N}}, \quad i = 1, \ldots, N \quad (4.3)$$

where $K = n/N$.

These vectors, called "snapshots", are statistically independent and have the form

$$\mathbf{x}_i = \mathbf{n}_i + \sum_{p=1}^{P} a_{i,p} \mathbf{s}_p$$
where \( v_i \) is the vector of noise samples, \( a_{i,p} \) is a complex number representing the amplitude and relative phase of the \( p \)th signal for the \( i \)th snapshot, and \( s_p \) is the usual narrowband signal representation in the frequency domain.

If \( \eta(t) \) is a second-order process and none of the signals are completely correlated with any of the other signals, then all of the angles-of-arrival can be determined from the spatial correlation matrix \( R = E[xx'] \), where \( x \) is the random process whose samples are the \( x_i \) and \( x' \) denotes conjugate transpose. The usual estimate of \( R \) is

\[
\hat{R} = \frac{1}{N} \sum_{i=1}^{N} x_i x_i',
\]

which is also the maximum likelihood estimate of \( R \) if the \( x_i \) are Gaussian random vectors. Additionally, if \( x \) is Gaussian, using \( R \) to find the angles-of-arrival is, with respect to some criteria, optimal. Even if the data is non-Gaussian, the spatial correlation matrix still contains all of the information about the angles, and the usual bearing estimation algorithms can still be used if \( R \) can be estimated accurately. However, if \( x \) is non-Gaussian, \( \hat{R} \) can be a very poor estimate of the spatial correlation matrix. Thus, we have concentrated on the estimation of \( R \) in the presence of non-Gaussian noise.

### 4.1 Estimating \( R \) When the Noise is Non-Gaussian

Previous work on the estimation of covariance matrices of non-Gaussian processes has assumed that the process is known to belong to a certain class of non-Gaussian processes [13,17,25,34]. Unfortunately, robust methods require the noise amplitude densities to be contained within a class of densities and do not allow them to be arbitrary. Additionally, this class of densities must be known in order for the algorithms to work properly. Because of the special nature of narrowband array processing algorithms and the Central Limit Theorem, we can define the class and use robust algorithms regardless of the amplitude distribution of the noise.
Let's consider the process of finding array output samples in the frequency domain. Data from each sensor is broken into blocks, windowed, and Fourier transformed to find the spectral value at one particular frequency. If the original data is assumed to be statistically independent in time and to have finite variance, then the Fourier transform at one frequency can be considered to be a sum of independent, non-identically distributed random variables with finite variance; hence, the Central Limit Theorem applies. In fact, if we consider equation 4.3, each vector $\xi_i$ is weighted by a finite quantity by the transform and then all of them are summed. Thus, the multivariate Central Limit Theorem applies, and, as the number of terms in the Fourier sum tends to infinity, the sum becomes Gaussian. For finite sums, the deviation of the sum's distribution from the Gaussian distribution is bounded by

$$\sup_{\vec{x} \in \mathbb{R}^{2M}} |F(\vec{x}) - \Phi(\vec{x})| < \epsilon$$  \hspace{1cm} (4.4)

where $\epsilon$ is in general a function of the number of sensors $M$, the Fourier transform weights, the variances and absolute third moments of the data, and $K$ (the number of terms being summed). The random vector $\vec{x}$ is a real vector created by concatenating the real and imaginary parts of $x$. Equation 4.4 gives a bound on the maximum Kolmogorov distance between the distribution of $\vec{x}$ and the Gaussian distribution. Thus, the Kolmogorov distance naturally characterizes the class to be used by robust algorithms in narrowband array processing problems. This Central Limit Theorem bound enables us in a practical sense to use robust techniques to estimate the covariance matrix.

Unfortunately, the bound on the maximum Kolmogorov distance between $F(\vec{x})$ and $\Phi(\vec{x})$ in equation 4.4 depends on being able to determine $\epsilon$. Values of $\epsilon$ have been found which are tight for the univariate case [29] but become extremely loose for the multivariate case [1,2,4]. Two multivariate bounds from [1] and [4] respec-
tively are

\[ \sup_{\bar{x} \in \mathbb{R}^{2M}} |F(\bar{x}) - \Phi(\bar{x})| < c_1(M)\Gamma_{0n} \quad \text{and} \quad \sup_{C \in \mathcal{C}} |F(C) - \Phi(C)| \leq c_2(M) \frac{\rho_3}{\rho_2^{3/2}} n^{-1/2} \]  

(4.5)

where \( c_1 \) and \( c_2 \) are only functions of \( M \) the dimension of \( \bar{x} \), \( \Gamma_{0n} \) and \( \rho_3/\rho_2^{3/2} \) in their notation are functions of the absolute third and second moments of the \( \xi_i \), and \( C \) is the set of all Borel-measurable convex sets in \( \mathbb{R}^{2M} \). \( F(C) \) and \( \Phi(C) \) denote the integrals of their respective density functions over the set \( C \). Because \( c_1 \) and \( c_2 \) increase rapidly as \( M \) increases (for example \( c_1(M) \approx \frac{4}{\sqrt{3\pi}} (2M)^{9/2} (\ln((2M)^2))^{1/2} \)), these bounds can get extremely large as the number of sensors increases and are useless for an array of any length. Having a bound which is greater than one is ridiculous as, in equation 4.4, \( \epsilon = 1 \) includes all distributions. We were unable to find any multivariate bounds which were reasonable for larger \( M \) and conjecture that they may not exist unless more information is known about the distributions of the \( \bar{x}_i \)'s. This proviso is the key to determining a more precise value for \( \epsilon \).

### 4.2 A Bound on the Rate of Convergence of the Multivariate Central Limit Theorem

We can improve on previous multivariate Central Limit Theorem bounds if we restrict ourselves to the robust covariance matrix estimation problem. In [34], we showed that to minimize the asymptotic variance of the estimator we should design for the worst case density, which corresponds to the one minimizing Fisher information. Suppose we have a value for the bound in equation 4.5 and wish to find the worst case density. The actual value of \( R \) is not a factor in determining the worst case density, and we can without loss of generality assume that the worst case density is white [17]. Then, for any given \( R \), the worst case density will be the same as this whitened density except with covariance \( R \). We now show that the worst case density is always spherically symmetric.
Theorem 4.1 If we assume that $E[\overline{x}\overline{x}^T] = I$, $E[\overline{x}] = 0$, and the function $F(C)$ is the integral of the density $f(\overline{x})$ over the set $C$, then the density minimizing Fisher information over the set of densities

$$\sup_{C \in \mathcal{C}} |F(C) - \Phi(C)| < \varepsilon$$

where $C$ contains all Borel-measurable convex sets in $\mathbb{R}^{2M}$ is always spherically symmetric.

Proof The proof of this theorem is given in section 2.2. □

As the worst case density is spherically symmetric, we need only concern ourselves with finding a bound on the rate of convergence of the multivariate Central Limit Theorem when the sum has a spherically symmetric density. Having more information about the distribution enables us to find a tighter multivariate bound than those mentioned above. The following theorem is a special case of the bound on the rate of convergence of the multivariate Central Limit Theorem presented by Bhattacharya and Rao [4].

