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The comparative effects of a non-neutral land value tax in a two period model of urban development

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Rice University, 1987
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THE COMPARATIVE EFFECTS OF A NON-NEUTRAL LAND VALUE TAX
IN A TWO PERIOD MODEL OF URBAN DEVELOPMENT

by

DUCK-HO LIN

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

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July, 1986
Abstract

The Comparative Effects of a Non-Neutral Land Value Tax in a Two Period Model of Urban Development

by

Duck-Ho Lim

I analyse a two period model of urban development in which a non-neutral land value tax accelerates city growth by encouraging the development of vacant land in the first period and decreases the size of the city in the second period. In contrast, a wage tax decreases the size of the city in both periods by increasing the cost of labor. At the level of an individual city a non-neutral land value tax results in a smaller dead weight loss relative to a wage tax of equal yield.

However, this result is not generally true when we study the relative efficiency of a land value tax and a wage tax in a system of identical cities. For this case a wage tax is relatively more efficient under a set of plausible restrictions. This analysis demonstrates the danger of making national policy prescriptions on the basis of results for an individual city.
Acknowledgements

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I must also express gratitude towards the other members of my committee. Dr. George Zodrow's questions and comments helped me strengthen weak arguments. Also, I am grateful to Dr. Rick Wilson for his careful reading of the dissertation and for his comments.

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# Table of Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Tables</td>
<td>iv</td>
</tr>
<tr>
<td>List of Figures</td>
<td>iv</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1. The Purpose of the Study</td>
<td>1</td>
</tr>
<tr>
<td>1.2. The Review of Literature</td>
<td>3</td>
</tr>
<tr>
<td>1.3. Overview of the Study</td>
<td>11</td>
</tr>
<tr>
<td>II. Land Speculation and Taxation in a Single Open City</td>
<td>15</td>
</tr>
<tr>
<td>2.1. Introduction</td>
<td>15</td>
</tr>
<tr>
<td>2.2. The model of Urban Development</td>
<td>16</td>
</tr>
<tr>
<td>2.3. The Effects of a Land Value Tax on Urban Development</td>
<td>31</td>
</tr>
<tr>
<td>2.4. The Effects of a Wage Tax on Urban Development</td>
<td>36</td>
</tr>
<tr>
<td>III. The Relative Efficiency of a Wage Tax and a Land Value Tax - A Single Open City</td>
<td>40</td>
</tr>
<tr>
<td>3.1. Introduction</td>
<td>40</td>
</tr>
<tr>
<td>3.2. For a Single Tax System</td>
<td>41</td>
</tr>
<tr>
<td>3.3. For a Simultaneous Tax System</td>
<td>50</td>
</tr>
<tr>
<td>IV. Land Speculation and Taxation in a System of Cities</td>
<td>63</td>
</tr>
<tr>
<td>4.1. Introduction</td>
<td>63</td>
</tr>
<tr>
<td>4.2. The Model</td>
<td>64</td>
</tr>
<tr>
<td>4.3. The Effects of a Wage Tax</td>
<td>73</td>
</tr>
<tr>
<td>4.4. The Effects of a Land Value Tax</td>
<td>80</td>
</tr>
<tr>
<td>4.5. The Relative Efficiency of a Wage Tax and a Land Value Tax</td>
<td>84</td>
</tr>
</tbody>
</table>
V. Conclusions 98
Appendix 4-1 100
Bibliography 104

List of Tables

Table 2-1 The Effects of Taxes on Urban Development in the Case of a Single Open City 39

List of Figures

Figure 2-1 The Pattern of Urban Development in One Period Model 18
Figure 2-2 The Pattern of Urban Development in a Two Period Model 20
Figure 2-3 The Locational Equilibrium Land-Rent Functions 29
Figure 4-1 The Pattern of Urban Development in the Case of an Exogenous Population 65
Figure 4-2 The Pattern of Urban Development in the Case of a System of Cities where a Land Value Tax is imposed 80
Chapter 1
Introduction and Review of Literature

1.1. The Purpose of The Study

The purpose of this dissertation is to study the effects of a land value tax on urban development in an intertemporal setting, to analyse the relative efficiency of land value taxation and taxes on wages. The main objective of the thesis is to determine the practical importance of the challenge to the classical view\(^1\) that a tax on land value is borne entirely by land owners and has no effects on resource allocation.

Once the non-neutrality of a land value tax is established it is important to develop a model that analyses the comparative, or relative, efficiency of a land value tax and alternative sources of finance such as a wage tax. I adopt a model of urban structure, developed by David Mills (1981)\(^2\) to carry out a comparative analysis for two cases. The first model is for a single open city where the level of real income is predetermined by national conditions and the supply of labor to the city is perfectly elastic. The second

\(^1\) In analysis of the effects of taxes on the distribution of income Ricardo, Smith, and Mills argued that a tax on land ownership is the ideal tax, since the supply of land is fixed, and thus its ownership can be taxed without real effects.

model is for a system of cities, where the nation is characterized by a number of identical cities. The key difference between the analysis of a single city and the national perspective is that the supply of labor is much more elastic at the level of a single open city.

Economists and policy makers have long recognized the apparent advantage of land value taxation and have argued that since the supply of land is perfectly inelastic a land value tax will be allocatively neutral, and less distortionary relative to other taxes imposed by local governments.

However, in an intertemporal setting, where the timing of urban development is endogenous a land value tax is non-neutral.\(^1\) We shall show that at the level of a single city a land value tax continues to be efficient relative to a wage tax of equal yield.\(^2\) But this result is not generally true in a situation where all cities on the nation or a large number of cities choose between a wage tax and a land value tax. This finding casts doubt on the relative efficiency of a land value tax, and also demonstrates that results on efficiency derived for a single city do not generalize to a system of cities where all jurisdictions carry out the same policy.\(^3\) This conclusion has direct bearing tax assignment of different taxes to various levels of government in a federal system.

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1 For details see Chapter II.
2 For details see Chapter III.
3 For details see Chapter IV.
1.2. The Review of Literature

Land value taxation has received considerable attention in recent years as scholars and practitioners have written about the theoretical advantages and disadvantages of this tax, the administrative feasibility, and tax revenue potential of this tax system. Ever since the publication of Henry George's, Progress and Poverty (1879), many writers have advocated adoption of taxes on land on equity considerations, and on the grounds that these taxes have no effect on resource allocation.

With the qualifications noted below, it is generally agreed that a tax on land rents is non-distortionary. Stiglitz (1977) and Hochman (1981) argued that besides Pigovian corrective taxes the only tax necessary to finance local government activities efficiently, is a tax on land rents, and that this is the only source of revenue that is allocatively neutral.¹

It is well established that the real property tax is non-neutral since the supply of structures in a particular jurisdiction is not fixed and the imposition of a tax on capital distorts capital-land use decisions. Oates (1969) examined the degree to which property taxes are capitalized and reflected in

property values. Also, Mieszkowski (1972) and Grieson (1974) have studied the allocative effects of a property tax on reproducible capital. The analysis for a single jurisdiction shows that the property tax will be shifted back to land owners if wages and the return to capital remain unchanged by the imposition of a tax in a single jurisdiction. However, when we adopt national perspective, and the overall supply of capital is fixed, a tax on capital imposed by local governments decreases the overall return to capital by the average rate of tax in the nation as a whole, and changes the supply price of capital to different cities according to relationship of the specific rates relative to the mean rate of property tax. Changes in the cost of capital lead to a reallocation of residential and industrial activities which in turn influence site value and so the real property tax is distortionary. In contrast, a tax on land rents is borne entirely by land owners.

A modern analysis of the land (site) value tax has been carried out by Brueckner (1986), who analysed both the long-run effects and the distribution of the short-run gains and losses of this tax. Brueckner analysed the effects of lowering the tax on improvements and increasing the land value tax in a particular tax

zone. He concluded that when the tax zone comprises a negligible portion of the housing market, gradation of the tax system by lowering the tax on improvements leaves the price of housing unchanged while raising both the level of improvements per acre and the value of land. When the tax zone, the area where gradation occurs, encompasses the entire housing market, gradation reduces the price of housing, raises the level of improvements and lowers the value of land. So, long-run effects are shown to depend crucially on the relative size of the tax zone, where gradation occurs, and the overall housing market.¹

In a well known article Feldstein (1977) analyzed the effects of a tax on land rents in an overlapping-generation model in which land and capital are assets held by households maximizing lifetime utility.² In this model a tax on pure land rent can change the growth rate of the economy by changing the rate of savings. After the imposition of the tax land owners suffer a capital loss and increase their savings. This increase in the holdings of real capital means that owners of existing capital stocks bear part of the burden of a tax on land.

Dealing with a very different issue, Donald Shoup (1970) demonstrated how the optimal timing of urban land development is

changed as population and income grow. In a static situation the
criterion for the profit maximizing allocation of land among
different uses is that the value of marginal product in all uses
should be equal. But in a growing city increasing demand for land
will lead to more intensive use of the land. The development
decision must trade off between lower density development in the
current period versus higher density and higher land value on the
site in a future period.¹

More recently the non-neutrality of a tax on current land
value was demonstrated by a number of writers including Skouras
(1976), David Mills (1979) and especially Brian Bentick (1979).²
They agreed that taxes on land rents are neutral. Also, a tax on
"standard" land value defined not in terms of current market value
but in terms of the value of a "physically" defined standard state
for tax purposes, as suggested by Vickrey (1970), is neutral.³
However, a tax on current land value is non-neutral as it creates

¹ D. C. Shoup, "The Optimal Timing of Urban Land Development,"
Papers of the Regional Science Assoc. 25, PP. 33-44.
² Brian Bentick demonstrated a counter-example to the neutrality
proposition by showing that a tax on current land value favors land
uses with early-payoff income streams. For the details see Brian
Bentick, "The Impact of Taxation and Valuation Practices on the
Timing and Efficiency of Land Use," Journal of Political Economy 87,
1979, PP. 859-68.
³ W. Vickrey, "Defining Land Value for Tax Purposes," in the
Assessment of Land Value, ed. by D. M. Holland, PP. 25-36, Madison:
University of Wisconsin Press, 1970.
an incentive for the more rapid development of land. The explanation for their result is that current land value reflects both current and future land rents, and a tax on land value will decrease the incentive of holding land idle for more profitable future development. Their non-neutrality argument developed in general model of investment, further examined by Tideman (1982) and Wildasin (1982), is correct.

In analyzing the effects of taxes on urban development and the relative efficiency of a land value tax and other taxes it is important to determine whether the holding of vacant land for more profitable future development is socially efficient. In a perfect-foresight model of urban development the amount of vacant land in the current period will affect the size of the city in the future period, and the net social product will be the sum of the present value of total land rents.

Land speculation is closely related to urban sprawl, as discussed by Harvey and Clark (1965). While Clawson (1962) has emphasized the importance of land speculation as a determinant of

---

2 For the details see Chapter II.
sprawl,\(^1\) Archer (1973) has argued that sprawl is caused by the failure of the market to account for the full costs of development.
Landowners who hold their land idle in the current period increase travel costs to the Central Business District (CBD) and increase the cost of supplying public goods.\(^2\)

Land speculation also accelerates the increase in land value by decreasing the supply of land for current construction. Thus, land speculation is condemned because it causes a scattered pattern of urban development and increases the current prices of housing.

However, urban land development is a dynamic process and market efficiency should be evaluated by dynamic criteria. Boyce (1963) suggested that urban sprawl provides flexibility in urban development carried out under conditions of uncertainty and imperfect knowledge. Land speculation carried out for more profitable future development might produce an urban structure that is more efficient in the long run.\(^3\) Ohls and Pines (1975) suggested additional reasons for discontinuous development. Their arguments are based on the tradeoff between living space and accessibility over time, and on the fact that the development of retail and commercial services near the urban fringe must often await the

development of markets large enough to exhaust economies of scale. They suggested that speculation in land markets is a substitute for government intervention as a mechanism for ensuring economic efficiency. They argued that when a piece of land has the potential for being highly productive at some future date, it will be in the economic interest of potential owners to buy the land, hold it vacant for a while, and then either to develop the land themselves or to sell the land in the future period at a high price which reflects its productivity at that time. If a number of potential owners are aware of the possibilities of future profits, they will bid up current price of land to a value that reflects its future profitability. The high current price of the land discourages premature development and reserves the land for more productive future use.¹

David Mills (1981) also criticized the idea that land speculation is inefficient. He stressed that the conversion of urban land is dynamic process and that in a growing city inferior parcels of urban land are sometimes kept vacant in the current period and preserved for future uses as the conversion of land from one use to another is so expensive. Also, while Mills agreed that leapfrogging, scattered, and mixed development will increase public sector costs he blames these effects, associated with sprawl, on the failure of

local governments to price their services at marginal cost.¹

Using a two-period model of urban development, developed in the next chapter, we conclude that taxes on vacant land are inefficient. The two period model assumes perfect foresight. Land owners allocate land to different uses in the two periods so as to maximize the present value of land rents. In the absence of externalities and increasing return to scale, net social product is maximized when land values or land rents are maximized. Consequently, a tax on vacant land is inefficient as it distorts the intertemporal development of land. In this model the holding of vacant land in the first period is socially productive as with land speculation output will be larger in the second period, and the present value of social output is maximized.

The holding of vacant land in the first period increases the cost of residential housing in this period and the consideration of public expenditures implies that vacant land should be taxed to offset the higher costs of public services. For example, if the land is held vacant and a road has to be built through the vacant land then taxes should be imposed on the vacant land so that timing of development is not affected by the public investment on the vacant land. One way of approximating benefit taxation is to impose development fees on vacant land or to require developers to provide

for some sort of the cost of the main roads as well as the access roads.

1.3. Overview of the Study

In a growing city the conversion of land from rural to urban use, or from a lower to higher productive use continuously takes place, and some land is temporarily held vacant, awaiting more profitable development in the future. In a static situation the profit maximizing allocation of land among different uses will be satisfied when the value of marginal product in all uses is equal; if the marginal product of land in one use is higher than in alternative, land should be developed for the higher valued use, up to the point where the present value of the two marginal products are equal. But in a growing city the optimal time for the development of a particular piece of land may not be the current period.

To carry out the differential analysis of the effects of taxes on urban development and the relative inefficiency between taxes we use a simple model of urban growth first presented by David Mills (1981). This model is a two-period model that assumes uniform spatial density for two activities, the production of an industrial good and the production of residential housing services. The exogenous growth mechanism in the model is an increase in export demand for the industrial good which is produced in the city. It is assumed that the industrial good is shipped through the CBD and that
workers commute daily to the CBD for recreation. Assumptions are made so that land close to the CBD is more valuable as industrial land. The essential land allocation decision is whether to develop land as residential land in the current period or to hold the land vacant and to develop it as industrial land in the future. Land owners will find it profitable to hold some land situated between the developed industrial zone and the residential zone vacant in the first period, thus providing an example of leapfrog development. In the second period this land is developed for industrial use to accommodate increased industrial production. The basic model modified for tax parameters is presented in Chapter II.

