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Simulation of viscous fingering during miscible displacement in nonuniform porous media

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SIMULATION OF VISCOS FINGERING DURING MISCEBLE DISPLACEMENT IN NONUNIFORM POROUS MEDIA

by

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Simulation of Viscous Fingering during Miscible Displacement in Nonuniform Porous Media

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ABSTRACT

Numerical simulation is used to study the effect of different factors on unstable miscible displacement and to clarify the mechanisms of finger growth and interaction. The two-dimensional equations of miscible displacement in a rectangular slab are dedimensionalized and the factors that affect their solution are combined into dimensionless parameters. These parameters are the viscosity ratio, the aspect ratio (ratio of longitudinal to transverse dimension), Peclet numbers for molecular, longitudinal and transverse dispersion and the gravity number.

To study the effect of the structure of the porous medium, simulations are performed on different random permeability fields, generated by a statistical method, so that they have a given coefficient of permeability variation and a given correlation length.

The concentration equation is solved by an implicit finite element modified method of characteristics, which performs backward characteristic tracking. A mixed finite element method is used for the solution of the pressure
The initial number and locations of fingers are dictated by the permeability distribution near the inflow end. The initial number of fingers is reduced by shielding and merging to a smaller number of "active fingers". Large viscosity ratio, aspect ratio, correlation length and coefficient of permeability variation facilitate merging and reduce the number of active fingers. With these parameters fixed, the latter is largely independent of the specific permeability distribution.

Growth of individual fingers is approximately linear in time, provided they do not interact with other fingers. Root mean square (RMS) length also grows linearly. The RMS growth rate increases with increasing viscosity ratio and seems to approach an asymptotic value as the viscosity ratio tends to infinity.

As the correlation length increases, RMS growth rate passes through a maximum. Large heterogeneity of the medium results in large RMS growth rate; the effect of heterogeneity increases with increasing correlation length.

For large gravity numbers, the displacement is dominated by gravity override. In that case a gravity tongue forms and fingering is suppressed. The tongue breaks through early and recovery efficiency after breakthrough is greatly reduced. The effect of gravity weakens as the aspect ratio increases.
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ΣΤΟΥΣ ΓΟΝΕΙΣ ΜΟΥ
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INTRODUCTION

Oil recovery processes are often plagued by an instability of flow, known as viscous fingering, which is due to the fact that the displacing fluid is less viscous than oil. This phenomenon is especially important in miscible enhanced oil recovery processes. Extensive viscous fingering greatly reduces the efficiency of the displacement, because most of the injected fluid flows through the fingers that form, bypassing much of the reservoir oil. Since many of the fluids used in tertiary oil recovery are expensive chemicals, a sufficiently accurate prediction of the performance of the flood, which takes viscous fingering into account, may be critical in decisions on the applicability of a particular recovery process.

Small fingers are generated by heterogeneities in the porous medium. If the flow is unstable they grow and dominate the displacement. The extent of viscous fingering, and consequently the performance of miscible displacement, depend on a number of factors, some of which are associated with the structure of the porous medium and some related to the properties of the fluids and the flow.

The objective of this work is to investigate the effect of these factors on unstable incompressible miscible displacement by means of numerical simulation. The
mathematical model of miscible displacement consists of a system of two coupled differential equations, one for solvent concentration and one for pressure and velocity. The concentration equation, which is a convection-diffusion equation, dominated by the convection term, is solved by a finite-element modified method of characteristics. A mixed finite element method is used for the solution of the elliptic pressure-velocity equation.

In Chapter 1 the equations that describe a two-dimensional incompressible miscible flood in linear geometry are formulated. They are then dedimensionalized, to identify the dimensionless numbers on which the solution depends. A brief literature review follows, explaining viscous fingering and covering various models and methods that have been proposed to model the phenomenon. In Chapter 2 the numerical method that is used for the simulations is described and tested for convergence, conservation of mass and artificial dispersion. The effect of the parameters that were obtained in Chapter 1 on unstable miscible displacement is studied in Chapter 3. A separate section deals with the effect of gravity. In Chapter 4 random permeability fields having given mean, standard deviation and correlation length are synthesized. They are then used in simulations to investigate the effect of the structure of the porous medium on viscous fingering. Results are summarized and discussed in Chapter 5.
CHAPTER 1

A MATHEMATICAL MODEL OF MISCIBLE DISPLACEMENT

Oil reservoirs are underground sedimentary formations of porous rocks, whose pores contain oil. In most reservoirs, upon the opening of a well, high reservoir pressure or other natural mechanisms associated with the reservoir help drive to the wellbore a portion of the oil; this stage of production, called "primary recovery", typically accounts for recovery of 5 to 20% of the oil [51]. Secondary and tertiary recovery involve injection of a fluid at certain wells in the reservoir, with the intention of displacing oil from the pores and driving it to the production wells.

Displacement processes are characterized as immiscible or miscible, depending on the miscibility of the displacing fluid with oil. Waterflooding is a typical immiscible process, used for secondary recovery. In immiscible displacement processes, high interfacial tension between oil and the injected fluid results in incomplete displacement, leaving a large amount of residual oil in the swept volume of the reservoir. In miscible displacement the interfacial tension is much lower or null, which leads to significantly lower
residual oil saturations.

The term "miscible displacement" usually includes two
different types of injection fluids: The fluids of the
first type, called "first-contact miscible", can mix with
oil in all proportions, forming a single phase. The fluids
belonging to the other category are not miscible with oil at
first contact. However in-situ mass transfer of components
between these fluids and oil forms a displacing phase with a
transition zone of fluid compositions, ranging from oil to
injection fluid composition with all compositions in this
transition zone being contiguously miscible. This type of
miscibility is called "dynamic miscibility" or "multiple
contact miscibility". In this study only first-contact mis-
cible injection fluids will be considered. Miscible EOR
processes have been developed and tested since the early
1950's. Injection fluids that have been considered include
hydrocarbon solvents, flue gas, nitrogen and carbon dioxide.
The CO₂ process is relatively recent (since 1970) and, be-
aving of several attractive features, is the subject of in-
tense research interest and testing [51].

1.1 DERIVATION OF THE EQUATIONS OF MISCIBLE DISPLACEMENT

A mathematical model of the miscible displacement pro-
cess will be constructed, with the following assumptions :
- Both the invading and the displaced fluid will be considered as mixtures of two components, which will be referred to as "solvent" and "oil"; the invading fluid is richer in solvent than the displaced fluid.
- Oil and solvent mix perfectly in all proportions forming a single phase.
- Both components and their mixtures are incompressible, which means that density is independent of pressure.
- Flow is two-dimensional.

Clearly the above assumptions constitute an idealization of the physical situation. Reservoir oil is too complicated chemically to be considered a two-component mixture and so are many of the chemicals used in oil recovery. Furthermore, there is always water in oil reservoirs, which may flow along with the solvent-oil mixture, so that the one-phase flow assumption does not hold. In fact most reservoirs have been waterflooded before miscible displacement processes are applied, so that the emergence of more than one phase is unavoidable. The incompressibility assumption may be quite realistic under a wide range of conditions. On the other hand, flow in reservoirs is tridimensional; the assumption of bidimensional flow may be realistic when one of the dimensions of the reservoir is very small, compared with the other two.

With the above assumptions, all properties of a solvent-oil mixture are functions of its composition only.
The composition of a mixture will be expressed by the volume fraction of the solvent in the mixture, denoted by \( C \). A mass balance for the solvent yields:

\[
\frac{\partial (\phi C)}{\partial t} + \nabla \cdot (uC) - \nabla \cdot (D \nabla C) = Q
\]

(1.1.1)

where \( \phi \) is the porosity of the porous medium, \( u \) is the superficial velocity, defined as the flowrate per unit cross sectional area, \( Q \) is a term representing injection and production wells and \( D \) is the dispersion tensor, explained below. The first term in the left hand side of equation (1.1.1) represents the accumulation of solvent, the second term relates to convective transport and the third term to diffusive and dispersive transport of solvent.

The dispersion tensor \( D \) is a combination of molecular diffusion and hydrodynamic dispersion. Hydrodynamic dispersion describes the additional mixing resulting from flow through the tortuous and irregular network of interconnected pores forming the porous medium. It has been experimentally observed [3] that flow channels in a pack of sand grains separate and meet again; the result is that different fluid elements travel different distances and the fluid is thus dispersed. This phenomenon can be modeled by a diffusion-like coefficient. Since the amount of dispersion depends on the distance traveled per unit time, this dispersion coefficient should be proportional to velocity. Laboratory experiments with core samples of reservoir rocks indicated
that the dispersion coefficient is proportional to velocity raised to the power 1.2 [39].

The mechanical mixing of fluid is different in the direction of flow and in the direction normal to the flow. Hence two dispersion coefficients are necessary, a longitudinal and a transverse dispersion coefficient. The transverse coefficient may be thirty times smaller than the longitudinal one. For a two-dimensional system the dispersion tensor is given by [38]:

\[
D = \begin{bmatrix}
D_{11} & D_{12} \\
D_{12} & D_{22}
\end{bmatrix}
\]

with:

\[
D_{11} = \phi D_m + \frac{d_1}{|u|} u^2 + \frac{d_t}{|u|} v^2
\]

\[
D_{12} = \frac{(d_1 - d_t)}{|u|} uv
\]

\[
D_{22} = \phi D_m + \frac{d_1}{|u|} v^2 + \frac{d_t}{|u|} u^2
\]

where \( u \) and \( v \) are the two components of the superficial velocity vector, \( d_1 \) and \( d_t \) are the longitudinal and transverse dispersion mixing lengths respectively, \( D_m \) is the molecular diffusion coefficient and \( |u| = \sqrt{u^2 + v^2} \).

The combined diffusive-dispersive flux, which appears in equation (1.1.1) is:

\[
J = - D \nabla C
\]

In summary, introduction of the dispersion tensor
accounts for the effect of heterogeneity on the pore scale, where a continuum description of the displacement is not possible and allows the use of quantities averaged over the scale of a few pores, which are necessary for a continuum description. In flow through porous media dispersion is the dominant mixing mechanism and in many cases molecular diffusion is insignificant.

On a macroscopic level, fluid flow in porous media is dominated by the viscous properties of the fluid, by the resistance of the porous rock to flow and by gravity. Fluid velocity is related to the pressure gradient through Darcy's law:

\[ u = - \frac{k}{\mu} (\nabla p - \rho g \nabla h) \]  

(1.1.3)

where: \( u \) is the superficial (or Darcy) velocity,
\( p \) is the pressure,
\( k \) is the permeability of the rock,
\( \mu \) is the fluid viscosity,
\( \rho \) is the fluid density,
\( g \) is the acceleration of gravity and
\( h \) is the depth, which is a function of position \( (x) \).

The quotient \( \Lambda = k/\mu \) is known as the mobility of the fluid.

It must be noted that equation (1.1.3) is valid only on the macroscopic scale. In the pore scale the convection mechanisms become very complex and do not obey Darcy's law.
Darcy's law is derived from the pore scale Navier-Stokes equations by a homogenization process of local volume averaging [49]. The characteristic length for averaging is much larger than the pore length (~10^2 microns) but much smaller than the macroscopic dimension of the porous medium (~1 to 10^3 meters). Thus the permeability k is meaningful only on a scale that comprises a large number of pores and it varies as a function of position within the porous medium. In fact permeability may be different in different directions and k should be a tensor. In this study permeability will be assumed isotropic, in which case the permeability tensor can be replaced by a scalar k.

For an incompressible fluid the equation of continuity is:
\[ \nabla \cdot \mathbf{u} = \tilde{q} \]  
(1.1.4)
where \( \tilde{q} \) is a flowrate associated with injection and production wells.

Equations (1.1.3) and (1.1.4) can be combined in a single equation:
\[ -\nabla \cdot \left[ \frac{k}{\mu} (\nabla p - \rho g \mathbf{h}) \right] = \tilde{q} \]  
(1.1.5)

The coupled system consisting of the "pressure equation" (1.1.5) and the "concentration equation" (1.1.1) constitutes a mathematical model for miscible displacement, under the assumptions made above. The boundary conditions depend on the geometry of the problem.
1.2 SIMPLIFICATION AND DEDIMENSIONALIZATION OF THE MISCIBLE DISPLACEMENT EQUATIONS FOR LINEAR GEOMETRY

The equations (1.1.1) to (1.1.5) can be simplified in some special cases. Assume a slab of a porous medium in the shape of a rectangular parallelepiped, with one dimension being much smaller than the other two, so that it can be considered practically two-dimensional. Let \( x \) and \( y \) denote the spatial coordinates and let \( L_x \) and \( L_y \) be the corresponding dimensions of the slab. The porosity \( \phi \) of the slab is assumed constant. The pore space is initially filled with resident fluid (\( C=C_0 \)). The slab is flooded at the side \( x=0 \) with injection fluid (\( C=C_1 \)), starting at \( t=0 \). So \( x \) is the principal flow direction and \( y \) is the direction transverse to the flow. Under these assumptions, equations (1.1.1), (1.1.3) and (1.1.4) can be written in scalar mode as follows:

\[
\begin{align*}
    u & = \frac{k}{\mu} \left( \frac{\partial p}{\partial x} - \rho g h_x \right) \quad (1.2.1) \\
    v & = \frac{k}{\mu} \left( \frac{\partial p}{\partial y} - \rho g h_y \right) \quad (1.2.2) \\
    \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & = 0 \quad (1.2.3) \\
    \phi \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} & = \frac{\partial}{\partial x} \left( D_{11} \frac{\partial C}{\partial x} + D_{12} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left( D_{22} \frac{\partial C}{\partial x} + D_{22} \frac{\partial C}{\partial y} \right) \quad (1.2.4)
\end{align*}
\]

where \( h_x = \partial h / \partial x \) and \( h_y = \partial h / \partial y \) are constants, whose values depend on the geometrical orientation of the slab, so that \( h = h_x x + h_y y \). The components of the dispersion tensor are...
given by (1.1.2).

The initial condition is: \( C = C_0(x,y) \)

(1.2.5)

The boundary conditions are:

At \( x=0 \): \( C = C_i \)

(1.2.6)

\( p = p_0 + \rho gh \)

(1.2.7)

At \( x=L_x \): \( \phi \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = 0 \)

(1.2.8)

\( p = p_1 + \int_0^y \rho g h_y ds \)

(1.2.9)

At \( y=0 \) and \( y=L_y \): \( \frac{\partial C}{\partial y} = 0 \)

(1.2.10)

\( v = 0 \)

(1.2.11)

Any function \( C_0(x,y) \) can be specified as initial condition. Equation (1.2.6) is self-explanatory. Equations (1.2.10) and (1.2.11) simply indicate that there is no flow or transport across the boundary at \( y=0 \) and \( y=L_y \). Equations (1.2.7) and (1.2.9) state that the pressure is equal to a constant plus the hydrostatic pressure. \( p_0 \) and \( p_1 \) can be functions of time. For incompressible flow only the difference \( p_0 - p_1 \) is important. Hence, without loss of generality, it can be assumed that \( p_1 = 0 \). In coreflooding experiments either \( p_0 - p_1 \) is kept constant throughout the experiment or the flowrate per unit area \( q \) is kept constant and the values of \( p_0 \) and \( p_1 \) vary accordingly. The latter approach will be adopted here. Equation (1.2.8) is a "reflection boundary condition"; its physical interpretation is that there is no dispersion of fluid across the outflow boundary and it is
derived by a balance of solvent in a control volume at the boundary, under this assumption.

To complete the statement of the problem, equations of state are needed, to relate density and viscosity to concentration. It will be assumed that mixing is ideal, i.e. that volume is conserved, so that:

$$\rho = \rho_s (1-C) \rho_o$$

(1.2.12)

where $\rho_s$ and $\rho_o$ are the solvent and oil densities respectively. Viscosity of an oil-solvent mixture will be assumed to follow the quarter-power mixing law, commonly applicable to hydrocarbon mixtures [26]:

$$\mu = \left[ \frac{C \mu_s - 0.25 + (1-C) \mu_o - 0.25}{2} \right]^{-4}$$

(1.2.13)

where $\mu_s$ and $\mu_o$ are the viscosities of solvent and oil respectively. The choice of the particular equations (1.2.12) and (1.2.13) is not restrictive and any other equations of state can be used instead.