Theorem 4.2 Let $y_1, \ldots, y_K$ be independent random vectors such that their sum $\overline{x}$ is a spherically symmetric distribution $f_{\overline{x}_i}(\overline{x}_i) = f(||\overline{x}_i||) = f(r_i)$. If $E[y_k] = 0$, $E[||y_k||^2] = \sigma_k^2 > 0$, $E[||y_k||^3] = \beta_k < \infty$,

and $\Phi(r) = \int_{0 \leq ||x|| \leq r} \cdots \int \phi(x) dx$, then

$$\sup_{r} \left| \Pr \left( \left\| \frac{1}{(\sum \sigma_k^2)^{1/2}} \sum_{k=1}^{K} y_k \right\| \leq r \right) - \Phi(r) \right| \leq \frac{C (\sum \beta_k)}{(\sum \sigma_k^3)^{3/2}}$$

where $C$ is a constant upper bounded by 1.3213.

The upper limit for the value of $C$ is calculated in appendix D. As the bound in theorem 4.2 is an upper bound for any possible distributions the $y_k$ may have,
it provides a conservative estimate of the class that contains \( \bar{x} \). Having such a general bound allows us to use robust estimation techniques no matter what the original noise amplitude densities were. As long as the noise has finite variance, we know to which class of densities the transform of the signal plus noise samples belongs; all that is needed now is a method to estimate \( R \) for this class of densities.

### 4.3 Use of the Central Limit Theorem in Narrowband Array Processing

The bound in theorem 4.2 can be used to give us the needed \( \epsilon \), and, hence, the p.d.f. class for use in a robust estimation algorithm. If we assume that the \( \xi_i \) in equation 4.3 all have the same second and absolute third moments denoted by \( E[\|\xi_i\|^2] = \sigma_x^2 \) and \( E[\|\xi_i\|^3] = \beta_x \) and that

\[
\begin{align*}
ya_{l,i} &= \cos(2\pi k_0 l/K)\xi_{l,i} \\
ya_{l,i+M} &= \sin(-2\pi k_0 l/K)\xi_{l,i}, \quad i = 1, \ldots, M
\end{align*}
\]

then \( \|y_i\| = \sqrt{\cos^2(2\pi k_0 l/K) + \sin^2(-2\pi k_0 l/K)}\|\xi_i\| \) which implies that

\[\epsilon = \frac{C\beta_x}{\sigma_x^3} \cdot \frac{1}{\sqrt{K}}\]

Note that this value is scale invariant; furthermore, the general bound in theorem 4.2 is also scale invariant. Scale invariance is due to the rate of convergence to the Gaussian distribution function should not change when the data is multiplied by a constant gain.

To find the worst case density for the multivariate Kolmogorov class implied by theorem 4.2, we must find the density which minimizes the Fisher information with respect to variance. To do this we followed the method used by Huber for the \( \epsilon \)-contamination class [17]. Huber's method involves determining what properties the worst case density must have, guessing at a solution, and then showing that this guess actually is the worst case density. For the class of densities in theorem 4.2,
the worst case density is derived in appendix A. For the special case of $M = 5$ and $\epsilon = 0.1$ we have

\[
f_0(r) = \begin{cases} 
\phi(a) \left( \frac{a}{r} \right)^{k_1^2} \cos^2 \left( \frac{\omega_1}{2} \frac{r}{\sqrt{2M}} \right), & 0 \leq r \leq a \\
\phi(a) \left( \frac{a}{r} \right)^{2M} \cos^2 \left( \frac{\omega_1}{2} \frac{a}{\sqrt{2M}} \right), & a \leq r \leq \sqrt{2M} \\
\phi(b) \left( \frac{b}{r} \right)^{2M} \cos^2 \left( \frac{\omega_2}{2} \frac{r}{\sqrt{2M}} \right), & \sqrt{2M} \leq r \leq b \\
\phi(b) \left( \frac{b}{r} \right)^{k_2^2}, & r \geq b
\end{cases}
\]

with $a = 2.267227$, $b = 4.167853$, $\omega_1 = 4.071497$, $\omega_2 = 5.025638$, $k_1^2 = 6.725245$, and $k_2^2 = 14.180138$.

Previously we have shown how to find robust maximum likelihood estimates of structured spatial correlation matrices [34]. A maximum likelihood estimate of a structured matrix involves maximizing the likelihood function over just those matrices which have the properties that the true matrix is known to have. Often in array processing applications we know that the matrix is symmetric or maybe even Toeplitz. Additionally, we could define the required structure to be that of the matrix specified by the signal and noise parameters and estimate the signal parameters directly [34]. In our earlier paper, the estimator was generalized from Gaussian processes to spherically symmetric processes and then to a robust estimator. The robust estimate was the asymptotically efficient estimate which minimized the maximum asymptotic estimate variance over the class of probability densities. For the particular class of densities implied by theorem 4.2 we have the optimization problem

\[
\max_{R^* \in \mathcal{R}} \left\{ -\frac{N}{2} \ln[\det(R^*)] + \sum_{i=1}^{N} \ln f_0 \left[ (\text{tr}(R^{*-1} \mathbf{x}_i \mathbf{x}_i^T))^{1/2} \right] \right\}
\]
where $f_0$ is our worst case density and $\mathcal{R}$ is the set of all possible matrices possessing the desired structure. The performance of this estimator for the examples in [34] show that little performance is lost when these estimators are used in Gaussian noise rather than the usual estimators. For the worst case density, much is gained by using this estimator over the usual one based on Gaussian statistics. Using this estimator and theorem 4.2, narrowband array processing is possible when all that is known about the distributions of the $\xi_i$ is that they are second order.
Chapter 5

Conclusions

We have proposed a robust estimator for structured covariance matrices and shown examples of its use in array processing applications. This estimator yields a better estimate of \( \mathbf{R} \) than the more conventional estimate, \( \hat{\mathbf{R}} \), formed by averaging the outer products of the sample vectors even in Gaussian noise. In addition, our estimate, \( \mathbf{R}^* \), shows much less degradation when the noise becomes non-Gaussian. The performance shown in figures 2.5 and 2.6 is actually the worst performance for any noise p.d.f. in their classes with \( \epsilon = 0.1 \). Any noise density in this class results in a better estimate than the one for the worst case density. Although there is some increase in computational complexity when finding the robust structured estimate rather than the Gaussian structured estimate, this increase is small compared to the amount of computation needed to invert \( \mathbf{R}^* \) in both cases. So the increase in complexity caused be going to the robust estimator is negligible compared to the amount of computation which is common to both estimators.

The improvement in the matrix estimate is much more evident in the eigenvector method than in Capon's method. This improvement could be related to the greater sensitivity to time-bandwidth product of the eigenvector method over Capon's method when \( \hat{\mathbf{R}} \) is used. The eigenvector method's performance improves tremendously with increasing \( N \) as \( \hat{\mathbf{R}} \) approaches \( \mathbf{R} \), while Capon's method shows no improvement after \( N \gg M \) [10]. As the maximum likelihood Toeplitz estimate of \( \mathbf{R} \) also generates an estimate closer to \( \mathbf{R} \) but without needing more samples, we can expect the improvement in the performance of the beamforming algorithms.
caused by using $R^*$ instead of $\hat{R}$ to be similar to their improvements when the
time-bandwidth product increases.

The estimators presented here also demonstrate the advantages of utilizing all
of the structural information known about the matrix being estimated. It is clear in
the figures that the estimates obtained by using more structural information yield
better performance when used by narrowband array algorithms than the estimates
which use less information.

