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Finite element solution methods for linear and nonlinear beam-on-foundation problems

Stephens, Denny Robert, Ph.D.
Rice University, 1989

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FINITE ELEMENT SOLUTION METHODS
FOR LINEAR AND NONLINEAR
BEAM-ON-FOUNDATION PROBLEMS

by

DENNY R. STEPHENS

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IN PARTIAL FULFILLMENT OF THE
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FINITE ELEMENT SOLUTION METHODS
FOR LINEAR AND NONLINEAR
BEAM-ON-FOUNDATION PROBLEMS

DENNY R. STEPHENS

ABSTRACT

The objective of this research is to develop finite elements and iterative techniques for numerical solution of linear and nonlinear beam-on-foundation problems encountered in structural engineering. Although closed-form solutions are available for analyzing simple linear load cases of beam-on-foundation problems, complicated loading combinations or beam-on-nonlinear-foundation problems generally require sophisticated numerical solution methods. This research establishes new finite elements that yield exact solutions at the nodes for linear beam-on-foundation problems and new iterative techniques for rapid solution convergence of beam-on-nonlinear-foundation problems.

The approach taken here in solving linear beam-on-foundation problems is unique in that the solution of the homogeneous portion of the fourth-order differential equation is used for the finite element shape functions. Furthermore, the complex form of the solutions is used rather than the real form. Elements that result from these shape functions are real and yield several major advantages for solving this class of problems, including
• Achieving exact solutions at the finite element nodes for linear, self-adjoint problems

• Providing two, easily implemented, complex shape function elements that address the majority of linear beam-on-foundation problems.

Nonlinear foundation problems are divided into two classes for solution here: those in which the foundation response and its first derivative are continuous functions of displacement ($C^1$-continuous functions), and those in which the foundation response can be modeled as a piecewise-linear function ($C^0$-continuous functions).

An example of the first case is the lateral motion of a beam buried in soil where restraint forces vary continuously with deflection. This class of problems is addressed using numerical integration to compute element stiffnesses and quasi-Newton methods to perform iterative solution.

An example of the second (piecewise-linear) class of foundations is that of a "compression only" foundation in which a beam may "liftoff" and lose contact with the foundation. A simple and straightforward nonlinear solution method is developed for this type of problem which exhibits rapid convergence properties.
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I want to sincerely thank my wife Karen, and my parents, Bob and Evelyn Stephens, for their support and encouragement through the long years. Special thanks to Karen who willingly conceded the evenings and weekends I needed to devote to this all consuming task.
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NOMENCLATURE

$a, a_k$ approximate derivative used in quasi-Newton solution of nonlinear equations

$A$ Cross sectional area of the beam $(\text{length}^2)$

$A$ used for integration over the beam cross section

$A$ Approximation of a Jacobian matrix, $J$

$B, B_y, B_z$ matrix containing the values of interpolation function $\phi(x)$ at element nodes

$B$ nonlinear differential operator

$C, C_1, C_2, C_3, C_4$ constants used in homogenous solutions, element shape functions, nonlinear foundations, and other functions

$C$ vector of interpolation function constants

$dx$ differential beam element

$\mathbf{d^e}$ three-dimensional beam element displacement vector

$E$ modulus of elasticity $(\text{force/length}^2)$

$f$ function in the domain $\Gamma$ (independent of displacement $u$)

$f$ nonlinear function of one variable

$F$ system of nonlinear equations

$g$ function in the domain $\Gamma$ (independent of displacement $u$)

$g$ nonlinear function of one variable (solved by direct iteration)

$G$ shearing modulus of elasticity $(\text{force/length}^2)$

$G$ system of nonlinear equations (solved by direct iteration)
NOMENCLATURE (Continued)

\( h, h_k \) finite difference step length used in quasi-Newton solution of nonlinear equations

\( I, I_y, I_z \) second moment of area of the beam, commonly known as moment of inertia (length\(^4\))

\( I_{yz} \) product of inertia of the beam (length\(^4\))

\( J \) polar moment of inertia of the beam (length\(^4\))

\( J \) Jacobian matrix of a system of nonlinear equations

\( k^0, k^0_x, k^0_y, k^0_{x,y} \) element stiffness matrix

\( k, k_x, k_y, k_{x,y} \) coefficient of foundation reaction per unit length (force/length\(^2\))

\( k_t, k_s \) tangent and secant stiffness of one-dimensional nonlinear foundation stiffness functions (force/length\(^2\))

\( K \) global stiffness matrix

\( K_s \) secant stiffness matrix of a system of nonlinear equations

\( K_t \) tangent stiffness matrix of a system of nonlinear equations

\( \xi, \xi_0, \xi_k \) location of the interface between free beam and beam-on-foundation system in beam liftoff problems (length)

\( \xi, \xi_0 \) vector of interface locations between free beam and beam-on-foundation system in beam liftoff problems

\( L \) length of beam finite element (length)

\( \mathcal{L} \) self-adjoint linear differential operator in the domain \( \Gamma \)

\( m_i \) solution constants of the characteristic equation of a homogeneous differential equation (dimensionless)
NOMENCLATURE (Continued)

\( M, M_x, M_y, M_z \)  beam internal bending moment (force-length)
\( M^*, M^*_x, M^*_y, M^*_z \)  concentrated external bending moment (force-length)
\( M_k \)  local model of a one-dimensional nonlinear equation
\( M_k \)  local model of a system of nonlinear equations
\( n \)  constant in power law nonlinear foundation model (dimensionless)
\( N \)  beam internal normal axial force (force)
\( N^* \)  concentrated external axial force (force)
\( p, p_x, p_y, p_z \)  distributed foundation restraint force (force/length)
\( p_{\text{max}} \)  maximum soil restraint force (force/length)
\( p \)  nonlinear differential operator
\( q^e \)  element load vector
\( q, q_x, q_y, q_z \)  distributed body forces (force/length)
\( q_1 \)  element load (body force) at \( x=0 \) (force/length)
\( q_2 \)  element load (body force) at \( x=L \) (force/length)
\( Q \)  global load vector
\( r \)  distance form the centroidal beam axis (length)
\( \mathcal{X} \)  Galerkin weighted residual
\( S, S_y, S_z \)  beam internal shear force (force)
\( S^*, S^*_y, S^*_z \)  concentrated external shear force (force)
\( s \)  used for integration over the beam surface
NOMENCLATURE (Continued)

t time
u axial (x direction) displacement of the beam (length)
u\textsuperscript{o} vector of nodal x direction displacements, axial degrees of freedom
\ddot{\mathbf{u}}\textsuperscript{o}, \dddot{\mathbf{u}}\textsuperscript{o} nodal x direction velocity and acceleration vectors
\ddot{\mathbf{u}}(x) approximate Galerkin solution of differential operator
u, u unknown solution of differential equation
u\textsubscript{e} exact solution of differential equation
u\textsubscript{n} continuous function of vector \mathbf{v}\textsubscript{n}, containing exact nodal solutions
u\textsubscript{d} difference between functions u\textsubscript{e} and u\textsubscript{n}
v vertical (y direction) displacement of the beam (length)
v\textsuperscript{o} vector of nodal y direction displacements and first derivatives (rotations), vertical degrees of freedom
\ddot{\mathbf{v}}\textsuperscript{o}, \dddot{\mathbf{v}}\textsuperscript{o} nodal y direction velocity and acceleration vectors
\ddot{\mathbf{v}} global solution vector
\mathbf{v}\textsubscript{n} vector containing exact differential equation solutions at the finite element nodes
\mathbf{v}\textsubscript{k}, \mathbf{v}\textsubscript{*} solution iteration vector for a system of nonlinear equations
w lateral (z direction) displacement of the beam (length)
w\textsuperscript{o}, \mathbf{w}\textsuperscript{o} vector of nodal z direction displacements and first derivatives (rotations), lateral degrees of freedom
\ddot{\mathbf{w}}\textsuperscript{o}, \dddot{\mathbf{w}}\textsuperscript{o} nodal z direction velocity and acceleration vectors
NOMENCLATURE (Continued)

\( W_i \)  Gaussian quadrature weight constants
\( x, y, z \)  Cartesian coordinate system with \( x \) directed along beam axis
\( x, x_k, x_* \)  variable in one dimensional nonlinear function
\( x \)  vector containing finite element node locations
\( \alpha \)  overrelaxation constant (dimensionless)
\( \gamma \)  Lipshitz constant (dimensionless)
\( \varepsilon_x \)  beam strain in the \( x \) direction (length/length)
\( \varepsilon_x, \varepsilon_y, \varepsilon_z \)  normal beam strains in the \( x, y, \) and \( z \) planes, respectively (length/length)
\( \varepsilon_{xy} \)  torsional shear strain about the centroidal beam axis (length/length)
\( \varepsilon_{yz}, \varepsilon_{zx} \)  beam shear strains in the \( y-z \) and \( x-z \) planes, respectively (length/length)
\( S_1, S_2, S_3, S_4, S_5, S_6 \)  vector of independent nodal weight functions
\( \eta \)  characteristic of an axially loaded beam system = \( \sqrt{\frac{N}{E I}} \) (1/length)
\( \theta \)  rotation of the plane perpendicular to the beam axis about its z axis (rotation)
\( \lambda \)  characteristic of a beam on elastic foundation system = \( \sqrt{\frac{k}{4E I}} \) (1/length)
\( \mu \)  measure of nonlinearity of a function (dimensionless)
\( \nu \)  Poisson's ratio (dimensionless)
NOMENCLATURE (Continued)

\( \xi \) location of zero deflection of the beam in liftoff problems (length)

\( \rho \) radius of curvature of beam in bending (length)

\( \rho \) lower bound on function derivative

\( \phi \) rotation of the plane perpendicular to the beam axis about its y axis (rotation)

\( \phi \) element interpolation vector derived from the homogeneous solution of a differential equation or arbitrary function

\( \sigma_x, \sigma_y, \sigma_z \) normal beam stresses in the x, y, and z planes, respectively (force/length²)

\( \sigma_{xy} \) torsional shear stress about the centroidal beam axis (force/length²)

\( \sigma_{yz}, \sigma_{zx} \) beam shear stresses in the y-z and z-x planes, respectively (force/length²)

\( \tau_x, \tau_y, \tau_z \) surface tractions on the beam exerted by foundation restraints (force/length³)

\( \psi_1, \psi_2, \psi_3 \) vector of independent nodal shape functions

\( \psi^0_1, \psi^0_2, \psi^0_3 \) individual terms of the shape function vectors \( \psi^0 \)

\( \omega \) rotation about the centroidal x beam axis (rotation)

\( \omega^0 \), vector of nodal torsion rotation, torsion degrees of freedom

\( \Gamma \) mass per unit length of the beam

\( \Gamma \) boundary of domain \( \Omega \)
NOMENCLATURE (Continued)

Σ represents a direct assembly process of the contribution of
discrete elements to a global matrix expression

φ viscous damping constant per unit length of the beam

Ω,Ω^⊕ domain of differential operators

|| denotes absolute value or a matrix norm
...tum miraculum occidit...
FINITE ELEMENT SOLUTION METHODS
FOR LINEAR AND NONLINEAR
BEAM-ON-Foundation PROBLEMS

1 PROBLEM DESCRIPTION AND INTRODUCTION

Railroad track and buried pipelines are two of the most common examples of beam-on-foundation structures. In each of these cases a beam is supported continuously along its length by a foundation support which restrains the vertical motion of beam. The foundation responds to beam deflections by exerting a normal pressure corresponding to the amount of deflection. The beam-on-foundation problem is illustrated in two dimensions in Figure 1.1. Here a straight beam under tension is supported along its entire length by an elastic medium and subjected to vertical forces. The applied forces cause the beam to deflect, producing continuously distributed reaction forces in the supporting medium. The reaction forces are assumed to be acting vertically, opposing the deflection of the beam. The objective of the research presented here is to develop a series of finite elements and nonlinear iteration techniques for solving complex linear and nonlinear beam-on-foundation problems.

Closed-form solutions are available\(^{(1,2)}\) for analyzing simple linear load cases of beam-on-foundation problems. These solutions are generally limited to problems of local loads on infinite or semi-infinite length beams. Complicated loading combinations or problems
involving nonlinear foundation characteristics, however, require more sophisticated approaches and generally must be solved using numerical techniques. Typically, the numerical solution schemes used for the problems rely on finite difference approaches(3,4) or finite element approaches based upon assumed polynomial shape functions(5).

The specific focus of this investigation is upon application of the Galerkin weighted residual method to beam-on-foundation problems that can be addressed by the fourth-order differential equation

$$\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} - \frac{dN}{dx} \frac{dv}{dx} + k(v)v = q \quad (1.1)$$

where

$\v$ is lateral deflection of the beam

$E$ is the beam modulus of elasticity

$I$ is the second moment of area of the beam, commonly known as moment of inertia

$N$ is the axial force in the element beam, positive for a tensile force and negative for a compressive force

$k(v)$ is the nonlinear coefficient of foundation reaction per unit length

$q$ is the distributed vertical loading on the beam.

The approach to solving linear beam-on-foundation problems taken here is unique in that the solution of the homogeneous portion of Equation 1.1 is used for the finite element shape functions. Furthermore, the complex form of the solutions is used rather than the
real form. The elements that result from these shape functions are real and offer a number of advantages for solving this class of problems.

The first advantage is that the elements yield exact solutions at finite element nodes. A mathematical proof is given which shows that use of the exact solution of the applicable homogenous differential equation as the shape function for the elements will result in exact solutions at the finite element nodes for linear, symmetric (self-adjoint) problems. In contrast with typical polynomial finite elements, additional elements do not have to be added to a finite element model for the sole purpose of improving solution accuracy. This approach permits solution of applicable problems with the minimum number of elements necessary for its geometric description.

The second advantage is that two complex shape function elements address the majority of linear beam-on-foundation problems, whereas six elements are necessary to address the same class of problems if real shape functions are used. Fewer element types simplify the implementation of finite element solution methods for these problems.

The finite elements developed here may be used for exact nodal solution of complex problems involving the following structures:

- Beams on elastic foundation
- Beam-columns and tensioned beams
- Beam-columns and tensioned beams on elastic foundation
- Beams on two-parameter foundations.
There are other engineering problems for which this fourth-order differential equation and resulting finite elements are applicable, such as cylindrical tubes under axially symmetric loading\(^1\), but this effort is focused on conventional beams on foundation.

The complex shape function elements are also used here for solving beam-on-nonlinear-foundation problems. Nonlinear foundation problems are divided into two classes: those in which the foundation response and its first derivative are continuous functions of displacement (\(C^1\)-continuous functions), and those in which the foundation response can be modeled as a piecewise-linear function (\(C^0\)-continuous functions). An example of the first case is the lateral motion of a beam buried in soil, where restraint forces vary continuously with deflection. This class of problems is addressed using numerical integration to compute element stiffnesses and variations on Newton's method for iterative solutions.

A primary example of the second (piecewise-linear) class of foundations is that of a "compression-only" foundation in which the foundation does not exert tensile loads. In this case a beam may "liftoff" and lose contact with the foundation. This problem is unique in that it becomes one of a conventional freely deforming beam, coupled to a beam-on-foundation system. The free, noncontact beam and the beam-on-foundation system are each represented by a linear differential equation. The location of the interface between the two components is a nonlinear function of the problem variables. Nonlinear solution methods are required to find the location of the interface "liftoff points". A
new, straightforward method is developed here for solving this type of nonlinear problem that exhibits rapid convergence properties.

Background discussion is presented in Chapter II including discussion of the fourth-order differential equation and its special cases for beam-on-foundation problems as well as assumptions and implementation of the Galerkin weighted residual method. Chapter III presents a derivation of fourth-order differential equations for the problem and application of the Galerkin method to derive the associated residual equations for element development. Chapter III concludes with a mathematical proof showing that the use of homogeneous-solution shape functions will result in exact problem solutions at the finite element nodes for linear, symmetric systems. Chapter IV presents the application of the residual expression to the homogeneous-solution shape function for the general problem and its special cases for development of finite elements. Elements are derived for finite length beams as well as for semi-infinite length beams. Chapter V develops a nonlinear solution method for beam on piecewise-linear foundation and proof of its q-superlinear rate of convergence. Chapter VI applies the Galerkin method and variations of Newton's method to the solution of beam-on-foundation problems whose foundations are $C^1$ continuous. Each chapter provides examples of application of the elements and solution methods developed and comparisons with results reported in literature.

The motivation for this work is a need on the part of the author for a collection of beam-on-foundation finite elements and solution strategies for accurate solution of linear and nonlinear buried pipe and railroad track analysis problems. The elements developed here
may be implemented easily and used to solve a broad range of problems accurately and efficiently. The ability to achieve exact solutions in the linear case for direct comparison with simple closed-form solutions provides an improved level of confidence in the solution of complex linear and nonlinear problems for which closed-form solutions do not exist. Material developed here also provides a tool to explore the relationship of empirical approaches to structural finite element analysis with the theoretical basis of numerical solution of linear and nonlinear differential equations.
II BACKGROUND

This chapter presents background information needed for developing finite element solutions of fourth-order beam-on-foundation problems. The chapter begins with a discussion of fourth-order beam equations and previously developed finite elements for their solution. Beam equations for a number of common structural engineering problems are introduced and shown to be special cases of Equation 1.1. In the second section of the chapter, typical nonlinear foundations models used for beam-soil interaction problems are explained in preparation for later development of nonlinear solution techniques. The chapter concludes with a description of the Galerkin weighted residual method, which is used here for solution of beam-foundation interaction problems using the finite element method.

Fourth-Order Beam and Beam-on-Foundation Problems

The fourth-order differential equation that is the focus of this study,

$$\dfrac{d^2}{dx^2} \left[ EI \dfrac{d^2 y}{dx^2} \right] - N \dfrac{d^2 y}{dx^2} - \dfrac{dN}{dx} \dfrac{dy}{dx} + kv = q \tag{2.1}$$

has four terms that describe beam deflection, $y$, and its derivatives in response to the transverse loading, $q$. The first, fourth-order term reflects the bending stiffness of the beam itself. The second order
term results from mean axial load $N$ in the beam, while the first order
term is a consequence of change in axial load. The last, linear term
results from the contribution of the foundation. Equation 2.1
implicitly includes a number of special cases in which one or more of
the terms is zero. Each of the special cases represents a unique beam
or beam-on-foundation problem geometry with a corresponding homogeneous
solution. Each case is important because it represents a different mode
of beam-on-foundation behavior and because the finite elements developed
here must solve each case as well as the general problem. Following is
a discussion of the special cases of Equation 2.1, their homogeneous
solutions, and finite elements that have been developed for numerical
solutions. The cases are discussed in order of increasing complexity.

**Conventional Cubic Beam Elements**

Clearly, the first and most simple special case of Equation
2.1 is that in which all but the fourth order term is zero. This is the
equation for the classical Euler-Bernoulli beam under transverse loading
given by the expression

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right) = q .
\]  

(2.2)

The homogeneous solution of this equation is the cubic polynomial
where

\[ v = C_1 + C_2 x + C_3 x^2 + C_4 x^3 \]  

(2.3)

This equation results from the Euler-Bernoulli assumption for a beam in bending that plane sections remain plane(6) and that strains remain small. As illustrated in Figure 2.1, the neutral axis of this simple beam is assumed to be bent from an initially straight line into a curve of radius \( \rho \).

Przemieniecki(7) illustrates the development of stiffness matrices for numerical solution of beam problems from closed-form
solutions of Equation 2.3 for specific boundary conditions. The beam element matrix that results from such solutions, given in Table 2.1(c), is found in numerous texts (8,9) and finite element programs. Fundamental mathematical approaches using variational analysis and assumed cubic displacement (shape) functions (8) have been shown on numerous occasions to yield identical finite element matrices. The finite element solutions for simple beams that result from these "cubic" finite elements are observed to be exact at the finite element nodes.

The cubic polynomial of Equation 2.3 is commonly used as the shape function for more complicated beam problems because it is simple and meets the continuity requirements for fourth-order finite elements. Table 2.2 summarizes the finite element matrices that result from each term of Equation 2.1 when cubic polynomials are assumed as the shape functions. These matrices yield approximate solutions at the finite element nodes.

**Beam-on-Elastic-Foundation Elements**

The next special case of Equation 2.1 is the equation for a beam resting on an elastic foundation which arises when Euler-Bernoulli beams are supported by an elastic restraining medium. The differential equation for this case is

\[
\frac{d^2}{dx^2} \left[ EI \frac{d^2v}{dx^2} \right] + kv = q .
\]  

(2.4)

The homogeneous solution of this equation is commonly written in terms of hyperbolic and trigonometric functions as
Table 2.1 Finite Element for Conventional Cubic Beam

<table>
<thead>
<tr>
<th>a) Differential Equation</th>
<th>( \frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) = q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>b) Shape Function</td>
<td>( v = C_1 + C_2x + C_3x^2 + C_4x^3 )</td>
</tr>
<tr>
<td>(Homogeneous Solution)</td>
<td></td>
</tr>
<tr>
<td>c) Stiffness Matrix</td>
<td>( k^e = \frac{EI}{L^3} \begin{bmatrix} 12 &amp; 6L &amp; -12 &amp; 6L \ 4L^2 &amp; -6L &amp; 2L^2 \ 12 &amp; -6L \ \text{symm.} &amp; 4L^2 \end{bmatrix} )</td>
</tr>
<tr>
<td>d) Load Vector</td>
<td>( q^e = \frac{L}{60} \begin{bmatrix} 30q_1 + 9(q_2-q_1) \ 5q_1L + 2(q_2-q_1)L \ 30q_1 + 21(q_2-q_1) \ -5q_1L - 3(q_2-q_1)L \end{bmatrix} )</td>
</tr>
<tr>
<td>(Trapezoidal load assuming ( q = q_1 + (q_2-q_1)x ))</td>
<td></td>
</tr>
</tbody>
</table>
Table 2.2 Cubic Finite Element Matrices for Fourth Order Differential Equation

a) Differential Equation
\[ \frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right) - N \frac{d^2y}{dx^2} - \frac{dN}{dx} \frac{dy}{dx} + ky = q \]

b) Shape Function
\[ y = C_1 + C_2x + C_3x^2 + C_4x^3 \]

c) Beam Stiffness Matrix\(^{(7,8)}\)
\[ \left\{ \frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right) \right\} \]
\[ k_b = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 4L^2 & -6L & 2L^2 & \\ \text{symm.} & & 4L^2 & \end{bmatrix} \]

d) Axial Stiffness Matrix\(^{(7)}\)
\[ \left\{ -N \frac{d^2v}{dx^2} \right\} \]
\[ k_n = \frac{N}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 4L^2 & -3L & -L^2 & \\ \text{symm.} & & 4L^2 & \end{bmatrix} \]
e) Axial Variation Stiffness Matrix\((10)\)  
\begin{align*}
\{ -\frac{dN}{dx} \frac{dv}{dx} \} & \quad k_{MN}^2 \frac{dN}{dx} \frac{1}{60} \\
\begin{bmatrix}
-30 & 6L & 30 & -6L \\
-6L & 0 & 6L & -L^2 \\
-30 & -6L & 30 & 6L \\
6L & L^2 & -6L & 0
\end{bmatrix}
\end{align*}

f) Foundation Stiffness Matrix\((5)\)  
\begin{align*}
\{ kv \} & \quad k_f = \frac{k}{420} \\
\begin{bmatrix}
156L & 22L^2 & 54L & -13L^2 \\
4L^3 & 13L^2 & -3L^3 & \\
\text{symm.} & 156L & -22L^2 & 4L^3
\end{bmatrix}
\end{align*}

g) Load vector\((10)\)  
(assuming \(q = q_1 + (q_2-q_1)\frac{x}{L}\))  
\begin{align*}
\{ q \} & \quad q^n = \frac{L}{60} \\
& \quad \begin{bmatrix}
30q_1 + 9(q_2-q_1) \\
5q_1L + 2(q_2-q_1)L \\
30q_1 + 21(q_2-q_1) \\
-5q_1L - 3(q_2-q_1)L
\end{bmatrix}
\end{align*}
\[ v = C_1 \cosh(\lambda x) \cos(\lambda x) + C_2 \cosh(\lambda x) \sin(\lambda x) \\
+ C_3 \sinh(\lambda x) \cos(\lambda x) + C_4 \sinh(\lambda x) \sin(\lambda x) \]
\[ \lambda = \sqrt[4]{\frac{k}{AE\ell}}. \] 

The term \(\lambda\), known as the "characteristic" of the beam on elastic foundation, is a structural stiffness parameter which includes the flexural rigidity of the beam as well as the elasticity of the supporting medium.

This equation results from the assumption that the beam is in contact with a foundation that reacts to beam displacement with a force proportional at every point to the deflection of the beam at that point. The problem is conceptualized as a beam lying on a series of adjacent, independently acting, linear springs, as illustrated in Figure 2.2. This formulation was introduced by Winkler(11) in 1867 and was the basis of a comprehensive text on the subject by Hetenyi(1), published in 1946. This equation has proven to be a very useful approximation for engineering solution of a wide variety of soil-structure interaction problems.

In 1977, Schmidt(5) developed a finite element for the beam on elastic foundation given by Equation 2.4 in which he used the cubic polynomial of Equation 2.3 as the element shape function. He found that the fourth-order beam term results in the same matrix shown in Table 2.1(c) and in Table 2.2(c). The linear foundation term yields the matrix given in Table 2.2(f).

Eisenberger and Yankelevsky(12) and Ting and Mockry(13) have independently developed an "improved" finite element for the beam-on-elastic-foundation problem using the homogeneous solution of
Figure 2.2  Beam on Elastic Foundation Model

Equation 2.5 as the shape function. The resulting finite element
stiffness matrix, as presented by Eisenberger and Yankelevsky, is given
in Table 2.3. The beam-on-elastic-foundation solutions that result from
this matrix have proven to be exact at the finite element nodes.

Beam-Column and Tensioned Beam Elements

The next special case, illustrated in Figure 2.3, represents a
conventional beam under transverse loading combined with either axial
tension or compression loading. This case is defined by the equation

\[
\frac{d^2}{dx^2} \left( \frac{EI}{dx^2} \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} = q .
\]  

(2.6)
Table 2.3 Homogeneous Solution Finite Element for Beam on Elastic Foundation

a) Differential Equation
\[ \frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right) + kv = q \]

b) Homogeneous Solution
\[ v = C_1 \cosh(\lambda x) \cos(\lambda x) + C_2 \cosh(\lambda x) \sin(\lambda x) \]
\[ + C_3 \sinh(\lambda x) \cos(\lambda x) + C_4 \sinh(\lambda x) \sin(\lambda x) \]
\[ \lambda = \sqrt[4]{\frac{k}{4EI}} \]

\[ k = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ \text{symm.} & & & k_{44} \end{bmatrix} \]

\[ k_{11} = k \frac{\sinh(\lambda L) \cos(\lambda L) + \cos(\lambda L) \sin(\lambda L)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} \]
\[ k_{12} = k \frac{\sin^2(\lambda L) + \sin^2(\lambda L)}{2\lambda^2 \sinh^2(\lambda L) - \sin^2(\lambda L)} \]
Table 2.3 (Continued)

\[
\begin{align*}
    k_{13} &= - \frac{k}{\lambda} \frac{\sinh(\lambda L) \cos(\lambda L) + \cosh(\lambda L) \sin(\lambda L)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} \\
    k_{22} &= \frac{k}{2\lambda^3} \frac{\sinh(\lambda L) \cosh(\lambda L) - \cos(\lambda L) \sin(\lambda L)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} \\
    k_{24} &= \frac{k}{2\lambda^3} \frac{\cosh(\lambda L) \sin(\lambda L) - \sin(\lambda L) \cosh(\lambda L)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} \\
    k_{14} &= \frac{k}{\lambda^2} \frac{\sinh(\lambda L) \sin(\lambda L)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} \\
    k_{23} &= \frac{k}{\lambda^2} \frac{\sinh(\lambda L) \sin(\lambda L)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} \\
    k_{33} &= k_{11} \\
    k_{34} &= -k_{12} \\
    k_{44} &= k_{22}
\end{align*}
\]
The axial loading on the beam results in a second order contribution to the differential equation. Two solutions exist for the equation, depending upon the sign of \( N \). If we define \( \eta = \sqrt{\frac{|N|}{EI}} \), the solution of Equation 2.6 for \( N \) positive (tensioned beam) is

\[
v = C_1 \cosh(\eta x) + C_2 \sinh(\eta x) + C_3 x + C_4
\]  

(2.7)

and the solution for \( N \) negative (beam column) is

\[
v = C_1 \cos(\eta x) + C_2 \sin(\eta x) + C_3 x + C_4
\]  

(2.8)

Finite elements based upon the above differential equation have been developed using both cubic shape functions(7) and homogeneous solutions of the differential equation(8,14). As before for a cubic
shape function, the fourth-order beam bending term yields the stiffness matrix given in Table 2.2(c). The second-order axial term yields the stiffness matrix in Table 2.2(d) for this case.

