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Studies of soil-structure and fluid-structure interaction

Prasad, Anumolu Meher, Ph.D.
Rice University, 1989
RICE UNIVERSITY

STUDIES OF SOIL-STRUCTURE
AND FLUID-STRUCTURE INTERACTION

by

ANUMOLU MEHER PRASAD

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

[Signature]
A. S. Veletsos, Brown & Root
Professor in the Department of
Civil Engineering, Chairman

[Signature]
P. C. Dakoulas, Assistant Pro-
fessor in the Department of
Civil Engineering

[Signature]
P. D. Spanos, L. B. Ryon Chair
in Engineering

Houston, Texas

May, 1989
ABSTRACT

STUDIES OF SOIL-STRUCTURE AND
FLUID-STRUCTURE INTERACTION

by

ANUMOLU MEHER PRASAD

This dissertation deals with two distinct topics: (1) The effects of soil-structure interaction, both kinematic and inertial, on the dynamic response of a variety of base-excited foundations and of simple structures supported on such foundations; and (2) the effects of fluid-structure interaction for relatively simple structural systems subjected to forces induced by waves and currents.

A fundamental step in the analysis of a base-excited structure-foundation system is the evaluation of the transfer functions of its foundation motion. Defined for harmonically excited massless foundations, these functions relate the amplitudes of the components of foundation motion to those of the free-field ground motion at some reference or control point. These functions are evaluated for surface-supported circular and rectangular rigid foundations and for embedded square foundations considering a spatially varying, horizontal free-field ground motion. Consideration is also given to more complex ground motions defined stochastically by a local power spectral density function and a spatial incoherence function. An approximate analyses based on the Iguchi-Scanlan averaging
technique is employed. The structures examined are considered to have one lateral and one torsional degree of freedom in their fixed-base condition.

The response quantities examined include the ensemble means of the peak values of the lateral and torsional components of the foundation input motion and of the associated structural deformations. These responses are evaluated over wide ranges of the parameters involved and are compared with those obtained for no soil-structure interaction and for kinematic interaction only. Simple, physically motivated interpretations are given for the observed differences.

The studies of fluid-structure interaction include comprehensive analyses of the differences in the responses of simple models of offshore structures computed by the standard and extended versions of Morison's equation for the hydrodynamic forces, and of the effects and relative importance of the numerous parameters involved. The responses are also evaluated by the equivalent linearization technique and Penzien's decoupling technique, and the interrelationship and accuracy of these approaches are elucidated. In addition, the decoupling technique is generalized to include consideration of a current of constant velocity, and a simple modification is proposed which improves the accuracy of this procedure. A simple approximation is included for the hydrodynamic modal damping values of multi-degree-of-freedom, stick-like systems.
ACKNOWLEDGMENTS

I sincerely express my gratitude and appreciation to my advisor, Professor Anestis S. Veletsos, not only for motivation, guidance and support provided during the development of this work, but also for his setting an example as a dedicated researcher, an excellent educator and a fine person. My sincere thanks are extended to Professor P. D. Spanos for his interest as a member of dissertation committee. I acknowledge the encouragement provided by Dr. Dakoulas, who also agreed to serve on thesis committee.

I appreciate and acknowledge the help provided by Drs. Yu Tang and G. Hahn by way of dialogue and comments. Appreciation is also extended to Dr. Senthil, Subir and Malhotra for their suggestions at various stages of this work. Many thanks are due to the faculty members and my fellow graduate students, whose interests and insights provided continuing stimulus to my intellectual curiosity. I always remember and cherish their making my life at Rice a learning experience.

Assistance rendered by Ms. Rhonda L. Moore in preparing the manuscript is gratefully acknowledged.

Finally, I wish to thank my parents for their moral support and encouragement throughout my academic pursuits.
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I INTRODUCTION

This dissertation deals with the following aspects of the dynamic response of structures: (1) The effects of soil-structure interaction for structures subjected to earthquake ground motions; and (2) The effects of fluid-structure interaction for structures subjected to wave-induced forces. The value, justification and scope of the studies reported herein are summarized briefly in the following sections. Additional information is given in the Introductions of the chapters in which the individual studies are reported.

1.1 STUDY OF SOIL-STRUCTURE INTERACTION

It is generally recognized that the motion experienced by the foundation of a structure may differ substantially from the free-field ground motion, which is the motion that the ground would experience at the foundation-soil interface in the absence of the structure. Two factors are responsible for this difference: (1) The inability of a rigid foundation to conform to the generally non-uniform, spatially varying, free-field ground motion; and (2) the interaction or coupling between the vibrating structure, its foundation, and supporting soils.

The soil-structure interaction effects are normally evaluated in two distinct steps: In the first, one considers the so-called kinematic interaction effects, which provide
for the non-uniformity of the free-field ground motion and the rigidity of a massless foundation. The non-uniformity of the free-field ground motion may be due to differences in the times of arrival of the individual wave trains and/or to lack of coherence in these waves. The second step of the analysis considers the so-called inertial interaction effects, which provide for the coupling between the vibrating structure, foundation and underlying soils, and for the capacity of the latter to dissipate energy by radiation of waves and by hysteretic action.

Comprehensive accounts of the status of current methods of analysis and of recent contributions are available in Johnson (1980), which also includes an extensive list of references. A relatively simple, design oriented procedure for incorporating the effects of soil-structure interaction in building structures is given in the ATC-NEHRP recommended provisions (ATC 1978; FEMA 1986) and in several other publications (Veletsos 1977, 1978; Veletsos and Meek 1974; Veletsos and Verbic 1973; Veletsos and Nair 1974). In the latter studies, the structure is assumed to be supported on a rigid mat foundation, and to be excited by a uniform horizontal free-field ground motion. This approach effectively disregards the kinematic interaction effects.
Field records of the seismic response of structures with foundations of extended plan dimensions show that the reduction in the response at high frequencies cannot be fully attributed to inertial interaction effects, and that the spatial variation of the free-field ground motion is also a significant contributor to this reduction. The effects of the spatial variation of the ground motion associated with the propagation of obliquely incident plane waves are known as the wave passage effects, whereas those due to the other, generally random, factors are known as the ground motion incoherence effects.

The deterministic wave passage effects on the dynamic response of massless foundations have been studied by Scanlan (1976), Luco and Sotiropoulos (1980) and Iguchi (1984) using an approximate averaging technique, and exactly by Luco and Wong (1980) for rectangular foundations and by Luco and Mita (1987) for circular foundations. The ground motion incoherence due to the random factors is characterized stochastically by a coherence function, which is essentially a normalized cross power spectral density function for the free-field ground motions at two arbitrary points of the foundation-soil interface. From analyses of field data, Harichandran and Vanmarcke (1986), Loh (1982, 1985) and others (Smith et al. 1982, Der 1988) proposed several different expressions for this function.
Using specific coherence functions, Matsushima (1977), Hoshiya and Ishii (1983) made exploratory studies of the effects of ground motion incoherence on the lateral response of structures. Luco and Wong (1986) and Luco and Mita (1987) studied the effects of ground motion incoherence on translational and torsional responses of square foundations and circular foundations, respectively. Mita and Luco (1986) studied the effect of such incoherence on the response of a structure of the shear-beam type. Pais and Kausel (1985; 1986) interpreted the ground motion incoherence to be due to the effect of uncorrelated, obliquely incident SH waves arriving simultaneously from several directions, and evaluated the response of simple structures over a range of the parameters involved.

With these studies in perspective, this dissertation makes a more critical assessment of the nature of the kinematic interaction effects for structures subjected to earthquakes. Answers to the following specific questions are sought: How different assumptions concerning the nature of the impinging waves at a site affect the resulting foundation motions, and how the critical responses of the structure are influenced by the difference between the actual foundation input motion and the stipulated motion at a reference or control point of the ground.
The objectives of this part of the study are: (1) To provide improved insight into the effects of kinematic interaction on the seismic response of relatively simple, mat-supported structures of extended plan dimensions by assessing the effects and relative importance of the numerous factors that influence the response of the systems involved; (2) to assess the adequacy of an approximate method of analyzing the problem; and (3) to assess the relative importance of kinematic interaction and inertial interaction, and to provide a basis for the formulation of a rational procedure for accounting the kinematic interaction effects.

1.2 STUDY OF FLUID-STRUCTURE INTERACTION

As used herein, the term fluid-structure interaction refers to the difference in the responses of a structure computed by the standard and the extended versions of Morison's equation for the hydrodynamic forces. In the standard version of Morison's equation, the drag forces are considered to be proportional to the square of the fluid particle velocities, whereas in the extended version, they are considered to be proportional to the square of the relative velocities. In a direct, numerical evaluation of the response of a structure as a function of time, there is no special difficulty in providing for these effects. This approach is generally too tedious and costly, however, for preliminary design purposes
simpler procedures have been sought. Two such procedures are available: (1) the Malhotra-Penzien extension (Malhotra and Penzien 1970; Foster 1970) of Borgman's linearization technique for the drag component of the exciting force (Borgman 1967); and (2) Penzien's decoupling technique (Penzien 1976; Penzien and Tseng 1978). Notwithstanding these and several other contributions (Dao an Penzien 1982; Gudmestad and Connor 1983; Krolikowski and Gay 1980; Nagashima 1981; Shyam Sunder and Connor 1982; Spanos and Chen 1981), there is a need to reexamine the problem from a unified point of view, and to assess critically the effects of the numerous factors involved. This study is intended to be responsive to this need. Although the reliability of both the standard and the extended versions of Morison's equation has been questioned (Laya et al. 1984; Sarpkaya and Isaacson 1981), the use of both variants of this equation is so widespread in practice that it is considered important that the interrelationship of the responses obtained from them be clarified.

The objectives of this part of the study are to: (1) Elucidate the nature of the fluid-structure interaction phenomenon; (2) assess the interrelationship, accuracy and ranges of applicability of previously proposed simple procedures for evaluating its effects; and (3) where necessary, to recommend appropriate improvements.
1.3 SCOPE OF WORK

The first part of the dissertation, reported in Chapters II through V, deals with the effects of soil-structure interaction. Subject matters considered include the effects of spatially varying free-field ground motions on the response of rigid massless foundations of different geometries, and the corresponding effects on the response of simple structures supported on such foundations.

Chapter II deals with the determination of the interrelationship of a spatially varying horizontal free-field ground motion and the foundation input motion for surface-supported, circular massless foundations. The spatial variability is specified stochastically by a coherence function, which is considered to be of the form proposed by Luco and Wong (1986). The spectral characteristics of the foundation input motion are evaluated approximately by the Iguchi-Scanlan averaging technique considering both wave passage and incoherence effects. The accuracy of this procedure is examined by comparing the results obtained for limiting cases with those computed by the more nearly exact formulation of Luco and Mita (1987a, 1987b). For the special case of vertically incident incoherent waves, simple closed-form expressions are presented for the transfer functions of the foundation; these functions relate the amplitudes of the components of foundation input motion to the amplitude of the free-field ground motion at some
reference or control point. The responses of simple structures to a spatially varying incoherent ground motions are also evaluated considering both kinematic and inertial interaction effects. The results are computed over wide ranges of the parameters involved and are compared with those obtained for no soil-structure interaction and for kinematic interaction only.

Chapter III deals with the computation of the transfer functions for surface supported, rectangular foundations subjected to non-uniform horizontal motions. Two different forms are considered for the ground motion incoherence function. The solution is based, as before, on the Iguchi-Scanlan averaging technique, and results in relatively simple, closed-form expressions. The effects of the primary parameters are displayed graphically.

Chapter IV deals with the evaluation of the transfer functions for rigid square foundations embedded in an elastic half-space. The spatially varying non-uniform ground motion in this case is due to vertically incident, horizontally polarized incoherent shear waves. The effects of embedment ratio and of the incoherence parameter on the amplitudes of the foundation motions are studied comprehensively.

Chapter V deals with the response to an ensemble of spatially varying horizontal ground motion of simple structures that are embedded in an elastic half-space. The response quantities
examined include the ensemble mean of the peak values of the foundation input motions and of the deformations of the structures. The structural responses are displayed in terms of pseudo-velocity response spectra over wide ranges of parameters involved. Both kinematic and inertial interaction effects are considered. The simple approximate procedure for the computation of inertial interaction effects employed in Chapter II for surface-supported foundations is extended to embedded foundations, and its accuracy is assessed.

The second part of the dissertation, reported in Chapter VI, deals with the effects of fluid-structure for structures subjected to waves. Comprehensive parametric studies are made of the exact maximum responses induced in simple mass-spring-dashpot systems by several different combinations of a simulated wave train and a constant-velocity current, and the results are compared with those obtained by the previously proposed approximate procedures or by appropriate extensions of them. The problem parameters examined include the natural frequency and the percentage of critical damping of the system, the relative magnitudes of the drag and inertia components of the exciting force, the ratio of current velocity to the peak value of the wave induced fluid particle velocity, and a dimensionless measure of the importance of fluid-structure interaction. The results are displayed in the form of response spectra for the absolute maximum displacement of the system.
The accuracy of the approximate techniques is evaluated over wide ranges of the parameters involved, and simple modifications are suggested to minimize the discrepancies and to improve the reliability of these techniques. In addition to the response of single-degree-of-freedom systems, the response of multi-degree-of-freedom, stick-like systems are examined, and a simple approximation is proposed for the effective modal damping of such systems.

In an effort to present the material of each chapter in a self-contained manner, some ideas are repeated in the sections that follow. It is hoped that this repetition may prove helpful to the reader.

1.4 NOTATION

The symbols used in this dissertation are defined when first introduced in the text, and those used extensively are summarized in Appendix E. The first section of this Appendix contains the symbols used in the studies of soil-structure interaction presented Chapters II through Chapter V, whereas the second section contains the symbols used in the study of fluid-structure interaction presented in Chapter VI.
II SEISMIC INTERACTION OF STRUCTURES ON CIRCULAR FOUNDATIONS

2.1 INTRODUCTION

In evaluating the response of structures to earthquakes, it is normally assumed that all points of the ground surface beneath the foundation are excited synchronously and experience the same free-field motion (ATC 1978; FEMA 1986; Veletsos 1977); the latter term refers to the motion which would be induced at the foundation-soil interface if no structure were present. The assumption of synchronous interface free-field ground motions is strictly valid only for vertically propagating coherent wave fields; in reality, the motions may vary from one point to the next (Abrahamson and Bolt 1985; Harichandran and Vanmarcke 1986; Loh 1985; Yamahara 1970). Even when the wave front is plane and propagates in a perfectly homogeneous medium, it may impinge the foundation at a finite angle, leading to motions at neighboring points which in the words of Kausel and Pais (1987) are "delayed replicas" of each other. Known as the wave passage effect, the consequences of such action have been the subject of numerous previous studies (Bogdanoff et al. 1965; Luco and Mita 1987a; Morgan et al. 1983; Newmark 1969; Roesset 1980; Scanlan 1976; Veletsos et al. 1976; Werner et al. 1979) and are reasonably well understood.
Several additional factors contribute to the spatial variability of the free-field ground motion. The individual wave trains may emanate from different points of an extended source and may impinge the foundation at different instants and with different angles of incidence, or they may propagate through paths of different physical properties and may be affected differently in both amplitude and phase by the characteristics of the travel paths and by reflections from, and diffractions around, the foundation. The spatial variability of the ground motion due to these factors will be referred to as the ground motion incoherence effect. This effect, which would exist even for horizontally polarized vertically propagating shear waves, has been the subject of only exploratory recent studies (Hoshiya and Ishii 1983; Luco and Mita 1987b; Luco and Wong 1986; Matsushima 1977; Mita and Luco 1987; Novak and Suen 1987; Pais and Kausel 1985).

The motion experienced by a rigid foundation is clearly different from the free-field ground motion. The actual motion may conveniently be evaluated in two steps. First, the so-called foundation input motion is computed; this is defined as the motion which would be experienced by the foundation if both it and the superimposed structure were massless.

Computed with due provision for the rigidity of the foundation, the foundation input motion includes both horizontal and torsional components even for a purely horizontal
free-field ground shaking. The difference in the responses of the structure computed for the foundation input motion and the free-field motion at some reference or control point of the ground surface is known as the kinematic interaction effect. The greater the degree of ground motion incoherence or the plan dimensions of the foundation in comparison to the length of the dominant seismic waves, the more important this effect is likely to be.

The actual motion of the foundation is also influenced by its own inertia and the inertia of the structure, and by the interaction or coupling between them and the supporting soils. For a structure subjected to a purely horizontal free-field ground shaking, not only are the horizontal and torsional components of the actual foundation motion different from those of the corresponding input motion, but the actual motion may also include rocking components about horizontal axes. Contributed by the overturning tendency of the superstructure, the latter components may be particularly prominent for tall slender structures and for soft soils. These factors are provided for in the second step of the evaluation process.

The term inertial interaction effect refers to the difference in structural responses computed for the actual motion of the foundation and the foundation input motion. The total soil structure interaction is clearly the sum of the kinematic and inertial interaction effects.
Although the inertial interaction effects have been the subject of numerous studies (FEMA 1986; Roesset 1980; Veletsos 1977; Veletsos 1978; Veletsos and Meek 1976), they have generally been examined at the exclusion of the kinematic interaction effects, and the interrelationship of the two effects has not been adequately assessed. The objectives of this paper are: to elucidate the nature of both types of interaction for seismically excited simple structures; to assess the effects and relative importance of the numerous parameters involved; and to present information and concepts with which the effects of the principal parameters may be evaluated readily. Primary emphasis is placed on the kinematic interaction effects.

The structures investigated are presumed to have one lateral and one torsional degree of freedom in their fixed-base condition and to be excited by obliquely incident, horizontally polarized, incoherent shear waves. The temporal variation of the free-field ground motion is expressed stochastically by a local power spectral density (psd) function, and its spatial variability is specified by a cross psd function. The response quantities examined include the ensemble means of the peak values of the lateral and torsional components of the foundation input motion and of the corresponding structural deformations. These deformations are displayed in the form of the pseudo-velocity response spectra and compared, over wide ranges of
the parameters involved, with those obtained for no soil-structure interaction and for kinematic interaction only. Simple, physically motivated interpretations are given for the observed differences.

A fundamental step in the analysis of a structure-foundation-soil system is the evaluation of the transfer functions of the foundation. Defined for harmonically excited massless foundations, these functions relate the amplitudes of the horizontal and torsional components of foundation input motion to the amplitude of the free-field ground motion. The relevant functions are evaluated herein by a relatively simple, approximate procedure, and their accuracy is assessed through comparisons with available exact solutions for special cases. In addition, simple closed-form expressions are presented for these functions for the important special case of vertically incident, incoherent waves.

2.2 SYSTEM CONSIDERED

The system investigated is shown in Figure (2.1). It is a linear structure of mass \( m \) and height \( h \), which is supported through a foundation of mass \( m_f \) at the surface of a homogeneous elastic half-space. The circular natural frequencies of lateral and torsional modes of vibration for the structure when fixed at its base are denoted by \( \rho_x = 2\pi f_x \) and \( \rho_\theta = 2\pi f_\theta \), respectively, in which \( f_x \) and \( f_\theta \) are the associated frequencies.
in cycles per unit of time; and the corresponding percentages of critical damping are denoted by $\zeta_x$ and $\zeta_\theta$, respectively. The foundation mat is idealized as a rigid circular plate of negligible thickness and radius $R$ which is bonded to the half-space so that no uplifting or sliding can occur, and the columns of the structure are presumed to be massless and axially inextensible. Both $m$ and $m_1$ are assumed to be uniformly distributed over identical circular areas. The supporting medium is characterized by its mass density, $\rho$, shear wave velocity, $u_s$, and Poisson's ratio, $\nu$. This structure may be viewed either as the direct model of a single-story building frame or, more generally, as the model of a multistory, multimode structure that responds as a system with one lateral and one torsional degrees of freedom in its fixed-base condition.

The free-field ground motion for all points of the foundation-soil interface is considered to be a unidirectional excitation directed parallel to the horizontal $x_1$-axis, as shown in Figure (2.1), with the detailed histories of the motions varying from point to point. Such motions may be induced by horizontally polarized, incoherent shear waves propagating either vertically or at an arbitrary angle with the vertical, $\alpha_\nu$. The intense portions of the motions are represented by a stationary random process of limited duration, $t_0$, and a space-invariant, local psd function, $S_\nu = S_\nu(\omega)$, in
which \( \omega \) = the circular frequency of the motions. The spatial variability of the motions is defined by a cross psd function, \( S(\vec{r}_1, \vec{r}_2, \omega) \), in which \( \vec{r}_1 \) and \( \vec{r}_2 \) are the position vectors for two arbitrary points.

A decreasing function of the frequency \( \omega \) and of the distance between the two points, \(|\vec{r}_1 - \vec{r}_2|\), the function \( S(\vec{r}_1, \vec{r}_2, \omega) \) is taken in the form suggested by Harichandran and Vanmarcke (1986) as

\[
S(\vec{r}_1, \vec{r}_2, \omega) = \Gamma(|\vec{r}_1 - \vec{r}_2|, \omega) \exp \left[ -i \omega \frac{d_1 - d_2}{c} \right] S_\sigma(\omega) \tag{2.1}
\]

in which \( \Gamma \), referred to as the incoherence function, is a dimensionless, decreasing function of \(|\vec{r}_1 - \vec{r}_2|\); \( i = \sqrt{-1} \); \( d_1 \) and \( d_2 \) are the components of \( \vec{r}_1 \) and \( \vec{r}_2 \) in the direction of propagation of the wave front (see Figure 2.1b); and \( c \) = the apparent horizontal velocity of the front. The latter quantity is related to the angle of incidence of the waves, \( \alpha_v \), by

\[
c = \frac{v_s}{\sin \alpha_v} \tag{2.2}
\]

The product of the exponential term in equation (2.1) and \( S_\sigma \) represents the wave passage effect, whereas the product \( \Gamma S_\sigma \) represents the effect of ground motion incoherence. The peak value of \( \Gamma \) is unity and occurs at \( \vec{r}_1 = \vec{r}_2 \).

Several different expressions have been suggested for the incoherence function (e.g., Refs. Harichandran and Vanmarcke 1986; Hoshiya and Ishii 1983; Loh 1985; Luco and Wong 1986;
Mita and Luco 1987), and there is no general agreement at this time on the form that may be the most appropriate for realistic earthquakes. In this study, the single-parameter, second order function recommended by Mita and Luco (1987) is used,

\[ \Gamma(|\vec{r}_1 - \vec{r}_2|, \omega) = \exp \left[ - \left( \frac{\gamma \omega |\vec{r}_1 - \vec{r}_2|}{v_s} \right)^2 \right] \]  

(2.3)

in which \( \gamma \) is a dimensionless factor, taken between zero and 0.5.

A different approach to the study of this problem has been taken by Pais and Kausel (1985). They have attributed the ground motion incoherence to arrays of uncorrelated, obliquely incident waves arriving from different directions within a sector of the supporting medium. The kinematic interaction effects in this approach are represented by weighted averages of the component wave passage effects.

2.3 KINEMATIC INTERACTION EFFECTS

2.3.1 Spectral Characterization of Foundation Input Motion

Let \( S_x \) be the psd function of the horizontal component of the foundation input displacement, and \( S_y \) be the corresponding function for the circumferential or tangential displacement component along the periphery of the foundation. Further, let \( S_{xy} \) for the cross spectral density function for the component displacements. Whereas \( S_x \) and \( S_y \) are real-valued, \( S_{xy} \) is generally complex-valued.
These functions were evaluated from the cross spectral density function, \( S(\vec{r}_1, \vec{r}_2, \omega) \), by application of the averaging technique employed by Iguchi (1984) and Scanlan (1976) in their studies of wave propagation effects. This approach leads to

\[
S_x = \frac{1}{A^2} \int_A \int_A S(\vec{r}_1, \vec{r}_2, \omega) dA_1 dA_2 \tag{2.4a}
\]

\[
S_y = \frac{R^2}{l_0} \int_A \int_A d_1 d_2 S(\vec{r}_1, \vec{r}_2, \omega) dA_1 dA_2 \tag{2.4b}
\]

\[
S_{xy} = \frac{R}{l_0 A} \int_A \int_A d_2 S(\vec{r}_1, \vec{r}_2, \omega) dA_1 dA_2 \tag{2.4c}
\]

in which \( dA_1 \) and \( dA_2 \) are elemental areas of the foundation; \( A = \pi R^2 \) = the area of the foundation; and \( l_0 = AR^2/2 \) = its polar moment of inertia about a vertical centroidal axis.

As is true of the corresponding exact expressions presented by Luco and Mita (1987), equations (2.4) represent weighted averages of \( S(\vec{r}_1, \vec{r}_2, \omega) \). However, whereas the weighting functions in the exact formulation are the complex distributions of the actual tractions at the foundation-soil interface, in the procedure employed herein they are taken as linear functions. This is tantamount to representing the restraining action of the supporting medium by a series of mutually independent springs of the Winkler type (Scanlan 1976). There are two main advantages to the use of the approximate formulation over the exact formulation: (a) it
reduces the number of independent parameters that must be considered, thereby simplifying the interpretation of the results; and (b) for important special cases, it leads to simple, closed-form expressions for the desired quantities. Additionally, the results are generally of good accuracy.

For the circular foundations examined herein, it is convenient to express \( \tilde{r}_1 \) and \( \tilde{r}_2 \) in equations (2.1) and (2.3) in terms of polar coordinates. On substituting equation (2.1) into equations (2.4), and making use of the appropriate coordinate transformation, one obtains

\[
\frac{S_x}{S_o} = \frac{1}{\pi^2} \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \xi_1 \xi_2 \exp(-b_0^2 \Delta_1) \cos(c_0 \Delta_2) d\theta_1 d\theta_2 d\xi_1 d\xi_2 \tag{2.5a}
\]

\[
\frac{S_y}{S_o} = \frac{4}{\pi^2} \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \xi_1 \xi_2^2 \exp(-b_0^2 \Delta_1) \cos(c_0 \Delta_2) 
\cos\theta_1 \cos\theta_2 d\theta_1 d\theta_2 d\xi_1 d\xi_2 \tag{2.5b}
\]

\[
\frac{S_{xy}}{S_o} = -\frac{2i}{\pi} \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \xi_1 \xi_2^2 \exp(-b_0^2 \Delta_1) \sin(c_0 \Delta_2) 
\cos\theta_2 d\theta_1 d\theta_2 d\xi_1 d\xi_2 \tag{2.5c}
\]

in which

\[
\Delta_1 = \xi_1^2 + \xi_2^2 - 2\xi_1 \xi_2 \cos(\theta_1 - \theta_2) \tag{2.6a}
\]

\[
\Delta_2 = \xi_1 \cos\theta_1 - \xi_2 \cos\theta_2 \tag{2.6b}
\]

\( \xi_1 \) and \( \xi_2 \) are the radial distances of the two points normalized with respect to the radius, \( \tilde{R} \); \( \theta_1 \) and \( \theta_2 \) are the corresponding angular coordinates measured from the direction of wave
propagation, as shown in Figure (2.1b); and \( b_0 \) and \( c_0 \) are dimensionless parameters related to the well known frequency parameter, \( a_0 = \omega R / u_x \), as follows:

\[
b_0 = \gamma a_0 \tag{2.7}
\]

\[
c_0 = \frac{u_x}{c} a_0 = (\sin \alpha) a_0 \tag{2.8}
\]

In the exact formulation of the problems presented in Refs. Luco and Mita (1987a) and Luco and Mita (1987b), the quantities \( \gamma \), \( a_0 \) and \( u_x / c \) appear independently.

**Integration of Equations.** For vertically incident incoherent waves, \( c_0 = 0 \) and the interrelationship of the free-field ground motion and the foundation input motion is defined by the single parameter \( b_0 \). Equations (2.5) in this case can be integrated exactly to yield

\[
S_x = \frac{1}{b_0^2} (1 - \exp(-2b_0^2)) \left[ I_0(2b_0^2) + I_1(2b_0^2) \right] S_y \tag{2.9a}
\]

\[
S_y = \frac{1}{b_0^2} (1 - \exp(-2b_0^2)) \left[ I_0(2b_0^2) + 2I_1(2b_0^2) + I_2(2b_0^2) \right] S_y \tag{2.9b}
\]

\[
S_{xy} = 0 \tag{2.9c}
\]

in which \( I_0, I_1 \) and \( I_2 \) are modified Bessel functions of the first kind of the order indicated by the subscript. Equation (2.9c) indicates that the horizontal and torsional components of the foundation input motion are statistically uncorrelated. The derivation of these expressions is given in Appendix A.

For obliquely incident coherent waves, for which \( \gamma = b_0 = 0 \),
the interrelationship of the two motions is defined completely by \( c_0 \), and equations (2.5) can again be integrated exactly to yield

\[
S_x = \left[ 2 \frac{J_1(c_0)}{c_0} \right]^2 S_0 \quad \text{(2.10a)}
\]

\[
S_y = \left[ 4 \frac{J_2(c_0)}{c_0} \right]^2 S_0 \quad \text{(2.10b)}
\]

\[
S_{xy} = \left[ 8 \frac{J_1(c_0)J_2(c_0)}{c_0^2} \right] S_0 \quad \text{(2.10c)}
\]

in which \( J_1 \) and \( J_2 \) are Bessel functions of the first kind of order one and two, respectively. The latter expressions have been presented previously by Pais and Kausel (1985). Note that \( S_{xy} \) is purely imaginary, indicating that there is a 90° phase angle in this case between the horizontal and torsional components of foundation input motion.