Additionally, we have presented a method of using these robust estimates for
narrowband array processing when little is known about the noise amplitude distri-
bution. By developing a new multivariate bound on the rate of convergence of the
multivariate Central Limit Theorem, we were able to arrive at a class of densities
for which robust covariance matrix estimation techniques can be designed. The
only requirement on the noise amplitude density is that it have finite variance;
a condition which in application is always met by the noise process after it has
been sampled by any realizable system. The performance of our algorithm for the
worst case density indicates that its performance is satisfactory over all allowable
densities.

By using real vectors instead of complex ones, the previous limit on the number
of signals which could be estimated with an $M$ sensor uniform linear array has been
surpassed. Whereas before it was often desirable to make algorithms independent
of signal phase offsets, we have shown that there are algorithms which are phase
offset dependent and can estimate $2M - 1$ signals instead of only $M - 1$.

The fact that $2M - 1$ angles-of-arrival can actually be estimated is demon-
strated by a simple maximum likelihood parametric estimation example. A non-
linear least-squares problem is solved to find the parameters of the signals, and
the angles-of-arrival are extracted from these parameters. The estimation error is
low but increases as the maximum number of signals is approached. As the num-
ber of signals increases, the least-squares routine must estimate a large number of
parameters for the given amount of data, and the accuracy of the results is not as good as when there are fewer signals.

The algorithm presented for estimating $2M - 1$ signals is seen to perform extremely well for small $M$. The least-squares problem for real vectors does not simplify like the one for complex vectors does, and the resulting optimization problem can be computationally expensive. For a two sensor array with 25 samples and three signals, the least-squares routine must estimate 81 parameters. Although the computation cost increases with the size of the array, there is also not as much interest in estimating more signals with larger arrays. As $M$ increases, the spacings of the sources rather than the number of sources becomes the limiting factor on resolution as it becomes difficult to estimate even $M - 1$ well-spaced sources. Additionally, for instances that sensors are expensive, it is desirable to estimate as many signals as possible with as few sensors as possible. There is also the possibility that using real signal vectors may result in improvements in other performance measures, such as being able to resolve signals which are closer together or have lower signal-to-noise ratios. Analysis of gains in these areas remains to be done.
Appendix A

The Worst Case Distribution for the Multivariate Kolmogorov Class

In this section we will determine the density in the multivariate Kolmogorov class of equation 2.12 which is worst case with respect to estimating the covariance matrix. The method used here is analogous to Huber's method for finding the worst case density for the ε-contamination class of equation 2.11 and the initial steps are the same [17, page 231]. As was shown in section 2.2 the worst case density is spherically symmetric and can be found by determining which spherically symmetric density in the class minimizes Fisher information.

Let's consider the family of elliptically symmetric, zero-mean distributions

\[ f(x| R) = (\det R)^{-1/2} f \left( \| R^{-1/2} x \| \right), \quad x \in \mathbb{R}^{2M} \]

If we assume that \( R \) depends differentiably on some real parameter \( \theta \), then the Fisher information with respect to \( \theta \) at \( R = I \) is

\[ I(f) = E \left\{ \left[ \frac{\partial}{\partial \theta} \ln f(x| R) \right]^2 \right\} \]

Because of symmetry it suffices to treat this special case of white noise. As Huber shows [17, page 231] this is equivalent to

\[ I(f) = \gamma E[u(\|x\|)^2] - (2M)^2 \beta^2 \]

where \( u(r) = - \frac{f'(r)}{f(r)} r, \quad \beta \|x\|^2 = E \left[ x^T \left( \frac{\partial R^{-1/2}}{\partial \theta} \right) x \|x\| \right], \quad \text{and} \quad \gamma \|x\|^4 = E \left[ x^T \left( \frac{\partial R^{-1/2}}{\partial \theta} \right)^2 x \|x\| \right]. \]

In order to minimize \( I(f) \) over the class of distributions \( \mathcal{F} \), it is sufficient to minimize

\[ J(f) = E \left[ u(\|x\|)^2 \right] = \frac{2(\pi)^M}{\Gamma(M)} \int_0^\infty \frac{f'(r)^2}{f(r)} r^{2M+1} dr \]
As can be seen $J(f)$ is independent of the choice of the parameter $\theta$. Thus, the worst case density will be worst case for estimating $R$ with respect to any parameter which $R$ depends upon. Consequently, we are able to work with just a univariate Fisher information rather than the multivariate Fisher information matrix. Taking the variation of $J(f)$ as $\left[ \frac{d}{dt} J(tf_0 + (1-t)f_1) \right]_{t=0} = \delta J(f)$ gives

$$
\delta J(f) = \frac{2(\pi)^M}{\Gamma(M)} \int_0^\infty (-u^2 + 4Mu + 2ru')r^{2M-1}\delta(f)dr
$$

where $\delta(f) = f_1 - f_0$. As we know that $\frac{2(\pi)^M}{\Gamma(M)} \int_0^\infty r^{2M-1}\delta f dr = 0$ because the distributions of $f_1$ and $f_0$ both go to 1 as $r$ goes to $\infty$, we obtain that the $u$ corresponding to the minimizing $f_0$ satisfies

$$
2ru' + 4Mu - u^2 = c \quad (A.1)
$$

for those $r$ where $f_0$ can be varied freely. For the ranges of $r$ where $f_0$ cannot be chosen freely, $f_0(r) \propto \phi(r)$.

We are interested in the class of distributions $\mathcal{F} = \{ f | \sup_r |F(r) - \Phi(r)| < \epsilon, \}$ and $f$ is a $2M$-dimensional spherically symmetric density where $\Phi(r) = \frac{2(\pi)^M}{\Gamma(M)} \int_0^r (2\pi)^{-M} e^{-r^2/2} d\alpha$ is the distribution function of the multivariate, zero mean, unit variance Gaussian random vector of dimension $2M$ and $F(r) = \frac{2(\pi)^M}{\Gamma(M)} \int_0^r f(\alpha)\alpha^{2M-1} d\alpha$ is the distribution function of the spherically symmetric density $f(r)$. From the work done by Huber on finding the worst case density for the Kolmogorov class of equation 2.8 with respect to estimation of the mean [17, page 86] and also from the relationship between the worst case densities for the $\epsilon$-contamination class for the estimation of the mean and the estimation of the covariance matrix, we note that the density we are looking for has certain properties. The density will be divided into five possible regions with the following properties:

1. $F_0(r) - \Phi(r) \geq 0, f_0(r) \geq \phi(r)$, and is monotone increasing until the end of this region when $F_0(r) - \Phi(r) = \epsilon$. 

2. \( f_0(r) = \phi(r) \)

3. \( f_0(r) \leq \phi(r) \) and at the end of this region \( F_0(r) - \Phi(r) = -\epsilon \)

4. \( f_0(r) = \phi(r) \)

5. \( f_0(r) \geq \phi(r) \) and at the end of this region \( F_0(\infty) = \Phi(\infty) = 1 \)

We also require that \( f_0(r) \) and \( u_0(r) = -\frac{f_0(r)}{f_0(r)} \) both be continuous. For sufficiently large \( \epsilon \), regions 2 and 4 will not exist and the differential equation A.1 will be piecewise constant for all values of \( r \). The worst case density which minimizes \( J(f) \) for different ranges of \( \epsilon \) follows.