The finite element stiffness matrix given by Cook(8) for the homogeneous solutions of Equation 2.6 are given in Table 2.4. These elements have also been shown to result in exact solutions at the finite element nodes for axially loaded beam problems.

**Beam on Foundation Under Axial and Transverse Load**

In recent years a combination of the beam-on-elastic-foundation and the beam-column problems have been discussed in the literature. The differential equation for these problems is

\[
\frac{d^2}{dx^2}\left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} + kv = q .
\]

(2.9)

There are six solutions cases for the homogeneous portion of this equation; which case is selected depends upon the relative values of the coefficients EI, N, and k. The six cases and their homogeneous solutions are given in Table 2.5.

This differential equation arises when tension is considered in two different contexts. First, it occurs when the beam is subjected to an axial load while restrained by an elastic medium(1). Second, it occurs when there is interaction between the "Winkler springs" of the medium. The first case, illustrated in Figure 2.4, is discussed by Hetenyi in his classic text on beams on elastic foundation in which he
Table 2.4 Homogeneous Solution Finite Element for Beam-Column and Tensioned Beam

<table>
<thead>
<tr>
<th>a) Differential Equation</th>
<th>b) Homogeneous Solution</th>
<th>c) Stiffness Matrix(8)</th>
</tr>
</thead>
</table>
| \( \frac{d^2}{dx^2} \left( \frac{\text{EI}}{dx^2} \right) - N \frac{d^2v}{dx^2} = q \) | N positive (Tensioned Beam): \( v = C_1 \cosh(\eta x) + C_2 \sinh(\eta x) + C_3 x + C_4 \) | \( \text{EI} \)
| | N negative (Beam-column): \( v = C_1 \cos(\eta x) + C_2 \sin(\eta x) + C_3 x + C_4 \) | \[ \begin{bmatrix} 12\phi_1 & 6L\phi_2 & -12\phi_1 & 6L\phi_2 \\ 4L^2\phi_3 & -6L\phi_2 & 2L^2\phi_4 & \\ 12\phi_1 & -6L\phi_2 & \\ \text{symm.} & 4L^2\phi_3 \end{bmatrix} \]
| | \( \eta = \sqrt{\frac{\text{EI}}{N}} \) |
Table 2.4 (Continued)

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
<th>( \phi_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\eta L)^3 \sinh(\eta L) )</td>
<td>( (\eta L)^2 { \cosh(\eta L) - 1 } )</td>
<td>( (\eta L) { (\eta L) \cosh(\eta L) - \sinh(\eta L) } )</td>
<td>( (\eta L) { \sinh(\eta L) - \eta L } )</td>
</tr>
<tr>
<td>( 12 { 2 - 2 \cosh(\eta L) + (\eta L)\sinh(\eta L) } )</td>
<td>( 6 { 2 - 2 \cosh(\eta L) + (\eta L)\sinh(\eta L) } )</td>
<td>( 4 { 2 - 2 \cosh(\eta L) + (\eta L)\sinh(\eta L) } )</td>
<td>( 2 { 2 - 2 \cosh(\eta L) + (\eta L)\sinh(\eta L) } )</td>
</tr>
</tbody>
</table>

N negative:

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
<th>( \phi_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\eta L)^3 \sin(\eta L) )</td>
<td>( (\eta L)^2 { 1 - \cos(\eta L) } )</td>
<td>( (\eta L) { \sin(\eta L) - (\eta L)\cos(\eta L) } )</td>
<td>( (\eta L) { (\eta L) \sin(\eta L) } )</td>
</tr>
<tr>
<td>( 12 { 2 - 2 \cos(\eta L) - (\eta L)\sin(\eta L) } )</td>
<td>( 6 { 2 - 2 \cos(\eta L) - (\eta L)\sin(\eta L) } )</td>
<td>( 4 { 2 - 2 \cos(\eta L) - (\eta L)\sin(\eta L) } )</td>
<td>( 2 { 2 - 2 \cos(\eta L) - (\eta L)\sin(\eta L) } )</td>
</tr>
</tbody>
</table>
Table 2.5 Homogeneous Solution Cases for Beam Under Combined Axial and Transverse Load(1)

Differential Equation
\[
\frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right) - N \frac{d^2y}{dx^2} + ky = q
\]

Solution Cases:

1) \( 0 < N < 2 \sqrt{EI} \)
\[
v = C_1 \cosh(ax) \cos(\beta x) + C_2 \sinh(ax) \cos(\beta x) + C_3 \cosh(ax) \sin(\beta x) + C_4 \sinh(ax) \cos(\beta x)
\]

2) \( N = 2 \sqrt{EI} \)
\[
v = C_1 \sinh(ax) + C_2 \cosh(ax) + C_3 x + C_4
\]

3) \( N > 2 \sqrt{EI} \)
\[
v = C_1 \sinh(a+\beta)x + C_2 \cosh(a+\beta)x + C_3 \sinh(a-\beta)x + C_4 \cosh(a-\beta)x
\]

4) \( N < 0 ; |N| < 2 \sqrt{EI} \)
\[
v = C_1 \cosh(ax) \cos(\beta x) + C_2 \sinh(ax) \cos(\beta x) + C_3 \cosh(ax) \sin(\beta x) + C_4 \sinh(ax) \cos(\beta x)
\]

5) \( N < 0 ; |N| = 2 \sqrt{EI} \)
\[
v = C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 x + C_4
\]
Table 2.5  (Continued)

6) \( N < 0 \); \(|N| > 2 \sqrt{EI} \)

\[
v = C_1 \sin(a+\rho)x + C_2 \cos(a+\rho)x
+ C_3 \sin(a-\rho)x + C_4 \cos(a-\rho)x
\]

\[
a = \sqrt{\frac{k}{4EI} + \frac{N}{4EI}}
\]

\[
\rho = \sqrt{\frac{k}{4EI} - \frac{N}{4EI}}
\]
Figure 2.4 Beam on Foundation Under Transverse and Axial loads

derives the differential equation, and discusses a number of closed-form solutions for various loading configurations and boundary conditions.

The second case in which Equation 2.9 arises is the two-parameter foundation problem in which the second order term represents an additional foundation response due to interaction between the "Winkler springs". The behavior of a continuous foundation with interaction is compared with that of the Winkler foundation in Figure 2.5. The foundation response in this case is no longer independent of the response of adjacent points on the foundation. Two different sets of assumptions have been used to represent this type of continuous foundation (see Kerr(15)): 1) Filonenko-Borodich assumed the top ends of
the spring elements were connected to a membrane under tension $N$, as illustrated in Figure 2.6a; and 2) Pasternak provided for shear interactions between the spring elements by connecting the ends of the springs to a layer of incompressible vertical elements which deform only by transverse shear, as illustrated in Figure 2.6b. As shown in Figure 2.6b the shear modulus of the layer is $N$. Although developed from different assumptions, both approaches result in the differential equation given by Equation 2.9.

In 1983, Zhaohua and Cook(16) presented a lengthy closed-form finite element matrix for solving Equation 2.9 for the case in which $0 < N < 2 \sqrt{E}$. The problem solutions found from this intricate
matrix also yield exact solutions at the finite element nodes for example problems.

**General Fourth-Order Differential Equation**

Equation 2.1 contains an additional first order term which incorporates the change in axial load, \( dN \), over the beam. This case arises in beam-on-foundation problems when there is a distributed reaction force restraining axial motion. This term provides a nonsymmetric contribution to the stiffness matrix that is frequently assumed to be negligible in relation to the other terms. It is included in this analysis for completeness and to maintain generality.
The homogeneous solution of Equation 2.1 is the sum of exponentials given by

\[ v = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} \]  \hspace{1cm} (2.10)

where the arguments \( m_i \) are the solution of the quartic characteristic equation

\[ E I m^4 - N m^2 - N' m + k = 0 . \]  \hspace{1cm} (2.11)

The constants \( m_i \) from this equation may be either real or complex, depending upon the values of the coefficients. Consequently, the general solution of this problem may also be real or complex.

At the time of this study, no references were found in the literature to finite elements developed for general solution of Equation 2.1. One objective of this investigation is to develop a method for computing finite element matrices for this expression, using the homogeneous solution, Equation 2.10.

The special case solutions given by Equations 2.5, 2.7, and 2.8 and Table 2.5 are written in terms of real functions as they are most often reported in the literature. The solutions, however, may also be written in terms of complex-exponential functions as shown in Table 2.6. From this table, it is clear that two complex-exponential functions encompass all special case solutions other than the cubic beam in Equation 2.2. The two equations are

\[ v = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} \]  \hspace{1cm} (2.12)

and
<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Complex Exponential Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) + kv = q$</td>
<td>$v = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + C_4e^{m_4x}$</td>
</tr>
<tr>
<td></td>
<td>$m_i = \pm(1 \pm i) \sqrt{\frac{k}{4EI}}$ ; $i = 1,2,3,4$</td>
</tr>
<tr>
<td>$\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} = q$</td>
<td>$v = C_1e^{mx} + C_2e^{-mx} + C_3x + C_4$</td>
</tr>
<tr>
<td></td>
<td>$m = \sqrt{\frac{N}{EI}}$</td>
</tr>
<tr>
<td></td>
<td>for $</td>
</tr>
<tr>
<td></td>
<td>$v = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + C_4e^{m_4x}$</td>
</tr>
<tr>
<td></td>
<td>$m_i = \pm(a \pm i\beta)$ ; $i = 1,2,3,4$</td>
</tr>
</tbody>
</table>
Table 2.6 (Continued)

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Complex Exponential Solution</th>
</tr>
</thead>
</table>

\[ a = \sqrt{\frac{k}{4EI}} + \frac{N}{4EI} \]

\[ \beta = \sqrt{\frac{k}{4EI}} - \frac{N}{4EI} \]

for |N| = 2 \sqrt{EI}:

\[ v = C_1 e^{mx} + C_2 e^{-mx} + C_3 x + C_4 \]

\[ m = (a+i\beta) \]

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} - \frac{dN}{dx} \frac{dv}{dx} + kv = q \]

\[ v = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} \]

\[ m_i \text{ are the roots of} \]

\[ EI m^4 - N m^2 - N' m + k = 0 \]
\[ v = C_1e^{\alpha_1x} + C_2e^{\alpha_2x} + C_3x + C_4. \] (2.13)

The number of homogeneous-solution finite element matrices required for complete solution of Equation 2.1 and its special cases is significantly reduced by using the complex-exponential form of the solution.

**Homogeneous-Solution Shape Functions**

Typically, polynomial displacement fields are used for finite element solutions because they can be manipulated easily and integrations can be carried out in closed form. The cubic displacement function is the most commonly used shape function for beam problems because it is the simplest function that meets the continuity requirements.

The complex-exponential functions given by Equations 2.12 and 2.13 are used in this investigation for development of finite elements for fourth-order beam-on-foundation problems. This approach was selected because the exponential function remains relatively simple to manipulate and the resulting elements offer several advantages over polynomial functions with regard to accuracy, element length, and general application. The following discussion and example provide background information and insight into this effort.

**Example Problem Solution**

In describing the special cases it was noted that whenever the solution of a homogeneous differential equation is used as the shape
function for a finite element, the resulting numerical solutions are exact at the nodes. An original proof, given later in Chapter III, shows that this is always the case for linear, symmetric problems. An example application of an "homogeneous-solution" shape function is instructive as to the accuracy and limitations of such elements.

For this example, consider the straightforward Euler-Bernoulli beam from Equation 2.2

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) = q \tag{2.14}
\]

which, as noted, has the cubic polynomial as its solution

\[
v = C_1 + C_2 x + C_3 x^2 + C_4 x^3. \tag{2.15}
\]

Applying this expression as the finite element shape function results in the beam stiffness matrix found in Table 2.1(c).

Compare the finite element matrix solution with the exact solution of the deflection of the cantilever beam having an applied triangular shaped load defined by (see Figure 2.7)

\[
EI \frac{d^4 v}{dx^4} = - \frac{x}{L}. \tag{2.16}
\]

The solution of this problem is

\[
EI v = - \frac{C x^5}{120 L} + C L x^3 \frac{C}{12} - \frac{C L^2 x^2}{6}. \tag{2.17}
\]

where the end point deflection and rotation (v') are, respectively,

\[
v = - \frac{11 C L^4}{120 E I}, \quad v' = - \frac{C L^3}{8 E I}. \tag{2.18}
\]
Solution of the same problem using the stiffness matrix of Table 2.1(c) with a consistent load matrix having two node points, one at the fixed boundary and one at the free end, requires the solution of the matrices

$$\begin{bmatrix} \frac{EI}{L^3} & 12 & -6L \\ -6L & 4L^2 & 0 \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix} = \begin{bmatrix} \frac{7CL}{20} \\ -\frac{CL^2}{20} \end{bmatrix}.$$  \hspace{1cm} (2.19)

This yields the exact solution at the free node:

$$v = -\frac{11CL^4}{120EI} \hspace{1cm} v' = -\frac{CL^3}{8EI}.$$  \hspace{1cm} (2.20)
For this example problem the solution at the one free node was found to be exact, although using the cubic shape functions to interpolate between nodal points cannot exactly model the fifth-order variation present in the actual solution. For this case, it is then apparent that homogeneous-solution shape function elements do not yield an accurate solution everywhere along the length of the beam, but can be used to achieve accurate solution at the finite element nodes.

**Element Convergence and Size Considerations**

As a direct consequence of the fact that homogenous-solution shape functions yield exact solutions at the finite element nodes, there is no theoretical limit on the size of these elements required to achieve a desired accuracy. Basic texts on finite element analysis discuss the fact that, for approximate elements meeting convergence requirements, exact answers will be approached as more elements are used in finer meshes. Because homogeneous-solution elements yield accurate results, they are "converged" and it is not necessary to limit their size to achieve accurate solutions. Furthermore, because long elements may be used, fewer elements are necessary for a given problem.

**Number of Solution Cases**

If a differential equation is solved using combinations of real functions, as is usually the case, it is common that more than one general solution case exists. This is true for the problem of beam
under combined axial and transverse loading, given in Equation 2.6, which has six different solution cases. If one desires to use real homogeneous-solution shape functions, then six different elements must be developed to address the six cases. The complex-exponential functions (Equations 2.12 and 2.13) that are selected as the basis of the elements developed here have the advantage that they solve the general case and most special cases. Consequently, two element formulations can be used to solve most problems. Additionally, the elements based upon the exponential function can be readily integrated in closed form so that numerical integration is not required.

**Foundation Models**

The elastic foundation hypothesized by Winkler reacts with a force proportional at every point to the deflection of the beam at that point. This results in the model in Figure 2.2 in which the beam is supported by a series of adjacent linear springs which act independently. The elastic restraint yields a relatively simple beam-on-elastic-foundation problem that can be solved in closed form for many load and boundary condition cases. This approach has proven adequate for solving problems in which the nonlinear foundation effects are of secondary importance.

In actual practice, however, beams are usually in contact with soil or rock foundations whose load deflection characteristics differ considerably from the Winkler spring model. Actual foundation reactions are nonlinear functions of beam deflection and of other parameters such
as the beam surface area in contact with the foundation. Iterative finite element techniques are generally required to solve complex, nonlinear foundation restraint problems.

In addition to developing linear finite elements for beam-on-foundation analysis, another objective of this investigation is to develop iterative techniques that can be used to solve a wide variety of nonlinear foundation problems. Two types of nonlinear foundation models are of interest here, those in which the foundation response model is piecewise linear and those in which the foundation response and its first derivative are continuous functions. In the following chapters a unique and efficient solution technique is developed for the first class of problems. For the second class of problems, Newton-type iterative solution methods are established in mathematical terms to suggest optimum solution approaches for different foundation responses. The foundation models are described in more detail below in preparation for development of solution techniques.

**Beam-on-Piecewise-Linear-Foundation Problems**

Some nonlinear foundation restraints are piecewise linear (C° continuous) by their physical characteristics; others use a piecewise-linear model for a smooth, nonlinear foundation response curve. A common example of a piecewise-linear beam-on-foundation problem is that of a beam physically lifting off from a linear-compression foundation, as shown in Figure 2.8. The foundation model for this is bilinear with the result that two linear differential equations apply to different
Figure 2.8 Beam Liftoff From a Compression-Only Foundation
parts of the beam. The differential equation for the beam and foundation in contact has a linear foundation response term and is given by

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) + kv = q. \]  

(2.21)

The differential equation of the noncontact, freely deflecting beam does not have the foundation response term

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) = q. \]  

(2.22)

For this example the point of liftoff is the boundary between the two differential equations and its location is a nonlinear function of the properties of the beam, foundation, and applied loadings.

A piecewise-linear segment may also occur in an otherwise continuous nonlinear response when the foundation has a "strength cutoff". Lateral soil restraint of vertical piles and buried pipelines, as shown in Figure 2.9, has such a strength cutoff that is frequently modeled as a bilinear, elastic-perfectly plastic foundation response. Observe, for these problems, that up to the point at which maximum strength level is attained, the differential equation of the beam-on-foundation is

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) + kv = q \]  

(2.23)

whereas at the point of the maximum loading the foundation responds with only a constant load, and the differential equation no longer has the linear foundation term, but is now
\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) = q - P_{\text{max}}.
\] (2.24)

This is the equation of a freely deforming beam under uniform lateral load without a foundation. These two differential equations are basically the same as those that apply to the beam-liftoff problem. The only difference between the two problems is the value of beam-on-foundation deflection at which the equation boundary matching point occurs. Hence, nonlinear pile and pipeline analysis problems may be governed by multiple linear differential equations in the same fashion as the beam-liftoff problem.

Some engineering applications that may be addressed by multiple linear differential equations are shown in Figures 2.10 through
2.12. Single point liftoff problems such as that shown in Figure 2.8 are frequently encountered in pipeline analysis in cases such as lifting a pipeline during installation or repair. Liftoff must be considered for offshore pipeline installation when a pipeline is lowered to the seabed under tension as shown in Figure 2.10. Although the general problem of offshore pipeline installation is large deflection in nature, the pipeline near the seabed undergoes small deflections.

More complex piecewise-linear problems arise when multiple liftoffs of a beam from a foundation are encountered, such as in analyzing the deflection of railroad track under loading of a train car (Figure 2.11). In this problem the railroad car applies multiple point loads, unevenly spaced along the track. Depending upon the relative stiffness characteristics of track and foundation, the track may liftoff from the foundation at multiple locations. The "equivalent" foundation response and liftoff behavior is also influenced by the speed of the railcars. Prior to this investigation, two point loads were the maximum number of loads for which the multiple liftoff problem had been solved in the literature.

Complex nonlinear foundation response behavior is commonly modeled as a piecewise-linear function. Such problems may also be treated as a set of linear differential equations as illustrated in Figure 2.12. Each segment of the beam-on-foundation is governed by a different linear differential equation, depending upon which segment of the nonlinear response applies. The bilinear elastic-perfectly plastic foundation response mentioned earlier is a subset of this type of foundation response model.
Figure 2.10  Beam Liftoff Problem During Offshore Pipeline Installation
\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_1 v = q \\
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_2 v = q \\
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_3 v = q \\n\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_2 v = q \\
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_1 v = q \\
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_3 v = q \\
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_2 v = q \\
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + k_1 v = q
\]
Individual solution methods have been proposed in the
literature for each of these piecewise-linear-foundation problems. In
Chapter V a uniform and consistent formulation is developed for the
entire class of problems. Based upon this formulation a unique solution
strategy is established which yields rapid convergence to the solution
of both simple and complex geometry problems.

Smooth Nonlinear Force-Deflection Foundation Models

The restraining medium in many beam-on-foundation problems is
typically a nonlinear soil. The constitutive relationships are
generally complicated functions of the type of soil, the moisture
content, soil homogeneity, confining pressure, etc. As a direct
consequence, the restraint relationship for a beam bearing on a soil
foundation cannot be readily predicted analytically. In actual
practice, the foundation reactions are developed experimentally and
given as empirical force-deflection curves.

Figure 2.13 illustrates the two most common $C^1$-continuous
nonlinear foundation response models used in analysis of beam-foundation
interaction for buried pipelines, foundation piles, and railroad
track(18,19,20,21). The power law relationship for foundation reaction
force, $p(v)$, is given by

$$p(v) = - C v^n$$

(2.25)

and the hyperbolic model for foundation reaction force is given by

$$p(v) = - \frac{v}{C_1 + C_2 v}.$$  

(2.26)
Figure 2.13 $C_1$-Continuous Nonlinear Foundation Models

Each of these expressions has two constants which are defined by empirical curve fits to experimental data.

Observe that the models of Equations 2.25 and 2.26 exhibit relatively high stiffness (slope) for small displacement and a significant decrease in stiffness for large displacement levels. This highly nonlinear behavior makes it difficult to identify an optimum solution method that yields rapid convergence for both small and large displacement problems. In Chapter VI Newton-type iterative solution methods are established in mathematical terms for solution of general beam-on-nonlinear-foundation problems.
Galerkin Method

The Galerkin weighted residual method has the advantage over variational approaches that it can be applied directly to solving differential equations without the need to establish a variational principle, and also that it can be applied to nonlinear as well as linear problems. A brief review of the Galerkin methodology is presented below as a conclusion to this chapter.

In general terms, an unknown solution $u$ is sought that satisfies the nonlinear differential equation represented by

$$\mathcal{P}(u) - f = 0$$  \hspace{1cm} (2.27)

in a domain $\Omega$, together with certain boundary conditions

$$\mathcal{B}(u) - g = 0$$  \hspace{1cm} (2.28)

on the boundary $\Gamma$ of the domain. Here $\mathcal{P}$ and $\mathcal{B}$ are nonlinear differential operators and $f$ and $g$ are functions that are independent of $u$. For the Galerkin method, the solution is approximated by $\tilde{u}(x)$, defined by the expansion

$$\tilde{u}(x) = \Psi(x)u.$$  \hspace{1cm} (2.29)

Here, $\Psi(x)$ is a vector of independent shape (interpolation) functions that are prescribed in terms of the variable $x$ and that satisfy the essential boundary conditions. The term $u$ is a vector containing parameters, all or part of which are unknown. For conventional matrix structural analysis the parameters $u$ correspond to degrees of freedom of the element.
Now let the domain $\Omega$ be divided into a collection of discrete elements $\Omega^e$. Over an element domain, $\tilde{u}$ is given by the expression

$$\tilde{u}(x) = \varphi^e(x)u^e; \ x \in \Omega^e$$  \hspace{1cm} (2.30)

where $\varphi^e$ and $u^e$ are subsets of the vectors $\varphi$ and $u$, defined over the element domain. Each shape function vector $\varphi^e$ is zero outside its specific element domain $\Omega^e$. Hence, the approximation for $\tilde{u}$ over the global domain $\Omega$ may be written as the summation of the contributions of the local element functions such as

$$\tilde{u} = \sum_{\Omega} \varphi^e u^e = \varphi u.$$  \hspace{1cm} (2.31)

The summation term $\Sigma$ represents a direct assembly process of the contribution of discrete elements to a global matrix expression. The summation results in vectors $\varphi$ and $u$.

When $\tilde{u}$ is substituted into Equations 2.27 and 2.28 the residuals, $\mathcal{R}_p$ and $\mathcal{R}_g$, respectively, are obtained because $\tilde{u}$ is not exact. The residuals are functions of $x$ and of the coefficient vector $u$ and are given by

$$\mathcal{R}_p(x,u) = \tilde{\alpha}u - f = \varphi(p)u - f$$  \hspace{1cm} (2.32)

$$\mathcal{R}_g(x,u) = \tilde{\beta}u - g = \varphi(g)u - g.$$  \hspace{1cm} (2.33)

The residual can only vanish completely for all $x$ in $\Omega$ if $\tilde{u} = u$.

According to the method of weighted residuals then, it is presumed that $\tilde{u}$ is a good approximation of the solution if the residuals are small.

As implied by the name, the method of weighted residuals seeks a solution of Equations 2.27 and 2.28 by requiring that the residuals of
Equations 2.32 and 2.33 vanish, in a weighted average sense, over the domain $\Omega$. Consequently, given a global vector of weighting functions $\varsigma(x)$ over the domain $\Omega$, the weighted residual method may be written as

$$\int_{\Omega} \varsigma(x) \mathcal{R}_p(x,u) \, d\Omega + \int_{\Gamma} \varsigma(x) \mathcal{R}_b(x,u) \, d\Gamma = 0 .$$  \hspace{1cm} (2.34)

Substituting for the residuals from Equations 2.32 and 2.33 yields

$$\int_{\Omega} \varsigma \left( p(\psi u) - f \right) \, d\Omega + \int_{\Gamma} \varsigma \left( b(\psi u) - g \right) \, d\Gamma = 0 .$$  \hspace{1cm} (2.35)

For the discretized domain this becomes

$$\sum_{\Omega^e} \int_{\Omega^e} \varsigma^e \left( p(\psi^e u^e) - f \right) \, d\Omega^e$$

$$+ \sum_{\Gamma^e} \int_{\Gamma^e} \varsigma^e \left( b(\psi^e u^e) - g \right) \, d\Gamma^e = 0 ,$$  \hspace{1cm} (2.36)

where $\varsigma^e$ is the subset of nonzero weighting functions from vector $\varsigma$ applicable over the local domain $\Omega^e$.

The Galerkin method is a special case of the weighted residual method in which the weighting functions $\varsigma^e$ are defined to be equal to the shape functions such that

$$\varsigma^e = \psi^e .$$  \hspace{1cm} (2.37)

The Galerkin solution of Equations 2.27 and 2.28 can be written as

$$\sum_{\Omega^e} \int_{\Omega^e} \psi^e \left( p(\psi^e u^e) - f \right) \, d\Omega^e$$

$$+ \sum_{\Gamma^e} \int_{\Gamma^e} \psi^e \left( b(\psi^e u^e) - g \right) \, d\Gamma^e = 0 \hspace{1cm} (2.38)$$
which represents a nonlinear expression for numerical solution. The resulting element matrices may be assembled using procedures developed for standard discrete systems\(^7,22,23\) to establish the nonlinear global matrix expression given by

\[
K(u) u = Q \tag{2.39}
\]

where

- \(K(u)\) is the nonlinear global stiffness matrix
- \(Q\) is the constant term.

The solution to this set of equations ensures that the weighted average of the residual vanishes globally, although the weighted residual for a local element may not vanish.

Restrictions that must be placed upon the continuity of approximation functions \(\bar{u}\) depend upon the order of differentiation implied in the equation \(P(u)\). Clearly, \(P(\bar{u})\) must be continuous in order for the integral of Equation 2.35 to be finite. If the highest order derivative of \(\bar{u}\) in \(P\) is \(n\), then, by the Fundamental Theorem of Calculus \(\int \bar{u}^{(n)} \, d\bar{u}\) is finite if \(\bar{u}^{(n-1)}\) is continuous. Consequently, solving Equation 2.27 by the Galerkin method requires that the approximate solution, and therefore the shape function, be \(C^{n-1}\) continuous. On occasion it is possible to perform integration by parts on Equation 2.35 in order to reduce the continuity requirements on \(\bar{u}\) to a lower order. This is achieved, however, at the expense of higher order continuity requirements on the weight functions \(\zeta\).

In the special case when \(P(u)\) represents a linear differential operator, the Galerkin method may be compared with the conventional
variational approach to finite elements. If both differential equations and a variational principle exist, the Galerkin method and stationary functional methods result in identical solutions when both use the same approximating function \( \tilde{u} \). Furthermore, the Galerkin method will yield a symmetric system of equations if the differential operator \( P(u) \) is self-adjoint\(^8,24\). In the current context, this implies that \( P \) must satisfy

\[
\int_{\Omega} \xi P(\psi u) \, d\Omega = \int_{\Omega} \psi u P(\xi) \, d\Omega .
\]

(2.40)

Observe that Equations 2.34 and 2.35 are orthogonality conditions. In addition to requiring that the residual vanish, the weighted residual methods constrain the residual to be orthogonal to a "subspace" spanned by the weight functions, \( \xi \). According to Theorem 10.8 of Oden\(^24\), it is necessary and sufficient that the quantity \((u - \tilde{u})\) be orthogonal to the subspace of weight functions, \( \xi \), in order for the distance between the exact solution and the approximate solution, \( ||u - \tilde{u}|| \), to be a minimum. This means, then, that for a given set of shape functions the Galerkin method will yield the best solution that can be achieved.

Further discussion of the Galerkin method and its theoretical basis can be found in advanced texts on the finite element method\(^8,23,24,25,26\).
III DERIVATION OF GLOBAL ANALYSIS EQUATIONS

The theoretical basis for finite element matrix solution of beam-on-foundation problems is developed in this chapter through application of the Galerkin weighted residual method to the differential equations. The chapter has three sections.

In the first section, the system of four differential equations that define the three-dimensional beam-on-foundation interaction problem is derived. Two of the differential equations that result are fourth order and represent the beam deflection in the lateral y and z directions. The other two equations resulting from the derivation are second order and represent the axial and torsional motion of the beam.