For the more general case involving combinations of wave passage and incoherence effects, formal integration of equations (6.5) has not proved possible, and the relevant expressions were integrated numerically.

**Presentation of Results.** The quantities \( \sqrt{S_x/S_0} \) and \( \sqrt{S_y/S_0} \) define the transfer functions for the amplitudes of the horizontal and rotational components of the foundation input motion, and the magnitude of \( S_{xy}/\sqrt{S_xS_y} \) define the degree of correlation or coherence of the components of the motion. A numerical value of unity for the latter quantity indicates
that the component motions are fully correlated, while a zero value indicates that they are uncorrelated. These quantities are plotted conveniently in Figures (2.2) and (2.3) as functions of the modified frequency parameter,

$$\tilde{a}_0 = \sqrt{b_0^2 + c_0^2} = \sqrt{\gamma^2 + \sin^2 \alpha \nu \alpha_0}$$

(2.11)

and the modified incoherence parameter,

$$\tilde{\gamma} = \frac{b_0}{c_0} = \frac{\gamma}{\sin \alpha \nu}$$

(2.12)

For incoherence effects only, $\alpha \nu = 0$, $\tilde{\gamma} = \infty$ and $\tilde{a}_0$ reduces to $a_0 = b_0$. Similarly, for wave passage effects only, $\gamma = \tilde{\gamma} = 0$ and $\tilde{a}_0$ reduces to $a_0 \sin \alpha \nu = c_0$.

Note that whereas the transfer function for the lateral component of the foundation input motion, $\sqrt{S_x/S_y}$, decreases monotonically in Figure (2.2) with increasing $a_0$, the corresponding function for the torsional component, $\sqrt{S_y/S_x}$, increases from zero to a peak and then decreases monotonically.

**Accuracy of Solutions.** As a measure of the accuracy of the reported data, the results computed for incoherence effects only and for wave passage effects only are compared in Figure (2.4) with the corresponding exact solutions of Luco and Mita (1987a,b). Since the factors $\gamma$ and $\sin \alpha \nu = v_z/c$ appear independently in the exact solutions, several different values are considered for these parameters. No comparisons are made for combinations of incoherence and wave passage as the exact
solutions are not available in this case.

Considering the uncertainties that are inherent in the definition of the incoherence function and in the choice of the parameter $\gamma$, the degree of the agreement in the two sets of results displayed in Figure (2.4) is deemed to be quite satisfactory. Note should also be taken of the fact that, excepting the narrow frequency ranges where the curves for wave passage only exhibit notch-like trends, the approximate solutions overestimate the amplitudes of foundation input motions.

Other Meanings for Results. Although defined specifically for the displacement histories of the foundation input motion, the spectral density ratios $S_x/S_\theta$, $S_y/S_\theta$ and $S_{xy}/S_\theta$ also define the ratios $S_x/S_\theta$, $S_y/S_\theta$, $S_{xy}/S_\theta$ and $S_x/S_\theta$, $S_y/S_\theta$, $S_{xy}/S_\theta$ of the corresponding velocity and acceleration histories. Recall that the psd function for the first derivative of a process is given by the product of $(2\pi f)^2$ and the psd function of the original process.

2.3.2 Spectral Characterization of Structural Response

With the psd functions of the foundation input motion established, the corresponding functions of the structural response can be obtained by well-established procedures [e.g., Lin (1976)]. Let $S_u$ be the psd function of the structural deformation, $u$, induced by the lateral component of the
foundation input motions; and let $S_v$ be the corresponding function of the deformation, $u = \psi R$, induced along the perimeter of the structure by the torsional component of response. The quantity $\psi$ represents the angular deformation of the structure. These functions are related to the psd functions of the foundation input accelerations, $S_\hat{x}$ and $S_y$, by

$$S_u = |H_u|^2 S_\hat{x}$$ (2.13)

and

$$S_v = |H_u|^2 S_y$$ (2.14)

in which $H_u$ = the transfer function for lateral response, given by

$$H_u = -\frac{1}{p_x^2} \frac{1}{1-(\omega/p_x)^2+i2\zeta_x(\omega/p_x)}$$ (2.15)

$H_u$ = the corresponding function for torsional response, obtained from equation (2.15) by replacing $p_x$ by $p_\theta$ and $\zeta_x$ by $\zeta_\theta$; and vertical bars indicate the modulus of the enclosed quantity. Similarly, the psd function $S_w$ for the total deformation at the most highly stressed point on the periphery of the structure, $w = u + v$, is given by Lin (1976)

$$S_w = S_u + S_v + 2|\Re(H_uH_v^*S_{\hat{x}y})|$$ (2.16)

in which $S_{\hat{x}y}$ = the cross psd function of the lateral and circumferential components of the foundation input acceler-
ations; \( \mathfrak{R} \) denotes the real part of the indicated quantity; and a star superscript denotes the complex conjugate of the quantity to which it is attached.

2.3.3 Characterization of Free-Field Earthquake Ground Motions

The local psd function for the set of acceleration traces considered in the remainder of this paper is taken in the form

\[
S_0 = \begin{cases} 
\frac{f^4}{0.5 + f^4} (1 - \frac{f^2}{f_0^2}) S_0 & \text{for } f \leq f_0 \\
0 & \text{for } f \geq f_0
\end{cases} \quad (2.17)
\]

in which \( S_0 \) = a constant; \( f = \omega / 2\pi \) = the exciting frequency in cps; and \( f_0 \) = the cut-off frequency, taken as 15 cps. Of the same general form as that employed in a related study by Pais and Kausel (1985), the function \( S_0 \), along with the associated functions for ground velocity and ground displacement, are plotted in Figure (2.5), with all peaks normalized to a unit value. As would be expected, the psd function for velocity decays much more rapidly with frequency than that for acceleration, and the corresponding displacement function decays even faster.

Let \( \bar{X}_a \) be the mean of the absolute maximum peaks of the acceleration traces, and \( \bar{X}_v \) and \( \bar{X}_d \) be the corresponding means of the velocity and displacement traces. These values were computed from Der Kieureghian's empirical expressions (Der Kieureghian 1980) summarized in Appendix A, considering the
duration of the intense portion of the excitation to be \( t_0 = 20 \) sec. The resulting values are \( \dot{X}_p = 26.17 \sqrt{S_0} \), \( \ddot{X}_p = 1.417 \sqrt{S_0} \) and \( X_p = 0.2468 \sqrt{S_0} \).

### 2.3.4 Foundation Input Motion

Before examining the response of the structure, it is desirable to compute the mean peak values of the acceleration, velocity and displacement traces of the horizontal and circumferential components of the foundation input motion. The relevant values for the horizontal component of motion are denoted by \( \dddot{X}, \dddot{Y} \) and \( X, Y \), and those for the circumferential component along the periphery of the foundation are denoted by \( \dddot{\psi}, \dddot{\gamma} \) and \( \psi, \gamma \). Computed by Der Kiureghian's approximation from the appropriate psd functions, these values are plotted in Figure (2.6) normalized with respect to the mean peak values of the corresponding histories of the free-field ground motion.

For the multi-frequency, transient excitation considered in this section, the solution is controlled by the effective transit time,

\[
\tau = \sqrt{\gamma^2 + \sin^2 \alpha \tau}
\]

in which \( \tau = R/v_z \) = the time required for the shear wave to traverse the radius of the foundation; and by the modified incoherence parameter, \( \tilde{\gamma} \), defined by equation (2.12).

The following observations may be made and inferences drawn from the data presented in Figure (2.6):
1. The reduction in the horizontal component of the foundation input motion and the corresponding increase in the rotational component are greatest for acceleration, much smaller for velocity, and almost negligible for displacement. Since the foundation filters the high-frequency wave components more effectively than the low-frequency wave components, the acceleration traces of the ground motion, which are richer in high-frequency content than the velocity and displacement traces, are influenced more than the latter traces.

2. Considering that the response of high-frequency systems is acceleration-sensitive whereas that of low-frequency systems is displacement-sensitive, it should be clear that the effects of kinematic interaction would be important for high-frequency systems and inconsequential for low-frequency systems. Furthermore, medium-frequency systems which are velocity-sensitive would be expected to be affected moderately. That this is indeed the case is confirmed by the data presented in the following sections.

2.3.5 Effects of Ground-Motion Incoherence on Structural Response

Let \( U_x \) = the mean of the maximum values of the structural deformations induced by the ensemble of lateral components of the foundation input motions, and \( U_y \) = the corresponding mean of the deformations induced at the periphery of the deck by
the torsional components. These quantities have been evaluated for vertically propagating incoherent shear waves ($\gamma = \infty$), and the results are displayed in Figure (2.7) in the form of tripartite response spectra. The solid curves in the upper part of the figure refer to lateral response, and the lower curves refer to torsional response. Several values of the effective transit time parameter, $\tilde{\tau}$, are considered, including the limiting value of $\tilde{\tau} = 0$ for which there is no kinematic interaction. The damping factors for both modes of response are taken as $\zeta_x = \zeta_\theta = 0.02$.

The left-hand diagonal scale in the upper part of Figure (2.7) represents $U_x$ normalized with respect to the mean peak value of the free-field displacement, $X_\phi$; the vertical scale represents the corresponding pseudo-velocity, $V_x = p_x U_x$, normalized with respect to $\dot{X}_\phi$; and the right-hand diagonal scale represents the corresponding pseudo-acceleration, $A_x = p_x V_x$, normalized with respect to $\ddot{X}_\phi$. In an analogous manner, the three scales in the lower part of the figure represent the deformation ratio, $U_y/X_\phi$; the pseudo-velocity ratio, $V_y/\dot{X}_\phi$, in which $V_y = p_\theta U_y$; and the pseudo-acceleration ratio, $A_y/\ddot{X}_\phi$, in which $A_y = p_\theta V_y = p_\theta^2 U_y$.

As anticipated from examination of the peak values of the foundation motions, the lateral component of the response of high-frequency systems in Figure (2.7) is affected materially by ground incoherence, and this effect is particularly large
in the practically important region of the response spectrum within which the pseudo-acceleration attains its maximum value. For $\gamma = 0.2$ and $\tilde{t} = 0.02$ sec. (a value corresponding to, say, $R = 100$ ft. and $v_s = 1000$ ft/sec), the maximum value of $A_x$ is 78 percent of that obtained for a fully coherent, uniform free-field ground motion; for $\tilde{t} = 0.05$ sec., the corresponding ratio is 55 percent. The reductions are significantly less pronounced for medium-frequency systems and practically negligible for low-frequency systems. For systems of very high-frequency, for which $A_x$ may be considered to be equal to the mean peak value of the foundation input acceleration, the percentage reductions are, of course, identical to those indicated in Figure (2.6) for the foundation input acceleration.

The general trends of the response spectra for the torsional deformation in Figure (2.7) are consistent with those of the corresponding curves for the foundation input motion presented in Figure (2.6). Specifically, in the low-frequency, displacement-sensitive region, the response increases with increasing values of the effective transit time, $\tilde{t}$, whereas in the high-frequency, acceleration-sensitive region, the response values for $\tilde{t} = 0.02$ sec. are higher than those for the higher values of considered. Furthermore, the percentage changes in response are comparable to those for the controlling values of the foundation input motion.
The component of the response contributed by the rotation of the foundation is generally small, and the combined effect of lateral and torsional responses is generally only slightly greater than that due solely to lateral response. The mean maximum values of the total deformation for the most highly stressed column along the periphery of the structure were evaluated considering $p_\theta/p_x = 1.5$, and the results are shown by the dashed lines in Figure (2.7). These results were computed by Der Kieureghian's approximation making use of equation (2.16) for the psd function of the combined motion.

Comparison of Incoherence and Wave Passage Effects. Some of the response spectra for the incoherent ground motions presented in Figure (2.7) are compared in Figure (2.8) with those computed considering only wave passage effects, and combinations of wave passage and incoherence represented by a value of $\tilde{\gamma} = 1$. It should be clear that the results are not particularly sensitive to the choice of the parameter $\tilde{\gamma}$, and that this insensitivity is fully compatible with that observed in Figure (2.6) for the peak values of the foundation input motions. Indeed, the ratio of the low-frequency limiting values of $U_x$ for $\tilde{\gamma} = 0$ and $\tilde{\gamma} = \infty$ in Figure (2.8) is almost identical to that of the peak values of the lateral component of the foundation input displacements in Figure (2.6), and the ratio of the corresponding values of $U_y$ is almost identical to the displacement ratio of the torsional component of the
foundation input motion. Similarly, the ratios of the high-frequency limits of $A_x$ and $A_y$ in Figure (2.8) are identical to those obtained from Figure (2.6) for the mean peak values of the lateral and torsional components of the foundation input accelerations. It follows that, to the degree of approximation represented by the differences in the results displayed in Figure (2.8), the effects of ground motion incoherence may be replaced by those of wave passage and vice versa. This possibility has also been suggested by Luco and Wong (1986) from examination of the relevant foundation transfer functions. In implementing this replacement, it is important that the value of $\tilde{\tau}$ be the same in the two cases.

2.4 INERTIAL INTERACTION EFFECTS

The inertial interaction effects are now evaluated by a simple modification of the procedure used in previous studies in which the effects of kinematic interaction were neglected (e.g., Veletsos 1977; Veletsos and Meek 1976). For each mode of excitation, it is only necessary to replace the free-field motion by the appropriate component of the foundation input motion.

The following steps are involved in the analysis: First, the harmonic response of the system is evaluated making use of the appropriate complex-valued foundation impedance functions. Next, the psd functions of the torsional and
lateral components of structural response are determined. The desired mean peak values of the responses are finally computed from Der Kiureghian's approximation. Additional details are given in Appendix A.

The foundation impedances for the torsional mode of vibration were computed from the approximate closed-form expressions of Veletsos and Nair (1974), and those for the horizontal and rocking motions were computed from the corresponding expressions of Veletsos and Verbic (1973). The cross coupling terms between horizontal and rocking motions were presumed to be negligible.

The principal parameters that influence the response of the system are the characteristics of the free-field ground motion; the fixed-base natural frequencies of the structure, \( f_x \) and \( f_\theta \), and the associated damping factors, \( \zeta_x \) and \( \zeta_\theta \); the height to base radius ratio, \( h/R \); the mass density ratio for the structure, defined conveniently as \( \delta = m/(\pi \rho R^2 h) \), in which the denominator represents the total mass of the structure when filled with the supporting soil; and the wave transit times, \( \tau \) and \( \bar{\tau} \). It is important to note that whereas the kinematic interaction effects are defined completely by \( \bar{\tau} \), the evaluation of the inertial interaction effects requires the separate specification of the parameters \( \gamma \) and \( \tau \). Other parameters affecting the response of the system are Poisson's ratio for the supporting medium, \( \nu \); the mass ratio of the
foundation and super-structure, \( m_f/m_t \); the ratio \( I_f/I \) of the mass moments of inertia of the foundation and structure about horizontal centroidal axes; and the ratio \( J_f/J \) of the corresponding polar moments of inertia. For the solutions presented herein, \( \zeta_x = \zeta_\theta = 0.02; \delta = 0.15; \nu = 1/3; \) and \( m_l \) (and hence \( I_f \) and \( J_f \)) are considered to be negligible.

2.4.1 Results for Vertically Propagating Incoherent Waves

Figure (2.9) shows response spectra for lateral and torsional response obtained for vertically propagating incoherent waves, taking \( \gamma = 0.4 \) and \( \tau = R/v_x = 0.05 \) sec. Three sets of solutions are presented: (a) making no provision for soil-structure interaction, i.e., considering the foundation motion to be equal to the free-field ground motion; (b) providing only for the kinematic interaction effects, i.e., using as base excitation the foundation input motions; and (c) providing for both kinematic and inertial interaction effects; i.e., analyzing the structure-foundation-soil system exactly as a coupled system. In the analysis of the inertial interaction effects, two values of \( h/R \) are used: a unit value, corresponding to short stubby structures, and a value of 3, corresponding to taller, more slender structures. Solutions (a) and (b) are independent of \( h/R \), and solutions (b) are valid for all combinations of \( \gamma \) and \( \tau \) for which \( \tilde{\tau} = \gamma \tau = 0.02 \) sec.
Previous studies of soil-structure interaction involving only inertial interaction effects (ATC 1978; FEMA 1986; Veletsos 1977, 1978) have shown that these effects may be evaluated to a high degree of approximation using the free-field ground motion as the foundation input motion and merely modifying the relevant natural frequency and damping of the structure. The modified frequency and damping are taken such that, for each mode of vibration, the magnitude and location of the resonant peak of the relevant harmonic response are identical for the actual and replacement systems. For structure for which the kinematic interaction effects are important, this approach would require that the response of the structure be evaluated for the horizontal and torsional components of the foundation input motion rather than for the free-field ground motion.

The mean maximum values of the responses obtained by this approximate procedure are shown by the dashed lines in Figure (2.9), and the values of the modified natural frequencies and damping factors are identified in Figure (2.10). Denoted with a tilde superscript, the modified frequencies are, of course, lower than the corresponding fixed-base frequencies, and the modified damping factors are higher than the value of $\zeta = 0.02$ assumed for the fixed-base structure.

The following trends should be observed in these figures:
1. Like kinematic interaction (KI), inertial interaction (II) may affect significantly the responses of systems in the medium- and high-frequency spectral regions.

2. The II effects are generally more important than the KI effects.

3. Unlike kinematic interaction which generally reduces the lateral response, inertial interaction may increase the corresponding response of tall, slender structures in the high frequency region of the response spectrum. Such structures, however, typically fall in the middle-frequency region of the spectrum, for which the interaction effects are relatively small.

4. The II effects for low-frequency, highly compliant structures are negligible because such systems "see" the half-space as a very stiff effectively rigid medium.

5. Provided the base excitation for the structure is taken equal to the foundation input motion rather than the free-field ground motion, the concept of modifying the fixed-base natural frequencies and associated damping values of the system provides a simple and highly reliable practical means for assessing the II effects.

It may be surprising that the values of $f_0$ and $\xi_0$ in Figure (2.10) are functions of the ratio $h/R$. This is due to the
fact that with the value of the mass ratio, $\delta$, fixed in these solutions, the polar mass moment of inertia of the system, $J$, is different for different values of $h/R$.

2.5 CONCLUSIONS

1. The information and concepts presented herein provide valuable insight into the nature of kinematic and inertial interaction effects for simple structures subjected to earthquakes, and into the effects and relative importance of the numerous parameters involved.

2. In the approximate method of analysis employed, the kinematic interaction effects are defined completely by the effective transit time, $\tilde{t}$, and the modified incoherence parameter, $\tilde{\gamma}$.

3. Even for vertically propagating waves, kinematic interaction may reduce significantly the critical responses of high-frequency systems. These reductions are generally smaller than, but of approximately the same order of magnitude as, those due to inertial interaction.

4. Reliable estimates of the effects of kinematic interaction on the peak values of structural response may be obtained from knowledge of the corresponding values of the acceleration, velocity and displacement traces of the foundation input motion. The latter quantities may be computed from analyses of the response of the massless foundation to the free-field
ground motion.

5. Insofar as the mean maximum values of the responses are concerned, the kinematic interaction effects due to ground motion incoherence are similar to those due to wave passage, and the two effects may be interrelated.

6. An excellent approximation to the inertial interaction effects may be obtained by a previously recommended simple procedure (ATC 1978; FEMA 1986; Veletsos 1977, 1978) using as base excitation the foundation input motion rather than the free-field motion. The inertial interaction effects in this approach are expressed by changes in the natural frequency of vibration and the associated damping of the structure for the mode of vibration considered.
Fig. 2.1 System Considered
Fig. 2.2 Magnitudes of Transfer Functions between Free-Field Ground Motion and Foundation Input Motions
Fig. 2.3 Normalized Cross PSD Function for Horizontal and Torsional Components of Foundation Input Motion
Fig. 2.4 Comparison of Approximate and Exact Magnitudes of Foundation Transfer Functions
Normalized Values of $S_g$, $S_{g_0}$ and $S_{g_f}$

Fig. 2.5 Normalized PSD Functions for Free-Field Ground Motions Considered
Fig. 2.6 Normalized Mean Peak Values of Lateral and Torsional Components of Foundation Input Accelerations, Velocities and Displacements
Fig. 2.7 Effects of Ground Motion Incoherence on Maximum Deformations of Structures with $\zeta_x = \zeta_\theta = 0.02$ for Vertically Propagating Waves.
Fig. 2.8 Effects of Ground Motion Incoherence and Wave Passage on Maximum Deformations of Structures with $\tau_x = \tau_\theta = 0.02$
Fig. 2.9 Comparison of Effects of Kinematic and Inertial Interaction on Maximum Deformations of Structures with $\zeta_x = \zeta_\theta = 0.02$
Fig. 2.10  Natural Frequencies and Damping of Modified Systems in Approximate Analysis of Inertial Interaction Effects
III TRANSFER FUNCTIONS FOR RECTANGULAR FOUNDATIONS

3.1 INTRODUCTION

As indicated in Chapter II, a fundamental step in the dynamic analysis of a base-excited structure-foundation system is the evaluation of the transfer functions of its foundation motion. Defined for harmonically excited massless foundations, these functions relate the amplitudes of the components of foundation input motion (FIM) to those of the free-field ground motion at some reference or control point.

In the preceding chapter, these functions were evaluated for surface-supported, circular rigid foundation excited by obliquely incident, horizontally polarized, incoherent shear waves. An approximate analysis, utilizing the averaging technique employed by Iguchi (1984) and Scanlan (1976) in their studies of wave passage effects for plane waves, was employed. In this chapter, the same approximate method of analysis is extended to surface-supported, rigid, rectangular foundations.

As before, the spatial variation of the free-field ground motion is specified stochastically by means of an incoherence function, for which two different forms are used. The response quantities examined are effectively the amplitudes of the translational and rotational components of the foundation input motion. These quantities are evaluated over wide ranges
of the parameters involved, and the more important trends are displayed graphically. For the special case of square foundations, it is shown that the results are in general agreement with those obtained previously for an equivalent circular foundation.

The problem considered herein has been examined previously by Luco and Wong (1987), who formulated the more nearly exact integral relations for the transfer functions of the foundation, and have presented numerical solutions for square foundations excited by vertically incident, incoherent shear waves. However, no solutions appear to have been presented for rectangular foundations. The advantages over the exact Luco-Wong formulation of the approximate method of analysis employed here are: (1) It leads to relatively simple, closed-form expressions for the transfer functions for both wave-passage and incoherence effects; and (2) it reduces the number of independent parameters that must be considered, thereby simplifying the interpretation of results. In addition, the resulting solutions are believed to be of sufficiently good accuracy for practical purposes.

3.2 STATEMENT OF PROBLEM

A rectangular, rigid, massless foundation supported at the surface of an elastic half-space is considered. The half-space is characterized by its shear modulus, $G$, Poisson's ratio, $\nu$,
and mass density, \( \rho \). The shear wave velocity for the medium is then given by \( v_s = \sqrt{C/\rho} \). The foundation is referred to a set of rectangular coordinate axes with its origin taken at the center of the foundation, as shown in Figure (3.1), and it is considered to be bonded to the supporting medium so that no sliding or uplifting may occur. The lengths of sides of the foundations along the \( x \) and \( y \) coordinate axes are denoted by \( 2a \) and \( 2b \), respectively.

The free-field ground motion for all points of the foundation-soil interface is presumed to be directed along the \( x \)-axis, with the wave front propagating along the positive \( y \)-axis, as shown in Figure (3.1). The motion at any point is specified stochastically by means of a local power spectral density (psd) function, \( S_\phi(\omega) \), in which \( \omega \) = the circular frequency of the harmonic component of the motion under consideration, and the interrelationship of the motions at two arbitrary points, defined by the position vectors \( \vec{x}_1 \) and \( \vec{x}_2 \), is specified by the cross power spectral density function,

\[
S_\phi(\vec{x}_1,\vec{x}_2,\omega) = \Gamma(|\vec{x}_1 - \vec{x}_2|, \omega) \exp\left[-i\omega\left(y_1 - y_2\right) / c_y\right] S_\phi(\omega) \quad (3.1)
\]

in which \( \Gamma \), the so-called incoherence function, which is a dimensionless, decreasing function of \( \omega \) and of the distance between the two points, \( |\vec{x}_1 - \vec{x}_2| ; i = \sqrt{-1} \); and \( c_y \) = the apparent horizontal velocity of propagation of the wave front; \( y_1 \) and \( y_2 \) are the components (projections) of \( \vec{x}_1 \) and \( \vec{x}_2 \) in the direction
of propagation of the wave front. As already noted in Chapter II, the product of \( S_\omega(\omega) \) and the exponential term in equation (3.1) represents the wave passage effect, whereas the product \( \Gamma S_\omega \) represents the effect of ground motion incoherence. The peak value of \( \Gamma \) is unity and occurs at \( x_1 = x_2 \). For the solutions presented herein, the incoherence function is taken in the form proposed by Der et al. (1988) as

\[
\Gamma(|x_1 - x_2|, \omega) = \exp \left[ -\left( \frac{\omega}{\nu_s} \right)^2 [\gamma_x^2 (x_1 - x_2)^2 + \gamma_y^2 (y_1 - y_2)^2] \right] \quad (3.2)
\]

in which \( \gamma_x \) and \( \gamma_y \) are dimensionless coefficients, normally taken in the range between zero and 0.5. For \( \gamma_x = \gamma_y = \gamma \), equation (3.2) reduces to the expression proposed by Mita and Luco (1987) and employed in Chapter II. Of primary interest in this Chapter is the evaluation of the spectral density functions of the components of input motion for foundations of arbitrary ratio of sides.

### 3.3 Foundation Input Motion

#### 3.3.1 Spectral Characterization of Motion

Let \( S_x \) be the psd function of the horizontal component of the foundation displacement, and \( S_\omega \) be the corresponding function of the circumferential or tangential component. Denoted by \( s(t) \), the latter component is defined by

\[
s(t) = b \theta(t) \quad (3.3)
\]
in which $\theta(t)$ = the instantaneous rotation of the foundation about a vertical centroidal axis. Further, let $S_x$, be the cross psd function of these displacements.

These functions were evaluated from the cross psd function of the free-field ground motion by application of the averaging technique employed by Iguchi (1984) and Scanlan (1976) in their studies of wave passage effects for plane waves. The approach leads to

$$S_x = \frac{1}{A^2} \int_A \int_A S(\bar{x}_1, \bar{x}_2, \omega) dA_1 dA_2$$  \hspace{1cm} (3.4a)

$$S_\tau = \frac{b^2}{I_0 A} \int_A \int_A y_1 y_2 S(\bar{x}_1, \bar{x}_2, \omega) dA_1 dA_2$$  \hspace{1cm} (3.4b)

$$S_{x\tau} = \frac{b}{I_0 A} \int_A \int_A y_2 S(\bar{x}_1, \bar{x}_2, \omega) dA_1 dA_2$$  \hspace{1cm} (3.4c)

in which $dA_1$ and $dA_2$ are elemental areas of the foundation; $A = 4ab$ = the area of the foundation; and $I_0 = \frac{1}{3} A(a^2 + b^2)$ = the polar moment of inertia of the foundation about a vertical centroidal axis.

On introducing the dimensionless distances $\xi_1 = x_1/a$, $\xi_2 = x_2/a$, $\eta_1 = \eta_1/b$ and $\eta_2 = \eta_2/b$, and making use of equation (3.1), equations (3.4a) through (3.4c) may be rewritten as

$$\frac{S_x}{S_\tau} = \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \eta_1 d\eta_1 d\eta_2$$  \hspace{1cm} (3.5a)

$$\frac{S_\tau}{S_\tau} = \left(\frac{b^2 A}{I_0} \right)^2 \left\{ \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \eta_1 \eta_2 \bar{f} d\xi_1 d\xi_2 d\eta_1 d\eta_2 \right\}$$  \hspace{1cm} (3.5b)
\[
\frac{S_{xx}}{S_\vartheta} = \left( \frac{b^2 A}{I_0} \right) \left( \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \eta_2 \bar{\Gamma} \, d\xi_1 \, d\xi_2 \, d\eta_1 \, d\eta_2 \right)
\] (3.5c)

in which

\[
\bar{\Gamma} = \exp \left( - \left[ b_{ox} \frac{a}{b^2} (\xi_1 - \xi_2)^2 + b_{oy} (\eta_1 - \eta_2)^2 \right] - \left[ i \frac{b_{oy}}{\gamma_y} (\eta_1 - \eta_2) \right] \right) \] (3.6a)

\[
b_{ox} = \gamma_x \frac{\omega b}{v_s}
\] (3.6b)

\[
b_{oy} = \gamma_y \frac{\omega b}{v_s}
\] (3.6c)

and

\[
\tilde{\gamma}_y = \gamma_y \frac{c_y}{v_s}
\] (3.6d)

In the exact formulation of the problem (Luco and Mita 1987b) the quantities \(\gamma_x, \gamma_y, c_y/v_s,\) and \(\omega b/v_s\) would appear independently as four distinct parameters.