For small \( \epsilon \):

\[
f_0(r) = \begin{cases} 
\phi(a) \left( \frac{a}{r} \right)^{a^2} & , \ 0 \leq r \leq a \\
\phi(r) & , \ a \leq r \leq b \\
\phi(b) \left( \frac{b}{r} \right)^{2M} \cos^2 \left( \frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}} \right) & , \ b \leq r \leq \sqrt{2M} \\
\phi(r) \frac{\cos^2 \left( \frac{\omega_1}{2} \ln \frac{b}{\sqrt{2M}} \right)}{\cos^2 \left( \frac{\omega_1}{2} \ln \frac{b}{\sqrt{2M}} \right)} & , \ \sqrt{2M} \leq r \leq c \\
\phi(c) \left( \frac{c}{r} \right)^{2M} \cos^2 \left( \frac{\omega_2}{2} \ln \frac{r}{\sqrt{2M}} \right) & , \ c \leq r \leq d \\
\phi(d) \left( \frac{d}{r} \right)^{d^2} & , \ r \geq d 
\end{cases}
\]
The parameters are determined by requiring that \( f_0(r) \) and

\[
  w_0(r) = \frac{f_0'(r)}{f_0(r)} r = \begin{cases}
    a^2 & , \quad 0 \leq r \leq a \\
    r^2 & , \quad a \leq r \leq b \\
    2M + \omega_1 \tan\left(\frac{\omega_1}{2} \ln \frac{b}{\sqrt{2M}}\right) & , \quad b \leq r \leq \sqrt{2M} \\
    2M + \omega_2 \tan\left(\frac{\omega_2}{2} \ln \frac{c}{\sqrt{2M}}\right) & , \quad \sqrt{2M} \leq r \leq c \\
    r^2 & , \quad c \leq r \leq d \\
    d^2 & , \quad r \geq d
  \end{cases}
\]

be continuous. Also we require that \( F_0(a) - \Phi(a) = \epsilon, \Phi(c) - F_0(c) = \epsilon \), and \( F_0(\infty) = 1 \). Thus we have six parameters \((a, b, c, d, \omega_1, \text{and} \, \omega_2)\) and six nonlinear equations to solve for them.

\[
  \frac{\phi(b) b^{2M}}{\cos^2\left(\frac{\omega_1}{2} \ln \frac{b}{\sqrt{2M}}\right)} = \frac{\phi(c) c^{2M}}{\cos^2\left(\frac{\omega_2}{2} \ln \frac{c}{\sqrt{2M}}\right)} \tag{i}
\]

\[
  b^2 = 2M + \omega_1 \tan\left(\frac{\omega_1}{2} \ln \frac{b}{\sqrt{2M}}\right) \tag{ii}
\]

\[
  c^2 = 2M + \omega_2 \tan\left(\frac{\omega_2}{2} \ln \frac{c}{\sqrt{2M}}\right) \tag{iii}
\]

\[
  \int_0^a \phi(a) \left(\frac{a}{r}\right)^2 r^{2M-1}dr - \Phi(a) = \epsilon \tag{iv}
\]

\[
  \Phi(c) - \Phi(b) = \left\{ \int_b^{\sqrt{2M}} \phi(b) \left(\frac{b}{r}\right)^{2M} \frac{\cos^2\left(\frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}}\right)}{\cos^2\left(\frac{\omega_1}{2} \ln \frac{b}{\sqrt{2M}}\right)} r^{2M-1}dr + \right. \\
  \left. \int_{\sqrt{2M}}^c \phi(c) \left(\frac{c}{r}\right)^{2M} \frac{\cos^2\left(\frac{\omega_2}{2} \ln \frac{r}{\sqrt{2M}}\right)}{\cos^2\left(\frac{\omega_2}{2} \ln \frac{c}{\sqrt{2M}}\right)} r^{2M-1}dr \right\} = 2\epsilon \tag{v}
\]
\[ \int_{d}^{\infty} \phi(d) \left( \frac{d}{r} \right)^{2r^{2M-1}dr + 1 - \Phi(d) = \epsilon} \]  

For large \( \epsilon \):

\[
f_0(r) = \begin{cases} 
\phi(a) \left( \frac{a}{r} \right)^{k_1^2}, & 0 \leq r \leq a \\
\phi(a) \left( \frac{a}{r} \right)^{2M \cos^2 \left( \frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}} \right)}, & a \leq r \leq \sqrt{2M} \\
\phi(b) \left( \frac{b}{r} \right)^{2M \cos^2 \left( \frac{\omega_2}{2} \ln \frac{r}{\sqrt{2M}} \right)}, & \sqrt{2M} \leq r \leq b \\
\phi(b) \left( \frac{b}{r} \right)^{k_2^2}, & r \geq b 
\end{cases}
\]

Again, the parameters are determined by requiring that \( f_0(r) \) and

\[
u_0(r) = -\frac{f_0'(r)}{f_0(r)} r = \begin{cases} 
k_1^2, & 0 \leq r \leq a \\
2M + \omega_1 \tan \left( \frac{\omega_1}{2} \ln \frac{a}{\sqrt{2M}} \right), & a \leq r \leq \sqrt{2M} \\
2M + \omega_2 \tan \left( \frac{\omega_2}{2} \ln \frac{b}{\sqrt{2M}} \right), & \sqrt{2M} \leq r \leq b \\
k_2^2, & r \geq b
\end{cases}
\]

be continuous. Also we require that \( F_0(a) - \Phi(a) = \epsilon, \Phi(b) - F_0(b) = \epsilon, \) and \( F_0(\infty) = 1 \). Thus we have six parameters \( a, b, k_1^2, k_2^2, \omega_1, \) and \( \omega_2 \) and six nonlinear equations to solve for them.

\[
\frac{\phi(a) a^{2M}}{\cos^2 \left( \frac{\omega_1}{2} \ln \frac{a}{\sqrt{2M}} \right)} = \frac{\phi(b) b^{2M}}{\cos^2 \left( \frac{\omega_2}{2} \ln \frac{b}{\sqrt{2M}} \right)} \tag{1}
\]

\[
k_1^2 = 2M + \omega_1 \tan \left( \frac{\omega_1}{2} \ln \frac{a}{\sqrt{2M}} \right) \tag{2}
\]

\[
k_2^2 = 2M + \omega_2 \tan \left( \frac{\omega_2}{2} \ln \frac{b}{\sqrt{2M}} \right) \tag{3}\]
\[
\int_0^a \phi(a) \left( \frac{a}{r} \right)^k r^{2M-1} dr - \Phi(a) = \epsilon \\
\Phi(b) - \Phi(a) = \left\{ \int_{\sqrt{2M}}^b \phi(a) \left( \frac{a}{r} \right)^{2M} \frac{\cos \left( \frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}} \right)}{\cos \left( \frac{\omega_2}{2} \ln \frac{a}{\sqrt{2M}} \right)} r^{2M-1} dr + \int_{\sqrt{2M}}^b \phi(b) \left( \frac{b}{r} \right)^{2M} \frac{\cos \left( \frac{\omega_1}{2} \ln \frac{r}{\sqrt{2M}} \right)}{\cos \left( \frac{\omega_2}{2} \ln \frac{b}{\sqrt{2M}} \right)} r^{2M-1} dr \right\} = 2\epsilon \\
\int_0^\infty \phi(b) \left( \frac{b}{r} \right)^k r^{2M-1} dr + 1 - \Phi(b) = \epsilon
\]
Appendix B