By rendering dimensionless the above equations (1.2.1) through (1.2.13), the factors that affect their solution can be isolated in the form of dimensionless parameters. The dimensionalization is achieved by the introduction of dimensionless quantities (denoted with a prime), as follows:

$$x = \frac{L_x x'}{q}, \quad y = \frac{L_y y'}{q}, \quad C = C_i C', \quad C_o = C_i C_0,'$$

$$\rho = \rho_s \rho', \quad k = \bar{k} k', \quad \mu = \mu_o \mu', \quad h = L_x h', \quad \bar{h} = \bar{h}', \quad h_y = ah'$$
\[ u=qu', \quad v=qv', \quad p=\frac{q\mu_o L_x p'}{\bar{k}}, \quad p_0=\frac{q\mu_o L_x p'}{\bar{k}}. \]

\[ D_{11}=qL_x D_{11}', \quad D_{12}=\frac{qL_x}{a} D_{12}', \quad D_{22}=\frac{qL_x}{a^2} D_{22}'. \]

where \( \bar{k} \) is the average permeability of the porous medium.

The various parameters that govern the problem are grouped in the following dimensionless numbers:

- **Aspect ratio**: \( a = \frac{L_x}{L_y} \),

- **Viscosity ratio**: \( M = \frac{\mu_o}{\mu_s} \),

- **Peclet numbers**: 
  - **Molecular**: \( Pe_m = \frac{qL_x}{\phi D_m} \),
  - **Longitudinal**: \( Pe_l = \frac{L_x}{d_l} \),
  - **Transverse**: \( Pe_t = \frac{L_y}{d_t} \),

- **\( \eta \)**: \( \eta = \frac{\rho_o - \rho_s}{\rho_s} \),

- **\( G \)**: \( G = \frac{\rho_g q\bar{k}}{\rho_o q\mu_o} \).

Let also the gravity number \( N_G \) be defined as:

\[ N_G = \eta G = \frac{(\rho_o - \rho_s)q\bar{k}}{\rho_o q\mu_o}. \]

The gravity number is the ratio of gravity forces to viscous
forces and it is a measure of the importance of gravity, relative to convection. The Peclet numbers \( Pe_m, Pe_l \) and \( Pe_t \) give the ratios of convective effects to those of molecular diffusion, longitudinal dispersion and transverse dispersion respectively. Both flow in oil reservoirs and the flows that result during coreflooding experiments are convective-dominated, characterized by high Peclet numbers.

In terms of dimensionless quantities, equations (1.2.1) to (1.2.13) are written as follows (Since all quantities and parameters are dimensionless, the primes are dropped for simplicity):

**Concentration Problem**

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{\partial}{\partial x} \left( D_{11} \frac{\partial C}{\partial x} + D_{12} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left( D_{21} \frac{\partial C}{\partial x} + D_{22} \frac{\partial C}{\partial y} \right) \tag{1.2.14}
\]

Initial condition: \( \text{At } t=0 : \quad C = C_0(x,y) \) \tag{1.2.15}

Boundary conditions:

\( \text{At } x=0 : \quad C = 1 \) \tag{1.2.16}

\( \text{At } x=1 : \quad \frac{\partial C}{\partial x} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = 0 \) \tag{1.2.17}

\( \text{At } y=0 \text{ and } y=1 : \quad \frac{\partial C}{\partial y} = 0 \) \tag{1.2.18}

**Pressure - Velocity Problem**

\[
u = -\frac{k}{\mu} \left( \frac{\partial p}{\partial x} - Gp_{h_x} \right) \tag{1.2.19}
\]

\[
u = -a^2 \frac{k}{\mu} \left( \frac{\partial p}{\partial y} - Gp_{h_y} \right) \tag{1.2.20}
\]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  
\hspace{1cm} (1.2.21)

Boundary conditions:

At \( x=0 \): \( p = p_0 + G \phi h \)  \hspace{1cm} (1.2.22')

At \( x=1 \): \( p = G \int_0^y \phi y \, du \)  \hspace{1cm} (1.2.23)

At \( y=0 \) and \( y=1 \): \( v = 0 \)  \hspace{1cm} (1.2.24)

Density function: \( \rho = 1 + \eta(1-C_i C) \)  \hspace{1cm} (1.2.25')

Viscosity function: \( \mu = \left[1+(M^{0.25}-1)C_i C\right]^{-4} \)  \hspace{1cm} (1.2.26')

The last assumption that will be made is that \( C_i = 1 \). This assumption does not affect the generality of the problem, since it does not change the nature of any of the above equations. However it allows the reference to the displaced fluid as oil and the displacing fluid as solvent and it simplifies somewhat equations (1.2.22'), (1.2.25') and (1.2.26'):

\[ p = p_0 + Gh \]  \hspace{1cm} (1.2.22)

\[ \rho = 1 + \eta(1-C) \]  \hspace{1cm} (1.2.25)

\[ \mu = \left[1+(M^{0.25}-1)C\right]^{-4} \]  \hspace{1cm} (1.2.26)

Equations (1.2.14) to (1.2.26) constitute a model of two-dimensional miscible displacement under all the assumptions made above.
1.3 MODELING OF VISCOUS FINGERING

Under certain conditions the displacement of fluids in porous media is unstable and results in the formation of long protrusions of the displacing fluid, which grow through the displaced fluid. This phenomenon is referred to as viscous fingering. When extensive fingering occurs in oil recovery, the recovery efficiency of the displacement is severely reduced, because the displacing fluid is produced in significant amounts long before all the oil has been displaced. Fingering may occur in both immiscible and miscible displacement processes.

In flow through porous media small instabilities are generated by heterogeneities of the porous medium. Such heterogeneities are present at all scales, from the pore scale to the reservoir scale. The growth or damping of these instabilities is a result of the combined effect of several factors, such as relative mobility, gravity, dispersion, interfacial tension and the heterogeneity of the porous medium. The mechanism of the instability may vary from case to case but in most cases it is linked to viscosity variations between the displaced and the displacing fluid.

A simple one-dimensional model (Hill, [19]), explaining the mechanism of an instability driven by mobility variations and/or gravity consists of a vertical displacement of
a fluid of viscosity $\mu_1$ and density $\rho_1$ by a miscible fluid of viscosity $\mu_2$ and density $\rho_2$ in a medium of constant permeability $k$. A narrow mixing zone separates the two fluids. Both fluids are incompressible. Darcy's law has to be satisfied on both sides of the mixing zone. Hence:

$$\frac{dp_1}{dx} = -\frac{\mu_1}{k} u + \rho_1 g$$ (1.3.1)

$$\frac{dp_2}{dx} = -\frac{\mu_2}{k} u + \rho_2 g$$ (1.3.2)

Assume now that a small finger of the displacing fluid has penetrated the displaced fluid, due to a very small permeability variation; let $\delta x$ be the length of the finger. The net force on the displaced fluid that results from this small instability is:

$$\delta p = p_2 - p_1 = \left[\left(\frac{\mu_1 - \mu_2}{\mu_2}\right)\frac{u}{k} + (\rho_2 - \rho_1)g\right] \delta x$$ (1.3.3)

If $\delta p$ has the same sign as $u$, the finger will grow and the displacement is unstable. From equation (1.3.3) it is obvious that an unfavorable (larger than unity) mobility ratio $M = \frac{\mu_1}{\mu_2}$ always tends to destabilize the displacement. Gravity may be either a stabilizing or a destabilizing force, depending on the sign of $u\Delta \rho$, where $\Delta \rho = \rho_2 - \rho_1$. For example, for an upward displacement ($u<0$) of a denser, more viscous fluid by a less dense, less viscous one: $M>1$, $\Delta \rho<0$ and both gravity and viscosity are destabilizing. If the same displacement is downward ($u>0$), viscosity is again destabilizing while gravity is stabilizing, leading to a
critical velocity
\[ u_c = \frac{(\rho_2 - \rho_1)gk}{\mu_1 - \mu_2} \] (1.3.4)
above which the displacement is unstable. Clearly two more cases can be considered.

The above analysis is also applicable to immiscible fluids, with the difference that the effect of interfacial tension must be considered. Moreover the effect of dispersion must be considered in the miscible case. Chouke et al. [5] and Saffman and Taylor [45] conducted a stability analysis for immiscible displacement, taking interfacial tension into account. Similar stability analyses for miscible displacement, which consider the effect of dispersion have been performed [17,40,46,55]. These mechanisms have a stabilizing effect, damping instabilities of wavelength smaller than a certain cutoff wavelength and leading thus to a fastest growing wavelength, which determines the width of the fingers. The main problem in considering dispersion or interfacial tension in a stability analysis is that the base state solution is time dependent, which makes the stability characteristics also time dependent. Usually a quasi-steady-state assumption is invoked and stability analyses are conducted at different times [23]. The analyses mentioned above concerned linear geometry. Analyses for radial displacement were also performed for both the immiscible [36,52] and the miscible [2] case.
All these stability analyses are linear and predict the tendency of fingers to grow or decay and their initial growth rates. They are applicable only in the early stages of the instability, when displacements are small and linear theory is valid. Analysis of the later stages of the instability is considerably more difficult. One way is to perform a higher order analysis, keeping nonlinear terms, but this has not been attempted, except by Outmans [35]. A different approach consists in the study of the growth of a single large finger in a displacement with adverse mobility ratio [21, 33, 45].

A simple phenomenological model of viscous fingering was suggested by Koval [26]. His model assumes that oil is being displaced by a mixing zone, where fingering occurs. At each position the fingers occupy a certain fraction \( S_f \) of the pore space and oil the rest of it. A Buckley-Leverett type analysis is then performed to calculate the "fractional flow" of fingers and to predict effluent history and recovery efficiency. The mixing zone is assumed to have an intermediate composition between oil and solvent, so that the "effective viscosity ratio" is much smaller than the actual one. Koval suggested the empirical relation for the effective viscosity ratio:

\[
E = \left(0.78 + 0.22 M^{1/4}\right)^4
\]

(1.3.5)

This relation is based on the assumption that the composition of the mixing zone is 78% oil and 22% solvent. By
multiplying $E$ by a factor $H$, which accounts for the heterogeneity of the medium, a single parameter $K$ is obtained, which is used the model. Koval's model satisfactorily predicts effluent histories and recovery efficiencies, under a wide range of conditions. Dougherty [11] developed a somewhat refined version of Koval's work, introducing three additional parameters, which account for dispersive mixing. Fayers [15] introduced a similar model with physically interpretable parameters.

The concept of fractals provides an alternative way of studying viscous fingering in highly unstable systems. Witten and Sander [54] introduced the diffusion-limited aggregation (or DLA) process. DLA is a growth process. In its two-dimensional version, a lattice is superimposed in the plane, a site on the lattice is declared occupied and a random walker is released from the circumference of a circle centered at the occupied site. If the walker comes to a site neighboring an occupied one, it sticks and contributes to the growth of the aggregate. Assuming a steady flux of walkers, the probability $p$ that a walker is at a certain point $x$ follows the Laplace equation:

$$\nabla^2 p = 0 \quad (1.3.6)$$

The method can be easily extended in more dimensions. DLA simulations have given rise to extremely ramified aggregates, which can be described by fractal geometry, with a fractal dimension of about 1.70 (for the 2-D case). The
fact that similar structures have emerged in porous media displacements with very high mobility ratios, motivated the application of DLA to describe the phenomenon. Paterson [37] indicated that DLA applies in the limit $M \rightarrow \infty$, in the absence of gravity, diffusion and surface tension. Under these conditions the pressure in the displacing fluid is uniform and in the displaced fluid it satisfies equation (1.3.6). Miscible displacement experiments with very high viscosity ratio revealed fractal fingers, characterized by a fractal dimension of approximately 1.70, which is consistent with DLA [8]. Simulations were also conducted [4,7,37]. DLA predicts a sweep efficiency of 0.11 at breakthrough, for infinite mobility ratio. Extension of this approach to more realistic conditions is a subject of current research interest [25,47]. Modifications are necessary to include the effects of finite mobility ratio, nonuniform permeability and mechanisms like dispersion and interfacial tension.

Numerical simulation is the most rigorous modeling technique currently available, since it solves directly the equations that describe the displacement, without making any additional simplifications. Another important advantage of simulation is that it can model directly complex physical phenomena that occur during the displacement as well as any geological information that may be available about the porous medium (such as the existence of faults or fractures), which may have an important effect on the
displacement. Several numerical methods have been applied but almost all industrial simulators use finite-difference schemes. The size of the grid used in the simulation of unstable displacements depends on the mobility ratio and the dispersion coefficients. For high mobility ratios and high Peclet numbers, the grid must be very fine and this requires a large amount of computer time. Recently, the emergence of supercomputers has permitted the performance of large scale computations.
CHAPTER 2

DESCRIPTION OF THE NUMERICAL METHODS USED FOR THE
SOLUTION OF THE MISCELLOUS DISPLACEMENT EQUATIONS

The purpose of numerical simulation of miscible displacement is the numerical approximation of solvent concentration in the porous medium, as a function of space and time. This task requires the numerical solution of the coupled system of the pressure equation (1.1.5) and the concentration equation (1.1.1). One numerical method must be applied for the solution of each equation. Since the two equations are coupled, they may be solved either simultaneously or sequentially, involving an iterative predictor-corrector procedure.

The concentration equation is a parabolic partial differential equation; however for high Peclet numbers, as is the case for miscible displacement, the convection term is dominant and the solution exhibits quasi-hyperbolic behavior. Step initial conditions propagate in space and time in the form of traveling, sharp, slowly diffusing fronts. If the dispersion tensor is exactly zero, equation (1.1.1) becomes a first order hyperbolic equation and its
solution consists of the initial condition propagating along characteristic curves. The basic idea of the method of characteristics is to track concentration along the characteristics of the hyperbolic problem, obtained by ignoring dispersion, then use finite elements to treat the dispersion term. In this way each physical process is treated with a well-suited numerical scheme. The method was first introduced by Douglas and Russell [12] for one space variable and extended to miscible displacement by Russell [42].

Because characteristic tracking requires accurate velocity computation, a mixed finite element method is used for the solution of the pressure equation. The method seeks a functional approximation for pressure and velocity simultaneously, rather than solving for pressure and obtaining velocity by numerical differentiation of the pressure solution. The latter would result in the loss of one order of accuracy in the velocity approximation.

The modified method of characteristics and the mixed finite element method have been combined for the solution of miscible displacement problems [13,43,44].

2.1 THE MODIFIED METHOD OF CHARACTERISTICS

For a rectangular slab flooded on the side x=0, solvent concentration is the solution to the problem described by
equations (1.2.14) to (1.2.18). These equations are rewritten here in vector mode:

$$\frac{\partial C}{\partial t} + u \cdot \nabla C - \nabla \cdot (D \nabla C) = 0 \quad \forall \Omega, 0 < t < T \quad (2.1.1)$$

Initial condition: \( C = C_0(x,y) \quad \forall \Omega, t = 0 \quad (2.1.2) \)

Boundary conditions:

$$C = 1 \quad \forall \Omega_1 \quad (2.1.3)$$

$$\frac{\partial C}{\partial t} + u \cdot \nabla C = 0 \quad \forall \Omega_2 \quad (2.1.4)$$

$$D \nabla C \cdot \mathbf{n} = 0 \quad \forall \Omega_3 \quad (2.1.5)$$

Here \( \mathbf{x}=(x,y) \) is the position vector, \( \mathbf{v} \) is the outward unit normal vector, \( \Omega=(0,1) \times (0,1) \), \( \Omega_1 = \{(0,y) \mid 0 < y < 1\} \), \( \Omega_2 = \{(1,y) \mid 0 < y < 1\} \), \( \Omega_3 = \{(x,0) \mid 0 < x < 1\} \cup \{(x,1) \mid 0 < x < 1\} \) and \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \). The velocity field \( \mathbf{u} = \mathbf{u}(\mathbf{x},t) \) will be assumed known in this section.

The unit vector in the characteristic direction is defined in \((\mathbf{x},t)\) coordinates as follows:

$$\tau(\mathbf{x},t) = \frac{1}{\sqrt{1 + |\mathbf{u}(\mathbf{x},t)|^2}} \begin{pmatrix} \mathbf{u}(\mathbf{x},t), 1 \end{pmatrix} \quad (2.1.6)$$

Then the characteristic derivative is given by:

$$\sqrt{1 + |\mathbf{u}|^2} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad (2.1.7)$$

Equation (2.1.1) can be written:

$$\sqrt{1 + |\mathbf{u}|^2} \frac{\partial C}{\partial t} = \nabla \cdot (D \nabla C) \quad (2.1.8)$$

Now suppose that the solution is known at time \( t^n \) and the solution at time \( t^{n+1} = t^n + \Delta t \) is sought. A backward
difference time discretization scheme is used to approximate the characteristic derivative:

\[ \sqrt{1+|u|^2} \frac{\partial C}{\partial t} (x, t^{n+1}) \approx \sqrt{1+|u|^2} \frac{C(x, t^{n+1}) - C(x, t^n)}{\Delta t} \]

\[ = \frac{C(x, t^{n+1}) - C(x, t^n)}{\Delta t} \sqrt{1+|u|^2} \]

\[ = \frac{C(x, t^{n+1}) - C(x, t^n)}{\Delta t} \quad (2.1.9) \]

where: \( \bar{x} = x - u^*(x) \Delta t \) \quad (2.1.10)

and \( u^*(x) \) is an approximate average velocity over \([t^n, t^{n+1})\].