In a two-period model of urban development with perfect foresight, a tax on land rent is neutral but a tax on current land value is non-neutral. ¹ Some land is held vacant for more profitable future development, earns zero land rents in the first period, and earns higher land rents in the second period. For taxes on land rents land owners who hold land vacant in the first period pay tax only in the second period. However, for a land value tax the land owners pay taxes in both periods as current land value reflects both current and future land rents. So, a land value tax is a form of double taxation. This tax encourages the development of land in the first period and the tax is distortionary. This premature development decreases industrial output and city size in

¹ For the details see Chapter II.
the second period as less land is available for industrial development.

The imposition of a wage tax in a city where the supply of labor is perfectly elastic increases the cost of labor and discourages the production of the industrial good in both the first and second periods. We will analyze the effects of a land value tax and a wage tax on urban development in detail in Chapter II.

In contrast, in a system of cities where the supply of labor is inelastic relative to a single open city the imposition of a wage tax will have no impact on the development of urban land except to extent that a wage tax decreases the overall supply of labor.\(^1\) So, the distortionary effect of a wage tax in a system of cities appears to be much smaller than those of a wage tax imposed by one city.

For a system of cities the imposition of a land value tax encourages the development of land in the first period, and though overall employment is little affected by this tax, the timing of development changes city structure, decreases the amount of vacant land in the first period, and results in a less efficient leapfrog form of development. The effects of taxes for a system of cities will be analyzed in detail in Chapter IV.

In an intertemporal setting, the effect of a non-neutral land value tax on the timing of urban development is more efficient than a wage tax of equal yield at the level of a single open city. This result was expected and it suggests that the finding that land value

\(^1\) For the details see Chapter IV.
taxes are distortionary is of little practical significance. The analysis for the relative efficiency of a land value tax and a wage tax for a single open city is presented in Chapter III.

The conclusion that a land value tax is efficient relative to a wage tax under very weak restrictions does not generalize to a system of cities where all cities adopt the same policy. For this case we are able to develop quite reasonable sufficient conditions under which a wage tax system is relatively more efficient. This results illustrate the danger of making national policy recommendations on the basis of results obtained for a single open city. The analysis for the system of cities is presented in Chapter IV.
Chapter II
Land Speculation and Taxation in a Single Open City

2.1. Introduction

In this chapter we study the basic model of urban development and develop the model that allows for an analysis of the comparative efficiency of a land value tax and a wage tax for a single open city. To carry out the differential analysis of the effects of taxes on urban development we use a model of urban growth developed by David Mills. This two-period model is modified for tax parameters. It assumes uniform spatial density for two activities, the production of an industrial good and the production of residential housing services.

We define land speculation as the leapfrog development which will occur when land is withheld from the development in the first period. Standard arguments imply that, with perfect foresight, a speculator will hold the land idle only if the increase in market value is large enough to offset the opportunity cost incurred during the holding period.

In this chapter the effects of a land value tax and a wage tax on urban development are analyzed for a single city, for a two-period model in which the timing of urban development is endogenous. For the analytical simplicity we start from a non-distorted equilibrium.
2.2. The Model of Urban Development

In a metropolitan area labor is combined with land in producing goods and services. The city lies on a featureless plane topographically and it is usually marked by a circle of radius \( x \) with a circumference of \( 2\pi x \) and an area of \( \pi x^2 \), and has a central business district (CBD). However, for analytical simplicity we consider only one straight line from the CBD to the edge of the city.

There are two production activities, the production of a composite commodity, \( Q \), (the industrial activity) and the production of housing services. The price of the industrial good which must be shipped through the CBD is fixed, exogenously at the CBD. Also, to simplify the structure of the model, we assume that workers travel daily to the CBD to shop and for recreation. However, in contrast to some urban models, employment is not concentrated at the CBD. Instead, workers are located in an industrial zone of uniform density extending in a line leading from the CBD. For simplicity we assume the wage rate is constant through the industrial zone. In determining the wage rate no allowance is made for differential commuting costs from the edge of the residential zone to different parts of the industrial zone.

The production function for \( Q \) has fixed factor-proportions and each unit of the commodity is produced by \( m \) units of land, and \( u \) units of labor. There is no capital in the model. Land rents and
wages are endogenous to the model. Also, the production technology does not change between periods.\footnote{If we allow for the change of the technology in production between periods, in the second period the amount of labor per unit of output will be reduced.}

Every household provides one unit of labor and workers are in perfectly elastic supply to the city at a constant real wage. For simplicity we suppress the substitutability of consumption goods and assume that each household consumes exactly a units of Q and one unit of residential land in each period.

To illustrate the industrial and the residential structure of the model consider a one period version. By assumption the cost of transporting a unit of the industrial good, per unit of distance, is $t$ and the cost of transporting a worker per unit of distance is $T$. We assume that $t$ is much greater than $T$, and for this reason the industrial zone is located next to the CBD and the residential zone is located beyond the industrial zone. The pattern of development is shown by Figure 2-1. At a distance $x$ from the CBD output will be equal to $1/x$ and after the payment of wages the product is divided between land rents (pure profits) and transportation costs. Thus, the land rent in the industrial zone at distance $x$ from the CBD is

$$\frac{1}{x}(P-tx-\omega)$$
where,

\[ P = \text{price per unit of output}, \]
\[ w = \text{wage per unit of output}, \]
\[ u = \text{labor per unit of output}, \]
\[ \pi = \text{land per unit of output}, \]
\[ t = \text{transportation cost, per unit of } x, \text{ for the industrial good.} \]

The wage rate, \( w \), is the gross wage, and is high enough to pay for three types of expenditures by the worker. Wages are used to finance the units of the export good, an expenditure \( P q \); second wages are also used to finance the purchase of the services of one unit of land, and to pay for transportation costs. Total expenditures on residential land rents plus transportation costs for workers are constant at all residential locations.

Figure 2-1 The Pattern of Urban Development in a One Period Model

- Industrial
- Residential

\[ x' \quad x'' \]
\[ \text{CBD} \quad P_0 \quad P_1 \]
Households who live on the outer edge of the residential zone pay no land rents, as we ignore the opportunity cost of land in agriculture. As all workers are assumed to travel to the CBD, expenditures on transportation of these workers are $T(x'+x'')$ where $T$ is the transportation cost per unit distance for a household. So, residential land rents at distance $x'$ from the CBD will be $Tx'$ and this must equal industrial land rents at $x'$.

To transform this one period model into a two period situation with an intertemporal allocation structure we assume an exogenous increase in the price of the industrial good.\(^1\) So, $P_2 > P_1$, where $P_1$, $P_2$ are the prices of the industrial good in the first and second periods, respectively. The cost of transporting the industrial good is greater than the cost of transporting the worker and land developed as residential land in the first period will remain residential in the second period since the cost of land conversion is assumed to be very high. Consequently, a pattern of 'leapfrog' development will occur. Under this pattern of development some land relatively close to the industrial zone is held vacant during the first period and is developed as industrial land in the second period.

\(^1\) The mechanism of city growth in this model is an exogenous increase in the price of commodity between two periods. In the case of inelastic labor supply city growth may result from an exogenous increase in population.
Figure 2-2 The Pattern of Urban Development in a Two Period Model

The land shown as vacant in period 1 in Figure 2-2 could be developed as residential land in the first period and could earn residential land rents in both periods. But land owners keep the land idle in the first period, and wait for the more profitable development in the second period. The tradeoff is between two periods of residential land rents versus one period of higher industrial land rents. Leapfrogging of the industrial zone to the area beyond the residential zone is not permitted by the assumption that $t$, the cost of transporting the industrial good, is sufficiently large relative to $T$, the cost of transporting worker. Land owners are assumed to have perfect foresight. Thus, the decision on the amount of vacant land determines the sizes of the industrial and residential zones in the second period.
Finally, we assume that the agents who make land-use decisions in the model are nonresidents. The nonresidency means land rents do not reappear in the analysis as household income.\footnote{For a system of cities we assume that the agents who make land-use decisions in the model are residents, and so land rents appear in the analysis as household's lump-sum rental income. See Chapter IV.}

The model, modified for tax parameters, consists of six unknowns, the wage rates in the two periods, $w_1$ and $w_2$, and four distances from the CBD, or boundaries.

\begin{align*}
x_0 &= \text{the length of the industrial zone in the first period,} \\
x_1 &= \text{the length of the industrial zone in the second period,} \\
x_2 &= \text{the length of the residential zone in the first period,} \\
x_3 &= \text{the length of the residential zone in the second period.}
\end{align*}

The six equations that determine these variables are as follows

\begin{align*}
w_1 &= p_1q + T(x_1 + x_2) \\ w_2 &= p_2q + T(x_1 + x_3)
\end{align*}
Equations (2-1) and (2-2) are wage equations that relate that the gross wage in each period to the purchase of a quantity $q$ of the industrial good and the payment of transportation costs from the outer edge of the residential zone (where land rents are zero) to the CBD.\footnote{If we consider the exogenous opportunity costs in agriculture, the two wage equations will be shown by 

\[ w_1 = P_1 q + T(x_1 + x_2) + OC_1 \]

\[ w_2 = P_2 q + T(x_1 + x_3) + OC_2 \]

and the opportunity costs, $OC_1$ and $OC_2$ will be equal to residential land rents on the outer edge of the residential zone in each period.}

\[
\frac{1}{w} (P_1 - tx_0 - uw_1) = 0
\]  

(2-3)

Equation (2-3) is a product exhaustion condition that says that in the first period land rents on the outer edge of the industrial zone are zero. If land rents were positive, land owners would develop the land industrially. The value of output is divided between the per-unit wage, $uw_1$, and firm's transportation costs, $tx_0$. The fourth equilibrium condition is that land on the outer edge of the industrial zone in the second period, boundary $B_1$ in Figure 2-2, is equally profitable as industrial (vacant) land in the second (first) period and as residential land in the first and second periods. This condition is written as
\[
\frac{1}{i} (P_2 - tx_1 - w_2) = \frac{Tx_2}{1+i} + \frac{Tx_3}{1+i} \quad (2-4)
\]

where \(i\) is the discount rate. In equation (2-4) the left hand side measures the present value of industrial land rents in the second period and the right hand side shows residential land rents, \(Tx_2\), and \(Tx_3\), in the first and second periods, respectively. In calculating residential land rents we make use of the fact that these rents are zero on the outer edge of the residential zone in each period.

The final two equations are implied by the fixed coefficient structure of production, the assumption that each unit of output is produced by \(x\) units of land and \(u\) units of labor. This implies that there is a proportional relationship in the sizes of the industrial and the residential zones in each time period.

\[
\begin{align*}
x_2 &= \frac{u}{x} x_0 \quad (2-5) \\
x_3 &= \frac{u}{x} x_1 \quad (2-6)
\end{align*}
\]

Equations (2-5) and (2-6) state that the residential zone is \(u/x\) times the size of the industrial zone in each period.

We can eliminate \(x_2\) and \(x_3\) from the system by substituting
(2-5) and (2-6) into (2-1), (2-2), and (2-4). This yields four equations in four unknowns, \( w_1 \), \( w_2 \), \( x_0 \), and \( x_1 \)

\[
\begin{align*}
    w_1 &= P_1q + Tu_{w_1}x_0 + Tx_1 \quad (2-1)' \\
    w_2 &= P_2q + T(1 + u/w)x_1 \quad (2-2)' \\
    P_1 - tx_0 - uw_1 &= 0 \quad (2-3)' \\
    \frac{P_2 - tx_1 - uw_2}{1+i} &= Tu_{x_0} + \frac{Tux_1}{1+i} \quad (2-4)'
\end{align*}
\]

Using equations (2-1)' and (2-3)' we eliminate \( w_1 \) to obtain

\[
(1/u - q)P_1 = (Tu/w + t/u)x_0 + Tx_1 \quad (2-5)
\]

This equation has a clear-cut interpretation. Since \( 1/u \) is the output per worker, the term \( (1/u-q)P_1 \) is always positive and is a measure of resources available for transportation costs. The terms \( Tu_{w_1}x_0 \) and \( Tx_1 \) measure the cost of transporting the workers who in the first period live on the outer edge of the residential zone. The term \( t/u \) \( x_0 \) is the cost of transporting the industrial good which is produced at distance \( x_0 \) from the CBD.

We use equations (2-2)' and (2-4)' to eliminate \( w_2 \) to obtain
\[
\frac{(1/u-q)P_2}{1+i} = T[x_0 + x_1/(1+i)] + \frac{[t/u+T(1+u/\pi)]}{1+i} x_1
\] (2-6)

Equation (2-6) has the same structure and interpretation as equation (2-5). The term \(T[x_0 + x_1/(1+i)]\) represents the industrial land rent for \(\pi\) units of land located on the boundary between the industrial zone and the residential zone in the second period. The term \(t/u x_1\) is the cost of transporting the industrial good produced at distance \(x_1\) from the CBD and \(T(1+u/\pi)x_1\) is the cost of transporting the worker who lives on the outer edge of the residential zone in the second period.

Solving (2-5) and (2-6) for \(x_0\) and \(x_1\), we measure the size of the industrial zone in each period in terms of the exogenous variables as

\[
x_0 = \frac{(1-qu)[(t/T + (2 + u/\pi)]P_1 - P_2}{[t/T + u(2 + u/\pi)](t/u+Tu/\pi)-(1+i)Tu}
\] (2-7)

\[
x_1 = \frac{(1-qu)[(t/Tu+u/\pi)P_2-(1+i)P_1]}{[t/T+u(2+u/\pi)](t/u+Tu/\pi)-(1+i)Tu}
\] (2-8)

From equations (2-5) and (2-6) it follows that \(1-qu>0\). In equations (2-7) and (2-8) the denominator is positive as the cost of
transporting the industrial good, it is greater than the cost of transporting worker, \( T \), and \( u/w > 1 \) as the amount of land required to produce a unit of the industrial good is smaller than residential land required to house the labor used to produce the good. Thus, the numerator in equation (2-7) must be positive since \( x_0 \), the size of the industrial zone in the first period is positive. It follows that

\[
P_2 < \left[ \frac{t}{Tu + (2 + u/w)} \right] P_1
\]

As the city is growing, \( x_1 \), the industrial zone in the second period is larger than the industrial zone in the first period: \( x_1 > x_0 \). From equations (2-7) and (2-8) we obtain

\[
x_1 > x_0; \left[ 1 + (2+i)/(1+t/Tu+u/w) \right] P_1 < P_2
\]

Therefore, the range of \( P_2 \) is

\[
\left[ 1+(2+i)/(1+t/Tu+u/w) \right] P_1 < P_2 < \left[ \frac{t}{Tu+(2+u/w)} \right] P_1 \quad (2-9)
\]

To analyze the relative efficiency of a wage tax and a land value tax we measure total net social product (pure profit) by the sum of the present value of land rents in the first and second periods. The first step in this calculation is to derive land-rent
functions in each period.

Every household residing in the city has the same real wage, and faces the same price for the industrial good in each period. Therefore, each household's total expenditures on residential land rents and transportation costs must be equal. Since we assume uniform density in the residential zone and zero land rents on the outer edge of the residential zone, it follows that the residential land-rent function is linear:

\[ R_1^H(x) = T(u/\pi x_0 + x_1) - Tx \]
\[ R_2^H(x) = T(1 + u/\pi)x_1 - Tx \]

where \( R_i^H(x), i=1,2, \) are the residential land rents at each location in the first and second periods, respectively. In the growing city the price of the industrial good, the wage rate, population, and residential and industrial land rents all increase in the second period. As we show in Figure 2-3, residential land rents in the second period increase by an amount of \( Tu/\pi(x_1-x_0) \) at every location.