The Independence of Zero-Phase Real Signal Vectors

We will show that, assuming that the $\theta_p$ are different, signal vectors of the form

$$[1 \ \cos \theta_p \ \cdots \ \cos(M-1)\theta_p \ \ 0 \ \sin \theta_p \ \cdots \ \sin(M-1)\theta_p]^T, \ p = 1, \ldots, P,$$

are linearly independent for $P \leq 2M - 1$. These vectors are linearly independent if the rank of the matrix formed from the signal vectors is $P$. We can reorder the rows of this matrix and ignore the row which is all zeros without affecting the rank of this matrix.

$$
\begin{bmatrix}
1 & \cdots & 1 \\
\cos \theta_1 & \cdots & \cos \theta_P \\
\sin \theta_1 & \cdots & \sin \theta_P \\
\cos 2\theta_1 & \cdots & \cos 2\theta_P \\
\sin 2\theta_1 & \cdots & \sin 2\theta_P \\
\vdots & \ddots & \vdots \\
\cos(M-1)\theta_1 & \cdots & \cos(M-1)\theta_P \\
\sin(M-1)\theta_1 & \cdots & \sin(M-1)\theta_P
\end{bmatrix}
$$

After doing so, multiply the signal matrix on the left side by

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & j & 0 & \cdots & 0 \\
0 & 1 & -j & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & j \\
0 & 0 & 0 & \cdots & 0 & 1 & -j
\end{bmatrix}
$$
and use the fact that \( \text{rank}(AB) \leq \min[\text{rank}(A), \text{rank}(B)] \) [27]. As the matrix which we are multiplying the signals by is obviously non-singular, the rank of the resulting matrix

\[
\begin{bmatrix}
1 & \cdots & 1 \\
e^{j\theta_1} & \cdots & e^{j\theta_P} \\
e^{-j\theta_1} & \cdots & e^{-j\theta_P} \\
\vdots & & \vdots \\
e^{j(M-1)\theta_1} & \cdots & e^{j(M-1)\theta_P} \\
e^{-j(M-1)\theta_1} & \cdots & e^{-j(M-1)\theta_P}
\end{bmatrix}
\]

is less than or equal to the rank of the matrix we started with. Switching the rows of this matrix and multiplying each column by \( e^{j(M-1)\theta_p} \) will not change its rank. The resulting matrix is

\[
\begin{bmatrix}
1 & \cdots & 1 \\
e^{j\theta_1} & \cdots & e^{j\theta_P} \\
e^{j2\theta_1} & \cdots & e^{j2\theta_P} \\
\vdots & & \vdots \\
e^{j2(M-1)\theta_1} & \cdots & e^{j2(M-1)\theta_P}
\end{bmatrix}
\]

This matrix is a Vandermonde matrix and will have full rank if all of the \( \theta_p \) are different and \( P \leq 2M - 1 \). Therefore, our original matrix has rank \( P \) and our vectors are linearly independent for \( P \leq 2M - 1 \).
Appendix C

The Independence of Nonzero-Phase Real Signal Vectors

We will now show that $2M$ signal vectors of the form $[\cos(\alpha_p) \; \cos(\alpha_p + \theta_p) \; \cdots \; \cos(\alpha_p + (M-1)\theta_p) \; \sin(\alpha_p) \; \sin(\alpha_p + \theta_p) \; \cdots \; \sin(\alpha_p + (M-1)\theta_p)]^T$, $p = 1, \ldots, 2M$ are linearly independent with probability one. Again, we will prove linear independence by showing that the matrix composed of these vectors

\[
\begin{bmatrix}
\cos \alpha_1 & \cdots & \cos \alpha_{2M} \\
\sin \alpha_1 & \cdots & \sin \alpha_{2M} \\
\cos(\alpha_1 + \theta_1) & \cdots & \cos(\alpha_{2M} + \theta_{2M}) \\
\sin(\alpha_1 + \theta_1) & \cdots & \sin(\alpha_{2M} + \theta_{2M}) \\
\vdots & & \vdots \\
\cos(\alpha_1 + (M-1)\theta_1) & \cdots & \cos(\alpha_{2M} + (M-1)\theta_{2M}) \\
\sin(\alpha_1 + (M-1)\theta_1) & \cdots & \sin(\alpha_{2M} + (M-1)\theta_{2M})
\end{bmatrix}
\]

has full rank. Multiplying this matrix by

\[
\begin{bmatrix}
1 & j & 0 & 0 & \cdots & 0 \\
1 & -j & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & j & 0 & \cdots & 0 \\
0 & 0 & 1 & -j & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & j \\
0 & 0 & \cdots & 0 & 1 & -j
\end{bmatrix}
\]
and each column of the resulting matrix by $e^{j((M-1)\theta_p+\alpha_p)}$ yields
\[
\begin{bmatrix}
1 & \cdots & 1 \\
e^{j\theta_1} & \cdots & e^{j\theta_2M} \\
\vdots & \ddots & \vdots \\
e^{j(M-1)\theta_1} & \cdots & e^{j(M-1)\theta_2M} \\
e^{j2\alpha_1} & \cdots & e^{j2\alpha_2M} \\
e^{j(2\alpha_1+\theta_1)} & \cdots & e^{j(2\alpha_2M+\theta_2M)} \\
\vdots & \ddots & \vdots \\
e^{j(2\alpha_1+(M-1)\theta_1)} & \cdots & e^{j(2\alpha_2M+(M-1)\theta_2M)}
\end{bmatrix}
\] (C.1)

We now divide this $2M \times 2M$ matrix into four $M \times M$ matrices:
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

and find that $A$ and $B$ are Vandermonde matrices, $C = AT$, and $D = BS$ where
\[
T = \begin{bmatrix}
e^{j2\alpha_1} & 0 \\
\vdots & \ddots \\
0 & e^{j2\alpha_M}
\end{bmatrix}, \quad S = \begin{bmatrix}
e^{j2\alpha_{M+1}} & 0 \\
\vdots & \ddots \\
0 & e^{j2\alpha_M}
\end{bmatrix}
\]

The $2M \times 2M$ matrix has full rank if its determinant is non-zero. The value of the determinant remains unchanged if a multiple of one row is added to another row.