The calculation of \( u^* \) and \( \bar{x} \) is described in the following paragraph.

2.1.1 Characteristic Tracking

The main difference between the modified method of characteristics and other front tracking methods is that the characteristics are traced backward in time. The question asked is not where a mesh point will go at the next time level, but where it was at the previous one. The backward tracing eliminates the need for a moving grid and makes possible the use of a fixed grid throughout the simulation, although a changing grid may also be used.

Since velocity is a continuous function of time, its direction may also change continuously with time. Thus characteristics are in general curvilinear. The curved
characteristic is approximated by a sequence of line segments corresponding to sub-time steps. Each of these segments is determined by a predictor-corrector procedure. The algorithm follows:

The velocity field at \( t^{n+1} \) is estimated first by linear extrapolation from the latest two time steps:

\[
    u(x,t^{n+1}) = u(x,t^n) + \left[ u(x,t^n) - u(x,t^{n-1}) \right] \frac{t^{n+1}-t^n}{t^n-t^{n-1}} \tag{2.1.11}
\]

for \( n>1 \) and

\[
    u(x,t^1) = u(x,t^0) \tag{2.1.12}
\]

For uniform time steps equation (2.1.11) becomes:

\[
    u(x,t^{n+1}) = 2u(x,t^n) - u(x,t^{n-1}) \tag{2.1.11'}
\]

Now for any intermediate time \( t \in [t^n,t^{n+1}] \), the velocity field can be estimated by linear interpolation between \( u(x,t^n) \) and \( u(x,t^{n+1}) \).

A number of sub-time steps \( \Delta t_k, k=1,2,\ldots,m \) are chosen such that:

\[
    \Delta t = \sum_{k=1}^{m} \Delta t_k
\]

Tracking begins at \( t=t^{n+1} \) and the intermediate time levels are defined by:

\[
    t_k = t^{n+1} - \sum_{i=1}^{k} \Delta t_i
\]

with \( t_0 = t^{n+1}, t_m = t^n \). To evaluate \( x \), an iterative procedure is performed:

\[
x_0 = x
\]
\[ x'_k = x_{k-1} - u(x_{k-1}, t_{k-1}) \Delta t_k \] (predictor)
\[ x_k = x_{k-1} - \frac{1}{2} u(x_{k-1}, t_{k-1}) + u(x'_k, t_k) \Delta t_k \] (corrector)
\[ \bar{x} = x_m \]

Finally \( u^* \) can be calculated from (2.1.10). The tracking procedure is illustrated in Fig. 2.1.

It remains to determine how the sub-steps \( \Delta t_k \) are chosen. In this study each \( \Delta t_k \) is defined in an ad hoc fashion, requiring that the corresponding line segment be not larger than a predefined length \( d_{\text{max}} \). Empirically it was found that a choice of \( d_{\text{max}} \) equal to 1/5 of the grid block diagonal is sufficient to yield accurate results. If the line segment crosses the boundary \( \partial \Omega_3 \), the no-flow boundary condition (2.1.5) is used to reflect the point \( x_k \) back to \( \Omega \). Because of the nature of the problem (flow in the positive \( x \) direction), no line segment can cross \( \partial \Omega_2 \). Crossing \( \partial \Omega_1 \) is expected and dealt with by setting \( C(\bar{x}, t^n) = 1 \) in equation (2.1.9).

### 2.1.2 Spatial Discretization

A substitution of (2.1.9) in (2.1.8) yields:

\[
\frac{C^{n+1} - C^n}{\Delta t} - \nabla \cdot (D^{n+1} \nabla C^{n+1}) = 0 \tag{2.1.13}
\]

where \( C^{n+1} = C(x, t^{n+1}) \), \( \bar{C}^n = C(\bar{x}, t^n) \) and \( D^{n+1} = D(u(x, t^{n+1})) \). To obtain equation (2.1.13) an implicit time discretization
Figure 2.1 Characteristic tracking
scheme is used.

A standard Galerkin finite element method is used for the spatial discretization. Piecewise bilinear functions are used as test and trial functions. A description of the method follows. Notation is simplified by setting $C = C^{n+1}$, $\overline{C} = \overline{C}^n$, $D = D^{n+1}$ and $dx = dxdy$.

Let $\theta(x,y)$ be a sufficiently smooth function on $\Omega$. Multiplication of equation (2.1.13) by $\theta$ and integration over $\Omega$ yields:

$$\int_{\Omega} \left[ \frac{C-\overline{C}}{\Delta t} \theta - \nabla \cdot (D \nabla C) \theta \right] dx = 0$$

Integrating by parts and rearranging terms:

$$\int_{\Omega} \left[ \frac{C}{\Delta t} \theta + (D \nabla C) \cdot \nabla \theta \right] dx - \int_{\partial \Omega} (D \nabla C) \cdot \nabla \theta ds = \int_{\Omega} \frac{\overline{C}}{\Delta t} \theta dx$$

or, in scalar mode:

$$\int_{0}^{1} \left[ \frac{C}{\Delta t} \theta + \left( D_{11} \frac{\partial \theta}{\partial x} + D_{12} \frac{\partial \theta}{\partial y} \right) \frac{\partial \theta}{\partial x} + \left( D_{12} \frac{\partial \theta}{\partial x} + D_{22} \frac{\partial \theta}{\partial y} \right) \frac{\partial \theta}{\partial y} \right] dxdy +$$

$$\int_{0}^{1} \left[ \frac{\partial (D_{11} \theta + D_{12} \theta)}{\partial x} \right]_{x=0}^{1} dy - \int_{0}^{1} \left[ \frac{\partial (D_{11} \theta + D_{12} \theta)}{\partial x} \right]_{x=1}^{0} dy$$

$$= \int_{0}^{1} \frac{\overline{C}}{\Delta t} \theta dxdy$$

(2.1.14)

Boundary condition (2.1.5) was taken into account here, by setting the boundary integral over $\partial \Omega_3$ equal to zero.

Before defining the space of trial functions and test functions, some notation must be introduced: Let $\Delta_x$ and $\Delta_y$
be subdivisions of the interval \([0,1]\):

\[
\Delta_x = \{0 = x_0 < x_1 < \ldots < x_M = 1\} \quad (2.1.15a)
\]

\[
\Delta_y = \{0 = y_0 < y_1 < \ldots < y_N = 1\} \quad (2.1.15b)
\]

Let \(\Delta x_i = x_i - x_{i-1}\) and \(\Delta y_j = y_j - y_{j-1}\). Let \(v_i(x), 0 \leq i \leq M\) and \(w_j(y), 0 \leq j \leq N\) be piecewise linear functions, for which:

\[
v_i(x_m) = \delta_{im} \quad \text{and} \quad w_j(y_k) = \delta_{jk}.
\]

Analytically \(v_i\) are given by:

\[
v_i(x) = \begin{cases} 
\frac{x-x_{i-1}}{\Delta x_i} & x \in [x_{i-1}, x_i], \ 1 \leq i \leq M \\
\frac{x_{i+1}-x}{\Delta x_{i+1}} & x \in [x_i, x_{i+1}], \ 0 \leq i \leq M-1 \\
0 & \text{otherwise}
\end{cases} \quad (2.1.16)
\]

The functions \(w_j(y)\) are given by similar relations. The functions \(v_i, 0 \leq i \leq M\) are linearly independent and they span a space \(M(\Delta x)\). Likewise the functions \(w_j, 0 \leq j \leq N\) span a space \(M(\Delta y)\). Now let:

\[
\Theta_{ij}(x,y) = v_i(x)w_j(y) \quad (2.1.17)
\]

and \(\Theta = \{\Theta_{ij}, 0 \leq i \leq M, 0 \leq j \leq N\}\). It is clear that \(\Theta\) is a basis for the tensor product space \(\mathcal{W} = M(\Delta x) \times M(\Delta y)\).

\(\mathcal{W}\) is used as the space of both test functions and trial functions. This means that an approximation \(c(x,y)\) of the solution \(C\) of (2.1.14) is sought in \(\mathcal{W}\), which must satisfy equation (2.1.14) for all test functions \(\Theta \in \mathcal{W}\) or, equivalently, for all the basis functions \(\Theta_{ij} \in \Theta\). Since \(c \in \mathcal{W}\), it has the following form:
\[ c(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} \gamma_{ij} v_i w_j \] (2.1.18)

The expressions for \( c \) and \( \theta \) from (2.1.17) and (2.1.18) are substituted in (2.1.14) and the resulting integrals are evaluated numerically. Because the characteristics defining \( \chi \) may cross element boundaries, an integration quadrature that includes points on the boundaries is desired; a Gauss quadrature might fail to obtain information from an adjacent element, which should be accounted. The natural choice is Lobatto quadrature. A three-point Lobatto quadrature (Simpson's rule) is used. Simpson's rule is of sufficient order to integrate exactly continuous bilinears. Thus, if the components of the dispersion tensor are constant, integration will give exact results.

The ensuing system of equations, which is of order \((M+1)(N+1)\), is solved for the coefficients \( \gamma_{ij} \) with a preconditioned conjugate gradient method.

2.1.3 Treatment of Boundary Conditions

Boundary condition (2.1.5) has already been taken into account in the formulation of (2.1.14). The other two boundary conditions (2.1.3) and (2.1.4) must also be considered.

To implement the boundary condition \( C=1 \) at the inflow end \((x=0)\), all coefficients \( \gamma_{0j} \), \( 0 \leq j \leq N \) must be forced to take the value one. This can be done directly, by transferring the corresponding terms to the right hand side of the
system, or indirectly, by setting the diagonal entry and the right hand side of each equation corresponding to \( \theta_{0j} \) equal to a very large number \( G \). The latter approach is used here.

The boundary condition (2.1.4) at the outflow end can be written, in terms of the characteristic derivative:

\[
\frac{\partial C}{\partial \tau} = 0 \tag{2.1.19}
\]

This can be implemented simply by setting:

\[C^{n+1}(1,y) = \bar{C}^n(1,y)\]. Again this is done indirectly, by setting the diagonal entry of each equation corresponding to \( \theta_{Mj} \) equal to a very large number \( G \) and the right hand side equal to \( G\bar{C}^n(1,y) \).

2.1.4 Theoretical Analysis

The modified method of characteristics has been analyzed by Douglas and Russell [12] for one space variable; the analysis was extended to multidimensional problems by Dawson et al. [9]. The truncation error in the discretization is bounded by \( K((\Delta x)^2 + \Delta t) \), where \( \Delta x \) is the maximum grid spacing and \( K \) is a constant, independent of \( \Delta x \) and \( \Delta t \).

The time discretization error is of order \( \left| \frac{\partial^2 C}{\partial \tau^2} \right| \Delta t \). This replaces the term \( \left| \frac{\partial^2 C}{\partial \tau^2} \right| \Delta t \), that appears with standard backward-difference time-stepping procedures. In general \( \left| \frac{\partial^2 C}{\partial \tau^2} \right| \ll \left| \frac{\partial^2 C}{\partial t^2} \right| \), because concentration changes much more slowly along characteristics that follow the flow than at a fixed point in space. Therefore, for a given spatial
discretization, much larger time steps can be taken with this method than with standard implicit procedures, without loss of accuracy.

Another advantage of the method is that its treatment of convection does not introduce appreciable numerical dispersion. A mesh of the order of $Pe_1^{-1/2}$ is sufficient to resolve the front [44]. This is an improvement over the order $Pe_1^{-1}$ that is needed in upstream weighting schemes.

The major drawback of the method is that it does not conserve mass exactly. The error depends on the precision in the approximation of the curved characteristics and consequently on the velocities upon which this approximation is based. Inaccurate velocities may cause substantial mass-balance errors. Hence the modified method of characteristics requires a method that gives accurate velocities to be used for the solution of the pressure equation. This is the motivation for the use of a mixed finite element method, which is described in the following section.

2.2 FORMULATION OF THE MIXED FINITE ELEMENT METHOD

Fluid velocity and pressure in the porous medium are determined by solving the problem described by equations (1.2.19) to (1.2.26). Since both density and viscosity are functions of concentration, the concentration equation must
be solved first. In this section it will be assumed that concentration at time $t^n$ is a known function $C(x)$. Pressure and velocity are sought at the same time level $t^n$. The mixed finite element method that is used for the solution is described below:

Let $W=L^2(\Omega)$ be the set of square integrable functions on $\Omega$. Let $V=H(div; \Omega)$ be the set of two-dimensional vector functions $f \in (L^2(\Omega))^2$ such that $\nabla \cdot f \in L^2(\Omega)$. Let $Z$ be a subset of $V$ and let $\text{div}(Z)=\{\theta \in W : \theta = \nabla \cdot f, \text{ for some } f \in Z\}$. Let $f=(F,H)$ be a function in $V$. By multiplying (1.2.19) by $F(x,y)$, integrating over $\Omega$ and integrating by parts, the following equation is obtained:

$$\int_{0}^{1} \int_{0}^{1} \left[ \Lambda_x^{-1} u F - \frac{\partial F}{\partial x} \right] dx dy = \int_{0}^{1} \int_{0}^{1} \left[ \frac{\partial F}{\partial x} \right]_{x=0}^{x=1} dy$$

Similarly, from equation (1.2.20):

$$\int_{0}^{1} \int_{0}^{1} \left[ \Lambda_y^{-1} v H - \frac{\partial H}{\partial y} \right] dx dy = \int_{0}^{1} \int_{0}^{1} \left[ \frac{\partial H}{\partial y} \right]_{y=0}^{y=1} dx$$

Finally, multiplication of (1.2.21) by a function $\theta \in W$ and integration yields:

$$\int_{0}^{1} \int_{0}^{1} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \theta dx dy = 0$$

where $\Lambda_x = \frac{k}{\mu}$ and $\Lambda_y = a^2 \frac{k}{\mu}$. The system (2.2.1) to (2.2.3) constitutes the weak form of the pressure-velocity problem.

Before defining the approximating spaces, the Raviart-Thomas spaces [41] must be defined. Let $\Delta_x$ and $\Delta_y$ be subdivisions of $[0,1]$ in the $x$ and $y$ direction respectively,
defined as in (2.1.15). Note that $\Delta_x$ contains $M$ intervals and $\Delta_y$ $N$ intervals. The space $M^r_k(\Delta_x)$ is defined as follows:

$$M^r_k(\Delta_x) = \{ \psi \in C^k[0,1] : \psi|_{[x_{i-1}, x_i]} \in P_r[x_{i-1}, x_i], 1 \leq i \leq M \}$$

where $C^k(S)$ is the set of functions with $k$ continuous derivatives on $S$ and $P_r(S)$ is the set of all polynomials on $S$ of degree not higher than $r$. $k=-1$ denotes discontinuous functions and $k=0$ continuous functions with a discontinuous derivative.

The next-lowest order mixed finite element method is defined by choosing the following spaces for both test and trial functions:

- $U_h = M^2_0(\Delta_x) \times M^1_{-1}(\Delta_y)$
- $V_h = M^1_{-1}(\Delta_x) \times M^2_0(\Delta_y)$
- $W_h = M^1_{-1}(\Delta_x) \times M^1_{-1}(\Delta_y)$

Note that $\tilde{U}_h = U_h \times V_h \subset V_h \subset W_h$ and $\text{div}(\tilde{U}_h) = W_h$.

Let $\Delta x_i = x_i - x_{i-1}, i=1,2,\ldots, M$. Let $\sigma_1$ and $\sigma_2$ be the two Gauss points in $[0,1]$ and let

$$\sigma_{1i} = x_{i-1} + \sigma_1 \Delta x_i \quad \text{and} \quad \sigma_{2i} = x_{i-1} + \sigma_2 \Delta x_i.$$  Then a basis for $M^1_{-1}(\Delta_x)$ is \{${w_i(x) : 1 \leq i \leq 2M}$, where:
\[
\omega_{2i-1}(x) = \begin{cases} 
\frac{\sigma_{2i-1}}{\sigma_{2i-1} - \sigma_{1i}} x \in [x_{i-1}, x_i], \quad 1 \leq i \leq M \\
0 \quad \text{otherwise}
\end{cases} \tag{2.2.4}
\]

\[
\omega_{2i}(x) = \begin{cases} 
\frac{x - \sigma_{1i}}{\sigma_{2i} - \sigma_{1i}} x \in [x_{i-1}, x_i], \quad 1 \leq i \leq M \\
0 \quad \text{otherwise}
\end{cases}
\]

A basis for \( M_0^2(\Delta_x) \) is \( \{v_i(x) : 0 \leq i \leq 2M\} \), where:

\[
v_{2i-1}(x) = \begin{cases} 
\frac{4(x - x_{i-1})(x_i - x)}{\Delta x_i^2} x \in [x_{i-1}, x_i], \quad 1 \leq i \leq M \\
0 \quad \text{otherwise}
\end{cases}
\]

\[
v_{2i}(x) = \begin{cases} 
\frac{x - x_{i-1}}{\Delta x_i} x \in [x_{i-1}, x_i], \quad 1 \leq i \leq M \\
\frac{x_{i+1} - x}{\Delta x_{i+1}} x \in [x_i, x_{i+1}], \quad 0 \leq i \leq M - 1 \\
0 \quad \text{otherwise}
\end{cases} \tag{2.2.5}
\]

The basis functions \( \omega_i \) and \( v_i \) are drawn in Figs. 2.2 and 2.3 respectively.