As markets are competitive firms earn zero profits, and at every location have the same expenditures on industrial land rents plus transportation costs. Also, industrial land rents on the outer edge of the industrial zone are zero in the first period since otherwise, land owners would develop the land industrially in the
first period. Thus, the industrial land-rent function in the first period is

$$\bar{R}_1^Q(x) = \frac{t}{\pi} x_0 - \frac{t}{\pi} x$$

where $\bar{R}_1^Q(x)$ is the industrial land rent at each industrial location in the first period.

Since the present value of land rents on the boundary between the industrial and the residential zones in the second period must be equal in the two alternative uses of the land, the boundary condition shown below can be used to calculate the industrial land rent function in the second period.

$$\bar{R}_2^Q(x) = R_1^H(x_1) + R_2^H(x_1) = \frac{Tu}{\pi} x_0 + Tux_1^*/\pi(1+i)$$

This relationship and the zero profit condition yield the industrial land-rent function in the second period.

$$\bar{R}_2^Q(x) = (1+i)Tu/\pi x_0 + (Tu/\pi + t/\pi)x_1 - t/\pi x$$

where $\bar{R}_2^Q(x)$ is the industrial land rent at each industrial location.

Firms located at the CBD don't pay transportation costs and pay higher industrial land rents relative to other industrial sites.
They pay \( \frac{t}{x_0} x_0 \) and \( (1+i) \frac{Tu}{\pi} x_0 + \left( \frac{Tu}{\pi} t/\pi \right) x_1 \) as industrial land rents in each period, respectively.

**Figure 2-3 Locational Equilibrium Land-Rent Functions**

However, firms located on the outer edge of the industrial zone in the first period do not pay industrial land rents in the first period and pay higher transportation costs to ship the industrial good to the CBD. On the boundary between the industrial zone and the
residential zone in the second period the industrial land rent in the second period is equal to the sum of the residential land rents in the first and second periods, \((1+i)Tu/\pi x_0 + Tu/\pi x_1\). On the outer edge of the residential zone in each period residential land rents are zero in each period, ignoring the opportunity costs in agriculture. This information is summarized in the graph above.

We measure net social product by means of the land-rent functions. In the nondistorted equilibrium total net social product (NSP) can be measured by the sum of land rents in urban area in each period.

\[
\begin{align*}
\text{NSP} &= \text{NSP}_1 + \text{NSP}_2/(1+i) = \int_0^{x_0} R_1^Q(x) \, dx + \int_{x_0}^{x_1} R_1^H(x) \, dx \\
&+ \frac{1}{(1+i)} \int_0^{x_1} R_2^Q(x) \, dx + \frac{1}{(1+i)} \int_{x_1}^{(1+u/\pi)x_1} R_2^H(x) \, dx \\
&= \frac{1}{2}(t/\pi + Tu^2/\pi^2)x_0^2 + \frac{1}{(1+i)} \left( \frac{1}{2}t/\pi + \frac{1}{2}Tu^2/\pi^2 + Tu/\pi \right)x_1^2 + Tu/\pi x_0 x_1
\end{align*}
\]

(2-10)

In Equation (2-10) the first term represents the present value of total net social product in the first period, and the second and third terms represent the present value of total net social product in the second period. The larger is the size of the industrial zone, the greater is the value of net social product.
2.3. The Effects of a Land Value Tax on Urban Development

A tax on annual rental income is neutral. Land owners on the boundary \( B_1 \) in Figure 2-2 who develop their land industrially in the second period after holding the land idle in the first period earn zero rents in the first period. They also earn higher industrial land rents in the second period, relative to residential land rents. Thus, if a tax on rental income is imposed, these land owners pay no tax in the first period, and pay a higher tax in the second period on industrial land rents. If they develop the land residually in the first period, they will earn residential land rents in the first and second periods, and pay tax in both periods. The equality between the present value of land rents at this boundary will not be affected by the imposition of a tax on rental income, and this tax is neutral as it does not change the optimal timing of land development: The equilibrium condition is rewritten as

\[
\frac{(1-\theta)R_2^0(B_1)}{1+i} = \frac{(1-\theta)R_1^H(B_1) + (1-\theta)R_2^H(B_1)}{1+i} \tag{2-11}
\]

where \( \theta \) is the tax rate of a tax on rental income. In equation (2-11), the term \((1-\theta)\) appears on both sides of the equation and so the tax does not change the boundary condition.

In contrast, a land value tax is non-neutral as this tax changes the optimal timing of land development and affects the boundary condition between the two land zones. Current land value is
the summation of the present value of land rents in each period. Thus, even if land is vacant in the first period and earns zero rents during this period it has a positive value as the land will be developed as industrial land in the second period. When a tax based on current land value is imposed, land owners who hold their land vacant in the first period pay a land value tax in both the first and second periods even though they earn zero land rents in the first period. A land value tax by taxing future as well as present rents is a form of double taxation.

In this model when a land value tax is imposed, the present value of net after tax residential land rents on the outer edge of the industrial zone in the second period becomes

$$V^H(B_1) = R_1^H(B_1) + R_2^H(B_1)/(1+i) - \theta_{LV}V^H(B_1) - \theta_{LV}R_2^H(B_1)/(1+i)$$

where \(\theta_{LV}\) is an ad valorem land value tax, \(V^H(B_1)\) is the present value of net after tax residential land rents on the outer edge of the industrial zone in the second period, \(\theta_{LV}V^H(B_1)\) is the measure of the capitalization of the land value tax into land value, and \(\theta_{LV}R_2^H(B_1)/(1+i)\) shows the effect of imposition of a land value tax in the second period. Rewriting this equation, the present value of net after tax residential land rents is

$$V^H(B_1) = R_1^H(B_1)/(1+\theta_{LV}) + (1-\theta_{LV})R_2^H(B_1)/(1+i)(1+\theta_{LV})$$
The present value of net after tax industrial land rents on the outer edge of the industrial zone in the second period is

$$U^Q(B_1) = R_2^Q(B_1)/(1+i) - \theta_{LV}U^Q(B_1) - \theta_{LV}R_2^Q(B_1)/(1+i)$$

where $U^Q(B_1)$ is the present value of net after tax industrial land rents on the boundary $B_1$ in Figure 2-2. Rearranging this equation, we obtain

$$U^Q(B_1) = (1-\theta_{LV})R_2^Q(B_1)/(1+i)(1+\theta_{LV})$$

When the land value tax is imposed, the present value of net after tax land rents of the land on the outer edge of the industrial zone in the second period must be equal in the two alternative uses of the land: $U^I(B_1) = U^Q(B_1)$. Thus, the equal profit boundary condition, distorted by the land value tax, is

$$R_2^Q(B_1)/(1+i) = R_1^H(B_1)/(1-\theta_{LV}) + R_2^H(B_1)/(1+i)$$

Rewriting this equation in the same form as equation (2-4)',

$$(P_2 - tx_1 - um_2)/(1+i) = Tux_0/(1-\theta_{LV}) + Tux_1/(1+i) \quad (2-12)$$

The only difference between equations (2-4)' and (2-12) is the first
term of the right hand side of the two equations. Since \( T_u x_0 / (1 - \theta_{LV}) \) is larger than \( T_u x_0 \), equation (2-12) shows that a land value tax encourages residential development on the original boundary between the two land zones. After the imposition of a land value tax the present value of net after tax residential land rents is larger than the present value of net after tax industrial land rents on the original outer edge of the industrial zone in the second period. Therefore, land owners on the outer edge will develop their vacant land residentially in the first period.

This tax induced increase in the supply of residential land in the first period leads to a fall in residential land rents, increases labor supply, and decreases the nominal wage rate. The land value tax expands industrial output in the first period, and accelerates city growth in this period. But a land value tax decreases the size of the city in the second period as a smaller amount of land is available for industrial development in this period as the result of the premature residential development in the first period.

To demonstrate these propositions more formally we substitute equation (2-2)' into equation (2-12), eliminate \( u_2 \), totally differentiate the resulting expression, and set the initial level of the tax equal to zero to obtain

\[ \text{For the analytical simplicity we start from zero tax to investigate the effects of a land value tax on urban development.} \]
\[-Tu \, dx_0 - 1/(1+i)[t + Tu(2+u/\pi)]dx_1 = Tx_0 \, d\theta_{LU}\]

Substituting (2-1)' into (2-3)' to eliminate \( u \), and totally differentiating to obtain

\[-(t+Tu^2/\pi)dx_0 - Tu \, dx_1 = 0\]

Solving these two equations for \( dx_0 \) and \( dx_1 \) we obtain

\[
dx_0 = \frac{(1+i)Tu \, x_0 \, d\theta_{LU}}{[t/T+u(2+u/\pi)](t/u+Tu/\pi)-(1+i)Tu}\quad (2-13)\]

\[
dx_1 = \frac{-(t/Tu+u/\pi)(1+i)Tu \, x_0 \, d\theta_{LU}}{[t/T+u(2+u/\pi)](t/u+Tu/\pi)-(1+i)Tu}\quad (2-14)\]

The denominators of equations (2-13) and (2-14) are positive. As the numerator of equation (2-13) is positive, it follows that \( dx_0/d\theta_{LU} > 0 \), which means that the imposition of a land value tax increases the sizes of the industrial and residential zones in the first period.

In equation (2-14), the numerator is negative. So, \( dx_1/d\theta_{LU} < 0 \), which means that the imposition of a land value tax decreases the size of the industrial zone in the second period by inducing residential development in the first period. The decrease of output
in the second period depends on the magnitude of the tax induced residential development.

2.4. The Effects of a Wage Tax on Urban Development

A wage tax has quite different effects on urban development. The impact of a wage tax is to decrease the real wage and to induce some workers in the city to move out, and to decrease land rents. This emigration will lead to an increase of the gross wage rate and to decrease in output in each of the two periods. The demand for industrial land will decrease and industrial land rents will fall.

For a wage tax we can show quite generally that the tax decreases the amount of output and the size of the city in both periods. After the introduction of this tax the wage equations become

\[ w_1(1 - \theta_w) = P_1 q + Tu/y x_0 + Tx_1 \]  \hspace{1cm} (2-15)
\[ w_2(1 - \theta_w) = P_2 q + T(1+u/y)x_1 \]  \hspace{1cm} (2-16)

where \( \theta_w \) is the tax rate of a wage tax. Equations (2-15) and (2-16) state that the wage tax needs to be offset by a decrease in transportation costs. Such a decrease in transportation costs means that the residential zone will be moved closer to the CBD in the first period through the conversion of vacant land to residential land.
Substituting equation (2-15) into (2-3)', totally differentiating, and setting \( \theta_0 = 0 \) as the initial value of the tax rate we obtain

\[-(t+Tu^2/\pi)dx_0 - Tu \ dx_1 = u(P_1q+Tu/\pi x_0 +Tx_1) d\theta_0\]

Substituting (2-16) into (2-4)' to eliminate \( \theta_1 \), totally differentiating the resulting expression and again setting \( \theta_0 = 0 \) we obtain

\[-Tu dx_0 - 1/(1+i)[t+Tu(2+u/\pi)] dx_1 = u/(1+i)[P_2q+T(1+u/\pi)x_1] d\theta_0\]

Solving these two equations for \( dx_0 \) and \( dx_1 \) to determine how a wage tax affects industrial output and city growth in each period we obtain

\[
dx_0 = \frac{(uP_2 - [t/T+u(2+u/\pi)](qP_1+Tu/\pi x_0) - (t+Tu) x_1) d\theta_0}{[t/T+u(2+u/\pi)](t/u+Tu/\pi) - (1+i)Tu} \quad (2-17)
\]

\[
dx_1 = \frac{-u[P(t/Tu+u/\pi)P_2-(1+i)P_1]d\theta_0}{[t/T+u(2+u/\pi)](t/u+Tu/\pi) - (1+i)Tu}
\]

\[
-\frac{2[(t/Tu+u/\pi)(1+u/\pi)-(1+i)]Tu x_1}{[t/T+u(2+u/\pi)](t/u+Tu/\pi) - (1+i)Tu} \quad (2-18)
\]

In equations (2-17) and (2-18) the denominator is positive. In equation (2-17) the numerator is negative, since the condition that
t/π is greater than (2+i)T is necessary for land speculation to take place;¹ the discount rate i is less than 1; uq < 1 is a measure of resources available for transportation costs; u/π > 1 by the assumption that the residential zone is larger than the industrial zone in each period, and the range of P₂ is shown by equation (2-9).

Thus, dx₀/dθ₀ < 0, which means that the imposition of a wage tax decreases the sizes of the industrial and residential zones.² In equation (2-18) the numerator is also negative and dx₁/dθ₀ < 0. Therefore, a wage tax also decreases industrial output and the size of the city in the second period. Table 2-1 summarizes the effects of taxes on urban development at the level of a single city.

¹ The necessary condition for land speculation to take place in a two-period model was proved by David Mills. See Journal of Urban Economics 10, 1981, PP. 223-24.

² In equation (2-17) we know that the denominator is positive. So, we determine dx₀/dθ₀ < 0 if the numerator is negative. From equation (2-9) we know the range of P₂. Substituting the upper bound of equation (2-9) into P₂ in the numerator of equation (2-17), the numerator is shown to be negative

\[-(t/T+u(2+u/π))/π - (t+Tu)x₁\]

Therefore, the numerator in equation (2-17) is negative and dx₀/dθ₀ is negative.
Table 2-1 The Effects of Taxes on Urban Development
in the Case of a Single Open City

<table>
<thead>
<tr>
<th>Period</th>
<th>Variables</th>
<th>Wage Tax</th>
<th>Land Value Tax</th>
</tr>
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<tbody>
<tr>
<td>First</td>
<td>Nominal wage rate</td>
<td>increase</td>
<td>decrease</td>
</tr>
<tr>
<td></td>
<td>Residential rents</td>
<td>decrease</td>
<td>decrease</td>
</tr>
<tr>
<td></td>
<td>Industrial rents</td>
<td>decrease</td>
<td>increase</td>
</tr>
<tr>
<td></td>
<td>Output</td>
<td>decrease</td>
<td>increase</td>
</tr>
<tr>
<td></td>
<td>City size</td>
<td>decrease</td>
<td>increase</td>
</tr>
<tr>
<td>Second</td>
<td>Nominal wage rate</td>
<td>increase</td>
<td>decrease</td>
</tr>
<tr>
<td></td>
<td>Residential rents</td>
<td>decrease</td>
<td>decrease</td>
</tr>
<tr>
<td></td>
<td>Industrial rents</td>
<td>decrease</td>
<td>increase</td>
</tr>
<tr>
<td></td>
<td>Output</td>
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<td>decrease</td>
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<tr>
<td></td>
<td>City size</td>
<td>decrease</td>
<td>decrease</td>
</tr>
</tbody>
</table>
Chapter III
The Relative Efficiency of A Wage Tax and
A Land Value Tax - A Single Open City

3.1. Introduction

In this chapter we analyze the relative efficiency of a wage tax and a land value tax at the level of a single open city by comparing net social products for different tax regimes of equal yield; a land value tax is less distortionary and relatively more efficient if the total net social product is larger for this tax relative to a wage tax. We investigate the relative efficiency of the two taxes when a local government imposes a single tax and when the two taxes are imposed simultaneously.