Therefore,
\[
\det \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \det \begin{bmatrix}
A & B \\
0 & D - (ATA^{-1})B
\end{bmatrix}
\]

Now that one of the submatrices is all zeros, the determinant of the entire $2M \times 2M$ matrix simplifies to the product of the determinants of the two matrices on the main diagonal [27], i.e.
\[
\det \begin{bmatrix}
A & B \\
0 & D - (ATA^{-1})B
\end{bmatrix} = \det(A) \det(D - (ATA^{-1})B)
\]
As $A$ is non-singular, $\det(A) \neq 0$, and we only need to show that $\det(D - (\text{ATA}^{-1})B) \neq 0$. Using the facts that $D = BS$ and, if $W$ is non-singular, $\det(WV) = \det(W)\det(V) \neq 0$ if and only if $\det(V) \neq 0$, we have the requirement

$$\det(I - S^{-1}B^{-1}\text{ATA}^{-1}B) \neq 0$$

which is equivalent to requiring the matrix $-S^{-1}B^{-1}\text{ATA}^{-1}B$ to have no eigenvalues equal to $-1$. So we will set the value of $T$ and see if any values of $S$ exist such that there is an eigenvalue of $-1$. Define $U = -B^{-1}\text{ATA}^{-1}B$. We will look at the determinant of $(S^{-1}U - \lambda I)$ with $\lambda = -1$ and see if it can equal zero. Let

$$S^{-1} = \begin{bmatrix}
e^{-j2\alpha_{M+1}} & 0 & \cdots \\
0 & e^{-j2\alpha_{M+1}} & \cdots \\
\vdots & \ddots & \ddots
\end{bmatrix} = \begin{bmatrix}s_1 & 0 \\
\cdots & \ddots \\
0 & s_M
\end{bmatrix}$$

and $U = \begin{bmatrix}u_1 \\
\vdots \\
u_M\end{bmatrix}$

where $u_i, i = 1, \ldots, M$, are length $M$ row vectors, and then find the values of $s_1, \ldots, s_M$ such that

$$\det(S^{-1}U + I) = \det\left(\begin{bmatrix}s_1 u_1 \\
\vdots \\
s_M u_M\end{bmatrix} + \begin{bmatrix}1 & 0 \\
\cdots & \ddots \\
0 & 1\end{bmatrix}\right) = 0 \quad (C.2)$$

We now have a multivariate polynomial in $s_1, \ldots, s_M$ which is of degree $\leq 1$ for each $s_i$.

Let $f(s_1, s_2, \ldots, s_M) = \det(S^{-1}U + I)$. We will show that, if we assume that the phases are uniformly distributed from $[0, 2\pi)$, the set of $s_i$'s such that $f(\cdot) = 0$ is of measure zero. Having independent phase offsets between signals is a reasonable assumption as we are already assuming the signals to be independent, and there is no reason for the phase offset of one signal to depend upon the phase offset of
another. Set the values of \( s_2, \ldots, s_M \) to allowable random values and consider what \( s_1 \) needs to be in order for \( f(\cdot) \) to be zero. We will denote this function of \( s_1 \) with \( s_2, \ldots, s_M \) fixed as \( f(s_1|s_2, \ldots, s_M) \). This function is a first order polynomial in \( s_1 \) and can be written

\[
f(s_1|s_2, \ldots, s_M) = c_1 s_1 + c_0
\]

for some complex values \( c_0 \) and \( c_1 \) depending upon \( s_2, \ldots, s_M \). If \( c_0 \) and \( c_1 \) are not both zero, then there is at most one allowable value of \( s_1 = e^{-j2\alpha_{M+1}} \) such that \( f(s_1|s_2, \ldots, s_M) = 0 \). If there is no value of \( \alpha_{M+1} \) such that the function is zero, then the function is never zero for the given \( s_2, \ldots, s_M \). If there is one value of \( \alpha_{M+1} \) such that \( f(s_1|s_2, \ldots, s_M) = 0 \), then, because \( \alpha_{M+1} \) is uniform on \([0, 2\pi)\), \( f(s_1|s_2, \ldots, s_M) \neq 0 \) with probability one.

Another case which must be considered is when \( c_0 = c_1 = 0 \) and \( f(s_1|s_2, \ldots, s_M) = 0 \) for all \( s_1 \). If we can show that this case occurs with probability zero, then we will have shown that \( f(s_1, s_2, \ldots, s_M) \) is non-zero with probability one. Obviously, \( Pr(c_0 = 0) \geq Pr(c_0 = c_1 = 0) \) so if \( Pr(c_0 = 0) = 0 \) then we are done. For \( c_0 \) to be zero we need

\[
\det \left( \begin{bmatrix} s_2 u_2^{[2]} \\ \vdots \\ s_M u_M^{[2]} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \vdots & \ddots \\ 0 & 1 \end{bmatrix} \right) = 0
\]

where \( u_i^{[2]} \) is the length \( M - 1 \) vector formed by eliminating the first element of \( u_i \).

Therefore, we again have the problem in equation C.2 and can proceed as before by varying only the element \( s_2 \) and seeing what the probability of this determinant being zero is.

We can continue in this manner until we are considering the case \( \det(s_M u_{MM} + 1) = 0 \). For this equation to be true we need \( s_M = e^{-j2\alpha_{2M}} = -1/u_{MM} \) which may or may not be possible. If it is possible, there is only one value of \( \alpha_{2M} \) such that \( s_M = -1/u_{MM} \) and, because \( \alpha_{2M} \) has a continuous distribution, this
has probability zero. Therefore, all the possible cases of equation C.2 being true have measure zero. Therefore, our original $2M \times 2M$ matrix has full rank with probability one. So $2M$ of these signal vectors form a linearly independent set with probability one.
Appendix D

A Bound on the Rate of Convergence of the Multivariate Central Limit Theorem

The purpose of this appendix is to show that, if $K$ random vectors $y_1, y_2, \cdots, y_K$ become a spherically symmetric random vector when summed, then the sum of these random vectors converges to the Gaussian distribution as quickly as in the univariate case. More specifically, we will prove theorem 4.2 with a value of $C = 1.3213$.

For the univariate case, both Berry [3] and Esseen [14] gave upper bounds on the value of $C$ in their original papers on the Berry-Esseen theorem ($C < 1.88$ for Berry and $C < 7.5$ for Esseen). Berry had the sharper bound, but there was an error in his calculations. The majority of the work done since then on sharpening the upper bound on $C$ has been done by Zolotarev [36,37,38,39] who refined Berry’s methods. While van Beek [29,30], by improving on Zolotarev’s work, has provided the sharpest bound so far for $C$ ($C < 0.7975$). We have been able to show that, if the sum of $K$ non-identically distributed random vectors is a spherically symmetric random vector, then the earlier work by Zolotarev can be generalized to include this multivariate case and we will have $C < 1.3213$. In what follows we will show the changes which must be made to Zolotarev’s work to generalize it. For each lemma we provide a reference to the corresponding univariate lemma in Zolotarev’s papers.

The basic idea behind finding the constant $C$ is to bound $\sup_r |F(r) - \Phi(r)|$ by some function which depends upon $\epsilon = \frac{\sum E [\|y_k\|^3]}{(\sum E [\|y_k\|^2])^{3/2}} = \frac{\sum \beta_k}{(\sum \sigma_k^2)^{3/2}}$ but is
independent of $r$. Then this function is maximized over $\varepsilon$ to yield an upper bound on the value of $C$.

**Lemma D.1** ([87, Lemma 3, page 476]) Let $G(r)$ be a spherically symmetric distribution function which is calculated by integrating the corresponding density over the sphere of radius $r$, $H$ a function of bounded variation such that $H(-\infty) = 1 - H(\infty)$, $H(x) = H(||x||)$,

$$q = \sup_r |H'(r)| < \infty$$

and $\rho(r) = \rho(||x||)$ some spherically symmetric distribution with $p(r)$ being the density of the radius of $\rho(r)$.