Basis functions \( \omega_j(y), 1 \leq j \leq 2N \) and \( v_j(y), 0 \leq j \leq 2N \) for the spaces \( M_0^1(\Delta_y) \) and \( M_0^2(\Delta_y) \) respectively are defined by
Figure 2.2 Basis functions for $M_{-1}^1(\Delta x)$

Figure 2.3 Basis functions for $M_0^2(\Delta x)$
analogous relations.

The basis functions for the tensor product spaces $U_h$, $V_h$ and $W_h$ are products of basis functions in the one-dimensional Raviart-Thomas spaces. Thus the basis functions for $U_h$ are $F_{ij}(x,y) = v_i(x)w_j(y)$, for $V_h$ $H_{ij}(x,y) = w_i(x)v_j(y)$ and for $W_h$, $\Theta_{ij}(x,y) = w_i(x)w_j(y)$. An approximation $(U, V, P)$ to the solution $(u, v, p)$ is sought, with $U \in U_h$, $V \in V_h$ and $P \in W_h$. $U$, $V$ and $P$ can be expanded in terms of the basis functions:

$$U = \sum_{i=0}^{2M+1} \sum_{j=1}^{2N} \beta_{ij}^{(1)} v_i(x)w_j(y) \quad (2.2.6a)$$

$$V = \sum_{i=0}^{2M} \sum_{j=1}^{2N-1} \beta_{ij}^{(2)} w_i(x)v_j(y) \quad (2.2.6b)$$

$$P = \sum_{i=0}^{2M} \sum_{j=1}^{2N} \alpha_{ij} w_i(x)w_j(y) \quad (2.2.6c)$$

The no-flow boundary condition (1.2.24) was used to eliminate entries with $j=0$ or $j=2N$ in (2.2.6b). $V$ must vanish at $y=0$ and $y=1$, which requires that coefficients $\beta_{i0}^{(2)}$ and $\beta_{i,2N}^{(2)}$ be zero for all values of $i$; this also suggests that test functions $H_{i0}$ and $H_{i,2N}$ need not be considered.

The expressions for $U$, $V$ and $P$ from (2.2.6) are substituted in equations (2.2.1) to (2.2.3), which must be satisfied for all basis functions. The integrals are computed numerically. A three-point Gauss quadrature is applied when functions of the form $v_i(x)$ or $v_j(y)$ are used as test functions. A two-point Gauss quadrature is applied when
functions of the form \( w_i(x) \) or \( w_j(y) \) are used. In each case
the quadrature used is the lowest-order quadrature that inte-
tegrates the resulting polynomials exactly. Thus the result
of the numerical integration is exact, if \( \Lambda \) and \( \rho \) are con-
stants.

2.2.1 Treatment of Boundary Conditions

The boundary conditions are implemented in the boundary
integrals, at the right hand sides of (2.2.1) and (2.2.2). By substituting the values for pressure at the boundaries
\( \partial \Omega_1 \) and \( \partial \Omega_2 \) from (1.2.22) and (1.2.23), the last term of
(2.2.1) becomes (after some manipulations):

\[
-\int_0^{y_{1,0}} \left[ \begin{array}{l}
\left( p_0 + Gh \right) w_j(y) dy - \left( \int_0^{y_{1,0}} \rho_h du \right) w_j(y) dy
\end{array} \right]
\]

Both integrals can be easily evaluated numerically, using a
two-point Gauss quadrature.

The integral \( \int_0^{y=1} \left[ \begin{array}{l}
[\rho H]_{y=0} \end{array} \right] \) dx, which appears in (2.2.2),
vanishes, since all test functions \( H_{ij} \) vanish at \( y=0 \) and
\( y=1 \).

It should be noted that, if the flowrate \( q \) is to be
kept constant throughout the displacement, \( p_0 \) changes in
general with time. Since the value of \( p_0 \) is unknown before
the solution of the pressure equation, an iterative pro-
cEDURE is necessary. The value of \( p_0 \) is adjusted in each
iteration, until \( \int_0^1 u(1,y)dy \) reaches the value one. It was found that three iterations were always sufficient for convergence.

For horizontal displacement, \( h_x = h_y = 0 \) and equations (2.2.1) and (2.2.2) can be simplified as follows:

\[
\iint_{00} [\Lambda^{-1}_x u F_{ij} - p \frac{\partial F_{ij}}{\partial x}] dx dy = \frac{1}{2} \delta_{i,0} \Delta y_k p_0 \tag{2.2.7}
\]

\[
\iint_{00} [\Lambda^{-1}_y v H_{ij} - p \frac{\partial H_{ij}}{\partial y}] dx dy = 0 \tag{2.2.8}
\]

where \( j = 2k - 1 \) or \( j = 2k \).

In this case the solution \((u,v,p)\) is proportional to \( p_0 \) and no iterative solution is necessary.

### 2.2.2 Solution of the Linear System

When all integrals have been computed, a linear system of algebraic equations ensues, which must be solved to determine the coefficients \( \beta_{ij}^{(1)}, \beta_{ij}^{(2)}, \) and \( \alpha_{ij} \). In matrix form, the system can be written as follows:

\[
\begin{bmatrix}
M_1 & 0 & -N_1 \\
0 & M_2 & -N_2 \\
N_1^T & N_2^T & 0 \\
\end{bmatrix}
\begin{bmatrix}
[\beta^{(1)}] \\
[\beta^{(2)}] \\
[\alpha] \\
\end{bmatrix}
= 
\begin{bmatrix}
R_1 \\
R_2 \\
RR \\
\end{bmatrix}
\tag{2.2.9}
\]

\( M_1 \) is a block diagonal matrix, consisting of \( 2N \) blocks. Each of these blocks is a tridiagonal matrix of order \( 2M+1 \). Similarly, \( M_2 \) is a block diagonal matrix, consisting of \( 2M \) tridiagonal blocks of order \( 2N-1 \). \( N_1 \) is not a square
matrix. However, it has a block diagonal form, containing 2N blocks, each of which is a matrix of size (2M+1) \times 2M. N_2 can be made to have the same form, by permutation of rows and corresponding columns. N_2 contains 2M diagonally arranged blocks, each of which has size (2N-1) \times 2N. Moreover the elements of the matrices N_1 and N_2 are constants, independent of time and of the problem parameters.

Eliminating \( \beta^{(1)} \) and \( \beta^{(2)} \) from (2.2.9) the equivalent system is obtained:

\[
\begin{align*}
\beta^{(1)} &= M_1^{-1}(R_1 + N_1 \alpha) \quad (2.2.10a) \\
\beta^{(2)} &= M_2^{-1}(R_2 + N_2 \alpha) \quad (2.2.10b) \\
(N_1^T M_1^{-1} N_1 + N_2^T M_2^{-1} N_2) \alpha &= -N_1^T M_1^{-1} R_1 - N_2^T M_2^{-1} R_2 \quad (2.2.11)
\end{align*}
\]

The system (2.2.11), which is of order 4MN, is solved for \( \alpha \). Then \( \beta^{(1)} \) and \( \beta^{(2)} \) are determined by solving (2.2.10).

A preconditioned conjugate gradient method is used for the solution of (2.2.11). The preconditioner used is based on the block-centered finite difference matrix and it is described in [53].

2.2.3 Theoretical Analysis

For the next lowest order mixed finite element method, the approximation errors for both pressure and velocity are globally of order \( O(h^2) \), where \( h = \max_{i,j}(\Delta x_i, \Delta y_j) \) is the maximum grid spacing [14,41]. Moreover the Gauss points used for the numerical integration are superconvergence points, where
pressures and velocities are third order accurate [34]. In comparison, second order methods that combine equations (1.2.14) to (1.2.16) and solve the resulting elliptic equation for pressure, provide only first order velocities, because one order of accuracy is lost in the numerical differentiation. The high order of the velocity approximation allows the use of a reasonably coarse grid in order to provide an accurate velocity field, which is necessary for the characteristic tracking.

2.3 TIME-STEPPING PROCEDURE

The pressure and the concentration equations are solved sequentially at each time step. The concentration at time \( t=0 \) is known from the initial condition. The velocity field at \( t=0 \) can be computed by solving the pressure equation. Then, for every time level \( t^n, n=1,2,\ldots \) the concentration equation is solved. The velocities used for the tracking of characteristics are extrapolated values from the latest two time levels \( t^{n-1} \) and \( t^{n-2} \), as described in section 2.1. The solution of the concentration equation is used to solve the pressure equation at \( t=t^n \). Since the velocities used in the concentration equation are not exact, this equation should be solved again at time \( t=t^n \), this time using the computed velocity field and more iterations should be performed, if
necessary, until convergence. However a non-iterative scheme has been established with this method and has been used with very satisfactory results [13,43,44]. This non-iterative scheme has been adopted in this study. The non-iterative procedure is compared to the iterative one in the following section.

2.4 NUMERICAL TESTS

Most of the numerical tests presented in this section were conducted on simulations that are described in Chapters 3 and 4. All these were performed on an 80 X 80 uniform grid and with a time step of 0.01 PVI. Thus all simulations referred to in this section have the same grid and time step, unless otherwise mentioned.

2.4.1 Convergence under grid refinement

To test the convergence of the numerical method under grid refinement, three simulations using different meshes were performed on the same porous medium ($\lambda_D=8$, Real. 1, $CV=0.2$, Chapter 4) and with the same set of parameters ($M=10$, $Pe_l=500$, $Pe_t=20000$, $a=1$). The grids used were a 40 X 40 uniform grid (A), an 80 X 80 uniform grid (B) and a 90 X 80 nonuniform grid (C). The latter had finer spacing
in the left half of the domain \(x<0.5\) and was equivalent to a uniform 160 X 80 grid (with 160 blocks in the longitudinal direction) until some time before the injected fluid reaches \(x=0.5\); in the case studied \(t=0.22\) PVI is a safe limit and the simulation on grid C was run only until that time. A simulation on an 160 X 160 uniform grid was not feasible, because of computer storage limitations.

Concentration profiles at \(t=0.2\) PVI (contours of all points \((x,y)\) for which \(C=0.5\) at \(t=0.2\) PVI) for the three simulations are plotted together in Fig. 2.4. Fig. 2.5 shows the root mean square length (RMS) of the \(C=0.5\) concentration profile (defined in equation 3.0.1, Chapter 3) for the three cases. The three curves in Fig. 2.5 differ somewhat at early times but at later times their difference diminishes. So, after \(t=0.18\), the solutions provided by grids B and C are almost identical, as is also seen from the concentration profiles at \(t=0.2\) PVI (Fig. 2.4). The solution given by grid A deviates more from the other two. However the difference is not very large, suggesting that, although 40 X 40 is not a sufficiently fine grid to provide an accurate solution, not much further refinement is necessary and convergence has probably been attained for an 80 X 80 grid.
2.4.2 Material balance

A global mass (or volume) balance for the solvent yields:

\[ \text{SIP} = S_{\text{in}} - S_{\text{out}} \]  \hspace{1cm} (2.4.1)

where SIP (solvent in place) is the total volume of solvent in the porous medium, \( S_{\text{in}} \) is the total volume of solvent that has flowed into the medium and \( S_{\text{out}} \) is the total volume of solvent that has flowed out of the medium. To be acceptable a simulator must conserve mass, i.e. equation (2.4.1) must be satisfied at all times, within a good approximation. One way to test this is to form the quotient

\[ B = \frac{\text{SIP} + S_{\text{out}}}{S_{\text{in}}} \]  \hspace{1cm} (2.4.2)

at each time step and compare its value to one.

Since the modified method of characteristics is not conservative, it must be demonstrated from simulation results that material balance is satisfactory. Fig. 2.6 shows \( B \) versus time for two simulations with mobility ratios 7.5 and 75 (Runs No. 1 and No. 6 in Table 3.1 respectively). These results are typical of most simulations conducted in this study. The mass balance error (\(|B-1|\)) is less than 1%, except at the first time step (when both the numerator and the denominator in equation (2.4.2) are small numbers and small errors may significantly affect the value of \( B \)). There is always a discontinuity in \( B \) at the time step when
breakthrough occurs. This discontinuity usually improves the mass balance, as in the case shown in Fig. 2.6. In a small number of simulations (especially with very heterogeneous media and high mobility ratios) the mass balance error was larger. In all cases it was smaller than 2% except possibly for the first one or two time steps.

2.4.3 Test of non-iterative versus iterative procedure

Both procedures are defined in section 2.3. Here the non-iterative scheme is tested against a "predictor-corrector" scheme, i.e. an iterative procedure with one iteration. The rationale is that, if the iterative scheme yields significantly different results, a large difference should be manifest after the first iteration. The case chosen is the most difficult simulation run in this study with the following values of the parameters:

Porous medium: \( \lambda_D = 8 \), Real. 1, CV=0.5, a=1 (Chapter 4).

Flow parameters: \( M = 41 \), \( Pe_{\|} = 500 \), \( Pe_{\perp} = 20000 \).

Fig. 2.7 shows the RMS length (equation 3.0.1) versus time for the two schemes. The difference between the two curves is rather small. Moreover, since this is the "worst" case simulated, in the rest of the runs this difference is expected to be even smaller. In general the non-iterative procedure is judged satisfactory, at least for the simulations performed in this study.
2.4.4 Numerical dispersion

For a homogeneous porous medium (i.e. with constant permeability) the solution of the convection-dispersion problem can be obtained analytically:

\[
C = \frac{1}{2} \left[ 1 - \operatorname{erf} \frac{x}{\sqrt{4t/Pe}} \right]
\]  \hspace{1cm} (2.4.3)

The artificial dispersion generated by any numerical method depends on the number of grid blocks used in the simulation. In order to estimate the numerical dispersion, three simulations with \( M=1 \) and \( Pe_1=500 \) (molecular diffusion was set to zero) were performed on a homogeneous porous medium using uniform 20 X 20, 40 X 40 and 80 X 80 grids. The three solutions are compared to the analytical solution (2.4.3) at \( t=0.5 \) PVI in Fig. 2.8. Clearly the 80 X 80 grid does not generate any measurable numerical dispersion for \( Pe_1=500 \) and is sufficient to resolve the front. The 40 X 40 grid also provides a reasonably good approximation, while the 20 X 20 grid fails. This is consistent with the result mentioned in section 2.1, according to which the modified method of characteristics requires a mesh size of the order of \( Pe_1^{-1/2} \) to resolve the front; for \( Pe_1=500 \) this yields 23 grid blocks in the longitudinal direction.

The simulation on the 80 X 80 grid was also performed for \( M=\frac{1}{41} \) and \( M=41 \). In the former case the concentration profile was exactly the same as for \( M=1 \) (Fig. 2.9). For
M=41 the profile was essentially the same but it contained small perturbations, arising from computational errors and enhanced because of the instability of the displacement. However, as seen in Fig. 2.10, their amplitude is too small to pose a threat to the numerical solution, especially when flow in heterogeneous media is simulated; in that case permeability variations generate fingers that are at least one order of magnitude larger.
Fig. 2.4 Test for convergence under grid refinement  
C=0.5 profiles at t=0.2 for different grids
Fig. 2.5 Test for convergence under grid refinement
RMS lengths for solutions with different mesh sizes
Fig. 2.6 Test for material balance
Fig. 2.7 Comparison of non-iterative and iterative schemes
Fig. 2.8 Test for artificial dispersion
Concentration profiles at $t=0.5$
for different mesh sizes
Fig. 2.9 Concentration profiles at t=0.4

M=1, Pe₁=500
Fig. 2.10  Concentration profiles at $t=0.4$

$M=41, \text{ Pe}_1=500$
CHAPTER 3

PARAMETRIC ANALYSIS OF MISCELLY VISCOS FINGERING

The dedimensionalization of the miscible displacement problem that was performed in chapter 1 combined the factors that influence its solution in a number of dimensionless parameters. These parameters are the viscosity (mobility) ratio, the aspect ratio, three Peclet numbers for molecular, longitudinal and transverse dispersion and the two parameters η and NG that describe the effect of gravity. It must be noted that this set of parameters does not include the effect of the heterogeneity of the porous medium. The latter will be studied in chapter 4.