For a single tax system investigated in Chapter II the introduction of a wage tax decreases the industrial output in both periods, but the imposition of a land value tax increases the amount of output in the first period and decreases it in the second period. In comparing the net social product for a land value tax with the net social product for a wage tax regime of equal yield we make use of the results derived in Chapter II, and we start from the non-distorted equilibrium.

For a simultaneous tax system where the local government imposes the wage tax and the land value tax simultaneously we start from a positive wage tax and a zero value for the land value tax; the local government is financing expenditures by means of a wage
tax and substitutes a small land value tax for a portion of the wage tax.\(^1\)

3.2. For A Single Tax System \(- \Theta_w = \Theta_{LV} = 0\)

The effects of the taxes on urban development in the first period are shown by equations (2-13) and (2-17), respectively. Rewriting these two equations, we have

\[
dx_0/d\Theta_{LV}d\Theta_w = A/B \cdot d\Theta_{LV} \quad (3-1)
\]

where,

\[
A = (1+i)Tu x_0, \quad A > 0
\]

\[
B = [t/T + u(2+u/\pi)](t/u+Tu/\pi) - (1+i)Tu, \quad B > 0
\]

\[
dx_0/d\Theta_w = C/B \cdot d\Theta_w \quad (3-2)
\]

where \(C = quP_2 - [t/T + u(2+u/\pi)](qP_1 + Tu/x_0) - (t+Tu)x_1, \quad C > 0\)

Rewriting equations (2-14) and (2-18) which illustrate the effects

\(^1\) Also, we can start from the positive land value tax and the zero wage tax. However, we face complexity in analyzing the relative efficiency of the two taxes by carrying out differential analysis due to the capitalization of a land value tax into land value.
of the two taxes on urban development in the second period, we obtain

\[ \frac{dx_1}{d\theta_W} = -D_B \frac{d\theta}{d\theta_W} \]  

(3-3)

where \( D = \frac{(t/Tu+u/\pi)(1+i)Tu}{x_0} \), \( D > 0 \)

\[ \frac{dx_1}{d\theta_W} = -E_B \frac{d\theta}{d\theta_W} \]  

(3-4)

where \( E = \left[ \left( \frac{t/Tu+u/\pi}{\pi} \right)^2 - (1+i)P_1 \right] q \left[ \left( \frac{t/Tu+u/\pi}{\pi} \right) Tu(1+u/\pi) \right. \]

\[ \left. - (1+i)Tu \right] x_1 - (1+i)Tu^2 \frac{1}{\pi} x_0, \quad E > 0 \]

To investigate how net social products are decreased for the two tax regimes we totally differentiate equation (2-10).

\[ d\text{NSP} = F \, dx_0 + G \, dx_1 \]

where,

\[ F = \left( \frac{t/\pi + Tu^2/\pi^2}{\pi} \right) x_0 + Tu/\pi x_1, \quad F > 0 \]

\[ G = \frac{1}{(1+i)} \left[ \left( \frac{t/\pi + Tu^2/\pi^2 + 2Tu/\pi} \right) x_1 + (1+i)Tu/\pi x_0 \right], \quad G > 0. \]

Therefore, the change in net social product resulting from the imposition of a wage tax is

\[ \frac{d\text{NSP}}{d\theta_W} = F \frac{dx_0}{d\theta_W} + G \frac{dx_1}{d\theta_W} \]  

(3-5)
From equations (2-17) and (2-18) we know that both $dx_0/d\theta_w$ and $dx_1/d\theta_w$ are negative. Thus, the introduction of a wage tax decreases the net social product in the first and second periods.

On the other hand, the change of net social product resulting from the imposition of a land value tax is

$$d\text{NSP}/d\theta_{LU} \cdot d\theta_{LU} = F \cdot dx_0/d\theta_{LU} \cdot d\theta_{LU} + G \cdot dx_1/d\theta_{LU} \cdot d\theta_{LU} \quad (3-6)$$

From equations (2-13) and (2-14) we know that $dx_0/d\theta_{LU} > 0$ and $dx_1/d\theta_{LU} < 0$. Thus, the imposition of a land value tax increases the net social product in the first period but decreases it in the second period. However, we see that the total net social product in both periods will decrease, i.e., $d\text{NSP}/d\theta_{LU} < 0$ since we know that $G > F$ and $|dx_1/d\theta_{LU} \cdot d\theta_{LU}| > |dx_0/d\theta_{LU} \cdot d\theta_{LU}|$.

To investigate the relative efficiency of the two taxes we must control for different tax bases. To do this we impose an equal yield condition. Suppose a local government finances expenditure through either a wage tax or a land value tax and the total present value of expenditures is fixed over two periods. The local government can borrow between periods and has to retain budget balance over two periods. The government budget constraint is

$$P_1Q_1g + P_2Q_2/(1+i) = \theta_w[(\bar{U}_1 + \bar{U}_2)/(1+i)]$$
\[ Q_{i}^{g} = \text{the quantity of the industrial good purchased by the local government in each period,} \]

\[ W_{i} = \text{the total gross wage in the city in each period, in the initial equilibrium,} \]

\[ V_{i} = \text{the total land value in the city in each period, in the initial equilibrium.} \]

The total gross wage in the city in each period can be measured by multiplying the gross wage rate by the total number of households in the city in each period as each worker earns the same wage rate. The present value of total before tax wage income in the city in two periods is

\[ W_{1} + W_{2} / (1+i) = u_{s} x_{0} [P_{1} q + T u_{s} x_{0} + T x_{1}] + x_{1} [P_{2} q + T (1+u/s)x_{1} ] / (1+i) \]

where \( u_{s} x_{i} (i=0,1) \) is the number of households in the first and second periods, respectively. We assume that every household consumes one unit of residential land in each period and \( (P_{1} q + T u_{s} x_{0} + T x_{1}) \) and \( [P_{2} q + T (1+u/s)x_{1}] \) are the gross wage rates in each period. The total present value of land in two periods can be measured as follows:
\[ U_1 + U_2/(1+i) = \int_{x_0}^{x_1} [R_1^H(x) + R_2^H(x)/(1+i)] \ dx + \int_{x_0}^{x_1} (1+u/x) x_1 \ R_2^H(x)/(1+i) \ dx \]

\[ + R_2^H(x)/(1+i)] dx + \int_{x_0}^{x_1} R_2^Q(x)/(1+i) \ dx + \int_{x_0}^{x_1} (1+u/x) x_1 \ R_2^H(x)/(1+i) \ dx \]

\[ = 1/2\left(\frac{t}{\pi} + Tu^2/\pi^2\right) x_0^2 + 1/(1+i) \left(\frac{t}{\pi} + Tu^2/\pi^2 + 2Tu/\pi\right) x_1^2 + 2Tu/\pi x_0 x_1 \]

Therefore, the equal yield condition for these two tax regimes in two periods is

\[ \Theta_u/u = \left\{x_0(P_1 T_a + Tu/x_0 + Tx_1) + x_1[P_2 T_b + (1+u/x)x_1]/(1+i)\right\} = \frac{\Theta_{LU}}{1/2 (t/\pi + Tu^2/\pi^2) x_0^2 + 1/(1+i) \left(\frac{t}{\pi} + Tu^2/\pi^2 + 2Tu/\pi\right) x_1^2 + 2Tu/\pi x_0 x_1} \]  

(3-7)

Totally differentiating equations (3-7) and setting \( \Theta_u = \Theta_{LU} = 0 \) as the initial value of the two taxes, and solving for \( d\Theta_{LU} \),

\[ d\Theta_{LU} = \frac{1}{H} \ d\Theta_u \]  

(3-8)

where,

\[ l = u/x \left\{x_0(P_1 T_a + Tu/x_0 + Tx_1) + x_1[P_2 T_b + (1+u/x)x_1]/(1+i)\right\}, \quad l > 0 \]

\[ H = 1/2 \left(\frac{t}{\pi} + Tu^2/\pi^2\right) x_0^2 + 1/(1+i) \left(\frac{t}{\pi} + Tu^2/\pi^2 + 2Tu/\pi\right) x_1^2 + 2Tu/\pi x_0 x_1, \quad H > 0 \]
Substituting equations (3-2) and (3-4) into equation (3-5),

\[ \frac{d\text{NSP}}{d\theta_u} d\theta_u = (FC-GE)/B \ d\theta_u \]  

(3-5)'

Substituting equations (3-1) and (3-3) and (3-8) into equation (3-6),

\[ \frac{d\text{NSP}}{d\theta_LV} d\theta_LV = (1+FA-GD)/BH \ d\theta_u \]  

(3-6)'

Subtracting equation (3-5)' from equation (3-6)',

\[ \frac{d\text{NSP}}{d\theta_LV} d\theta_LV - \frac{d\text{NSP}}{d\theta_u} d\theta_u = \frac{[G(EH-D1)+1FA-FCH]}{BH} \ d\theta_u \]  

(3-9)

If equation (3-9) has a positive value, a land value tax is relatively more efficient than a wage tax. We have to determine the sign of coefficient on the right hand side of equation (3-9). Since BH, IFA and G are positive and FCH is negative (because FH>0 and C<0), a land value tax is more efficient than a wage tax at the level of a single open city if EH ≥ DI. So, we have to prove that (EH-D1) has a positive value. The proof is as follows.

Proof 1)

\[
EH-D1=\{(t/Tu+u/\pi)P_2-(1+i)P_1\}q+(t/Tu+u/\pi)Tu(1+u/\pi)
-\{(1+i)Tu\}x_1-(1+i)Tu^2/\pi x_0\{1/2(t/\pi+Tu^2/\pi^2)x_0^2
+1/(1+i)(t/\pi+Tu^2/\pi^2+2Tu/\pi)x_1^2+2Tu/\pi x_0 x_1\}-(1+i)Tu^2/\pi (t/Tu
\]
\[ +u/\pi x_0 ((P_1 q + Tu/\pi x_0 + Tx_1)x_1 + [P_2 q + T(1+u/\pi)x_1]x_1/(1+i)) \] (3-10)

Rewriting equation (3-10),

\[ \text{EH-DI} = qu(1/2(t/\pi + Tu^2/\pi^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2 + 2Tu/\pi)x_1^2 + 2Tu/\pi x_0 x_1) [P_2 - (1+i)P_1] + (t/Tu + u/\pi - 1) qu P_2 (1/2(t/\pi + Tu^2/\pi^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2 + 2Tu/\pi)x_1^2 + 2Tu/\pi x_0 x_1) - qu P_1 (1+i) Tu/\pi (t/Tu + u/\pi)x_0^2 + \{(t/Tu + u/\pi) Tu(1+u/\pi) - (1+i) Tu\} x_1 \\
- (1+i) Tu^2/\pi x_0^2 (1/2(t/\pi + Tu^2/\pi^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2 + 2Tu/\pi)x_1^2 + 2Tu/\pi x_0 x_1) - (1+i) Tu^2/\pi (t/Tu + u/\pi)x_0 [T Tu + x_0^2 + Tx_0 x_1] + T(1+u/\pi)x_1^2/(1+i)) \] (3-10)'

In the right hand side of equation (3-10)' the first term is positive if we assume

\[ P_2 > (1+i)P_1 \]

From the second term and the third and fourth terms in equation (3-10)', if \( P_2 > (1+i)P_1 \) and if we assume

\[ u/\pi > 1, \text{ and } x_1 > (1+i)x_0, \]

we can determine the following:
\[(t/Tu+u/\pi-1)quP_2\left(1/2(t/\pi+ Tu^2/\pi^2)x_0^2+(t/\pi+ Tu^2/\pi^2+2Tu/\pi)x_1^2/(1+i)\right)\]
\[+2Tu/\pi x_0x_1\} > \{quP_2(t/Tu+u/\pi)Tu/\pi x_0x_1+quP_1(1+i)Tu/\pi(t/Tu+u/\pi)x_0^2\}\]

(3-11)

In equation (3-11), \((t/Tu+u/\pi-1)quP_2\left(1/2(t/\pi+ Tu^2/\pi^2)x_0^2+(1+i)x_0\right)\) is greater than \(quP_2\left(t/\pi+ Tu^2/\pi^2\right)x_0x_1\) if \(u/\pi>1\) and \(x_1>(1+i)x_0\), and we also know that \((t/Tu+u/\pi-1)quP_2\left(1/2(t/\pi+ Tu^2/\pi^2)x_0^2+2Tu/\pi x_1^2/(1+i)\right)\)
\[+2Tu/\pi x_0x_1\} is greater than \(quP_1(1+i)Tu/\pi(t/Tu+u/\pi)x_0^2\) if \(u/\pi>1\) and \(P_2>(1+i)P_1\) and \(x_1>(1+i)x_0\). When we assume that \(x_1>(1+i)x_0\), it follows from the remaining terms in equation (3-10)' that

\[\{(t/Tu+u/\pi)(1+u/\pi)-(1+i)Tu\}x_1-(1+i)Tu^2/\pi x_0^2\]
\[+1/(1+i)(t/\pi+ Tu^2/\pi^2+2Tu/\pi)x_1^2+2Tu/\pi x_0x_1\} > (1+i)Tu^2/\pi(t/Tu\]
\[+u/\pi)x_0[Tu/\pi x_0^2+Tx_0x_1+T(1+u/\pi)x_1^2/(1+i)]\]

(3-12)

Rewriting equation (3-12) to obtain

\[\{(t+Tu^3/\pi^2)+tu/\pi-(1+i)Tu\}x_1+Tu^2/\pi[x_1-(1+i)x_0]\}1/2(t/\pi+ Tu^2/\pi^2)x_0^2\]
\[+1/(1+i)(t/\pi+ Tu^2/\pi^2+2Tu/\pi)x_1^2+2Tu/\pi x_0x_1\} > (1+i)Tu^2/\pi(t/Tu\]
\[+u/\pi)x_0[Tu/\pi x_0^2+Tx_0x_1+T(1+u/\pi)x_1^2/(1+i)]\]

(3-12)'

In the left hand side of equation (3-12)' we know that \([tu/\pi-(1+i)Tu]=u[t/\pi-(1+i)T]\) is positive from the necessary condition, \(t/\pi>(2+i)T\), for land speculation to take place and that
\( x_1 > (1+i)x_0 \). So, if 
\[
(1/2)(t/v+Tu^2/v^2)x_0^2 + 1/(1+i)(t/v+Tu^2/v^2) + 2Tu/v)x_1^2 + 2Tu/v)x_0x_1 \]
is greater than 
\[
(t/v+Tu^2/v^2)x_0^2 + 1/(1+i)(tu/v+Tu^2/v^2)x_0[Tu/v)x_1^2 + T(1+u/v)x_1^2/(1+i)]
\]
equation (3-12) is true.