3.3.2 Integration of Equations

Equations (3.5) can be integrated exactly in terms of the standard error function of complex argument. The details of integration are given in Appendix B. The results may be stated as

\[
\frac{S_x}{S_\vartheta} = f_1(x, \psi) g_1(\phi)
\] (3.7a)

\[
\frac{S_z}{S_\vartheta} = \frac{9}{\left[ 1 + (a/b)^2 \right]^2} f_2(x, \psi) g_1(\phi)
\] (3.7b)
\[
\frac{S_{xs}}{S_v} = \frac{3}{1+(a/b)^2} f_3(\chi, \psi) g_1(\phi)
\]  

(3.7c)

In which

\[\phi = 2 \frac{a}{b} b_0 x = 2 \gamma_x \frac{\omega a}{u_s}\]  

(3.8a)

\[\chi = 2 b_0 \gamma_y\]  

(3.8b)

\[\psi = \frac{1}{2 \gamma_y}\]  

(3.8c)

The functions \(f_1, f_2, f_3\) and \(g_1\) are given by

\[f_1(\chi, \psi) = B_1(\chi, \psi) - B_3(\chi, \psi) - \frac{\psi}{\chi} B_2(\chi, \psi)\]  

(3.9a)

\[f_2(\chi, \psi) = \frac{1}{3} \left[ B_1(\chi, \psi) - B_3(\chi, \psi) - \frac{2}{\chi^2} (1 - B_3(\chi, \psi)) \right] - \frac{\psi(3\chi^2 + 2\psi^2 - 3)}{\chi^3} B_2(\chi, \psi) - 2 \frac{\psi^2}{\chi^2} (B_3(\chi, \psi) - 2 B_4(\chi, \psi))\]  

(3.9b)

\[f_3(\chi, \psi) = i \left( \frac{\psi}{\chi} B_1(\chi, \psi) - B_3(\chi, \psi) \right) + \frac{1 - 2\psi^2}{2\chi^2} B_2(\chi, \psi)\]  

(3.9c)

\[g_1(\phi) = f_1(\phi, 0) = \left\{ \sqrt{\frac{\pi}{\phi}} \Phi(\phi) - \frac{[1 - \exp(-\phi^2)]}{\phi^2} \right\}\]  

(3.9d)

And

\[B_1(\chi, \psi) = \sqrt{\frac{\pi}{\chi}} \exp(-\psi^2) \Re[\Phi(\chi + i \psi)]\]  

(3.10a)

\[B_2(\chi, \psi) = \sqrt{\frac{\pi}{\chi}} \exp(-\psi^2) \Im[\Phi(\chi + i \psi) - \Phi(i \psi)]\]  

(3.10b)
\[ B_3(\chi, \psi) = \frac{1 - \exp(-\chi^2) \cos(2\chi\psi)}{\chi^2} \quad (3.10c) \]

\[ B_4(\chi, \psi) = \frac{\exp(-\chi^2) \sin(2\chi\psi)}{2\chi\psi} \quad (3.10d) \]

The function \( \Phi(z) \) in equations (3.10a) and (3.10b) is the error function with complex argument \( z \), defined by

\[ \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt \quad (3.11) \]

and the symbols \( \Re[.] \) and \( \Im[.] \) represent the real and imaginary parts of the bracketed quantity.

For vertically incident incoherent waves, for which \( c_y = \tilde{\gamma}_y = 0 \), equations (3.7) for \( S_x, S_z \) and \( S_{xz} \) reduce to

\[ \frac{S_x}{S_0} = g_1(\chi)g_1(\phi) \quad (3.12a) \]

\[ \frac{S_z}{S_0} = \frac{3}{[1+(a/b)^2]^2} \left( g_1(\chi) - \frac{2}{\chi^2}[1 - B(\chi)] \right) g_1(\phi) \quad (3.12b) \]

\[ S_{xz} = 0 \quad (3.12c) \]

in which

\[ B(\chi) = B_3(\chi, 0) = \frac{1 - \exp(-\chi^2)}{\chi^2} \quad (3.13) \]

For obliquely incident coherent waves, for which \( \gamma_y = \gamma_x = 0 \), equations (3.7) can be written in terms of a frequency parameter, \( c_{0y} \), defined by
\[ c_{oy} = \frac{\omega b}{c_y} \]  
\( (3.14) \)

The expressions for \( S_x, S_z \) and \( S_{xs} \) in this case reduce to

\[ \frac{S_x}{S_y} = \left( \frac{\sin(c_{oy})}{c_{oy}} \right)^2 \]  
\( (3.15a) \)

\[ \frac{S_z}{S_y} = \frac{9}{[1+(a/b)^2]^2} \left( \frac{1}{c_{oy}} \left[ \frac{\sin(c_{oy})}{c_{oy}} - \cos(c_{oy}) \right] \right)^2 \]  
\( (3.15b) \)

\[ \frac{S_{xs}}{S_y} = \frac{3i}{[1+(a/b)^2]} \left( \frac{\sin(c_{oy})}{c_{oy}} \right) \left( \frac{1}{c_{oy}} \left[ \frac{\sin(c_{oy})}{c_{oy}} - \cos(c_{oy}) \right] \right) \]  
\( (3.15c) \)

The latter expressions have been presented previously by Luco and Sotiropoulos (1980).

3.3.3 Results for Vertically Incident Waves with Two-Dimensional Incoherence

The quantities \( \sqrt{S_x/S_y} \) and \( \sqrt{S_z/S_y} \) represent the transfer functions of the foundation, namely, the ratios of the amplitude of the horizontal and torsional components of foundation motion to the amplitude of the free-field control point motion of the ground. These quantities are displayed in Figure (3.2) for vertically incident waves with two-dimensional isotropic incoherence (i.e., \( \gamma_x = \gamma_y \)). This is essentially Luco's (1987b) incoherence function considered previously in Chapter II. The frequency parameters \( b_x \) and \( b_y \) in this case are equal and \( \tilde{\gamma}_y = 0 \). The results are plotted as a function of \( b_y \) for several different values of the aspect ratio, \( a/b \), in the range between
zero and 4.

The following trends are worth noting in Figure (3.2):

1. For a fixed value of $a/b$, the transfer functions for the lateral component of foundation motion decreases monotonically with increasing value of the frequency parameter, $b_0$, whereas the torsional component increases from zero to a peak and then decreases monotonically. These trends are consistent with those of the corresponding curves for circular foundations presented in Chapter II.

2. Increasing the aspect ratio, $a/b$, decreases both $\sqrt{S_x/S_y}$ and $\sqrt{S_z/S_y}$. These trends may be appreciated by noting that the length $b$ in these plots is considered to be the same for the different curves. Increasing $a/b$, increases the length 'a' over which the incoherence of the ground motion must be averaged, and this increase decreases the effective or weighted values of both the translational and torsional components of the resulting foundation motion.

The reductions with increasing $a/b$ for the torsional component of foundation input motion are greater than for the horizontal component because increasing 'a' increases the moment of inertia of the foundation about the vertical centroidal axis more rapidly than does the area of the foundation. The increase in moment of inertia is effectively represented
by the factor \([1+(a/b)^2]^2\) in the denominator of equation (3.7b).

For the conditions considered in this section and in section 3.3.4, \(S_{xx} = 0\), indicating that the lateral and torsional components of the foundation motion are uncorrelated in this case.

Properties of Transfer Functions for Lateral Components of Response. Examination of equations (3.5a) and (3.7a) reveals that the value \(S_x/S_g\) for a rectangular foundation is equal to the geometric mean of the values obtained for square foundations with sides equal to each of the sides of the rectangular foundations. Specifically, if \((S_x/S_g)_{a,b}\) is the value of \(S_x/S_g\) for a rectangular foundation with sides \(2a\) and \(2b\), and \((S_x/S_g)_a\) and \((S_x/S_g)_b\) are the corresponding values for square foundations with sides \(2a\) and \(2b\), respectively, then

\[
\left( \frac{S_x}{S_g} \right)_{a,b} = \sqrt{\left( \frac{S_x}{S_g} \right)_a \left( \frac{S_x}{S_g} \right)_b} 
\]  

(3.16)

Equation (3.16) may also be used to establish a simple relationship between the \(S_x/S_g\) values of a strip foundation (i.e. a foundation with \(a/b \to 0\)) and a square foundation. On noting that for \(a/b \to 0\), \((S_x/S_g)_a\) tends to unity, and that \((S_x/S_g)_b\) in equation (3.16) effectively refers to a square plate with sides \(2b\), one obtains

\[
\left( \frac{S_x}{S_g} \right)_{a/b \to 0,b} = \sqrt{\left( \frac{S_x}{S_g} \right)_b} 
\]  

(3.17)
It should be recalled that the length of the foundation, $2b$, normal to the direction of motion is considered to be fixed in this comparison.

3.3.4 Solutions for Vertically Incident Waves with One-Dimensional Incoherence

For the one-dimensional incoherence function considered in this section, $b_{ox} = \phi = 0$ and

$$g_1(0) = B_1(0,0) - B_3(0,0) = 2 - 1 = 1$$

Equations (3.7a) and (3.7b) then reduce to

$$\frac{S_x}{S_g} = f_1(\chi, \psi) \quad (3.18a)$$

and

$$\frac{S_z}{S_g} = \frac{9}{[1 + (a/b)^2]^2} f_2(\chi, \psi) \quad (3.18b)$$

Now, consider a strip footing with $a/b \to 0$ and the two-dimensional, isotropic incoherence. In this case, $\phi$ also equals zero, and equations (3.7a) and (3.7b) reduce to

$$\frac{S_x}{S_g} = f_1(\chi, \psi) \quad (3.19a)$$

and

$$\frac{S_z}{S_g} = 9 f_2(\chi, \psi) \quad (3.19b)$$

The similarity of equations (3.18) and (3.19) permit the transfer functions for the two systems to be interrelated as
follows. Let the subscripts \((a/b,1)\) identify a rectangular foundation with arbitrary ratio of sides subjected to a ground motion with one-dimensional incoherence, and \((a/b \to 0,2)\) identify a strip foundation subjected to a motion associated with a two-dimensional, isotropic incoherence. Then

\[
\left( \frac{S_x}{S_y} \right)_{a/b,1} = \left( \frac{S_x}{S_y} \right)_{a/b \to 0,2} \tag{3.20a}
\]

and

\[
\left( \frac{S_x}{S_y} \right)_{a/b,1} = \frac{1}{[1+(a/b)^2]} \left( \frac{S_x}{S_y} \right)_{a/b \to 0,2} \tag{3.20b}
\]

3.3.5 Solutions for Combination of Wave Passage and Ground Motion Incoherence

Consideration is now given to the effect of obliquely incident incoherent waves propagating in the \(y\)-direction. The incoherence of the free-field motion is defined by the two-dimensional function with values of \(\gamma_x = \gamma_y\). The foundation input motion in this case depends on the aspect ratio of the foundation, \(a/b\); the frequency parameter, \(b_{0y}\); and the modified incoherence factor, \(\tilde{\gamma}_y\).

In Figure (3.3) are presented the transfer functions for the lateral and torsional components of the input motion for square foundations. The results are plotted not against the frequency parameter, \(b_{0y}\), but rather against the modified frequency parameter, \(\tilde{a}_{0y}\), defined by
\[ \tilde{a}_{0y} = \sqrt{1 + \frac{1}{\bar{V}_y^2}} \quad b_{0y} = \sqrt{1 + \bar{V}_y^2} \quad c_{0y} \]  \hspace{1cm} (3.21)

When displayed in this format, the results for wave passage only can be plotted along with those for the combined effects of wave passage and incoherence. For wave passage only, \( \bar{V}_y = 0 \), and \( \tilde{a}_{0y} \) reduces to \( c_{0y} \). Similarly, for incoherence only, \( \bar{V}_y = \infty \), and \( \tilde{a}_{0y} \) reduces to \( b_{0y} \).

The trends of the curves in Figure (3.3) are similar to those of the curves presented in Figure (2.2) of Chapter II for circular foundations. This should not be surprising, of course. In fact, the transfer functions for the lateral components of motion for circular and square foundations may be approximated with reasonable accuracy by equating the areas of the two foundations. This leads to the following relationship between the radius, \( R \), of the circular foundation and the half-length of side, \( b_* \), of the equivalent square foundation:

\[ b_* = \sqrt{\frac{\pi}{2}} R \]  \hspace{1cm} (3.22)

Similarly, the transfer functions for the torsional components of motion for circular and square foundations may be interrelated by equating the moments of inertia of the foundations about their vertical centroidal axes. The interrelationship of \( R \) and \( b_* \) in this case is

\[ b_* = \frac{(3\pi)^{1/4}}{2} R \]  \hspace{1cm} (3.23)
Inasmuch as no closed form expressions could be developed in Chapter II for the transfer functions of circular foundations subjected to the combined effects of wave passage effects and ground motion incoherence, Equations (3.7a) and (3.7b), along with the solutions for rectangular foundations examined herein, may be used to deduce approximate expressions for these functions.

Figure (3.4) displays plots similar to those presented in Figure (3.3) for a one-dimensional incoherence function with non-zero \( \gamma_y \) and \( \gamma_x = 0 \). The differences in the two sets of results are substantial only for large values of \( \tilde{\gamma}_y \), for which the effects of ground motion incoherence are dominant. As would be expected, the reduction with increasing \( b_{0y} \) of the lateral component of foundation motion is substantially less for the one-dimensional incoherence function than for the two-dimensional function. Similarly, the increase in the rotational component of motion is substantially greater for the one-dimensional function. With decreasing value of \( \tilde{\gamma}_y \), the differences between the two sets of result decrease, and for \( \tilde{\gamma}_y = 0 \), which corresponds to the propagation of plane coherent waves, the two sets of results, naturally, coincide.

The cross spectral density function, \( S_{x\omega} \), of the foundation input motion for the conditions considered in Figures (3.3) and (3.4) is plotted in Figure (3.5), normalized with respect to \( i\sqrt{S_x S_\omega} \). As indicated in Chapter II, the magnitude of
defines the degree of correlation or coherence between the lateral and torsional components of the foundation input motion. A numerical value of unity for this quantity indicates that the component motions are fully correlated, whereas a zero value indicates that they are uncorrelated.

When normalized in this form, it is simple matter to show from equations (3.7) that the plots in Figure (3.5) are applicable to both the one-dimensional and two-dimensional coherence functions examined. It can further be shown that the results are independent of the aspect ratio, \( a/b \), and are therefore, applicable to both square and rectangular foundations.

### 3.4 CONCLUSIONS

The information presented provide valuable insight into the nature of kinematic interaction for surface supported, rigid rectangular foundations, and into the effects and relative importance of the various parameters involved. With the transfer functions for the lateral and torsional components of the foundation input motion established, the response of such foundations to more complex free-field control point ground motions, and of structures supported on such foundations may be evaluated by the procedures employed in Chapter II.
Fig. 3.1 System Considered
Fig. 3.2 Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Rectangular Foundations Subjected to Vertically Incident Incoherent Waves; ($\gamma_x = \gamma_y = \gamma$)
Fig. 3.3 Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Square Foundations Subjected to Obliquely Incident Incoherent Waves; ($\gamma_x = \gamma_y = \gamma$)
Fig. 3.4 Transfer Functions for Lateral and Torsional Components of Foundation Input Motion for Square Foundations Subjected to Obliquely Incident Incoherent Waves; (One-Dimensional Incoherence with $y = 0$)
Fig. 3.5 Normalized Cross PSD Function for Horizontal and Torsional Components of Foundation Input Motion; Rectangular Foundations Subjected to Obliquely Incident Incoherent Waves
IV TRANSFER FUNCTIONS FOR EMBEDDED SQUARE FOUNDATIONS

4.1 INTRODUCTION

In Chapters II and III, the transfer functions for surface-supported circular and rectangular foundations subjected to spatially varying free-field ground motions were evaluated approximately using the Iguchi-Scanlan averaging scheme. In this chapter, the same general approach is used to evaluate the corresponding functions for rigid square foundations embedded in a homogeneous half-space.

The free-field ground motion in this evaluation is presumed to be due to vertically incident, horizontally polarized, incoherent waves. This motion is described stochastically by a space invariant, local power spectral density function and an incoherence function. For the limiting case of surface-supported foundations, the incoherence function considered reduces to those used in the preceding two chapters.

The response quantities examined are effectively the six components of the resulting foundation motion. The interrelationship of this motion to the free-field, control point motion is evaluated over wide ranges of the parameters involved, and the results are displayed graphically.

The first attempt at solving this problem appears to have been made by Hoshiya and Ishii (1983) who considered a simpler form of incoherence than the one employed here. However, their
analyses ignored the contribution of the free-field contact tractions along the foundation-soil interface, which in related studies of vertically incident plane waves had been shown by Iguchi (1984) and Luco (1986) to be quite important. The solutions presented herein give due consideration to the effects of these forces.

4.2 STATEMENT OF PROBLEM

A massless, rigid square foundation embedded in a homogeneous elastic half-space is considered. Points on the foundation-soil interface are referred to a Cartesian coordinate system, the origin of which is taken at the ground surface immediately above the foundation center and the \( x \) and \( y \) coordinate axes are taken parallel to the sides of foundation, as shown in Figure (4.1). The width of the foundation and the depth of embedment are denoted by \( 2a \) and \( e \), respectively. The foundation is presumed to be bonded such that no separation may result between them. The medium is characterized by its shear modulus of elasticity, \( G \), mass density, \( \rho \), Poisson's ratio, \( \nu \). The velocity of shear wave propagation for the medium is given by \( v_s = \sqrt{G/\rho} \).

The free-field ground motion for all points of the foundation-soil interface in this chapter is presumed to be directed along \( y \)-axis, as shown in Figure (4.1). (Note that this direction is not consistent with that considered in
Chapter III) The intense portion of this motion is represented stochastically by a space invariant local power spectral density function, $S_{gy} (\omega)$, and a cross power spectral density function, $S_{gxy} (\vec{x}_1, \vec{x}_2, \omega)$, in which $\vec{x}_1$ and $\vec{x}_2$ are the position vectors for two arbitrary points. The latter is taken in the form

$$S_{gxy} (\vec{x}_1, \vec{x}_2, \omega) = \Gamma (|\vec{x}_1 - \vec{x}_2|, \omega) \cos \left( \frac{\omega}{v_s} z_1 \right) \cos \left( \frac{\omega}{v_s} z_2 \right) S_{gy} (\omega) \tag{4.1}$$

with the incoherence function, $\Gamma (.)$, taken as

$$\Gamma (|\vec{x}_1 - \vec{x}_2|, \omega) = \exp \left\{ - \left( \frac{\omega}{v_s} \right)^2 \left[ \gamma_x^2 (x_1 - x_2)^2 + \gamma_y^2 (y_1 - y_2)^2 \right] \right\} \tag{4.2}$$

The quantities $x_1$, $y_1$, $z_1$ and $x_2$, $y_2$, $z_2$ in these expressions represent the components (projections) of $\vec{x}_1$ and $\vec{x}_2$ along the coordinate axes. Only two limiting cases of the incoherence function are considered. They include the isotropic incoherence function of Mita and Luco (1987) for which $\gamma_x = \gamma_y = \gamma$, and the one-dimensional incoherence corresponding to $\gamma_y = 0$ and $\gamma_x = \gamma$. The product of $S_{gy} (\omega)$, and the cosine terms in equation (4.1) represents the wave passage effect for the vertically incident waves, whereas the product $\Gamma S_{gy}$ represents the effect of ground motion incoherence for surface supported foundations. As indicated in previous chapters, the peak values of $\Gamma$ is unity and occurs at $\vec{x}_1 = \vec{x}_2$. Referred to as incoherence parameter, the dimensionless factor $\gamma$ is typically taken between 0 and 0.5.
Even for the relatively simple free-field ground motion considered, an embedded foundation will experience three translational and three rotational components of motion along and about each coordinate axis. The translational component along the \( y \)-axis and the torsional components about the vertical \( z \)-axis would be expected to be the more important. Also of importance would be expected to be the rocking component of foundation motion about a horizontal axis perpendicular the direction of motion. The objective is to evaluate all six components of the foundation motion.

4.3 FUNDAMENTAL RELATIONS

The following relations are presented for the general case in which the free-field ground motion has components along each of the three coordinate axes. The presentation is based on the work of Luco (1986) and makes use of the same notation as that employed in his work.

4.3.1 Harmonic Ground Motion

Let \( U_{g_x} = U_{g_x}(\vec{x}, \omega) \) be the amplitude along the \( x \)-axis of a free-field harmonic displacement at a point defined by the position vector \( \vec{x} \), and \( U_{g_y} = U_{g_y}(\vec{x}, \omega) \) and \( U_{g_z} = U_{g_z}(\vec{x}, \omega) \) be the corresponding displacement amplitudes along the \( y \) and \( z \) axes, respectively. Similarly, let \( T_{g_x} = T_{g_x}(\vec{x}, \omega) \) be the \( x \)-component of the interface tractions associated with the free-field ground motion at point \( \vec{x} \), and \( T_{g_y} = T_{g_y}(\vec{x}, \omega) \) and \( T_{g_z} = T_{g_z}(\vec{x}, \omega) \)
be the corresponding components along the $y$ and $z$ axes. The $3 \times 1$ vectors of these displacements and tractions are denoted compactly as

$$\{U_g(\vec{x}, \omega)\} = (U_{gx}, U_{gy}, U_{gz})^T$$  \hspace{1cm} (4.3)

and

$$\{T_g(\vec{x}, \omega)\} = (T_{gx}, T_{gy}, T_{gz})^T$$  \hspace{1cm} (4.4)

in which the $T$ superscript denotes the transpose of the vector to which it is attached. It should be emphasized that the free-field tractions refer to the complete half-space before the excitation for the foundation.

Further, let $U_{ox}$, $U_{oy}$ and $U_{oz}$ be the amplitudes of the translational components of the foundation in put motion along the $x$, $y$ and $z$ axes, respectively and $\Theta_x$, $\Theta_y$ and $\Theta_z$ be the amplitudes of the rotational components about the same three axes. Finally, let $\{U_0(\omega)\}$ be the $6 \times 1$ generalized foundation input displacement vector, defined as

$$\{U_0(\omega)\} = (U_{ox}, a\Theta_y, U_{oy}, a\Theta_x, U_{oz}, a\Theta_z)^T$$  \hspace{1cm} (4.5)

It has been shown by Luco (1986) that $\{U_0(\omega)\}$ is related to $\{U_g(\vec{x}, \omega)\}$ and $\{T_g(\vec{x}, \omega)\}$ by

$$\{U_0(\omega)\} = \int_A [F(\vec{x}, \omega)]^T \{U_g(\vec{x}, \omega)\} dA$$

$$- [C(\omega)] \int_A [\alpha(\vec{x})]^T \{T_g(\vec{x}, \omega)\} dA$$  \hspace{1cm} (4.6)
$dA$ represents the elemental area about position $\vec{x}$; $[F(\vec{x}, \omega)]$ = a $3 \times 6$, generally complex-valued, matrix of the amplitudes of the tractions along the foundation-soil interface induced by unit forces and moments acting at the center of foundation, these forces and moments being taken in the order of the corresponding displacement components $\{U_0(\omega)\}$; and $[C(\omega)]$ = the $6 \times 6$ foundation compliance matrix. The elements of the latter matrix give the complex-valued amplitudes of the foundation displacements and rotations induced by harmonic forces and moments of unit amplitude applied to the center of the foundation. Finally [$\alpha(\vec{x})$] is a $3 \times 6$ rigid-body motion influence matrix for the generalized displacements, given by

$$
[\alpha(\vec{x})] = \begin{bmatrix} 
1 & \frac{z-z_0}{a} & 0 & 0 & 0 & -\frac{y}{a} \\
0 & 0 & 1 & -\frac{z-z_0}{a} & 0 & \frac{x}{a} \\
0 & -\frac{x}{a} & 0 & \frac{y}{a} & 1 & 0 
\end{bmatrix}
$$

(4.7)

The quantity $z_0$ in the latter equation represents the $z$-coordinate of the centroid of the contact area of the foundation. (See Fig. 4.1)

For surface-supported foundations the vector of the interface tractions $\{T_0(\vec{x}, \omega)\}$ is zero, and the right most term in the equation (4.5) vanishes. This term is generally non-zero for embedded foundations.
4.3.2 Extension to Stochastic Ground Motion

The three components of the free-field motion in this section are specified stochastically in terms of the PSD matrix 
$[S_e(\vec{x}_1, \vec{x}_2, \omega)]$, in which the diagonal elements, $S_{gxx}$, $S_{gyy}$ and 
$S_{gzz}$, represent the cross PSD functions for each of the three 
components of the ground motion at points $\vec{x}_1$ and $\vec{x}_2$, and the 
off-diagonal terms represent the cross PSD functions for pairs 
of different components of the motion at these points. In 
particular, $S_{gxz}$ represents the cross PSD function of the 
$x$-component of motion at point $\vec{x}_1$ and the $z$-component of motion 
at point $\vec{x}_2$.

Now, let $[S_T(\vec{x}_1, \vec{x}_2, \omega)]$ be the $3 \times 3$ matrix of the cross PSD 
functions for the components of the free-field tractions for 
points $\vec{x}_1$ and $\vec{x}_2$ of the half-space along the foundation-soil 
interface, and $[S_{Tg}(\vec{x}_1, \vec{x}_2, \omega)]$ be the $3 \times 3$ matrix of the cross 
PSD functions for the tractions at point $\vec{x}_1$ and the associated 
motions at point $\vec{x}_2$.

Finally, let $[S(\omega)]$ be the power spectral density matrix 
of size $6 \times 6$ for the foundation input motion with elements 
$S_{pq}(\omega)$, in which $p$ and $q$ are taken, for convenience, as integers 
with values of 1 to 6. The diagonal elements of this matrix 
represent the PSD functions of the three translational and 
three rotational components of the foundation input motion, 
whereas the off-diagonal elements represent the cross PSD 
functions of these motions. More specifically, $S_{11}$, $S_{33}$ and
\( S_{ss} \) refer to the translational components of motion along the \( x, y \) and \( z \) axes, respectively, whereas \( S_{22}, S_{44}, \) and \( S_{66} \) refer to the rotational components about the \( y, x \) and \( z \) axes, respectively. The point of reference for these motions is at the centroid of the foundation-soil interface at a depth \( z_0 \).

The relationship of \([S_\varphi(\overline{x}_1, \overline{x}_2, \omega)], [S_\tau(\overline{x}_1, \overline{x}_2, \omega)]\) and \([S_\tau(\overline{x}_1, \overline{x}_2, \omega)]\) to \([S(\omega)]\) may then be obtained by application of the expression (Vanmarcke 1983)

\[
[S(\omega)] = \frac{\pi}{t_0} E[\{U_0(\omega)\}^*\{U_0(\omega)\}^T]
\]  

(4.8)

in which a star superscript denotes the complex conjugate of the quantity to which it is attached; \( E[.] \) = the expected value of the bracketed quantity; and \( t_0 \) represents the duration of the stationary motion under consideration. On substituting equation (4.6) into equation (4.8) and noting that the matrices \([S_\varphi(\overline{x}_1, \overline{x}_2, \omega)], [S_\tau(\overline{x}_1, \overline{x}_2, \omega)]\) and \([S_\tau(\overline{x}_1, \overline{x}_2, \omega)]\) are related to the vectors \(\{U_\varphi\}\) and \(\{T_\tau\}\) by expressions analogous to equation (4.8), one obtains

\[
[S(\omega)] = [P] - [C][Q] - [Q^*][C^*]^T + [C][R][C^*]^T
\]

(4.9)

in which

\[
[P] = \int_A \int_A [F(\overline{x}_1, \omega)]^T [S_\varphi(\overline{x}_1, \overline{x}_2, \omega)][F^*(\overline{x}_2, \omega)]dA_1dA_2
\]

(4.10)

\[
[Q] = \int_A \int_A [\alpha(\overline{x}_1)]^T [S_\tau(\overline{x}_1, \overline{x}_2, \omega)][F^*(\overline{x}_2, \omega)]dA_1dA_2
\]

(4.11)

and
\[
[R] = \int_A \int_A [\alpha(\vec{x}_1)]^T [S_T(\vec{x}_1, \vec{x}_2, \omega)] [\alpha(\vec{x}_2)] dA_1 dA_2 \tag{4.12}
\]

The matrix \([P]\) arises solely from the first integral in the right hand member of equation (4.6), whereas the remaining terms stem from the second or the first and second integrals in this equation. The latter terms are present only for embedded foundations.