The effect of the above parameters on unstable displacement is studied by means of numerical simulation. To this aim, a base case is chosen, corresponding to a certain set of values of these parameters and a simulation run is performed for the base case. More simulation runs are performed for cases in which the values of one or two of the parameters have been changed from the base case. The comparison of the results of the various runs show the effect of each parameter on the number of fingers, their width,
their locations, breakthrough time and the root mean square growth rate of the concentration profile. The root mean square length of the C=0.5 profile is a quantitative measure of the instability of the displacement before breakthrough and it is defined as follows:

$$\text{RMS} = \left[ \int_0^1 (x_{0.5} - \bar{x})^2 dy \right]^{1/2}$$  \hspace{1cm} (3.0.1)

where \((x_{0.5}, y_{0.5})\) are all the points \((x,y)\) of the domain for which \(C(x,y) = 0.5\) and \(\bar{x}\) is the mean advance of the profile:

$$\bar{x} = \int_0^1 x_{0.5} dy$$  \hspace{1cm} (3.0.2)

In this chapter and the following one RMS is often plotted against time for a particular simulation. All such plots stop at breakthrough time, which can thus easily be inferred.

Since the effect of the heterogeneity of the porous medium is not studied in this chapter, the same porous medium is considered for all runs. This medium, which will be referred to henceforth as the ARCO slab, is a Berea sandstone slab in the shape of a rectangular parallelepiped of dimensions 2ft X 2ft X 1/2in. It was used by Giordano, Salter and Mohanty [16] for a study including flooding experiments and simulation. They established a 40 X 40 grid, by drawing lines on one of the large faces of the slab, and measured the permeability in each of the 1600 small parallelepipeds that comprised the slab. Permeability was
measured in the direction perpendicular to the slab. The resulting permeability map, which is shown schematically in Fig. 3.1, is used in all simulations. Permeability was assumed isotropic, with a constant value in each block. The statistical parameters of the permeability distribution are:

\[
\text{Mean (arithmetic) permeability } \bar{k} = 430 \text{ md and}\]

\[
\text{Standard deviation } \sigma = 0.133 \bar{k}.
\]

The values of the physical constants that are used in the simulation of the base case were the ones in the experiments by Giordano et al. [16].

\[
\phi=0.195 \quad q=2 \text{ ft/day} \quad M=7.5 \text{ and } 75 \quad .
\]

\[
D_m = 7.5 \times 10^{-6} \text{ cm}^2/\text{s} \quad d_1 = 2.76 \times 10^{-3} \text{ ft} \quad d_t = 7.73 \times 10^{-5} \text{ ft}.
\]

The effect of gravity is studied in section 3.5. In all other sections the displacement is assumed horizontal for all simulations and the parameters \( \eta \) and \( N_G \) are not considered.

The values of the dimensionless parameters for the runs that were conducted (excluding the runs with gravity effects) are listed in Table 3.1. Runs 1 and 6 correspond to the base case.

All simulations were run on a uniform 80 X 80 grid (i.e. one permeability grid block corresponds to 2 X 2 simulation grid blocks), with a time step of 0.01 pore volumes injected (PVI).

The concentration profiles (contours of constant
Fig. 3.1 Permeability map of the ARCO slab

Bright areas correspond to high permeability
Table 3.1 Simulation runs for parametric analysis

<table>
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<tr>
<th>Run No.</th>
<th>( M )</th>
<th>( a )</th>
<th>( \text{Pe}_m )</th>
<th>( \text{Pe}_1 )</th>
<th>( \text{Pe}_2 )</th>
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<td>725</td>
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<td>750</td>
<td>1</td>
<td>29409</td>
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<td>25873</td>
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</table>

Concentration) at different times are plotted in Fig. 3.2a-d for run No. 1 and in Fig. 3.3a-d for run No. 6. From Fig. 3.3a it can be observed that, at \( t=0.1 \) PVI, the profile is quite irregular, with a fairly large number of small fingers. At later times (\( t=0.2 \), Fig. 3.3b) some of the initial fingers (in this case five) grow faster than the rest of the fingers and they eventually dominate the
displacement. These long fingers suppress the growth of smaller fingers (this phenomenon is known as shielding). Later the smaller fingers merge with larger ones (Fig. 3.3c,d), leaving unswept areas that may be fairly extensive. This process results in a number of large fingers, which grow quite independently of each other, at least until breakthrough. These fingers will be referred to as "active" fingers. The qualitative behavior exposed here was present in all runs that were conducted. However the number of active fingers and the time of their formation vary with mobility ratio, dispersion and aspect ratio, as will be explained in the following sections. A third phenomenon, which does not occur in all of the runs, is the splitting of the tip of a finger (for example the top finger in Fig. 3.3d).

3.1 EFFECT OF MOBILITY RATIO

The qualitative difference in the displacement at moderate, high and very high mobility ratios is manifest by comparing Figs. 3.2, 3.3, 3.4 and 3.5. At moderate mobility ratios (M=2,7.5) no fingers grow much faster than the rest. In fact for M=2 (Fig. 3.4) there is very little interaction among fingers; no shielding or merging is apparent and it is very difficult to discern active fingers. For M=7.5
Fig. 3.2a Run No. 1 (base case), t=0.1
M=7.5, Peₜ=725, Peₜ=25873, a=1
Fig. 3.2b  Run No. 1 (base case), t=0.2
M=7.5, Pe₁=725, Pe₉=25873, a=1
Fig. 3.2c  Run No. 1 (base case), t=0.3
M=7.5, Pe_j=725, Pe_t=25873, a=1
Fig. 3.2d  Run No. 1 (base case), t=0.4
M=7.5, Pe_l=725, Pe_t=25873, a=1
Fig. 3.3a  Run No. 6 (base case), t=0.1
M=75, Pe₁=725, Pe₂=25873, a=1
Fig. 3.3b  Run No. 6 (base case), $t=0.2$

$M=75, \ Pe_l=725, \ Pe_t=25873, \ a=1$
Fig. 3.3c  Run No. 6 (base case), t=0.3
M=75, Pe_l=725, Pe_t=25873, a=1
Fig. 3.3d  Run No. 6 (base case), $t=0.4$
$M=75$, $Pe_1=725$, $Pe_t=25873$, $a=1$
Fig. 3.4 Run No. 11, t=0.3
M=2, Pe₁=725, Pe₉=25873, a=1
Fig. 3.5 Run No. 16, t=0.3
M=750, Pe₁=725, Peₜ=25873, a=1
active fingers do emerge (Fig. 3.2d) but a longer time is needed for them to outgrow the others than for M=75 (Fig. 3.3); shielding and merging occur to a lesser extent, resulting in a larger number of active fingers, whose growth rates do not vary much. On the other hand, at very high mobility ratios (M=750, Fig. 3.5) the active fingers start to outgrow the others earlier than for M=75. More shielding and merging occur and this results in a smaller number of active fingers. So seven active fingers emerge for M=7.5, five active fingers for M=75 and and three active fingers for M=750 (with the one at the top dominating).

It can also be noticed that, as the mobility ratio increases, fingers become unstable. Thus, for M=7.5 the fingers are relatively smooth, for M=75 they are irregularly shaped and exhibit some splitting and for M=750 they are very irregular and exhibit large scale splitting, growing secondary fingers. There are two causes for the increasing instability of the fingers with increasing mobility ratio. First, at high mobility ratios the few fingers present grow wider (in the y-direction). For sufficiently wide fingers there exist permeability variations of a scale smaller than the finger width, which introduce small perturbations on the shape of the fingers. Subsequent growth of these perturbations is favored by the high mobility ratio, so that secondary fingering may ensue. Second, the thickness of the displacement front (represented in the plots as the distance
between different concentration contours) decreases as the mobility ratio increases. As stability analysis indicates [23], the sharp front, which appears at high mobility ratios, is more susceptible to instabilities than the relatively diffuse front that exists at moderate mobility ratios.

The fact that fingers grow wider at high mobility ratios can be explained by comparing the pressure profiles (lines of constant pressure) for $M=7.5$ and $M=75$ at $t=0.2$ PVI (Figs. 3.6 and 3.7 respectively). Streamlines are everywhere perpendicular to iso-pressure lines. For a uniform displacement the latter would be straight lines, parallel to the $y$ axis. For $M=7.5$ it can be seen that there is little distortion of the pressure profiles; this means that fluid flow is almost entirely in the $x$ direction with very little lateral growth of the fingers. In comparison, for $M=75$ pressure profiles around the fingertips follow the shape of the fingers. Thus for high mobility ratios the invading fluid that is within the fingers tends to flow outwards, causing considerable lateral finger growth and more interaction among fingers. The apparent inversion of the pressure profiles upstream is due to the presence of the tips of shorter fingers. Further upstream, near the base of the fingers, the pressure distribution causes the displacing fluid to flow laterally into the fingers.

The root mean square length of the profile is plotted
Fig. 3.6 Pressure profile, Run No. 1, t=0.2
M=7.5, Pe_l=725, Pe_t=25873, a=1
Fig. 3.7 Pressure profile, Run No. 6, t=0.2
M=75, Pe₁=725, Pe₂=25873, a=1
versus time for different mobility ratios in Fig. 3.8. Table 3.2 shows the average root mean square growth rate RMS for runs 1, 6, and 11-16. RMS (dropping the overbar) is the average slope of the RMS versus time curve, obtained by a least square line fit between t=0.1 and 0.5 PVI or breakthrough, whichever occurs first. The correlation coefficient r is also listed in Table 3.2.

The effect of mobility ratio on the root mean square growth rate seems more pronounced at low values of M than at high ones. As M tends to infinity, RMS seems to approach an asymptotic value; this could however be a numerical effect because a finer grid is needed for the accurate simulation of displacements with very high mobility ratios.

As seen in Table 3.2, RMS grows almost linearly in time. The fit, which is very good for moderate mobility ratios, deteriorates slightly as M increases, due to nonlinear phenomena like merging and tip-splitting. Growth of individual fingers was found to be linear before breakthrough, as long as the finger does not merge and is not shielded. When fingers merge, growth of the resulting finger is accelerated. From Fig. 3.8 it is clear that linear growth does not include the initial stage of the displacement. In fact an abrupt change in slope is observed at approximately 0.04 PVI. This change in slope may be a result of the different effect of the permeability distribution of the porous medium on the growth of fingers at early and later times.
Fig. 3.8 Effect of mobility ratio on RMS length
<table>
<thead>
<tr>
<th>Run No.</th>
<th>M</th>
<th>RMS'</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>2</td>
<td>0.1418</td>
<td>0.9985</td>
</tr>
<tr>
<td>12</td>
<td>3.75</td>
<td>0.2206</td>
<td>0.9997</td>
</tr>
<tr>
<td>1</td>
<td>7.5</td>
<td>0.3146</td>
<td>0.9994</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>0.3463</td>
<td>0.9998</td>
</tr>
<tr>
<td>14</td>
<td>25</td>
<td>0.4586</td>
<td>0.9991</td>
</tr>
<tr>
<td>6</td>
<td>75</td>
<td>0.5822</td>
<td>0.9981</td>
</tr>
<tr>
<td>15</td>
<td>300</td>
<td>0.6882</td>
<td>0.9982</td>
</tr>
<tr>
<td>16</td>
<td>750</td>
<td>0.7276</td>
<td>0.9985</td>
</tr>
</tbody>
</table>

The RMS versus time curves exhibit small wiggles, whose magnitude decreases as the mobility ratio increases. These wiggles account for the slightly lower value of the correlation coefficient \( r \) for \( M=2 \). The wiggles are caused by permeability variations downstream, whose effect diminishes as the mobility ratio increases. Effects of the permeability distribution will be discussed in chapter 4.

3.2 EFFECT OF ASPECT RATIO

In runs No. 2 and No. 7 the aspect ratio of the slab was changed to 2. The permeability distribution remained the same but the 40 x 40 permeability grid blocks were changed to rectangles of aspect ratio 2. Likewise, the
aspect ratio was changed to 4 in runs No. 3 and No. 8. The concentration profiles for runs 2, 3, 7 and 8 at \( t=0.3 \) PVI are given in Figs. 3.9, 3.10, 3.11 and 3.12 respectively. As it can be seen, when the slab is made narrower, the initial fingers are closer to each other and consequently their interaction is stronger. The result is that shielding and merging occur in the initial stages of the displacement, yielding a small number of active fingers, even at early times. Thus, for \( M=7.5 \), seven active fingers form for \( a=1 \), two or three for \( a=2 \) and two for \( a=4 \). For \( M=75 \), five active fingers form for \( a=1 \), two for \( a=2 \) and one single active finger for \( a=4 \).

Fig. 3.13 shows a substantial increase in the root mean square growth rate with increasing aspect ratio. This increase is due to the fact that more merging occurs at large aspect ratios early in the displacement, causing acceleration of the few active fingers that form. However, eventually the longest finger for \( a=1 \) reaches the same length as the one for \( a=2 \) and \( a=4 \) (Fig. 3.14) just before breakthrough, so that breakthrough time is the same for all aspect ratios. The ordinate in Fig. 3.14 is the \( x \) coordinate of the tip of the longest finger reduced by the mean advance of the profile \( \bar{x} \), given by (3.0.2).
Fig. 3.9 Run No. 2, t=0.3
M=7.5, Pe_1=725, Pe_t=25873, a=2

LEGEND

---

C=0.25
---

C=0.50
---

C=0.75
---

Fig. 3.10 Run No. 3, t=0.3
M=7.5, Pe_1=725, Pe_t=25873, a=4
Fig. 3.11 Run No. 7, t=0.3
M=75, Pe_1=725, Pe_t=25873, a=2

Fig. 3.12 Run No. 8, t=0.3
M=75, Pe_1=725, Pe_t=25873, a=4
Fig. 3.13 Effect of aspect ratio on RMS length
Fig. 3.14 Effect of aspect ratio on length of longest finger
3.3 EFFECT OF LONGITUDINAL DISPERSION

In runs No. 4 and No. 9 the values of all Peclet numbers are five times lower than in the base case. The concentration profiles at $t=0.3$ corresponding to these runs are plotted in Figs. 3.15 and 3.16 respectively. These figures, compared to Figs. 3.2c and 3.3c, illustrate the effect of increased longitudinal dispersion (although both longitudinal and transverse dispersion are increased, transverse dispersion is still relatively small and increasing longitudinal dispersion is the dominant effect). Fig. 3.17 shows its effect on RMS.

Increased dispersion means that better mixing of the fluid takes place in all directions. As expected, a thicker (more diffuse) displacement front results, which makes the fingers less prone to small wavelength disturbances. In both runs the value of longitudinal dispersion is high enough to make the fingers virtually free of small scale instabilities.

Concerning the instability of the displacement, the effect of increased longitudinal dispersion appears to be different at the two mobility ratios studied. For the high mobility ratio (75) the waves (concentration profiles) corresponding to small values of $C$ grow faster, resulting in earlier breakthrough and in less efficient displacement. The profile is so diffuse that it is difficult to define
Fig. 3.15 Run No. 4, t=0.3
M=7.5, Pe₁=145, Pe₇=5175, a=1
Fig. 3.16 Run No. 9, t=0.3
M=75, Pe₁=145, Pe₉=5175, a=1
Fig. 3.17 Effect of longitudinal dispersion on RMS length
finger lengths in this case. Although for the C=0.5 concentration profile RMS seems unaffected by longitudinal dispersion, it is found that the root mean square growth rate increases with longitudinal dispersion for C<0.5 and decreases for C>0.5. For the moderate mobility ratio (7.5), the better mixing resulting from high longitudinal dispersion clearly causes substantial suppression of finger growth and a more efficient displacement.

3.4 EFFECT OF TRANSVERSE DISPERSION

In runs No. 5 and No. 10 a value of the transverse Peclet number ten times lower than in the base case was used. The effect of increased transverse dispersion on the displacement is illustrated in Figs. 3.18 and 3.19; its effect on RMS is shown in Fig. 3.20.

Larger transverse dispersion for both mobility ratios results in shorter and wider fingers and in a thicker displacement front. Fingers are also smoother (the effect of the diffuse front more than compensates for the effect of wider fingers). The number of active fingers has not been affected by transverse dispersion in the cases studied. However merging of small fingers with longer ones occurs faster when transverse dispersion is larger. Even greater amounts of transverse dispersion are expected to cause much
Fig. 3.18  Run No. 5, t=0.3
M=7.5, Pe_l=725, Pe_t=2587, a=1
Fig. 3.19  Run No. 10, t=0.3
M=75, Peₐ=725, Peₜ=2587, a=1
Fig. 3.20 Effect of transverse dispersion on RMS length
more merging, resulting in a smaller number of active fingers.