Proof 11)
\[
(1/2)(t/v+Tu^2/v^2)x_0^2 + 1/(1+i)(t/v+Tu^2/v^2)x_1^2
+ 2Tu/v)x_0x_1 - (1+i)(tu/v+Tu^2/v^2)x_0[Tu/v)x_0^2 + Tx_0x_1
+ T(1+u/v)x_1^2/(1+i)] > 0
\]
(3-12)*

In equation (3-12)* we know that 
\[
(1/2)(t/v+Tu^2/v^2)x_0^2 + 2Tu/v)x_0x_1
\]
is greater than 
\[
[Tu/v)x_0^2 + Tx_0x_1 + T(1+u/v)x_1^2/(1+i)]
\]
and that 
\( x_1 \) is greater than 
\( (1+i)x_0 \). Therefore, we can determine that equation (3-12)* is true since 
\[
tx_1(1/2)(t/v+Tu^2/v^2)x_0^2 + 1/(1+i)(t/v
+ Tu/v)(2+u/v)x_1^2 + 2Tu/v)x_0x_1 + 1/2 Tu^2/v^2 x_0^2 x_1 \]
is greater than 
\[
(1+i)tx_0[Tu/v)x_0^2 + Tu/v)x_0x_1 + Tu/v)(1+u/v)x_1^2/(1+i)].
\]

Therefore, we have three sufficient conditions that ensure that a land value tax is relatively more efficient than a wage tax at the level of an individual city: they are

1. \( x_1 > (1+i)x_0 \),
2. \( P_2 > (1+i)P_1 \),
3. \( u/v > 2 \)

The first condition says that the length of the industrial zone in
the second period is greater than \((1+i)\) times the industrial zone in the first period, where \(i\) is the discount rate. This condition must be satisfied in a growing city. The second condition implies that the price of the industrial good is \((1+i)\) times greater in the second period than in the first period, and this condition is also satisfied. The third sufficient condition means that the size of the residential zone is more than two times greater than the industrial zone in each period. This condition will typically be satisfied; in the city the residential zone is much larger than the industrial zone. Therefore, we conclude that at the level of a single open city a land value tax is relatively more efficient than a wage tax under very weak restrictions.

3.3. For A Simultaneous Tax System - \(\theta_U > 0, \theta_L = 0\)

We now consider the case where the government imposes the wage tax and the land value tax simultaneously. We assume that the total expenditure of the government is fixed over the two periods and the government can borrow between periods. When the local government finances expenditures through the imposition of both a wage tax and a land value tax, the budget constraint of the government is

\[
P_1Q_1 + P_2Q_2/(1+i) = \theta_L[U_1 + U_2/(1+i)] + \theta_L[U_1 + U_2/(1+i)]
\]
From the budget constraint, we know that the increase of one tax leads to a decrease of the other tax. Substituting the present values of total wage and total land value in the city in two periods into this budget constraint, we obtain

\[
P_1v_1 + P_2v_2/(1+i) = \theta_u u + \{(P_1q + T_u/\pi)x_0 + Tx_1\}x_0
- \{[P_2q + T(1+u/\pi)x_1/(1+i)] + \theta_Lu\{1/2(t/T + Tu^2/\pi^2)x_0^2
+ 1/(1+i)(t/T + Tu^2/\pi^2 + 2Tu/\pi)x_1^2 + 2Tu/\pi x_0 x_1}\} (3-13)
\]

From equation (2-10) we know net social product in each period is determined by the amount of the industrial output (the size of the industrial zone, \(x_0, x_1\)). Therefore, if the substitution of a land value tax for a wage tax increases the size of the industrial zone (and the size of the city) in both periods, then the net social product will be increased by the tax substitution. This analysis for a simultaneous tax system will strengthen the results derived for the single tax system.

For a simultaneous tax system in which the local government finances expenditures by means of a land value tax and a wage tax, we have four equations and four unknowns.

\[
\varphi_1(1-\theta_u) = P_1q + T_u/\pi x_0 + Tx_1 \tag{2-15}
\]

\[
\varphi_2(1-\theta_u) = P_2q + T(1+u/\pi)x_1 \tag{2-16}
\]

\[
P_1 - tx_0 - u = 0 \tag{2-3}\
\]
\begin{equation}
(P_2 - tx_1 - u w_2)/(1+i) = T u x_0/(1-\theta_{LV}) + T u x_t/(1+i) \tag{2-12}
\end{equation}

From equations (2-15) and (2-3) eliminating \( w_1 \),

\begin{equation}
[1/u-q/(1-\theta_{u})]P_1 = [T u/x(1-\theta_{u}) + t/u]x_t + T x_t/(1-\theta_{u}) \tag{3-14}
\end{equation}

Equation (3-14) has the same interpretation as equation (2-5). When a wage tax and a land value tax are both introduced, the surplus or remainder that is left over for the payment of transportation costs, the term \([1/u-q/(1-\theta_{u})]P_1 \) shrinks. Note also that when the two taxes are imposed simultaneously the cost of transporting workers increases, i.e., both the terms, \( T u/x(1-\theta_{u}) \) and \( T/(1-\theta_{u}) \) increase.

This suggests that both \( x_0 \) and \( x_t \) (the industrial zone in each period) will decrease to satisfy equation (3-14). From equations (2-16) and (2-12) after eliminating \( w_2 \), we obtain

\begin{equation}
[1/u-q/(1-\theta_{u})]P_2/(1+i) = T[x_0/(1-\theta_{LV}) + x_t/(1+i)]
+ [t/u + T(1+u/x)/(1-\theta_{u})]x_t/(1+i) \tag{3-15}
\end{equation}

Equation (3-15) has essentially the same structure and interpretation as equation (2-6). When a wage tax and a land value tax are both introduced, the remainder that is left over for the payment of transportation costs and industrial rents on the outer
edge of the industrial zone in the second period, \( [1/u-q/(1-\theta_u)]P_2 \) decreases. However, the industrial rents in the second period, \( T[x_0/(1-\theta_{LU})+x_1/(1+i)] \), and the cost of transporting the worker, \( T(1+u/\pi)/(1-\theta_u) \) increase. This implies that both \( x_0 \) and \( x_1 \) will decrease to satisfy equation (3-15).

Solving equations (3-14) and (3-15) for \( x_0 \) and \( x_1 \), we have

\[
x_0 = \frac{[1/u-q/(1-\theta_u)]((1-\theta_u)[t/Tu+(1+u/\pi)/(1-\theta_u)+1]P_1-P_2)}{(1-\theta_u)[t/Tu+(1+u/\pi)/(1-\theta_u)+1][Tu/\pi(1-\theta_u)+t/u]-(1+i)T/(1-\theta_{LU})}
\]  

(3-16)

\[
x_1 = \frac{(1-\theta_u)[1-u-q/(1-\theta_u)]([u/\pi(1-\theta_u)+t/Tu])P_2-(1+i)P_1/(1-\theta_{LU})}{(1-\theta_u)[t/Tu+(1+u/\pi)/(1-\theta_u)+1][Tu/\pi(1-\theta_u)+t/u]-(1+i)T/(1-\theta_{LU})}
\]  

(3-17)

where \( 1/u-q/(1-\theta_u) \) is positive as a measure of resources available for transportation costs. In equations (3-16) and (3-17) the denominator is positive since \( i<1 \). Thus, we can determine the range of \( P_2 \) in the same way as equation (2-9).  

\[
\frac{(1-\theta_u)[t/Tu+(1+u/\pi)/(1-\theta_u)+1+(1+i)/(1-\theta_{LU})]P_1}{(1-\theta_u)[u/\pi(1-\theta_u)+t/Tu]+1} < P_2
\]

\[
P_2 < (1-\theta_u)[1-u-q/(1-\theta_u)][t/Tu+(1+u/\pi)/(1-\theta_u)+1]P_1 \quad (3-9)'
\]
To investigate how the substitution of one tax for the other tax affects output and city growth in the first period we totally differentiate equation (3-16), set $\theta_{LV}=0$ as the initial value of the land value tax, and solve for $dx_0$,

$$dx_0 = (a \, d\theta_u + b \, d\theta_{LV})/e \quad (3-18)$$

where,

$$a = 1/(1-\theta_u)^2 \left\{ qP_2 + T(1+u/\tau)x_1 - (1-\theta_u)[t/Tu+(1+u/\tau)/(1-\theta_u)] \\
+ 1\right\} (qP_1 + Tu/\tau x_0 + Tx_1)$$

$$b = (1+i)Tx_0, \quad b > 0$$

$$e = (1-\theta_u)[t/u+T(1+u/\tau)/(1-\theta_u)+T][(1+u/\tau)/(1-\theta_u)+t/Tu]-(1+i)T,$$

$$e > 0.$$

In equation (3-18) $a$ is negative. In equation (3-16) $x_0$, the industrial zone in the first period is positive and as the denominator is positive, it follows that the numerator is also positive. Therefore, from the numerator of equation (3-16) we can determine that $qP_2 < (1-\theta_u)[t/Tu+(1+u/\tau)/(1-\theta_u)+1]qP_1$. So, the term $a$ is negative.

From equation (3-18) we can analyze the effects of the tax substitution on the development of urban land in the first period.

$$dx_0/d\theta_{LV}d\theta_{LU} = [(a \, d\theta_u/d\theta_{LV}d\theta_{LU} + b \, d\theta_{LU})/e \quad (3-19)$$
Since \( a<0, \frac{d\theta_u}{d\theta}\cdot 0, b>0, \) and \( e>0, \) it follows that \( dx_0/d\theta_{LV}>0. \) So, the substitution of a land value tax for a wage tax increases industrial output and net social product in the first period.

In the second period the effects of the tax substitution can be analyzed by making use of the budget constraint. Totally differentiating equation (3-17), setting \( \theta_{LV}=0 \) as the initial value and solving for \( dx_1, \) we obtain

\[
dx_1 = -f/e \cdot d\theta_u - g/e \cdot d\theta_{LV} \tag{3-20}
\]

where,

\[
f = 1/(1-\theta_u)2\{[(1+u/w)+t(1-\theta_u)/Tu][qP_2+T(1+u/w)x_1]
-(1+i)(1-\theta_u)(qP_1+Tu/x_0+Tx_1)}, \quad f > 0
\]

\[
g = (1+i)[Tu/x+t(1-\theta_u)/x_0], \quad g > 0
\]

In equation (3-20) \( f \) is positive since the range of \( P_2 \) is restricted as in equation (3-9). In equation (3-20) we have two different tax bases and so, we measure the value of \( d\theta_u \) in terms of \( d\theta_{LV} \) by making use of the budget constraint of the local government. Totally differentiating equation (3-13) and setting \( \theta_{LV}=0, \)

\[
0 = u/s\{(P_1q+Tu/x_0+Tx_1)x_0+[P_2q+T(1+u/w)x_1]x_1\}d\theta_u
+\theta_u/s\{(2Tu/x_0+Tx_1)dx_0+[\theta_u/u/Tx_0+2T(1+u/w)x_1/(1+i)]dx_1
\]
\[
+\left(1/2\left(t/\pi+Tu^2/\pi^2\right)x_0^2+1/(1+i)\left(t/\pi+Tu^2/\pi^2+2Tu/\pi\right)x_1^2\right)
+2Tu/\pi x_0x_1\right)\,d\theta_\nu
\]  
(3-21)

Substituting equation (3-19) into equation (3-21) and solving for \(d\theta_\nu\),

\[
d\theta_\nu = -k/j \, d\theta_{LV} - h/j \, dx_1
\]  
(3-22)

where,

\[
h = \theta_\nu u/\pi Tx_0 + 2T(1+u/\pi)x_1/(1+i), \quad h > 0
\]
\[
j = u/\pi \left\{(P_1 q + Tu/\pi x_0 + Tx_1)x_0 + [P_2 q + T(1+u/\pi)x_1]x_1/(1+i)\right\}
\]
\[
+ \theta_\nu u/\pi (2Tu/\pi x_0 + Tx_1)a/e,
\]
\[
k = \theta_\nu u/\pi (2Tu/\pi x_0 + Tx_1)b/e + (1/2\left(t/\pi+Tu^2/\pi^2\right)x_0^2
\]
\[
+1/(1+i)\left(t/\pi+Tu^2/\pi^2+2Tu/\pi\right)x_1^2+2Tu/\pi x_0x_1), \quad k > 0
\]

Substituting equation (3-22) into equation (3-20) and solving for \(dx_1\),

\[
dx_1 = (fk-gj)/(ej-fh) \, d\theta_{LV}
\]  
(3-23)

In equation (3-23) if the coefficient is positive, the substitution of a land value tax for a wage tax increases industrial output and net social product. In equation (3-23) if both the numerator and the denominator are positive, then the coefficient is positive. First, we prove that the numerator is positive.
Proof 111)

\[\begin{align*}
f(k-g) & = 1/(1-\theta_1)^2 \left\{ \left[ 1+u/\pi+t(1-\theta_1)/Tu \right] \{ qP_2 + T(1+u/\pi)x_1 \right. \\
& \left. - (1+i)(1-\theta_1) \{ qP_1 + Tu/x_0 + Tx_1 \} \} \theta_0 u/\pi \{ (2Tu/x_0 + Tx_1) b/e \right. \\
& \left. + [1/2(t/\pi^2 + u^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2)x_1^2 + 2Tu/\pi x_1 x_1 \} \right. \\
& \left. - (1+i)u/(\pi+t(1-\theta_1)/Tu) x_0 \{ (2Tu/\pi x_0 + Tx_1) x_0 \right. \\
& \left. + (P_2 + T(1+u/\pi)x_1 x_1/(1+i)) \theta_0 u/\pi \{ (2Tu/\pi x_0 + Tx_1) a/e \} \right. \\
& \left. (3-24) \right. 
\end{align*}\]

Rewriting equation (3-24), we obtain

\[\begin{align*}
f(k-g) & = 1/(1-\theta_1)^2 \left\{ 1/2(t/\pi + Tu^2/\pi^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2)x_1^2 \\
& + 2Tu/\pi x_1 x_1 \right. \\
& \left. \{ qP_2 - (1+i)(1-\theta_1) P_1 \right. \\
& \left. + [1/(1-\theta_1)^2 \left\{ u/(\pi+t(1-\theta_1)/Tu) \left\{ 1/2(t/\pi^2 + u^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2) \\
& + 2Tu/\pi x_1^2 + 2Tu/\pi x_0 x_1 \right. \\
& \left. - (1+i)(u/(\pi+t(1-\theta_1)/Tu) x_0 \{ (2Tu/\pi x_0 + Tx_1) x_0 + T(1+u/\pi)x_1^2/(1+i) \right. \\
& \left. + 1/(1-\theta_1)^2 \left\{ ((1+u/\pi) + t(1-\theta_1)/Tu) \right. \\
& \left. \left\{ qP_2 + T(1+u/\pi)x_1 \right. \\
& \left. - (1+i)(1-\theta_1) \{ qP_1 \\
& + Tu/\pi x_0 + Tx_1 \} \theta_0 u/\pi \{ (2Tu/\pi x_0 + Tx_1) b/e \\
& \left. - (1+i)(u/(\pi+t(1-\theta_1)/Tu) x_0 \{ (2Tu/\pi x_0 + Tx_1) a/e \} \right. \\
& \left. (3-24)' \right. 
\end{align*}\]

In the right hand side of equation (3-24)', the first term is positive if we assume
\[ P_2 > (1+i)(1-\theta_\omega)P_1 \]

We know that the second term is positive if we assume

\[ u/\pi > 1 \]

The third term minus the fourth term is positive if we assume

\[ x_1 > (1+i)x_0 \]

Rewriting the third term minus the fourth term in the right hand side of equation (3-24)',

\[
\frac{1}{(1-\theta_\omega)^2} \left\{ T(1+u/\pi)x_1-(1+i)(1-\theta_\omega)(Tu/\pi x_0 + Tx_1) + T(1+u/\pi)x_1[u/\pi + t(1-\theta_\omega)/Tu] \right\}
\]
\[
\frac{1}{2}(t/\pi + Tu^2/\pi^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2 + 2Tu/\pi)x_1^2
\]
\[
+ 2Tu/\pi x_0 x_1 - (1+i)Tx_0[u/\pi + t(1-\theta_\omega)/Tu][u/\pi (Tu/\pi x_0 + Tx_1)x_0
\]
\[ + T(1+u/\pi)x_1^2/(1+i)] \]

and we know \( T(1+u/\pi)x_1, (1+i)(1-\theta_\omega)(Tu/\pi x_0 + Tx_1) \) and \( T(1+u/\pi)x_1[1/2(t/\pi + Tu^2/\pi^2)x_0^2 + 1/(1+i)(t/\pi + Tu^2/\pi^2 + 2Tu/\pi)x_1^2 + 2Tu/\pi x_0 x_1] \) is greater than \( (1+i)Tx_0[u/\pi (Tu/\pi x_0 + Tx_1)x_0 + T(1+u/\pi)x_1^2/(1+i)] \) if we assume that \( u/\pi \) is greater than 1 and \( x_1 \) is greater than \( (1+i)x_0 \). Therefore, \( f_k-g_j \) is positive if we assume
\[ u/\pi > 1, \quad P_2 > (1+i)(1-\theta_e)P_1, \quad x_i > (1+i)x_0 \]

Next, we prove that \( ej - fh \) is positive.