### 4.4 FOUNDATION INPUT MOTION

For the unidirectional free-field motion under consideration, the matrices \([S_\varphi(\vec{x}_1, \vec{x}_2, \omega)], \ [S_{T\varphi}(\vec{x}_1, \vec{x}_2, \omega)]\) and \([S_T(\vec{x}_1, \vec{x}_2, \omega)]\) are given by

\[
[S_\varphi(\vec{x}_1, \vec{x}_2, \omega)] = \begin{bmatrix}
0 & 0 & 0 \\
0 & S_{\varphi y}(\vec{x}_1, \vec{x}_2, \omega) & 0 \\
0 & 0 & 0
\end{bmatrix} \tag{4.13a}
\]

\[
[S_{T\varphi}(\vec{x}_1, \vec{x}_2, \omega)] = \begin{bmatrix}
0 & S_{T\varphi y y}(\vec{x}_1, \vec{x}_2, \omega) & 0 \\
0 & S_{T\varphi y y}(\vec{x}_1, \vec{x}_2, \omega) & 0 \\
0 & S_{T\varphi z y}(\vec{x}_1, \vec{x}_2, \omega) & 0
\end{bmatrix} \tag{4.13b}
\]

\[
[S_T(\vec{x}_1, \vec{x}_2, \omega)] = \\
\begin{bmatrix}
S_{Txx}(\vec{x}_1, \vec{x}_2, \omega) & S_{Txy}(\vec{x}_1, \vec{x}_2, \omega) & S_{Txz}(\vec{x}_1, \vec{x}_2, \omega) \\
S_{Tyx}(\vec{x}_1, \vec{x}_2, \omega) & S_{Tyy}(\vec{x}_1, \vec{x}_2, \omega) & S_{Tyz}(\vec{x}_1, \vec{x}_2, \omega) \\
S_{Tzx}(\vec{x}_1, \vec{x}_2, \omega) & S_{Tzy}(\vec{x}_1, \vec{x}_2, \omega) & S_{Tzz}(\vec{x}_1, \vec{x}_2, \omega)
\end{bmatrix} \tag{4.13c}
\]

The elements of the last two matrices are evaluated as follows: (1) The strain-field associated with \(U_{\varphi y}(\vec{x}, \omega)\) is determined from the strain-displacement relations; (2) use is then made of the stress-strain relations to obtain the
associated stress-field; These free-field stresses, represented by the vector \( T_0(\vec{x}, \omega) \), are determined for each of the five sides of the foundation-soil interface; (3) Finally, use is made of the definition of cross PSD function (analogous to one given by equation (4.8)) and of the linear properties of the differential and expectation operators to relate the elements of \([S_{T_0}(\vec{x}_1, \vec{x}_2, \omega)]\) and \([S_T(\vec{x}_1, \vec{x}_2, \omega)]\) to \(S_{yy}(\vec{x}_1, \vec{x}_2, \omega)\). A more detailed description of these steps is given in Appendix C.

In the Iguchi-Scanlan averaging technique, the supporting medium is effectively represented by a series of linear springs of the Winkler type, and the contact traction matrix \([F(\vec{x}, \omega)]\) is given by

\[
[F(\vec{x}, \omega)] = \begin{bmatrix}
1 & \frac{z-z_0}{a} \left( \frac{a^2 A}{I_y} \right) & 0 & 0 & 0 & -\frac{y}{a} \left( \frac{a^2 A}{I_z} \right) \\
\frac{1}{A} & 0 & 0 & 1 & -\frac{z-z_0}{a} \left( \frac{a^2 A}{I_x} \right) & 0 & \frac{x}{a} \left( \frac{a^2 A}{I_z} \right) \\
0 & -\frac{x}{a} \left( \frac{a^2 A}{I_y} \right) & 0 & y \left( \frac{a^2 A}{I_x} \right) & 1 & 0
\end{bmatrix}
\] (4.14)

in which \(A\) = the total contact area for the foundation; and \(I_x\), \(I_y\) and \(I_z\) are its moments of inertia about the centroidal axes parallel to the \(x\), \(y\) and \(z\) axes, respectively. The expressions for these quantities are given in Appendix C.
4.4.1 Elements of $[P]$, $[Q]$, $[R]$

On substituting $[S_0(\vec{x}_1, \vec{x}_2, \omega)]$ into equation (4.10), it can readily be shown that the several of the elements of $[P]$ are zero and that the non-zero elements are given by

$$P_{33} = \frac{1}{A^2} \int_A \int_A S_{yy}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2$$  \hspace{2cm} (4.15a)

$$P_{34} = P_{43} = \frac{-\alpha}{l_x} \int_A \int_A (z_2 - z_0) S_{yy}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2$$  \hspace{2cm} (4.15b)

$$P_{44} = \frac{a^2}{l_x^2} \int_A \int_A (z_1 - z_0)(z_2 - z_0) S_{yy}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2$$  \hspace{2cm} (4.15c)

$$P_{66} = \frac{a^2}{l_x^2} \int_A \int_A x_1 x_2 S_{yy}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2$$  \hspace{2cm} (4.15d)

Similarly, on substituting $[S_{T_0}(\vec{x}_1, \vec{x}_2, \omega)]$ into equation (4.11) and making use of equation (4.14), one obtains the following expressions for the non-zero elements of $[Q]$:

$$Q_{33} = \frac{1}{A} \int_A \int_A S_{T_{yy}}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2$$  \hspace{2cm} (4.16a)

$$Q_{34} = \frac{-\alpha}{l_x} \int_A \int_A (z_2 - z_0) S_{T_{yy}}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2$$  \hspace{2cm} (4.16b)

$$Q_{43} = \frac{-1}{\alpha A} \left( \int_A \int_A (z_1 - z_0) S_{T_{yy}}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2 \right.$$ \hspace{2cm}

$$\left. - \int_A \int_A y_1 S_{T_{yy}}(\vec{x}_1, \vec{x}_2, \omega) dA_1 dA_2 \right)$$  \hspace{2cm} (4.16c)
\[ Q_{44} = \frac{1}{I_x} \left\{ \int_A \int_A (z_1 - z_0)(z_2 - z_0) S_{T_{yy}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A y_1(z_2 - z_0) S_{T_{zy}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.16d) \]

\[ Q_{66} = \frac{1}{I_x} \left\{ \int_A \int_A x_1 x_2 S_{T_{yy}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_B y_1 x_2 S_{T_{zy}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.16e) \]

Finally, on substituting \([S_T(x_1, x_2, \omega)]\) and \([\alpha(x)]\) into equation (4.12), the following expressions are obtained for the non-zero elements of \([R]\):

\[ R_{11} = \int_A \int_A S_{T_{xx}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \quad (4.17a) \]

\[ R_{12} = R_{21}^* = \frac{1}{a} \left\{ \int_A \int_A (z_2 - z_0) S_{T_{xx}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A x_2 S_{T_{xz}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.17b) \]

\[ R_{22} = \frac{1}{a^2} \left\{ \int_A \int_A (z_1 - z_0)(z_2 - z_0) S_{T_{xx}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A x_2(z_1 - z_0) S_{T_{xz}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A x_1(z_2 - z_0) S_{T_{xz}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. + \int_A \int_A x_1 x_2 S_{T_{zz}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.17c) \]

\[ R_{33} = \int_A \int_A S_{T_{yy}}(x_1, x_2, \omega) \, dA_1 \, dA_2 \quad (4.17d) \]
\[ R_{34} = R_{43}^* = -\frac{1}{\alpha} \left\{ \int_A \int_A (z_2 - z_0) \{ x_1, x_2, \omega \} \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A y_2 S_{Tyz}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.17e) \]

\[ R_{44} = \frac{1}{\alpha^2} \left\{ \int_A \int_A (z_1 - z_0) \{ z_2 - z_0 \} S_{Tyy}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A y_2 (z_1 - z_0) S_{Tyz}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. + \int_A \int_A y_1 (z_2 - z_0) S_{Txy}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.17f) \]

\[ R_{55} = \int_A \int_A S_{Tzz}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \quad (4.17g) \]

\[ R_{66} = \frac{1}{\alpha^2} \left\{ \int_A \int_A y_1 y_2 S_{Txx}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. - \int_A \int_A x_2 y_1 S_{Txy}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. + \int_A \int_A y_2 x_1 S_{Tyx}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right. \]
\[ \left. + \int_A \int_A x_1 x_2 S_{Tyy}(\vec{x}_1, \vec{x}_2, \omega) \, dA_1 \, dA_2 \right\} \quad (4.17h) \]

### 4.4.2 Boundary Condition on Normal Stress

For the unidirectional free-field ground motion considered, the normal stress \( \sigma_z \) under free-field conditions at any depth is given by

\[ \sigma_z = \mu G \frac{\partial U_{gy}}{\partial y} \quad (4.18) \]
in which

\[ \mu = \frac{2\nu}{1-2\nu} \]  \hspace{1cm} (4.19)

Now considering that the incoherence function defined by equation (4.2) depends on both \( x \) and \( y \), it should be clear that the associated motion will generally violate the stress-free boundary condition at the surface. This discrepancy affects only the terms in equation (4.9) involving the matrices \([Q]\) and \([R]\). Inasmuch as these matrices did not appear in the solutions for the surface-supported foundations presented in Chapters II and III, this discrepancy was of no consequence in those solutions. However, when these matrices are considered, it will affect the solution of the embedded foundation as the limiting case of no embedment is approached. For the solutions presented in the following section, the free-field normal tractions at the base of the foundation-soil interface are taken as zero. It should be noted that this discrepancy does not arise for the one-dimensional incoherence function, as the stress-free boundary condition is automatically satisfied in this case.

4.4.3 Integration of Equations for \([P]\), \([Q]\), \([R]\)

For the incoherence function defined by equation (4.2), equations (4.15), (4.16) and (4.17) can be integrated exactly to yield closed form expressions in terms of standard error
function. In the following sections the matrices $[P]$, $[Q]$, $[R]$ will be normalized with respect to $S_{\gamma\gamma}(\omega)$, and will be denoted with a bar superscript.

**Expressions for Isotropic Incoherence Function.** The non-zero elements of $[\bar{P}]$ are obtained by integration of equations (4.15). The results are

$$
\bar{P}_{33} = \frac{\alpha^4}{A^2} \{M_1^2D_1^2 + 8M_1M_2D_1D_2 + 4(M_1M_3 + 2M_2^2)D_2^2\} \quad (4.20a)
$$

$$
\bar{P}_{34} = \bar{P}_{43} = -\frac{\alpha^6}{Al_x} \{M_1^2D_1D_3 + 4M_1M_2(D_1D_4 + D_2D_3) + 4(M_1M_3 + 2M_2^2)D_2D_4\} \quad (4.20b)
$$

$$
\bar{P}_{44} = \frac{\alpha^8}{I_x^2} \{M_1^2D_3^2 + 8M_1M_2D_3D_4 + 4(M_1M_3 + 2M_2^2)D_4^2\} \quad (4.20c)
$$

$$
\bar{P}_{66} = \frac{\alpha^8}{I_z^2} \{M_1M_4D_1^2 + 4(M_2M_4 + M_1M_5)D_1D_2 + 2(M_1M_6 + 4M_2M_5 + M_3M_4)D_2^2\} \quad (4.20d)
$$

in which

$$
M_1 = 4[2K_1 - K_2] \quad (4.21a)
$$

$$
M_2 = 2K_1 \quad (4.21b)
$$

$$
M_3 = [1 + \exp(-4b_0^2)] \quad (4.21c)
$$

$$
M_4 = \frac{4}{3}(2K_1 - K_2 - K_3) \quad (4.21d)
$$

$$
M_5 = 2(K_1 - K_2) \quad (4.21e)
$$

$$
M_6 = [1 - \exp(-4b_0^2)] \quad (4.21f)
$$
\[ K_1 = \frac{\sqrt{\pi}}{4b_0} \phi(2b_0) \]  
(4.22a)

\[ K_2 = \frac{[1 - \exp(-4b_0^2)]}{4b_0^2} \]  
(4.22b)

\[ K_3 = \frac{1}{2b_0^2} \left( 1 - \frac{1 - \exp(-4b_0^2)}{4b_0^2} \right) \]  
(4.22c)

\[ D_1 = \cos\left( \frac{e}{a} \alpha_0 \right) \]  
(4.23a)

\[ D_2 = \frac{1}{\alpha_0} \sin\left( \frac{e}{a} \alpha_0 \right) \]  
(4.23b)

\[ D_3 = \frac{h}{a} \frac{(e - z_0)}{a} \cos\left( \frac{e}{a} \alpha_0 \right) \]  
(4.23c)

\[ D_4 = \frac{h}{a} \left[ \frac{1}{\alpha_0} \sin\left( \frac{e}{a} \alpha_0 \right) \right] - \frac{1}{\alpha_0^2} \left[ 1 - \cos\left( \frac{e}{a} \alpha_0 \right) \right] \]  
(4.23d)

The error function, \( \phi(\cdot) \), is defined by equation (3.11); and \( h = e - z_0 \) is the height of the centroid of the contact area, measured from the center of the base of the embedded foundation.

The non-vanishing elements of matrix, \( \bar{Q} \) are similarly given by

\[ \bar{Q}_{33} = \left( \frac{a^2}{A} \right) G \alpha \left[ M_1^2 D_1 E_1 + 2(2M_2 E_1 + (3 + \mu) M_6 D_1) M_1 D_2 + 2(3 + \mu)(M_1 N_1 + 2M_2 M_6) D_2^2 \right] \]  
(4.24a)

\[ \bar{Q}_{34} = -\left( \frac{a^4}{I_x} \right) G \alpha \left[ M_1^2 D_3 E_1 + 2(2M_2 D_4 E_1 + (3 + \mu) M_6 D_2 D_3) M_1 + 2(3 + \mu)(M_1 N_1 + 2M_2 M_6) D_2 D_4 \right] \]  
(4.24b)
$$\overline{Q}_{43} = -\left( \frac{a^2}{A} \right) (G\alpha) \{(M_1^2 D_1 F_3 + 2(2M_2 D_2 F_3 + (3+\mu)M_6 D_1 D_4)M_1 + 2(3+\mu)(M_1 N_1 + 2M_2 M_6)D_2 D_4) - \left[ 2M_1 M_2 D_1 F_2 + 2(M_1 M_3 + 2M_2^2)D_2 F_2 \right]\} \quad (4.24c)$$

$$\overline{Q}_{44} = \left( \frac{a^4}{I_x} \right) (G\alpha) \{(M_1^2 D_3 F_3 + 2(2M_2 D_3 F_3 + (3+\mu)M_6 D_1)M_1 D_4 + 2(3+\mu)(M_1 N_1 + 2M_2 M_6)D_4^2) - \left[ 2M_1 M_2 D_3 F_2 + 2(M_1 M_3 + 2M_2^2)D_4 F_2 \right]\} \quad (4.24d)$$

$$\overline{Q}_{66} = \left( \frac{a^4}{I_x} \right) (G\alpha) \{(M_1 M_4 D_1 F_1 + 2(M_5 E_1 + N_2 E_1)M_1 D_2 + 2(M_2 E_1 + (2+\mu)M_6 D_1)M_4 D_2 + 2((2+\mu)M_4 N_1 - M_1 N_1 + 2M_2 N_2 + 2(2+\mu)M_5 M_6)D_2^2) - 4[(-M_2 + \mu M_5)M_5 D_1 D_2 + ((M_2 M_5 - \mu M_5 N_2) - (M_3 - \mu M_5)M_5 D_2^2)]\} \quad (4.24e)$$

in which

$$N_1 = 4b_0^2 \exp(-4b_0^2) \quad (4.25a)$$

$$N_2 = (M_3 - 2K_1) \quad (4.25b)$$

$$E_1 = a_0 \sin\left( \frac{e}{a} a_0 \right) \quad (4.26a)$$

$$E_2 = \left[ 1 - \cos\left( \frac{e}{a} a_0 \right) \right] \quad (4.26b)$$

$$E_3 = \frac{h}{a} \left[ a_0 \sin\left( \frac{e}{a} a_0 \right) \right] \quad (4.26c)$$

Finally, the non-vanishing elements of matrix $[\overline{R}]$ are given by
\[ \bar{R}_{11} = (G\alpha)^2 \{4(1+\mu)^2 M_6^2 D_2^2 \} \quad (4.27a) \]
\[ \bar{R}_{12} = \bar{R}_{21} = (G\alpha)^2 \{4[(1+\mu)^2 M_6 D_4 + (1+\mu)M_6 D_2]M_6 D_2 \} \quad (4.27b) \]
\[ \bar{R}_{22} = (G\alpha)^2 \{4(1+\mu)^2 M_6^2 D_4^2 + 8(1+\mu)M_5 M_6 D_4 E_2 + 2M_4 M_6 E_2^2 \} \quad (4.27c) \]
\[ \bar{R}_{33} = (G\alpha)^2 \{M_1^2 E_1^2 + 4(3+\mu)M_1 M_6 D_2 E_1 + (2(1+(2+\mu)^2)M_1 N_3 + 8(2+\mu)M_6^2 D_2^2 \} \quad (4.27d) \]
\[ \bar{R}_{34} = \bar{R}_{43} = (G\alpha)^2 \{-[M_1^2 E_1 E_3 + 2(3+\mu)(D_4 E_1 + D_2 E_3)M_1 M_6 + (2(1+(2+\mu)^2)M_1 N_3 + 8(2+\mu)M_6^2 D_2 D_4) + [2M_1 M_2 E_1 E_2 + (2(2+\mu)M_1 N_1 + 4M_2 M_6)D_2 E_2] \} \quad (4.27e) \]
\[ \bar{R}_{44} = (G\alpha)^2 \{[M_1^2 E_3^2 + 4(3+\mu)M_1 M_6 D_4 E_3 + (2(1+(2+\mu)^2)M_1 N_3 + 8(2+\mu)M_6^2 D_4^2)] + [2M_1 M_2 E_2 E_3] - 2[2M_1 M_2 E_1 E_3 + (2(2+\mu)M_1 N_1 + 4M_2 M_6)D_4 E_2] \} \quad (4.27f) \]
\[ \bar{R}_{55} = (G\alpha)^2 \{2M_1 M_6 E_2^2 \} \quad (4.27g) \]
\[ \bar{R}_{66} = (G\alpha)^2 \{(M_1 M_4 E_1^2 + 4(M_1 N_2 + (2+\mu)M_4 M_6)D_2 E_1 + (2M_1 N_4 + 2(2+\mu)M_5 M_6 N_3 + 8(2+\mu)M_6 N_2)D_2^2)] + [2(2+\mu)M_6 N_6 + 2M_3 M_6 + 4\mu M_6 N_2)D_2^2)] - 8[(-M_2 + \mu M_5)M_5 D_2 E_1 + (-2(1+\mu)M_5 N_1 + M_2 N_1 + \mu(2+\mu)M_5 N_5)D_2^2] \} \quad (4.27h) \]

in which
\[ N_3 = 2b_0^2(M_6 - 2M_7) \quad (4.28a) \]
\[ N_4 = 2b_0^2(M_3 - 2M_7) \quad (4.28b) \]
\[ N_5 = (M_6 - M_7) \quad (4.28c) \]
\[ \bar{Q}_{43} = -\left( \frac{a^2}{A} \right) (Ga) \{ [4M_1D_1E_3 + 4(M_1 + 2M_2)D_2E_3 + 8M_6D_1D_4 + 8(M_6 + N_1)D_2D_4] - [4M_1D_1E_2 + 4(M_1 + 2M_2)D_2E_2] \} \]  
(4.30c)

\[ \bar{Q}_{44} = \left( \frac{a^4}{I_x} \right) (Ga) \{ [4M_1D_3E_2 + 2(M_1 + 2M_2)D_4E_3 + 8M_6D_3D_4 + 8(M_6 + N_1)D_4^2] - [4M_1D_3E_2 + 4(M_1 + 2M_2)D_4E_2] \} \]  
(4.30d)

\[ \bar{Q}_{66} = \left( \frac{a^4}{I_z} \right) (Ga) \{ [4M_4D_1E_1 + 2(M_4 + 2M_5)D_2E_1 + 8N_2D_1D_2 + 8(N_2 - N_1)D_2^2] - [-8M_6D_1D_2 + 8(M_6 - M_5)D_2^2] \} \]  
(4.30e)

The non-vanishing elements of the matrix [\( \bar{R} \)] are

\[ \bar{R}_{33} = (Ga)^2[4M_1E_1^2 + 16M_6D_2E_1 + 8N_3D_2^2] \]  
(4.31a)

\[ \bar{R}_{34} = \bar{R}_{43} = (Ga)^2\{ -[4M_1E_1E_3 + 8M_6(D_4E_1 + D_2E_3) + 8N_3D_2D_4] \} + [4M_1E_1E_2 + 8M_6D_2E_2] \]  
(4.31b)

\[ \bar{R}_{44} = (Ga)^2\{ [4M_1E_3^2 + 16M_6D_4E_3 + 8N_3D_4^2] + [4M_1E_2^2] - 2[4M_1E_2E_3 + 8M_6D_4E_2] \} \]  
(4.31c)

\[ \bar{R}_{66} = (Ga)^2\{ [4M_4E_1^2 + 16N_2D_2E_1 + 8N_4D_2^2] + [8M_6D_2^2] \} - 2[-8M_6D_2E_1 + 8N_1D_2^2] \} \]  
(4.31d)

It should be noted that unlike the results for the two-dimensional, isotropic incoherence that are functions of Poisson's ratio for the medium, those for the one-dimensional incoherence are independent of \( v \).
4.4.4 Compliance Matrix \([C]\)

This matrix was determined from the approximate expressions for the foundation impedances presented by Pais and Kausel (1988). Inasmuch as the force-displacement relations in the latter solutions are given for a point at the center of the foundation base, these relations must first be transformed to the center of gravity of the foundation, located at point \((0, 0, z_0)\). The transformed expressions are given in Appendix C. These expressions are functions of the frequency parameter, \(\alpha_0\); the embedment ratio, \(e/\alpha\), and Poisson's ratio for the soil medium, \(\nu\).

4.4.5 Computation of Foundation Input Matrix, \([S]\)

With the matrices \([P]\), \([Q]\), \([R]\) and \([C]\) established, the matrix \([S]\) for the foundation input motion is determined from equation (4.9). It should be recalled that this matrix is defined for the centroid of the foundation soil interface. When referred to the center of the foundation base at \((0, 0, e)\), the elements of the resulting matrices identified with the a prime superscript, may be determined from

\[
S'_{11} = S_{11} + \frac{h}{\alpha} (S_{12} + S_{21}) + \left( \frac{h}{\alpha} \right)^2 S_{22} \tag{4.32a}
\]

\[
S'_{12} = S'_{21} = S_{12} + \frac{h}{\alpha} S_{22} \tag{4.32b}
\]
\[ S'_{33} = S_{33} - \frac{\bar{n}}{a} (S_{34} + S_{43}) + \left( \frac{\bar{n}}{a} \right)^2 S_{44} \]  
\[ S'_{34} = S'_{43} = S_{34} - \frac{\bar{n}}{a} S_{44} \]

The elements \( S'_{22}, S'_{44}, S'_{55} \) and \( S'_{66} \) in this case will be the same as \( S_{22}, S_{44}, S_{55} \) and \( S_{66} \), and other off-diagonal terms of \([S']\) like those of \([S]\) will be zero.

4.4.6 Presentation of Numerical Solutions

For the isotropic incoherence function considered, the matrices \([P]\), \([Q]\) and \([R]\) in equations (4.10-4.12) depend on the modified frequency parameter, \( \gamma a_0 \), the embedment ratio, \( e/a \), and Poisson's ratio for the supporting medium, \( \nu \). On the other hand, the compliance matrix, \([C]\), depends on \( a_0, e/a \) and \( \nu \). Accordingly, the foundation input matrices, \([S]\) and \([S']\), depend separately on \( \gamma, a_0, e/a \) and \( \nu \). For the solutions presented herein, Poisson's ratio for the soil is taken as \( \nu = 0.25 \).

The quantities \( \sqrt{S'_{pp}/S_{yy}} \) (\( p = 1, 2, \ldots, 6 \)) represent the transfer functions for the foundation input displacements. Of the six displacement components involved, the three most important of these functions are for: (1) the lateral component along an axis parallel to the \( y \)-axis, \( \sqrt{S'_{33}/S_{yy}} \); (2) the rocking component about the \( x \)-axis, \( \sqrt{S'_{44}/S_{yy}} \); and (3) the torsional component about the \( z \)-axis, \( \sqrt{S'_{66}/S_{yy}} \). The rocking component is induced
by the reactive pressures along the vertical faces of the foundation that are normal to the direction of free-field ground motion, whereas the torsional component is induced mainly by the incoherence of the ground motion at the interface of the foundation base and soil.

Figures (4.2) and (4.3) display the transfer function, $\sqrt{S_{00}/S_{0y}}$, for the translational component of foundation motion in the direction of the free-field ground motion for two-dimensional isotropic incoherence. The results are plotted against the frequency parameter, $a_0$, in two different formats. In Figure (4.2) they are displayed for six different values of the incoherence parameter, $\gamma$, with embedment ratio, $e/a$ taken as 0.5 and 1, whereas in Figure (4.3) they are displayed for five different values of $e/a$ with $\gamma$ taken as 0.2 and 0.4. The curves for $\gamma=0$ represent the wave passage effect for vertically propagating coherent waves, whereas the differences between these curves and those corresponding to non-zero values of $\gamma$ represent the effects of ground motion incoherence.

As would be expected, the lateral component of foundation input motion generally decreases with increasing values of $a_0$, $e/a$ and $\gamma$. Note further that the effect of ground motion incoherence decreases with increasing embedment ratio, and that increasing $\gamma$ reduces the oscillations of the resulting curves. By contrast, increasing $e/a$ increases the oscillations.
Figures (4.4) and (4.5) present the transfer functions for the rocking component of foundation input motion about x-axis, \( \sqrt{S_{44}/S_{yy}} \). This component increases with increasing \( \alpha_0 \), and reaches a peak at frequency that seems to correspond to a constant value of

\[
\alpha_0 = \frac{\omega e}{\nu_s}
\]

If the results in Figure (4.5) were plotted against this parameter, rather than \( \alpha_0 \), the curves would have been closer together. The same would also be true of the curves for the lateral component of motion presented in Figure (4.3). The peaks of the curves in Figures (4.4) and (4.5) decreases with increasing \( \gamma \) and increases with increasing \( e/\alpha \), and the greater the value of \( e/\alpha \), the more oscillatory the resulting curves are.

The transfer functions for the torsional components of foundation motion, \( \sqrt{S_{66}/S_{yy}} \), are displayed in Figures (4.6) and (4.7). The peaks of the curves in this case decreases with increasing \( e/\alpha \) and increases with the increasing \( \gamma \). All the other trends are generally similar to those for rocking components presented in Figures (4.4) and (4.5).

Figures (4.8), (4.10) and (4.9) display the transfer functions for the translational motions along the x and z axes, and for the rocking components about the y-axis. The motions represented by these functions are generally small.
Even for the largest value of $\gamma = 0.5$ considered, the peak amplitudes are only 10 percent of the corresponding amplitude of the free-field, control point motion.

The magnitude of $S_{34}/\sqrt{S_{33}S_{44}}$ defines the degree of correlation or coherence between the lateral component of the foundation motion along the $y$-axis and rocking component of motion about a horizontal axis through the center of the foundation base parallel to the $x$-axis. A numerical value of unity for the latter quantity indicates that the component motions are fully correlated, while a zero value indicates that they are uncorrelated. Note that the coherence between these components decreases with increasing $\gamma$.

4.4.7 Results for One-Dimensional Incoherence Function

Figures (4.12-4.14) present the plots similar to those presented in Figures (4.4), (4.6) and (4.8) for a one-dimensional incoherence function (i.e., $\gamma_x = \gamma_y = 0$). For this form of incoherence, the component of motion along the $y$-axis, the rocking component about the $x$-axis and the torsional component about the vertical $z$-axis, are the only components of foundation motion induced. A comparison of the curves for lateral and rocking components of the foundation motion in Figures (4.12) to (4.13) with the corresponding curves in Figures (4.4) and (4.6) reveals that the effects of ground motion incoherence for the one-dimensional function are
substantially less than those for two-dimensional, isotropic function. By contrast, a comparison of curves presented in Figure (4.14) with corresponding curves in Figure (4.8) reveals that opposite is true for torsional component of foundation motion. The differences are largest for surface-supported foundations and decrease with increasing embedment ratio.

4.5 CONCLUSIONS

Relatively simple, closed-form expressions and extensive numerical data have been presented for the transfer functions of square, embedded foundations subjected to a spatially varying free-field horizontal ground motion. These results presented elucidate the effects of the numerous parameters involved.
Fig. 4.1 System Considered
Fig. 4.2 Transfer Functions for $y$-Component of Foundation Motion; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.3 Effect of Embedment Ratio on Transfer Functions for $y$-Component of Foundation Motion; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.4 Transfer Functions for Rocking Component of Foundation Motion about x-Axis; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.5 Effect of Embedment Ratio on Transfer Functions for Rocking Component of Foundation Motion About x-Axis; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.6 Transfer Functions for Torsional Component of Foundation Motion about z-Axis; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.7 Effect of Embedment Ratio on Transfer Functions for Torsional Component of Foundation Motion About z-Axis; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.8 Transfer Functions for Foundation Motion Along x-Axis; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.9 Transfer Functions for Rocking Component of Foundation Motion about y-Axis; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.10 Transfer Functions for z-Component of Foundation Motion; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.11 Normalized Cross PSD Functions for Horizontal and Rocking Components of Foundation Motion; Isotropic Incoherence Function, $\gamma_x = \gamma_y = \gamma$
Fig. 4.12 Effect of Embedment Ratio on Transfer Functions for $y$-Component of Foundation Motion; One-Dimensional Incoherence Function, $\gamma_y = 0$
Fig. 4.13 Effect of Embedment Ratio on Transfer Functions for Rocking Component of Foundation Motion About x-Axis; One-Dimensional Incoherence Function, $\gamma_y = 0$
Fig. 4.14 Effect of Embedment Ratio on Transfer Functions for Torsional Component of Foundation Motion About z-Axis; One-Dimensional Incoherence Function, $\gamma_y = 0$
V SEISMIC INTERACTION OF SIMPLE STRUCTURES ON SQUARE EMBEDDED FOUNDATIONS

5.1 INTRODUCTION

This chapter is concerned with the effects of foundation embedment on the response of simple structures subjected to earthquake ground motions. Both kinematic and inertial interaction effects are examined. The structures are considered to have one lateral and one torsional degree of freedom in their fixed-base condition and to be excited by a non-uniform, horizontal free-field ground motion. This motion is specified stochastically by means of a space-invariant, local PSD function and an incoherence function.