Let, moreover, $\omega(\tau)$ denote the absolute value of the characteristic function which corresponds to $p(r)$ and let $g(t)$ and $h(t)$ be the Fourier transforms of the derivatives of the functions $G$ and $H$. We set $\delta = |g - h|,

$$V(x) = x \int_0^x p(u) du, \quad Q(y) = \frac{y}{2\pi q} \int_0^\infty \omega\left(\frac{\tau}{y}\right) \delta(\tau) d\tau$$

where $\tau = ||t||$, and denote by $\chi$ the only positive root of the equation

$$4V(x) = x$$

Then the following inequality holds for all $y > 0$ and all $x > \chi$:

$$\Delta = \sup_r |G(r) - H(r)| \leq 2q \frac{x[V(x) + Q(y)]}{y[4V(x) - x]} = I(x, y) \quad (D.1)$$

**Proof** Without reducing the generality, we can assume that a value of the argument $||x|| = ||x_1||$ of the function $D(||x||) = G(||x||) - H(||x||)$ can be found such that $D(||x_1||) = -\Delta$. Let us choose some number $2 < \lambda < 4$. Using the monotonicity of $G$ and the boundedness of $H'$, we obtain, for values $||y|| < \gamma = \frac{4 - \lambda}{2q} \Delta$, $\Delta$,

$$D(x_1 - \gamma + y) \leq D(||x_1||) + (\gamma - ||y||)q$$

$$= \frac{2 - \lambda}{2} \Delta - ||y||q$$
where, if \( a_1 \) is the unit vector in the \( x_1 \) direction, \( \gamma = \gamma a_1 \). Hence by virtue of the symmetry of \( \rho(\|y\|) \):

\[
\int_{\|y\|>\gamma} D(x_1 - \gamma + y)T \rho(yT)dy \leq \frac{2 - \lambda}{2} \Delta \int_{\|y\|>\gamma T} \rho(y)dy
\]

Moreover, because \(|D| \leq \Delta\), it is obvious that

\[
\left| \int_{\|y\|>\gamma} D(x_1 - \gamma + y)T \rho(yT)dy \right| \leq \Delta \int_{\|y\|>\gamma T} \rho(y)dy
\]

Therefore

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D(x_1 - \gamma + y)T \rho(yT)dy \geq \Delta \left\{ \frac{\lambda - 2}{2} \int_{\|y\|<\gamma T} \rho(y)dy - \int_{\|y\|>\gamma T} \rho(y)dy \right\} = \Delta W(\gamma T, \lambda)
\]

where \( W(u, \lambda) = \frac{\lambda}{u} V(u) - 1 \).

The integral under the absolute value sign on the left side of inequality (D.2) is the convolution of the functions \( D \) and \( \rho \), and therefore can be represented as the inverse Fourier transform of the product of the Fourier transforms of \( D \) and \( \rho \). Using this relationship between the Fourier transforms in addition to some properties of spherically symmetric densities [23], it is possible to show that

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D(x_1 - \gamma + y)T \rho(yT)dy \leq \frac{1}{\pi} \int_{0}^{\infty} \omega \left( \frac{\tau}{T} \right) \delta(\tau) \frac{d\tau}{\tau} = \frac{2q}{T} Q(T)
\]

so that

\[
\gamma T W(\gamma T, \lambda) \leq (4 - \lambda) Q(T)
\]

Now let \( x \) be some positive number. Then either \( \gamma T < x \) or \( \gamma T \geq x \). In the second case, because of the monotonicity of the function \( W \), we have \( W(x, \lambda) \leq W(\gamma T, \lambda) \). Hence we have

\[
\gamma T \leq \frac{(4 - \lambda) Q(T)}{W(x, \lambda)}
\]
Then, for any \( x > 0 \),
\[
\gamma T \leq \max \left\{ x, \frac{(4 - \lambda) Q(T)}{W(x, \lambda)} \right\}
\]
so that
\[
\Delta \leq \frac{2q}{T} \max \left\{ \frac{x}{4 - \lambda}, \frac{Q(T)}{W(x, \lambda)} \right\}
\]
(D.3)

Up to this point, \( \lambda \) was a free parameter restrained only by the condition \( 2 < \lambda < 4 \). We want to select \( \lambda \) so that, for fixed values of \( x \) and \( T \), the right side will be minimized. Since the first term under the maximum sign is monotonically increasing with \( \lambda \), while the second one is monotonically decreasing, a minimum of the right side will be attained for a \( \lambda \) for which both terms are equal to one another, i.e., for
\[
\lambda = \frac{4Q(T) + x}{Q(T) + V(x)}
\]
by substituting this value of \( \lambda \) into inequality D.3, we obtain inequality D.1. \( \square \)

As is easy to see, inequality D.1 leaves a lot of freedom in selecting values of the variables \( x \) and \( y \) for the already selected function \( p \). We shall now consider the question of minimizing the right side of inequality D.1, introducing into consideration the following functions:
\[
P(y) = -\frac{1}{2\pi q} \int_0^\infty \omega'\left(\frac{\tau}{y}\right) \delta(\tau) d\tau
\]
where \( \tau = ||t|| \) and \( \omega'(\tau) = \frac{d}{d\tau} \omega(\tau) \)
\[
R(y) = Q(y) + P(y)
\]
\[
W(x) = \frac{4V(x) - x[V(x) + x^2p(x)]}{4xp}
\]

**Lemma D.2 ([36, Lemma 3, page 97])** The function \( I(x, y) \) attains its minimal value for \( x = x^*, y = y^* \) given as solutions to the equations
\[
P(y) - V(x), \quad R(y) = W(x)
\]
(D.4)
Proof: This follows directly by requiring that
\[ \frac{\partial I}{\partial x} = 0, \quad \frac{\partial I}{\partial y} = 0 \]
and simplifying the expressions. \( \square \)

As a consequence of this lemma we see that if \( x^*, y^* \) is a nontrivial solution of equations D.4, then
\[ I(x^*, y^*) = 2q \frac{R(y^*)}{y^*} \frac{x^*}{[4V(x^*) - x^*]} = 2q \frac{x^*W(x^*)}{y^* [4V(x^*) - x^*]} \]

Lemma D.3 ([36, Lemma 4, page 97]) Let the function \( \delta(t) = \delta(t, \theta) \) depend on some parameter \( \theta \) in such a way that
\[ \delta(t, \theta_1) \leq \delta(t, \theta_2) \text{ if } \theta_1 \leq \theta_2 \]
and let \( x^* = x^*(\theta) \) and \( y^* = y^*(\theta) \) be the values of the arguments realizing the minimum of the function \( I(x, y) = I(x, y, \theta) \). Then the minimal value of this function \( I(x^*, y^*, \theta) \) will also be a monotonically increasing function of \( \theta \).