**Proof (II)**

\[
ej - fh = u/\pi \{(qP_1 + Tu/\pi x_0 + Tx_1)x_1 + [qP_2 + T(1+u/\pi)x_1]/(1+i)\}[[t/u
+ T(1+u/\pi)/(1-\theta_e) + T][/(1+u/\pi) + t(1-\theta_e)/Tu] - (1+i)T]

+ \theta_e u/(1-\theta_e)2\pi(2Tu/\pi x_0 + Tx_1)}[qP_2 + T(1+u/\pi)x_1 - (1-\theta_e)[t/Tu
+(1+u/\pi)/(1-\theta_e) + 1]}[qP_1 + Tu/\pi x_0 + Tx_1)] - 1/\{(1+u/\pi)
+t(i-\theta_e)/Tu][qP_2 + T(1+u/\pi)x_1] - (1+i)(1-\theta_e)(qP_1 + Tu/\pi x_0
+ Tx_1)]\theta_e u/\pi Tx_0 + 2T(1+u/\pi)x_1/(1+i)\]

(3-25)

Rewriting equation (3-25), we obtain

\[
ej - fh = [qP_2 + T(1+u/\pi)x_1]([[1+u/\pi] + t(1-\theta_e)/u][T(1+u/\pi)u/\pi x_1/(1+i)(1-\theta_e)]
- 2T(1+u/\pi)x_1/(1+i)(1-\theta_e)^2} + [qP_2 + T(1+u/\pi)x_1]([[(1+u/\pi)
+ t(1-\theta_e)/u][Tu/\pi x_1/(1+i) - \theta_e u/\pi x_0/(1-\theta_e)^2] - Tu/\pi x_1} + [qP_2 + T(1
+ u/\pi)x_1][[(1+u/\pi) + t(1-\theta_e)/Tu]t/u/\pi x_1/(1+i)
+(qP_2 + Tu/\pi x_0 + Tx_1}/Tu/\pi x_0] [t/Tu + (1+u/\pi)/(1-\theta_e) + 1]([1+u/\pi)
+ t(1-\theta_e)/Tu - 2\theta_e u/\pi (1-\theta_e)] - (1+i)Tu/\pi x_0 + (1+i)\theta_e/(1-\theta_e)\}

+(qP_1 + Tu/\pi x_0 + Tx_1)Tx_1[2(1+u/\pi)/(1-\theta_e) - \theta_e u/(1-\theta_e)]t/Tu}
+(1+u/\pi)/(1-\theta_u)+1))} \quad (3-25)'

In equation (3-25)' the first term is positive if we assume

u/\pi > 2/(1-\theta_u)

Rewriting the second term in equation (3-25)',

[qP_2+T(1+u/\pi)x_1]/[(1+u/\pi)+t(1-\theta_u)/u][T(u/\pi-1)x_1/(1+i)
\theta_u Tu/\pi x_0/(1-\theta_u)^2]+Tx_1/(1+i)[(1+u/\pi)+t(1-\theta_u)/u]-Tu/\pi x_1]

Therefore, the second term is also positive if we assume that
T(u/\pi-1)x_1/(1+i) is greater than \theta_u Tu/\pi x_0/(1-\theta_u). This sufficient condition is satisfied if we assume

x_1 > (1+i)x_0, \quad u/\pi > 1/[1-\theta_u/(1-\theta_u)^2]

We know that the fourth term in equation (3-25)' is positive. Finally, from the third and fourth terms we can derive the following.

[[(1+u/\pi)/(1-\theta_u)+t/Tu]((1-\theta_u)[qP_2+T(1+u/\pi)x_1]/\pi x_1/(1+i)
-(qP_1+Tu/\pi x_0+Tx_1)\theta_u Tu/\pi x_1/(1-\theta_u))+(qP_1+Tu/\pi x_0+Tx_1)Tx_1[2(1
+u/\pi)/(1-\theta_u)-\theta_u u/\pi(1-\theta_u)]]

(3-26)
Since we know that \(2(1+u/\pi)\theta_0 u/\pi\), so if \((1-\theta_0)u_2 t/\pi(1+i)\) is greater than \(\theta_0 u_1 Tu/\pi(1-\theta_0)\) where \(u_1 = qP_1 + Tu/\pi x_0 + Tx_1\) and \(u_2 = qP_2 + (1+u/\pi)x_1\), equation (3-26) is positive. Rewriting this sufficient condition, we obtain

\[u_2 > [\theta_0 u/(1+i)Tu_1]/[(1-\theta_0)^2 t/\pi]\]

Thus, we have three sufficient conditions so that \(e_j-f_h\) is positive:

\[u/\pi > 2/(1-\theta_0), \ x_1 > (1+i)x_0, \ u_2 > [\theta_0 u/(1+i)Tu_1]/[(1-\theta_0)^2 t/\pi]\]

Therefore, four sufficient conditions establish that the coefficient of equation (3-23) is positive:

(1) \(u/\pi > 2/(1-\theta_0)\), (2) \(P_2 > (1+i)(1-\theta_0)P_1\), (3) \(x_1 > (1+i)x_0\),

(4) \(u_2 > [\theta_0 u/(1+i)Tu_1]/[(1-\theta_0)^2 t/\pi]\)

Consequently, if these four sufficient conditions are satisfied then the substitution of a land value tax for a wage tax increases the industrial output and city size in the second period. The first condition states that the size of the residential zone is greater than \(2/(1-\theta_0)\) times the industrial zone in each period. This condition will typically be satisfied as the industrial zone is
expected to be small relative to the residential zone. The second and third sufficient conditions must be satisfied in a growing city. The fourth sufficient condition will also be satisfied as the nominal wage rate is greater in the second period, \( \omega_2 > \omega_1 \); \( \pi/u \) is less than 1 as the industrial zone is smaller than the residential zone, and \( t/\pi \) must be larger than \((2+i)\)T in order for land speculation to take place. A land value tax is relatively more efficient than a wage tax at the level of an individual city under very weak restrictions, where the wage tax is initially positive and the land value tax is initially zero.

In the case of a single open city we arrive at three conclusions. A land value tax increases industrial output and accelerates city growth in the first period by encouraging the development of vacant land. But in the second period output and the size of the city are decreased by the premature residential development in the first period. Second, a wage tax decreases output and the size of the city in both periods by increasing the cost of labor. Finally, in a two-period model of urban development a land value tax affects the optimal timing of urban development but is more efficient than a wage tax under very weak restrictions.
Chapter IV

Land Speculation and Taxation in a System of Cities

4.1. Introduction

In the previous chapters we analyzed the effects of taxes on urban development from the perspective of a single open city located in a large national economy. In this chapter we study the effects and the relative efficiency of a land value tax and a wage tax from the standpoint of a system of cities, or the nation as a whole. The key difference between the single city analysis and the national perspective is that supplies of labor and capital are more elastic to a single open city, relative to the national economy.¹

In a system of cities a wage tax will have no impact on development except to extent that a particular city imposes a relatively high or low tax, or that the overall supply of labor in the nation is changed by the imposition of a system of wage taxes. So, the distortionary effects of a wage tax system appear much smaller than those of a wage tax imposed by one city. The imposition of a land value tax by all cities will encourage the development of land in the first period, and while the nation's overall employment will be little affected by this tax, the decrease of the amount of land held vacant in the first period will result in less efficient

¹ For the analytical simplicity there is no capital in this model.
leapfrog development in the second period.\footnote{If we assume that in each period population is exogenous and every household consumes a fixed amount of residential land and there exists fixed coefficient of production between labor and land, the non-neutral land value tax decreases vacant land in the first period and this results in less efficient leapfrog development beyond the residential zone in the first period.}

At the level of a single open city we showed that a land value tax is relatively more efficient than a wage tax under very weak restrictions. However, for a system of cities this result is not necessarily ensured because the supply of labor is relatively inelastic to a system of cities. In analyzing the relative efficiency of the two taxes we compare the change of social welfare\footnote{In this model social welfare function is measured by the sum of individual utility in the city, and we use a Cobb-Douglas utility function.} for a wage tax and for a land value tax when a single tax is imposed; and when both taxes are imposed simultaneously, and a tax substitution is made.

4.2. The Model

Consider a number of identical cities in the nation. In each city population is fixed exogenously in each period. The residential zone is also fixed exogenously, as we assume that every household in the city consumes fixed amount of residential land. The price of
manufactured commodity is also fixed exogenously in both periods; we assume $P_1 = P_2 = 1$ as a numéraire. In a system of cities a city's growth depends on exogenous population growth rather than price change. As we assume fixed factor-proportions in production, the sizes of the industrial zones are proportional to the sizes of the residential zones. In Figure 4-1 $x_0 = y_0 / N$ and $x_1 = y_1 / N$ where $N$ is workers per unit of industrial land and $x_0$ and $x_1$ are the sizes of the industrial zones and $y_0$ and $y_1$ are the sizes of the residential zones in the first and second periods, respectively.

*Figure 4-1 The Pattern of Urban Development in the Case of an Exogenous Population*

Every household supplies the same work hours at all employment sites in the city. Workers who reside close to the CBD enjoy more leisure as they save on travel time to the CBD, relative to workers
who live further from the CBD. These workers pay more in residential rents and give up some amount of the industrial good in return for additional leisure.

Suppose a worker who resides on the boundary $B_1$ in Figure 4-1 consumes $L_1(B_1)$ hours of leisure and $C_1(B_1)$ units of the industrial good in period one and two respectively. Then, work hours are

$$H_1 = \left[ T_{01} - L_1(B_1) - T_t y_1/N \right] \text{ in the first period,}$$
$$H_2 = \left[ T_{02} - L_2(B_1) - T_t y_1/N \right] \text{ in the second period.}$$

where,

$H_i$ ($i=1,2$) = work hours in each period,

$T_{0i}$ = total time allocated between leisure, work hours, and travel time in each period,

$T_t$ = travel time taken by workers per unit distance and fixed exogenously in both of two periods.

Since every worker in the city works the same hours regardless of residential location, workers who live on the outer edge of the residential zones, $B_2$ in the first period and $B_3$ in the second period consumes less leisure because of longer travel time, relative to workers who reside on the boundary $B_1$ in Figure 4-1. The consumption of leisure for workers who live on the outer edge of the residential zone in the first period is shown by
\[ L_1(B_2) = L_1(B_1) - T_y y_0 \]

The leisure for workers who live on the outer edge of the residential zone in the second period is shown by

\[ L_2(B_3) = L_2(B_1) - T_y y_1 \]

Workers who live on the outer edge of the residential zone in each of the two periods pay zero residential land rents, and they consume more of the industrial good at the cost of leisure, relative to workers who reside close to the CBD.

All workers in the city have the same income structure as they supply the same work hours, and earn the same hourly wage rate at all employment sites and receive the same lump-sum rental income. The lump-sum rental income is calculated by dividing total land rents by the number of households in the city, and is distributed equally to workers. Workers at every residential location maximize utility subject to their budget constraint in each period, and have the same level of utility at each location. So, we solve the utility maximization problem for a worker who lives on the boundary between the industrial zone and the residential zone. We assume a Cobb-Douglas utility function.

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1 In the case of a system of cities land owners are profit-maximizing residents and lump-sum rental income appears in household's income equations as the city is characterized as the closed city.
2 This is not an overlapping-generation model. Workers maximize their utility in each period
\[ \max \quad U_i(B_i) = \left[ C_i(B_i) \right]^\beta \left[ L_i(B_i) \right]^{1-\beta}, \quad 0 < \beta < 1 \]
\[ s/t. \quad w_i H_i + TR_i / y_j = C_i(B_i) + R_i H(B_i) + T a y_i / H \quad (4-1) \]

where,
\[ i=1,2, \quad j=0,1. \]

\( C_i(B_i) \) = the amount of the industrial good purchased by workers who live on the boundary between the industrial zone and the residential zone,

\( w_i \) = hourly wage rate,

\( TR_i \) = total rental income,

\( y_j \) = the number of households in the city in the first and second periods, respectively,

\( T \) = money costs per unit distance in household's transportation costs,

\( R_i H(B_i) \) = residential land rents at distance \( y_i / H \) from the CBD in each period.

\( H \) = workers per unit of industrial land.

From the first order conditions we obtain

\[ C_i(B_i) = \beta / (1-\beta) \quad w_i L_i(B_i) \quad (4-2) \]

Equation (4-2) shows that if leisure increases then at the given hourly wage rate consumption also increases. This condition implies
that the return from lump-sum rental income increases both leisure and consumption.

While we use Cobb-Douglas utility function, the social welfare function \( S.W \) is shown by multiplying total number of households in each period by the level of an individual utility as all workers in the city have the same level of utility at every residential location. Substituting equation (4-2) into household's utility function we obtain

\[
S.W = \frac{\beta}{(1-\beta)} [y_0 \omega_1 \beta L_1(B_1) + y_1 \omega_2 \beta L_2(B_1)]
\]  

(4-3)

Equation (4-3) illustrates that social welfare depends on the real wage rate and leisure in each period.