The response quantities examined include the ensemble means of the peak values of the lateral and rocking components of foundation input motion and of the resulting structural deformations. These deformations are displayed in the form of pseudo-velocity response spectra and compared, over wide ranges of the parameters involved, with those obtained for no soil-structure interaction and for kinematic interaction only.

As indicated in Chapter IV, foundation embedment tends to reduce the torsional component of input motion induced by the incoherence of ground motion, and this reduction will also reduce the torsional response of the structure. The latter response is not examined herein.
5.2 SYSTEM CONSIDERED

The system investigated is shown in Figure (5.1). It is similar to the one considered in Chapter II, except that it is square in plan, with sides $2a \times 2a$, and its foundation is embedded to a depth $e$. The total height of the structure is denoted by $h$. The structural and foundation masses, $m$ and $m_f$, are assumed to be uniformly distributed over identical areas. The supporting medium is characterized by its mass density, $\rho$, shear wave velocity, $v_s$, and Poisson's ratio, $\nu$.

The free-field ground motion is conceived to be due to horizontally polarized, vertically incident, incoherent wave trains. Following the approach followed in previous chapters, this motion is specified stochastically in terms of a local power spectral density (psd) function, $S_\omega(\omega)$, and a cross psd function, $S(x_1,x_2,\omega)$. The local psd function for the acceleration traces of the free-field ground is taken in the form defined by equation (2.17) of Chapter II, with a cut-off frequency taken as $f_0 = 15$ cps. The duration of the intense portion of the seismic excitation was presumed to be $t_0 = 20$ seconds. The cross psd function is taken in the form defined by equations (4.1-4.2) in Chapter IV. The numerical solutions presented in this chapter are mainly for the two-dimensional, isotropic incoherence function, although some results are also included for the one-dimensional function.
5.3 KINEMATIC INTERACTION EFFECTS

5.3.1 Foundation Input Motion

Before examining the lateral response of the structure, it is desirable to evaluate the ensemble peak values of the horizontal and rocking components of the foundation input motion. The rocking component, \( \phi_0(t) \), will be expressed by the linear displacement \( z(t) = a\phi_0(t) \).

Let \( X \) and \( Z \) be the ensemble means of the absolute maximum values of the horizontal component of foundation input displacement, \( x(t) \), and of the displacement \( z(t) \), respectively, and let \( X_v \) be the corresponding value of the free-field control point displacement. These quantities may be computed from their corresponding psd functions making use of Der Kiureghian's approximation (1980).

Figure (5.2) displays the results obtained for a square foundation with \( e/a = 0.5 \) and an isotropic incoherence function, considering several different values of \( \gamma \). The results are normalized with respect to \( X_v \), and plotted against the transit time, \( \tau = a/v_z \). Also shown are the ratios of the mean peak values of the corresponding velocity and acceleration histories. The latter quantities are identified with one and two dot superscripts, respectively.

The plots in Figure (5.2) are similar to those presented in Figure (2.6) for surface-supported circular foundations,
and the conclusions and inferences drawn from the previous plots also apply in this case. Specifically, the reduction in the lateral component of foundation input motion and the corresponding increase in the rocking component are greatest for acceleration, much smaller for velocity, and almost negligible for displacement. Considering that high-frequency systems are acceleration-sensitive, the effects of foundation embedment and ground motion incoherence would be expected to be most pronounced for high-frequency systems. It is of some interest to note that, for small values of transit time, ground motion incoherence may increase the mean peak value of the rocking component of foundation input motion.

Figure (5.3) shows the effect of embedment ratio, $e/a$, on the values of $\ddot{x}/\ddot{x}_g$ and $\ddot{z}/\ddot{x}_g$ for three different transit times and a value of $\gamma = 0.2$. Note that, for a fixed transit time, $\ddot{x}/\ddot{x}_g$ decreases with increasing $e/a$, whereas $\ddot{z}/\ddot{x}_g$ increases.

Figure (5.4) displays similar results for a value of $\tau = 0.05$ sec and three different values of $\gamma$. It is clear from these plots that the relative reduction due to incoherence in the mean peak values of the lateral component of foundation input motion is higher for shallow foundations than for deep foundations. It is also observed that the mean peak acceleration of the rocking component of motion for shallow foundations ($e/a < 0.25$) is small and practically independent of $\gamma$, whereas for deep foundations it decreases with increasing $\gamma$. 
5.3.2 Spectral Characterization of Structural Response

Let \( S_\xi \) and \( S_\zeta \) be the psd functions of the accelerations for the lateral and rocking components of FIM, respectively, and let \( S_{\xi\zeta} \) be the corresponding cross-psd function. These functions are related to the functions \( S'_{33}(\omega) \), \( S'_{44}(\omega) \) and \( S'_{34}(\omega) \) presented in Chapter IV as follows.

\[
S_\xi = \omega^4 S'_{33}(\omega) \hspace{1cm} (5.1a)
\]
\[
S_\zeta = \omega^4 S'_{44}(\omega) \hspace{1cm} (5.1b)
\]
\[
S_{\xi\zeta} = \omega^4 S'_{34}(\omega) \hspace{1cm} (5.1c)
\]

Further, let \( S_u \) be the psd function of the lateral structural deformation, \( u \), induced by the lateral and rocking components of the foundation input motion. The latter function may then be determined from (Lin 1976)

\[
S_u = |H_u|^2 \left[ S_\xi + 2(h/a)R(S_{\xi\zeta}) + (h/a)^2 S_\zeta \right] \hspace{1cm} (5.2)
\]

in which \( R \) denotes the real part of the quantity that follows, and \( H_u \), the transfer function for lateral response, is given by equation (2.15). The vertical bars indicate the modulus of the enclosed quantity.

5.3.3 Effects of Ground-Motion Incoherence on Structural Response

Let \( U_\alpha \) be the ensemble mean of the maximum values of the structural deformations induced by the lateral and rocking
components of foundation input motion, and \( V_x \) be the corresponding pseudo-velocity, defined as \( V_x = p_x U_x \), in which \( p_x = 2\pi f_x \) = the fixed-base circular natural frequency of the system in the lateral mode of vibration, and \( f_x \) = the associated frequency in cycles per second.

Figure (5.5) presents response spectra for \( V_x \), normalized with respect to the mean peak value of the ground velocity of the control point, \( V_0 \). Two values of the embedment ratios, \( e/a \), and two values of the incoherence parameter, \( \gamma \), are considered. The damping factor for the structure in its fixed-base condition in these solutions is taken as \( \zeta_x = 0.02 \), and the transit time, \( \tau \), is taken as 0.05 seconds. The latter value may correspond to, say, \( a = 50 \text{ ft} \) and \( u_s = 1000 \text{ ft/sec} \). The solid curves, corresponding to \( \gamma = 0 \), represent the results obtained for coherent ground motions. The spectra in the upper part of the figure are for relatively short structures with \( h/a = 1 \), whereas those in the lower part are for structures with \( h/a = 3 \).

As anticipated from the information presented in Figure (5.2), kinematic interaction has a negligible effect on the response of systems in the low-frequency, displacement-sensitive region of the spectrum; only a moderate effect on the response of systems in the medium-frequency, velocity-sensitive spectral region; and a generally significant effect on the response of systems in the high-frequency,
acceleration-sensitive spectral region, particularly close to frequencies for which the pseudo-acceleration, \( A_x = p_x V_x \), attains its maximum value. For example, for a system with \( h/a = 1 \) and values of \( \gamma = 0, e/a = 0.25 \) and \( \tau = 0.05 \) sec., the maximum value of \( A_x \) is 84 percent of that obtained for a surface-supported structure in the absence of any kinematic interaction. For \( \gamma = 0.4, e/a = 0.25 \) and \( \tau = 0.05 \) sec., and for \( \gamma = 0.4, e/a = 0 \) and \( \tau = 0.05 \) sec., the corresponding values are 62 and 71 percent, respectively. These percentages are comparable to the normalized mean peak values of the lateral component of foundation input accelerations presented in Figure (5.4).

**Effect of embedment ratio.** Figure (5.6) displays response spectra similar to those presented in Figure (5.5) for four different values of the embedment ratio, \( e/a \). Included for purposes of comparison are also the results obtained for \( \gamma = e/a = 0 \), i.e., surface-supported structures subjected to the free-field, control point motion. It is important to note that, for relatively stubby structures with \( h/a = 1 \), the spectral value of structural response decreases consistently with increasing embedment ratio, and the largest reductions being attained at frequencies close to those for which the pseudo-acceleration for the non-interacting system attains its maximum value. By contrast, for more slender structures with \( h/a = 3 \), the reductions are considerably less pronounced,
and there is a cross over of the spectra for $e/\alpha = 0.5$ and $e/\alpha = 1.0$. This trend is a consequence of the increasing importance of the rocking component of foundation input motion for deeply embedded foundations and tall, slender structures.

For systems with $h/\alpha = 3$, $\gamma = 0.4$, $e/\alpha = 0.5$ and $\tau = 0.05$ sec, the maximum value of the pseudo acceleration, $A_x$, is 62 percent of that obtained for a similar surface-supported structure without interaction. It is noteworthy that this value is close to the mean peak value of the combination of the lateral component of foundation input accelerations and $h$ times the rocking component of the corresponding accelerations. Estimated by the RMS rule the combined value turns out to be 59.5 percent of the free-field acceleration at the control point.

**Sensitivity of Response to Ground Motion Incoherence Function.** As a measure of the sensitivity of the mean maximum response of the structure to the form of the incoherence function employed, the response spectra for the isotropic incoherence presented in Figure (5.5) are compared in Figure (5.7) with those obtained for a one-dimensional function with same value of $\gamma$. As would be expected, the reductions in response are smaller for the one-dimensional incoherence function. In particular, the reduction in the absolute maximum value of the pseudo-acceleration decreases from 39% for the two-dimensional function to 28% for the one-dimensional function.
5.4 INERTIAL INTERACTION EFFECTS

The inertial interaction effects were evaluated by the procedure outlined in the latter part of Chapter II, using the approximate, closed form expressions for the foundation impedances presented by Pais and Kausel (1988) and summarized in Appendix C.

The dimensionless parameters governing the response of the system have already been identified in Chapter II, and will not be repeated here. It should be noted, however, that one must also consider the embedment ratio, \( e/a \). The mass density ratio for the structure in this case is defined as \( \delta = m/4\pi a^2 h \), in which the denominator represents the total mass of the structure when filled with the supporting soil. For the solutions presented herein, the following values of the dimensionless parameters are used: \( \zeta_x = 0.02 \), \( \delta = 0.15 \), \( \nu = 0.25 \), \( \tau = 0.05 \) sec., and \( m_f \) (and hence \( l_f \)) are considered negligible.

As a measure of the relative importance of kinematic and inertial interaction effects, in Figure (5.8) are shown the response spectra for the lateral response obtained for systems with \( h/a = 1 \) and 3, \( e/a = 0.25 \), and \( \gamma = 0.4 \).

The following three sets of solutions are displayed: (1) making no provision for soil-structure interaction, i.e., considering the foundation motion to be the same as the free-field control point motion; (2) providing only for the kinematic interaction effects, i.e., using as base excitation
the foundation input motion and analyzing the superstructure without regard for the flexibility of the supporting medium; and (3) providing for both kinematic and inertial interaction effects, i.e., analyzing the structure-foundation soil system exactly as a coupled system. Also included in Figure (5.8) are the results obtained by the simplified analysis presented in the latter part of Chapter II. The base excitation in the latter analysis is defined by the translational and rocking components of the foundation input motion, and the modified or effective natural frequency and damping of the system are determined appropriately in the manner indicated in Chapter II. The latter frequencies and damping factors are identified with a tilde superscript, and their values for the systems considered are displayed in Figure (5.9). The trends in these plots are similar to those presented in Figure (2.10) and need no elaboration.

From the data presented in Figure (5.8), the following conclusions, also deduced in Chapter II from analyses of structures supported on circular foundations, are drawn: (1) The response of systems in the medium- and high- frequency spectral regions may be influenced significantly not only by kinematic interaction but by inertial interaction as well; (2) The inertial interaction effects are generally more important than kinematic interaction effects; and (3) the
simplified method of analysis, in which the inertial inter-
action effects are provided simply by changing the fixed-base
natural frequency and the associated damping of the system,
is also applicable to structures with embedded foundations.

5.5 CONCLUSIONS

The principal conclusions of the study reported in this
chapter may be summarized as follows:

1. The information presented herein provide valuable
insights into the effects of foundation embedment and ground
motion incoherence on the response of simple structures
subjected to earthquake ground motions.

2. For short, stubby structures for which the response is
dominated by the translational component of foundation input
motion, kinematic interaction reduces the lateral response of
the structure, the magnitude of reduction increasing with
increasing embedment and increasing ground motion incoherence.
In particular, the reduction due to a two-dimensional, iso-
tropic incoherence function is greater than that due to a
one-dimensional function having the same incoherence factor.
By contrast, for more slender, tall structures the reductions
in response are generally smaller, and an increase in foundation
embedment may well increase the response. This is due to the
increasing importance of the foundation rocking in such
structures.
3. The kinematic interaction effects for structures with embedded foundations may be estimated from the peak values of the foundation input motion in a manner analogous to that described in Chapter II for surface-supported structures.

4. The simple technique of approximating the effects of inertial interaction by changes in the effective natural frequency and the associated damping of the structure has shown to be applicable to structures with embedded foundations as well.
Fig. 5.1 System Considered
Fig. 5.2 Normalized Mean Peak Values of Lateral and Rocking Components of Foundation Input Accelerations, Velocities and Displacements; (e/a = 0.5)
Fig. 5.3 Normalized Mean Peak Values of Lateral and Rocking Components of Foundation Input Acceleration; ($\gamma = 0.2$)
Fig. 5.4 Normalized Mean Peak Values of Lateral and Rocking Components of Foundation Input Acceleration; ($\tau = 0.05$ sec)
Fig. 5.5 Effects of Ground Motion Incoherence on Maximum Deformation of Structures with $\varepsilon_x = 0.02$; $\tau = 0.05$ sec
Fig. 5.6 Effects of Embedment Ratio on Maximum Deformation of Structures with $x = 0.02; \tau = 0.05$ sec
Fig. 5.7 Sensitivity of Maximum Deformation of Structures with $\zeta_x = 0.02$; to the forms of Incoherence Function; ($\gamma = 0.4$)
Fig. 5.8 Comparison of Effects of Kinematic and Inertial Interaction on Maximum Deformations of Structures with $\zeta_x = 0.02$; $e/a = 0.25$
Fig. 5.9 Natural Frequencies and Damping of Modified Systems in Approximate Analysis of Inertial Interaction Effects (Systems with e/a = 0.25)
VI FLUID-STRUCTURE INTERACTION EFFECTS FOR OFFSHORE STRUCTURES

6.1 INTRODUCTION

Wave forces on an offshore structure are normally computed by one of the following variants of Morison's equation: (1) the standard version, in which the structure is effectively presumed to be rigid and the drag components of the wave forces are taken proportional to the square of the fluid particle velocities; and (2) the generalized or extended version, in which the drag force components are considered to be proportional to the square of the relative velocities of the fluid and structure. The difference in the responses of a structure computed by these two representations of the forcing function will be referred to herein as the fluid-structure interaction effect.

In a direct, numerical evaluation of the response of the structure as a function of time, there is no fundamental difficulty in providing for these effects. This approach, however, is generally too tedious and costly for preliminary design purposes, and simpler techniques are needed to define the conditions under which the effects are of sufficient importance to warrant their consideration in design and to evaluate them reliably and cost-effectively. Two such techniques have already appeared in the literature. They are: (1)
the Malhotra-Penzien extension (Malhotra and Penzien 1970) of Borgman's linearization technique for the drag component of the exciting force (Borgman, 1967); (2) Penzien's decoupling technique (Penzien 1976; Penzien and Tseng 1978).

Notwithstanding these and several other contributions (Foster 1970; Dao and Penzien 1982; Gudmestad and Connor 1983; Krolikowski and Gay 1980; Nagashima 1981; Shyam Sunder and Connor 1982; Spanos and Chen 1981), there is need for a re-examination of the problem from a unified point of view, and for a critical assessment of the effects of the numerous factors involved. This study is intended to be responsive to this need. Its objectives are: (1) to elucidate the nature of the fluid-structure interaction phenomenon; (2) to assess the interrelationship, accuracy and ranges of applicability of the previously proposed simple procedures for evaluating its effects; and (3) where necessary, to recommend appropriate improvements.

Comprehensive parametric studies are made of the exact maximum responses induced in simple mass-spring-dashpot systems by different combinations of a simulated wave train and constant velocity current, and the results are compared with those obtained by the previously proposed approximate procedures or appropriate extensions of them. The problem parameters examined include the natural frequency and percentage of critical damping of the system, the relative
magnitudes of the drag and inertia components of the exciting force, the ratio of current velocity to the peak value of the wave-induced fluid particle velocity, and a dimensionless measure of importance of fluid-structure interaction. The results are displayed in the form response spectra for the absolute maximum displacement of the system. It is shown that the accuracy of the approximate techniques depends importantly on the relative magnitudes of the drag and inertia components of wave loading, and that these procedures may lead to substantial errors under certain conditions. The reasons for these discrepancies are identified, and simple modifications are proposed which improve the reliability of these techniques.

In addition to the response of single-degree-of-freedom systems, the response of multi-degree-of-freedom cantilever systems is examined, and a simple approximation is proposed for the effective modal damping of such systems.

Although the reliability of both the standard and extended versions of Morison's equation has been questioned (Laya et al. 1984; Sarpkaya and Isaacson 1982), their use in practice is so widespread that it is considered important that the interrelationship of the responses obtained from two forms of this equation be clarified.

6.2 STATEMENT OF PROBLEM

Consideration is first given to the response of single-
degree-of-freedom, mass-spring-dashpot systems submerged in an oscillating fluid for which the particle velocity is \( \dot{u}(t) = \dot{\bar{u}} \) and the associated acceleration is \( \ddot{u}(t) = \ddot{\bar{u}} \). Let \( m \) be the effective mass of the system, including the contribution of added fluid mass, and let \( k \) and \( c \) be its stiffness and coefficient of damping, respectively.

With the hydrodynamic force defined by the generalized version of Morison's equation, the equation of motion for the system may conveniently be expressed in the form

\[
m \ddot{x} + c \dot{x} + kx = P_i \frac{\ddot{u}}{\bar{u}_0} + P_d \frac{|\ddot{u} - \dot{x}|}{\bar{u}_0} \frac{(\ddot{u} - \dot{x})}{\bar{u}_0} \tag{6.1}
\]

in which \( x \) denotes the displacement of the system; a dot superscript denotes differentiation with respect to time; \( \bar{u}_0 \) and \( \dot{\bar{u}}_0 \) are the absolute maximum values of \( \ddot{u} \) and \( \dot{u} \); and \( P_i \) and \( P_d \) are the corresponding values of the inertia and drag components of the exciting force computed without regard for interaction. The latter values are given by

\[
P_i = C_m \rho V \ddot{u}_0 \tag{6.2a}
\]

and

\[
P_d = \frac{1}{2} C_d \rho A \dot{u}_0^2 \tag{6.2b}
\]

in which \( C_m \) and \( C_d \) are the inertia and drag force coefficients; \( \rho \) is the mass density of the fluid; and \( V \) and \( A \) are the volume and projected area of the system, respectively. The coefficients \( C_m \) and \( C_d \) are presumed to be constants.
Of concern here is the difference in the responses of the system computed on the following bases: (1) considering \( P(t) \) to be defined by the right-hand member of equation (1); and (2) using the following simpler expression for it:

\[
P(t) = P_i \frac{\ddot{u}}{\dot{u}_0} + P_d \frac{|\ddot{u}| \ddot{u}}{\dot{u}_0 \dot{u}_0}
\]  \hspace{1cm} (6.3)

Of particular interest is the difference in the absolute maximum values of the resulting displacements, \( |x_{\text{max}}| \). These displacements will be normalized with respect to \( x_{st} \), the static displacement induced by the peak value of the wave loading. The interaction effects for more complex, multi-degree-of-freedom systems are examined at the end of the chapter.

6.3 PROBLEM PARAMETERS

The normalized response of the system to a prescribed sea state depends on the following parameters: (1) the relative magnitudes of the drag and inertia components of the exciting force, expressed conveniently by the factor

\[
\alpha = \frac{P_d}{P_i + P_d}
\]  \hspace{1cm} (6.4)

(2) the dynamic properties of the system, including its natural frequency, \( f \), and the percentage of critical damping, \( \zeta \); and (3) the dimensionless interaction parameter defined by
\[ \delta = \frac{\omega (x_{st})_d}{u_0} \]  

(6.5)

in which \( \omega = 2\pi f \) is the circular natural frequency of the system; and \( (x_{st})_d = P_d/k \) is the static displacement induced by the peak value of the drag component of the wave loading. The parameter \( \delta \) is deduced readily from equation (6.1) by expressing the latter in terms of the dimensionless time, \( \tau = \omega t \), and by normalizing \( x \) in terms of \( (x_{st})_d \). A value of \( \delta = 0 \) refers to a non-interacting system. One of the distinguishing features of the present study, the use of the dimensionless factors \( \alpha \) and \( \delta \), greatly simplifies the interpretation and use of the data presented herein.

6.4 SEA STATE CONSIDERED

A simulated sea state, generated from a one-dimensional Pierson-Moskowitz wave spectrum in combination with linear wave theory, is considered. The total depth of water is presumed to be 800 ft; the significant height and mean period of the waves are taken as 40 ft and 12.4 s, respectively; the ordinates of the wave spectrum for frequencies in excess of 0.4 cps are assumed to be zero; and the phase angles for the component harmonics are taken as random numbers uniformly distributed between zero and \( 2\pi \). This approach effectively leads to a sea state represented by a stationary, ergodic, Gaussian process with zero mean. As a result, no distinction is made in the
following developments between temporal and ensemble averages of fluid kinematics, and the two measures are used interchangeably.

The wave spectrum considered was sampled at increments of 0.004883 cps for a total of 2048 points. The resulting wave trains are thus defined at time increments of 0.1 sec and repeat at intervals of 204.8 s. All wave forces were computed from this sea state at a depth of 40 ft beneath the mean water level. The relevant histories of the horizontal components of the fluid particle velocity and acceleration are shown in Figure (6.1) normalized to a unit peak value.

6.5 PRINCIPAL EFFECTS OF INTERACTION

The natural and most direct means of assessing the consequences of fluid-structure interaction would be to compare the histories of the exciting forces and of the corresponding responses computed from equations (6.1) and (6.3).

Such comparisons are presented in Figures (6.2) and (6.3) for systems with $f = 0.1$ cps and $0.2$ cps subjected to a purely drag component of wave loading ($\alpha = 1$). The damping factor of the system in these solutions was taken as $\zeta = 0.02$. The solid lines in the top two diagrams in each of these figures represent the solutions obtained without regard for interaction, and the dashed lines represent the corresponding solutions obtained with due provision for interaction, assuming that $\delta = 0.10$. Also
shown in expanded scales are the histories of the differences in the two sets of results. The force histories are normalized with respect to \( P \), the peak value of the total exciting force for no interaction, and the displacement histories are normalized with respect to the corresponding static displacement, \( x_s \). All system responses were evaluated by numerical integration of governing equation of motion for a single cycle of the forcing function, using an integration step of 0.1 s and considering the structure to be initially at rest.

It can be seen that fluid-structure interaction modifies both the exciting force and the resulting response, generally reducing the absolute maximum values of these quantities. The change in the exciting force, \( \Delta P(t) \), is generally quite small, particularly when the natural frequency of the system is substantially different from the dominant frequency of the excitation, as is the case in Figure (6.3). By contrast, the change in response, \( \Delta x(t) \), is generally significant. This is due to fact that \( \Delta P(t) \) is an oscillatory, nearly periodic force component with dominant frequency equal to the natural frequency of the system under consideration and it induces a resonant-like component of response.

Critical analysis of these data further reveals that there is approximately a 90° phase difference between \( \Delta P(t) \) and \( \Delta x(t) \). This suggests that the mechanism of fluid-structure interaction is similar in its effect to that of linear viscous damping,
and that it tends to reduce the absolute maximum displacement of the system, \( |x_{\text{max}}| \). It further suggests that the reduction in response would be particularly significant at natural frequencies close to dominant frequency of the exciting force.

Although comparative studies of the type presented in Figures (6.2) and (6.3) provide valuable insight into the mechanism of fluid-structure interaction, they are not particularly convenient for quantifying the effect of this action on the maximum response of the system, and the alternative approaches examined in the following sections are preferable. Clearly, there is no simple correlation between the change in the exciting force and the corresponding change in the response. In particular, the change in the peak value of the exciting force is generally a poor indicator of the corresponding change in peak response. This is clearly shown in Table (6.1), in which dimensionless measures of \( |P_{\text{max}}| \) and \( |x_{\text{max}}| \) for several different values of \( f \) and \( \alpha \) are listed for both non-interacting systems \((\delta = 0)\) and for interacting systems with \( \delta = 0.10 \). Note that the effect of interaction on both \( |P_{\text{max}}| \) and \( |x_{\text{max}}| \) is highly sensitive to the values of \( f \) and \( \alpha \) involved.

### 6.6 Equivalent Linearization Technique

The velocity squared term in the expression for the drag component of the exciting force in this approach is approximated by a linear term as
\[(u - \dot{x})(\ddot{u} - \ddot{x}) \approx 2b_0 \dot{u}_0 (\ddot{u} - \ddot{x}) \quad (6.6)\]

in which \(b_0\) is a dimensionless factor determined so as to minimize the temporal average of the square of the resulting error. The latter factor is given by

\[
b_0 = \frac{1}{2u_0} \frac{<|\dot{u} - \dot{x}|^3>}{<(\dot{u} - \dot{x})^2>} \quad (6.7a)\]

in which \(\langle \cdot \rangle\) denotes the temporal average of the enclosed quantity. The coefficient 2 on the right-hand member of equation (6.6) is included for the convenience in relating the results of this approximation to those of the decoupling technique presented later. For a normal random wave (one for which the fluid kinematics is represented by a Gaussian random process), equation (6.7a) reduces to

\[
b_0 = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sigma_{u - \dot{x}}}{\dot{u}_0} \quad (6.7b)\]

in which \(\sigma_{u - \dot{x}}\) is the standard deviation of the relative velocity of the fluid and structure. Implicit in the development of equation (6.7b) is the assumption that the relative velocity is Gaussian.

On introducing the approximation defined by equation (6.6) into equation (6.1), and transposing to the left-hand member of the equation the term involving the structural velocity, one obtains

\[
m\ddot{x} + (c + c_0)\dot{x} + kx = P_i \frac{\ddot{u}}{\dot{u}_0} + 2b_0 P_d \frac{\dot{u}}{\dot{u}_0} \quad (6.8)\]
in which
\[ c_0 = \frac{2b_0 P_d}{u_0} \]  
(6.9)

Fluid-structure interaction according to this approximation has a two-fold effect: (1) it modifies the drag component of the exciting force to a linear function of the fluid particle velocity; and (2) it increases the damping of the system.

When expressed in per cent of the critical damping coefficient, \( c_{cr} = 2m\omega \), the effective damping of the system, \( \xi = (c+c_0)/c_{cr} \), is given by
\[ \xi = \zeta + \xi_0 = \zeta + b_0 \delta \]  
(6.10)
in which use has been made of equation (6.5). The quantities \( \zeta \) and \( \xi_0 \) represent the contributions of the structural damping and of the added hydrodynamic damping, respectively.

The dimensionless factor \( b_0 \) in equations (6.8), (6.9) and (6.10) is a function of the response of the system, and must therefore be computed by iteration. The process is typically started with the value of \( b_0 \) corresponding to \( \dot{x} = 0 \). The response of the system is then computed and a new value of \( b_0 \) is determined. These steps are repeated until the difference between the starting and derived values is less than a prescribed tolerance.

6.6.1 Results for Non-interacting Systems

Valuable insight into the accuracy of the linearization
technique may be gained from an analysis of the response of non-interacting systems, for which the procedure reduces to that originally proposed by Borgman (1967). In this case, the structural velocity in equation (6.6) and hence the term involving the damping coefficient \( c_0 \) in equation (6.8) vanish, and equations (6.7a) and (6.7b) reduce to

\[
b_0 = \frac{1}{2u_0} \frac{\langle |\dot{u}|^3 \rangle}{\langle (\dot{u})^2 \rangle}
\]  

(6.11a)

and

\[
b_0 = \sqrt{\left( \frac{2}{\pi} \frac{\sigma_\dot{u}}{u_0} \right)}
\]  

(6.11b)

respectively.