Proof: This proof is unchanged from that of Zolotarev [36]. \( \square \)

Now we must determine an upper bound for the function \( \delta(\tau) \) which in turn is an upper bound for \( I(x, y) \). Let us denote by \( f \) the characteristic function of the normed sum \( (y_1 + \cdots + y_K)/(\sum \sigma^2_k)^{1/2} \) and set
\[ \delta(\tau) = \delta(\|t\|) = |f(t) - e^{-t^2/2}| \]
We construct an estimate \( \delta^* \), for the function \( \delta \), which depends only on \( t \) and \( \epsilon \) and increases monotonically with increasing \( \epsilon \). Then, using the function \( \delta^* \) and the function \( p(\tau) = \frac{1 - \cos(\tau)}{\pi \tau^2} \) for which
\[ \omega(\tau) = \begin{cases} (1 - \tau), & \text{for } 0 \leq \tau \leq 1 \\ 0, & \text{for } \tau > 1 \end{cases} \]
we construct the quantity \( I = I(x^*, y^*, \epsilon) \) and then maximize \( I \) over \( \epsilon \).
Lemma D.4 ([96, Lemma 5, page 98])

1. In the interval \( \|t\| < \sqrt{2\varepsilon^{-1/3}} \)

\[
\delta(\|t\|) \leq \delta_1^* = \exp \left( -\frac{t^T t}{2} \right) \left[ \exp(L(\|t\|\varepsilon^{-1/3})) - 1 \right]
\]

where \( L(x) = -\frac{x^T x}{2} + \frac{\|x\|^3}{6} - \ln \left( 1 - \frac{x^T x}{2} \right) \)

2. For all values of \( t \)

\[
\delta(\|t\|) \leq \delta_2^* = \exp \left( -\frac{t^T t}{2} \right) \left[ \exp \left( \frac{1}{3}\|t\| \right) + 1 \right]
\]

3. For all values of \( t \)

\[
\delta(\|t\|) \leq \delta_3^* = 1 + \exp \left( -\frac{t^T t}{2} \right)
\]

Proof

1. Let \( r_k = f_k - 1, f_k = f_k(t/\sigma) \) where \( f_k \) is the characteristic function of \( y_k \) and \( \sigma^2 = \sum \sigma_k^2 \). We have

\[
|r_k| = \left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \exp \left( j t^T x \right) - 1 - j t^T x \right) dF_k(x) \right|
\]

\[
\leq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{(t^T x)^2}{2} dF_k(x)
\]

\[
= \frac{\sigma_k^2 t^T t}{\sigma^2} < e^{2/3} \frac{t^T t}{2}
\]

(D.5)

where \( F_k \) is the distribution of \( y_k \). Consequently, in the interval \( \|t\| < \sqrt{2\varepsilon^{-1/3}}, |f_k| > 0 \) for all \( k \). This gives us the possibility of considering \( \ln f_k \) without having to fear branching points. We have

\[
\ln(1 + r_k) = r_k + \theta_1 \{ \ln(1 - |r_k|) + |r_k| \}
\]

which, for a set value of \( r_k \), is true for some value of \( |\theta_1| \leq 1 \). Using equation D.5 we also have

\[
\ln f_k = -\frac{\sigma_k^2 t^T t}{\sigma^2} + \theta_2 \frac{t^T t}{\sigma^3 \varepsilon} L \left( \|t\| \varepsilon^{1/3} \right)
\]
for some $|\theta_2| \leq 1$. Summing over all $k$ term by term, we obtain
\[
\ln f = -\frac{t^T t}{2} + \theta_3 L \left( \|t\|^{1/3} \right)
\]
for some $|\theta_3| \leq 1$. Since for any complex $z$,
\[
|e^z - 1| \leq e^{|z|} - 1
\]
we have in the interval of variation of $t$ under consideration
\[
\delta(\|t\|) \leq \exp(-t^T t/2) \left| \exp \left( \frac{t^T t}{2} + \ln f \right) - 1 \right| \leq \delta^*_1
\]

2. In the univariate case this is a consequence of [37, lemma 4]. Let $y_k'$ be a
random vector independent of $y_k$ but having the same distribution. Thus,
the characteristic function of the random vector $y_k - y_k'$ is $|f_k(t)|^2$. We have
\[
E\{\|y_k - y_k'\|^3\} \leq E\{\|y_k - y_k'\|^2(\|y_k\| + \|y_k'\|)\}
= 2(\beta_k + \sigma_k^2 E\{\|y_k\|\}) \leq 4\beta_k
\]
Therefore, for some $\theta < 1$
\[
|f_k(t)|^2 = 1 - \sigma_k^2 t^T t + \theta E\{\|y_k - y_k'\|^3\} \frac{\|t\|^3}{6}
\leq 1 - \sigma_k^2 t^T t + \frac{2}{3} \beta_k \|t\|^3
\]
Hence,
\[
|f(t/\sigma)|^2 \leq \prod_k I_k, \quad I_k = 1 - \frac{\sigma_k^2}{\sigma^2} t^T t + \frac{2}{3} \beta_k \|t\|^3
\]
Since
\[
\sum_k I_k = K - t^T t + \frac{2}{3} \|t\|^3
\]
is a constant for fixed values of $t$ and $\epsilon$, the product $\prod I_k$ attains its greatest
value when the factors are equal. For $\|t\| \leq 1.5/\epsilon$ we have
\[
\left| f \left( \frac{t}{\sigma} \right) \right|^2 \leq \left[ 1 - \frac{t^T t}{K} \left( 1 - \frac{2}{3} \epsilon \|t\| \right) \right]^K
\leq \exp \left( -t^T t \left( 1 - \frac{2}{3} \epsilon \|t\| \right) \right)
\]
So for $\|t\| \leq 1.5/\epsilon$,

$$
\delta(\|t\|) = \left| f(t) - \exp \left( -\frac{t^T t}{2} \right) \right| \leq \exp \left( -\frac{t^T t}{2} \right) \left[ \exp \left( \frac{\epsilon \|t\|^3}{3} \right) + 1 \right]
$$

3. For all values of $t$, $|f(t)| \leq 1$. Therefore,

$$
\delta(\|t\|) \leq \delta_3^* = 1 + \exp \left( -\frac{t^T t}{2} \right)
$$

□

As the required estimate $\delta^* = \delta^*(\|t\|, \epsilon)$ we naturally select $\delta^* = \min(\delta_1^*, \delta_2^*, \delta_3^*)$. Since each of the estimates $\delta_i^*$ is monotonic in $\epsilon$, $\delta^*$ will also have this property. It is not difficult to see that in addition the $\delta_i^*$ will be piece-wise monotonic functions of the variable $\|t\|$.

If we denote by $\alpha = 1.302866$ the only positive root of the equation

$$
\exp (L(x)) - 1 = \exp \left( \frac{x^3}{3} \right) + 1
$$

then the estimate $\delta^*$ can be written in the form

$$
\delta^* = \begin{cases} 
\delta_1^* & \text{if } \|t\| < \alpha \epsilon^{-1/3} \\
\delta_2^* & \text{if } \alpha \epsilon^{-1/3} \leq \|t\| < 1.5 \epsilon^{-1} \\
\delta_3^* & \text{if } \|t\| \geq 1.5 \epsilon^{-1}
\end{cases}
$$

So we have for our case that

$$
Q^*(y) = \frac{1}{\sqrt{2\pi}} \int_0^y \left( \frac{y}{\tau} - 1 \right) \delta^*(\tau, \epsilon) d\tau
$$

and

$$
I(x, y, \epsilon) = \frac{x \left[ \sqrt{2\pi} V(x) + \sqrt{2\pi} Q(y) \right]}{y \left[ 4\pi V(x) - \pi x \right]}
$$

These are \emph{exactly} the same equations and are subject to the same conditions as those obtained by Zolotarev [36]. From these he is able to show that $\frac{I(x, y, \epsilon)}{\epsilon} < 1.3213$ which yields an upper bound for $C$.

There are sharper bounds for $C$ obtained later by Zolotarev [39] and by van Beek [29,30]. We speculate that these bounds may also apply to our case of interest but, as of yet, have no results in this direction.
Bibliography