For large transverse dispersion, better mixing in the transverse direction causes suppression of finger growth. Hence the root mean square growth rate decreases as transverse dispersion increases (Fig. 3.20). The effect seems to be slightly more pronounced at low values of $M$. It should be noted that, although in the two runs studied here transverse dispersion was increased by a factor of ten, its value remains relatively small and the displacement continues to be dominated by convection. In this range, the suppression of finger growth caused by increased transverse dispersion is not very important.

3.5 EFFECT OF GRAVITY

When the displacement is not horizontal, gravitational forces must be considered. The effect of gravity on the displacement depends on the ratio of gravity to viscous forces, expressed by the gravity number (section 1.2):

$$
N_G = \frac{(\rho_o - \rho_s)g\bar{K}}{q\mu_o}
$$

(3.5.1)

The effect of gravity depends also on the main direction of flow relative to the vertical direction. In vertical
displacement gravity may be a stabilizing or a destabilizing mechanism, as discussed in section 1.3.

When the main direction of flow is horizontal but vertical flow is also possible, gravity has a different effect. A set of simulations was performed to study this effect in the case of displacement with unfavorable mobility ratio and its dependence on the gravity number $N_G$, the dimensionless density difference $\eta$, the mobility ratio and the aspect ratio. In all these simulations the ARCO slab is assumed oriented in such a way that gravity acts in the negative $y$ direction. The slab is flooded at the side $x=0$, so that flow is mainly in the positive $x$ direction. In all cases the solvent is assumed lighter than the oil. The values of $N_G$, $\eta$, $M$ and $a$ for each simulation are listed in Table 3.3. The base case values $\text{Pe}_l=725$ and $\text{Pe}_e=25873$ are used in all simulations but $\text{Pe}_m$ varies from run to run and is also given in the table. However, for all the simulations listed, $\text{Pe}_m$ is sufficiently large that its effect on the displacement is not important; moreover the simulations for which results are compared have the same or similar $\text{Pe}_m$. The last column in Table 3.3 shows the breakthrough time for each case.

The values of the physical constants for the first two cases were taken from two examples given in Stalkup [51] as typical of a vaporizing gas drive and a CO$_2$ flood respectively. According to Stalkup’s calculations, the former flood is dominated by gravity tonguing and the latter by
Table 3.3 Simulations with gravity effects

<table>
<thead>
<tr>
<th>Run No.</th>
<th>M</th>
<th>a</th>
<th>$Pe_m$</th>
<th>$N_G$</th>
<th>n</th>
<th>$t_{PT}(PVI)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>25</td>
<td>1</td>
<td>6323</td>
<td>0.479</td>
<td>1.000</td>
<td>0.27</td>
</tr>
<tr>
<td>G2</td>
<td>25</td>
<td>1</td>
<td>2206</td>
<td>0.006</td>
<td>0.143</td>
<td>0.50</td>
</tr>
<tr>
<td>G3</td>
<td>25</td>
<td>1</td>
<td>2206</td>
<td>0.056</td>
<td>0.143</td>
<td>0.44</td>
</tr>
<tr>
<td>G4</td>
<td>25</td>
<td>1</td>
<td>2206</td>
<td>0.169</td>
<td>0.143</td>
<td>0.35</td>
</tr>
<tr>
<td>G5</td>
<td>25</td>
<td>1</td>
<td>2206</td>
<td>0.563</td>
<td>0.143</td>
<td>0.26</td>
</tr>
<tr>
<td>G6</td>
<td>25</td>
<td>2</td>
<td>2206</td>
<td>0.169</td>
<td>0.143</td>
<td>0.41</td>
</tr>
<tr>
<td>G7</td>
<td>25</td>
<td>4</td>
<td>2206</td>
<td>0.169</td>
<td>0.143</td>
<td>0.48</td>
</tr>
<tr>
<td>G8</td>
<td>10</td>
<td>1</td>
<td>24213</td>
<td>0.080</td>
<td>0.800</td>
<td>0.50</td>
</tr>
<tr>
<td>G9</td>
<td>75</td>
<td>1</td>
<td>19372</td>
<td>0.080</td>
<td>0.800</td>
<td>0.37</td>
</tr>
<tr>
<td>G10</td>
<td>300</td>
<td>1</td>
<td>4843</td>
<td>0.080</td>
<td>0.800</td>
<td>0.35</td>
</tr>
</tbody>
</table>

viscous fingering.

The results of the simulations for the cases of Table 3.3 indicate that the character of the displacement is different for different values of $N_G$. For very small $N_G$ gravity is unimportant and viscous fingering occurs as in horizontal displacement (run G2, Fig. 3.21). For larger values of $N_G$ (run G3, Fig. 3.22) there is viscous fingering but gravity influences the growth rates of the individual fingers. Since solvent is lighter than oil, it tends to flow upwards (in the positive y direction). So gravity causes more fluid to flow into the finger in the upper part of the slab (near y=1), resulting in a faster growth of that finger, while growth of the rest of the fingers is somewhat
suppressed, partly because of gravity drainage and partly because of shielding. In the case of the ARCO slab, where the longest finger grows near \( y=1 \) even in horizontal displacement, this phenomenon clearly results in early breakthrough and reduced sweep efficiency. If, however, the solvent were heavier than the oil (or the slab were reversed), it is conceivable that, for a range of values of \( N_G \), gravity might improve the sweep efficiency by delaying breakthrough of the fastest-growing finger.

For even larger \( N_G \) (run G4, Fig. 3.23) a "gravity tongue" forms and grows at the top of the medium \( (y=1) \). In addition to this tongue, viscous fingering occurs near \( y=1 \), while fingering in the rest of the medium is substantially suppressed. In this flow regime both "gravity override" and viscous fingering are important and affect the displacement. In the case studied here (G4) the former mechanism eventually prevails, since the gravity tongue moves faster than any of the fingers and breaks through first. After breakthrough the injected fluid flows preferentially through the tongue, because of the lower resistance to flow. Thus the rate of subsequent oil recovery is substantially reduced (Fig. 3.26), while the lower right part of the porous medium is probably left unswept. For very large values of \( N_G \) (Runs G5, Fig. 3.24 and G1, Fig. 3.25) gravity override completely dominates the displacement, suppressing any fingers that may form at early times. The gravity tongue breaks through very
early and the rate of oil recovery after breakthrough is very low. Recovery curves (fraction of oil recovered as a function of time) for cases G1 to G5 are shown in Fig. 3.26. It is apparent that, as \( N_G \) increases, breakthrough time decreases and the rate of oil recovery after breakthrough (slope of the recovery curve) also decreases.

The reason that RMS is not used here is that the root mean square length must be calculated with respect to a reference concentration profile, which corresponds to uniform stable displacement. For horizontal displacement this concentration profile is parallel to the \( y \) axis. This is not true when gravity effects are present (and especially when they are dominant); in this case the reference concentration profile depends on \( N_G \), \( M \) and \( a \) and the comparison of root mean square lengths is unclear.

The cases G1 and G5 have very different values of \( \eta \) but about the same \( N_G \) and the same \( M \) and \( a \). The similarity of recovery curves and concentration profiles for the two cases suggests that \( \eta \) (or the actual density difference \( \rho_o - \rho_s \)) does not affect the displacement as an independent parameter but only as a factor of the gravity number \( N_G \).

For the geometry presently considered the aspect ratio \( a \) is the ratio of the horizontal to the vertical dimension of the medium. Runs G6 and G7 have \( a=2 \) and \( a=4 \) respectively and all the other parameters the same as in run G4. Concentration profiles for runs G6 and G7 are plotted in
Figs. 3.27 and 3.28. As seen in these plots, both a long finger and a gravity tongue form but, in contrast to run G4, (Fig. 3.23) here fingering is the dominant mechanism. As discussed in section 3.3, a large aspect ratio accelerates merging and the formation of active fingers. Thus a large aspect ratio favors fingering over gravity override, so that a larger value of $N_G$ is needed for gravity override to dominate. Since fingers grow less rapidly than the gravity tongue, breakthrough is delayed and oil recovery improves (Fig. 3.29).

Runs G8, G9 and G10 illustrate the effect of gravity at various mobility ratios. Concentration profiles for runs G8 and G9 are plotted in Figs. 3.30 and 3.31 respectively and recovery curves for G9 and G10 are plotted in Fig. 3.32. $N_G$ is in the range where both fingering and gravity override occur but fingering dominates. A large finger (two for $M=10$), favored by gravity, forms and grows near $y=1$. The effect of gravity in this flow regime seems to be more pronounced for high mobility ratios. The reason is that the actual ratio of gravity to viscous forces is inversely proportional to the viscosity of the fluid currently present in the porous medium; this ratio is equal to $N_G$ only at $t=0$. As the displacement proceeds and more lower viscosity fluid enters the medium at a constant rate, the ratio of gravity to viscous forces increases and gravity becomes progressively more important. This phenomenon is more prominent at
high mobility ratios.
Fig. 3.21 Run No. G2 (CO₂ flood example), t=0.4
M=25, a=1, N₆=0.006
Fig. 3.22 Run No. G3, t=0.4
M=25, a=1, \( N_g = 0.06 \)
Fig. 3.23a  Run No. 64, t=0.2
M=25, a=1, N_G=0.17
Fig. 3.23b  Run No. 64, t=0.4
M=25, a=1, N_g=0.17
Fig. 3.24a  Run No. 65, t=0.2
M=25, a=1, N_G=0.56
Fig. 3.24b  Run No. G5, t=0.4  
M=25, a=1, N_G=0.56
Fig. 3.25 Run No. G1 (Vaporizing gas drive), t=0.2
M=25, a=1, N_G=0.48
Fig. 3.26 Effect of gravity on oil recovery

M=25, a=1
Fig. 3.27  Run No. G6, t=0.4
M=25, a=2, N_G=0.17

LEGEND

--- C=0.25
--- C=0.50
--- C=0.75

Fig. 3.28  Run No. G7, t=0.4
M=25, a=4, N_G=0.17
Fig. 3.29 Effect of gravity on oil recovery at different aspect ratios
\( N_G = 0.17, \ M = 25 \)
Fig. 3.30  Run No. GB, t=0.3
M=10, a=1, N_G=0.08
Fig. 3.31  Run No. 69, $t=0.3$
$M=75, \ a=1, \ N_{G}=0.08$
Fig. 3.32 Effect of gravity on oil recovery for different mobility ratios

\[ a=1 \]
CHAPTER 4

EFFECT OF HETEROGENEITY OF THE POROUS MEDIUM ON UNSTABLE MISCELL BE DISPLACEMENT

Fluid displacement in a porous medium depends on the structure of the medium. This structure can be very complex, containing heterogeneities at all length scales, from the scale of kilometers down to the pore scale. In flow modeling by numerical simulation, heterogeneities of scale smaller than the spatial grid size are modeled by the introduction of an effective dispersion coefficient. Larger scale heterogeneities are modeled by assigning a set of averaged parameters to each grid block. Permeability is the most important of these parameters.

From field observations it has been established that permeability in a porous medium usually follows a log normal distribution [6]; this distribution can be characterized by its mean $\bar{k}$ and its standard deviation $\sigma$. The ratio

$$CV = \frac{\sigma}{\bar{k}}$$  \hspace{1cm} (4.0.1)

is called coefficient of permeability variation (or simply permeability variation) [6] and is used widely in reservoir engineering as a measure of the heterogeneity of the porous
medium.

However these statistics do not provide any information on the spatial correlation of permeability, which is also known to influence fluid flow \[27\]. A look at Fig. 3.1 provides a strong impression that the permeability of the ARCO slab is indeed spatially correlated, in the sense that the permeability of each grid block is dependent on the permeability of neighboring blocks, albeit not in a deterministic way. (Compare with Fig. 4.2, which shows an uncorrelated random permeability field). Statistically the correlation is described by the permeability autocovariance function. The range over which permeability is correlated varies in different porous media. A measure of this range of correlation is the correlation length. The autocovariance and the correlation length will be defined in the following section.

Realistic description of porous media for numerical simulation is a subject of recent intense research interest. Conditional simulation \[10,24,27\] involves generation of "synthetic" porous media, that are compatible with the available statistical information. This synthesis is a stochastic process, resulting in a single "realization" of the porous medium, which is input to a flow simulation. To obtain statistically meaningful results, several simulations must be performed for different realizations of the same porous medium and the results must be averaged. A number of
techniques, which generate realizations of a porous medium from raw statistics, are discussed in the literature [1,32,48,50]. Of particular interest is the Turning Bands Method [30,31], which generates realizations having a given autocovariance function. Recently Hewett [18] has argued that permeability variations have a fractal character, including long range correlations, and has suggested a method to synthesize fields with a fractal correlation structure. Lasseter et al. [28] introduced a three-stage scale-up procedure, which takes into account the effect of small and medium scale heterogeneities in large scale flow simulations.

In this study a simple method is used to generate realizations of a log normally distributed two-dimensional permeability field with two scales of heterogeneity and a given correlation length. The fields thus generated are then used in simulations of miscible displacement with adverse mobility ratios to evaluate the effect of the above parameters on unstable miscible displacement.

4.1 GENERATION OF PERMEABILITY FIELDS

Before describing the algorithm used for the synthesis of permeability fields, a few statistical definitions must be given [22].
Let \( z(x) \) be a random field, i.e. a random function of a vector \( x \). The autocovariance function of \( z \) is defined as:

\[
\gamma(x_1, x_2) = E\left[ \left( z(x_1) - E\left( z(x_1) \right) \right) \left( z(x_2) - E\left( z(x_2) \right) \right) \right] \tag{4.1.1}
\]

where \( E \) is the expectation operator.

A field is called stationary, when statistical properties do not depend on the point \( x \). A necessary condition for stationarity is that the statistical parameters of the distribution are independent of \( x \). So, for a stationary random field \( z \), the expected value \( E(z(x)) \) is a constant \( \tilde{z} \), the variance \( \sigma^2 \) is also constant and the autocovariance given by (4.1.1) depends only on the distance between \( x_1 \) and \( x_2 \) and not on the position of the individual points \( x_1 \) and \( x_2 \). Thus for a stationary field:

\[
\gamma(u) = E\left[ \left( z(x+u) - \tilde{z} \right) \left( z(x) - \tilde{z} \right) \right] \tag{4.1.2}
\]

and its value depends only on \( u \), which is called the lag distance. All subsequent discussion concerns only stationary fields.

Setting \( u = 0 \) in (4.1.2):

\[
\gamma(0) = E\left[ \left( z(x) - \tilde{z} \right)^2 \right] = \sigma^2 \tag{4.1.3}
\]

The autocorrelation function is defined as the autocovariance normalized by the variance \( \sigma^2 \):

\[
p(u) = \frac{\gamma(u)}{\gamma(0)} \tag{4.1.4}
\]

Let now \( z_i = z(x_i) \), \( 0 \leq i \leq N \) be a set of values of the function \( z \). The autocovariance of the field can be estimated by the sample autocovariance function:
\[ \gamma^*(u) = \frac{1}{N'} \sum_{i=1}^{N'} [z(x+u) - \bar{z}^*][z(x) - \bar{z}^*] \]  

(4.1.5)

where \( \bar{z}^* \) is the sample mean and \( N' \) is the number of pairs of data points separated by distance \( u \). It should be noted that, for a small lag distance \( u \), \( N' \) is a large number and \( \gamma^*(u) \) is a good estimator of the field autocovariance \( \gamma(u) \). However, for values of \( u \) approaching half the total length of the domain being modeled, \( N' \) is small and \( \gamma^*(u) \) becomes unreliable in estimating \( \gamma(u) \) [18]. Thus, the autocovariance will only be calculated for lag distances up to half of the domain. With this consideration, the asterisks in (4.1.5) will be dropped and no distinction between field properties and their estimates will be made in the subsequent analysis.

The true autocorrelation function \( \rho \) for a stationary field is equal to unity for \( u=0 \); as \( u \) increases, \( \rho \) decreases and drops to zero for a certain value of \( u \). The range where \( \rho(u) \) is substantially different from zero, is the range over which \( z \) is correlated. The correlation length is heuristically defined as the distance over which the autocovariance "drops to near zero" [27], and various ways of estimating it are described in the literature. For an exponentially decaying autocorrelation function \( (\rho = \exp(-u/\lambda)) \), the correlation length is the parameter \( \lambda \). For discrete functions, a rough estimate of the correlation length [29] can be computed by fitting an exponential curve
to the five smallest lag distances and using the fitted value of $\lambda$, or, alternatively, by taking the value of $u$ for which $p$ equals $1/e$. The latter estimation criterion is used in this study.