The exact and approximate values of \( |x_{\text{max}}| \) induced in systems with \( \zeta = 0.02 \) by several different combinations of the drag and inertia components of the simulated wave loading are compared in Figure (6.4). The results are displayed in the form of response spectra. Not to be confused with a wave spectrum which characterizes the excitation, a response spectrum defines the maximum response to a specified excitation of a family of single-degree-of-freedom systems having different natural frequencies. A rather broad range of natural frequencies and values of \( \alpha \) in the range between 0.25 and unity are considered. The normalizing displacement, \( x_n \), in these plots, as in Figures (6.2) and (6.3), represents the static displacement induced by the peak value of the actual wave force, not of its linearized
approximation.

The exact responses were computed by direct integration of equation (1) considering the system to be initially at rest, and the approximate responses were obtained by the linearization technique using the value of \( b_0 = 0.266 \) determined from equation (6.11a). For comparison, the value obtained from equation (6.11b) for a normal random wave is the same, whereas the value obtained for a single harmonic wave is \( b_0 = 0.424 \).

As would be expected, the accuracy of the approximate solution depends importantly on the value of the load factor, \( \alpha \). The larger this factor, the greater is the part of the response contributed by the drag component of the exciting force, and hence the greater is the consequence of the approximations involved in the linearization technique. For a specified value of \( \alpha \), the agreement between the approximate and exact solutions can be seen to be excellent in the central region of the response spectrum which is associated with a resonant-like response and large amplification factors. The agreement also is good in the low-frequency spectral region for which the response is generally not sensitive to the amount of damping involved. By contrast, there are significant and consistent differences in the practically important, right-hand region of the spectrum which covers the typical range of natural frequencies for fixed-base offshore platforms.

The latter differences stem from the inability of the
linearization technique to represent adequately the high-frequency force components to which high-frequency systems are sensitive. This fact is also reflected in the high-frequency limits of the response spectra. Considering that very stiff systems respond as if they were statically loaded, the right-hand limits of the spectra must be proportional to the absolute maximum value of the exciting force under consideration, $|P_{\text{max}}|$.

The exact and approximate values of this force are listed in Columns 2 and 3 of Table (6.2) for several different values of the load factor, $\alpha$. The results are normalized with respect to $P_i + P_d$, the numerical sum of the peak values of the component forces, with $P_d$ taken as the peak value of the exact drag force. Note that the difference between the exact and approximate values of $|P_{\text{max}}|$ increases with increasing $\alpha$. It is of interest to note further that $|P_{\text{max}}|$ is substantially less than $P_i + P_d$. This is, of course, due to the fact that the peak values of the inertia and drag force components occur at different times.

6.6.2 Result for Interacting Systems

The results for the non-interacting systems presented in Figure (6.4) suggest that the linearization technique should lead to similar errors for interacting systems as well. That this is indeed the case is demonstrated by the spectra in
Figure (6.5) which refer to systems with $\zeta = 0.02$ and three different values of the load factor, $\alpha$. The interaction parameter in these solutions is taken as $\delta = 0.10$; in reality, low-frequency systems are likely to be associated with larger values of $\delta$ than high-frequency systems.

Similar data are presented in Figure (6.6) for systems which in addition to the wave loading are acted upon by a current of constant velocity, $\dot{u}_c = 0.5 \dot{u}_0$. The normalizing displacement, $x_{st}$, in these plots represents, as in Figure (6.4), the static displacement induced by the peak value of the wave component of loading, not of the combination of wave and current. Similarly, the value of $\delta$ is expressed in terms of the $(x_{st})_{d}$ corresponding to the wave loading only.

The solutions by the linearization technique were obtained in the manner indicated in Reference (Spanos and Chen 1981) by: (1) replacing the wave-induced fluid velocity, $\dot{u}$, in equation (6.8) by the total velocity

$$\dot{u} = \dot{u}_c + \dot{u}$$  \hspace{1cm} (6.12)

(2) determining the factor $b_0$ in equations (6.8) and (6.9) from the following generalized version of equation (6.7a),

$$b_0 = \frac{1}{2 \dot{u}_0} \frac{<|\dot{u} - \dot{x}|(\dot{u} - \dot{x})(\dot{u} - \dot{x})>}{<(\dot{u} - \dot{x})^2>}$$  \hspace{1cm} (6.13)

and (3) interpreting $x$ to be the displacement measured from the mean value of the resulting total displacement. Denoted by $x_0$, the mean displacement is given by
\[ x_0 = \frac{\langle \dot{u} - \ddot{x} | (\ddot{x} - \dot{x}) \rangle}{\dddot{u}_0} (x_{st})_d \]  

(6.14)

With \( x \) computed in this manner, the total displacement is obtained by superimposing the value \( x_0 \). The exact solutions were computed from equation (6.1) by replacing \( \dot{u} \) by \( \ddot{u} \); and to avoid the spurious oscillations in the response due to the initial discontinuity of the current loading, the equation was rewritten in terms of \( x - x_c \), in which \( x_c \) is the static displacement due to the current-induced loading.

Note that even for \( \dot{u}_c/\dddot{u}_0 = 0.5 \), for which only a relatively small fraction of the total loading gets approximated in the linearization technique, the errors in response may be significant in the high-frequency region of the response spectra. The maximum values of the exact and linearized versions of the exciting forces are listed in Columns 4 and 5 of Table (6.2) along with those corresponding to \( \dot{u}_c/\dddot{u}_0 = 1 \). The high-frequency limits of the response spectra in Figure (6.6) are, of course, proportional to these force values.

6.6.3 Convergence of Procedure

The iterating process required to compute the factor \( b_0 \) in equations (6.8) and (6.9) generally converges rapidly. In Table (6.3) are listed the values of \( b_0 \) obtained for several different combinations of the parameters involved, along with the number of cycles, \( N \), required to compute the associated
values of $\zeta_0$ within a tolerance of 0.001. Note that convergence in all cases is achieved in one to three cycles.

6.7 ADJUSTED PROCEDURE

The data presented in the preceding section reveal that the inaccuracy of the linearization technique stems from the term representing the modified drag component of loading rather than the term representing the additional viscous damping. It would be reasonable, therefore, to expect that the reliability of this technique could be improved by retaining the added damping term in the form that has been presented but replacing the linearized drag force by the corresponding force for a non-interacting system [extreme right-hand member of equation (6.3)].

When adjusted in this manner, the equivalent linearization technique is intimately related to, and under certain conditions reduces to, Penzien's decoupling technique examined in the next section.

6.8 DECOUPLING TECHNIQUE GENERALIZED

Formulated originally for systems subjected to a wave loading only, the decoupling technique is generalized in this section for a wave acting in combination with a current of constant velocity, $\dot{u}_c$.

The equation of motion of the system in this case is given by
\[ m\ddot{x} + c\dot{x} + kx = P_i \frac{\ddot{u}}{u_0} + P_d \frac{|\dot{u} - \dot{x}|(\dot{u} - \dot{x})}{\dot{u}_0^2} \]  
(6.15)

in which \( x \) is the total displacement measured from the position of rest; and \( \dot{u} \), \( P_i \) and \( P_d \) are as previously defined. Recall that \( P_d \) represents the peak value of the drag component of the force due to the wave only, not the combination of wave and current.

On expanding the expression for the drag force, neglecting the term involving the square of \( \dot{x} \), taking \( \text{sgn}(\dot{u} - \dot{x}) = \text{sgn}(\dot{u}) \), and transferring to the left member of the equation the term involving the product of \( \dot{u} \) and \( \dot{x} \), one obtains

\[ m\ddot{x} + (c + c'_0)\dot{x} + kx = P_i \frac{\ddot{u}}{u_0} + P_d \frac{|\dot{u}|\dot{u}}{\dot{u}_0^2} \]  
(6.16)

in which \( c'_0 \), the viscous damping coefficient for the added damping approximating the effect of fluid-structure interaction, is a time-dependent quantity given by

\[ c'_0 = \left[ 2\text{sgn}(\dot{u}) \frac{\dot{u}}{u_0} \right] \frac{P_d}{\dot{u}_0} \]  
(6.17)

In his treatment of the effect of wave loading, Penzien (1976) replaced the time-dependent product \( u\text{sgn}(\dot{u}) \) by the temporal mean of the absolute value of the fluid velocity, \( <|\dot{u}|> \). The use of the same approximation for the generalized fluid kinematics considered here leads to

\[ c'_0 = 2 \frac{<|\dot{u}|> P_d}{u_0} \]  
(6.18)
When expressed in per cent of the critical coefficient of damping, \( c_{cr} \), the resulting hydrodynamic damping factor, \( \zeta_0 \), is given by

\[
\zeta_0 = \frac{c_0}{c_{cr}} = b_0 \delta
\]  \hspace{1cm} (6.19)

in which

\[
b_0' = \frac{\langle |\dot{v}| \rangle}{u_0} \hspace{1cm} (6.20)
\]

and \( \delta \) is defined by equation (6.5). The factor \( b_0' \) is the counterpart of the factor \( b_0 \) in the linearization technique.

For a normal random wave acting in combination with a current of constant velocity, it is a simple matter to show (Papoulis 1965) that \( b_0' \) is given by

\[
b_0' = \sqrt{\left( \frac{2}{\pi} \right) \frac{\sigma_u}{u_0} \exp \left( \frac{-\dot{u}_c^2}{2\sigma_u^2} \right) + \frac{\dot{u}_c}{u_0} \text{erf} \left( \frac{\dot{u}_c}{\sqrt{2}\sigma_u} \right)} \]  \hspace{1cm} (6.21)

in which \( \text{erf} \) stands for the error function. For \( \dot{u}_c = 0 \), equation (6.21) reduces to the following expression presented by Penzien (1976):

\[
b_0' = \sqrt{\left( \frac{2}{\pi} \right) \frac{\sigma_u}{u_0}} \hspace{1cm} (6.22)
\]

Equation (6.21) and the corresponding expression for \( \zeta_0 \) are identical to those obtained by the equivalent linearization technique (equation 15) in Spanos and Chen (1981) considering the structural velocity, \( \ddot{x} \), to be zero.
6.9 INTERRELATIONSHIP AND ACCURACY OF PROCEDURES

The interrelationship of the decoupling and linearization techniques may now be summarized as follows.

1. The drag component of loading in both the decoupling technique and the adjusted version of the linearization technique is identical to that for a non-interacting system. By contrast, in the original linearization technique, it is a linear function of the fluid velocity.

2. Fluid-structure interaction increases the effective damping of the system and reduces the response. In the decoupling technique, the added damping is independent of the structural response and may be computed directly from the history of fluid particle velocity, whereas in the linearization techniques, it is also a function of the structural velocity and must be evaluated by iteration.

3. Provided that the added damping of the system in the linearization technique is determined assuming $\dot{x}=0$, the adjusted version of this technique for a random normal wave acting in combination with a constant-velocity current is identical to the decoupling technique.

Representative response spectra computed by the decoupling technique are compared in Figures (6.7) and (6.8) with the corresponding exact spectra. Figure (6.7) refers to systems excited by the simulated wave loading only, whereas Figure (6.8) also incorporates the effect of a current with $\dot{u}_c = 0.5\dot{u}_o$. 
Three different values of $\alpha$ in the range between 0.5 and unity, and single values of $\zeta$ and $\delta$ are considered. No data are presented for values of $\alpha$ less than 0.5 as the accuracy of the decoupling technique is quite high in this case.

Comparison of these data with the corresponding data presented in Figures (6.5) and (6.6) reveals the following.

1. For systems with natural frequencies substantially higher than those for which the response spectra attain their absolute maximum values, the decoupling technique, and hence the adjusted linearization technique, are superior to the original version of the latter procedure. In the spectral regions associated with the absolute maximum responses, the decoupling technique is inferior to the linearization technique but the differences are generally small in this case.

2. The decoupling technique generally underestimates the added damping and overestimates the maximum response. The errors increase with increasing $\alpha$, i.e. increasing contribution of the drag component of loading.

### 6.10 PROPOSED MODIFICATION OF DECOUPLING TECHNIQUE

#### 6.10.1 Effect of Load Factor, $\alpha$

It is desirable to examine at this stage the influence that this factor has on the characteristics of the exciting forces computed without regard for interaction. Representative histories for these forces are shown in Figure (6.9) for
several different values of $\alpha$ in the range between zero and unity. The following trends should be observed.

1. Whereas the inertia force ($\alpha=0$) is characterized by oscillations of nearly equal peak values which are more or less uniformly distributed along the entire record, the major oscillations in the drag force ($\alpha=1$) are significantly fewer in number and generally widely separated. A consequence of the squaring of the velocity trace which tends to suppress the contributions of the smaller amplitude oscillations, the reduced repetitiveness of the drag force leads to a less severe resonant-like response and to reduced amplification factors for systems with natural frequencies close to the dominant frequency of the excitation than does the inertia force. This can clearly be seen from the response spectra for ($\alpha=0$) and ($\alpha=1$) compared in Figure (6.10).

2. Because of the differences noted under item 1 and the fact that the peak ordinates of the inertia force component correspond to zero ordinates of the drag force, the inertia force history generally dominates the characteristics of the total force and of the corresponding response spectra. For example, for ($\alpha=0.5$) for which the peak values of the component forces are the same, the history of the combined force and the associated response spectrum are much closer to those of the inertia force than those of the drag force.

These observations suggest that, whereas for small values
of the response of the system is influenced more or less uniformly by all pulses of the forcing function, for values of close to unity, it is dominated by the small number of pulses with the large amplitudes. It follows that the replacement of equation (6.17) by equation (6.18) would not be a good approximation for the larger values of $\alpha$, and that this approach would tend to underestimate the value of $b_0$ and the associated value of $\zeta_0$.

Based on these considerations, it is recommended that for systems subjected to a purely drag component of loading, only those pulses in the fluid velocity history whose amplitudes exceed the 70 per cent level of the absolute maximum velocity be considered in the averaging process. It is further recommended that the proposed threshold limit be considered to decrease linearly from the indicated value for ($\alpha=1$) to zero for ($\alpha=0$). If $z$ represents the appropriate limit in per cent of the maximum fluid velocity, then $z=0.7\alpha$.

This modification of the decoupling technique is tantamount to replacing the quantity $\langle |\dot{v}| \rangle$ in equation (6.20) by $\langle |\dot{v}_x| \rangle$, the temporal mean of the absolute value of those pulses in the fluid velocity trace whose amplitudes exceed the specified threshold limit, $z$.

6.10.2 Random Wave

For a normal random wave without any current, the proposed
approximation leads to the following expression (see Appendix D) for the factor $b'_0$ in equation (6.19):

$$b'_0 = \frac{1-F}{1-z} \sqrt{\left(\frac{2}{\pi}\right)\frac{\sigma_u}{\bar{u}_0}}$$  \hspace{1cm} (6.23)

in which $F$ = the chi-square probability distribution function of three degrees of freedom, given by

$$F = \frac{1}{\sqrt{2\pi s}} \int_0^s \sqrt{\eta} \exp\left(-\frac{\eta}{2}\right) d\eta$$  \hspace{1cm} (6.24)

and $s = 2 \ln[1/(1-z)]$. The values of $F$ corresponding to different values of $z = 0.7\alpha$ were computed by use of a standard IMSL subroutine (IMSL 1982) and the results led to the following linear approximation for $b'_0$:

$$b'_0 = [1 + 0.61\alpha] \sqrt{\left(\frac{2}{\pi}\right)\frac{\sigma_u}{\bar{u}_0}}$$  \hspace{1cm} (6.25)

Note that, for $\alpha = 1$, the values of $b'_0$ and of the associated damping factor, $\xi'_0$, are approximately 61 per cent larger than those obtained from Penzien's original proposal. The difference in the two approaches naturally decreases with decreasing $\alpha$.

6.10.3 Wave Combined with Current

For a sea state represented by a random wave in combination with a current, the exact expressions for $b'_0$ are derived in Appendix D. Inasmuch as the evaluation of these expressions
is tedious, the use of the following simpler approximation, obtained by modifying the first term of equation (6.21), is recommended instead:

\[ b' = (1 + 0.61\alpha)\sqrt{\left(\frac{2}{\pi}\right)\frac{\sigma_u}{\bar{u}_0}} \exp\left(-\frac{\bar{u}_0^2}{2\sigma_u^2}\right) + \frac{\bar{u}_c}{\bar{u}_0} \text{erf}\left(\frac{\bar{u}_c}{\sqrt{2}\sigma_u}\right) \]  

(6.26)

The values of \( b' \) determined from this approximation are plotted as a function of \( \bar{u}_c/\bar{u}_0 \) in Figure (6.11), where they are also compared with those obtained from the corresponding exact expression.

6.10.4 Accuracy of Procedure

The exact response spectra for the simulated sea state without any current considered previously are compared in Figure (6.12) with those computed by the proposed modification of the decoupling technique. As before, the structural damping factor in these solutions is taken as \( \zeta = 0.02 \), and two different values of the loading factor, \( \alpha \), and several values of the interaction parameter, \( \delta \), are considered. The agreement between the two sets of results, excluding those corresponding to \( \alpha = 1 \) and \( \delta = 0.5 \), is considered to be quite good. Comparable agreements have been obtained for several other sea states.

The results for \( \alpha = 1 \) and \( \delta = 0.5 \) correspond to an unrealistic combination of the parameters and should be viewed as an extreme test of the accuracy of the proposed approximation.
Even in this case, however, this approximation is superior within the high-frequency region of the response spectrum to those presented previously (see Figure 6.13).

6.11 APPLICATION TO MORE COMPLEX SYSTEMS

The application of both the linearization and decoupling techniques to the analysis of multi-degree-of-freedom systems has already been described in the literature (Malhotra and Penzien 1970; Penzien 1976; Penzien and Tseng 1978; Foster 1970; Shyam Sunder and Connor 1982). The proposed modification of the decoupling technique can be implemented in a similar manner, except that it is necessary to use for the damping coefficients, $b_0$, the values corresponding to the fluid kinematics at the particular water depth under consideration.

The steps involved in the application of these techniques may be summarized as follows:

1. The wave and current forces exerted at various nodes of the structure are first computed from the fluid kinematics and the relevant tributary volumes and areas of the structure.

2. The drag components of these forces are then approximated in the manner indicated for single-degree-of-freedom systems, and two subsets of forces are obtained. The first, which is a function of the fluid particle velocities only, is retained on the right-hand member of the equations as modified drag forces, and the second, which is proportional to the structural
velocities, is transferred to the left-hand member and interpreted as added damping forces.

3. The matrix for the overall damping of the system obtained in this manner has no relationship to either the mass or stiffness matrix of the system, and system damping is of the non-classical type. To obviate the need for using complex-valued natural modes of vibration, the transformed modal damping matrix of the system is replaced by a diagonal matrix, the elements of which are determined by minimizing the square of the error between the damping forces associated with the original and the approximating matrices. Being functions of the response of the system, the elements of the diagonalized damping matrix must be computed by iteration.

4. With the overall damping of the system approximated in this manner, the analysis is implemented by use of the classical modal superposition method.

In the decoupling techniques, step 1 to 4 are carried out only once, whereas in the linearization technique, they are repeated until the modal damping values corresponding to the starting and the derived sets of structural velocities agree within a prescribed tolerance.

6.11.1 Proposed Simplification

While straightforward in principle, these procedures are generally time consuming and tedious. A simpler approximation
for the modal damping values of the system may be obtained by deleting the off-diagonal terms of the transformed damping matrix referred to in item 4 of the preceding section. This approximation is presented in the following paragraphs for discrete, stick-like systems having a total of \( n \) submerged nodes.

Let \( P_{dj} \) be the maximum value of the drag component of the wave force acting on the \( j \)th node of the system, and \( \dot{u}_{0j} \) be the corresponding value of the wave-induced fluid particle velocity. Further, let \( <|\dot{u}_{zj}|> \) be the relevant temporal mean of the absolute value of the total fluid velocity at that depth. (In the proposed modification of the decoupling technique, only those pulses with amplitudes in excess of the specified threshold limit, \( z = 0.7 \alpha \), are considered.) Finally, let

\[
\dot{b}_{0j} = \frac{<|\dot{u}_{zj}|>}{\dot{u}_{0j}} \quad (6.27)
\]

and

\[
\delta_{ij} = \frac{\omega_i P_{dj}}{k_i \dot{u}_{0j}} \quad (6.28)
\]

in which \( \omega_i \) = the \( i \)-th circular frequency of vibration of the system; \( k_i = \omega_i^2 m_i \) = the generalized or effective stiffness of the \( i \)th natural mode; and \( m_i \) = the corresponding effective mass. For a normal random wave, the factors \( \dot{b}_{0j} \) may be determined
from Figure (6.11) or equation (6.26) using the values of 
\( \alpha, \frac{\dot{u}_c}{\dot{u}_0} \) and \( \sigma_a \) that are appropriate to the particular node 
under consideration.

With these parameters and the approximation referred to, it is a simple matter to show that the percentage of the 
hydrodynamic damping for the \( i \)th mode of vibration, \( \zeta_{0i} \), is given by

\[
\zeta_{0i} = \sum_{j=1}^{n} b_{0j} \delta_{ij} \phi_{ij}^2
\]  
(6.29)

in which \( \phi_{ij} \) = the ordinate at node \( j \) of the \( i \)th mode of vibration. If \( \zeta_i \) represents the corresponding percentage of structural 
damping, then the total damping factor for the \( i \)th mode of vibration, \( \xi_i \), is given by

\[
\xi_i = \zeta_i + \zeta_{0i}
\]  
(6.30)

Equations (6.29) and (6.30) are generalized versions of 
equations (6.9) and (6.10), respectively.

The computation of the factors \( b_{0j} \) and of the quantities 
\( P_{dj} \) and \( \dot{u}_{0j} \) in equation (6.28) still entails considerable effort, and an even simpler approximation for \( \zeta_{0i} \) is desirable. Considering that the response of the structure is dominated 
by the forces acting on its upper parts, it is recommended that the quantities \( P_{dj} \) and \( \dot{u}_{0j} \) in equation (6.28) be reinter- 
terpreted to be those corresponding to the instant, \( t_0 \), for 
which the drag component of the wave force at the uppermost, 
fully submerged node of the structure attains its maximum
value. This node is typically located at a depth below mean water level approximately equal to the maximum surface wave height. It is further proposed that the factors $b_0$, in equation (6.27) be replaced by a constant value, $b_0^*$, determined from the fluid kinematics at the depth of the resultant of the drag forces at time $t_0$. This approximation has been tested for a number of structural systems and has been found to yield results of high accuracy.

As an indication of the differences in the values of $\zeta_0$ that may result from the use of the different approximations, in Table (6.4) are listed the results obtained for the first three modes of vibration of an offshore guyed-tower model in 1600 ft of water. The characteristics of the structure are given in reference by Hahn and Veletsos (1985). Its first three natural periods are 29.1, 4.88 and 2.33 s, respectively. The results refer to a simulated sea state with the same surface wave characteristics as those considered previously and no current. Also listed in the table are the approximate and exact values of selected maximum responses.

As would be expected from the data presented in Figure (6.5), the linearization technique for this highly compliant structure gives excellent results. However, this would not be the case for fixed-base structures which are associated with higher natural frequencies.
The proposed modification of the decoupling technique is clearly superior to the original version, and the simpler version of the proposed modification provides excellent approximations to the modal damping values. The latter values may be determined by this approach at a fraction of the time required by the linearization technique.

6.12 CONCLUSION

With the formation and concepts that have been presented, the effects of fluid-structure interaction on the maximum response of simple models of offshore structures can be estimated readily. The proposed approximation for the hydrodynamic modal damping factors of multi-degree-of-freedom systems should prove particularly useful in preliminary design decisions requiring an estimate of these quantities.

The equivalent linearization technique has been shown to lead to substantial errors for structures for which the drag component of the exciting force is dominant and for which the fundamental natural frequency of vibration is substantially higher than the dominant frequency of the wave loading. Penzien's decoupling technique is superior in this case and the proposed modification further improves its accuracy.
Table (6.1)  Peak values of exciting force and resulting response for systems without and with interaction

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\( f \) & \( \alpha = 1.0 \) & \( \alpha = 0.75 \) & \( \alpha = 0.50 \) & \( \alpha = 0.25 \) \\
\hline
cps & \( \delta = 0 \) & \( \delta = 0.1 \) & \( \delta = 0 \) & \( \delta = 0.1 \) & \( \delta = 0 \) & \( \delta = 0.1 \) \\
\hline
\hline
Values of \( |P_{\text{max}}|/P \) & & & & & & \\
\hline
0.04 & 1.000 & 0.943 & 1.000 & 0.913 & 1.000 & 0.912 & 1.000 & 0.979 \\
0.08 & 0.720 & 0.694 & 0.775 & 0.929 \\
0.10 & 0.783 & 0.816 & 0.815 & 0.954 \\
0.20 & 0.941 & 1.008 & 1.068 & 0.978 \\
0.333 & 0.960 & 0.939 & 1.004 & 1.006 \\
0.50 & 1.012 & 1.012 & 0.984 & 1.004 \\
\hline
Values of \( |x_{\text{max}}|/x_{st} \) & & & & & & \\
\hline
0.04 & 0.809 & 0.691 & 0.851 & 0.712 & 0.929 & 0.740 & 0.862 & 0.691 \\
0.08 & 3.452 & 2.212 & 3.744 & 2.526 & 4.705 & 3.237 & 5.004 & 3.118 \\
0.10 & 2.499 & 1.856 & 2.358 & 1.976 & 3.389 & 2.560 & 4.340 & 2.762 \\
0.20 & 1.628 & 1.202 & 1.306 & 1.038 & 1.352 & 1.618 & 1.530 & 1.735 \\
0.333 & 1.537 & 1.363 & 1.417 & 1.271 & 1.064 & 1.074 & 1.199 & 1.219 \\
0.50 & 1.127 & 1.085 & 1.177 & 1.133 & 1.135 & 1.107 & 1.050 & 1.061 \\
\hline
\end{tabular}

Table (6.2)  Comparison of maximum values of forces for systems without interaction computed exactly and by the linearization technique

\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{Value} & \multicolumn{4}{|c|}{Values of \( |P_{\text{max}}|/(P_i+P_d) \)} \\
\hline
\multicolumn{2}{|c|}{of} & \( \dot{u}_c/\dot{u}_0 = 0 \) & \( \dot{u}_c/\dot{u}_0 = 0.5 \) & \( \dot{u}_c/\dot{u}_0 = 1.0 \) \\
\hline
\( \alpha \) & Exact & Linearized & Exact & Linearized & Exact & Linearized \\
\hline
(1) & (2) & (3) & (4) & (5) & (6) & (7) \\
\hline
1 & 1.000 & 0.532 & 2.250 & 1.420 & 4.000 & 3.154 \\
0.75 & 0.773 & 0.481 & 1.702 & 1.108 & 3.011 & 2.388 \\
0.50 & 0.645 & 0.595 & 1.218 & 0.938 & 2.068 & 1.710 \\
0.25 & 0.768 & 0.785 & 0.948 & 0.924 & 1.301 & 1.232 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}
<table>
<thead>
<tr>
<th>( f )</th>
<th>( \dot{u}_c / \dot{u}_0 = 0 )</th>
<th>( \dot{u}_c / \dot{u}_0 = 0.5 )</th>
<th>( \dot{u}_c / \dot{u}_0 = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 1 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
</tr>
<tr>
<td>0.04</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>0.10</td>
<td>0.255</td>
<td>2</td>
<td>0.524</td>
</tr>
<tr>
<td>0.25</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>0.50</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>( \alpha = 0.75 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
</tr>
<tr>
<td>0.04</td>
<td>0.254</td>
<td>2</td>
<td>0.512</td>
</tr>
<tr>
<td>0.10</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>0.25</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>0.50</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>( \alpha = 0.50 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
</tr>
<tr>
<td>0.04</td>
<td>0.240</td>
<td>2</td>
<td>0.509</td>
</tr>
<tr>
<td>0.10</td>
<td>0.300</td>
<td>2</td>
<td>0.524</td>
</tr>
<tr>
<td>0.25</td>
<td>0.283</td>
<td>2</td>
<td>0.536</td>
</tr>
<tr>
<td>0.50</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
<tr>
<td>( \alpha = 0.25 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
<td>( \dot{u}_0 )</td>
</tr>
<tr>
<td>0.04</td>
<td>0.204</td>
<td>2</td>
<td>0.502</td>
</tr>
<tr>
<td>0.10</td>
<td>0.408</td>
<td>3</td>
<td>0.575</td>
</tr>
<tr>
<td>0.25</td>
<td>0.323</td>
<td>2</td>
<td>0.562</td>
</tr>
<tr>
<td>0.50</td>
<td>0.266</td>
<td>1</td>
<td>0.524</td>
</tr>
</tbody>
</table>
Table (6.4) Comparision of solutions for a guyed-tower model in 1600 ft of water