In the second section of the chapter, the Galerkin method is applied to the four differential equations to derive the weighted residual equations for discrete finite elements. These residual equations are used in the next chapter to derive the finite element matrices for numerical solution of beam-on-foundation problems.

In the last section of the chapter, the accuracy of the finite elements developed in this investigation is addressed. The finite elements developed here are unique in that they are based upon shape functions which are homogeneous solutions of the applicable differential equations. A mathematical proof is given in the last section which shows that finite elements based upon homogeneous-solution shape
functions will yield exact solutions at the finite element nodes for linear, self-adjoint problems. It establishes the theoretical accuracy of the finite elements developed for beam-on-foundation problems.

Derivation of Differential Equations

The fourth-order differential equation, Equation 1.1, that is the focus of this study may arise in different physical problems, such as that of a cylindrical tube under axially symmetric loading\(^1\) or in the steady state propagation of a crack in a cylinder\(^2\). For this investigation the fourth-order differential equation is developed from the physical problem of a beam resting on a foundation or a beam surrounded by a restraining medium. The physical problem of a restrained beam is three dimensional in nature and is generally defined by four coupled differential equations: two that are fourth-order equations, similar to Equation 1.1, and two that are second-order equations. Finite element methods are generally necessary to solve the fully coupled system, and to solve complex geometry and loading conditions for beam and foundation interaction problems, without the need for simplifying assumptions. For completeness, the coupled three-dimensional differential equations are developed below.

Begin the derivation by considering a beam, surrounded by a reactive foundation medium, as illustrated in Figure 3.1. The coordinate system for local description of the beam is located with the x direction located along the centroidal axis. Displacements u, v, and w occur in the x, y, and z directions, respectively, and may be
functions of time as well as location. Also, torsional rotation \( \omega \) may occur about the centroidal \( x \) axis. Invoke the Euler-Bernoulli assumption that plane sections in the beam remain plane and further assume that deflections and strains in the beam remain small.

The reactive medium exerts surface tractions on the beam, \( \tau_x \), \( \tau_y \), and \( \tau_z \), shown in Figure 3.2, which are arbitrary functions of the displacements \( u \), \( v \), and \( w \) and their derivatives. For conventional beam-on-foundation analysis the tractions are integrated over the beam surface and resolved into distributed forces, \( p_x \), \( p_y \), and \( p_z \), acting through the centroidal axis of the beam as shown in the figure. It is assumed here that no moments are generated by the surface tractions. Formally, \( p_x \), \( p_y \), and \( p_z \) are defined by
Figure 3.2  Surface Traction Resulting from the Restraining Medium

\[ p_x \ dx = \int_{S} \tau_x \ ds \ dx \]  \hspace{1cm} (3.1)

\[ p_y \ dx = \int_{S} \tau_y \ ds \ dx \]  \hspace{1cm} (3.2)

\[ p_z \ dx = \int_{S} \tau_z \ ds \ dx \]  \hspace{1cm} (3.3)

where \( \int_{S} ds \) implies integration over the circumference of the beam.

In addition to the surface tractions resulting from the restraining medium, the beam in Figure 3.1 is subject to a system of externally generated body forces that must be in equilibrium with
internal axial, shear, and moment forces. The system of internal and external forces acting in each plane of the differential beam element is shown in Figure 3.3. Body forces are resolved into the components \( q_x \), \( q_y \), and \( q_z \) acting through the beam centroidal axis in the positive \( x \), \( y \), and \( z \) directions, respectively, and may vary with time. The beam has mass \( \Gamma \) per unit length, which results in an inertial force, and internal viscous damping constant \( \phi \) per unit length, which results in a damping force (proportional to velocity).

The internal reaction forces shown in the beam are bending moments, \( M_x \), \( M_y \) and \( M_z \), shear forces, \( S_y \) and \( S_z \), and normal axial force \( N \). The axial force is defined as positive when the beam is in tension and negative when the beam is in compression.

The differential equations for beams on foundation explicitly define the relationship between beam deflections \( u \), \( v \), and \( w \), and the applied external, inertial, and damping loads. These equations are developed below from the internal force-deflection relationship and equilibrium of the internal and external forces.

**Internal Force-Deflection Relationships**

The relationship between internal forces and beam deflections is constructed from the appropriate compatibility, constitutive and equilibrium expressions for an elastic beam. Compatibility expressions relate local strain to beam displacement. Constitutive relationships define stress as a function of local strain. Lastly, equilibrium requirements yield internal force and stress relationships. Once
Figure 3.3 System of Internal and External Forces Acting on a Differential Element of the Beam
c) System of Lateral Forces in x-z Plane

Figure 3.3 (Continued)
defined individually, these three expressions are combined to define the internal force-deflection relationship for a beam.

**Compatibility Relationships.** Consider the differential beam element of length $dx$, shown in Figure 3.4, subjected to bending and axial tension. Recalling the assumption that plane sections remain plane, impose upon the element a uniform axial displacement, $du$, over its length, and a small rotation, $d\theta$, about its $z$ axis. At distance $y$ measured from the centroid of the beam, the total displacement, $du_t$, is the sum of the uniform displacement and the displacement resulting from the rotation. This is written formally as

$$du_t = du - y \tan d\theta \approx du - y d\theta$$

(3.4)

![Figure 3.4 Deformation of a Beam Element in Tension and Bending](image-url)
where \( \tan \, d\theta = d\theta \) for small rotations. If a similar rotation \( \phi \) about the \( y \) axis is also imposed, the total displacement at a point \( (y,z) \) on the beam surface becomes

\[
d_{u_t} = du - y \, d\theta - z \, d\phi.
\] (3.5)

For small rotations, \( \theta \) and \( \phi \) are related to the change in displacements by the expressions

\[
\theta = \frac{dv}{dx} \quad \quad \quad \phi = \frac{dw}{dx}.
\] (3.6)

and

\[
d\theta = \frac{d^2v}{dx^2} \, dx \quad \quad \quad d\phi = \frac{d^2w}{dx^2} \, dx.
\] (3.7)

The \( x \) direction strain, \( \varepsilon_x \), is defined as the rate of change of total displacement, \( u_t \),

\[
\varepsilon_x = \frac{\delta u_t}{\delta x}.
\] (3.8)

Substitution from Equations 3.5 and 3.7 yields the compatibility formula for axial strain in the beam

\[
\varepsilon_x = \frac{\delta u}{\delta x} - y \frac{\delta^2 v}{\delta x^2} - z \frac{\delta^2 w}{\delta x^2}.
\] (3.9)

Partial derivatives are used because \( u \), \( v \), and \( w \) are functions of time as well as displacement.

Torsional shear strain, \( \varepsilon_{xy} \), in the beam results from the
rotation \( \omega \) about the centroidal \( x \) axis. The compatibility expression is given by

\[
\epsilon_{xy} = r \frac{d\omega}{dx}
\]  

(3.10)

where \( r \) is a measure of the distance from the centroid given by

\[
r = \sqrt{y^2 + z^2}.
\]

**Constitutive Relationships.** The normal and shear strains in the beam are related to their respective stresses by means of Hooke's law

\[
\epsilon_x = \frac{1}{E} \left( \sigma_x - \nu(\sigma_y + \sigma_z) \right) \quad \epsilon_{xy} = \frac{1}{G} \sigma_{xy}
\]

\[
\epsilon_y = \frac{1}{E} \left( \sigma_y - \nu(\sigma_x + \sigma_z) \right) \quad \epsilon_{yz} = \frac{1}{G} \sigma_{yz}
\]

\[
\epsilon_z = \frac{1}{E} \left( \sigma_z - \nu(\sigma_x + \sigma_y) \right) \quad \epsilon_{zx} = \frac{1}{G} \sigma_{zx}
\]

where

- \( E \) is the modulus of elasticity
- \( \nu \) is Poisson's ratio
- \( G = \frac{E}{2(1+\nu)} \) is the shear modulus of elasticity.

In typical beam analyses, it is assumed that the normal stresses \( \sigma_y \) and \( \sigma_z \), and the shear stresses \( \sigma_{yz} \) and \( \sigma_{zx} \) are negligible in comparison with \( \sigma_x \) and \( \sigma_{xy} \). This is not the case, however, for the common beam problem of a thin-wall pipe under internal pressure. The
circumferential stress in this case can be as great or greater than the axial stresses. As a consequence, in this analysis it is assumed that the vector quantity \((\sigma_y + \sigma_z)\) is constant over the beam cross-section, as would be expected for a pressurized pipe. It is shown in subsequent derivations that this assumption does not modify the resulting differential equations for beam motion.

From Hooke's law then the expression for axial stress, \(\sigma_x\), in the beam can be written as

\[
\sigma_x = E\epsilon_x - \nu(\sigma_y + \sigma_z)
\]  

(3.12)

**Equilibrium Conditions.** The internal forces within the beam must be in equilibrium with the system of local stresses. The equilibrium expressions for internal forces are given by

\[
N = \int_A \sigma_x \, dA
\]  

(3.13)

\[
M_x = \int_A r\sigma_{xy} \, dA
\]  

(3.14)

\[
M_y = -\int_A y\sigma_x \, dA
\]  

(3.15)

\[
M_z = \int_A z\sigma_x \, dA
\]  

(3.16)

where \(\int dA\) implies integration over the cross-sectional area of the beam.
The difference in sign in the expressions for $M_y$ and $M_z$ result from the definition of positive moments and curvature with respect to positive $y$ and $z$ directions shown in Figure 3.3.

**Synthesis of Internal Force-Deflection Expressions.** Now substituting the constitutive expressions for stresses $\sigma_x$ and $\sigma_{xy}$ into Equations 3.13 through 3.16 yields the following equations for internal forces in terms of local strain:

\[
N = \int_A E \varepsilon_x - \nu (\sigma_y + \sigma_z) \, dA \tag{3.17}
\]

\[
M_x = G \int_A r \varepsilon_{xy} \, dA \tag{3.18}
\]

\[
M_y = - \int_A y (E \varepsilon_x - \nu (\sigma_y + \sigma_z)) \, dA \tag{3.19}
\]

\[
M_z = \int_A z (E \varepsilon_x - \nu (\sigma_y + \sigma_z)) \, dA \tag{3.20}
\]

But by definition of the centroid

\[
\int_A y \, dA = 0 \quad \int_A z \, dA = 0 \tag{3.21}
\]

Therefore

\[
M_y = - E \int_A y \varepsilon_x \, dA \tag{3.22}
\]

\[
M_z = E \int_A z \varepsilon_x \, dA \tag{3.23}
\]
Substituting compatibility expressions for strain finally yields the internal force-deflection relationships. Substitution in the expression for internal axial load, \( N \), gives

\[
N = \int_A E \left( \frac{\partial u}{\partial x} - y \frac{\partial^2 v}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2} \right) dA - \nu (\sigma_y + \sigma_z) \int_A dA
\]

\[
= \left( E \frac{\partial u}{\partial x} - \nu (\sigma_y + \sigma_z) \right) \int_A dA - \frac{\partial^2 v}{\partial x^2} \int_A y dA - \frac{\partial^2 w}{\partial x^2} \int_A z dA .
\]  

(3.24)

Now defining \( A \) as the cross-sectional area of the beam, and recalling the definition of centroid of the beam, yields the following axial force-deflection expression:

\[
N = EA \frac{\partial u}{\partial x} - \nu A (\sigma_y + \sigma_z) .
\]  

(3.25)

Substituting the compatibility expression for strain \( \epsilon_{xy} \) in the expression for torsion moment, \( M_x \), gives

\[
M_x = G \frac{dw}{dx} \int_A r^2 dA .
\]  

(3.26)

Define the \( J \) as the polar moment of inertia given by

\[
J = \int_A r^2 dA .
\]  

(3.27)

The torsional moment-rotation expression is finally given by

\[
M_x = GJ \frac{dw}{dx} .
\]  

(3.28)
Lastly, substitute the compatibility relationship (Equation 3.9) into the bending moment expressions. Equation 3.19 for bending moment \( M_y \) becomes

\[
M_y = -E \int_A y \left( \frac{\partial u}{\partial x} - y \frac{\partial^2 v}{\partial x^2} - z \frac{\partial^2 w}{\partial x^2} \right) dA
\]

\[
= -E \frac{\partial u}{\partial x} \int_A y \, dA + E \frac{\partial^2 v}{\partial x^2} \int_A y^2 \, dA + E \frac{\partial^2 w}{\partial x^2} \int_A yz \, dA . \tag{3.29}
\]

Define the terms \( I_y \) and \( I_{yz} \) as

\[
I_y = \int_A y^2 \, dA \tag{3.30}
\]

\[
I_{yz} = \int_A yz \, dA . \tag{3.31}
\]

\( I_y \) is the second moment of the area, most commonly known as the area moment of inertia, and \( I_{yz} \) is the product of inertia. Recalling the definition of the centroid, \( M_y \) is now given by

\[
M_y = EI_y \frac{\partial^2 v}{\partial x^2} + EI_{yz} \frac{\partial^2 w}{\partial x^2} . \tag{3.32}
\]

Now define the \( y \) and \( z \) axes as the principal axes of the beam, such that \( I_{yz} \) becomes zero. The equation for the moment-curvature in the \( x-y \) plane becomes

\[
M_y = EI_y \frac{\partial^2 v}{\partial x^2} . \tag{3.33}
\]

Similarly, the bending moment \( M_z \) in the \( x-z \) plane is given by the moment-curvature expression
\[ M_z = -EI_z \frac{\partial^2 w}{\partial x^2} \quad (3.34) \]

where

\[ I_z = \int_A z^2 \, dA . \quad (3.35) \]

The difference in signs for \( M_y \) and \( M_z \) is again a result of the definitions of positive moments and curvature given in Figure 3.3.

**Equilibrium of Internal and External Forces**

The differential equations of motion for the beam may now be derived by establishing equilibrium of forces in the principal directions and substituting the appropriate displacement expressions for internal forces. Summation of forces in the \( x \) direction results in a second-order equation for axial displacements of the centroid of the beam. Summation of the moments about the centroidal \( x \) axis also results in a second-order equation for the torsion of the beam. Summation of forces and moments in the \( y \) direction yields a fourth-order equation for lateral displacement of the beam in the \( x-y \) plane; similar summation in the \( z \) direction yields a fourth-order equation for the displacement in the \( x-z \) plane.

**Axial Displacement Equation.** Begin by summation of forces in the \( x \) direction from Figure 3.3a.
\[
\left( N + \frac{\partial N}{\partial x} \right) \cos \left( \theta + \frac{\partial \theta}{\partial x} \right) - N \cos \theta + p_x \ dx + q_x \ dx \\
- \Gamma \ dx \ \frac{\partial^2 u}{\partial t^2} - \phi \ dx \ \frac{\partial u}{\partial t} = 0
\]

(3.36)

From the small rotation assumption, \( \cos \theta = 1 \) and second order effects are negligible. The equation reduces to

\[
\frac{\partial N}{\partial x} + p_x + q_x - \Gamma \frac{\partial^2 u}{\partial t^2} - \phi \frac{\partial u}{\partial t} = 0.
\]

(3.37)

The force-displacement relationship for axial load in Equation 3.25 gives

\[
\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left( EA \ \frac{\partial u}{\partial x} - \nu A (\sigma_y + \sigma_z) \right).
\]

(3.38)

Because the expression \( (\sigma_y + \sigma_z) \) is assumed to be constant along the length of the beam, it does not contribute to the variation in \( N \). Hence, the second-order partial differential equation for axial motion finally becomes

\[
\frac{\partial}{\partial x} \left( EA \ \frac{\partial u}{\partial x} \right) + p_x + q_x - \Gamma \frac{\partial^2 u}{\partial t^2} - \phi \frac{\partial u}{\partial t} = 0.
\]

(3.39)

The variation in axial load along the length of the beam, given by Equation 3.38, is important because the term \( \frac{\partial N}{\partial x} \) also arises in the subsequent derivation of lateral displacement equations, yielding a direct coupling of the axial motion to lateral deflection.

**Torsion Equation.** Summation of moments about the \( x \) axis of the beam yields the equation
\[ M_x + \frac{\partial M_x}{\partial x} \, dx \] - \[ M_x = 0 \] \quad (3.40)

which reduces to

\[ \frac{\partial M_x}{\partial x} = 0 . \] \quad (3.41)

Substituting the moment-rotation expression Equation 3.28 results in the differential equation for torsion of

\[ \frac{d}{dx}(GJ \frac{dw}{dx}) = 0 . \] \quad (3.42)

**Lateral Deflection Equations.** The differential equation for lateral deflection of the beam is developed by summation of forces and moments in the x-y plane of the beam. From Figure 3.3b summation of forces in the y direction results in the equation:

\[
\left[ S_y + \frac{\partial S_y}{\partial x} \, dx \right] \cos \left[ \theta + \frac{\partial \theta}{\partial x} \, dx \right] - S_y \cos \theta +
\left[ N + \frac{\partial N}{\partial x} \, dx \right] \sin \left[ \theta + \frac{\partial \theta}{\partial x} \, dx \right] - N \sin \theta
+ p_y \, dx + q_y \, dx - \Gamma \, dx \frac{\partial^2 v}{\partial t^2} - \phi \, dx \frac{\partial v}{\partial t} = 0 .
\] \quad (3.43)

Assuming small deflections and small rotations yields the approximations that \( \theta = \frac{\partial v}{\partial x}, \frac{\partial \theta}{\partial x} = \frac{\partial^2 v}{\partial x^2}, \) \( \sin \theta = \theta, \) and that second order effects are negligible. Equation 3.43 therefore becomes

\[ \frac{\partial S_y}{\partial x} + N \frac{\partial^2 v}{\partial x^2} + \frac{\partial N}{\partial x} \frac{\partial v}{\partial x} + p_y + q_y - \Gamma \frac{\partial^2 v}{\partial t^2} - \phi \frac{\partial v}{\partial t} = 0 . \] \quad (3.44)
Similarly, summation of the forces in the z direction and simplification yields the equation

\[
\frac{\partial S_z}{\partial x} + N \frac{\partial^2 w}{\partial x^2} + \frac{\partial N}{\partial x} \frac{\partial w}{\partial x} + p_z + q_z - \Gamma \frac{\partial^2 w}{\partial t^2} - \phi \frac{\partial w}{\partial t} = 0 .
\]  

(3.45)

Two additional equations result from summation of bending moments about the y and z axes. Summation of y direction moments from Figure 3.3b yields

\[
\left( M_y + \frac{\partial M_y}{\partial x} dx \right) - M_y + S_y dx = 0
\]

(3.46)

which reduces to

\[
\frac{\partial M_y}{\partial x} + S_y = 0 .
\]

(3.47)

The derivative of this expression with respect to x yields the following differential expression:

\[
\frac{\partial^2 M_y}{\partial x^2} + \frac{\partial S_y}{\partial x} = 0 .
\]

(3.48)

Similar action in the z direction yields

\[
- \frac{\partial^2 M_z}{\partial x^2} + \frac{\partial S_z}{\partial x} = 0 .
\]

(3.49)

The difference in signs is a result of the definition of positive moment with respect to the direction of positive y and z.

Now in Equations 3.48 and 3.49, substitute the moment-curvature relationships (Equations 3.33 and 3.34) for \( M_y \) and \( M_z \) and
shear Equations 3.44 and 3.45 for $S_y$ and $S_z$. This yields the following pair of coupled fourth-order differential equations:

\[
\frac{\partial^2}{\partial x^2} \left( EI_y \frac{\partial^2 v}{\partial x^2} \right) - N \frac{\partial^2 v}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial v}{\partial x} - p_y - q_y \\
+ \Gamma \frac{\partial^2 v}{\partial t^2} + \phi \frac{\partial v}{\partial t} = 0
\]  
(3.50)

\[
\frac{\partial^2}{\partial x^2} \left( EI_z \frac{\partial^2 w}{\partial x^2} \right) - N \frac{\partial^2 w}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial w}{\partial x} - p_z - q_z \\
+ \Gamma \frac{\partial^2 w}{\partial t^2} + \phi \frac{\partial w}{\partial t} = 0
\]  
(3.51)

Equations 3.39, 3.42, 3.50, and 3.51 now comprise a set of four differential equations for axial, torsional, and lateral motion of a beam with surface tractions, coupled with the axial tension terms. The general foundation traction terms are specified below.

**Foundation Traction.** Typical foundation media, such as various types of soil, are complex engineering materials whose mechanical properties generally cannot be measured or predicted adequately to define the local surface tractions $\tau_x$, $\tau_y$, and $\tau_z$ on the beam. Consequently, the foundation reaction forces, $p_x$, $p_y$, and $p_z$, for beam-on-foundation problems are developed experimentally and given as empirically based force-deflection curves such as those discussed in Chapter II.

As noted earlier, it is assumed in this study that the foundation restraints can be resolved into distributed reaction forces
acting through the beam centroidal axis. The resulting force-deflection relationships for the foundation are of the form

\[ p_x = - k_x(u) u \]  \hspace{1cm} (3.52)

\[ p_y = - k_y(v) v - k_{yz}(v,w) w \]  \hspace{1cm} (3.53)

\[ p_z = - k_z(w) w - k_{yz}(v,w) v \]  \hspace{1cm} (3.54)

The functions \( k_x, k_y, \) and \( k_z \) are assumed to be nonlinear functions of beam displacement. Additionally, the foundation reactions in the \( y \) and \( z \) directions may be coupled through the \( k_{yz} \) term, such that \( y \)-direction motion can result in \( z \)-direction reactions and vice versa.

**Summary of Differential Equations**

The differential equations for the three-dimensional beam-on-foundation problem are now completed by substituting the foundation traction terms into Equations 3.39, 3.50, and 3.51. These substitutions result in the following four differential equations for a three-dimensional beam surrounded by a reactive medium:

\[ \frac{d}{dx} \left( EA \frac{du}{dx} \right) - k_x u + q_x - \Gamma \frac{d^2u}{dt^2} - \phi \frac{du}{dt} = 0 \]  \hspace{1cm} (3.55)

\[ \frac{d}{dx} \left( GJ \frac{dw}{dx} \right) = 0 \]  \hspace{1cm} (3.56)
\[
\frac{\partial^2}{\partial x^2} \left( EI_y \frac{\partial^2 v}{\partial x^2} \right) - N \frac{\partial^2 v}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial v}{\partial x} + k_y v + k_{yz} w - q_y \\
+ \Gamma \frac{\partial^2 v}{\partial t^2} + \phi \frac{\partial v}{\partial t} = 0
\]
(3.57)

\[
\frac{\partial^2}{\partial x^2} \left( EI_z \frac{\partial^2 w}{\partial x^2} \right) - N \frac{\partial^2 w}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial w}{\partial x} + k_z w + k_{yz} v - q_z \\
+ \Gamma \frac{\partial^2 w}{\partial t^2} + \phi \frac{\partial w}{\partial t} = 0
\]
(3.58)

**Boundary Conditions**

In addition to the distributed external loads acting on a beam element, concentrated external loads may also be acting at the boundaries. These concentrated loads are subject to equilibrium requirements as well. An additional set of equations is necessary to define these equilibrium requirements for concentrated boundary loadings. The concentrated, externally applied loadings correspond loosely with internal loads. Hence, the beam is subject to a concentrated axial load \( N^* \), concentrated shear loads \( S_x^* \) and \( S_z^* \), and concentrated moments \( M_x^* \), \( M_y^* \), and \( M_z^* \). Boundary equations define the required equilibrium relationships between concentrated external loads and beam boundary deflections.

Consider the beam boundary in the \( x-y \) plane, illustrated in Figure 3.5. Compare the internal forces in the deformed beam with the concentrated external forces corresponding to the undeformed
Figure 3.5 Internal and External Forces at the Beam Element Boundary

configuration. Equilibrium of forces in the x direction yields the expression

\[ N^* = N \cos \theta - S_y \sin \theta = N \]  \hspace{1cm} (3.59)

Here the shear term is second order and negligible in comparison with the axial force \( N \). The same is true for the contribution from shear force \( S_z \). Substituting for \( N \) from the axial force-deflection expression Equation 3.25 and assuming small rotations yields the axial load requirement at the boundary

\[ EA \frac{d^2 u}{dx^2} - \nu A (\sigma_y + \sigma_z) - N^* = 0 \]  \hspace{1cm} (3.60)
Next consider the shear forces shown in Figure 3.5. Here the concentrated force $S_y^*$ is applied parallel to the $y$ axis, while the shear force $S_y$ is normal to the deflected axis of the beam. Equilibrium of forces in the $y$ direction yields the expression

$$S_y^* = S_y \cos \theta + N \sin \theta.$$  \hspace{1cm} (3.61)

From the expression for $S_y$ in Equation 3.47 and assuming small rotations, the shear boundary condition is

$$\frac{\partial}{\partial x} \left( EI_y \frac{\partial^2 y}{\partial x^2} \right) - N \frac{\partial y}{\partial x} - S_y^* = 0. \hspace{1cm} (3.62)$$

Similarly, the boundary equilibrium for concentrated shear force $S_z^*$ requires that

$$\frac{\partial}{\partial x} \left( EI_z \frac{\partial^2 w}{\partial x^2} \right) - N \frac{\partial w}{\partial x} - S_z^* = 0. \hspace{1cm} (3.63)$$

The assumptions made in deriving these boundary conditions for shear are consistent with the assumptions made in deriving the differential equations. Timoshenko(28) compares the approach used here, in which the shear forces, $S_y$ and $S_z$, are directed normal to the deflected axis of the beam, with the alternative in which the shear forces are defined to be perpendicular to the $x$ axis. He shows that the two assumptions yield the same differential equation but they yield different boundary conditions for axial load and shear. His comparison clearly shows that consistency in the derivation of differential equations and boundary conditions is critical.
Moment equilibrium at beam boundaries may be found from Equations 3.28, 3.33, and 3.34. For concentrated boundary moments, $M_x^*$, equilibrium requires that

$$M_x = M_x^*.$$  \hfill (3.64)

The definition of $M_x$ from Equation 3.28 yields the boundary condition

$$GJ \frac{dw}{dx} - M_x^* = 0.$$  \hfill (3.65)

Similar equilibrium summations and the use of Equations 3.33 and 3.34 provide the boundary conditions for moments $M_y^*$ and $M_z^*$.

$$EI_y \frac{d^2v}{dx^2} - M_y^* = 0.$$  \hfill (3.66)

$$- EI_z \frac{d^2w}{dx^2} + M_z^* = 0.$$  \hfill (3.67)

The boundary equilibrium requirements given by Equations 3.60, 3.62, 3.63, 3.65, 3.66, and 3.67 are known as the natural or nonessential boundary conditions. Additional discussion of boundary conditions of beams and beam-columns is given by Timoshenko and Gere(28) Fung(29), Kerr(30), and Chen and Atsuta(31).

**Derivation of Galerkin Equations**

The weighted residual method is used to find an approximate solution to a set of differential equations and boundary conditions over a given domain. In this case the domain is a finite length beam which is discretized into elements of typical length $L$. Following is the
application of the Galerkin method to the differential equations and boundary conditions developed previously to define weighted residual equations for finite element solution of beam-on-foundation problems.

**Axial Displacement Equation**

From the differential equation for axial motion Equation 3.55, the interior residual $R_L$ is given by

$$R_L = \frac{\partial}{\partial x} \left( EA \frac{\partial u}{\partial x} \right) - k_x u + q_x - \Gamma \frac{\partial^2 u}{\partial t^2} - \Phi \frac{\partial u}{\partial t} .$$

(3.68)

The nonessential boundary conditions on the axial displacement yield the boundary residual

$$R_B = EA \frac{\partial u}{\partial x} - N^*$$

(3.69)

where $N^*$ is a prescribed value of axial load.

According to the Galerkin method, the domain of the beam-on-foundation is divided into "n" discrete elements of typical length $L$ and the summation of the weighted average of the residual contribution from each element is set to zero. The weighted residual expression for axial displacement takes the form

$$0 = \sum_{1}^{n} \left[ \int_{a}^{L} \phi_i^* \left( EA \frac{\partial u}{\partial x} \right) - k_x u + q_x - \Gamma \frac{\partial^2 u}{\partial t^2} - \Phi \frac{\partial u}{\partial t} \right] dx \right]$$

$$- \sum_{1}^{n} \left[ \phi_i^* \left( EA \frac{\partial u}{\partial x} - N^*\right) \right]_a^L .$$

(3.70)

In this expression $\phi_i^*$ is a vector of weight functions defined over the element length. The summation parameter, $\Sigma$, represents a direct
assembly process of discrete elements into a global weighted residual matrix expression.

The overall order of integration of Equation 3.70 may be reduced by integrating the first term of the residual expression by parts. The expression becomes

\[
0 = \sum_{i=1}^{n} \left[ \int_{s}^{L} \left( \frac{\partial \xi^x}{\partial x} \left( EA \frac{\partial u}{\partial x} \right) - \xi^x_k u + \xi^x_q \right) \right. \\
- \left. \xi^x_P \frac{\partial^2 u}{\partial t^2} - \xi^x_\psi \frac{\partial u}{\partial t} \right] \, dx \right] + \sum_{i=1}^{n} \left[ \xi^e N^e \right]_{s}^{L}.
\]

(3.71)

Observe that integration by parts yields a term that cancels the first boundary term of Equation 3.70.