The purpose of the method presented in this section is to generate realizations of a two-dimensional random permeability field. The logarithm of permeability, denoted by $z$, follows a normal distribution, with mean $\bar{z}$ (which will be assumed zero, without loss of generality), given standard deviation $\sigma$ and correlation characterized by a given correlation length $\lambda$. The domain will be taken to be the square $[0,1] \times [0,1]$, so $\lambda$ can be considered a dimensionless number. Permeability values are generated on the gridpoints of a rectangular grid, so that the field can be used directly as input to flow simulations. Two scales of heterogeneity are built into the permeability field, the "long" scale being of the order of the correlation length and the "short" scale of the order of the grid block. The description of the method follows:

A number $N = \frac{8}{\lambda^2}$ of points, which will be called "pilot points", are randomly generated in the domain. $\beta$ is a parameter that has to be adjusted. Lake [27] refers to these points as "Heller points" and uses $\beta = 1$. Given the different definitions of correlation length adopted in [27] and here, a value of $\beta = 0.5$ was found suitable. Thus $N = \frac{1}{2\lambda^2}$ pilot points are generated. Their locations are selected
randomly; each coordinate follows a uniform distribution in [0,1]. To avoid potential aggregation of pilot points in certain areas, which would leave other areas void, the distance between any two pilot points is not allowed to be smaller than \( \lambda \). A value of \( z = Z_i \) is assigned to each pilot point. The \( Z_i \), \( 0 \leq i \leq N \) are random numbers, following a normal distribution with mean zero and standard deviation \( \sigma \).

Let \( \mathbf{x} \) be the location of a gridpoint (or any other point in the domain). Let \( d_i \) be the distance between \( \mathbf{x} \) and the \( i \)-th pilot point, let \( d_{\text{min}} \) be the minimum of these distances and let \( D_i = \frac{d_i}{d_{\text{min}}} \). Define weighting factors:

\[
w_i = \frac{\exp(-\alpha D_i)}{\sum_{i=1}^{N} \exp(-\alpha D_i)} \quad (4.1.6)
\]

where \( \alpha \) is an adjustable parameter, taken here equal to 2.

The value of \( z \) at \( \mathbf{x} \) is:

\[
z = z_l + z_s \quad (4.1.7)
\]

The long scale contribution \( z_l \) is the weighted average of the values of \( z \) at the pilot points:

\[
z_l = \frac{\sum_{i=1}^{N} w_i Z_i}{\sum_{i=1}^{N} w_i} \quad (4.1.8)
\]

The short scale contribution \( z_s \) is essentially a random noise superimposed on \( z_l \):

\[
z_s = f \sigma (2R-1) \exp \left( 1 - \frac{\lambda}{d_{\text{min}}} \right) \quad (4.1.9)
\]
where $R$ is a random number uniformly distributed in $[0,1]$ and $f$ is an adjustable parameter. $f$ is a measure of the importance of the short range correlation, relative to the long range correlation and should be less than one. A value of $f=0.5$ was used in most of the performed simulations, but values of $f=0$ and $f=1$ have also been tried. The form of equation (4.1.9) is chosen in such a way that the maximum allowed amplitude of the noise $z_s$ decreases with $d_{\text{min}}$, being equal to $f\sigma$ for $d_{\text{min}}=\lambda$ and approaching zero as $d_{\text{min}} \to 0$.

Finally the permeability field is given by:

$$k = \exp(z)$$

(4.1.10)

In the present study the permeability fields were generated on an 80 X 80 spatial grid. The permeability of the $(i,j)$ grid block will be denoted by $k_{ij}$. The correlation length can be expressed more conveniently in grid block units, by setting $\lambda_D = 80\lambda$. The values of $\lambda_D$ tried were 0, 2, 4 and 8 grid blocks (respectively $\lambda = 0, 1/40, 1/20$ and $1/10$ of the porous medium). $\lambda = 0$ corresponds to uncorrelated random permeability fields, for which the autocorrelation function is the delta function. Two different permeability fields were generated for each correlation length, except $\lambda_D = 8$, for which three realizations were generated. For each of the resulting nine realizations the coefficient of permeability variation CV was allowed to take the values 0.1, 0.2 and 0.5. This gave a total of twenty-seven actual permeability distributions. The parameter $f$ was taken equal to
0.5 for all the fields described above. Autocorrelolograms of the first realization for each correlation length are plotted in Fig. 4.1. In all cases the correlation length obtained from the autocorrelologram agrees with the input value of the parameter $\lambda$, used in the synthesisation algorithm. Fig. 4.2 shows the permeability field for $\lambda_0=0$, realization No. 1 and Fig. 4.3 for $\lambda_0=8$, realization No. 1.

Simulations are performed for each realization for mobility ratios of 10 and 41. The rest of the physical parameters are:

$$P_e_m = 10753, \quad P_e_l = 500, \quad P_e_t = 20000, \quad a = 1$$

An 80 X 80 uniform grid (which coincides with the permeability grid) and a time step of 0.01 PVI are used for all simulations.

Concentration profiles at $t=0.1$ and $t=0.3$ PVI are plotted for some of the simulations in Figs. 4.7 to 4.14.

4.2 EFFECT OF THE PERMEABILITY DISTRIBUTION

4.2.1 Effect of permeability distribution upstream

To evaluate the effect of the permeability distribution near the inflow end ($x=0$) on the fingering pattern, the concentration profile at $t=0.1$ PVI is related to the permeability map near the inflow end. More precisely the quantities
Fig. 4.1 Autocorrelograms of generated permeability fields

- $\lambda_D=0$, Re. 1
- $\lambda_D=2$, Re. 1
- $\lambda_D=4$, Re. 1
- $\lambda_D=8$, Re. 1

$P$ vs lag $x$ (grid blocks)
Fig. 4.2 Permeability map of generated medium
\[ \lambda_D = 0 \], Realization No. 1

Bright areas correspond to high permeability
Fig. 4.3 Permeability map of generated medium $\lambda_D=8$, Realization No. 1

Bright areas correspond to high permeability
correlated are the average of $x_{0.5}(y)$ (distance from the inlet of location where C=0.5) in each grid block $j$ in the $y$ direction (denoted by $X_j$) and the harmonic mean permeability $K_{ij}^m$ over the first $m$ blocks in the $x$ direction, i.e.:

$$
\frac{m}{K_j^m} = \sum_{i=1}^{m} \frac{1}{K_{ij}}
$$

(4.2.1)

Harmonic averages of permeability are used, because harmonic averaging gives the "effective permeability" in the direction of flow.

Correlation coefficients are calculated for different values of $m$. Let $m_{\text{max}}$ be the value of $m$ that gives the highest correlation coefficient for a particular simulation and let $r_{\text{max}}$ be that coefficient. Values of $m_{\text{max}}$ and $r_{\text{max}}$ for some realizations with CV=0.2 are shown in Table 4.1. It is found that $m_{\text{max}}$ is independent of the mobility ratio, increases with the correlation length and is roughly the same for all realizations having the same correlation length.

The fact that $m_{\text{max}}$ increases with $\lambda_D$ indicates that large scale permeability variations are sensed over a longer range than small scale variations, which is expected. The relatively high values of $r_{\text{max}}$ suggest that the permeability distribution near the inflow end dictates the fingering pattern at early times. To illustrate this effect, the concentration profiles at $t=0.1$ FVI for $M=10$ and $M=41$ are
Table 4.1 Correlation of concentration profile at $t=0.1$

with the permeability distribution near the inflow end

<table>
<thead>
<tr>
<th>$\lambda_B$</th>
<th>Real.</th>
<th>$M$</th>
<th>$m_{\text{max}}$</th>
<th>$r_{\text{max}}$</th>
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juxtaposed to a sketch of $\bar{R}_j^m$ (plotted as a function of $j$, for $m=m_{\text{max}}$) for three runs in Figs. 4.4, 4.5 and 4.6. From these plots it is clear that the fingering pattern is independent of the mobility ratio, in the sense that, for $t=0.1$ PVI, the number of fingers, their locations and their relative lengths are the same for both mobility ratios studied (for the same realization), though the length of the
fingers is, of course, different. Simulations with the ARCO slab (Chapter 3), confirm this result for a wider range of mobility ratios.

For \( \lambda_D = 0 \) (Fig. 4.4) \( \bar{R}_j^m \) is a random function of \( j \). From the plot it is clear that the concentration profile is determined almost entirely by the profile of \( \bar{K}_j^m \) and specifically that a finger appears everywhere \( \bar{R}_j^m \) exhibits a (local) maximum. For \( \lambda_D = 8 \) the form of \( \bar{R}_j^m \) is the superposition of a rather smooth curve, resulting from long scale permeability variations and a random noise, due to short scale variations. In the case presented here (Fig. 4.6) the smoothed \( \bar{K}_j^m \) curve has three maxima and it is clear that these maxima dictate the areas where fingers will initially grow. It was found that, in each of these areas, the detailed concentration profile at very early times (before 0.1 PVI) is dictated by short scale permeability variations near \( x = 0 \), over a range of much less than \( m_{\max}^{(15)} \) blocks; however the fingers that form at that time merge very early, so that by 0.1 PVI the concentration profile no longer corresponds to short scale permeability variations. For \( \lambda_D = 4 \) the results are qualitatively similar to those for \( \lambda_D = 8 \). \( \lambda_D = 2 \) (Fig. 4.5) represents an intermediate case, in which the ranges of "long scale" and "short scale" variations are comparable and both variations influence the concentration profile.

Since the number of maxima in \( \bar{R}_j^m \) decreases with
Fig. 4.4 Effect of permeability distribution near the inflow end on viscous fingering
C=0.5 Concentration profiles at t=0.1
$\lambda_D=0$, CV=0.2, Real. 1
Fig. 4.5 Effect of permeability distribution near the inflow end on viscous fingering
C=0.5 Concentration profiles at t=0.1
\( \lambda_D = 2, \ CV = 0.2, \ Real. ~ 1 \)
Fig. 4.6 Effect of permeability distribution near the inflow end on viscous fingering
$C=0.5$ Concentration profiles at $t=0.1$
$\lambda_D=8$, $CV=0.2$, Real. 1
increasing correlation length, the number of fingers initially forming is a decreasing function of the correlation length. This fact can be verified from Figs. 4.7 to 4.14.

For \( t=0.2 \) PVI or higher the correlation between \( \tilde{X}_j \) and \( \tilde{K}_j^m \) breaks down. This correlation is good only during the initial stage of the displacement, whose duration is a function of \( CV \) and the mobility ratio. Nonlinear phenomena (mainly shielding, discussed in Chapter 3) are already present at \( t=0.1 \) PVI and this is why the correlation is not perfect. The occurrence of more nonlinear phenomena at higher mobility ratios explains the better correlation that was observed in all cases for \( M=10 \) than for \( M=41 \). For the same reason \( r_{\text{max}} \) also decreases with increasing \( CV \). Simulations with \( M=1 \) show an \( r_{\text{max}}>0.9 \) at all times before breakthrough, with \( m_{\text{max}} \) continuously increasing. This suggests that for \( M=1 \) the displacement is governed by permeability variations at all times, which is expected, since the displacement is stable and permeability variation is the only source of instability in the system.

4.2.2 Effect of permeability variations downstream

To study the importance of the permeability distribution downstream, a set of runs was performed, in which the downstream 80% of the permeability field was "homogenized", i.e. the permeabilities of the downstream 64 X 80 blocks
Fig. 4.7  C=0.5 concentration profiles
$\lambda_D=0$, Real. 1, CV=0.2, M=10
Fig. 4.8  C=0.5 concentration profiles
λ_D=0, Real. 1, CV=0.2, M=41
Fig. 4.9 C=0.5 concentration profiles
$\lambda_D=2$, Real. 1, CV=0.2, M=10
Fig. 4.10  C=0.5 concentration profiles
\( \lambda_D=2 \), Real. 1, CV=0.2, M=41
Fig. 4.11  C=0.5 concentration profiles

\[ \lambda_D = 4, \text{ Real. 1, CV=0.2, } M=10 \]
Fig. 4.12  C=0.5 concentration profiles
\( \lambda_D=4, \text{ Real. 1, CV}=0.2, M=41 \)
Fig. 4.13  C=0.5 concentration profiles
$\lambda_D=8$, Real. 1, CV=0.2, M=10
Fig. 4.14 C=0.5 concentration profiles
λ_D=8, Real. 1, CV=0.2, M=41
were replaced by their (arithmetic) average; in this way any effect of permeability variations downstream is removed. This was done for $\lambda_D=0$, $\lambda_D=2$ and $\lambda_D=8$ (one realization each) with CV=0.2. The resulting concentration profiles are plotted in Fig. 4.15 for $\lambda_D=0$ and Fig. 4.16 for $\lambda_D=8$.

There is little difference between the concentration profiles for $\lambda_D=0$, plotted in Figs. 4.15 and 4.8 (the corresponding base case); this difference consists mainly of the more irregular shape of the fingers in Fig. 4.15. The number, locations and lengths of fingers are the same. The root mean square growth rates for both simulations are also the same, as shown in Fig. 4.17. Thus, in random uncorrelated media, the variation of permeability downstream produces local (small wavelength) irregularities in the shape of fingers but does not affect the displacement in any significant way.

Conversely, comparison of Figs. 4.16 and 4.14 shows that significantly more finger growth has occurred in the latter plot, due to permeability variations downstream. Specifically, these variations cause additional merging near the centre of the medium, resulting in a longer and faster growing finger in that area, which causes somewhat stronger shielding of the fingers present in the top and the bottom of the medium. Thus, in correlated media, the permeability distribution downstream has a significant effect on the displacement, accounting for part of the growth rate of the
Fig. 4.15 C=0.5 concentration profiles
Permeability homogenized downstream
$\lambda_d=0$, Real. 1, CV=0.2, M=41
Fig. 4.16  C=0.5 concentration profiles
Permeability homogenized downstream
$\lambda_D=8$, Real. 1, CV=0.2, M=41
fingers. This effect is also manifest in the plot of RMS versus time (Fig. 4.18). Note that, for $M=10$, breakthrough time for the base case is 0.45 PVI, while for the homogenized case it is greater than 0.5 PVI. For $\lambda_D=2$ the behavior is qualitatively the same as for $\lambda_D=8$.

The cause for this different behavior is linked again with the scale of the permeability variations. For a finger to be significantly affected by a permeability variation, the range of the variation has to be of the same order as the finger width. This is true for media with nonzero correlation length, as can be seen in Figs. 4.9 to 4.14. In these media, high or low permeability streaks affect growth of the fingers mainly by causing additional merging, as discussed above. The fact that finger width increases with mobility ratio may be the reason why the effect of the permeability distribution downstream seems to be stronger for $M=10$ than for $M=41$. On the other hand, for $\lambda_D=0$, the (finite) width of the fingers will always be larger than the range of the permeability variations. Even though in the simulations with $\lambda_D=0$ the range of permeability variations is actually of the order of a grid block, the width of the fingers is substantially larger (Figs. 4.7, 4.8). When averaged over several blocks, the small scale heterogeneities cancel out and their net result is only short wavelength perturbations.

As seen in Figs. 4.17 and 4.18, downstream permeability
Fig. 4.17 Effect of permeability variations downstream
$\lambda_D=0$, Real. 1, CV=0.2
Fig. 4.18 Effect of permeability variations downstream
\( \lambda_D = 8 \), Real. 1, CV = 0.2
variations cause some wiggles in the RMS curves; considerably less pronounced wiggles exist in the curves corresponding to the homogenized cases. This justifies the explanation of the wiggles observed in the RMS curves for low mobility ratios (mainly \( M=2 \)), in Fig. 3.8 as due to downstream permeability variations.

4.2.3 Effect of small scale heterogeneities

As explained in the beginning of this chapter, the field \( z = \ln k \) is a superposition of a long scale contribution \( z_L \) and a short scale contribution \( z_S \) (equation 4.1.7). A series of simulations were conducted to investigate the effect of small scale permeability variations in media where long scale variations exist as well. The base case is \( \lambda_D = 8 \), Real. 1, CV=0.2; here it will be termed Real. 1.1. All permeability fields considered in these simulations had the same CV and exactly the same \( z_L \) field as the base case but their \( z_S \) fields differed, as is explained below.

For the second medium (Real. 1.0), \( z_S \) is identically zero, i.e. no short scale random variations exist. This is equivalent to setting \( f = 0 \) in equation (4.1.9). To generate \( z_S \) for the third medium (Real. 1.2), the same sequence of random numbers is used as for the base case but \( f \) is taken equal to 1. This results in short scale permeability variations of greater magnitude than in the base case. Finally,
for the fourth medium (Real. 1.3), f=0.5, as in the base case, but a different sequence of random numbers is chosen, resulting in a different $z_s$ field. All simulations were conducted with $M=41$.