<table>
<thead>
<tr>
<th>Linearization</th>
<th>Penzien's decoupling</th>
<th>Modified decoupling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact technique</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hydrodynamic modal damping factor, $\zeta_{ii}$, for $i = 1, 2$ and $3$, respectively</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.133</td>
<td>0.174</td>
<td>0.242</td>
</tr>
<tr>
<td>0.022</td>
<td>0.007</td>
<td>0.015</td>
</tr>
<tr>
<td>0.012</td>
<td>0.003</td>
<td>0.006</td>
</tr>
<tr>
<td>Maximum top displacement, in ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.38</td>
<td>14.62</td>
<td>17.25</td>
</tr>
<tr>
<td>Maximum base shear, in kip</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3840</td>
<td>3726</td>
<td>4481</td>
</tr>
<tr>
<td>Maximum base moment, in kip-ft $\times 10^{-6}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.233</td>
<td>0.243</td>
<td>0.286</td>
</tr>
<tr>
<td>Maximum moment at 520 ft. depth, in kip-ft $\times 10^{-6}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.833</td>
<td>1.772</td>
<td>2.038</td>
</tr>
</tbody>
</table>
Fig. 6.1 Normalized Fluid Kinematics
Fig. 6.2 Normalized Histories of Forces and Displacements for Systems with $\zeta = 0.02$ and $f = 0.1$ cps
Fig. 6.3 Normalized Histories of Forces and Displacements for Systems with $\zeta = 0.02$ and $f = 0.2$ cps
Fig. 6.4 Response Spectra for Non-Interacting Systems with \( \zeta = 0.02 \) Subjected to Wave Loading
Fig. 6.5 Response Spectra for Interacting Systems with $\delta = 0.10$ and $\zeta = 0.02$ Subjected to Wave Loading
Fig. 6.6  Response Spectra for Interacting Systems with $\delta = 0.10$ and $\zeta = 0.02$ Subjected to Combination of Wave and Current with $\tilde{u}_c = 0.5\tilde{u}_o$. 
Fig. 6.7 Comparison of Response Spectra Obtained by Exact Method and by Decoupling Technique; Systems with $\delta = 0.10$ and $\zeta = 0.02$ Subjected to Wave Only
Fig. 6.8 Comparison of Response Spectra Obtained by Exact Method and Decoupling Technique; Systems with $\delta = 0.10$ and $\xi = 0.02$ Subjected to Wave and Current with $\dot{u}_c = 0.5 \dot{u}_0$
Fig. 6.9 Effect of Load Factor, \( \alpha \), on Histories of Exciting Forces for Systems without Interaction
Fig. 6.10 Effect of Load Factor, $\alpha$, on Characteristics of Response Spectra for Non-Interacting Systems with $\zeta = 0.02$ Subjected to Wave Loading
Fig. 6.11 Values of Proposed Damping Factor Coefficient, $b_o'$
Fig. 6.12 Comparison of Response Spectra Computed by Exact Method and by Proposed Modification of Decoupling Technique; Interacting Systems with $\zeta = 0.02$ Subjected to Wave Loading
Fig. 6.13 An Extreme Comparison of Response Spectra Computed by Different Techniques; Systems with $\delta = 0.50$, $\zeta = 0.02$ and $\alpha = 1$ Subjected to Wave Loading
REFERENCES

References for Soil-Structure Interaction


References for Fluid-Structure Interaction


APPENDIX A

A.1 DERIVATIONS OF EQUATIONS 2.9

For incoherence effects only, the integrands in equations 2.5a and 2.5b are symmetric about \( \xi_1 = \xi_2 \). This symmetry may be provided for by multiplying the expressions by 2 and changing the upper limit of integration of \( \xi_2 \) from unity to \( \xi_1 \). On using the identity

\[
I_n(z) = \frac{1}{\pi} \int_0^\pi \exp(z\cos\theta)\cos n\theta d\theta
\]  

(A.1)
given as equation 9.6.19 in Abromowitz and Stegun (1970), the specialized form of equations 2.5a and 2.5b are integrated with respect to the circumferential co-ordinates to yield

\[
\frac{S_x}{S^*_g} = 8 \int_0^1 \int_0^{\xi_1} \xi_1 \xi_2 \exp[-b_0^2(\xi_1^2 + \xi_2^2)] I_0(2b_0^2 \xi_1 \xi_2) d\xi_1 d\xi_2
\]  

(A.2a)

\[
\frac{S_y}{S^*_g} = 16 \int_0^1 \int_0^{\xi_1} (\xi_1 \xi_2)^2 \exp[-b_0^2(\xi_1^2 + \xi_2^2)] I_1(2b_0^2 \xi_1 \xi_2) d\xi_1 d\xi_2
\]  

(A.2b)

The dummy variable \( \xi_2 \) in these equations is then expressed as \( \xi_2 = s \xi_1 \), and the resulting expressions are integrated with respect to \( s \) by making use of the identity (see equation 6.631.8 in Gradshteyn and Ryzik (1980).)

\[
\int_0^1 s^{n+1} \exp(-as^2)I_n(2as)ds = \frac{1}{2a} \left[ \exp(\alpha) - \exp(-\alpha) \sum_{\gamma=n}^{\infty} I_{\gamma}(2\alpha) \right]
\]  

(A.3)

to yield

\[
\frac{S_x}{S^*_g} = \frac{1}{b_0^2} \int_0^1 2\xi_1 [1 - \exp(-2b_0^2 \xi_1^2)] I_0(2b_0^2 \xi_1^2) d\xi_1
\]  

(A.4a)
\[
\frac{S_x}{S_y} = \frac{4}{b_0} \int_0^1 2\xi_1^2 \{1 - \exp(-2b_0^2\xi_1^2)[I_0(2b_0^2\xi_1^2) + 2I_1(2b_0^2\xi_1^2)]\} d\xi_1 \quad (A.4b)
\]

Finally, on letting \(a = 2b_0^2\xi_1^2\) and making use of the identities

\[
\int_0^z \exp(-a)a^nI_n(a)da = \frac{\exp(-z)z^{n+2}}{2n+1}[I_n(z) + I_{n+1}(z)] \quad (A.5)
\]

\[
\int_0^z a^nI_{n-1}(a)da = z^nI_n(z) \quad (A.6)
\]

\[
I_0(z) - I_2(z) = \frac{2}{z}I_1(z) \quad (A.7)
\]

given as equations 11.3.12, 11.3.25 and 9.6.26 in Abromowitz and Stegun (1970), equations A.4a and A.4b are integrated to yield equations 2.9a and 2.9b. Equation 2.9c follows from the fact that the integrand of equation 2.5c is anti-symmetric in this case.

For small values of \(b_0\), application of Taylor's series expansion to equations 2.9a and 2.9b yields

\[
S_x = \left[1 - b_0^2 + \frac{5}{6}b_0^4 + \ldots\right]S_y \quad (A.8a)
\]

\[
S_y = \left[\frac{1}{2}b_0^2 - \frac{2}{3}b_0^4 + \ldots\right]S_y \quad (A.8b)
\]

### A.2 EVALUATION OF PEAK VALUES OF INPUT AND RESPONSE

Let \(z(t)\) be a stationary, ergodic random Gaussian process with zero mean and limited duration, \(t_0\), and let \(Z\) be the
ensemble mean of its peak values. Further, let $G(\omega)$ be the one-sided power spectral density of the process, and $\lambda_0$, $\lambda_1$, and $\lambda_2$ be its first three moments, defined by

$$\lambda_n = \int_0^\infty \omega^n G(\omega)d\omega \quad n=0,1,2 \quad (A.10)$$

The value of $Z$ in Der Kiureghian's approach is evaluated conservatively from

$$Z = \left[ \sqrt{2\ln(\bar{\mu} t_0)} + \frac{0.5772}{2\ln(\bar{\mu} t_0)} \right]^{\lambda_0} \quad (A.11)$$

in which

$$\bar{\mu} t_0 = \begin{cases} 2.1 \text{ or } 2q\bar{\mu} t_0 \text{ if greater than } 2.1 & \text{for } q \leq 0.1 \\ (1.63q^{0.45} - 0.38)\bar{\mu} t_0 & \text{for } 0.1 \leq q \leq 0.69 \\ \bar{\mu} t_0 & \text{for } q \geq 0.69 \end{cases} \quad (A.12)$$

$\bar{\mu} = \sqrt{\frac{\lambda_2}{\lambda_0}}/\pi = \pi$ = the mean zero-crossing rate of the process; and $q = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}}$ = Vanmarcke’s (1975) band width parameter.

### A.3 HARMONIC RESPONSE OF SYSTEMS WITH INERTIAL INTERACTION

#### A.3.1 Torsionally excited systems

Let $\chi = \chi(t)$ and $\chi_i = \chi_i(t)$ be the torsional components of the foundation input displacement and the actual foundation displacement, respectively, and $\psi = \psi(t)$ be the resulting torsional deformation of the structure. The equations of motion for the system may then be written as

$$\ddot{\psi} + 2\xi_0 p_0 \dot{\psi} + p_0^2 \psi = -\ddot{\chi}_i \quad (A.13)$$

and
\[ J(\ddot{\psi} + \ddot{\chi}_f) + J_f \ddot{\chi}_f + Q_0(t) = 0 \]  
\text{(A.14)}

in which a dot superscript denotes differentiation with respect to time; \( J \) and \( J_f \) are the polar mass moments of inertia of the structure and foundation respectively; and \( Q_0(t) \) = the instantaneous value of the torque at the foundation-soil interface. Equation (A.13) expresses the dynamic equilibrium of the forces acting on the structural mass, whereas equation (A.14) expresses the fact that the sum of the torsional moments due to the inertia of the structure and the foundation equals the torque acting at the foundation-soil interface.

For the harmonic response considered

\[ \chi(t) = X e^{i\omega t} \]  
\text{(A.15a)}

\[ \chi_f(t) = X_f e^{i\omega t} \]  
\text{(A.15b)}

\[ \psi(t) = \Psi e^{i\omega t} \]  
\text{(A.15c)}

and

\[ Q_0(t) = K_0 (X_f - X) e^{i\omega t} \]  
\text{(A.16)}

in which \( X, X_f \) and \( \Psi \) are the complex valued amplitudes of \( \chi, \chi_f \) and \( \psi \), respectively; and \( K_0 \) = the complex-valued torsional impedance of the massless foundation.

Equations (A.13) and (A.14) are solved in three steps as follows: First, on substituting equations (A.15b) and (A.15c) and its derivatives into equation (A.13), the deformation
Amplitude of the structure, \( \Psi \), is expressed in terms of amplitude of the torsional component of the foundation acceleration, \( \ddot{X}_f \), as
\[
\Psi = H_v \ddot{X}_f  \tag{A.17}
\]
in which \( \ddot{X}_f = -\omega^2 X_f \), and \( H_v \) = the transfer function for torsional response, given by
\[
H_v = -\frac{1}{p_0^2} \frac{1}{1 - (\omega/p_0)^2 + i2\zeta_0(\omega/p_0)} \tag{A.18}
\]
Next, equation (A.16), along with equations (A.15a), (A.15b) are substituted into equation (A.14), and the resulting expression is solved for \( \ddot{X}_f \). This step yields
\[
\ddot{X}_f = T_v \ddot{X}  \tag{A.19}
\]
in which
\[
T_v = \frac{\kappa_0}{\kappa_0 - (TR)_0 + (J_f/J)\alpha_0^2}  \tag{A.20}
\]
\( \kappa_0 = K_0 R^2/(J_v^2) \); and \( (TR)_0 \) = the torsional transmissibility of the system, given by
\[
(TR)_0 = -(p_0^2 + i2\zeta_0 \omega p_0)H_v = \frac{1 + i2\zeta_0(\omega/p_0)}{1 - (\omega/p_0)^2 + i2\zeta_0(\omega/p_0)}  \tag{A.21}
\]
The expressions for \( \ddot{X}_f \), defined by equation (A.19) is finally substituted into equation (A.17) to yield
\[
\Psi = H_v \bar{T}_v \ddot{X}  \tag{A.22}
\]
from which it follows that the psd function of the deformation of the structure along its periphery, \( S_v \), is given by
\[ S_v = |H_v|^2 |T_v|^2 S_\gamma \]  
(A.23)

The factor \(|T_v|^2\) in the latter expression represents the effect of inertial interaction.

For circular foundation considered, \(K_\theta\) is defined by (Veletsos and Nair 1974)

\[ K_\theta = \frac{16}{3} GR^3 (\alpha_\theta + i \alpha_0 \beta_\theta) \]  
(A.23)

in which \(\alpha_\theta\) and \(\beta_\theta\) are the dimensionless functions of the frequency parameter \(\alpha_0 = \omega R/v_z\). On making use of the expression \(v_z = \sqrt{G/\rho}\), the dimensionless stiffness factor, \(\kappa_\theta\), in equation (A.20) may also be written as

\[ \kappa_\theta = \frac{16 \rho R^5}{3 J} (\alpha_0 + i \alpha_0 \beta_\theta) \]  
(A.25)

### A.3.2 Laterally Excited System

Let \(x = x(t)\) and \(x_f = x_f(t)\) be the lateral components of foundation input displacement and actual foundation displacement, respectively, and \(\phi(t)\) be the angular rocking displacement of the foundation. Further, let \(u = u(t)\) be the resulting lateral deformation of the structure. The equations of motion of the system may then be expressed as

\[ \ddot{u} + 2 \zeta_x p_x \dot{u} + p_x^2 = -(\ddot{x}_f + h \dot{\phi}) \]  
(A.26)

\[ m_f \ddot{x}_f + m(\ddot{x}_f + h \dot{\phi} + \ddot{u}) + Q_x(t) = 0 \]  
(A.27)
\[ I_T \ddot{\phi} + m h ( \dot{x}_f + h \dot{\phi} + \ddot{u} ) + Q_\phi(t) = 0 \]  

(A.28)

in which \( I_T \) = the total mass moment of inertia of the structure and its foundation about a horizontal axis through the centroid of the foundation, and \( Q_x(t) \) and \( Q_\phi(t) \) are the horizontal shear and overturning moment at the foundation-soil interface. Equation (A.26) expresses the dynamic equilibrium of the forces acting on the structural mass, whereas equations (A.27) and (A.28) express the equality of the interface forces to the total horizontal force and the base moment induced by the inertia forces acting on the structure and its foundation.

For the harmonic response considered,

\[ x(t) = X_0 e^{i\omega t} \]  

(A.29a)

\[ x_f(t) = X_f e^{i\omega t} \]  

(A.29b)

\[ \phi(t) = \Phi e^{i\omega t} \]  

(A.29c)

\[ u(t) = U e^{i\omega t} \]  

(A.29d)

and

\[
\begin{bmatrix}
Q_x(t) \\
Q_\phi(t)
\end{bmatrix} = [K_f] \begin{bmatrix}
X_f - X_0 \\
\Phi
\end{bmatrix} e^{i\omega t}
\]  

(A.30)

in which \([K_f]\) = a 2×2 complex-valued impedance matrix for the massless foundation.

The solution of equations (A.26) through (A.28) may be obtained in three steps in a manner analogous to that described for the torsionally excited system. First, equation (A.26) is
solved for $U$ in terms of $\ddot{X}_f + h\dot{\Phi}$, in which $\ddot{X}_f = -\omega^2 X_f$ and $\dot{\Phi} = -\omega^2 \Phi$. Second, the quantity $U$ is eliminated from equations (A.27) and (A.28) utilizing the result of the first step, and the resulting expressions are solved for $\ddot{X}_f$ and $\dot{\Phi}$ by making use of equation (A.30). Finally, the expressions for $\ddot{X}_f$ and $\dot{\Phi}$ are back substituted in the expression for $U$ obtained in the first step.

Implementation of the first step leads to

$$U = H_u(\ddot{X}_f + h\dot{\Phi}) \quad (A.31)$$

in which $H_u$ is defined by equation (2.15); and implementation of the second step leads to the following system of algebraic equations in $\ddot{X}_f$ and $\dot{\Phi}$

$$\omega^2 \begin{bmatrix} m(TR)_x + m_f & mh(TR)_x \\ mh(TR)_x & I_f + mh^2(TR)_x \end{bmatrix} \begin{bmatrix} \ddot{X}_f \\ \dot{\Phi} \end{bmatrix} - [K_f] \begin{bmatrix} \ddot{X}_f \\ \dot{\Phi} \end{bmatrix} = -[K_f] \begin{bmatrix} \ddot{X}_0 \\ 0 \end{bmatrix} \quad (A.32)$$

in which $(TR)_x$ is the transmissibility factor for lateral response, defined by

$$(TR)_x = -(p_x^2 + i2\zeta_x \omega p_x)H_u = \frac{1 + i2\zeta_x(\omega/p_x)}{1 - (\omega/p_x)^2 + i2\zeta_x(\omega/p_x)} \quad (A.33)$$

On solving equations (A.32) and substituting the resulting values of $\ddot{X}_f$ and $\dot{\Phi}$ into equation (A.31), one obtains the desired $U$.

For the solutions presented in this report, the off-diagonal terms of $[K_f]$ are presumed to be negligible, and the diagonal
terms are denoted by $K_x$ and $K_\phi$. On letting $\varepsilon_i = I_T / mh^2$, $\varepsilon_m = m_j / m$, $\kappa_x = K_x R^2 / (mu_0^2)$ and $\kappa_\phi = K_\phi R^2 / (mh^2 u_0^2)$, the solution for $U$ may be expressed in the form

$$U = H_u T_u \ddot{x}_0$$  \hspace{1cm} (A.34)

in which $\ddot{x}_0 = -\omega^2 x_0$; and $T_u = \text{the dimensionless factor that provides for the inertial interaction effects.}$ The latter factor is given by

$$T_u = \frac{B_3}{B_1 a_0^4 - B_2 a_0^2 + B_3}$$  \hspace{1cm} (A.35)

in which

$$B_1 = (\varepsilon_i + \varepsilon_m)(TR)_x + \varepsilon_i \varepsilon_m$$  \hspace{1cm} (A.36a)

$$B_2 = (\kappa_x + \kappa_\phi)(TR)_x + \varepsilon_m \kappa_\phi$$  \hspace{1cm} (A.36b)

$$B_3 = \kappa_x (\kappa_\phi - \varepsilon_i a_0^2)$$  \hspace{1cm} (A.36c)

For the circular foundations considered, the expressions for $K_x$ and $K_\phi$ are given by (Veletsos and Wei 1971)

$$K_x = \frac{8GR}{(2-\nu)}(\alpha_x + ia_0 \beta_x)$$  \hspace{1cm} (A.37)

$$K_\phi = \frac{8GR^3}{3(1-\nu)}(\alpha_\phi + ia_0 \beta_\phi)$$  \hspace{1cm} (A.38)

in which $\alpha_x$, $\beta_x$, $\alpha_\phi$ and $\beta_\phi$ are the dimensionless factors that depend on Poisson's ratio for the half-space material, $\nu$, and the dimensionless frequency parameter, $a_0$. On making use of the expression $u_s = \sqrt{G/\rho}$, the stiffness factors $\kappa_x$ and $\kappa_\phi$ in equations (A.36) may be written as
\[ \kappa_x = \frac{8}{(2 - \nu)} \frac{pR^3}{m} (\alpha_x + i\alpha_0 \beta_x) \quad (A.39) \]

\[ \kappa_\psi = \frac{8}{3(1 - \nu)} \frac{pR^5}{m\hbar^2} (\alpha_\psi + i\alpha_0 \beta_\psi) \quad (A.40) \]

The psd function for the deformation of the interacting system, \( S_u \), is then given by

\[ S_u = |H_u|^2 |T_u|^2 S_x \quad (A.41) \]
Evaluation of Functions $f_1, f_2, f_3$

The functions $f_1$, $f_2$ and $f_3$ in equations (3.5) are defined by the integrals

$$f_1(\chi, \psi) = \int_{-1}^{1} \int_{-1}^{1} f(\eta_1 - \eta_2) d\eta_1 d\eta_2 \quad (B.1)$$

$$f_2(\chi, \psi) = \int_{-1}^{1} \int_{-1}^{1} \eta_1 \eta_2 f(\eta_1 - \eta_2) d\eta_1 d\eta_2 \quad (B.2)$$

$$f_3(\chi, \psi) = \int_{-1}^{1} \int_{-1}^{1} \eta_2 f(\eta_1 - \eta_2) d\eta_1 d\eta_2 \quad (B.3)$$

in which

$$f(\eta_1 - \eta_2) = \frac{1}{4} \exp \left[ -\frac{\chi^2}{4} (\eta_1 - \eta_2)^2 - i \chi \psi (\eta_1 - \eta_2) \right] \quad (B.4)$$

On introducing the variable $w = \eta_1 - \eta_2$ and carrying out the indicated integrations with respect to $\eta_1$, equations (B.1-B.3) reduce to

$$f_1(\chi, \psi) = \int_{0}^{2} (2-w)(f(w) + f(-w)) dw \quad (B.5)$$

$$f_2(\chi, \psi) = \frac{1}{6} \int_{0}^{2} (4-6w+w^2)(f(w) + f(-w)) dw \quad (B.6)$$

$$f_3(\chi, \psi) = -\frac{1}{2} \int_{0}^{2} w(2-w)(f(w) - f(-w)) dw \quad (B.7)$$

The functions $f(w)$ and $f(-w)$ are then replaced by the expressions defined by equation (B.4) to yield
\[ f_1(x, \psi) = \frac{1}{2} \Re \left\{ \int_0^2 (2-w) \exp \left[ -\left( \frac{x^2}{4} w^2 + i \chi \psi w \right) \right] dw \right\} \quad (B.8) \]

\[ f_2(x, \psi) = \frac{1}{12} \Re \left\{ \int_0^2 (4-6w+w^3) \exp \left[ -\left( \frac{x^2}{4} w^2 + i \chi \psi w \right) \right] dw \right\} \quad (B.9) \]

\[ f_3(x, \psi) = -\frac{1}{4} \Im \left\{ \int_0^2 w(2-w) \exp \left[ -\left( \frac{x^2}{4} w^2 + i \chi \psi w \right) \right] dw \right\} \quad (B.10) \]

which, on letting \( z = \chi(w/2) + i \psi \), become

\[ f_1(x, \psi) = \Re \left\{ \frac{2e^{-\psi^2}}{x^2} \int_{i\psi}^{x+i\psi} [(\chi+i\psi) - z] \exp(-z^2) dz \right\} \quad (B.11) \]

\[ f_2(x, \psi) = \frac{1}{3} \Re \left\{ \frac{2e^{-\psi^2}}{x^4} \int_{i\psi}^{x+i\psi} [(\chi^3 + i\psi(3\chi^2 + 2\psi^2)) - 3(\chi^2 + 2\psi^2)z - 6i\psi z^2 + 2z^3] \exp(-z^2) dz \right\} \quad (B.12) \]

\[ f_3(x, \psi) = -i \Im \left\{ \frac{2e^{-\psi^2}}{x^3} \int_{i\psi}^{x+i\psi} [(\psi^2 - i\chi \psi) + (\chi + 2i\psi)z - z^2] \right. \]

\[ \left. \exp(-z^2) dz \right\} \quad (B.13) \]

Finally, use is made of the identities (Abromowitz and Stegun 1970)

\[ \int_0^z \exp(-t^2) dt = \frac{\sqrt{\pi}}{2} \Phi(z) \quad (B.14) \]

\[ \int_0^z t \exp(-t^2) dt = \frac{1}{2} [1 - \exp(-z^2)] \quad (B.15) \]

\[ \int_0^z t^2 \exp(-t^2) dt = \frac{1}{2} \left[ -z \exp(-z^2) + \frac{\sqrt{\pi}}{2} \Phi(z) \right] \quad (B.16) \]
\[
\int_0^z t^3 \exp(-t^2) dt = \frac{1}{2} [1 - (1 + z^2) \exp(-z^2)]
\]  \hspace{1cm} (B.17)

to obtain equations (3.9a-d).
APPENDIX C

C.1 Properties of Square Embedded Foundations

The total contact area, $A$, and the height of the centroid of contact area measured from the bottom of foundation, $\bar{h}$, for the square embedded foundation of width, $2a$, and depth of embedment, $e$, are given by

$$A = 4a^2[1 + 2(e/a)] \quad (C.1)$$

$$\bar{h} = a \frac{(e/a)^2}{[1 + 2(e/a)]} \quad (C.2)$$

The moments of inertia about $x$ and $y$ axes at the bottom center of foundations, $I_{bx}$ and $I_{by}$ are given by

$$I_{bx} = I_{by} = \frac{4a^4}{3} \left[2(e/a)^3 + 4(e/a) + 1\right] \quad (C.3)$$

The moments of inertia with respect to system of axes passing through centroid are given by

$$I_x = I_y = I_{bx} - A\bar{h}^2 \quad (C.4)$$

The torsional moment of inertia about $z$-axis is given by

$$I_z = \frac{8a^4}{3} \left[1 + 4(e/a)\right] \quad (C.5)$$

C.2 Free-field Contact Traction

The strain-displacement relations for the unidirectional free-field motion, $\{U_g(\mathbf{x})\} = (0, U_{gy}(\mathbf{x}), 0)^T$, are given by
\[ \varepsilon_x = \frac{\partial U_{yx}}{\partial x} = 0 \]  \hspace{1cm} (C.6a)

\[ \varepsilon_y = \frac{\partial U_{yx}}{\partial y} \]  \hspace{1cm} (C.6b)

\[ \varepsilon_z = \frac{\partial U_{yx}}{\partial z} = 0 \]  \hspace{1cm} (C.6c)

\[ \gamma_{xy} = \frac{\partial U_{yx}}{\partial x} + \frac{\partial U_{yx}}{\partial y} = \frac{\partial U_{yx}}{\partial x} \]  \hspace{1cm} (C.6d)

\[ \gamma_{xz} = \frac{\partial U_{yx}}{\partial x} + \frac{\partial U_{yx}}{\partial z} = 0 \]  \hspace{1cm} (C.6e)

\[ \gamma_{yz} = \frac{\partial U_{yx}}{\partial z} + \frac{\partial U_{yx}}{\partial y} = \frac{\partial U_{yx}}{\partial y} \]  \hspace{1cm} (C.6f)

in which \( \varepsilon_x \), \( \varepsilon_y \) and \( \varepsilon_z \) are the normal strains along \( x \), \( y \) and \( z \) axes, respectively; and \( \gamma_{xy} \), \( \gamma_{xz} \) and \( \gamma_{yz} \) are the shear strains. Making use of equations (C.6) and stress-strain relations, the stress-field can be related to free-field ground motion by

\[ \sigma_x = \mu G \varepsilon_y = \mu G \frac{\partial U_{yx}}{\partial y} \]  \hspace{1cm} (C.7a)

\[ \sigma_y = (2 + \mu) G \varepsilon_y = (2 + \mu) G \frac{\partial U_{yx}}{\partial y} \]  \hspace{1cm} (C.7b)

\[ \tau_{xy} = G \gamma_{xy} = G \frac{\partial U_{yx}}{\partial x} \]  \hspace{1cm} (C.7d)

\[ \tau_{yz} = G \gamma_{yz} = G \frac{\partial U_{yx}}{\partial z} \]  \hspace{1cm} (C.7e)
\[ \tau_{xz} = G \gamma_{xz} = 0 \] (C.7f)

in which \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) are the normal stresses; \( \tau_{xy} \), \( \tau_{xz} \) and \( \tau_{yz} \) are the shear stresses; and \( \mu \) is a function of Poisson's ratio of medium, \( \nu \), defined by equation (4.19).

The free-field contact traction vector at the bottom of embedded foundation (at \( z = \varepsilon \)) is given by

\[ \{ T_g(x) \} = -(\tau_{xz}, \tau_{yz}, \sigma_z)^T = - \left( 0, \frac{\partial U_{xy}}{\partial z}, \mu \frac{\partial U_{xy}}{\partial y} \right)^T \] (C.8)

The free-field contact traction vectors on the lateral surfaces defined by \( x = \pm \alpha \) are given by

\[ \{ T_g(x) \} = \pi (\sigma_x, \tau_{xy}, \tau_{xz})^T = \pi \left( \mu \frac{\partial U_{xy}}{\partial y}, \frac{\partial U_{xy}}{\partial x}, 0 \right)^T \] (C.9)

The free-field contact traction vectors on the lateral surfaces defined by \( y = \pm \alpha \) are given by

\[ \{ T_g(x) \} = \pi (\tau_{yx}, \sigma_y, \tau_{yz})^T = \pi \left( \frac{\partial U_{xy}}{\partial x}, (\mu + 2) \frac{\partial U_{xy}}{\partial y}, \frac{\partial U_{xy}}{\partial z} \right)^T \] (C.10)

Making use of equations (C.9) through (C.11) and the definition of cross-psd function, the elements of spectral matrices \([ S_T(x_1, x_2, \omega) ]\) and \([ S_{Tg}(x_1, x_2, \omega) ]\) can be written in terms of first and second spatial partial derivatives of the cross-psd for the free-field ground motion, respectively. For example, \( S_{Tyy} \) term on the bottom face of the foundation is given by (See equation 4.8)
\[ S_{Tyy} = \frac{H}{t_0} E[T_{vy} T_{vy}^*] \]  
(C.11)

On substituting the expression for the \( T_{vy} \) given by equation (C.8) and making use of the linear properties of differential and Expectation operators, the expression for the element, \( S_{Tyy} \) can be rewritten as

\[ S_{Tyy} = G^2 \frac{\partial^2}{\partial z_1 \partial z_2} \left( \frac{H}{t_0} E[U_{vy} U_{vy}^*] \right) = G^2 \frac{\partial^2}{\partial z_1 \partial z_2} S_{vy} \]  
(C.12)

Analogous expressions for the each element of \([S_T(\vec{x}_1, \vec{x}_2, \omega)]\) and \([S_{T\phi}(\vec{x}_1, \vec{x}_2, \omega)]\) on each face of the foundation-soil interface can be related to the ground motion cross-PSD function.