Now assume that the element displacement field, \( u(x,t) \), is described by the product of a vector of shape (interpolation) functions, \( \psi^o(x) \), and the vector of nodal displacements for the element, \( u^o(t) \), which are functions of displacement \( x \) and time \( t \), respectively. This is expressed formally as

\[
u(x,t) = \psi^o(x) u^o(t).
\]

(3.72)

Because the highest derivative in Equation 3.71 is one, \( \psi^o \) is required to be \( C^1 \) continuous or greater, as discussed in Chapter II.

The weighting function \( \xi^x \) in Equation 3.71 is defined by the Galerkin method to be equal to the shape function, and written as

\[
\xi^x(x) = \psi^x(x).
\]

(3.73)

Derivatives of the displacement, \( u \), and the weighting function, \( \xi^x \), therefore become
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \psi_x \frac{\partial^2 u^e}{\partial t^2} = \psi_x(x) \ddot{u}^e(t) \tag{3.77}
\]

Substituting these derivatives into Equation 3.71 yields the final weighted residual expression for axial displacement as follows

\[
0 = \sum_{i=1}^{n} \left\{ \left[ \int_0^L -\psi_x^e \psi_x^e \ T \ E A \ \psi_x^e \ dx - \int_0^L \psi_x^e \ T \ k_x \ \psi_x^e \ dx \right] u^e \right. \\
+ \int_0^L \psi_x^e \ T \ q_x \ dx + \\
\left. - \left[ \int_0^L \psi_x^e \ T \ \phi \ \psi_x^e \ dx \right] \ddot{u}^e - \left[ \int_0^L \psi_x^e \ T \ \phi \ \psi_x^e \ dx \right] \ddot{u}^e \right\} \\
+ \sum_{i=1}^{n} \left[ -\psi_x^e \ N^e \right]_0^L. \tag{3.78}
\]

**Torsion Equation**

The weighted residual expression for the second-order torsion equation is developed in a manner analogous to that of the expression for axial displacement. The resulting expression for torsion is given by
\[ 0 = \sum_{i=1}^{n} \left[ \int_{\theta}^{L} \mathbf{\psi}_{xy}^T \mathbf{GJ} \mathbf{\psi}_{xy}^T \, d\mathbf{\xi} \right] \mathbf{\omega}^0 + \sum_{i=1}^{n} \left[ \mathbf{\psi}_{xy}^T \mathbf{M}_{xy}^0 \right]_{\theta} \quad (3.79) \]

Here, \( \mathbf{\psi}_{xy}^0 \) is a \( C^0 \) or greater continuous function that is distinct from \( \mathbf{\psi}_{x}^0 \).

\textit{Bending Equations}

From the differential equation 3.57 the expression for the internal residual, \( \mathcal{R}_L \), for bending is

\[
\mathcal{R}_L = \frac{\partial^2}{\partial x^2} \left( EI_y \frac{\partial^2 v}{\partial x^2} \right) - N \frac{\partial^2 v}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial v}{\partial x} + k_y v + k_{yz} w - q_y
\]
\[
+ \int \frac{\partial^2 v}{\partial t^2} + \psi \frac{\partial v}{\partial t} = 0 \quad (3.80)
\]

The expressions for the boundary residual are found using Equations 3.62 and 3.66 and written as

\[
\mathcal{R}_{B1} = EI_y \frac{\partial^2 v}{\partial x^2} - M_y^* = 0 \quad (3.81)
\]

\[
\mathcal{R}_{B2} = \frac{\partial}{\partial x} \left( EI_y \frac{\partial^2 v}{\partial x^2} \right) - N \frac{\partial v}{\partial x} - S_y^* = 0 \quad (3.82)
\]

where \( M_y^* \) and \( S_y^* \) are prescribed values of moment and shear. The essential boundary conditions are prescribed values of \( v \) and \( \frac{\partial v}{\partial x} \).

As in the axial displacement case, the beam-on-foundation is divided into discrete elements of typical length \( L \) and the sum of the weighted average of element residuals is set to zero. This is written as
\begin{align*}
0 &= \sum_{i=1}^{n} \left[ \int_{\gamma}^{L} \left( \frac{\partial}{\partial x^2} \left[ E I_y \frac{\partial^2 v}{\partial x^2} \right] - N \frac{\partial^2 v}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial v}{\partial x} + k_y v + k_{yz} w \\
&\quad - q_y + \Gamma \frac{\partial^2 v}{\partial t^2} + \phi \frac{\partial v}{\partial t} \right) \, dx \right] \\
&\quad + \sum_{i=1}^{n} \left[ \frac{\partial S^*_i}{\partial x} \left( E I_y \frac{\partial^2 v}{\partial x^2} - M^*_y \right) \right]_{\gamma}^L \\
&\quad - \sum_{i=1}^{n} \left[ S^* \left( \frac{\partial}{\partial x} \left( E I_y \frac{\partial^2 v}{\partial x^2} - N \frac{\partial v}{\partial x} - S^*_y \right) \right]_{\gamma}^L (3.83)
\end{align*}

where $S^*_y$ is again a vector of weight functions defined over the element length.

The overall order of integration of the bending equations is reduced by integrating the first term by parts twice, and the second term by parts once. The resulting equation is

\begin{align*}
0 &= \sum_{i=1}^{n} \left[ \int_{\gamma}^{L} \left( \frac{\partial^2 S^*_i}{\partial x^2} \left( E I_y \frac{\partial^2 v}{\partial x^2} \right) + \frac{\partial S^*_i}{\partial x} N \frac{\partial v}{\partial x} - S^*_y \frac{\partial N}{\partial x} \frac{\partial v}{\partial x} \\
&\quad + S^*_y k_y v + S^*_y k_{yz} w - S^*_y q_y + S^*_y \Gamma \frac{\partial^2 v}{\partial t^2} + S^*_y \phi \frac{\partial v}{\partial t} \right) \, dx \right] \\
&\quad + \sum_{i=1}^{n} \left[ \frac{\partial S^*_i}{\partial x} M^*_y \right]_{\gamma}^L + \sum_{i=1}^{n} \left[ S^* S^*_y \right]_{\gamma}^L (3.84)
\end{align*}

Note that integration by parts has reduced the maximum order of the derivatives of $v$ to two, but has done so at the expense of increasing the order of derivatives of $S^*_y$ to two. As in the axial displacement case, the boundary terms are reduced to reflect only the concentrated nodal loads.

The element displacement field $v(x,t)$ is assumed to be described by the product of a vector of shape functions, $\Phi^*(x)$, and of the vector of nodal displacements and rotations for the element, $v^0(t)$. 
Because the highest order derivative in Equation 3.84 is two, the shape function \( \psi_y^o \) must be \( C^1 \) continuous or greater. The displacement field is given by the expression

\[
v(x,t) = \psi_y^o(x) v^o(t) .
\]  

(3.85)

The weighting function \( \zeta_y^o \) is now given as

\[
\zeta_y^o(x) = \psi_y^o(x) .
\]  

(3.86)

Derivatives of the displacement, \( u \), and the weighting function, \( \zeta_y^o \), therefore become

\[
\frac{\partial \zeta_y^o}{\partial x}(x) = \psi_y^o .
\]  

(3.87)

\[
\frac{\partial v}{\partial x}(x,t) = \psi_y^o v^o
\]  

(3.88)

\[
\frac{\partial^2 v}{\partial x^2}(x,t) = \psi_y^o v^o
\]  

(3.89)

\[
\frac{\partial v}{\partial t}(x,t) = \zeta_y^o \frac{\partial v^o}{\partial t} = \psi_y^o(x) \dot{v}^o(t)
\]  

(3.90)

\[
\frac{\partial^2 v}{\partial t^2}(x,t) = \zeta_y^o \frac{\partial^2 v^o}{\partial t^2} = \psi_y^o(x) \ddot{v}^o(t) .
\]  

(3.91)

Similarly, the displacement field \( w(x,t) \) using shape function \( \psi_z^o \) and nodal displacement vector \( w^o(t) \) becomes

\[
w(x,t) = \psi_z^o(x) w^o(t)
\]  

(3.92)
Observe here that although the form of the shape function $\psi_y^e$ and $\psi_z^e$ may be similar, their constants and hence their "shape" will be different because the beam, foundation, and axial load constants in the x-y and x-z planes will be different.

The weighted residual expression for the bending in the x-y plane is finally written as

$$
0 = \sum_{i=1}^{n} \left\{ \left[ \int_{\gamma} L \psi_{y}^{e_i T} EI \psi_{y}^{e_i} \, dx + \int_{\gamma} L \psi_{y}^{e_i T} N \psi_{y}^{e_i} \, dx \right. \right.
$$

$$
- \int_{\gamma} L \psi_{y}^{e_i T} \frac{\partial N}{\partial x} \psi_{y}^{e_i} \, dx + \int_{\gamma} L \psi_{y}^{e_i T} k_y \psi_{y}^{e_i} \, dx \right] v^e
$$

$$
+ \left[ \int_{\gamma} L \psi_{y}^{e_i T} k_{yz} \psi_{z}^{e_i} \, dx \right] w^e - \int_{\gamma} L \psi_{y}^{e_i T} q_y \, dx
$$

$$
- \left[ \int_{\gamma} L \psi_{y}^{e_i T} \Gamma \psi_{y}^{e_i} \, dx \right] \ddot{v}^e - \left[ \int_{\gamma} L \psi_{y}^{e_i T} \phi \psi_{y}^{e_i} \, dx \right] \ddot{v}^e
$$

$$
- \sum_{i=1}^{n} \left[ \psi_{y}^{e_i T} M_y^e \right]_\gamma + \sum_{i=1}^{n} \left[ \psi_{y}^{e_i T} S_y^e \right]_\gamma.
$$

(3.93)

Note that the $\frac{\partial N}{\partial x}$ term is not symmetric and will prevent the system from being self-adjoint when it is present. Consequently, according to the following section exact solutions at the finite element nodes can be assured only when this term is neglected.

The weighted residual expression for bending in the x-z plane, developed in a fully analogous fashion to Equation 3.93, is
\[ 0 = \sum_{i=1}^{n} \left\{ \int_{\Theta} \psi_{2i}^{eT} EI \psi_{2i}^{e} \, dx + \int_{\Theta} \psi_{2i}^{eT} N \psi_{2i}^{e} \, dx - \int_{\Theta} \psi_{2i}^{eT} \frac{\partial N}{\partial x} \psi_{2i}^{e} \, dx + \int_{\Theta} \psi_{2i}^{eT} k_{z} \psi_{2i}^{e} \, dx \right\} \omega^{e} \\
+ \left[ \int_{\Theta} \psi_{2}^{eT} k_{yz} \psi_{y}^{e} \, dx \right] \nu^{e} - \int_{\Theta} \psi_{2}^{eT} q_{z} \, dx \\
- \left[ \int_{\Theta} \psi_{2}^{eT} \Gamma \psi_{2}^{e} \, dx \right] \omega^{e} - \left[ \int_{\Theta} \psi_{2}^{eT} \phi \psi_{2}^{e} \, dx \right] \phi^{e} \right\} \\
- \sum_{i=1}^{n} \left[ \psi_{2i}^{eT} M_{z}^{e} \right]_{\Theta} + \sum_{i=1}^{n} \left[ \psi_{2}^{eT} S_{z}^{e} \right]_{\Theta}. \quad (3.94) \]

Equations 3.78, 3.79, 3.93, and 3.94 now comprise a set of four coupled residual expressions that define the finite element solution method for three-dimensional beam-on-foundation problems. The specific finite element matrices to be used for solving problems are found by introducing appropriate shape functions into the expressions and completing the integrations. The nature of the element shape function selected will define the accuracy of the resulting elements.

**Accuracy of Homogeneous-Solution Elements**

In Chapter II, a number of cases were given in which various authors demonstrated that using the homogeneous solution of a differential equation as the finite element shape function for a specific problem yields its exact numerical solution at the element nodes. In preparation for subsequent element development, a proof is given below that this will be the case for self-adjoint problems.
In 1969 Tong\(^{(32)}\) published a comparable proof based upon variational calculus methods. According to his analysis, when the homogeneous solutions of the Euler differential equations of a positive definite functional in one variable are used as shape functions, the solution values at the nodal points, based upon the functional, will be the exact solution values at those points, regardless of the number of elements used and the form of the particular solution. The following proof is similar to Tong's approach, but is unique in that it utilizes the principles of the Galerkin method to prove that use of the homogeneous solution will yield an exact solution at the nodes.

**Proof of Exact Solutions**

Consider the linear differential equation

\[ \mathcal{L}(u) - f = 0 \quad (3.95) \]

where \( \mathcal{L} \) is a self-adjoint linear differential operator in the domain \( \Omega \) of order \( 2n \) acting on \( u \), and \( f \) is a given function of \( x \) only. Define \( \varphi \) as a prescribed set of shape (interpolation) functions, defined over the domain \( \Omega \), and which also satisfy the homogeneous equation

\[ \mathcal{L}(\varphi) = 0 \quad . \quad (3.96) \]

For a discrete element analysis, the solution is approximated by \( \tilde{u}(x) \), a function of the shape functions \( \varphi(x) \), and undetermined coefficients \( v \), defined over domain \( \Omega \). This results in the approximation

\[ \tilde{u}(x) = \varphi(x)v = \Sigma \varphi^*(x)v^* \quad (3.97) \]
Applying the Galerkin method to seek an approximate solution of the differential equation given in Equation 3.95 results in the weighted residual expression:

$$0 = \int_{\Omega} \psi(L(\tilde{u}) - f) \, d\Omega$$

The order of this expression can be reduced by integrating by parts "n" times, yielding the differential operator $D$ of order $n$, such that

$$0 = \int_{\Omega} (D(\psi)D(\tilde{u}) - \psi f) \, d\Omega = \int_{\Omega} (D(\psi)D(\psi v) - \psi^* f) \, d\Omega$$

$$= \sum_{\Omega} \int_{\Omega^*} (D(\psi^*)D(\psi^* v^*) - \psi^* f) \, d\Omega^*.$$  (3.99)

Applying standard methods for assembly of discrete systems to this Galerkin expression readily yields the matrix expression

$$K \tilde{\nu} = Q$$  (3.100)

where $\tilde{\nu}$ is the solution vector for the expression.

Now, establish a set of definitions for exact and approximate solution functions. Define $u_e(x)$ as the exact solution of the differential equation (Equation 3.95). With $\tilde{\nu}$ as the solution of Equation 3.100, the corresponding continuous function $\tilde{u}$ is given by

$$\tilde{u}(x) = \psi(x) \tilde{\nu}.$$  (3.101)

Define $v_n$ as a vector with components equal to the exact solution (up to order $n-1$) of Equation 3.95 at the finite element nodes, and let $u_n(x)$ be the corresponding continuous function given by

$$u_n(x) = \psi(x)v_n.$$  (3.102)
Then, at the finite element nodes, we have that

\[ u_e(x_i) = u_n(x_i) = \nu_{ni} \quad (3.103) \]

for up to order \((n-1)\) derivatives. Here \(x_i\) is the location of the "ith" finite element node, and \(\nu_{ni}\) is the exact solution for the degrees of freedom at node \(i\).

Lastly, define \(u_d(x)\) as a continuous function of the difference between exact solution function \(u_e(x)\) and nodal solution function \(u_n(x)\) such that

\[ u_d(x) = u_e(x) - u_n(x) = u_e(x) - \Phi(x)\nu_n \quad (3.104) \]

Observe that \(u_d(x_i)=0\) (at the finite element nodes) for derivatives of order \((n-1)\) and less.

Linear manipulation of the nodal solution may be performed as follows:

\[
\mathcal{L}(u_n) = \mathcal{L}(\Phi \nu_n) \\
= \mathcal{L}(\Phi)\nu_n \\
\mathcal{L}(u_n) = 0.
\quad (3.105)
\]

Thus, the nodal solution given by Equation 3.102 is a solution of the homogeneous equation for the problem, when Equation 3.96 is satisfied.

Next, perform similar linear manipulation of the difference function, \(u_d\).

\[
\mathcal{L}(u_d) - f = \mathcal{L}(u_e - u_n) - f \\
= \mathcal{L}(u_e) - \mathcal{L}(u_n) - f \\
= \mathcal{L}(u_e) - f \\
\mathcal{L}(u_d) - f = 0
\quad (3.106)
\]

Thus, the difference function is a solution of the general differential equation (Equation 3.95).
Now consider that \( u_e \) must satisfy the weighted residual expression

\[
\int_\Omega \psi (L(u_e) - f) \, d\Omega = 0. \tag{3.107}
\]

This expression is integrated by parts \( n \) times yielding the differential operator \( D \) of order \( n \) in the resulting equation

\[
\int_\Omega (D(\psi)D(u_e) - \psi f) \, d\Omega = 0. \tag{3.108}
\]

No boundary terms are generated because the system is self-adjoint (symmetric). Linear manipulation of the reduced order system yields

\[
\int_\Omega (D(\psi)D(u_e) - \psi f) \, d\Omega = \int_\Omega (D(\psi)D(\psi v_n + u_d) - \psi f) \, d\Omega = 0
\]

\[
= \int_\Omega (D(\psi)D(\psi v_n) - \psi f) \, d\Omega + \int_\Omega D(\psi)D(u_d) \, d\Omega. \tag{3.109}
\]

The last term may be integrated by parts to give

\[
\int_\Omega D(\psi)D(u_d) \, d\Omega = \int_\Omega u_d L(\psi) \, d\Omega + \text{boundary terms.} \tag{3.110}
\]

Recalling Equation 3.96 and that \( u_d(x_i) = 0 \) on the boundary for derivatives of order \( (n-1) \), the right hand side of the expression is zero. Thus the weighted residual of the difference function is zero and Equation 3.109 now becomes

\[
\int_\Omega (D(\psi)D(\psi v) - \psi f) \, d\Omega = \sum_{\Omega^o} \int_{\Omega^o} (D(\psi^o)D(\psi^o v^o) - \psi^o f) \, d\Omega^o = 0 \tag{3.111}
\]
proving that the weighted residual of the nodal solution $v_n$ is exactly zero. As with Equation 3.99 this expression can also be written in matrix form as

$$K v_n = Q.$$  \hfill (3.112)

Because $\mathcal{L}$ is self-adjoint, $K$ is positive definite and solution $v_n$ is unique. Comparing Equations 3.99 and 3.111 shows that matrices $K$ and $Q$ which result from each expression, are identical. Therefore, because $v_n$ is unique, we must conclude that

$$\tilde{v} = v_n,$$  \hfill (3.113)

proving that the homogeneous-solution shape function yields exact solutions at the finite element nodes.

**Summary**

Equations 3.78, 3.79, 3.93, and 3.94, which have been developed in this chapter, comprise a set of four coupled residual expressions that define the finite element solution method for three-dimensional beam-on-foundation problems. A proof has also been presented showing that applying the homogeneous solution of the applicable differential equation in these expressions as the shape function will yield elements whose solutions are exact at finite element nodes for symmetric problems. This exercise will be carried out in the following chapter to establish the "exact solution" finite elements for linear beam-on-foundation problems.
IV FUTURE ELEMENT DEVELOPMENT FOR LINEAR BEAM-ON-FOUNDATION PROBLEMS

This chapter presents the implementation of the Galerkin equations developed in Chapter III and the establishment of finite elements for exact nodal solutions of linear, symmetric beam-on-foundation problems. The homogeneous solution of appropriate differential equations are employed here as shape functions in the Galerkin expressions to derive element matrices for numerical solution. Matrices are developed for fixed length beam-on-foundation elements, as well as for a semi-infinite beam-on-foundation element.

The advantages of using of homogeneous solutions as finite element shape functions have been discussed in Chapter II. The complex-exponential form of the homogeneous-solution equations are used here for development of the fourth-order bending elements resulting in compact and efficient expressions for element stiffnesses. For most cases, the finite elements consist of three complex matrices that are multiplied numerically to yield the element matrix. Any drawback that arises from this requirement for matrix multiplication is offset by the fact that these elements yield exact solutions with the minimum number of elements required to describe a problem.

This chapter is organized into three sections. The first section defines the degrees of freedom for beam elements and the algebraic manipulation necessary to develop nodal element shape functions from arbitrarily defined functions. This is followed by a
section describing derivation of fixed length elements and the semi-infinite element. The chapter concludes with a discussion of the behavior of the fourth-order shape functions and example problems.

Element Degrees of Freedom and Nodal Shape Functions

The derivation of differential equations for beam-on-foundation problems in Chapter III resulted in four equations: two second-order expressions and two fourth-order expressions. The second-order expressions address axial displacement and torsion (axial rotation) whereas the fourth-order expressions address bending, one equation for the x-y plane and one for the x-z plane. Consequently, a finite element for analysis of the problem will have two axial degrees of freedom, two torsional degrees of freedom, and eight bending degrees of freedom (four in each of the y and z directions) for a total of twelve degrees of freedom for the element. These degrees of freedom are selected to be nodal displacements and rotations (first derivatives) of the ends of the elements, consistent with the practices of matrix structural analysis. The conventional definition of degrees of freedom(7) are shown in Figure 4.1. Note that positive directions of the displacements and rotations are taken to be consistent with the positive directions of the local axes.

The complete stiffness matrix for a three-dimensional beam is 12 by 12. The axial, torsional, and bending degrees of freedom are uncoupled for most cases (i.e., when \( k_{yz}(v,w) = 0 \) in Equations 3.93 and 3.94) so that it is possible to construct this matrix from 2 by 2 and
4 by 4 submatrices. Table 4.1 defines the submatrices corresponding to the uncoupled differential equations in terms of the complete 12 by 12 beam element stiffness matrix, $k^e$, and complete 12 by 1 displacement vector, $d^e$. The 12 by 12 element can be readily assembled from these submatrices according to straightforward procedures developed for standard discrete systems\(^{(7,22,23)}\). For clarity of presentation, the element submatrices are derived separately.

For most engineering problems, the lateral motion of the beam in the x-y and x-z planes remains decoupled, however the residual expression Equations 3.93 and 3.94 include a foundation interaction term, $k_{yz}(v,w)$. This term is discussed later, following development of the uncoupled stiffness terms.
Table 4.1 Beam Element Submatrices

a) Three-dimensional Beam Element Matrices:

Stiffness Matrix - \( k^e = [k_{ij}], \ i,j = 1, \ldots, 12 \)

Displacement Vector - \( d^e = [d_i], \ i = 1, \ldots, 12 \)

b) Axial Displacement

\[
\begin{bmatrix}
u^e \\
u(L)
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
d_7
\end{bmatrix}
\quad \quad k^e_x =
\begin{bmatrix}
k_{11} & k_{17} \\
\text{symm.} & k_{77}
\end{bmatrix}
\]

c) Torsion

\[
\begin{bmatrix}
\omega^e \\
\omega(L)
\end{bmatrix} =
\begin{bmatrix}
d_4 \\
d_{10}
\end{bmatrix}
\quad \quad k^e_{xy} =
\begin{bmatrix}
k_{44} & k_{410} \\
\text{symm.} & k_{1010}
\end{bmatrix}
\]

d) Lateral Bending in x-y Plane

\[
\begin{bmatrix}
v^e \\
v'(0) \\
v(L) \\
v'(L)
\end{bmatrix} =
\begin{bmatrix}
d_2 \\
d_5 \\
d_8 \\
d_{11}
\end{bmatrix}
\quad \quad k^e_y =
\begin{bmatrix}
k_{22} & k_{25} & k_{28} & k_{211} \\
k_{55} & k_{58} & k_{511} \\
k_{88} & k_{811} \\
\text{symm.} & & & k_{1111}
\end{bmatrix}
\]

e) Lateral Bending in x-z Plane

\[
\begin{bmatrix}
w^e \\
w'(0) \\
w(L) \\
w'(L)
\end{bmatrix} =
\begin{bmatrix}
d_3 \\
d_8 \\
d_9 \\
d_{12}
\end{bmatrix}
\quad \quad k^e_z =
\begin{bmatrix}
k_{33} & k_{36} & k_{39} & k_{312} \\
k_{66} & k_{69} & k_{612} \\
k_{99} & k_{912} \\
\text{symm.} & & & k_{1212}
\end{bmatrix}
\]
The homogeneous solutions of the beam-on-foundation differential equations were given in Chapter II in terms of arbitrary constants, rather than nodal quantities. The remainder of this section presents the algebraic manipulations that are necessary to write general homogeneous solutions in terms of nodal displacements, for subsequent use as finite element shape functions.

The homogeneous solutions to be used as shape functions for \( v(x) \) are of the form

\[
v(x) = \sum_{i} C_i \phi_i(x) \quad i = 1, \ldots, n
\]  

(4.1)

where

\( \phi_i(x) \) are terms of the interpolation function (functions of \( x \))

\( C_i \) are constants determined by boundary conditions and loadings on the beam

\( n \) is the number of degrees of freedom in the expression.

This may also be written in vector form as

\[
v(x) = \phi(x) C
\]  

(4.2)

where bold notation is used to indicate vectors and matrices. The vector \( C \) is constant, therefore the first and second derivatives of \( v(x) \) are given respectively by

\[
v'(x) = \phi'(x) C
\]  

(4.3)

and

\[
v''(x) = \phi''(x) C .
\]  

(4.4)
The constant vector \( \mathbf{C} \) is to be replaced in these expressions by a vector function of element nodal parameters \( \mathbf{v}^e \). The vector of nodal parameters may be written as

\[
\mathbf{v}^e = \mathbf{B} \mathbf{C}
\]  

(4.5)

where

\( \mathbf{B} \) is an \( n \) by \( n \) matrix containing the values of \( \phi_i(x) \) at the element nodes.

In the case of a fourth-order shape function for beam bending, the nodal vector \( \mathbf{v}^e \) is given by

\[
\mathbf{v}^e = \begin{bmatrix}
v(0) \\
v'(0) \\
v(L) \\
v'(L)
\end{bmatrix}
\]  

(4.6)

and the matrix \( \mathbf{B} \) is

\[
\mathbf{B} = \begin{bmatrix}
\phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) \\
\phi_1'(0) & \phi_2'(0) & \phi_3'(0) & \phi_4'(0) \\
\phi_1(L) & \phi_2(L) & \phi_3(L) & \phi_4(L) \\
\phi_1'(L) & \phi_2'(L) & \phi_3'(L) & \phi_4'(L)
\end{bmatrix}
\]  

(4.7)

where \( L \) is the element length.

The inverse expression for constant vector \( \mathbf{C} \) is therefore given by

\[
\mathbf{C} = \mathbf{B}^{-1} \mathbf{v}^e
\]  

(4.8)
The element displacement, \( v(x) \), can now be given in terms of nodal values by the expression

\[
v(x) = \phi(x) B^{-1} v^e.
\] (4.9)

Recall that for a nodal shape function vector, \( \phi^e(x) \),

\[
v(x) = \phi^e(x) v^e.
\] (4.10)

Hence, it is clear that the nodal shape function is equivalent to

\[
\phi^e(x) = \phi(x) B^{-1}.
\] (4.11)

Its first and second derivatives of this row vector are

\[
\phi^e'(x) = \phi'(x) B^{-1}
\] (4.12)

and

\[
\phi^e''(x) = \phi''(x) B^{-1}.
\] (4.13)

Thus, Equation 4.11 yields a nodal finite element shape function developed from an arbitrary function of the form given by Equation 4.1 above. The simplest nodal shape functions result when the matrix \( B \) can be inverted in closed form. Although \( B \) must typically be inverted numerically when functions other than polynomials are used as interpolation functions, \( B \) is inverted in closed form in subsequent element derivations for all but the most general and complex case.

**Element Matrix Development**

The element matrices for beam-on-foundation elements developed from the residual expressions from Chapter III are presented in this
section, first for the axial degrees of freedom and second for the bending degrees of freedom.