Concentration profiles are plotted in Figs. 4.19, 4.20 and 4.21 for realizations 1.0, 1.2 and 1.3 respectively. Profiles for the base case (Real. 1.1) are shown in Fig. 4.14. The profiles at $t=0.1$ will be compared first. Since for Real. 1.0 there are no small scale heterogeneities which would generate individual fingers, the profile at $t=0.1$ (Fig. 4.19) shows three areas of potential finger growth, but there is only one well-defined finger. On the other hand, for Real. 1.2 (Fig. 4.20), the profile is essentially the same as in the base case, except that the fingers are slightly longer, as expected. For Real. 1.3 the profile at $t=0.1$ is very different from the base case in its details but it also shows finger growth in the same three areas. At $t=0.3$ however, after active fingers have developed, all four profiles look essentially the same. Each profile clearly shows two active fingers in the same locations (about $y=0.2$ and $y=0.55$), of which the latter is longer and has clearly formed by merging (except in Real. 1.0), and a number of shielded fingers between $y=0.8$ and $y=1$. The location where the merging occurs is a high permeability area. Of course the profile for Real. 1.0 is smoother and for Real. 1.2 more irregular than for the base case, because of the effect of
small scale heterogeneities. More important is the fact that there is some difference in the length of the longest finger. However, this difference is rather small and it becomes even smaller later in the displacement, except for Real. 1.2, which has a slightly earlier breakthrough (0.39 PVI as opposed to 0.41 PVI for the three other realizations). RMS growth rates for all four realizations are about the same (Fig. 4.22). In conclusion, it seems that small scale heterogeneities in media with a large correlation length do not significantly affect the displacement.

4.3 EFFECT OF THE CORRELATION LENGTH

In chapter 3 it was mentioned that, as the displacement proceeds, merging reduces the number of fingers initially formed to a small number of active fingers. As one might expect, this number of active fingers is a decreasing function of the correlation length. Two reasons account for this. First, the initial number of fingers (e.g. at \( t=0.1 \) PVI) is smaller for larger correlation lengths (as explained in the previous section). Second, the effect of the downstream permeability distribution is more significant for large \( \lambda_D \), resulting in more merging, which reduces the number of fingers. From the simulations conducted it was found that the number of active fingers was independent of
Fig. 4.19  C=0.5 concentration profiles
Effect of small scale permeability variation
$\lambda_D=8$, Real. 1.0, CV=0.2, M=41
Fig. 4.20  C=0.5 concentration profiles
Effect of small scale permeability variation
\( \lambda_D = 8 \), Real. 1.2, CV=0.2, M=41
Fig. 4.21 C=0.5 concentration profiles
Effect of small scale permeability variation
\( \lambda_D = 8, \text{ Real. } 1.3, CV=0.2, M=41 \)
Fig. 4.22 Effect of small scale heterogeneity

$\lambda_D=8$, Real. 1, CV=0.2, M=41
the specific permeability distribution, in the sense that different realizations having the same values of the parameters \( \lambda_D \), CV and M produced the same number of active fingers. Thus, for example for CV=0.2 and M=10, six or seven active fingers form for \( \lambda_D=0 \) (Fig. 4.7), four for \( \lambda_D=2 \) (Fig. 4.9), three for \( \lambda_D=4 \) (Fig. 4.11), two for \( \lambda_D=8 \) (Fig. 4.13).

RMS length for each of the cases simulated was calculated by averaging RMS values for different realizations. The agreement between realizations was reasonably good for \( \lambda_D=0,2 \) and 4 (Typical example in Fig. 4.23). For \( \lambda_D=8 \) however, the RMS growth rate for Real. 1 is much higher than for the other two realizations, which are in good agreement (Fig. 4.24). The reason for this behavior is that for \( \lambda_D=8 \) the correlation length is one tenth of the size of the domain, hence comparable to it. The domain contains only about fifty "permeability regions", equal to the number of pilot points. This number of regions is not sufficient to average out the permeability variations from region to region and flow channeling may occur to varying degrees, thus causing large differences between realizations. In order to obtain statistically meaningful results for large correlation length, simulations must be performed for a large number of realizations.

Average root mean square growth rates RMS' for the cases simulated are listed in Table 4.2; RMS' was
Fig. 4.23 RMS length for different realizations

$\lambda_D = 4, \ M = 10$
Fig. 4.24 RMS length for different realizations
\[ \lambda_D = 8, \ M = 10 \]
calculated by a linear least-square fit between \( t=0.1 \) and \( t=0.5 \) or breakthrough (whichever occurs first) and is averaged over different realizations. The correlation coefficient \( r \) of the fit is included in the Table. The range of breakthrough times \( t_{BT} \) for the various realizations is also listed. RMS is plotted versus time in Fig. 4.25 with \( \lambda_D \) as a parameter, for \( CV=0.2 \) and \( M=10 \). Fig. 4.26 shows the same for \( M=41 \). From both plots it is apparent that, for given degree of heterogeneity, the growth rate of fingers increases with the correlation length up to \( \lambda_D=4 \). An anomaly in this trend is observed for \( \lambda_D=8 \). The smaller RMS length for \( \lambda_D=8 \) is due to the fact that, for this high correlation length, a very long time is required for fingers to develop initially. (For both \( M=10 \) and \( M=41 \), the slope of the RMS curve near \( t=0 \) is smaller for \( \lambda_D=8 \) than for any other correlation length.) This slow initial growth is observed in all three realizations. This behavior is consistent with linear stability analysis (for example [23]), which predicts that growth rate of perturbations decreases with increasing wavelength. For large correlation lengths the permeability variation tends to generate fingers of large wavelengths, which grow relatively slowly. However, once these fingers grow past the zone described by linear stability analysis, it is easier for the invading fluid to develop flow channels and the RMS growth rate (slope of the RMS curve) is higher for large correlation length. (For example at \( t=0.35 \) the
slope for $\lambda_D=8$ is about the same as for $\lambda_D=4$). These two competing mechanisms (i.e. slower initial growth and faster subsequent growth as the correlation length increases) determine the actual RMS growth rates for different correlation lengths. In the cases simulated a maximum average RMS growth rate is observed for $\lambda_D=4$. However this correlation length for which maximum fingering occurs may depend on the degree of heterogeneity of the porous medium.

4.4 EFFECT OF THE COEFFICIENT OF PERMEABILITY VARIATION

The number of active fingers is a decreasing function of the coefficient of permeability variation CV. As the value of CV increases, the difference in permeability between a high permeability region and a low permeability region increases. In the initial stage of the displacement this difference favors the growth of the longest fingers, which grow in the high permeability regions. Thus, for highly heterogeneous media, the longest fingers grow much faster than the rest and dominate the displacement relatively early, shielding the smaller fingers. Later in the displacement the large difference in permeability from region to region facilitates merging. The result of these two mechanisms is reduction of the number of active fingers. Since almost all the invading fluid flows through one or a
Table 4.2 Average RMS' for the cases simulated

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<th>ΔD</th>
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Fig. 4.25 Average RMS length for different $\lambda_D$
CV=0.2, $M=10$
Fig. 4.26 Average RMS length for different $\lambda_D$

$CV=0.2, M=41$
few fingers, breakthrough occurs early and sweep efficiency is poor.

Conversely, for low values of CV, the heterogeneity of the medium is small. In this case all fingers grow at comparable speeds, less shielding and merging occur, the number of active fingers is larger and breakthrough is delayed.

All simulations performed with CV=0.5 resulted in a single fast growing active finger ($\lambda_D=8$, Fig. 4.29), two active fingers ($\lambda_D=2.4$, Fig. 4.28) or three active fingers ($\lambda_D=0$, Fig. 4.27). An example of a simulation with CV=0.1 is given in Fig. 4.30.

As expected, the RMS growth rate increases with CV. However the effect of CV increases dramatically with the correlation length. The difference in the effect of CV is especially great between the uncorrelated medium (Fig. 4.31) and the correlated media (Figs. 4.32 to 4.34). This quantitative difference is due to the fact that more flow channeling occurs as the correlation length increases. For large correlation lengths increased heterogeneity favors flow of the invading fluid through the channels, causing high finger growth rates and early breakthrough. For uncorrelated media, or media with small correlation length, there is little or no channeling and the effect of increased heterogeneity is not as pronounced.
Fig. 4.27  C=0.5 concentration profiles
Effect of large heterogeneity
λ₀=0, Real. 1, CV=0.5, M=41
Fig. 4.28  C=0.5 concentration profiles
Effect of large heterogeneity
$\lambda_D=2$, Real. 1, CV=0.5, M=41
Fig. 4.29  C=0.5 concentration profiles
Effect of large heterogeneity
$\lambda_D=8$, Real. 1, CV=0.5, M=41
Fig. 4.30  C=0.5 concentration profiles
Effect of small heterogeneity
\( \lambda_D=2 \), Real. 1, CV=0.1, M=41
Fig. 4.31  Average RMS length for different CV
\( \lambda_0 = 0 \) (uncorrelated medium)
Fig. 4.32 Average RMS length for different CV

$\lambda_D=2$
Fig. 4.33 Average RMS length for different CV
\( \lambda_D = 4 \)
Fig. 4.34 Average RMS length for different CV

$\lambda_D = 8$
CHAPTER 5

CONCLUSIONS

Numerical simulation was used to study unstable incompressible miscible displacement in two-dimensional linear geometry. The mechanisms of viscous fingering and the performance of the displacement under different conditions were investigated. These conditions were expressed by a set of dimensionless parameters, which arose from the dedimensionalization of the miscible displacement equations. The parameters whose effects were considered are the mobility ratio (M), the aspect ratio (defined as the ratio of the longitudinal to the transverse dimension of the medium) and Peclet numbers for molecular diffusion, longitudinal dispersion and transverse dispersion. Effects of gravity were also considered and were expressed by the gravity number.

The effect of the structure of the porous medium was also investigated; it was expressed by two parameters, the coefficient of permeability variation and the correlation length of the permeability distribution. The main results reached in this research are outlined below:
- The permeability distribution near the inflow end of the porous medium determines the initial number, locations and relative growth rates of the fingers; none of these depends on the mobility ratio. Locations where fingers form are dictated by maxima in the permeability near the inflow end. The initial number of fingers depends strongly on the correlation length of the permeability distribution. For highly correlated media permeability has fewer maxima and fewer fingers form.

- Subsequent merging reduces the initial number of fingers to a smaller number of "active fingers", which dominate the displacement. Smaller fingers are shielded (their growth is suppressed) and eventually merge with longer fingers. The number of fingers is thus reduced as the displacement proceeds. The number of "active" fingers is a decreasing function of the correlation length, the heterogeneity of the medium (i.e. the coefficient of permeability variation) the aspect ratio and the mobility ratio but, with those parameters fixed, it does not depend on the specific permeability distribution. The effect of downstream permeability variations, negligible for uncorrelated media, becomes more substantial as the correlation length increases. The effect is more pronounced for small mobility ratios.
- After a short initiation period, fingers grow linearly in time until they start interacting with other fingers. The root mean square growth rate (RMS') can be approximated by a constant, although the approximation deteriorates slightly at high mobility ratios. As the correlation length increases, RMS' passes through a maximum. RMS' also increases with increasing heterogeneity of the medium. This effect is much more pronounced in highly correlated media, since, in those, large heterogeneity facilitates channeling of flow.

- The root mean square growth rate increases with mobility ratio; the increase is more prominent at low mobility ratios. RMS' seems to reach an asymptotic value as $M \to \infty$. At low mobility ratios little merging occurs, resulting in a large number of thin active fingers with comparable growth rates. At high mobility ratios a smaller number of wider active fingers form. At high mobility ratios the fingers are unstable, exhibit splitting and sometimes form secondary fingers.

- A large aspect ratio (medium longer in the longitudinal direction) causes extensive merging early in the displacement and yields a smaller number of active fingers. At very large aspect ratios a single finger
dominates the displacement. RMS' increases with aspect ratio but breakthrough time is not affected.

- Increased longitudinal dispersion causes a thicker displacement front and makes the fingers smoother. Depending on the mobility ratio, the efficiency of the displacement may improve (for low mobility ratios) or deteriorate. Increased transverse dispersion causes earlier merging and results in wider and smoother fingers. Transverse dispersion is a stabilizing factor; it suppresses finger growth, delays breakthrough and improves the efficiency of the displacement.

- When there is a substantial density difference between the displaced and the displacing fluid, gravity forces acting in the transverse direction may cause separation of the two fluids. In media of high permeability and/or for low displacement velocity, gravity may dominate the displacement. In that case most of the displacing fluid flows through a gravity tongue that grows on the top or the bottom of the medium. After the tongue breaks through virtually all the flow is channeled through it and the recovery efficiency is drastically reduced. Gravity tonguing and viscous fingering may coexist or either mechanism may be dominant, depending on the value of the gravity
number. The effect of gravity is reduced when the aspect ratio increases.

An obvious extension of the present work is the application of the same analysis in radial and five-spot geometry. The effect of the structure of the porous medium may be further investigated by using different statistical methods to generate random porous media; media with large heterogeneities (e.g., fractures, impermeable rocks etc.) should be included, since, in that case, extensive flow channeling may result in a qualitatively different displacement. The combined effects of gravity and heterogeneity should be considered.

Numerical simulation can also be used to study viscous fingering in immiscible displacement. This requires an adjustment of the numerical method, to handle the nonlinear flux term that is present in the immiscible displacement saturation equation.

Finally an extension to three spatial dimensions may be performed; solving the problem in three dimensions is particularly important when gravity effects are present. In that case, it is conjectured that, allowing the displacement to take place in two directions rather than just one direction transverse to gravity, would give very different results, showing more gravity dominance.
NOMENCLATURE

Latin symbols

a Aspect ratio
B Mass balance test parameter
C Solvent concentration
C₀ Initial solvent concentration
CV Coefficient of permeability variation
D Dispersion tensor
Dm Molecular diffusion coefficient
d₁ Longitudinal dispersion mixing length
dₜ Transverse dispersion mixing length
E Koval's effective mobility ratio
f Parameter determining maximum amplitude of small scale permeability variations (Eqn. 4.1.9)
g Acceleration of gravity (981 cm/sec²)
h Depth
k Permeability
Lₓ Length of the domain (in the x direction)
Lᵧ Width of the domain (in the y direction)
M Mobility (viscosity) ratio
M Number of intervals in the x direction
N Number of intervals in the y direction
NG Gravity number
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$P_{el}$</td>
<td>Longitudinal Peclet number</td>
</tr>
<tr>
<td>$P_{em}$</td>
<td>Molecular Peclet number</td>
</tr>
<tr>
<td>$P_{et}$</td>
<td>Transverse Peclet number</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure</td>
</tr>
<tr>
<td>$q$</td>
<td>Flowrate (velocity per unit area)</td>
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<tr>
<td>RMS</td>
<td>Root mean square length of the C=0.5 profile</td>
</tr>
<tr>
<td>RMS'</td>
<td>Average root mean square growth rate of the C=0.5 profile</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$u$</td>
<td>Lag distance</td>
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<tr>
<td>$u_x$</td>
<td>Velocity in the x direction</td>
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<td>$u$</td>
<td>Velocity vector</td>
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<tr>
<td>$v$</td>
<td>Velocity in the y direction</td>
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<tr>
<td>$x$</td>
<td>Spatial coordinate</td>
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<tr>
<td>$x$</td>
<td>Position vector</td>
</tr>
<tr>
<td>$y$</td>
<td>Spatial coordinate</td>
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<tr>
<td>$z$</td>
<td>Logarithm of permeability</td>
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**Greek symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>$\gamma$</td>
<td>Permeability autocovariance function</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>Time step</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>Spatial grid in the x direction</td>
</tr>
<tr>
<td>$\Delta y$</td>
<td>Spatial grid in the y direction</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Dimensionless density difference</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Fluid mobility</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Correlation length (dimensionless)</td>
</tr>
</tbody>
</table>
\[ \lambda_D \quad \text{Correlation length (in grid block units)} \]

\[ \mu \quad \text{Viscosity} \]

\[ \nu \quad \text{Unit normal vector (outward)} \]

\[ \rho \quad \text{Density} \]

\[ \rho \quad \text{Permeability autocorrelation function} \]

\[ \sigma \quad \text{Standard deviation of logarithm of permeability} \]

\[ \tau \quad \text{Characteristic direction} \]

\[ \phi \quad \text{Porosity} \]

\[ \Omega \quad \text{Spatial domain} \]

\[ \partial \Omega \quad \text{Domain boundary} \]

**Subscripts**

\[ \text{BT} \quad \text{Breakthrough} \]

\[ i \quad \text{Grid index in the x direction} \]

\[ j \quad \text{Grid index in the y direction} \]

\[ k \quad \text{Sub-time step index} \]

\[ o \quad \text{Oil} \]

\[ s \quad \text{Solvent} \]

**Superscripts**

\[ \cdot \quad \text{Average quantity} \]

\[ n \quad \text{Time step index} \]
REFERENCES


[22] G.M. Jenkins and D.G. Watts, Spectral Analysis and its