To calculate lump-sum rental income in equation (4-1) we derive land-rent functions. Industrial land-rent functions can be derived from firm's zero profit condition. In a competitive market all firms in the industry will have zero profits. So, the product exhaustion condition is

\[
P_i H_i Q^H_i - t H_i Q^H_i x - H_i R_i Q^O_i - R_i Q^O(x) = 0
\]

where,

- \( Q^H_i \): the amount of the industrial good produced by a worker per unit hour and is assumed to be equal to 1 for the simplicity,
- \( t \): firm's transportation costs per unit distance, per unit of output,
\[ R_i^Q(x) = \text{industrial land rents at distance } x \text{ from the CBD.} \]

Solving this equation for \( R_i^Q(x) \), we obtain the industrial land-rent functions in each period.

\[ R_i^Q(x) = N_h (1 - tx - w_i) \]

Wage income is calculated by multiplying total work hours by the hourly wage rate. The price of leisure can be defined to be the opportunity cost of wage income. So, we assume that in each period workers who reside on the outer edge of the residential zone can consume more units of the industrial commodity at the cost of leisure. The income equation for a worker who lives on the outer edge of the residential zone in each period is

\[ w_i H_i + TR_i/y_j = C_i (B_1) + T_l y_j w_i + T_m (y_i/N + y_j) \quad (4-4) \]

Since every worker supplies the same work hours and earns the same wage rate and receives the same lump-sum rental income, the right hand sides of equations (4-1) and (4-4) must be equal in each period and residential land rents on the boundary \( B_1 \) is \( R_i^H(B_1) = (T_l u_i + T_m) y_j \). Since we assume uniform density and zero land rents on the outer edge of the residential zone, the residential land-rent functions can be approximated by
\[ R^H_i(x) = (T_u + T_m)(y_1 + y_j) - (T_u + T_m)x \] (4-5)

For analytical simplicity, we assume linear land-rent functions as the approximations for the rent gradient. The actual land-rent function will be non-linear as indifference curves are convex.\(^1\)

From the locational equilibrium land-rent functions we can measure lump-sum rental income in each period. In the first period lump-sum rental income is

\[
\begin{align*}
TR_1/y_0 &= 1/y_0 \left( \int_{y_0/2}^{y_0} R^Q_i(x) \, dx + \int_{y_0/2}^{y_0} (y_0 + y_0) R^H_i(x) \, dx \right) \frac{y_1}{y_0} \\
&= 1/2 [H_1(y_1 - y_1) + (T_u + T_m)y_0]
\end{align*}
\] (4-6)

In the second period lump-sum rental income is

\[
\begin{align*}
TR_2/y_1 &= 1/y_1 \left( \int_{y_1/2}^{y_1} R^Q_i(x) \, dx + \int_{y_1/2}^{y_1} (1+y_1) y_1 R^H_i(x) \, dx \right) \frac{y_1}{y_1} \\
&= 1/2 [H_2(2y_2 - y_1) + (T_u + T_m)y_1]
\end{align*}
\] (4-7)

For a system of cities we have four unknowns, \(L_1(B_1), L_2(B_1), w_1,\) and \(w_2\) and four equations. Substituting equations (4-2), (4-5), (4-6)

\(^1\) The value of the approximated linear land-rent is very close to the true value of the non-linear land-rent. See Appendix.
and (4-7) into equations (4-1), we have two income equations in the first and second periods.

\[ \frac{1}{2}(1 + \frac{\alpha_1}{\beta_1})H_1 = \beta/(1 - \beta) \frac{\alpha_1}{\beta_1}L_1(B_1) + 1/2(T_1 + T_a)Y_0 + T_aY_1/N \]  
\[ (1 - 1/2ty_1/N)H_2 = \beta/(1 - \beta) \frac{\alpha_2}{\beta_2}L_2(B_1) + 1/2(T_1 + T_a)Y + T_aY_1/N \]  
(4-8)  
(4-9)

The product exhaustion condition states that land rents on the outer edge of the industrial zone in the first period is zero, and is equal to

\[ NH_1[1 - ty_0/N - w_1] = 0 \]  
(4-10)

The boundary condition between two land zones, a location that is equally profitable as industrial land in the second period and as residential land in the first and second periods is

\[ NH_2[1 - ty_1/N - w_2]/(1 + i) \]

\[ = (T_a + T_1)Y_0 + (T_a + T_1)Y_1/(1 + i) \]  
(4-11)

In equation (4-11) the left hand side represents the present value of industrial rent and the right hand side shows the sum of the present value of residential rents on the boundary \( B_1 \) in Figure 4-1.

Since we have four unknowns and four equations, we can solve for the endogenous variables in terms of exogenous variables.
4.3. The Effects of A Wage Tax

For a system of cities the introduction of a wage tax decreases work hours (or increase leisure) and leads to a decrease of output. This tax decreases residential rents\(^1\) by decreasing the real wage rate and also decreases industrial rents.\(^2\) Also, the wage tax will decrease the amount of the industrial good purchased by household due to the decrease of the real wage rate. We assume that the local government returns the tax revenue as a lump-sum payment. So, when a wage tax is imposed, household budget constraint becomes

\[
\pi_i(1-\omega_i)H_i + TR_i(\omega_i)/y_j + TR_i(\omega_i)/y_j = C_i(B_1, \omega_i) + R_i^{H}(B_1, \omega_i) + T_m y_1/N \tag{4-1}'
\]

where \(TR_i(\omega_i)/y_j\) is lump-sum tax return. When a wage tax is introduced, lump-sum rental income is

\[
TR_i(\omega_i)/y_0 = 1/2[H_i(1-w_i) + [T_t w_i(1-\omega_i) + T_m]y_0] \tag{4-6}'
\]

In the second period lump-sum rental income is

\(^1\) The residential land-rent function becomes \([T_t w_i(1-\omega_i) + T_m][y_1/N + y_j - x]\).
\(^2\) Initially, firms pay the same wage rate when a wage tax is imposed and the gross wage rate is determined from the model.
\[ TR_2(θ_0)/y_1 = \frac{1}{2}(H_2(2-2m_2-ty_1/H)+[T_m+T_t(1-θ_0)m_2]y_1) \]  

where \( TR_i(θ_0) \) is total land rents when a wage tax is imposed. The lump-sum tax payment is calculated by dividing the total tax revenue by the number of workers in each period.

\[ TTR_i(θ_0)/y_j = θ_0w_iH_i \]

where \( TTR_i(θ_0) \) is the total tax revenue for a wage tax and \( w_i \) is the before tax wage rate. The wage tax decreases consumption of the industrial good. The new equilibrium condition is

\[ C_i(B_1,θ_0) = β/(1-β)u_i(1-θ_0)L_i(B_1) \]  

Substituting \((4-2)\) into the social welfare function \((4-3)\) we obtain

\[ SW(θ_0) = \left[ β/(1-β) \right] B y_0 [u_i(1-θ_0)] B L_i(B_1) + y_1 [w_2(1-θ_0)] B L_2(B_1) \]  

Substituting equations \((4-2)'\), \((4-6)'\) and \((4-7)'\) into \((4-1)'\), we obtain two income equations.

\[ \frac{1}{2}(1+w_1)H_i = \frac{β}{(1-β)}(1-θ_0)w_iL_i(B_1) \]
\[ +\frac{1}{2}[T_t(1-\theta_\omega)\omega_1 + T_\omega]y_0 + T_\omega y_1 / N \]  

(4-8)',

\[ (1-\frac{1}{2}ty_1 / H)H_2 = \beta/(1-\beta)(1-\theta_\omega)\omega_2 L_2(B_1) \]

\[ +\frac{1}{2}[T_t(1-\theta_\omega)\omega_2 + T_\omega]y_1 + T_\omega y_1 / N \]  

(4-9)',

The product exhaustion condition doesn't change since firms pay the before tax wage rate. The boundary condition between two land zones is changed by the wage tax as land rents are decreased.

\[ H(1-ty_1 / N-\omega_2)H_2 / (1+i) \]

\[ = [T_t(1-\theta_\omega)\omega_1 + T_\omega]y_0 + [T_t(1-\theta_\omega)\omega_2 + T_\omega]y_1 / (1+i) \]  

(4-11)',

To investigate how a wage tax affects leisure and wage rate in each period we totally differentiate equations (4-8)', (4-9)', (4-10) and (4-11)' and set \( \theta_\omega = 0 \) as the initial value of the wage tax.

\[ [\beta/(1-\beta) \omega_1 + 1/2(1+u_1)]dL_1(B_1) = [1/2H_1-\beta/(1-\beta)L_1(B_1)-1/2T_t y_0]d\omega_1 \]

\[ + [1/2T_t y_0 \omega_1 + \beta/(1-\beta) \omega_1 L_1(B_1)]d\theta_\omega \]  

(4-12)

\[ [\beta/(1-\beta) \omega_2 + (1-1/2ty_1 / H)]dL_2(B_1) = -[\beta/(1-\beta)L_2(B_1)+1/2T_t y_1]d\omega_2 \]

\[ + \omega_2[\beta/(1-\beta) L_2(B_1)+1/2T_t y_1]d\theta_\omega \]  

(4-13)
\[-NH_1dw_1-N(1-t_y_0/N-w_1)dl_1(B_1)=0 \tag{4-14}\]

\[N(1-t_y_1/N-w_2)dl_2(B_1)=-(NH_2+T_t y_1)dw_2-(1+i)T_t y_0dw_1 + [(1+i)T_t y_0w_1 + T_t y_1w_2]d\theta_\theta \tag{4-15}\]

From equation (4-10) we know that \(H_1=[T_0-L_1(B_1)-T_t y_t/N]\) is positive as work hours and \((1-t_y_0/N-w_1)=0\). So, from equation (4-14) it follows that \(dw_1=0\), which means that the imposition of a wage tax does not affect the wage rate in the first period. This result follows from the fact that while industrial land rents on the outer edge of the industrial zone are zero in the first period, if the wage rate increases (decreases) then industrial land rents would be negative (positive) on the outer edge of the industrial zone in the first period. However, this tax decreases work efforts. Solving equation (4-12) for \(dl_1(B_1)\), we obtain

\[dl_1(B_1) = a d\theta_\theta \tag{4-16}\]

where \(a=w_1[B/(1-\beta)L_1(B_1)+1/2T_t y_0]/[w_1\beta/(1-\beta)+1/2(1+w_1)]\) and \(a\) is positive. From equation (4-16) we determine \(dl_1(B_1)/d\theta_\theta>0\), which means that the introduction of a wage tax increases leisure in the first period. Consequently, when a wage tax is imposed, a firm's output, \(NH_1\) must be decreased in the first period by the decrease in work effort.
Solving equations (4-13) and (4-15) for $d\omega_2$, we have

$$d\omega_2 = \frac{(b\omega_2 - e)}{(b-c)} \, d\theta$$

(4-17)

where,

$$b = \frac{\beta}{(1-\beta) L_2(B_1) + 1/2 T_1 Y_1} \left( \frac{\beta}{(1-\beta) \omega_2 + 1/2 ty_1/H} \right), \quad b > 0$$

$$c = \frac{NY_2 + T_1 y_1}{(1-t)y_1/H - \omega_2}, \quad c > 0$$

$$e = \frac{(1+i) T_1 y_0 \omega_1 + T_1 y_1 \omega_2}{(1-t)y_1/H - \omega_2}, \quad e > 0$$

We know that $c > e$ since $H_2$ is work hours and $T_1 y_j$ is travel time through the residential zone in each period. To investigate how a wage tax affects the wage rate in the second period we must determine the sign of the coefficient of equation (4-17). If the coefficient has a negative value, we conclude that the imposition of a wage tax decreases the wage rate in the second period. In equation (4-17) the denominator is negative if we assume

$$\beta < H_2/[H_2 + (1-t)y_1/H - \omega_2] L_2(B_1)]$$

(4-18)

We prove this as follows:

Proof $\forall$)

$$b - c = \frac{\beta}{(1-\beta) L_2(B_1) + 1/2 T_1 Y_1} - \frac{H_2 + T_1 Y_1/H}{(1-t)y_1/H - \omega_2}$$

(4-19)
In equation (4-19) \((1-ty_1/N-w_2)\) is very small value relative to \(w_2\) as a large portion of firm's revenue is paid as wage income and \(ty_1/N\) is also relatively small. So, we conclude that since the first term's denominator in equation (4-19) is greater than 1, if \(H_2\) is greater than \(\beta/(1-\beta)(1-ty_1/N-w_2)L_2(B_1),\) \((b-c)\) is negative. Rearranging this inequality, we obtain the restriction on \(\beta\) as in equation (4-18).

In equation (4-17) the numerator is positive if we assume

\[
\frac{[L_2(B_1) - H_2]}{[L_2(B_1) - H_2 + w_2H_2]} < \beta
\]  \hspace{1cm} (4-20)

We prove this result as follows.

Proof VI)

\[
bw_2-e = \frac{[\beta/(1-\beta) L_2(B_1)+1/2ty_1]N(1-ty_1/N-w_2)w_2}{[\beta/(1-\beta) w_2 + 1-1/2ty_1/N]N(1-ty_1/N-w_2)}
\]

\[
-\frac{[1+i)T_1y_0w_1+T_1y_1w_2][\beta/(1-\beta) w_2 + 1-1/2ty_1/N]}{[\beta/(1-\beta) w_2 + 1-1/2ty_1/N]N(1-ty_1/N-w_2)} \hspace{1cm} (4-21)
\]

Since we know that the denominator in equation (4-21) is positive, \((bw_2-e)\) is positive if the numerator is positive. Also, we know from the boundary condition between two land zones that

\[
N(1-ty_1/N-w_2) = [(1+i)(T_1w_1+T_m)y_0+(T_tw_2+T_{m'})y_1]/H_2 \hspace{1cm} (4-22)
\]
Substituting (4-22) into the numerator in equation (4-21) and arranging for \( \beta \), we obtain the restriction on \( \beta \), equation (4-20), for equation (4-21) to be positive.

Therefore, the imposition of a wage tax decreases the wage rate in the second period if the two following restrictions are satisfied.

\[
\frac{L_2(B_1) - H_2}{[L_2(B_1) - H_2] + H_2 w_2} < \beta < \frac{H_2}{[H_2 + (1 - ty_1/N - w_2) L_2(B_1)]} \tag{4-23}
\]

As the utility function is Cobb-Douglas, the range of \( \beta \) is between zero and one. In the restriction (4-23) the upper and lower bounds should be satisfied. The upper bound is close to 1 since the value \((1 - ty_1/N - w_2)\) is very small. The lower bound is close to .5.

Solving equations (4-13) and (4-15) for \( dL_2(B_1) \), we obtain

\[
dL_2(B_1) = b(e - cw_2)/(b - c) \ d\theta_u \tag{4-24}
\]

In equation (4-24) we know that \((b - c) < 0, b > 0 \) and \((e - cw_2) < 0 \) as \( c > b \) and \( bw_2 > e \). Therefore, we conclude \( dL_2(B_1)/d\theta_u > 0 \), which means that the imposition of a wage tax increases (decreases) leisure (work effort).
1.1. The Effects of A Land Value Tax

For a system of cities the land value tax decreases the amount of vacant land in the first period by encouraging premature development of vacant land as residential land. Such a decrease of vacant land will result in less efficient leapfrog, industrial development in the second period beyond the outer edge of the residential zone.