**Axial Displacement Matrices**

The residual expression, from Chapter III, for axial displacement is

\[
0 = \sum_{1}^{n} \left\{ \left[ \int_{g}^{L} \psi_{x}^{e} \psi_{x}^{eT} EA \psi_{x}^{eT} dx - \int_{g}^{L} \psi_{x}^{eT} k_{x} \psi_{x}^{e} dx \right] \ddot{u}^{e} 
+ \int_{g}^{L} \psi_{x}^{eT} q_{x} dx + 
- \left[ \int_{g}^{L} \psi_{x}^{eT} \Gamma \psi_{x}^{e} dx \right] \dddot{u}^{e} - \left[ \int_{g}^{L} \psi_{x}^{eT} \phi \psi_{x}^{e} dx \right] \dddot{u}^{e} \right\} 
+ \sum_{1}^{n} \left[ \psi_{x}^{eT} N^{*} \right]^{L}_{g}. \tag{4.14}
\]

The homogeneous portion of the corresponding differential equation for axial motion is

\[
\frac{\partial}{\partial x} \left( EA \frac{\partial u}{\partial x} \right) - k_{x} u = 0 ; \quad k_{x} > 0 \tag{4.15}
\]

This expression assumes that the foundation acts as a linear elastic restraint in the axial direction. The solution of the expression is

\[
u(x) = C_{1} e^{mx} + C_{2} e^{-mx} \tag{4.16}
\]

where

\[
\frac{m}{EA} = \frac{k_{x}}{EA}.
\]
For the axial displacement case, the $B$ matrix is 2 by 2 and can be readily inverted in closed form. The resulting nodal shape function vector is
\[
\psi^e_i(x) = \begin{bmatrix} \frac{\sinh(m(L-x))}{\sinh(ml)} & \frac{\sinh(mx)}{\sinh(ml)} \end{bmatrix} \tag{4.17}
\]
where
\[
\sinh(mx) = \frac{e^{mx} - e^{-mx}}{2} \quad \sinh(m(L-x)) = \frac{e^{m(L-x)} - e^{-m(L-x)}}{2}. \tag{4.18}
\]
Table 4.2 presents the resulting element matrices for the axial case with a linearly elastic foundation response. The closed-form nodal shape functions yield compact and concise element expressions.

**Torsion Matrix**

The residual expression developed for torsion is
\[
0 = \sum_{i=1}^{n} \left[ \int_{a}^{L} -\psi^{e,T}_{xy} GJ \psi^{e}_y \, dx \right] \omega^{e} + \sum_{i=1}^{n} \left[ \psi^{e,T}_{xy} M^{e}_{x} \right]^{L}_{a}. \tag{4.19}
\]

The differential equation for element torsion, assuming that rotations are small and that no warping of planes occurs, is given by
\[
\frac{d}{dx} \left( GJ \frac{d\omega}{dx} \right) = 0. \tag{4.20}
\]
The solution to Equation 4.20 is the linear polynomial
\[
\omega(x) = C_1 + C_2 x. \tag{4.21}
\]
The nodal shape function vector is the simple expression
Table 4.2 Element Matrices for Axial Displacement

a) Nodal Shape Function

\[ \psi_\xi(x) = \left[ \frac{\sinh(m(L-x))}{\sinh(mL)} \quad \frac{\sinh(mx)}{\sinh(mL)} \right], \quad m = \frac{k_x}{EA}. \]

b) Element Stiffness Matrix

\[
\int_0^L \left( -\psi_\xi^T \right) \chi_k \psi_\xi \, dx - \int_0^L \psi_\xi^T \chi_k \psi_\xi \, dx = \frac{1}{(2m)\sinh^2(mL)} \begin{bmatrix}
(\kappa_+ \sinh(2mL) - \kappa_- mL) & (-\kappa_- \sinh(mL) + \kappa_+ \sinh(2mL)) \\
\text{symm.} & (\kappa_+ \sinh(2mL) - \kappa_- mL)
\end{bmatrix}
\]

\[ m = \frac{k_x}{EA}, \quad \kappa_+ = (EAm^2 + k_x), \quad \kappa_- = (-EAm^2 + k_x) \]

c) Element Load Vector

\[
\int_0^L \psi_\xi^T q_x \, dx = \begin{bmatrix}
\frac{q_1}{m} \frac{(\cosh(mL) - 1)}{\sinh(mL)} + \frac{(q_2 - q_1)}{m^2} \left( 1 - \frac{mL}{\sinh(mL)} \right) \\
\frac{q_1}{m} \frac{(\cosh(mL) - 1)}{\sinh(mL)} + \frac{(q_2 - q_1)}{m^2} \left( (mL) \coth(mL) - 1 \right)
\end{bmatrix}
\]

where

\[ q_x(x) = q_1 + \frac{(q_2 - q_1)x}{L} \]

d) Element Mass Matrix

\[
\int_0^L \psi_\xi^T \Gamma \psi_\xi \, dx = \frac{\Gamma}{2m \sinh^2(mL)} \begin{bmatrix}
(\sinh(2mL) - mL) & (\sinh(mL) + \sinh(2mL)) \\
\text{symm.} & (\sinh(2mL) - mL)
\end{bmatrix}
\]
Table 4.2 (Continued)

\( e) \quad \text{Element Damping Matrix} \)

\[
\begin{align*}
\int_{s}^{L} \psi_{s}^{T} \phi \psi_{s} \, dx &= \\
\dfrac{\phi}{2m \sinh^{2}(mL)} \begin{bmatrix} (\sinh(2mL) - mL) & (\sinh(mL) + \sinh(2mL)) \\ \text{symm.} & (\sinh(2mL) - mL) \end{bmatrix}
\end{align*}
\]

\( f) \quad \text{Concentrated Nodal Load Vector} \)

\[
\begin{bmatrix} \psi_{s}^{T} N^{s} \end{bmatrix}_{s}^{L} = \begin{bmatrix} N^{s}(0) \\ N^{s}(L) \end{bmatrix}
\]
\[ \psi^o_{xy}(x) = \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix}. \]  

(4.22)

The resulting element matrices are given in Table 4.3.

**Fourth-order Homogeneous-Solution Elements**

Although the axial degrees of freedom of beam motion play a role in beam analysis, the primary focus of this study is fourth-order equations for beam-on-foundation problems. From Chapter III the homogeneous fourth-order differential equation for beam-on-foundation problems was shown to be

\[
\frac{\partial^2}{\partial x^2} \left( EI_y \frac{\partial^2 v}{\partial x^2} \right) - N \frac{\partial^2 v}{\partial x^2} - BN \frac{\partial v}{\partial x} + k_y v = 0.
\]

(4.23)

The corresponding Galerkin weighted residual expression is

\[
0 = \sum_{1}^{n} \left\{ \left[ \int_{g}^{L} \psi^e_{y}^T \Gamma \psi^e_{y} \, dx + \int_{g}^{L} \psi^e_{y}^T N \psi^e_{y} \, dx 
- \int_{g}^{L} \psi^e_{y}^T BN \psi^e_{y} \, dx + \int_{g}^{L} \psi^e_{y}^T k_y \psi^e_{y} \, dx \right] v^e 
- \int_{g}^{L} \psi^e_{y}^T q_y \, dx 
\right. 
- \left[ \int_{g}^{L} \psi^e_{y}^T \Gamma \psi^e_{y} \, dx \right] \ddot{v}^e 
- \left[ \int_{g}^{L} \psi^e_{y}^T q_y \psi^e_{y} \, dx \right] \ddot{v}^e 
\right\} 
- \sum_{1}^{n} \left[ \psi^e_{y}^T M^e_y \right]_{g}^{L} + \sum_{1}^{n} \left[ \psi^e_{y}^T S^e_y \right]_{g}^{L}.
\]

(4.24)
Table 4.3 Element Matrices for Torsion

a) Nodal Shape Function

$$\psi_{xy}(x) = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix}$$

b) Element Stiffness Matrix

$$\int_0^L -\psi_{xy}^T GJ \psi_{xy} \, dx = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

c) Concentrated Nodal Load Vector

$$\left[ \begin{array}{c} \psi_{xy}^T \\ M_x^* \end{array} \right]_0^L = \left[ \begin{array}{c} M_x^*(0) \\ M_x^*(L) \end{array} \right]$$
For this development the nodal shape function, $\psi_y(x)$, is rewritten in terms of $B_y$ and $\phi_y(x)$ as given by Equation 4.11. The residual expression becomes

$$
0 = \sum_{i=1}^{n} \left\{ [B_y^{-1}]^T \left[ \int_{0}^{L} \phi_y^T EI \phi_y'' \, dx + \int_{0}^{L} \phi_y^T N \phi_y' \, dx ight. \
- \int_{0}^{L} \phi_y^T \frac{\partial N}{\partial x} \phi_y' \, dx + \int_{0}^{L} \phi_y^T k_y \phi_y \, dx \right] [B_y^{-1}] v^e \\
- [B_y^{-1}]^T \left[ \int_{0}^{L} \phi_y^T q_y \, dx \right] \\
- [B_y^{-1}]^T \left[ \int_{0}^{L} \phi_y^T \Gamma \phi_y \, dx \right] [B_y^{-1}] \ddot{v}^e \\
- [B_y^{-1}]^T \left[ \int_{0}^{L} \phi_y^T \phi_y \, dx \right] [B_y^{-1}] \dot{v}^e \\
- \sum_{i=1}^{n} [B_y^{-1}]^T \left[ \phi_y^T M_y \right]_{\theta}^{L} + \sum_{i=1}^{n} [B_y^{-1}]^T \left[ \phi_y^T S_y \right]_{\theta}^{L}. \right. \tag{4.25}
$$

The residual expression for lateral bending in the x-z planes corresponds directly to this expression above. Consequently, the matrices for lateral motion in the x-z plane correspond directly to those for the x-y plane with proper substitution of the function $\phi_z$.

The candidate interpolation functions $\phi_y(x)$ are selected here to be the homogeneous solutions of Equation 4.23 and its special cases in terms of complex exponentials, summarized in Table 4.4. The solution of the most general case, Equation 4.23, is given by

$$
v = C_1 e^{s_1 x} + C_2 e^{s_2 x} + C_3 e^{s_3 x} + C_4 e^{s_4 x}. \tag{4.26}
$$

The special cases of Equation 4.23 are addressed by two complex-exponential functions given by
Table 4.4 Complex Exponential Solutions for Fourth Order Differential Equations

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Complex Exponential Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) + kv = q$</td>
<td>$v = C_1e^{\alpha x} + C_2e^{\beta x} + C_3e^{-\alpha x} + C_4e^{-\beta x}$</td>
</tr>
<tr>
<td></td>
<td>$m_i = (1 \pm i) \frac{k}{4EI}; i = 1,2$</td>
</tr>
<tr>
<td>$\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} = q$</td>
<td>$v = C_1e^{\alpha x} + C_2e^{-\alpha x} + C_3x + C_4$</td>
</tr>
<tr>
<td></td>
<td>$m = \sqrt{\frac{N}{EI}}$</td>
</tr>
<tr>
<td>$\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} + kv = q$</td>
<td>for $</td>
</tr>
<tr>
<td></td>
<td>$m_i = (\alpha \pm i\beta); i = 1,2$</td>
</tr>
</tbody>
</table>
Table 4.4 (Continued)

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Complex Exponential Solution</th>
</tr>
</thead>
</table>
| \[
\frac{d^2}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - N \frac{d^2v}{dx^2} - \frac{dN}{dx} \frac{dv}{dx} + kv = q
\] |
| \[
\alpha = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{N}{4EI}}
\] |
| \[
\beta = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{N}{4EI}}
\] |
| \[
v = C_1 e^{\alpha x} + C_2 e^{-\alpha x} + C_3 x + C_4
\] |
| \[
m = (\alpha + i\beta)
\] |
| \[
v = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x}
\] |
| \[
m_i \text{ are the roots of }
\] |
| \[
EIm^4 - Nm^2 - N'm + k = 0
\] |
\[ v = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-x} + C_4 e^{-2x} \]  \hspace{1cm} (4.27)

and

\[ v = C_1 e^{x} + C_2 e^{-2x} + C_3 x + C_4 \]  \hspace{1cm} (4.28)

The general case is encountered infrequently in engineering problems, so that functions given by Equations 4.27 and 4.28 address the majority of engineering beam-on-foundation problems.

Finite elements based upon the three expressions in Equations 4.26, 4.27, and 4.28 will exactly solve symmetric special cases of the fourth-order differential equation (i.e. \( \frac{\partial N}{\partial x} = 0 \) in Equation 4.23) other than the most basic cubic beam. The finite element matrices resulting from each of these functions are given in Tables 4.5, 4.6, and 4.7. The \( B_y \) matrix resulting from functions in Equations 4.27 and 4.28 has been inverted in closed form, while, for the most general case (Equation 4.26), the \( B_y \) matrix must be inverted numerically. The element integrations have been carried out in closed form for all three cases.

In developing solutions for Equations 4.27 and 4.28 the terms \((e^{x} - e^{-x})\) and \((e^{x} + e^{-x})\) arise frequently. In the finite element matrices of Tables 4.6 and 4.7, the notation is simplified significantly by using the hyperbolic sine \((\sinh)\), hyperbolic cosine \((\cosh)\), and hyperbolic cotangent \((\coth)\) functions, defined by

\[ \sinh z = \frac{e^z - e^{-z}}{2} \quad \cosh z = \frac{e^z + e^{-z}}{2} \quad \coth z = \frac{e^z + e^{-z}}{e^z - e^{-z}} \]  \hspace{1cm} (4.29)

where \( z \) may be real, imaginary, or complex. Whereas these functions are typically limited to the real domain in the literature, here they are permitted to take on complex and imaginary values. It should also
Table 4.5 Element Matrices for General Fourth Order Bending Equations

a) Element Interpolation Function

\[ v = C_1 e^{x_1} + C_2 e^{x_2} + C_3 e^{x_3} + C_4 e^{x_4} \]

b) Function Transformation Matrix

\[
B_y = \begin{bmatrix}
1 & 1 & 1 & 1 \\
m_1 & m_2 & m_3 & m_4 \\
es_{*1L} & es_{*2L} & es_{*3L} & es_{*4L} \\
me_{*1L} & mve_{*2L} & mme_{*3L} & mmme_{*4L}
\end{bmatrix}
\]

c) Interpolation Stiffness Matrix

\[
\int_S \phi_y^T EI \phi_y \, dx + \int_S \phi_y^T N \phi_y \, dx - \int_S \phi_y^T \frac{\partial N}{\partial x} \phi_y \, dx \\
+ \int_S \phi_y^T k_y \phi_y \, dx = [k_{ij}], \ i,j = 1,2,3,4
\]

\[
k_{ij} = \frac{1}{m_i + m_j} \left( m_i^2 m_j^2 EI + m_i m_j N - m_i \frac{dN}{dx} + k_y \right) \left( e^{x_i-x_j}L-1 \right)
\]

d) Interpolation Load Vector

\[- \int_S \phi_y^T q_y \, dx = [q_i], \ i = 1,2,3,4
\]

\[
q_i = e^{x_i L} \left( \frac{q_2 (m_i L - 1) + q_1}{m_i^2 L} \right) + \left( \frac{q_2 - q_1 (m_i L + 1)}{m_i^2 L} \right)
\]

where

\[ q_y(x) = q_1 + \frac{(q_2-q_1)x}{L} \]
e) Interpolation Mass Matrix

\[ \int_{\Omega} \phi_i^T \Gamma \phi_j \, dx = [\Gamma_{ij}], \ i,j = 1, 2, 3, 4 \]
\[ \Gamma_{ij} = \frac{\Gamma}{m_i + m_j} (e^{(m_i + m_j)L} - 1) \]

f) Interpolation Damping Matrix

\[ \int_{\Omega} \phi_i^T \phi_j \, dx = [\Phi_{ij}], \ i,j = 1, 2, 3, 4 \]
\[ \Phi_{ij} = \frac{\Phi}{m_i + m_j} (e^{(m_i + m_j)L} - 1) \]

g) Concentrated Nodal Load Vector

\[ [B_y^{-1}]^T \left[ \phi_y^T M_y^* \right]_{\text{g}}^L + [B_y^{-1}]^T \left[ \phi_y^T S_y^* \right]_{\text{g}}^L = \begin{bmatrix} S_y^*(0) \\ M_y^*(0) \\ S_y^*(L) \\ M_y^*(L) \end{bmatrix} \]
Table 4.6 Fourth Order Element Matrices for $\frac{\partial N}{\partial x} = 0$ and $|N| \neq 2 \sqrt{EI}$

a) Element Interpolation Function

$$v(x) = C_1 \left( \frac{\sinh(m_1 (L-x))}{\sinh(m_1 L)} + \frac{\sinh(m_2 (L-x))}{\sinh(m_2 L)} \right) + C_2 \left( \frac{\sinh(m_1 (L-x))}{\sinh(m_1 L)} - \frac{\sinh(m_2 (L-x))}{\sinh(m_2 L)} \right)$$

$$+ C_3 \left( \frac{\sinh(m_1 x)}{\sinh(m_1 L)} + \frac{\sinh(m_2 x)}{\sinh(m_2 L)} \right) + C_4 \left( \frac{\sinh(m_1 x)}{\sinh(m_1 L)} - \frac{\sinh(m_2 x)}{\sinh(m_2 L)} \right)$$

b) Function Transformation Matrix, $B^{-1} = [B_{ij}]$, $i,j = 1,2,3,4$

$$B_{11} = 0.5 = B_{33}$$

$$B_{12} = 0 = B_{13} = B_{14} = B_{31} = B_{32} = B_{34}$$

$$B_{21} = 2m.m. (\sinh^2(m.L) - \sinh^2(m.L))/D = B_{43}$$

$$B_{22} = (m. \sinh(2m.L) - m. \sinh(2m.L))/D = -B_{44}$$

$$B_{23} = -2m_1 m_2 \sinh(m.L) \sinh(m.L)/D = B_{41}$$

$$B_{24} = (m_1 \sinh(m_2 L) - m_2 \sinh(m_1 L))/D = -B_{42}$$

$$D = 4(m^2 \sinh^2(m.L) - m^2 \sinh^2(m.L))$$

$$m_1 = \frac{(m_2 + m_1)}{2}, \quad m_2 = \frac{(m_2 - m_1)}{2}$$

c) Interpolation Stiffness Matrix

$$\int_0^L \phi_y^T EI \phi_y \, dx + \int_0^L \phi_y^T N \phi_y \, dx + \int_0^L \phi_y^T k_y \phi_y \, dx = [k_{ij}], \ i,j = 1,2,3,4 \ (symmetric)$$
Table 4.6 (Continued)

\[ k_{11} = (\kappa_1 \sinh(2m_1 L) - 2\kappa_2 m_1 L) + (\kappa_3 \sinh(2m_2 L) - 2\kappa_4 m_2 L) + 2(\kappa_5 m \cdot \sinh(2m_1 L) - \kappa_6 m \cdot \sinh(2m_2 L)) = k_{33} \]

\[ k_{12} = (\kappa_1 \sinh(2m_1 L) - 2\kappa_2 m_1 L) - (\kappa_3 \sinh(2m_2 L) - 2\kappa_4 m_2 L) = k_{34} \]

\[ k_{13} = 2(\kappa_1 m_1 L \cosh(m_1 L) - \kappa_2 \sinh(m_1 L)) + 2(\kappa_3 m_2 L \cosh(m_2 L) - \kappa_4 \sinh(m_2 L)) + 2((\kappa_5 m \cdot -\kappa_6 m \cdot) \sinh(m_2 L) - (\kappa_5 m \cdot +\kappa_6 m \cdot) \sinh(m_1 L)) \]

\[ k_{14} = 2(\kappa_1 m_1 L \cosh(m_1 L) - \kappa_2 \sinh(m_1 L)) - 2(\kappa_3 m_2 L \cosh(m_2 L) - \kappa_4 \sinh(m_2 L)) = k_{23} \]

\[ k_{22} = (\kappa_1 \sinh(2m_1 L) - 2\kappa_2 m_1 L) + (\kappa_3 \sinh(2m_2 L) - 2\kappa_4 m_2 L) - 2(\kappa_5 m \cdot \sinh(2m_1 L) - \kappa_6 m \cdot \sinh(2m_2 L)) = k_{44} \]

\[ k_{24} = 2(\kappa_1 m_1 L \cosh(m_1 L) - \kappa_2 \sinh(m_1 L)) + 2(\kappa_3 m_2 L \cosh(m_2 L) - \kappa_4 \sinh(m_2 L)) - 2((\kappa_5 m \cdot -\kappa_6 m \cdot) \sinh(m_2 L) - (\kappa_5 m \cdot +\kappa_6 m \cdot) \sinh(m_1 L)) \]

where

\[ \kappa_1 = \frac{m_1^4 EI + m_1^2 N + k_y}{4m_1 \sinh^2(m_1 L)} \]
\[ \kappa_2 = \frac{m_1^4 EI - m_1^2 N + k_y}{4m_1 \sinh^2(m_1 L)} \]
\[ \kappa_3 = \frac{m_2^4 EI + m_2^2 N + k_y}{4m_2 \sinh^2(m_2 L)} \]
\[ \kappa_4 = \frac{m_2^4 EI - m_2^2 N + k_y}{4m_2 \sinh^2(m_2 L)} \]
\[ \kappa_5 = \frac{m_1^2 m_2^2 EI + m_1 m_2 N + k_y}{4m_1 m \cdot \sinh(m_1 L) \sinh(m_2 L)} \]
\[ \kappa_6 = \frac{m_1^2 m_2^2 EI - m_1 m_2 N + k_y}{4m_1 m \cdot \sinh(m_1 L) \sinh(m_2 L)} \]
d) Interpolation Load Vector

\[- \int_{\theta}^{L} \phi^T \phi_y \, dx = \]

\[
\begin{bmatrix}
q_1 \left( \frac{(\cosh(m_1L) - 1)}{m_1 \sinh(m_1L)} + \frac{(\cosh(m_2L) - 1)}{m_2 \sinh(m_2L)} \right) + \frac{(q_2 - q_1)}{m_1L} \left( \frac{1}{m_1} - \frac{L}{\sinh(m_1L)} \right) + \frac{(q_2 - q_1)}{m_2L} \left( \frac{1}{m_2} - \frac{L}{\sinh(m_2L)} \right) \\
q_1 \left( \frac{(\cosh(m_1L) - 1)}{m_1 \sinh(m_1L)} - \frac{(\cosh(m_2L) - 1)}{m_2 \sinh(m_2L)} \right) + \frac{(q_2 - q_1)}{m_1L} \left( \frac{1}{m_1} - \frac{L}{\sinh(m_1L)} \right) - \frac{(q_2 - q_1)}{m_2L} \left( \frac{1}{m_2} - \frac{L}{\sinh(m_2L)} \right) \\
q_1 \left( \frac{(\cosh(m_1L) - 1)}{m_1 \sinh(m_1L)} + \frac{(\cosh(m_2L) - 1)}{m_2 \sinh(m_2L)} \right) + \frac{(q_2 - q_1)}{m_1L} \left( \text{Lcoth}(m_1L) - \frac{1}{m_1} \right) + \frac{(q_2 - q_1)}{m_2L} \left( \text{Lcoth}(m_2L) - \frac{1}{m_2} \right) \\
q_1 \left( \frac{(\cosh(m_1L) - 1)}{m_1 \sinh(m_1L)} - \frac{(\cosh(m_2L) - 1)}{m_2 \sinh(m_2L)} \right) + \frac{(q_2 - q_1)}{m_1L} \left( \text{Lcoth}(m_1L) - \frac{1}{m_1} \right) - \frac{(q_2 - q_1)}{m_2L} \left( \text{Lcoth}(m_2L) - \frac{1}{m_2} \right)
\end{bmatrix}
\]

where

\[q_y(x) = q_1 + \frac{(q_2 - q_1)x}{L} \]

e) Interpolation Mass Matrix

\[
\int_{\theta}^{L} \phi_y^T \Gamma \phi_y \, dx = [\Gamma_{ij}], \quad i,j = 1,2,3,4
\]

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\( \Gamma_{ij} = k_{ij} \) in (c) where
\[
\kappa_1 = \kappa_2 = \frac{\Gamma}{4m_1 \sinh^2(m_1L)} \quad \kappa_3 = \kappa_4 = \frac{\Gamma}{4m_2 \sinh^2(m_2L)} \quad \kappa_5 = \kappa_6 = \frac{\Gamma}{4m_\infty \sinh(m_1L) \sinh(m_2L)}
\]

f) Interpolation Damping Matrix

\[
\int_0^L \phi_y^T \Phi \phi_y \, dx = [\Phi_{ij}], \ i,j = 1,2,3,4
\]
\( \Phi_{ij} = k_{ij} \) in (c) where
\[
\kappa_1 = \kappa_2 = \frac{\phi}{4m_1 \sinh^2(m_1L)} \quad \kappa_3 = \kappa_4 = \frac{\phi}{4m_2 \sinh^2(m_2L)} \quad \kappa_5 = \kappa_6 = \frac{\phi}{4m_\infty \sinh(m_1L) \sinh(m_2L)}
\]

g) Concentrated Nodal Load Vector

\[
[B_y^{-1}]^T \left[ \phi_y^T \ M_y^* \right]_0^L + [B_y^{-1}]^T \left[ \phi_y^T \ S_y^* \right]_0^L = \begin{bmatrix} S_y^* (0) \\ M_y^* (0) \\ S_y^* (L) \\ M_y^* (L) \end{bmatrix}
\]
Table 4.7 Fourth Order Element Matrices for $\frac{\partial N}{\partial x} = 0$ and $|N| = 2 \sqrt{EIT}$

a) Element Interpolation Function

$$v(x) = C_1 \frac{\sinh(m(L-x))}{\sinh(mL)} + C_2 \frac{\sinh(mx)}{\sinh(mL)} + C_3x + C_4$$

b) Function Transformation Matrix, $B^{-1} = [B_{ij}], i,j = 1,2,3,4$

$$B_{11} = \frac{m(1 - \cosh(mL))}{D} = -B_{13} = -B_{21} = B_{23} = B_{32} = B_{34} = B_{43}$$
$$B_{12} = \frac{(\sinh(mL) - mL\cosh(mL))}{D} = -B_{24} = -B_{42}$$
$$B_{22} = \frac{(\sinh(mL) - mL)}{D} = -B_{14} = B_{44}$$
$$B_{33} = \frac{m^2}{D} = -B_{31}$$
$$B_{41} = \frac{m(1 - 2\cosh(mL) + mL\sinh(mL))}{D}$$
$$D = m(2 - 2\cosh(mL) + mL\sinh(mL))$$

c) Interpolation Stiffness Matrix

$$\int_0^L \phi_y^{T}EI \phi_y \, dx + \int_0^L \phi_y^{T} \mathbf{N} \phi_y \, dx + \int_0^L \phi_y^{T}k_y \phi_y \, dx = [k_{ij}], i,j = 1,2,3,4 \text{ (symmetric)},$$

$$k_{11} = \frac{\kappa \cdot \sinh(2mL) - \kappa \cdot mL}{2m \sinh^2(mL)} = k_{22} \quad k_{12} = \frac{-\kappa \cdot \sinh(mL) + \kappa \cdot \sinh(2mL)}{2m \sinh^2(mL)}$$
$$k_{13} = -N + \frac{k_y}{m^2}(1 - \frac{mL}{\sinh(mL)}) \quad k_{14} = \frac{k_y (\cosh(mL) - 1)}{m \sinh(mL)} = k_{24}$$
$$k_{23} = N + \frac{k_y}{m^2} (mL \coth(mL) + 1) \quad k_{33} = \frac{k_y L^3}{3} + NL$$
$$k_{34} = \frac{k_y L^2}{2} \quad k_{44} = k_y L$$

where

$$\kappa = (EIm^4 + Nm^2 + k_y) \quad \kappa_1 = (EIm^4 - Nm^2 + k_y)$$
d) Interpolation Load Vector

\[
- \int_0^L \phi_y^T q_y \, dx = \begin{bmatrix}
\frac{q_1}{m} \frac{(cosh(mL) - 1)}{sinh(mL)} + \frac{(q_2-q_1)}{m^2 L} \left( 1 - \frac{mL}{sinh(mL)} \right) \\
\frac{q_1}{m} \frac{(cosh(mL) - 1)}{sinh(mL)} + \frac{(q_2-q_1)}{m^2 L} \left( mL\coth(mL) - 1 \right) \\
\left[ \frac{q_1}{2} + \frac{(q_2-q_1)}{3} \right] L^2 \\
\left[ q_1 + \frac{(q_2-q_1)}{2} \right] L
\end{bmatrix}
\]

e) Interpolation Mass Matrix

\[
\int_0^L \phi_y^T \Gamma \phi_y \, dx = [\Gamma_{ij}], \, i,j = 1,2,3,4 \text{ (symmetric)},
\]

\[
\Gamma_{11} = \Gamma_{22} = \frac{\Gamma(\sinh(2mL) - mL)}{2m \sinh^2(mL)} \quad \Gamma_{12} = \frac{\Gamma(-\sinh(mL) + \sinh(2mL))}{2m \sinh^2(mL)}
\]

\[
\Gamma_{13} = \frac{\Gamma}{m^2} \left( 1 - \frac{mL}{\sinh(mL)} \right) \quad \Gamma_{14} = \frac{\Gamma}{m} \frac{\cosh(mL) - 1}{\sinh(mL)}
\]

\[
\Gamma_{23} = \frac{\Gamma}{m^2} \left( mL \coth(mL) + 1 \right) \quad \Gamma_{33} = \frac{\Gamma}{3} L^3
\]

\[
\Gamma_{34} = \frac{\Gamma}{2} L^2 
\]

\[
\Gamma_{44} = \Gamma L
\]

f) Interpolation Damping Matrix

\[
\int_0^L \phi_y^T \phi \phi_y \, dx = [\phi_{ij}], \, i,j = 1,2,3,4 \text{ (symmetric)},
\]

\[
\phi_{11} = \phi_{22} = \frac{\phi(\sinh(2mL) - mL)}{2m \sinh^2(mL)} \quad \phi_{12} = \frac{\phi(-\sinh(mL) + \sinh(2mL))}{2m \sinh^2(mL)}
\]

\[
\phi_{13} = \frac{\phi}{m^2} \left( 1 - \frac{mL}{\sinh(mL)} \right) \quad \phi_{14} = \frac{\phi}{m} \frac{\cosh(mL) - 1}{\sinh(mL)}
\]

\[
\phi_{23} = \frac{\phi}{m^2} \left( mL \coth(mL) + 1 \right) \quad \phi_{33} = \frac{\phi}{3} L^3
\]

\[
\phi_{34} = \frac{\phi}{2} L^2 
\]

\[
\phi_{44} = \phi L
\]
\[ \Phi_{23} = \frac{\Phi}{m^2} (mL \coth(mL) + 1) \quad \Phi_{33} = \frac{\Phi L^3}{3} \]
\[ \Phi_{34} = \frac{\Phi L^2}{2} \quad \Phi_{44} = \Phi L \]

\text{g) Concentrated Nodal Load Vector}

\[
[B_y^{-1}]^T \left[ \left[ \begin{array}{c} \Phi_y^T \\ M_y^* \end{array} \right]^L \right]_g + [B_y^{-1}]^T \left[ \left[ \begin{array}{c} \Phi_y^T \\ S_y^* \end{array} \right]^L \right]_g = \left[ \begin{array}{c} S_y^*(0) \\ M_y^*(0) \\ S_y^*(L) \\ M_y^*(L) \end{array} \right]
\]
be noted that the hyperbolic sinh and cosh functions are solutions of the special cases of Equation 4.23 in which \( \frac{\partial N}{\partial x} = 0 \), although they are not solutions of the general differential equation itself.