C.3 Compliance Matrix for Square Embedded Foundations

Let \( K_z, K_x, K_\phi \) and \( K_\theta \) be the complex-valued foundation impedances of massless foundations in vertical, lateral, rocking and torsional modes of vibration, respectively, and let the \( s \) superscript identify the corresponding static stiffnesses. The point of reference involved in the definitions for \( K_z, K_x, K_\phi \) and \( K_\theta \) is considered to be at the center of foundation base not the center of gravity of the foundation.

For the square embedded foundations considered, the approximate expressions for these dynamic impedances are given by (Pais and Kausel 1988)

\[ K_z = K_z^s (\alpha_z + i\alpha_0 \beta_z) \]  
(C.13)
\[ K_x = K_x^z (\alpha_x + i\alpha_0 \beta_x) \]  
\[ K_\phi = K_\phi^z (\alpha_\phi + i\alpha_0 \beta_\phi) \]  
\[ K_\theta = K_\theta^z (\alpha_\theta + i\alpha_0 \beta_\theta) \]  

in which the static stiffnesses, \( K_x^z \), \( K_\phi^z \), \( K_\theta^z \) and \( K_\theta^z \) are given by

\[ K_x^z = 4.7 \left[ 1 + 0.5 \left( \frac{e}{a} \right)^{0.8} \right] \frac{Ga}{(1-\nu)} \]  
\[ K_\phi^z = 9.2 \left[ 1 + \left( \frac{e}{a} \right)^{0.8} \right] \frac{Ga}{(2-\nu)} \]  
\[ K_\theta^z = 4 \left[ 1 + \frac{e}{a} + 1.1852 \left( \frac{e}{a} \right)^2 \right] \frac{Ga^3}{(1-\nu)} \]  
\[ K_\theta^z = 8.31 \left[ 1 + 2.62 \left( \frac{e}{a} \right)^{0.9} \right] Ga^3 \]

The coefficients \( \alpha_x \), \( \alpha_\phi \), \( \alpha_\theta \) are given by

\[ \alpha_x = 1.0 - \frac{0.6a_\theta^2}{10 + a_\theta^2} \]  
\[ \alpha_\phi = 1 - \frac{0.55a_\theta^2}{2 + a_\theta^2} \]  
\[ \alpha_\theta = 1 - \frac{0.33a_\theta^2}{0.8 + a_\theta^2} \]

The coefficients \( \beta_x \), \( \beta_\phi \), \( \beta_\theta \) are given by

\[ \beta_x = \frac{Ga}{K_x^z} \left[ 4 \left( \alpha + 2 \frac{e}{a} \right) \right] \]
\[ \beta_x = \frac{G\alpha}{K_x} \left[ 4\left(1 + (1 + \alpha)\frac{e}{a}\right) \right] \]  
(C.22b)

\[ \beta_\phi = w_1 \frac{a_0^2}{1.8 + a_0^2} + w_2 \frac{1.8}{1.8 + a_0^2} \]  
(C.22c)

\[ \beta_\theta = w_3 \frac{a_0^2}{1.4 + a_0^2} \]  
(C.22d)

in which
\[ \alpha = \sqrt{\frac{2(1-\nu)}{(1-2\nu)}} \]  
(C.23)

and
\[ w_1 = \frac{G\alpha^3}{K_x} \frac{4}{3} \left[ 4\left(\frac{e}{a}\right) + (1 + \alpha)\left(\frac{e}{a}\right)^3 + \alpha \right] \]  
(C.24a)

\[ w_2 = \frac{G\alpha^3}{K_x} \frac{4}{3} \left[ (1 + \alpha)\left(\frac{e}{a}\right)^3 \right] \]  
(C.24b)

\[ w_3 = \frac{G\alpha^3}{K_x} \frac{8}{3} \left[ 1 + (3 + \alpha)\left(\frac{e}{a}\right) \right] \]  
(C.24c)

The coupling term relating the horizontal and rocking dynamic impedances, denoted by \( K_{x\phi} \), is given by
\[ K_{x\phi} = \frac{e}{3} K_x \]  
(C.26)

When these impedances are referred to the centroid of the foundation-soil interface, only impedance terms corresponding to rocking and coupling terms relating the horizontal and rocking terms would change. The impedance terms at centroid of foundation are denoted by a bar superscript and are given
by

\[ \bar{K}_{x_\phi} = K_{x_\phi} - \bar{h}K_x \quad (C.27) \]

\[ \bar{K}_\phi = K_\phi - 2\bar{h}K_{x_\phi} + \bar{h}^2 K_x \quad (C.38) \]

in which \( \bar{h} \), the height of centroid of the contact area measured from the foundation base, is given by equation (C.2). The impedances \( \bar{K}_z, \bar{K}_x \) and \( \bar{K}_\theta \) will be same as \( K_z, K_x \) and \( K_\theta \).

Finally, the impedance matrix at centroid is inverted to yield desired compliance matrix.
APPENDIX D

D.1 Effects of wave only

For the normal random sea state considered, the probability density function of the wave amplitudes, \( \hat{u} \), of the velocity trace of the fluid motion is a Rayleigh distribution, given by

\[
p(\hat{u}) = \frac{\hat{u}}{\sigma_{\hat{u}}^2} \exp\left(\frac{-\hat{u}^2}{2\sigma_{\hat{u}}^2}\right)
\]  

(D.1)

If \( z \) represents a specified percentage of the absolute maximum velocity amplitude and \( \hat{u}_z \) represents the amplitude corresponding to that percentage, then the cumulative probability of the amplitudes with values greater than \( \hat{u}_z \) is given by

\[
P(\hat{u} \geq \hat{u}_z) = \int_{\hat{u}_z}^{\infty} p(\hat{u})d\hat{u} = 1 - z
\]  

(D.2)

On substituting equation (D.1) into equation (D.2), performing the indicated integration and taking the natural logarithms of the two members of the resulting expression, one obtains

\[
\hat{u}_z = 2\ln[1/(1-z)]\sigma_{\hat{u}}
\]  

(D.3)

Denoted by \( <|\hat{u}_z|> \), the average of the velocity pulses with values of the amplitudes in excess of \( \hat{u}_z \) is then given by

\[
<|\hat{u}_z|> = \frac{\frac{2}{\pi} \int_{\hat{u}_z}^{\infty} \hat{u} p(\hat{u})d\hat{u}}{\frac{2}{\pi(1-z)} \int_{\hat{u}_z}^{\infty} \hat{u}^2 \exp\left(\frac{-\hat{u}^2}{2\sigma_{\hat{u}}^2}\right)d\hat{u}}
\]  

(D.4)

which on integration yields
\[ |\bar{u}_z| = \left( 1 - F \right) \frac{1}{1 - z} \sqrt{\frac{2}{\pi \eta}} \sigma_u \]  

(D.5)

The quantity \( F \) in this expression is defined by equation (6.24), in which \( \eta = \dot{u}^2 / \sigma_u^2 \). The ratio of \( |\bar{u}_z| \) and \( \bar{u}_0 \) represents the factor \( b_0 \) defined by equation (6.23).

D.2 Effects of combinations of wave and current

For a random sea state that includes a current with a constant velocity \( \dot{u}_c \), the probability density function of the peaks of the total velocity trace, \( \dot{v} = \dot{u}_c + \ddot{u} \), is given by (Papoulis 1967)

\[ p(\dot{v}) = \frac{\dot{v}}{\sigma_u^2} I_0 \left( \frac{\dot{u}_c \dot{v}}{\sigma_u^2} \right) \exp \left( -\frac{\dot{v}^2 + \dot{u}_c^2}{2\sigma_u^2} \right) \]  

(D.6)

Proceeding as in the preceding section, the following relationship is obtained between the specified percentage of the absolute maximum value of the total velocity trace, \( z \), and the corresponding total velocity amplitude, \( \dot{v}_z \):

\[ 1 - z = \int_{\dot{u}_c / \sigma_u^2 \sigma_u^2}^{\infty} \frac{\dot{v}}{\sigma_u^2} I_0 \left( \frac{\dot{u}_c \dot{v}}{\sigma_u^2} \right) \exp \left( -\frac{\dot{v}^2 + \dot{u}_c^2}{2\sigma_u^2} \right) d\dot{u} \]  

(D.7)

which, on introducing the parameter \( \theta = \dot{v}^2 / (2\sigma_u^2) \) and properly adjusting the lower limit of integration, can also be written as

\[ 1 - z = \exp \left( -\frac{\dot{u}_c^2}{2\sigma_u^2} \right) \int_{\dot{u}_c / \sigma_u^2 \sigma_u^2}^{\infty} I_0 \left( \frac{\sqrt{2}\dot{u}_c \sqrt{\theta}}{\sigma_u} \right) \exp(-\theta) d\theta \]  

(D.8)
Unlike the corresponding expression in the preceding section from which a closed-form expression could be obtained for $\dot{u}_z$ by formal integration, in the present case this does not appear to be possible, and the value of $\dot{u}_z$ corresponding to $z$ must be computed iteratively by numerical integration. A first approximation to $\dot{u}_z$ may be determined from

$$ \dot{u}_z = 2 \ln \left[ \frac{1}{1 - z} \right] \sigma_u + \dot{u}_c \quad (D.9) $$

Let $\langle |\dot{u}_z| \rangle$ represent the average of the velocity pulses with amplitudes in excess of $\dot{u}_z$. When measured from the level of the current velocity, $\dot{u}_c$, the amplitudes of the velocity pulses have a Rayleigh distribution. Accordingly, the value of $\langle |\dot{u}_z| \rangle$ may be determined from the right-hand member of equation (D.4) noting that $\dot{u}_z = \dot{u}_c - \dot{u}^*$, and interpreting the quantity $(2/\pi)\dot{u}$ in this equation as the average of the absolute value of a velocity pulse of amplitude $\dot{u}$ that is superposed on the current velocity, $\dot{u}_c$. Denoted by $\dot{w}$, the latter quantity is given by

$$ \dot{w} = \frac{1}{2\pi} \int_0^{2\pi} |\dot{u}_c + \dot{u} \sin \tau| d\tau \quad (D.10) $$

in which $\tau$ denotes time. Integration of equation (D.10) yields

$$ \dot{w} = \begin{cases} \dot{u}_c & \text{for } \dot{u} \leq \dot{u}_c \\ \sqrt{\frac{2}{\pi}} \dot{u} \left( \frac{\dot{u}_c}{\dot{u}} \sin^{-1} \left( \frac{\dot{u}_c}{\dot{u}} \right) + \sqrt{1 - \left( \frac{\dot{u}_c}{\dot{u}} \right)^2} \right) & \text{for } \dot{u} > \dot{u}_c \\ \end{cases} \quad (D.11) $$
On substituting equation (D.11) in the reinterpreted version of equation (D.4) and evaluating the integrals involved, one obtains

$$<|\dot{v}_z|> = \exp\left(\frac{\dot{u}_z^2}{2\sigma_u^2}\right)\left\{\sqrt{\frac{2}{\pi}}\sigma_u\exp\left(-\frac{\dot{u}_e^2}{2\sigma_u^2}\right) + \dot{u}_e\left[1 + \text{erf}\left(\frac{\dot{u}_e}{\sqrt{2}\sigma_u}\right)\right]\right\} - \dot{u}_e$$

for $\dot{v}_z \leq 2\dot{u}_e$  \hspace{1cm} (D.12)

and

$$<|\dot{v}_z|> = \frac{2}{\pi}\left\{\dot{u}_e\sin^{-1}\left(\frac{\dot{u}_e}{\dot{u}_z}\right) + \dot{u}_z\sqrt{1 - \left(\frac{\dot{u}_e}{\dot{u}_z}\right)^2}\right\} + \exp\left(\frac{\dot{u}_z^2}{2\sigma_u^2}\right)$$

$$\left\{\sqrt{\frac{2}{\pi}}\sigma_u\exp\left(-\frac{\dot{u}_e^2}{2\sigma_u^2}\right) - \dot{u}_e\left[1 - \text{erf}\left(\frac{\dot{u}_e}{\sqrt{2}\sigma_u}\right)\right]\right\}$$

$$- \frac{2}{\pi}\int_{\dot{u}_e}^{\dot{u}_z}\frac{\sqrt{\dot{u}_z^2 - \dot{u}_e^2}}{\dot{u}}\exp\left(-\frac{\dot{u}_e^2}{2\sigma_u^2}\right)d\dot{u}$$

for $\dot{v}_z > 2\dot{u}_e$  \hspace{1cm} (6A.13)

The factor $b'_0$ is defined by the ratio of $<|\dot{v}_z|>$ and $\dot{u}_0$. For $\dot{u}_e = 0$, equation (D.13) reduces, as it should to equation (D.5).
APPENDIX E

E.1 Notation for Chapters II through V

\( a \)
half-length of rectangular foundation along the direction of free-field ground motion

\( a_0 \)
\( \omega R/v_z = \omega b/v_z = \) frequency parameter

\( a_{0x}, a_{0y} \)
modified frequency parameter for combined wave passage and incoherence effects, defined by Equation (2.11) and Equation (3.21), respectively

\( A \)
foundation contact area

\( A_x \)
\( p_x^2 U_x = \) pseudo-acceleration value of the mean maximum deformation induced by the lateral component of the foundation input motion

\( A_y \)
\( p_y^2 U_y = \) pseudo-acceleration value of the mean maximum deformation induced along the perimeter of the structure by the torsional component of the foundation input motion

\( b \)
half-length of rectangular foundation along the direction of propagation of wave front

\( b_0 \)
\( \gamma a_0 = \) modified frequency parameter for incoherence only

\( b_{0x}, b_{0y} \)
modified frequency parameter for incoherence only, defined by \( \gamma_x a_0 \) and \( \gamma_y a_0 \), respectively

\( B_1, B_2, B_3 \)
dimensionless parameters in expression for \( T_w \), defined by Equations (A.36)

\( B(\chi) \)
dimensionless function defined by equation (3.13)

\( B_1(\chi, \psi), B_2(\chi, \psi), B_3(\chi, \psi), B_4(\chi, \psi) \)
dimensionless functions defined by equations (3.10)
\( c \) or \( c_y \) \( \frac{v_s}{\sin \alpha_y} = \) apparent horizontal velocity of wave front

\([C(\omega)]\) \( (v_s/c)\alpha_0 = (\sin \alpha_y)\alpha_0 = \) modified frequency parameter for wave passage effect only

\( c_0 \) or \( c_{0y} \) components of \( \vec{r}_1 \) and \( \vec{r}_2 \) in the direction of propagation of the seismic wave front

\( d_1, d_2 \) depth of embedment

\([F(\vec{x}, \omega)]\) \( 3 \times 6 \) matrix in which each column corresponds to the traction vector at position \( \vec{x} \) on the foundation soil interface for unit harmonic generalized forces and moments applied to the rigid foundation in the same order as generalized displacement components of \( \{U_0(\omega)\} \)

\( f_0 \) cut-off frequency of excitation

\( f_1(\chi, \psi), f_2(\chi, \psi) \) dimensionless functions defined by equations (3.9)

\( f_x, f_x \) natural frequencies of rigidly and elastically supported structures in lateral mode of vibration, in cps

\( f_\theta, f_\theta \) natural frequencies of rigidly and elastically supported structures in torsional mode of vibration, in cps

\( G \) \( \rho v_s^2 = \) the shear modulus of supporting medium

\( G(\omega) \) one sided-power spectral density function for a stationary Gaussian random process

\( h \) height of structure

\( \bar{h} \) height of centroid of contact area measured from the foundation base
$H_u, H_v$  
transfer functions relating the lateral and torsional deformations of the structure to the corresponding components of the foundation input acceleration

$i$  
$\sqrt{-1}$

$I_0, I_1, I_2$  
modified Bessel functions of the first kind of order zero, one and two, respectively

$I, I_f$  
mass moments of inertia of structure and foundation about a horizontal centroidal axis

$I_0$  
polar area moment of inertia of foundation about a vertical centroidal axis

$I_x, I_y, I_z$  
area moments of inertia of foundation about centroidal axes parallel to $x, y$ and $z$ respectively

$J_1, J_2$  
Bessel functions of first kind of order one and two, respectively

$J, J_f$  
polar mass moments of inertia of structure and foundation about a vertical centroidal axis

$K_z, K_x, K_\theta, K_\phi$  
complex-valued foundation impedances of massless foundations in vertical, lateral, rocking and torsional modes of vibration with point of reference for force-displacement relations defined at center of foundation base

$m, m_f$  
mass of structure and foundation, respectively

psd  
power spectral density function

$p_x$  
$2\pi f_x = \text{fixed-base circular natural frequency of the structure in lateral mode of vibration}$

$p_\theta$  
$2\pi f_\theta = \text{fixed-base circular natural frequency of the structure in torsional mode of vibration}$

$[P], [Q], [R]$  
matrices defined by Equations (4.10-4.11)

$q$  
Vanmarcke's band width parameters
\( Q_x, Q_y, Q_\theta \) lateral force, overturning moment and torsional moment at foundation-soil interface

\( \vec{r}_1, \vec{r}_2 \) position vectors for two arbitrary points on foundation-soil interface

\( R \) radius of foundation

\( S(\vec{r}_1, \vec{r}_2, \omega) \) cross psd function for motions at points \( \vec{r}_1 \) and \( \vec{r}_2 \)

\( [S(\omega)] \) the power spectral density matrix of size \( 6 \times 6 \) for the foundation input motion at centroid of foundation.

\( [S'(\omega)] \) the power spectral density matrix of size \( 6 \times 6 \) for the foundation input motion at bottom center of embedded foundation.

\( S_0 \) constant in expression for psd function of the free-field ground acceleration

\( S_\theta, S_\phi, S_\phi \) local psd functions for the displacement, velocity and acceleration histories of the free-field ground motion

\( [S_x(\vec{x}_1, \vec{x}_2, \omega)] \) a \( 3 \times 3 \) matrix of the cross psd functions for the components of free-field ground for points \( \vec{x}_1 \) and \( \vec{x}_2 \) of the half-space.

\( [S_t(\vec{x}_1, \vec{x}_2, \omega)] \) a \( 3 \times 3 \) matrix of the cross psd functions of free-field tractions for points \( \vec{x}_1 \) and \( \vec{x}_2 \) of the half-space along the foundation-soil interface.

\( [S_{t_\theta}(\vec{x}_1, \vec{x}_2, \omega)] \) a \( 3 \times 3 \) matrix of the cross psd functions for the tractions at point \( \vec{x}_1 \) and the associated motions at point \( \vec{x}_2 \)

\( S_x, S_x, S_x \) psd functions for the displacement, velocity and acceleration histories of the lateral component of foundation input motion
$S_y, S_\dot{y}, S_\ddot{y}$  psd functions for the displacement, velocity and acceleration histories of the motion along the perimeter of the foundation induced by the torsional component of foundation input motion

$S_{xy}, S_{x\dot{y}}$  cross psd functions for the horizontal and torsional components of the foundation input displacement and foundation input acceleration, respectively

$S_{x\dot{x}}$  cross psd functions for the horizontal and rocking components of the foundation input acceleration

$S_u$  psd function for the structural deformation induced by the lateral component of foundation input motion

$S_v$  psd function for the deformation $v = \psi R$ induced at the periphery of the structure by the torsional component of the foundation input motion

$S_w$  psd function for the total deformation at the most highly stressed point on the periphery of the structure

$S_z$  psd function for the linear acceleration induced at height $\alpha$ by rocking component of foundation input motion.

$\{T_g(\vec{x})\}$  $(T_{gx}, T_{gy}, T_{gz})^T$ = the interface traction vector at position $\vec{x}$ associated with the free-field ground motion $\{U_g(\vec{x})\}$

$t_0$  duration of strong motion portion of earthquake

$T_u, T_v$  dimensionless transfer factors that provide for the effects of inertial interaction for laterally and torsionally excited systems

$(TR)_x$  transmissibility of laterally excited system defined by Equation (A.33)

$(TR)_\theta$  transmissibility of torsionally excited system defined by Equation (A.21)
\{U_g(\vec{x}, \omega)\} (U_{gx}, U_{gy}, U_{gz})^T = a 3 \times 1 vector of free-field ground motion with components along three principal directions x, y, z respectively

\{U_0(\omega)\} (U_{0x}, a\Theta_y, U_{0y}, a\Theta_x, U_{0z}, a\Theta_z)^T = a 6 \times 1 generalized foundation input motion vector at center of foundation with translational components along three principal directions and with normalized rotations about the three principal axes

\[ U_x \] mean value of maximum structural deformations induced by lateral component of foundation input motion

\[ U_y \] mean value of maximum deformations induced along the perimeter of the structure by the torsional component of foundation input motion

\[ \nu \] \[ \psi R \] = structural deformation induced along the perimeter of the structure by the torsional component of foundation input motion

\[ v_s \] shear wave velocity for the soil medium

\[ V_x \] \[ p_x U_x = \text{pseudo-velocity value corresponding to } U_x \]

\[ V_y \] \[ p_y U_y = \text{pseudo-velocity value corresponding to } U_y \]

\[ w \] \[ u + \nu = \text{total deformation at the most highly stressed point at periphery of the structure} \]

\[ x \] lateral component of foundation input displacement

\[ x_f \] lateral component of actual foundation displacement

\[ X, \dot{X}, \ddot{X} \] mean maximum values of the horizontal components of the displacement, velocity and acceleration histories of the foundation input motion

\[ X_0, \dot{X}_0, \ddot{X}_0 \] mean maximum values of the horizontal components of the displacement, velocity and acceleration histories of the free-field, control point ground motion

\[ X_0 \] amplitude of x for harmonic motion
\( Y, \dot{Y}, \ddot{Y} \) mean maximum values of the displacement, velocity and acceleration at the periphery of the foundation induced by the torsional component of the foundation input motion

\( Z, \dot{Z}, \ddot{Z} \) mean maximum values of the linear displacement, velocity and acceleration induced by the rocking component of the foundation input motion

\[ \alpha = \sqrt{2(1-\nu)/(1-2\nu)} \]

\( \alpha_v \) angle of incidence of seismic waves, measured from vertical axis

\[ [\alpha(\vec{x})] \] a matrix of size 3\( \times \)6 representing a rigid-body motion influence matrix for generalized displacements

\( \alpha_z, \alpha_x, \alpha_y, \alpha_0 \) dimensionless stiffness coefficients in expressions for foundation impedances \( K_z, K_x, K_y \) and \( K_0 \)

\( \beta_z, \beta_x, \beta_y, \beta_0 \) dimensionless damping coefficients in expressions for foundation impedances \( K_z, K_x, K_y \) and \( K_0 \)

\( \gamma \) dimensionless incoherence parameter

\( \tilde{\gamma} \) \( \gamma c/v_z \) = modified incoherence parameter

\( \Gamma \) spatial coherence function for the free-field ground motion

\( \delta \) mass density ratio for the structure

\( \Delta_1, \Delta_2 \) dimensionless distance parameters, defined by Equations (2.6)

\( \epsilon_i \) \( I_T/m_h^2 \) = dimensionless measure of mass moment of inertia of structure-foundation system about a horizontal centroidal axis

\( \epsilon_m \) \( m_f/m \) = mass ratio of foundation and structure

\( \zeta_x, \zeta_0 \) percentages of critical structural damping for fixed-base structure in lateral and torsional modes of vibration, respectively
\[ \xi_x, \xi_\theta \] effective structural damping factors for elastically supported system in lateral and torsional modes of vibration, respectively

\[ \theta_1, \theta_2 \] circumferential co-ordinates of points \( \tilde{r}_1 \) and \( \tilde{r}_2 \), respectively

\[ \kappa_x \] \[ K_x \frac{R^2}{(mv_x^2)} \] dimensionless measure of lateral foundation impedance

\[ \kappa_\theta \] \[ K_\theta \frac{R^2}{(Ju_\theta^2)} \] dimensionless measure of torsional foundation impedance

\[ \kappa_\phi \] \[ K_\phi \frac{R^2}{(mh^2v_\phi^2)} \] dimensionless measure of rocking foundation impedance

\[ \lambda_n \] \( n \)th moment of one-sided power spectral density given by Equation (A.10)

\[ \mu \] \[ \frac{2\nu}{(1-2\nu)} \]

\[ \bar{\mu} \] mean rate of zero crossings for stationary process

\[ \bar{\mu}_e \] effective mean rate of zero crossings

\[ \nu \] Poisson's ratio of soil medium

\[ \xi_1, \xi_2 \] normalized radial coordinates of points \( \tilde{r}_1 \) and \( \tilde{r}_2 \), respectively

\[ \rho \] mass density of soil medium

\[ \tau \] \[ \frac{R}{v_x} = \frac{a}{v_x} = \text{transit time} \]

\[ \tilde{\tau} \] \[ \sqrt{\gamma^2 + \sin^2 \alpha_v} \tau = \text{effective transit time} \]

\[ \phi \] actual rocking displacement of foundation

\[ \Phi \] complex-valued amplitude of \( \phi \) for harmonic motion

\[ \chi \] torsional component of foundation input motion

\[ \chi \] complex-valued amplitude of \( \chi \) for harmonic motion
\( \chi_f \) torsional component of actual displacement of foundation

\( X_f \) complex-valued amplitude of \( \chi_f \) for harmonic motion

\( \psi \) torsional deformation of structure

\( \Psi \) complex-valued amplitude of \( \psi \) for harmonic motion

\( \omega \) circular frequency of excitation and resulting motion

**E.2 Notation for Chapter VI**

\( b_0 \) constant in expression for the hydrodynamic damping determined by equivalent linearization technique

\( b'_0 \) constant in expression for the hydrodynamic damping determined by the decoupling techniques

\( b'_{0,j} \) value of \( b'_0 \) at \( j \)-th submerged node

\( c \) coefficient of viscous structural damping

\( c_0, c'_0 \) coefficients of added damping due to effect of fluid-structure interaction

\( F \) chi-square probability function with three degrees of freedom

\( f \) natural frequency of the structure, in Cps

\( k \) structural stiffness

\( k_i^* \) generalized structural stiffness for the \( i \)-th natural mode of vibration

\( m \) total mass of structure, including added mass due to hydrodynamic inertia effect

\( m_i^* \) generalized total mass for \( i \)-th natural mode of vibration
\( P \)  
maximum value of total hydrodynamic force due to wave only

\( P_d \)  
maximum value of drag component of hydrodynamic force due to wave only

\( P_{d_j} \)  
value of \( P_d \) for \( j \)-th submerged node

\( P_i \)  
maximum value of inertia component of hydrodynamic force due to wave only

\( \dot{u} \)  
fluid particle velocity due to wave only

\( \dot{u}_c \)  
velocity of current

\( \dot{u}_0 \)  
absolute maximum value of \( \dot{u} \)

\( \dot{u}_{0j} \)  
value of for \( j \)-th submerged node

\( \dot{u}_z \)  
threshold amplitude of fluid velocity used in computation of \( b^{'}_0 \)

\( \dot{u} \)  
\( \dot{u} + \dot{u}_z = \) total fluid particle velocity due to wave and current

\( \dot{u}_z \)  
threshold amplitude of total fluid particle velocity used in the computation of \( b^{'}_0 \)

\( \dot{u}_{zj} \)  
value of for \( j \)-th submerged node

\( x \)  
structural displacement, measured from undeflected position of structure

\( x_0 \)  
temporal mean of the displacement of the structure due to combination off the wave and current loadings

\( x_{st} \)  
static displacement of structure due to peak value of the total hydrodynamic force induced by the wave component of loading

\( (x_{st})_d \)  
static displacement of structure due to peak value of the drag force induced by the wave component of loading
\( \alpha \)  
threshold percentage of velocity amplitudes considered in computation of the temporal average of fluid particle velocity

\( \delta \)  
dimensionless fluid-structure interaction parameter defined by equation (6.5)

\( \delta_{ij} \)  
dimensionless fluid-structure interaction parameter for the \( j \)-th submerged node of a system vibrating in \( i \)-th natural mode; given by equation (6.28)

\( \Delta F(t) \)  
difference in effective exciting forces for interacting and non-interacting systems subjected to a wave loading only

\( \Delta x(t) \)  
difference in displacements for interacting and non-interacting systems subjected to a wave loading only

\( \{ \phi_i \} \)  
\( i \)-th modal vector

\( \zeta \)  
structural damping factor, in percent of critical damping

\( \xi \)  
\( \zeta + \zeta_0 = \) total system damping factor, in percent of critical damping

\( \xi_i \)  
value of \( \xi \) for \( i \)-th mode of vibration

\( \zeta_0 \)  
added damping factor in percent of critical damping, approximating effects of fluid-structure interaction

\( \zeta_{0i} \)  
value of \( \zeta_0 \) for \( i \)-th mode of vibration

\( \sigma_u \)  
standard deviation of fluid particle velocity due to wave only

\( \omega \)  
circular natural frequency of simple oscillator

\( \omega_i \)  
\( i \)-th circular natural frequency of system