Clearly, the component matrices that form the element stiffnesses resulting from Equations 4.26, 4.27, and 4.28 may be complex numbers, depending upon the coefficients \( m_i \). However, the displacements, derivatives, and applied loads that occur in beam-on-foundation problems are real values. Consequently, the element stiffnesses and load vectors that finally result from multiplication of the complex matrices have zero imaginary components and are therefore real.

**Foundation Interaction Matrix**

Up to this point the development of the element matrices has focused on planar motion of a beam-on-foundation. As mentioned earlier, the residual expressions (Equations 3.93 and 3.94) show that there may be a coupling of displacements between the two primary planes of motions due to a foundation interaction term \( k_{xy}(v,w) \). An example of such interaction is the case in which a beam is only partially surrounded by soil (Figure 4.2). When a lateral force is applied to the beam, the foundation yields both lateral and vertical reaction forces. The contribution of this coupled reaction term to the residual expression is

\[
\int_{0}^{L} \phi^e_i \mathbf{w}^e \quad dx \quad \mathbf{w}^e
\]

in the x-y plane and

\[
\int_{0}^{L} \phi^e_y \mathbf{k}_{xy} \phi^e_z \quad dx \quad \mathbf{w}^e
\]
Figure 4.2 Coupling of Lateral Beam Displacements

$$\int_{s}^{L} \mathbf{y}_z^e \mathbf{y}_y^e \mathbf{k}_{yz} dx \mathbf{v}_e$$

(4.31)

in the x-z plane. These expressions both include a contribution from the shape functions of both the x-y and x-z planes of motion. The resulting element matrix contributions from the expressions are given in Table 4.8.

**Semi-infinite Elements**

In physical beam-on-foundation problems, such as railroad tracks and buried pipelines, semi-infinite boundary conditions often exist in which the beam extends an infinite distance in one direction
Table 4.8 Element Matrix Contribution for Foundation Coupling Terms

a) Submatrices for coupled foundation
\[ k_{yz}^o = \begin{bmatrix} k_{23} & k_{26} & k_{29} & k_{212} \\ k_{53} & k_{56} & k_{59} & k_{512} \\ k_{83} & k_{86} & k_{89} & k_{812} \\ k_{113} & k_{116} & k_{119} & k_{1112} \end{bmatrix} \]
\[ k_{zy}^o = \begin{bmatrix} k_{32} & k_{35} & k_{38} & k_{311} \\ k_{62} & k_{65} & k_{68} & k_{611} \\ k_{92} & k_{95} & k_{98} & k_{911} \\ k_{122} & k_{125} & k_{128} & k_{1211} \end{bmatrix} \]

b) Interpolation functions
\[ v_y = C_1 e^{y_1 x} + C_2 e^{y_2 x} + C_3 e^{y_3 x} + C_4 e^{y_4 x} \]
\[ v_z = C_1 e^{z_1 x} + C_2 e^{z_2 x} + C_3 e^{z_3 x} + C_4 e^{z_4 x} \]

c) Coupled Foundation Stiffness Matrices
\[ \int_0^L \phi^T \phi y^o \ dx = [B_y^{-1}]^T[k_{ij}][B_z^{-1}], \ i,j = 1,2,3,4 \]
where
\[ k_{ij} = \frac{k_{yz}}{m_y+y_i+m_zy_j} (e^{y_i+y_j})^{L-1} \]
\[
B_y = \begin{bmatrix}
1 & 1 & 1 & 1 \\
\frac{m_y}{1} & \frac{m_y}{2} & \frac{m_y}{3} & \frac{m_y}{4} \\
e^{\frac{ym_1}{1}} & e^{\frac{ym_2}{2}} & e^{\frac{ym_3}{3}} & e^{\frac{ym_4}{4}} \\
m_y e^{\frac{ym_1}{1}} & m_y e^{\frac{ym_2}{2}} & m_y e^{\frac{ym_3}{3}} & m_y e^{\frac{ym_4}{4}}
\end{bmatrix}
\]

\[
B_z = \begin{bmatrix}
1 & 1 & 1 & 1 \\
\frac{m_z}{1} & \frac{m_z}{2} & \frac{m_z}{3} & \frac{m_z}{4} \\
e^{\frac{zm_1}{1}} & e^{\frac{zm_2}{2}} & e^{\frac{zm_3}{3}} & e^{\frac{zm_4}{4}} \\
m_z e^{\frac{zm_1}{1}} & m_z e^{\frac{zm_2}{2}} & m_z e^{\frac{zm_3}{3}} & m_z e^{\frac{zm_4}{4}}
\end{bmatrix}
\]

\[
\int_\theta^L \theta_z^T k_{yz} \theta_y^T \, dx = [B_z^{-1}]^T [k_{ij}] [B_y^{-1}]
\]
away from the area of immediate interest to the analysis. In such problems the presence of a foundation serves to limit and localize the effect of loadings, such that displacements at a significant distance away are uncoupled from the immediate area of concern. In these problems simple fixed or pivoted end conditions are inadequate and semi-infinite elements are necessary to accurately represent the physical boundary conditions.

Semi-infinite elements may be developed easily for functions which remain finite or go to zero at infinity, such as the expression

$$v = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{-m_1 x} + C_4 e^{-m_2 x}.$$  \hspace{1cm} (4.32)

The constants $C_1$ and $C_2$ in this function become zero when the boundary constraints $v(\infty) = 0$ and $v'(\infty) = 0$ are applied. The remaining terms remain finite and are well behaved as element length increases, so long as $m_1$ and $m_2$ have real components and are not strictly imaginary.

Equation 4.16 for axial displacement is also well behaved for semi-infinite boundary conditions. The semi-infinite elements resulting from second-order equations have one degree of freedom while those for fourth-order equations have two degrees of freedom. Matrices for semi-infinite elements are given for axial displacement and for beam bending problems solved by Equation 4.32 in Tables 4.9 and 4.10, respectively.

**Behavior of the Fourth-Order Shape Functions**

The fourth-order differential equation encompasses a broad range of behaviors for beam-on-foundation problems. Consequently, the behavior of the corresponding shape functions extends beyond the
Table 4.9 Element Matrices for Axial Displacement of a Semi-Infinite Beam

a) Nodal Shape Function

\[ \psi^o_x(x) = \left[ e^{-mx} \right] ; \quad m = \frac{k_x}{EA} \]

b) Element Stiffness Matrix

\[ \int_0^\infty - \psi^o_x \psi^o_x^T \cdot EA \cdot \psi^o_x \cdot dx - \int_0^\infty \psi^o_x \psi^o_x \psi^o_k \cdot dx = \begin{bmatrix} \frac{mEA}{2} + \frac{k_x}{2m} \end{bmatrix} \]

c) Element Load Vector

\[ \int_0^\infty \psi^o_x \psi^o_x \psi^o_q \cdot dx = \begin{bmatrix} q_x \end{bmatrix} ; \quad q_x = \text{constant} \]

d) Element Mass Matrix

\[ \int_0^\infty \psi^o_x \psi^o_x \psi^o_m \cdot dx = \begin{bmatrix} \frac{m}{2m} \end{bmatrix} \]

e) Element Damping Matrix

\[ \int_0^\infty \psi^o_x \psi^o_x \psi^o_\phi \cdot dx = \begin{bmatrix} \frac{\phi}{2m} \end{bmatrix} \]

f) Concentrated Nodal Load Vector

\[ \left[ \psi^o_x \psi^o_N \right]_0^\infty = \begin{bmatrix} N^*(0) \end{bmatrix} \]
Table 4.10 Element Matrices for Lateral Displacement of a Semi-Infinite Beam

a) Nodal Shape Function

\[ \psi_y^0(x) = \begin{bmatrix} \frac{m_1e^{2x} - m_2e^{x}}{m_1 - m_2} & \frac{e^{2x} - e^{x}}{m_1 - m_2} \end{bmatrix} \]

b) Element Stiffness Matrix

\[ \int_{\theta}^{\infty} \psi_y^{0T} EI \psi_y^{0} \, dx + \int_{\theta}^{\infty} \psi_y^{0} N \psi_y^{0} \, dx + \int_{\theta}^{\infty} \psi_y^{0T} k_y \psi_y^{0} \, dx = \]

\[ \begin{bmatrix} -EI \frac{m_1^3 m_2^3 - N m_1^2 m_2}{2m_1 m_2 (m_1 + m_2)} & \frac{EI m_1^2 m_2^2 - k}{2m_1 m_2} \\ \text{symm.} & -CEI \frac{m_1^3 m_2^3 - N m_1^2 m_2}{2m_1 m_2 (m_1 + m_2)} - k \end{bmatrix} \]

where

\[ C = m_1^2 + 3m_1 m_2 + m_2^2 \]

c) Element Load Vector

\[ - \int_{\theta}^{\infty} \psi_y^{0T} q_y \, dx = \begin{bmatrix} \frac{-m_1 - m_2}{m_1 m_2} \\ \frac{1}{m_1 m_2} \end{bmatrix} \]

d) Element Mass Matrix

\[ - \int_{\theta}^{\infty} \psi_y^{0T} \Gamma \psi_y^{0} \, dx = \begin{bmatrix} \frac{-\Gamma}{2m_1 m_2 (m_1 + m_2)} & \frac{\Gamma}{2m_1 m_2} \\ \text{symm.} & \frac{-\Gamma}{2m_1 m_2 (m_1 + m_2)} \end{bmatrix} \]
Table 4.10 (Continued)

e) Element Damping Matrix

\[
\int_{\gamma}^{\infty} \phi_y^T \phi_y^T \phi_y^T \phi_y^T \, dx = \begin{bmatrix}
-\frac{C\phi}{2m_1m_2(m_1+m_2)} & \frac{\phi}{2m_1m_2} \\
\text{symm.} & -\frac{C\phi}{2m_1m_2(m_1+m_2)}
\end{bmatrix}
\]

f) Concentrated Nodal Load Vector

\[
\left[ \phi_y^T M_y^* \right]_{\theta}^{\infty} + \left[ \phi_y^T S_y^* \right]_{\theta}^{\infty} = \begin{bmatrix}
S_y^*(0) \\
M_y^*(0)
\end{bmatrix}
\]
conventional quadratic or cubic behavior resulting from polynomial shape functions. A worthwhile exercise is to demonstrate the variation in element geometry over the range of possible fourth-order equation constants. This aids in understanding the behavior of the homogeneous-solution finite elements, as well as understanding the solutions of the fourth-order differential equation. The following discussion focuses on "element behavior" resulting from variation of the primary parameters $EI$ (beam stiffness), $N$ (axial load), and $k$ (foundation stiffness).

The primary differential equation controlling element behavior is given by

$$\frac{d^2}{dx^2}\left(\frac{EI}{d^2v}{dx^2}\right) - N\frac{d^2v}{dx^2} + kv = 0.$$ (4.33)

The constants controlling the solution of the equation are clearly $EI$, $N$, and $k$. In this review of element behavior, consider first the behavior in the simplest case, where both $N$ and $k$ are negligible. The differential equation for this case is the expression

$$\frac{d^2}{dx^2}\left(\frac{EI}{d^2v}{dx^2}\right) = q$$ (4.34)

which has the cubic polynomial solution

$$v = C_1 + C_2x + C_3x^2 + C_4x^3.$$ (4.35)

The shape functions here are the common Hermite cubic interpolation functions illustrated in Figure 4.3. For a unit displacement at $x=0$ and zero displacement at $x=L$ the simple beam assumes the shape defined by $\psi_1$ and has the slope defined by $\psi_2$. Similarly zero displacement at $x=0$ and unit displacement at $x=L$ yields $\psi_3$ and $\psi_4$. This basic cubic
behavior is modified significantly when $N$ and $k$ are no longer negligible.

Consider the case when the value of $k$ becomes significant. The differential equation is that of the beam on elastic foundation

$$
\frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) + kv = q
$$

(4.36)

which has the solution

$$
v = C_1 \cosh(\lambda x) \cos(\lambda x) + C_2 \cosh(\lambda x) \sin(\lambda x) + C_3 \sinh(\lambda x) \cos(\lambda x) + C_4 \sinh(\lambda x) \sin(\lambda x).
$$

(4.37)

$$
\lambda = \sqrt[4]{\frac{k}{4EI}}.
$$
This particular combination of hyperbolic and trigonometric functions leads to an exponentially decaying sine wave function illustrated in Figure 4.4. Figure 4.5 illustrates how the shape function $\psi_1^0$ varies in geometry as the value of $k$ increases relative to $EI$. Observe that the portion of the element with nonzero deflection diminishes as $k$ increases.

The contribution of the second order "N" term to Equation 4.33 is best understood by considering the solutions of the simple differential equation

$$- N \frac{d^2v}{dx^2} + v = 0 .$$

(4.38)

![Graph showing exponentially decaying sine wave](image-url)

Figure 4.4 Exponentially Decaying Sine Wave
Figure 4.5 Variation of Exponential Shape Function with Increasing $k$
For \( N \) positive the solution is
\[
v = C_1 \cosh(Nx) + C_2 \sinh(Nx) \tag{4.39}
\]
and the solution for \( N \) negative is
\[
v = C_1 \cos(|N|x) + C_2 \sin(|N|x) . \tag{4.40}
\]
The hyperbolic functions that result from \( N \) positive serve to
"straighten" and "stiffen" a shape function for increasing \( N \). For the
case of negative \( N \) the trigonometric functions serve to increase the
amplitude of sinusoidal displacements without the benefit of damping
observed in Equation 4.37. This type of contribution is observed when \( N \)
becomes significant in Equation 4.33. Figure 4.6 illustrates the
influence of the variation of \( N \) in Equation 4.33 on the shape function
\( \psi_i \) when \( k \) is zero. Figure 4.7 illustrates the variation of behavior of
\( \psi_i \) when both \( N \) and \( k \) are significant with respect to \( EI \).

It is important to observe that increasingly negative \( N \) leads
to large displacements along the element length. Such conditions
correspond to physically buckled mode shapes for beams. The stiffness
matrices that result from such "buckled" shape functions most often
result in singularities that have little physical meaning.

Hetenyi(1) provides a detailed discussion of the compressive
axial loads, \((-N)\) required to buckle beams on foundation under various
values of \( k \) and various boundary conditions. Such problems are
nonlinear; there is no simple expression to indicate when buckling and
associated ill-conditioned matrix problems may arise. Applying the
elements developed here to problems in which critical loads may be
Figure 4.6 Variation in Shape Function with $N$
Figure 4.7 Variation in Shape Function with N and k
exceeded should be done carefully, and should also include available screening criteria for the physical problem.

**Example Application of Homogeneous-Solution Elements**

Figure 4.8 presents four example beam-on-foundation problems for which exact, closed-form solutions are available from Hetenyi[1]. Tables 4.11 through 4.14 show comparisons of the closed-form solutions with finite element solutions using the homogeneous-solution (complex-exponential) elements developed here and with finite element solutions using cubic shape function elements. In all cases the homogeneous-solution elements yield exact solutions at the finite element nodes.

In these examples the cubic element solutions yield reasonable results with a limited number of elements for tensile axial loads in the beam. They perform poorly however, for compressive axial loads. In the case of a concentrated point load and a concentrated moment, the cubic elements yield solutions that may be in error by 100% or more if too few elements are used. This error is largely explained by the inability of cubic functions to represent the sinusoidal variations observed for large negative values of N in Figures 4.6 and 4.7.

The homogeneous-solution elements developed here overcome the difficulties found in using approximate elements. These example problems demonstrate that the exact solution to given problems can be obtained with the minimum number of elements necessary to describe the problem geometry. Consequently, optimum accuracy can be achieved
Figure 4.8 Example Beam-on-Foundation Problems from Hetenyi(1)
Table 4.11 Comparison with Hetenyi Solution for Concentrated Load*(1)

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<th>Homogeneous Element Solution</th>
<th>Cubic Element Solution</th>
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*E = 30,000,000.0, I = 0.1, k = 10.0, S' = 200.0
Table 4.12 Comparison with Hetenyi Solution for Concentrated Moment*(1)

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*(1) Where $N < 2\sqrt{KEI}$
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<th>Homogeneous Element Solution 1 Elem</th>
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*E = 30,000,000.0, I = 10.0, k = 0.1, M = 20000.0
Table 4.13 Comparison with Hetenyi Solution for Uniform Loading*(1)

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*E = 30,000,000.0, I = 0.1, k = 10.0, q = 20.0
Table 4.14 Comparison with Hetenyi Solution for Trapezoidal Loading*(1)

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</tr>
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<tr>
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</table>

*E = 30,000,000.0, I = 0.1, k = 10.0, q1 = 10.0, q2 = 0.0
without sacrifice of computational speed due to an excessive number of elements.

**Summary**

The development of a comprehensive series of finite elements to address beam-on-foundation problems and the fourth-order differential equation has been presented in this chapter. The elements developed are unique in that they utilize homogeneous solutions of the applicable differential equation as shape functions and, as proven earlier, the elements result in exact solutions for linear, symmetric problems at the finite element nodes. Although the elements require numerical multiplication of complex matrices, achieving exact solutions means that fewer elements are required for problem solving than would needed with conventional cubic elements, and there is no question of accuracy. The element matrices developed utilize complex mathematics to represent the solution of the fourth-order equation and are able to cover the majority of linear beam-on-foundation problems with two elements. Consequently, the elements developed in this chapter are comprehensive in the range of problems they address and are straightforward and efficient in their application. The following chapter utilizes these elements to solve some nonlinear problems that arise frequently in engineering analysis of beams-on-foundation.
V SOLUTION OF BEAM-LIFTOFF AND PIECEWISE-LINEAR-FOUNDATION PROBLEMS

While linear foundation models are often adequate for basic engineering solutions of beam-on-foundation problems, there is an increasing need to achieve more accurate solutions incorporating nonlinear foundation response. This chapter and the following chapter present and discuss iterative solution methods for beam-on-nonlinear-foundation problems. For the purposes of this study, nonlinear beam-foundation interaction problems are divided into two classes: those in which the foundation response and its first derivative are continuous functions of displacement, and those in which the foundation response is, or can be, modeled as a piecewise-linear function. This chapter addresses the latter. It presents a unique finite element solution methodology for piecewise-linear problems that is simple to implement and that performs well, both theoretically and practically. The new method permits straightforward solution of previously unsolved problems, such as multiple liftoff problems. Solution accuracy for the method is ensured by use of the finite elements developed in Chapter IV.

Piecewise-linear-foundation problems were introduced and described in Chapter II. They include both "liftoff" problems and piecewise-linear-foundation restraints. The following discussion presents the mathematical formulation of this class of problems, followed by development and discussion of solution methods. The chapter concludes with example solutions.
Formulation of the Piecewise-Linear-Foundation Problem

Although beam-liftoff problems and beam on piecewise-linear-foundation restraints are typically treated as unique problems, the assertion was made in Chapter II that they are both part of the same class of problems. Both problems have at least two regions governed by different differential equations. The equations have a common interface at which boundary conditions must be equivalent. The following formulation of this class of problems is unique to this study and presents the necessary framework for straightforward finite element solution. For clarity of presentation, the formulation is initially posed in terms of the liftoff problem, although it also applies to the general piecewise-linear problem.

The geometry of the liftoff problem, illustrated in Figure 5.1, is one in which a continuous beam is in partial contact with a "compression-only" foundation. Here $x$ and $y$ are the local coordinates of distance in the axial and vertical direction, respectively. The parameter $v$ represents the continuous deflection of both the contact and noncontact regions of the beam. Figure 5.1 illustrates the final solution of the nonlinear problem in which the liftoff location (where the foundation first contacts the beam) is also the point at which the beam has zero deflection.

The nonlinear liftoff problem must be solved as a series of linear steps, which converge to the solution. Therefore in each iteration the foundation must act linearly, in tension as well as in
Figure 5.1 Beam Lift-off from a Compression-Only Foundation
compression. Consequently, the contact interface and the point of zero deflection will not coincide until the final solution is achieved.

Consider the geometry of a linear solution step illustrated in Figure 5.2. Here, the parameter $\xi$ is defined as the location of the interface between the noncontact (unrestrained) beam and the contacting beam. The parameter $\xi$ is the location of the point of zero deflection. Figure 5.3 is an illustration of the fact that the point of zero deflection, $\xi$, is unique for each value of $\xi$. The solution objective for the liftoff problem is to find the interface location, $\xi$, for which $v=0$.

The beam deflection, $v$, is a unique function of the independent variables $x$ and $\xi$. The liftoff problem may be stated as

![Figure 5.2 Geometry of Linear Solution Step](image-url)
Figure 5.3 Illustration of Unique Dependence of $\xi$ on $\ell$
follows: given the function \( v(x, \mathcal{E}) \), find \( \mathcal{E} \), for which \( v(\mathcal{E}_*, \mathcal{E}_*) = 0 \). This description is the basis for implementing quasi-Newton methods for solving piecewise-linear problems. For other piecewise-linear problems, the solution \( \mathcal{E}_* \) may be required to satisfy \( v(\mathcal{E}_*, \mathcal{E}_*) = y_\mathcal{E} \), where \( y_\mathcal{E} \) is a value of deflection selected for the interface.

For a given value of \( \mathcal{E} \), finite element analysis yields the deflection \( v(x, \mathcal{E}) \) at discrete nodal locations, \( x_i \), rather than as a continuous function of \( x \). The finite element solution of beam deflections for the interface location \( \mathcal{E} \) is the vector \( v(x, \mathcal{E}) \) where \( x \) is the vector of nodal locations. The vector \( v \) is found from the matrix expression

\[
K(x, \mathcal{E}) \ v(x, \mathcal{E}) = Q(x, \mathcal{E}) \tag{5.1}
\]

where

- \( K(x, \mathcal{E}) \) is the finite element global stiffness matrix, and
- \( Q(x, \mathcal{E}) \) is the finite element global force vector.

For problems with multiple linear differential equations and therefore multiple interface locations, the parameter \( \mathcal{E} \) becomes a vector of locations, \( \mathcal{E} \). If needed, the continuous function \( v(x, \mathcal{E}) \) is readily found by interpolation formulas such as

\[
v(x, \mathcal{E}) = \psi(x) \ v(x, \mathcal{E}) \tag{5.2}
\]

where

- \( \psi(x) \) is a vector of interpolation shape functions.
For this formulation the locations of foundation contact interfaces, \( \mathcal{L} \), are defined to be coincident with selected finite element nodes. Furthermore, the interface nodes are permitted to vary their location on the x axis from iteration to iteration by corresponding extension and contraction of adjacent finite elements. Noninterface nodes remain fixed. The fundamental assumption of small displacements and rotations will require that the overall problem length and the distance between noninterface nodal points remain fixed.

The admission of variable nodal location and element length contrasts significantly with conventional structural finite element analysis. Variation is permitted here because the elements developed previously do not have maximum length limitations on accuracy and because of the simplicity of the one-dimensional nature of beam analysis. The approach of variable node location is adopted because it simplifies the analysis solution of the problem by the finite element method. Accurately solving this problem using other approaches requires development of special elements which permit an interface point to be located between finite element nodes. Such elements are difficult to develop and implement successfully, particularly for multiple interface problems. In the subsequent development, this variation on conventional finite elements proves to be a fundamental assumption which permits simple and straightforward solution of highly complex, liftoff and piecewise-linear problems.

The following section presents the development of a unique nonlinear solution method for finding the interface locations in piecewise-linear problems defined earlier.
Locally Convergent Solution of Piecewise-Linear-Foundation Problems

In contrast with the formulation in which one finds \( \xi_* \) for which \( v(\xi_*, \xi_*) = 0 \), consider that, at the solution, we also have \( \xi_* = \xi(\xi_*) \). This format is used for solution by direct iteration*, with the iterate \( \xi_{k+1} = \xi(\xi_k) \). Direct iteration is often selected intuitively and proves to converge reliably, although it has limitations.

One simple implementation of direct iteration solution of liftoff problems is the "foundation release" method. In this approach, the compression-only foundation is represented by nodal springs. Following a solution step, each element is checked to see if the foundation is in tension or compression. If the foundation is in tension, the foundation restraint is removed from the beam in the following step. The problem is solved iteratively until all foundation elements are in compression. This methodology for liftoff problems suffers from poor solution accuracy. The solution can only be as accurate as the length of the foundation elements.

The foundation release method can be improved easily. Consider the example case in Figure 5.4 of liftoff of a semi-infinite beam, with one interface and no axial tension. For a given value of \( \xi \), the displacement \( v(\xi) \) and first derivative (rotation), \( v'(\xi) \), may be found from linear finite element expressions. The location of \( \xi \) is a nonlinear function given by

* Refer to Appendix A for background discussion of nonlinear solution methods and convergence rates.
\[ \xi(\varepsilon) = \frac{1}{\lambda} \tan^{-1} \frac{-\lambda v(\varepsilon)}{(\lambda v(\varepsilon) + v'(\varepsilon))} \]  \hspace{1cm} (5.3)

where

\[ \lambda = \sqrt{\frac{k}{4EI}}. \]

For a finite element solution of the liftoff problem, this equation represents a nested nonlinearity that must be solved prior to the next iteration. For anything but this simple example, the nested nonlinearity complicates the solution strategy and will likely reduce computational speed.

Returning to the application of Newton's method where, for a given function \( v(x, \varepsilon) \), we wish to find \( \varepsilon_* \) such that \( v(\varepsilon_*, \varepsilon_*) = 0 \). As discussed previously, the interface location \( \varepsilon \) is a finite element
node. The Lagrangian description of motion is adopted such that we focus attention on the node where \( x = \ell \). The deflection at this location is \( v(x = \ell, \ell) \) which simplifies the problem to a one-dimensional application of Newton's method.

The Newton iteration for finding the root \( \ell \), of the nonlinear function \( v(x = \ell, \ell) = 0 \) is given by

\[
\ell_{k+1} = \ell_k - \frac{v(\ell, \ell)}{\frac{D}{D\ell} v(\ell, \ell)}.
\]  

(5.4)

Consistent with the Lagrangian description, the derivative in this expression is not the partial derivative of \( v \) with respect to \( \ell \) but is the total derivative given by

\[
\frac{D}{D\ell} v(x = \ell, \ell) = \frac{\partial v}{\partial x} \frac{dx}{d\ell} + \frac{\partial v}{\partial \ell}.
\]  

(5.5)

Because finite element analysis provides discrete values of \( v(\ell, \ell) \) and does not give the total derivative explicitly, the typical approach to solution would be to approximate the derivative by difference formulas.

Rather than pursue a finite difference approach, consider the individual terms of the total derivative expression. Because the point of interest at each iteration is \( x = \ell \), it follows that \( \frac{dx}{d\ell} = 1 \). Hence,

\[
\frac{D}{D\ell} v(x = \ell, \ell) = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial \ell}.
\]  

(5.6)

Observe that the partial derivative \( \frac{\partial v}{\partial x} \) is the finite element nodal value \( \frac{dv}{dx} \) at \( x = \ell \), given explicitly by the finite element analysis. As with the total derivative, the partial derivative \( \frac{\partial v}{\partial \ell} \) is not given explicitly.

Consider next the behavior of the function \( v \) and its derivatives at the zero deflection location \( \xi(\ell) \).
\[
\frac{D}{D\xi} v(x=\xi, \xi) = \frac{\partial v}{\partial x} \frac{dx}{d\xi} + \frac{\partial v}{\partial \xi} \\
= \frac{\partial v}{\partial x} \frac{dx}{d\xi} \frac{d\xi}{d\xi} + \frac{\partial v}{\partial \xi} 
\]  
(5.7)

But by definition, \( v(\xi(\xi), \xi) = 0 \) for all \( \xi \) and thus for all \( \xi \) as well; therefore at this location

\[
\frac{D}{D\xi} v(x=\xi, \xi) = 0 
\]  
(5.8)

and

\[
\frac{\partial}{\partial \xi} v(x=\xi, \xi) = \frac{\partial v}{\partial x} \frac{dx}{d\xi} = 0 . 
\]  
(5.9)

Equation 5.7 therefore yields

\[
\frac{\partial}{\partial \xi} v(x=\xi, \xi) = 0 . 
\]  
(5.10)

But at the solution \( \xi = \xi_* \),

\[
\frac{\partial}{\partial \xi} v(\xi, \xi) = 0 
\]  
(5.11)

which implies

\[
\frac{\partial}{\partial \xi} v(x=\xi, \xi) + 0 \text{ as } \xi + \xi_* . 
\]  
(5.12)

Hence, the partial derivative of \( v \) with respect to \( \xi \) converges to zero at the solution. This suggests that the total derivative in Equation 5.6 may be approximated near the solution by

\[
\frac{D}{D\xi} v(\xi, \xi) \approx \frac{\partial}{\partial x} v(\xi, \xi) . 
\]  
(5.13)

For this approximation the iteration expression, Equation 5.4, becomes
\[ \ell_{k+1} = \ell_k - \frac{v(\ell, \ell)}{\frac{\partial}{\partial x} v(\ell, \ell)} \]  

(5.14)

in which both the function evaluation \( v \) and its derivative are given explicitly by the finite element analysis at the node \( x=\ell \). As illustrated in Figure 5.5, this "partial derivative" Newton iteration is simple to implement in a finite element analysis in which the interface location is a finite element node as previously defined. The theoretical rate of convergence of this method remains to be proven before it can be proposed as a solution method for piecewise-linear-foundation problems.

---

Figure 5.5 Implementation of Partial Derivative Newton Method for Solution of Liftoff Problem
Convergence Rate

The rate of convergence of the proposed solution method can be derived from the results of the convergence rate discussion in Appendix A. Assume that the first partial derivatives of $v(x, \ell)$ are Lipschitz continuous with constant $\gamma$. Then Equation A.40 provides the following basic convergence rate expression for the proposed quasi-Newton solution method

$$
|\ell_{k+1} - \ell_*| \leq \frac{1}{\rho} \left[ \gamma |\ell_k - \ell_*| + 2 \left| \frac{\partial}{\partial \ell} v(\ell_k, \ell_k) - a_k \right| |\ell_k - \ell_*| \right] |\ell_k - \ell_*|; \frac{\partial v}{\partial \ell} \geq \rho \quad (5.15)
$$

where

- $\ell_k$ is the value of $\ell$ for the current iteration
- $\ell_{k+1}$ is the value of $\ell$ computed for the next iteration
- $\ell_*$ is the solution of the nonlinear expression
- $\rho$ is a lower bound on the partial derivative
- $a_k$ is the proposed approximation of $\frac{\partial v}{\partial \ell}$.

From Equation 5.6, the difference between the total derivative and the approximation is given by

$$
\frac{\partial}{\partial \ell} v(\ell_k, \ell_k) - a_k = \frac{\partial}{\partial \ell} v(\ell_k, \ell_k) - \frac{\partial}{\partial x} v(\ell_k, \ell_k) = \frac{\partial}{\partial \ell} v(\ell_k, \ell_k).
$$

Thus, the convergence rate then becomes

$$
|\ell_{k+1} - \ell_*| \leq \frac{1}{\rho} \left[ \gamma |\ell_k - \ell_*| + 2 \left| \frac{\partial}{\partial \ell} v(\ell_k, \ell_k) \right| |\ell_k - \ell_*| \right] |\ell_k - \ell_*| \quad (5.17)
$$
Recalling that $\frac{\partial}{\partial \xi} v(\xi, \xi) \to 0$ as $\xi \to \xi_*$, then it is clear that the term in square brackets converges to zero as $\xi \to \xi_*$. Therefore

$$|\xi_{k+1} - \xi_*| \leq c_k |\xi_k - \xi_*| ; \quad c_k \to 0 \quad (5.18)$$

proving that the proposed solution method achieves $q$-superlinear convergence properties for the piecewise-linear-foundation problem. Figure 5.6 illustrates an example of a beam-liftoff problem in which a continuous, infinitely long beam is lifted from its supporting foundation at a point. Table 5.1 presents the solution steps for analysis of this example by the partial derivative Newton method. The table shows that the parameter $c_k$ goes to zero for this example, experimentally confirming $q$-superlinear convergence.

![Figure 5.6 Beam Liftoff Example Problem](image-url)
Table 5.1 Convergence of Partial Derivative Newton Method

\[ c_k = \frac{|e_{k+1} - e_k|}{|e_k - e_0|} \]

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\*E = 30,000,000.0, I = 0.1, k = 10.0, q = 10.0, \( \delta = 100.0 \)

**Accuracy**

The nonlinear solution methods discussed here are all based upon convergence of a series of linear solution steps. In typical finite element solutions, the linear steps are finite element approximations to the intended theoretical step. As a result, the final solution cannot be more accurate than the finite element approximations.

As proven in Chapter III, the use of homogeneous-solution elements will result in exact solutions at the finite element nodes for linear problems. Consequently, when these elements are used for solving nonlinear problems, the results for each linear iteration are exact at the nodes. The accuracy of the nodal solutions for the piecewise-linear
problems under consideration therefore depends only upon the criteria used for terminating the nonlinear iterations.

**Local Solution of Multiple Interface Problems**

The local solution method developed for a single interface can be readily expanded to address multiple interface problems. In this case the variable for interface locations, $\mathbf{\ell}$, becomes the vector $\mathbf{\ell}$ whose "ith" component is the $x$ coordinate of the $i$th interface. The variable $x$ also becomes a vector and the displacement becomes a vector function of $x$ and $\mathbf{\ell}$. Now, given the vector function $\mathbf{v}(x, \mathbf{\ell})$, the solution objective is to find $\mathbf{\ell}_*$ such that $\mathbf{v}(\mathbf{\ell}_*, \mathbf{\ell}_*) = 0$. The Newton iteration for multiple interfaces is

$$\mathbf{\ell}_{k+1} = \mathbf{\ell}_k - \frac{\mathbf{D} \mathbf{v}(\mathbf{\ell}, \mathbf{\ell})}{\mathbf{D} \mathbf{v}(\mathbf{\ell}, \mathbf{\ell})} \cdot (5.19)$$

The total derivative is equivalent to the Jacobian in this case. The derivative may be written in terms of its partial derivatives as

$$\frac{\mathbf{D}}{\mathbf{D} \mathbf{\ell}} \mathbf{v}(x, \mathbf{\ell}) = \mathbf{J}(\mathbf{\ell}) = \frac{\partial}{\partial x} \mathbf{v}(x, \mathbf{\ell}) \frac{dx}{d\mathbf{\ell}} + \frac{\partial}{\partial \mathbf{\ell}} \mathbf{v}(x, \mathbf{\ell}) \cdot (5.20)$$

As in the single variable case the $\frac{\partial \mathbf{v}}{\partial \mathbf{\ell}}$ term goes to zero. Observe that because $x=\ell$,

$$\frac{dx}{d\mathbf{\ell}} = [1] \cdot (5.21)$$

The approximate total derivative is given by

$$\frac{\mathbf{D}}{\mathbf{D} \mathbf{\ell}} \mathbf{v}(x, \mathbf{\ell}) \approx \frac{\partial}{\partial x} \mathbf{v}(x, \mathbf{\ell}) \cdot (5.22)$$
For a given interface location, this derivative will be independent of the properties of other interface locations, so that the resulting matrix is diagonal. The resulting quasi-Newton iteration is

$$\mathbf{e}_{k+1} = \mathbf{e}_k - \frac{\mathbf{v}(\mathbf{e}, \mathbf{e})}{\frac{\partial}{\partial x} \mathbf{v}(\mathbf{e}, \mathbf{e})}.$$  \hspace{1cm} (5.23)

This proposed solution method is simple to implement in multiple dimensions.

**Global Solution Strategy for Multiple Interfaces**

In situations where the number of boundary points is known, careful reasoning can yield an initial problem configuration that will converge to the correct solution using the partial derivative Newton's method. This is not necessarily the case for problems in which the number of interface locations is unknown. For these problems a "more robust" global solution strategy must be used in combination with Newton's method. Because there is no ideal global solution strategy for all problems, an understanding of the problem geometry will be used here to suggest a solution strategy that complements the Newton strategy for the class of problems under consideration.

As formulated here, any change in the number of interface locations in a problem implies that there is a change in the number of equations in the finite element solution, Equation 5.1. To overcome the need for a complex global analysis that must redefine nodes and change the number of equations at each iteration, a method of constrained nodes is adopted as the basis of the global solution strategy. This strategy
is similar to that of finite element nonlinear gap and interface elements in which the problem is initially defined using more than an adequate number of nodes and degrees of freedom. The "extra" nodes are initially coincident and constrained to move with other nodes. If additional degrees of freedom are necessary these extra nodes are "released" and become additional element equations.

Implementation of this strategy for the liftoff problem is illustrated in Figure 5.7. In the initial configuration, the beam is entirely in contact with the foundation. Because the beam may liftoff from the foundation at the "peak", two coincident finite element nodes are located at that point. As long as the deflection of the point remains below the y=0 level, at which foundation contact is maintained,

![Diagram](attachment:image.png)

**Nodes n and n+1 coincident**

**a) Displacement Less Than Zero: Nodes Constrained**

**b) Displacement Greater Than Zero: Nodal Constraint Replaced by Beam Element Stiffness**

Figure 5.7 Illustration of Constrained Node Solution Strategy
the two nodes are constrained to move together. If the deflection exceeds \( y = 0 \) at some iteration, then the nodal constraint is released and the constraint equation between the nodes is replaced with that of a noncontact element. An initial estimate of the noncontact element length can be obtained a number of ways from the constrained node geometry prior to release. This solution strategy can also be applied to the piecewise-linear-foundation response problem and is implemented by adding multiple constrained nodes.

As can be seen in Figure 5.7, the constrained nodes must be located at the peak of the concave upward segment of the beam, in order to verify whether the nodes should remain constrained or be released. This is achieved by requiring as part of the solution method that the slope at the "peak" nodes be zero. For situations in which the peak node does not exceed the threshold for constraint release, the adjacent element lengths are modified to adjust the node location if its slope is not zero. The change in length is found by solving the Newton step for slope, \( \frac{dv}{dx} \), equal to zero. The derivative of the nodal slope is, of course, the curvature, \( \frac{d^2v}{dx^2} \), which can be obtained at each iteration from the recovered nodal moment, \( EI \frac{d^2v}{dx^2} \). The Newton step for relocating the node, is

\[
x_{k+1} = x_k - \frac{dv}{\frac{d^2v}{dx^2}}.
\]  \hspace{1cm} (5.24)

When used with the partial derivative Newton method, this global solution strategy can be loosely considered a part of a class of
solution methods known as model trust region methods(33). In model trust region methods a Newton or quasi-Newton method is used to develop a local model of the problem at each iteration, \( k \), and suggest a new step for the next iteration, \( k+1 \). If no change occurs in element contact status at the next iteration, then the Newton model can be "trusted" in that region and the step \( k+1 \) is accepted. If a status change occurs at \( k+1 \), then the Newton model is inappropriate, and a new model is formed using the information obtained at step \( k+1 \) by releasing or constraining a node. Newton's method can then proceed from the new model to search for a local solution. This class of methods is robust and yields rapid convergence because the Newton step is always taken first at each iteration to ensure that full advantage is taken of Newton's method near the solution.

The constrained node solution strategy as described above is simple and straightforward. Furthermore, it is compatible with the proposed modification of Newton's method in that decisions at each iteration can be made solely on the basis of nodal parameters. Although the overall problem can become highly complex, the solution iteration strategy is simple and efficient.

**Example Applications of Solution Method**

In 1980, Hobbs(34) published nonlinear solutions for lifting conventional cubic beams from an elastic foundation. He addressed the problems of lifting a continuous beam, illustrated in Figure 5.8a, and lifting one end of semi-infinite beam, illustrated in Figure 5.8b.
Table 5.2 presents a comparison of the solutions achieved using Hobbs' method and the partial derivative Newton method developed here. The solutions compare exactly up to the number of digits required by the solution tolerance.

Kerr and Bassler(35) presented an analysis of liftoff of train rails under the influence of a single wheel load in 1982. Their analysis involved development of two simultaneous nonlinear equations which they solved numerically using routines from the International Mathematics & Statistics Library (IMSL). Figure 5.9 provides a comparison of Kerr and Bassler's results with those from the partial derivative Newton method and a conventional beam-on-elastic-foundation analysis. It is clear in this figure that the liftoff analyses yield comparable results. The figure also shows experimentally measured rail deflections as presented by Kerr and Bassler from an ASCE-AREA test program(36) The experimental results compare well and lend further support to the analysis approach.

In both cases above, the authors have developed detailed and tedious nonlinear algebraic expressions for beam-liftoff problems which were solved numerically. The complexity of this approach generally limits an analysis to consideration of a single point load on a foundation. In some cases symmetry can be used to obtain analysis results for a maximum of two point loads. Figures 5.10 through 5.12 illustrate the additional versatility that can be obtained using the partial derivative Newton method for finite element solution of complex multiple liftoff problems.
Figure 5.8 Liftoff Geometries Studied by Hobbs

Table 5.2 Liftoff Length Solution Comparison for Pipeline Pickup

<table>
<thead>
<tr>
<th></th>
<th>P.D. Newton</th>
<th>Hobbs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous Beam</td>
<td>181.971</td>
<td>181.971</td>
</tr>
<tr>
<td>Severed Beam</td>
<td>130.238</td>
<td>130.238</td>
</tr>
</tbody>
</table>

\( \varepsilon = 30,000,000.0, I = 0.1, k = 10.0, q = 10.0, \delta = 100.0 \)
Figure 5.9 Solution Comparison of Partial Derivative Newton Method with Kerr and Bassler (35) Analysis Results
Figure 5.10 compares the results obtained from two analyses of a two point load rail-liftoff problem in which the train rail lifts off in three locations. The particular loads and spacing used for this analysis is representative of a hopper car resting on two axles, spaced 420 inches apart. The problem was solved first using a symmetrical model to obtain the results shown to the right of the centerline. The problem was then solved using a finite element model with two point loads to obtain the complete results and illustrate versatility of the analysis procedures.

In most cases rail cars rest on two "trucks", each of which have two axles. To further illustrate the versatility of the solution method, a finite element model using four point loads was developed and analyzed, yielding the comparison in Figure 5.11. The comparison shows that, although the total load on the rail may be the same, the maximum deflections are significantly reduced when it is distributed over four rather than two points. In this case the two point model would overestimate deflections for a four axle system.

The analysis of Hobbs and that of Kerr and Bassler are also limited in that they cannot consider the influence of axial loads upon beam-liftoff deflections, which can also be significant. Figure 5.12 presents a comparison of the four point load results in Figure 5.11 with those obtained when an axial tension and an axial compression are present in the rail. The height of vertical liftoff is modified significantly in this case by the presence of an axial load.
Figure 5.10 Symmetric and Full Model Results for Track Liftoff with Two Point Loads

- $I = 27 \text{ in}^4$
- $k = 1143 \text{ lb/in}^2$
- $q = 2.35 \text{ lb/in}$
Figure 5.11 Comparison of Track Liftoff Results for Two and Four Point Loads

Two point load results

Four point load results

Distance from centerline (inches)

Vertical displacement (inches)

$I = 27 \text{ in}^4$

$k = 1143 \text{ lb/in}^2$

$q = 2.35 \text{ lb/in}$
Figure 5.12 Comparison of Track Liftoff Results Under Compressive and Tensile Axial Loads
Summary

In this chapter a unique finite element formulation and solution method has been developed for beam-liftoff and piecewise-linear-foundation problems. The solution method is simple and straightforward to implement in finite element analyses. The method is an approximation of Newton's method and is shown to achieve q-superlinear convergence, both theoretically and for an example problem. A compatible global solution framework is suggested which is robust and takes full advantage of the new local solution method. The solution method is shown in example problems to achieve accurate solutions for single and multiple liftoff problems.
VI SOLUTION OF BEAM ON C\(^1\)-CONTINUOUS NONLINEAR FOUNDATION PROBLEMS

The second type of beam-on-nonlinear-foundation problems to be considered in this analysis are those whose foundation models have continuous first derivatives (i.e., C\(^1\) continuous). In contrast to the piecewise-linear foundations discussed in Chapter V, C\(^1\)-continuous foundation models are "smooth" and yield a single nonlinear differential equation over the problem domain. This chapter presents numerically integrated finite elements and iterative solution methods that are necessary for solving this class of beam-on-foundation problems.

The first section of the chapter presents iterative solution methods for nonlinear finite element problems, utilizing the mathematical framework from the advanced numerical methods approach to solution of systems of continuous nonlinear equations. Most engineering finite element texts present nonlinear solution techniques for discrete finite element equations on a graphical, intuitive basis. They generally compare different methods on the basis of the number of times the global stiffness matrix must be inverted. In contrast, advanced numerical methods texts\(^{(33,37)}\) provide detailed analyses of convergence of iterative solution methods for continuous nonlinear equations. The discussion here applies the mathematical science approach to discrete finite elements to establish a more consistent means of using and comparing different solution methods.
The second section of the chapter develops the numerical integration expressions for $C^1$-continuous foundations for computing their contribution to the finite element stiffnesses. The chapter concludes with example applications of the finite elements and solution methods.

Iterative Solution of Nonlinear Galerkin Residuals

For a given set of nonlinear equations, $F$, the objective of iterative solution methods is to find $v_*$ such that $F(v_*)=0$. In the case of a nonlinear Galerkin problem, the system of equations, $F(v)$, is equivalent to the discrete weighted residual, $R(v)$. Recall from Chapter II that the nonlinear residual expression may be written in terms of finite element matrices as

$$R(v) = -Q + K_s(v)v$$

(6.1)

where

- $R(v)$ is the nonlinear Galerkin weighted residual expression
- $K_s(v)$ is the secant stiffness matrix for the nonlinear functions
- $Q$ is the constant, applied loading term.

Here the secant stiffness matrix includes contributions from both the beam and the nonlinear foundation. For application of Newton and associated quasi-Newton solution methods to this problem, the function $F$ and its Jacobian, $J$, become
\[ F(v) = \mathcal{R}(v) = -Q + K_s(v)v \]  
\[ J(v) = \frac{d\mathcal{R}}{dv} = K_s(v) + \frac{d}{dv} K_s(v) v. \]  

Figure 6.1 illustrates the meaning of each term in the Jacobian expression for a one-dimensional system. Here the complete Jacobian is equivalent to the tangent stiffness matrix for the nonlinear function. From Equation 6.3 then, the tangent stiffness, \( K_t \), and secant stiffness, \( K_s \), are related by the expression

\[ K_t(v) = K_s(v) + \frac{d}{dv} K_s(v) v. \]  

Expressions for computing the tangent and secant stiffness matrices for two common \( C^1 \)-continuous foundation response models are developed later in this chapter.

\[ F \\
K_t = J(v) \\
\frac{dK_s}{dv} v \\
K_s \]

\[ v \]

Figure 6.1 One-Dimensional Illustration of Jacobian Terms
The following discussion presents the application of four common methods for iterative solution of Equation 6.2. Background descriptions of the methods and derivation of convergence rates are given in Appendix A.

**Newton Method**

At each iteration of the Newton method, a local model of the nonlinear function is constructed and solved for the root of the model. The Newton model at the "kth" iteration is given by

\[ M_k(v_k + s) = F(v_k) + J(v_k)s = \{ -Q + K_s(v_k)v_k \} + \{ K_t(v_k) \} s. \]  

(6.5)

As discussed in Appendix A, comparing this expression with the Taylor series for \( F \) at \( v_k \) shows that this is a quadratic model of the function. Solving the expression for \( s \) which yields \( M_k=0 \), gives the Newton iteration

\[ \{ K_t(v_k) \} s_k = \{ Q - K_s(v_k)v_k \} \]

\[ v_{k+1} = v_k + s_k. \]

(6.6)

The individual terms in this expression are illustrated in Figure 6.2a. Figure 6.2b illustrates the convergence of the terms to the solution for a specific nonlinear function.

As developed in the convergence analysis in Appendix A, Newton's method converges \( q \)-quadratically, the most rapid of the four approaches considered here. It has the drawback, however, that it may fail to converge when the tangent stiffness is small. For typical foundation models such as shown in Figure 6.1, the tangent stiffness
a) Terms in Newton Step

b) Convergence of Newton's Method

Figure 6.2 Illustration of Newton's Method
tends toward zero for large displacements. The nonlinear solution will then have large displacement increments at each iteration and may fail to converge in some circumstances.

Also observe in Equation 6.6 that the tangent stiffness matrix (Jacobian) must be recomputed and reinverted at each iteration for the classical Newton method. This can require significant computational time for large problems. Although simplifications of Equation 6.6 will result in slower convergence to the solution, overall improvements in computational efficiency can be achieved if the simplifications result in fewer updates and inversions of the Jacobian.

**Modified Newton Method**

In the single variable case, discussed in Appendix A, the modified Newton method utilizes the tangent stiffness matrix from the first iteration as an approximation of the derivative for all subsequent iterations. In applying this method to a multiple-variable problem the Jacobian is approximated by $A(v)$ given by

$$A(v_k) = K_t(v_k) = K_s(v_0) + \frac{d}{dv} K_s(v_0) v_0 = K_s(v_0)$$

$$= K_t(v_0).$$

Here the typical assumption is made that $v_0=0$ for the first iteration. The local model becomes

$$M_k(v_k + s) = F(v_k) + A(v_k)s$$

$$= \{-Q + K_s(v_k)v_k\} + \{K_t(v_0)\} s.$$  

Finally, the solution iteration for the modified Newton method is
\[ \{K_s(v_k)\} s_k = \{Q - K_s(v_k)v_k\} \]
\[ v_{k+1} = v_k + s_k. \]

As shown in Figure 6.3, this approach is also incremental in nature.

The modified Newton method has an advantage that the approximate tangent stiffness matrix must only be inverted once when using this solution method, at the price of slower, q-linear convergence. The element stiffness must be evaluated at each iteration, however, because it is a fundamental part of the function evaluation on the right side.

Figure 6.3 Convergence of the Modified Newton Method
Direct Iteration Method

For solving problems with "small" nonlinearities, engineers frequently use a method commonly known as the direct iteration or secant stiffness method. An alternative to Newton, this method is illustrated in one dimension in Figure 6.4. In this approach the system of equations $F$ is rewritten such that

$$F(v) = v - G(v) = 0$$
$$= v - K_s^{-1}(v) Q.$$  \hspace{1cm} (6.10)

The solution of this system is $v_*$ for which $v_* = G(v_*)$. The model at the $k$th iteration is given by

![Figure 6.4 Convergence of the Direct Iteration Method](image-url)
\[ M(v_{k+1}) = v_{k+1} - G(v_k) \]  
(6.11)

yielding an iterate of
\[ v_{k+1} = G(v_k) . \]  
(6.12)

From Equation 6.10 then, the finite element iterate becomes
\[ v_{k+1} = \{K_s(v_k)^{-1}\} Q \]  
(6.13)

or
\[ \{K_s(v_k)\} v_{k+1} = Q . \]  
(6.14)

For engineers, the direct iteration approach has the aesthetic appeal of solving directly for the total solution \( v_* \) at each iteration, rather than solving incrementally. The solution iterations are directly comparable in form to solving a linear system of equations. Although the method will reliably converge to the solution, it has the drawback that it is limited to \( q \)-linear convergence and that the stiffness matrix must be updated at each iteration.

**Direct Iteration with Overrelaxation**

In problems with significant nonlinearities, the direct iteration method is sometimes modified to permit acceleration of the iterations by a method often referred to as overrelaxation. In this modified approach the iterate of Equation 6.12 is revised as
\[ v_{k+1} = (1-\alpha)v_k + \alpha G(v_k) . \]  
(6.15)

Clearly for the solution, \( v_* \),
\[ v_1 = (1-\alpha)v_0 + \alpha G(v_0) = (1-\alpha)v_0 + \alpha v_0 = (1-\alpha)v_0 + \alpha v_0. \]

Equation 6.15 can be manipulated algebraically as follows

\[
\begin{align*}
\mathbf{v}_{k+1} &= (1-\alpha)G(v_{k-1}) + \alpha G(v_k) \\
&= \{ (1-\alpha)K_s(v_{k-1})^{-1} + \alpha K_s(v_k)^{-1} \} \mathbf{Q}. 
\end{align*}
\]  

(6.17)

The finite element expression for the problem is then

\[
\{ (1-\alpha)K_s(v_{k-1}) + \alpha K_s(v_k) \} \cdot \mathbf{v}_{k+1} = \mathbf{Q} 
\]

(6.18)

which may be rewritten as

\[
\{ K_s(v_k) + (1-\alpha)[K_s(v_{k-1}) - K_s(v_k)] \} \mathbf{v}_{k+1} = \mathbf{Q}. 
\]

(6.19)

The term \( \alpha \) represents a parameter which "accelerates" the change in stiffness by increasing the change more than would be indicated by the basic iteration. The optimum value for parameter \( \alpha \) is highly dependent upon the problem to be solved and therefore must be selected on the basis of experience.

The terms overrelaxation and underrelaxation come from problems in which the secant stiffness decreases with displacement \( \mathbf{v} \). In these cases \( \alpha > 0 \) will serve to "relax" the stiffness more than required by the basic iteration, hence the term overrelaxation. For \( \alpha < 0 \), the stiffness is increased rather than relaxed, hence the term underrelaxation.

The method of direct iteration with overrelaxation is likely the most widely used approach for solving beam-on-nonlinear-foundation problems because of its simplicity and solution stability.
Measure of Nonlinearity

Selection an optimum solution method for a given problem is a complex tradeoff among minimizing matrix inversions, maximizing convergence rate, and ensuring convergence to the solution. Typically texts on finite elements recommend solution methods on the basis of "small" nonlinearities or "large" nonlinearities but provide no measure for distinguishing small from large or for providing additional assistance. Beam-on-foundation problems are further complicated because the presence of the beam will significantly influence the degree of nonlinearity in a problem. Dennis and Schnabel\(^{33}\) suggest that the degree of nonlinearity of a problem can be quantified by the Lipshitz constant \(\gamma\). This suggestion is used below to establish a means for measuring nonlinearity.

For a function, \(f(x)\), whose derivative is Lipshitz continuous with constant \(\gamma\), we know that

\[
|f'(x) - f'(y)| \leq \gamma |x - y|
\]

and

\[
|f(y) - f(x) - f'(x)(y - x)| \leq \frac{\gamma}{2} |x - y|^2 .
\]

(6.20)

(6.21)

The derivatives in this case correspond to "tangent stiffnesses" (Jacobians in multiple dimensions). Equation 6.21 may be compared with the expression for Taylor series with error bound given by

\[
f(y) = f(x) + f'(x)(y-x) + E
\]

(6.22)
where
\[ E = \frac{f''(x)}{2} (y-x)^2 \quad \text{for} \quad x < x < y. \] (6.23)

The Lipschitz parameter compares directly with an error bound on the ability of linear Taylor series terms to represent a function. The parameter, \( \gamma \), is a bound on the absolute nonlinearity of a function when \( \gamma \) is computed from
\[ \gamma = \frac{|f'(x) - f'(y)|}{|x - y|}. \] (6.24)

When the derivatives at \( x \) and \( y \) are equal for monotonic functions, then the function is linear and \( \gamma = 0 \).

The parameter \( \gamma \), as defined in Equation 6.24, is dependent upon the scale of \( f' \) and of \( x \). Changes in the units of either will result in a change in \( \gamma \). A scale free measure of nonlinearity is the relative rate of change in \( f'(x) \), denoted by \( \mu \), and computed by
\[ \mu = \frac{|f'(x) - f'(y)|}{|f'(x)|}. \] (6.25)

If \( x_{\max} \) represents an upper bound on \( x \) for a given problem, then for monotonic function \( f(x) \), \( f'(x_{\max}) \) is a lower bound on the derivative \( f'(x) \). The nonlinearity parameter in this case, \( \mu(x_{\max}) \) represents a dimensionless upper bound on the nonlinearity of \( f(x) \) for \( x \) less than \( x_{\max} \).

The nonlinearity measure can be applied to solution of nonlinear foundation problems by observing that \( f'(x) \) corresponds to the tangent stiffness of \( f(x) \). For a maximum deflection, \( v_{\max} \), the upper bound on nonlinearity for a nonlinear foundation model with tangent stiffness \( k_t \) is
\[ \mu = \frac{|k_t(v_{\text{max}}) - k_t(v=0)|}{|k_t(v_{\text{max}})|}, \quad 0 \leq |v| \leq |v_{\text{max}}| \]  

(6.26)

Through experience and experimentation on specific classes of problems, this measure can be used as a basis for selecting appropriate solution methods. In some circumstances it may desirable to change solution methods during an analysis or to use different relaxation parameters for different elements depending upon "local nonlinearity". For this case, let \( x \) and \( y \) be the values of deflection \( v \) at the \( k \) and \( k-1 \) solution iterations, respectively. The relative nonlinearity thus becomes

\[ \mu = \frac{|k_t(v_k) - k_t(v_{k-1})|}{|k_t(v_k)|}. \]  

(6.27)

This definition establishes a relative measure of nonlinearity for a given solution iteration which can be used as the basis for deciding whether to accelerate or decelerate subsequent iterations.

**Integration of Nonlinear Foundation Stiffnesses**

For a nonlinear foundation response the foundation stiffness is a function of the local deflection. The stiffness for a given finite element, however, is dependent upon the deflection throughout the element length. Following is the development of the integration of finite element foundation stiffnesses for the hyperbolic and power law foundation models described in Chapter II.

Consider the Galerkin weighted residual for the tangent foundation stiffness, \( k_t \), of a one-dimensional foundation response model,
\[ R_f = \int_{a}^{L} \psi^o k_t(v) \psi^o \, dx \psi^o . \] (6.28)

The corresponding tangent stiffness matrix is given by

\[ k^o_t = \int_{a}^{L} \psi^o k_t(v) \psi^o \, dx . \] (6.29)

The stiffness \( k_t \) in these expressions is a function of displacement solution \( v_* \), which is unavailable. The displacement solution will therefore be approximated by

\[ v(x) = \psi^o(x) \psi^o \] (6.30)

so that Equation 6.28 becomes

\[ k^o_t = \int_{a}^{L} \psi^o k_t(\psi^o v^o) \psi^o \, dx . \] (6.31)

In general, this integration must be carried out numerically by methods such as Gaussian quadrature. For such methods, the integral expression in Equation 6.31 becomes a summation of terms consisting of function evaluations and weighting factors. Gaussian quadrature of the foundation integral with "n" points and weighting values \( W_i \) yields a stiffness matrix

\[ k^o_t = \sum_{i=1}^{n} W_i \left[ \psi^oT(x_i) k_t(\psi^o(x_i)v^o) \psi^o(x_i) \right] \, dx . \] (6.32)

Gaussian points and weight factors may be found in most basic numerical methods texts\(^{38,39,40}\). The secant stiffness matrices may be integrated in the same manner as the tangent stiffness matrix.
The tangent and secant stiffnesses for hyperbolic and power law foundation response models are given in Tables 6.1 and 6.2 in terms of Gaussian quadrature summations. The complete element stiffness for a beam on foundation will also include the beam and axial load contributions along with the foundation. The shape functions should be consistent. The homogenous-solution shape functions developed in Chapter IV are appropriate for use in solving of nonlinear foundation problems.

**Example Application of Solution Methods**

Figure 6.5a illustrates the problem of a point load on a nonlinear foundation used as an example for comparison of solution methods. The foundation restraint is hyperbolic as shown in Figure 6.5b. This problem was solved for seven different loads by Newton's method, modified Newton's method, and direct iteration, using the finite element stiffnesses from Table 6.1. The function corresponding to a beam-on-elastic-foundation model was used for the element shape functions. All solution methods yielded the same deflection results for the seven load levels, shown in Figure 6.6a. Figure 6.6b compares the maximum deflections with the applied load, clearly showing the nonlinearity of the problem.

Table 6.3 compares the number of iterations required for solution of the example problem by each of the methods. As shown, the nonlinearity measure increases as the load is increased. All three methods solve the problems with few iterations for small values of the
Table 6.1 Numerical Integration of Hyperbolic Foundation Stiffness

a) Foundation response model

\[ p(v) = -\frac{v}{(C_1+C_2 v)} \]

b) Secant stiffness and derivative

\[ k_s(v) = -\frac{p(v)}{v} = \frac{1}{(C_1+C_2 |v|)} \quad \frac{dk_s}{dv} = -\frac{C_2}{(C_1+C_2 |v|)^2} \]

c) Tangent stiffness

\[ k_t(v) = \frac{1}{(C_1+C_2 |v|)} - \frac{C_2 |v|}{(C_1+C_2 |v|)^2} = \frac{C_1}{(C_1+C_2 |v|)^2} \]

d) Gaussian quadrature formula for residual integration

\[ R_f = \sum_{i=1}^{n} W_i \left[ y^o(x_i) \ k \ (y^o(x_i)v^o) \ y^o(x_i) \right] dx \ v^o \]

e) Secant stiffness matrix, \( k_s^o \)

\[ \int_\varnothing y^o^T k_s(y^o v^o) y^o \ dx = \sum_{i=1}^{n} W_i \left[ y_j^o(x_i) \frac{1}{C_1+C_2 |y^o(x_i)v^o|} y_k^o(x_i) \right]_{j,k = 1,\ldots,4} \]

f) Tangent stiffness matrix, \( k_t^o \)

\[ \int_\varnothing y^o^T k_t(y^o v^o) y^o \ dx = \sum_{i=1}^{n} W_i \left[ y_j^o(x_i) \frac{C_1}{[C_1+C_2 |y^o(x_i)v^o|]^2} y_k^o(x_i) \right]_{j,k = 1,\ldots,4} \]
Table 6.2  Numerical Integration of Power Law Foundation Stiffness

a) Foundation response model

\[ p(v) = -Cv^n \]

b) Secant stiffness and derivative

\[ k_s(v) = -\left| \frac{p(v)}{v} \right| = C|v|^{n-1} \quad \frac{dk_s}{dv} = -(n-1)C|v|^{n-2} \]

c) Tangent stiffness

\[ k_t(v) = C|v|^{n-1} + (n-1)C|v|^{n-2}|v| = nC|v|^{n-1} \]

d) Gaussian quadrature formula for residual integration

\[ R_f = \sum_{i=1}^{n} W_i \left[ \psi^o(x_i) k (\psi^o(x_i)v^o) \phi^o(x_i) \right] dx^o \]

e) Secant stiffness matrix, \( k_s^o \)

\[ \int_{\partial} \psi^o^T k_s(\psi^o v^o) \psi^o \, dx = \sum_{i=1}^{n} W_i \left[ \psi^o_j(x_i) C|\psi^o(x_i)v^o|^{n-1} \phi^o_k(x_i) \right] \quad j, k = 1, \ldots, 4 \]

f) Tangent stiffness matrix, \( k_t^o \)

\[ \int_{\partial} \psi^o^T k_t(\psi^o v^o) \psi^o \, dx = \sum_{i=1}^{n} W_i \left[ \psi^o_j(x_i) nC|\psi^o(x_i)v^o|^{n-1} \phi^o_k(x_i) \right] \quad j, k = 1, \ldots, 4 \]
a) Problem Geometry

\[ p = -\frac{v}{C_1 + C_2 v} \]

- \( C_1 = 0.1 \)
- \( C_2 = 0.5 \)

b) Hyperbolic Foundation Response

Figure 6.5 Point Load on Nonlinear Foundation Example
a) Beam Deflection Results

b) Nonlinear Relationship of Maximum Deflections and Applied Loads

Figure 6.6 Solution of Nonlinear Foundation Example
Table 6.3  Comparison of Solution Iterations for Three Solution Methods

| Vertical Load | Maximum Deflection | $\mu = \frac{|k_t(v_{max}) - k_t(v=0)|}{|k_t(v_{max})|}$ | Solution Iterations |
|---------------|-------------------|-------------------------------------------------|---------------------|
|               |                   |                                                 | Modified Newton  |
| 20            | 0.11727           | 0.82893                                         | 5  12  9           |
| 40            | 0.31168           | 0.68815                                         | 6  20  12          |
| 60            | 0.62692           | 3.55670                                         | 6  32  15          |
| 80            | 1.09886           | 19.19873                                        | 7  46  17          |
| 100           | 1.72188           | 56.90296                                        | 7  57  17          |
| 120           | 2.45550           | 126.1820                                        | 7  64  17          |
| 140           | 3.25907           | 232.9477                                        | 7  69  16          |

*E = 30,000,000.0, I = 0.1, C_1 = 0.1, C_2 = 0.5*
nonlinearity parameter, but the modified Newton method shows a significant increase for larger nonlinearities. In contrast, Newton's method and direct iteration reach a plateau in the number of iterations, although the nonlinearity continues to increase. The direct iteration achieves relatively rapid convergence, although for larger linearities it requires more than double the number of iterations required by Newton's method.

**Summary**

This chapter presents nonlinear finite elements and iterative solution techniques necessary for solving problems of beams on $C^1$-continuous nonlinear foundation restraints. Four different iterative solution methods are presented in a consistent mathematical framework for use in comparison and selection, based upon problem constraints. The nonlinear finite elements that are developed utilize numerical integration to compute the contribution of nonlinear foundation restraints to element stiffnesses.
VII SUMMARY AND CONCLUSIONS

The objective of the research presented here is to develop a comprehensive series of finite elements and nonlinear iteration techniques for solving complex linear and nonlinear beam-on-foundation problems. The work has focused upon application of the Galerkin weighted residual method to beam-on-foundation problems that can be addressed by the fourth-order differential equation

\[
\frac{d^2}{dx^2}\left(\frac{d^2v}{dx^2}\right) - N \frac{d^2v}{dx^2} - \frac{dN}{dx}\frac{dv}{dx} + k(v)v = q \tag{7.1}
\]

The engineering problems that have been addressed with the elements and analysis techniques developed here include the following:

- Beams on linear foundations under combined axial and transverse loadings
- Beams that lift off from the foundation
- Beams on \(C^0\) piecewise-linear foundations
- Beams on \(C^1\)-continuous nonlinear foundations.

The finite element solutions developed here are applicable to a number of other fourth-order engineering problems as well.

A collection of linear finite elements has been developed for analyzing beam-on-foundation problems with varied combinations of beam, foundation, and loading parameters. The Galerkin weighted residual
method was applied as the basis for development of the finite element matrices for the following:

- Beams on elastic foundation with combined transverse and axial loads
- Semi-infinite beams on elastic foundation with combined transverse and axial loads
- Beam-columns and tensioned beams
- Beams on two-parameter foundations.

These elements include all terms of the differential equation of Equation 7.1 above.

The approach taken in this work is unique in that the exact complex-exponential solution of the homogeneous portion of the differential equation of Equation 7.1 is used as the shape function for the elements. These shape functions have a number of major advantages. First, as shown in a mathematical proof, use of the homogenous solution as the shape function for the elements results in exact solutions at the finite element nodes for linear, self-adjoint problems. Second, because complex-exponential functions will solve all cases of the differential equation, the elements developed using these functions will exactly solve all cases for symmetric, constant coefficient problems. Lastly, two unique special case elements, applicable to the majority of linear problems, are developed in closed form, yielding rapid solutions.

The same complex shape functions are used as the basis for developing elements for solving nonlinear foundation problems.
Nonlinear problems are divided into two basic groups for solution: one in which the foundation reaction is a $C^1$-continuous nonlinear function of deflection over the entire range of interest, and the other in which the foundation reaction is $C^p$ continuous (i.e., piecewise linear).

Lateral motion of a fully buried beam is an example of the first case where restraint force will vary with deflection. Numerically integrated nonlinear finite elements and quasi-Newton methods are established for iterative solution of this class of nonlinear problems.

A beam that lifts off its foundation surface is an example of the second class of piecewise-linear-foundation problems that are addressed. This liftoff problem is unique in that it degenerates into a conventional freely deforming beam in contact with a beam-foundation system. A straightforward and simple method is developed for solving of this class of problems. The method allows solution of a broad class of multiple liftoff and piecewise-linear-foundation problems which was not previously possible. Furthermore, the method is shown theoretically and experimentally to converge $q$-superlinearly to the solution.

The finite elements and solution techniques developed here provide a comprehensive framework for accurately solving linear and nonlinear buried pipeline and railroad track analysis problems. The elements developed can be implemented easily and used to solve a broad class of problems accurately and efficiently. The ability to achieve exact solutions in the linear case provides an improved level of confidence in the solution of complex beam and foundation interaction problems. The elements and iterative techniques developed and
implemented form a valuable resource for solving a broad range of beam-on-nonlinear-foundation problems.
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REFERENCES (Continued)


REFERENCES (Continued)


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APPENDIX A  NUMERICAL SOLUTION OF NONLINEAR EQUATIONS

While linear algebraic systems can be solved directly through matrix factorization methods such as Gaussian elimination or Cholesky decomposition, nonlinear systems of equations must be solved indirectly. Nonlinear systems are typically solved by repeated solution of local, linear models of the system. Beginning from an initial estimate, a linear model of the behavior of the nonlinear system is formed and solved to establish an improved estimate. At this point, a new model is developed and solved for further improvement. If the initial guess was close enough to the solution and if the model was properly formed, the repeated estimates will converge toward the correct solution of the nonlinear system.

Following is a brief review of iterative numerical methods for solutions of nonlinear equations. Different methods of solution are developed mathematically within the consistent framework suggested by advanced numerical methods. The convergence rates of these solutions are also derived to establish a consistent basis for comparison of the methods. The methods and convergence rates are presented in one variable. A discussion of solution of multi-variable problems is given in Chapter VI. More details and mathematical analysis of solution methods for nonlinear equations may be found in texts by Dennis and Schnabel(33) and Ortega and Reinbolt(37).
Solution of Problems in One Variable

Taylor series are the conventional starting point for development of a linear model for solving a nonlinear equation. For one variable the behavior of a function at a point \( x_k \) can be modeled by its Taylor series approximation,

\[
f(x) = f(x_k) + f'(x_k)(x-x_k) + f''(x_k)\frac{(x-x_k)^2}{2!} + \ldots \tag{A.1}
\]

\[
= \sum_{i=0}^{\infty} \frac{f^{(i)}(x_k)(x-x_k)^i}{i!} \tag{A.2}
\]

The first two terms of the series yield a linear model, \( M_k(x) \),

\[ M_k(x) = f(x_k) + f'(x_k)(x-x_k) . \tag{A.3} \]

This expression is the line with function value \( f(x_k) \) and slope \( f'(x_k) \). Solving the linear model for \( M_k(x_{k+1}) = 0 \), yields the Newton iterate

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} . \tag{A.4} \]

The use of the Taylor series to derive the Newton iteration unfortunately implies continuity of derivatives of higher order than those used in the iteration. This problem becomes more acute when the method is expanded to multi-variable nonlinear problems. An alternative derivation that does not include assumptions on derivatives comes from Newton's theorem, which states

\[ f(x) = f(x_k) + \int_{x_k}^{x} f'(z) \, dz . \tag{A.5} \]

If the integral is approximated by
\[
\int_{x_k}^{x} f'(z) \, dz = f'(x_k)(x-x_k) \quad \text{(A.6)}
\]

then solving \(f(x_{k+1})=0\) again gives the Newton iterate as

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \text{(A.7)}
\]

Newton's method for solving nonlinear equations is illustrated in Figure A.1. As shown in the figure, from an initial estimate of the solution, \(x_k\), an improved estimate is obtained by drawing the line that is tangent to \(f(x)\) at \(x_k\), and finding the point \(x_{k+1}\) where this line crosses the \(x\) axis. This improved estimate can serve as a new value for \(x_k\) and another tangent can be drawn to find a new value for \(x_{k+1}\). The

![Figure A.1 Convergence of Newton's Method](image)
solution, \( x_\ast \), is achieved when the difference between two subsequent estimates is smaller than some predefined criterion.

In many cases, the derivative of the function \( f \) is not available or is prohibitively difficult to compute, in which case the derivative, \( f'(x) \) must be approximated. For an approximate derivative, \( a(x_k) \), the quasi-Newton iterate is written as

\[
x_{k+1} = x_k - \frac{f(x_k)}{a(x_k)}.
\]

(A.8)

A number of approximation techniques may be used for \( a(x_k) \), depending upon the cost of function evaluations and the desired rate of convergence. Solution methods using approximations of the derivative are known as quasi-Newton methods. The more commonly used derivative approximations are the modified Newton method,

\[
a(x_k) = f'(x_k)
\]

(A.9)

the finite difference Newton method,

\[
a_k = \frac{f(x_k + h) - f(x_k)}{h}
\]

(A.10)

where

\( h \) is an arbitrarily selected difference step

and the secant method

\[
a_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}
\]

(A.11)

where

\( x_{k-1} \) and \( f(x_{k-1}) \) are the values of \( x \) and \( f(x) \) from the previous iteration.
Each of these methods is illustrated conceptually in Figure A.2. Observe that all of these methods are incremental.

The method of direct iteration is an alternative to Newton-based methods. For this approach, the nonlinear function is rewritten in the form

\[ f(x) = g(x) - x. \]  \hspace{1cm} (A.12)

At the solution \( f(x_*) = 0 \), then

\[ x_* = g(x_*). \]  \hspace{1cm} (A.13)

The iterations are given by the recursion relationship

\[ x_{k+1} = g(x_k). \]  \hspace{1cm} (A.14)

As discussed later, the function \( g \) must have certain properties in order to converge to the solution. This method is also known as the secant stiffness method and the method of successive substitution. Unlike Newton's method, the objective of direct iteration is to solve directly for \( x_* \) at each iteration rather than approach it incrementally.

**Solution Convergence Rates**

Both Newton's method and the method of direct iteration are part of a class of nonlinear solution approaches known as methods of successive approximation. Texts on theory and solution of nonlinear problems(23,33,37) provide proofs of existence of unique solutions and convergence to the solution from "close" initial estimates for this class. Such proofs are based upon the Contraction Mapping Principle.
a) Modified Newton Method

b) Finite Difference Newton Method

Figure A.2 Convergence of Quasi-Newton Methods
c) Secant Method

Figure A.2 (Continued)
The most rational basis of comparison of different solution methods is their rates of convergence. The following definitions from Dennis & Schnabel (33) characterize the convergence properties that are to be considered.

Let $x_*$ and $x_k$ (for $k=0, 1, 2, \ldots$) be members of the set of real numbers. Then the sequence $\{x_k\} = \{x_0, x_1, x_2, \ldots\}$ is said to converge to $x_*$ if

$$\lim_{k \to \infty} |x_k - x_*| = 0 \quad \text{(A.15)}$$

If there exists a positive constant $c < 1$ such that

$$|x_{k+1} - x_*| \leq c|x_k - x_*| \quad \text{(A.16)}$$

then the sequence $\{x_k\}$ is said to converge q-linearly to $x_*$. If a sequence $c_k$ converges to zero and

$$|x_{k+1} - x_*| \leq c_k|x_k - x_*| \quad \text{(A.17)}$$

then the sequence $\{x_k\}$ is said to converge q-superlinearly to $x_*$. If

$$|x_{k+1} - x_*| \leq c|x_k - x_*|^2 \quad \text{(A.18)}$$

the sequence $\{x_k\}$ is said to converge q-quadratically to $x_*$. Lastly, if

$$|x_{k+2} - x_*| \leq c|x_k - x_*|^2 \quad \text{(A.19)}$$

then the sequence $\{x_k\}$ is said to converge 2-step, q-quadratically to $x_*$.  

A function $f$ is Lipshitz continuous with constant $\gamma$ in a set if, for every $x$ and $y$ in the set,

$$|f(y) - f(x)| \leq \gamma |y - x| \quad \text{(A.20)}$$
Furthermore, Lemma 2.4.2 of Dennis and Schnabel(33) gives that, if \( f' \) is Lipshitz continuous with constant \( \gamma \), then

\[
|f(y) - f(x) - f'(x)(y-x)| \leq \frac{\gamma(y-x)^2}{2}
\]  

(A.21)

With these definitions, the following derivation of convergence rates for solutions in one variable may be given.

**Convergence of Newton's Methods**

Let the function \( f \) be a mapping from an open interval \( D \) to the real domain \( \mathbb{R} \) and let \( f' \) be Lipshitz continuous with constant \( \gamma \) in set \( D \). Assume for \( \rho > 0 \), that \( |f'(x)| \geq \rho \) for every \( x \) in set \( D \).

Taking the Newton iterate

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}
\]  

(A.22)

then

\[
(x_{k+1} - x_*) = (x_k - x_*) - \frac{f(x_k)}{f'(x_k)}.
\]  

(A.23)

Recalling that \( f(x_*) = 0 \), yields

\[
(x_{k+1} - x_*) = f'(x_k)^{-1}[f(x_*) - f(x_k) - f'(x_k)(x_* - x_k)].
\]  

(A.24)

Lipshitz continuity of \( f' \) gives

\[
|f'(y) - f'(x)| \leq \gamma |y - x|
\]  

(A.26)

and
\[ |f(y) - f(x) - f'(x)(y-x)| \leq \frac{\gamma(y-x)^2}{2}. \quad (A.27) \]

By comparison then
\[ |x_{k+1} - x^*| \leq \left[ \frac{\gamma}{2f'(x_k)} |x_k - x^*|^2 \right]. \quad (A.28) \]

Finally from the assumptions on \( f' \) we have
\[ |x_{k+1} - x^*| \leq \left[ \frac{\gamma}{2\rho} |x_k - x^*|^2 \right] \quad (A.29) \]

which clearly shows the \( q \)-quadratic convergence of Newton's method.

**Convergence of Quasi-Newton Methods**

Again let the function \( f \) be a mapping from an open interval \( D \) to the real domain \( \mathbb{R} \) and let \( f' \) be Lipshitz continuous with constant \( \gamma \) in set \( D \). Assume for \( \rho > 0 \), that \( |f'(x)| \geq \rho \) for every \( x \) in set \( D \). Now approximate the derivative \( f'(x) \) at each iteration by \( a_k \).

The quasi-Newton iterate is given by
\[ x_{k+1} = x_k - \frac{f(x_k)}{a(x_k)} \quad (A.30) \]

then
\[ (x_{k+1} - x^*) = (x_k - x^*) - \frac{f(x_k)}{a_k}. \quad (A.31) \]

Recalling that \( f(x^*) = 0 \), yields
\[ (x_{k+1} - x^*) = a_k^{-1} [f(x^*) - f(x_k) - a_k(x^* - x_k)] \quad (A.32) \]
\[ (x_{k+1} - x^*) = a_k^{-1} [f(x^*) - f(x_k) - f'(x_k)(x^* - x_k) + (f(x_k) - a_k)(x^* - x_k)]. \quad (A.33) \]
From Lipschitz continuity of $f'$

$$|x_{k+1} - x_*| \leq |a_k^{-1}| \left[ \frac{1}{2} |x_k - x_*|^2 + |f'(x_k) - a_k||x_k - x_*| \right]. \quad (A.34)$$

Now consider that the approximate derivative of each of the quasi-Newton methods can be written as a finite difference formula of the form

$$a_k = \frac{f(x_k + h_k) - f(x_k)}{h_k}. \quad (A.35)$$

By Lipschitz continuity

$$|f(x_k + h_k) - f(x_k) - h_k f'(x_k)| \leq \frac{\tau |h_k|^2}{2} \quad (A.36)$$

and therefore

$$\left| \frac{f(x_k + h_k) - f(x_k)}{h_k} - f'(x_k) \right| = |a_k - f'(x_k)| \leq \frac{\tau |h_k|}{2}. \quad (A.37)$$

Then for $h_k$ sufficiently small

$$2|a_k| \geq |f'(x_k)| \geq \rho \quad (A.38)$$

and

$$|a_k^{-1}| \leq 2\rho^{-1}. \quad (A.39)$$

Hence, we have the following expression for rate of convergence for an arbitrary quasi-Newton method

$$|x_{k+1} - x_*| \leq \frac{1}{\rho} \left[ \gamma |x_k - x_*| + 2|f'(x_k) - a_k||x_k - x_*| \right]. \quad (A.40)$$

The convergence rate depends upon the properties of the term $|f'(x_k) - a_k|$. 
Finite Difference Newton Method. For the finite difference Newton method

\[ a_k = \frac{f(x_k+h) - f(x_k)}{h} \]  \hspace{1cm} (A.41)

From Equation A.37 above

\[ \left| \frac{f(x_k+h) - f(x_k)}{h} - f'(x_k) \right| = \left| a_k - f'(x_k) \right| \leq \frac{\gamma |h|}{2} \]  \hspace{1cm} (A.42)

The convergence rate from Equation A.40 now becomes

\[ |x_{k+1} - x_*| \leq \frac{\gamma}{\rho} \left[ |x_k - x_*| + |h| + |x_k - x_*| \right] |x_k - x_*| \]  \hspace{1cm} (A.43)

Thus with \( h \) constant, the finite difference Newton method yields \( q \)-linear convergence.

Secant Method. For the secant method, the \( h \) in Equation A.43 is replaced by the term \( h_k = |x_k - x_{k-1}| \), yielding

\[ |x_{k+1} - x_*| \leq \frac{\gamma}{\rho} \left[ |x_k - x_*| + |x_k - x_{k-1}| \right] |x_k - x_*| \]  \hspace{1cm} (A.44)

and

\[ |x_{k+1} - x_*| \leq \frac{\gamma}{\rho} \left[ 2 |x_k - x_*| - |x_{k-1} - x_*| \right] |x_k - x_*| \]  \hspace{1cm} (A.45)

But \( |x_k - x_*| < |x_{k-1} - x_*| \) so that

\[ |x_{k+1} - x_*| \leq \frac{\gamma}{\rho} |x_{k-1} - x_*|^2 \]  \hspace{1cm} (A.46)

Finally

\[ |x_{k+2} - x_*| \leq \frac{\gamma}{\rho} |x_k - x_*|^2 \]  \hspace{1cm} (A.47)
which yields two-step q-quadratic convergence for the secant method.

**Modified Newton's Method.** For this method, $a_k = f'(x_k)$. From Equation A.40

$$|x_{k+1} - x_*| \leq \frac{1}{\rho} \left[ \gamma |x_k - x_*| + 2 |f'(x_k) - f'(x_\theta)| |x_k - x_*| \right]. \quad (A.48)$$

Recalling that $f(x_\theta) = 0$

$$|x_{k+1} - x_*| \leq \frac{1}{\rho} \left[ \gamma |x_k - x_*| + 2 \left( |f'(x_k) - f'(x_\theta)| - |f'(x_\theta) - f'(x_\star)| \right) \right] |x_k - x_*|. \quad (A.49)$$

Now by Lipshitz continuity

$$|x_{k+1} - x_*| \leq \frac{2}{\rho} \left[ |x_k - x_*| - |x_\theta - x_*| \right] |x_k - x_*|. \quad (A.50)$$

But $|x_\theta - x_*|$ is a constant so that the modified Newton's method yields q-linear convergence.

**Convergence of Direct Iteration Method**

The iterate for direct iteration is given by

$$x_{k+1} = g(x_k). \quad (A.51)$$

For this method $g(x)$ must be Lipshitz continuous with constant $\gamma < 1$ in set $D$. The iterate can be rewritten as

$$x_{k+1} - x_* = g(x_k) - x_*$$
$$= g(x_k) - g(x_\star). \quad (A.52)$$

By Lipshitz continuity,

$$|g(x_k) - g(x_\star)| \leq \gamma |x_k - x_*| \quad (A.53)$$
and

$$|x_{k+1} - x_*| \leq \gamma |x_k - x_*|$$  \hspace{1cm} (A.54)

which yields q-linear convergence for this method.

**Solution of Multi-variable Problems**

For multi-variable problems the Newton theorem can be readily expanded

$$F(x+s) = F(x) + \int_x^{x+s} F'(z) \, dz = F(x) + \int_x^{x+s} J(z) \, dz$$  \hspace{1cm} (A.55)

where

$$F'(x) = J(x) = \nabla F(x)^T$$

$$F'_{ij}(x) = \frac{\partial f_i(x)}{\partial x_j} .$$

Here $J(x)$ is known as the Jacobian matrix of $F$ at $x$, frequently shortened to "the Jacobian". The integral expression above suggests a linear model for $F(x_k+s)$ of the form

$$M_k(x_k+s) = F(x_k) + J(x_k)s .$$  \hspace{1cm} (A.56)

Solving for the step $s_k$ such that $M_k(x_k+s_k) = 0$ will yield the Newton step for multi-variable problems of

$$J(x_k)^{-1} s_k = -F(x_k)$$

$$x_{k+1} = x_k + s_k .$$  \hspace{1cm} (A.57)

As in the single variable case, the derivatives of the Jacobian may not be available or may be prohibitively difficult to
compute such that Jacobian approximations, $A(x)$, must be used for solution. Multi-variable matrix methods have been developed which correspond to the single variable quasi-Newton methods. These methods are described in Chapter VI and in most texts on finite elements(8,23). The convergence rates of these methods in multiple variable cases are also shown in texts on nonlinear solution theory(24,33,37) to be the same as those for the single variable cases.