Figure 4-2 The Pattern of Urban Development For a System of Cities Where a Land Value Tax Is Imposed.

In Figure 4-2 $x_2$ represents the conversion of vacant land into residential land in the first period following the imposition of a
land value tax. This decrease in vacant land results in leapfrog development, \( x_2' \) for industrial use in the second period. In Figure 4-2 \( x_2 \) must be equal to \( x_2' \) as population is fixed exogenously and the industrial land per worker is fixed. Before the imposition of the tax the outer edge of the residential zone in the first period was \( B_2 \) and land rents on this boundary were zero. When a land value tax is imposed, the outer edge moves toward the CBD by the conversion of vacant land into residential land, and because of land speculation land rents, \( (T_1 + T_2) x_2 \) are no longer zero. In the second period the outer edge of the residential zone does not change because of a fixed population. So, the residential land-rent functions don't change when a land value tax is imposed. However, industrial land rents in the second period will rise because firms will be located beyond the outer edge of the residential zone by leapfrog development. The increase in industrial land rents in the second period will decrease the wage rate in this period.

When a land value tax is imposed, we have the same equations as in the non-distorted equilibrium except for the change in the boundary condition between the industrial and the residential zones. When we start from the non-distorted equilibrium (no tax on land), we have four unknowns and four equations since \( x_2 = x_2' = 0 \) as \( \theta_{lv} = 0 \).

---

\(^1\) If we start from the positive land value tax (the distorted equilibrium), we have five unknowns including \( x_2 \) in Figure 4-2 because of leapfrog development beyond the residential zone and we have one more boundary condition between the industrial zone and the residential zone in the second period, \( B_2 \) in Figure 4-2. Also, the land value tax is capitalized into the price of land.
As we assume that the local government imposes taxes and returns the proceeds lump-sum, the sum of net lump-sum rental income after paying a land value tax and lump-sum tax return is exactly equal to the nondistorted lump-sum rental income. The land value tax does not change the income equations but this tax is still distortionary as it changes the optimal timing of urban development. When a land value tax is imposed, the boundary condition becomes

\[ N(1-ty_t/N-w_2)H_2/(1+i) \]
\[ = (T_t w_1 + T_t y_0)/(1-\theta_{LUV}) + (T_t w_2 + T_t y_1)/(1+i) \]  \hspace{1cm} (4-11)*

To investigate how a land value tax affects the wage rates and leisure we totally differentiate equations (4-8)-(4-10) and (4-11)* and set \( \theta_{LUV} = 0 \) as the initial value for the tax.

\[ \frac{\beta}{(1-\beta)} w_1 + 1/2(1+w_1) dL_1(B_1) \]
\[ = [1/2H_1 - \beta/(1-\beta) L_1(B_1) - 1/2T_t y_0] dw_1 \]  \hspace{1cm} (4-25)

\[ \frac{\beta}{(1-\beta)} w_2 + (1-1/2ty_t/N) dL_2(B_1) \]
\[ = -[\beta/(1-\beta) L_2(B_1) + 1/2T_t y_1] dw_2 \]  \hspace{1cm} (4-26)

\[ -N_H d\omega_1 - N(1-ty_0/N-w_1) dL_1(B_1) = 0 \]  \hspace{1cm} (4-14)

\[ N(1-ty_t/N-w_2) dL_2(B_1) = -N_H \omega_2 - T_t y_1 d\omega_2 - (1+i)T_t y_0 d\omega_1 \]
\[-(1+i)(T_L+T_Ly_L)y_0 \, d\theta_{LY} \]  \hspace{1cm} (4-27)

From equation (4-14) we know \( d\theta_L = 0 \). Substituting this value into

equation (4-25) we obtain \( dL_L(B_1) = 0 \). Even though the land value tax

is non-neutral with respect to the timing of urban development it
does not affect the wage rate or leisure in the first period. This
result is explained by the fact that the decrease of household's
transportation costs is exactly offset by the increase in
residential rents resulting from land speculation around the outer
edge of the residential zone.

Solving equations (4-26) and (4-27) for \( d\omega_2 \), we obtain

\[ d\omega_2 = f/(b-c) \, d\theta_{LY} \]  \hspace{1cm} (4-28)

where,

\[ f = (T_Ly_L + T_M)y_0 / N(1 - ty_L/N - \omega_2), \quad f > 0 \]

We know \((b-c)\) is negative and \( f \) is positive. So, we conclude

\[ d\omega_2 / d\theta_{LY} < 0, \] which means that the imposition of a land value tax
decreases the wage rate in the second period. This result is
explained by the leapfrog, industrial development which increases
industrial transportation costs in the second period. Solving

equations (4-26) and (4-27) for \( dL_L(B_1) \) to obtain

\[ dL_L(B_1) = -bf/(b-c) \, d\theta_{LY} \]  \hspace{1cm} (4-29)
From equation (4-16) we can determine $dL_2(B_1)/d\theta > 0$, which means that the increase of a land value tax increases leisure in the second period. Consequently, the imposition of a land value tax decreases the wage rate and work effort in the second period and decreases output.

4.5. The Relative Efficiency of A Wage Tax and A Land Value Tax

At the level of the national economy the introduction of a wage tax does not affect the before tax wage rate in the first period but increases the consumption of leisure. The imposition of a land value tax does not affect either the before tax or after tax real wage in the first period. In the second period both the land value tax and the wage tax decrease the wage rate and increase leisure.

'To analyze the change in social welfare for the imposition of a wage tax we totally differentiate equation (4-3)' and set $\theta_0 = 0$ as the initial value of this tax.

\[
\begin{align*}
\frac{dSW(\theta_0)}{d\theta} &= \beta \{ y_0 y_1 \rho dL_1(B_1) + \beta y_0 y_1 \rho^{-1} L_1(B_1) dL_1 + y_1 \omega_1 \beta dL_2(B_1) \\
&\quad + \beta y_1 \omega_2 \beta^{-1} L_2(B_1) dL_2 - [\beta y_0 y_1 \rho^{-1} L_1(B_1) + y_1 \omega_2 \beta^{-1} L_2(B_1)] d\theta_0 \}
\end{align*}
\]

Substituting equations (4-16), (4-17) and (4-24) into equation
(4-30) and setting $dw_1=0$, we obtain

$$dSW(\theta_w) = \beta y_0 w_1^{\beta-1} [\alpha w_1 - BL_1(B_1)]$$

$$+ y_1 w_2^{\beta-1} [(b-c)w_2b(c-w_2) + BL_2(B_1)(b-w_2) - (b-c)L_2(B_1)]$$

Similarly, to investigate how a land value tax affects social welfare we totally differentiate equation (4-3) and substitute equations (4-28) and (4-29) into the resulting expression and set $dw_1=dL_1(B_1)=0$ to obtain

$$dSW(\theta_{LU}) = \beta y_1 w_2^{\beta-1} [-w_2bf/(b-c)]$$

$$+ BL_2(B_1)f/(b-c)]d\theta_{LU}$$

(4-31)

From equations (4-30)' and (4-31) we determine which tax is relatively more efficient by comparing the change of social welfare resulting from the imposition of each of these two taxes. In analyzing the relative efficiency of the two taxes we have different tax bases. To equalize tax revenues between the land value tax and the wage tax we make use of an equal yield condition.

When a wage tax is imposed, total tax revenue in both periods is calculated by multiplying the tax rate by the sum of the present value of total wage income in each period.

$$TTR(\theta_w) = \theta_w[y_0 w_1 H_1 + y_1 w_2 H_2/(1+i)]$$
When a land value tax is imposed, total tax revenue is measured by multiplying this tax rate by the sum of the present value of land in both periods and the value of land is calculated in the same method way as for the case of a single open city as we showed in Chapter III, page 45.

\[
TTR(\Theta_{UL}) = \frac{1}{2} [y_0 (1-w_t) H_1 + (T_t w_t + T_m) y_0^2] + [(1-w_2)(y_t - y_0) - t/N (y_t^2 - 1/2y_0^2)] H_2 / (1+i) + (1/2y_0^2 + y_t^2 - y_0 y_t) (T_t w_2 + T_m)/(1+i)
\]

Using the equal yield condition, and totally differentiating and solving for \( d\Theta_{LU} \), we obtain

\[
d\Theta_{LU} = h/j \ d\Theta_g \tag{4-32}
\]

where \( h \) and \( j \) represent total wage income and total land value in the city, respectively.

Substituting equation (4-32) into equation (4-31) and subtracting the resulting expression from equation (4-30)', we have

\[
dSW(\Theta_g) - dSW(\Theta_{LU}) = [(\beta/(1-\beta))]^\delta A \ d\Theta_g \tag{4-33}
\]

where

\[
A = y_t w_t [aw_t - BL_1(B_1)] + y_t w_2 \beta^{-1} (b-c) b (e-cw_2) + \beta (b w_2 - e) L_2(B_t) - (b-c) L_2(B_t) + w_2 fbh/j - \beta L_2(B_1) fh/j \tag{4-34}
\]

From equation (4-33) we conclude that a wage tax is relatively more
efficient than a land value tax at the level of a system of cities only if \( A \) is positive. So, we demonstrate that \( A \) is positive under the following restrictions.

Proof VII)

In equation (4-21) the first term is positive if \( \alpha_1 > \beta L_1(B_1) \).

\[
\alpha_1 - \beta L_1(B_1) = \frac{\beta/(1-\beta)L_1(B_1)+1/2T_1y_0}{\beta/(1-\beta)+1/2(1+1/w_1)}L_1(B_1)
\]

Since the denominator is positive, if the numerator is positive then the first term is positive. From the numerator we determine that if \( \beta/(1-\beta) \alpha_1 > \beta[\beta/(1-\beta)+1/2(1+1/w_1)] \) is satisfied then the first term in equation (4-34) is positive. Rearranging this inequality for \( \beta \), we obtain

\[
(2\alpha_1-1-1/w_1)/(1/w_1-1) < \beta
\]

This restriction is always satisfied since the range of \( \beta \) is between zero and one and the left hand side of this restriction is always negative because the numerator, \( (2\alpha_1-1-1/w_1) \) is always negative as \( \alpha_1 \) is less than 1. Rewriting the second term of equation (4-34) to obtain

\[
y_1\alpha_1^{\beta-1}/(b-c)\{b\alpha_2(e-c\alpha_2+fh/j)L_2(B_1)[\beta(b\alpha_2-e)-(b-c)-\beta fh/j] \}
\]
In (4-35) we know (b-c) is negative. So, if both the first and second terms are negative in the bracket of (4-35), the second term in equation (4-34) is positive.

\[
e^{-c\omega_2} + fh/j = -[NH_2\omega_2 - (1+i)T_t y_0 \omega_1] / H(1-ty_1/\omega_2)
\]

\[
+ (T_t \omega_1 + T_m) y_0 h/j
\]

(4-36)

Equation (4-36) is negative if

\[
h/j < \frac{[NH_2\omega_2 - (1+i)T_t y_0 \omega_1]}{H(1-ty_1/\omega_2)(T_t \omega_1 + T_m)y_0}
\]

(4-37)

\[
\beta(b\omega_2 - e)-(b-c)-\beta fh/j
\]

\[
= \frac{A(\beta \omega_2 - 1) + B[H_2 + T_t y_1/N - \beta(1+i)T_t y_0 \omega_1 - T_t y_1 \omega_2]}{A}
\]

\[
- \beta fh/j
\]

(4-38)

where,

\[
A = \frac{\beta}{(1-\beta)} L_2(B_1) + 1/2 T_t y_1 N(1-ty_1/\omega_2)
\]

\[
B = \frac{\beta}{(1-\beta)} \omega_2 + 1 - 1/2 ty_1 / N
\]

We determine the restrictions for equation (4-38) to be negative. In the right hand side of equation (4-38) \(\beta \omega_2 - 1\) is negative since \(\omega_2\) is less than 1. In (4-38) \([T_t y_1/N - \beta(1+i)T_t y_0 \omega_1 - T_t y_1 \omega_2]\) is also negative. So, equation (4-38) is negative only if
\[ \frac{B h_2 / B f a}{h / j} < h / j \]

Rewriting this restriction we obtain

\[ \frac{[B/((1-\beta) w_2 + 1 - 1/2 ty_1 / H)]h_2}{\beta(t w_1 + T_m)y_0[B/(1-\beta)L_2(B_1)+1/2 Ty_1]} < \frac{h}{j} \]

(4-39)

Therefore, we have two restrictions for a wage tax to be relatively more efficient than a land value tax for a system of cities. The first restriction on the value of \( \beta \), equation (4-40) below establishes that \((b-c)\), the denominator in equation (4-35) is negative.

\[ 0 < \beta < H_2/[H_2/(1-ty_1/H-w_2)L_2(B_1)] \]

(4-40)

Restriction (4-40) is satisfied as we have explained in Proof V) on pages 77-78. The second restriction shown by equations (4-37) and (4-39) is on the value of the ratio of total tax revenue for a wage tax to the revenue for a land value tax. This restriction is likely to be satisfied. We expect that wage tax base is much larger than land value tax base. Equation (4-37) represents the upper bound for the ratio of these two tax bases and has a large value. In the numerator of equation (4-37) \( H_2w_2 \) represents wage income and is much larger than household's transportation costs, \((T w_1 + T_m)y_0\). Also, the term \((1-ty_1/H-w_2)\) in the denominator of equation (4-37)
represents industrial land rents in the second period and has a small value relative to wage income. The lower bound, equation (4-39), is small relative to the upper bound value as the denominator in (4-39) includes leisure, that is expected to be greater than the number of work hours. If the two restrictions are satisfied, a wage tax is more efficient than a land value tax for a system of cities; otherwise the result is ambiguous.

We now consider a simultaneous tax system where the local government is financing expenditures by means of a wage tax, and substitutes a small land value tax for a portion of a wage tax. The government budget is balanced over the two periods and the increase of one tax results in a decrease of the other tax. For analytical simplicity we consider the case where the wage tax is positive and the land value tax is initially zero.

For the simultaneous tax system we have four equations, the income equations (4-8)'and (4-9)', the product exhaustion condition, equation (4-10) and the new boundary condition between the industrial and residential zones in the second period, equation (4-41) below.

\[
H(1-ty_1/N-t_2)H_2/(1+i) \\
= [T_L w_1(1-\theta_L) + T_m y_0/(1-\theta_L y)] + [T_L w_2(1-\theta_L) + T_m y_1/(1+i)]
\]

\[
(4-41)
\]

To investigate how the substitution of a land value tax for a wage tax affects the wage rate and leisure in each period we totally differentiate equations (4-8)', (4-9)', (4-10) and (4-41), and set
$\theta_{LU}=0$ as the initial value of the tax rate for the land value tax to obtain

$$\left[\beta/(1-\beta) \cdot 1/(1-\omega) + \frac{1}{2}(1+\omega)\right]dL_1(B_1) = \frac{1}{2}H_1 + \beta/(1-\beta)(1-\omega)L_1(B_1)$$

$$-\frac{1}{2}(1-\omega)T_t y_0 d\omega + \frac{1}{2}T_t y_0 \omega_1 + \beta/(1-\beta) \cdot 1/(1-\omega) \cdot L_1(B_1) d\theta$$

(4-42)

$$\left[\beta/(1-\beta) \cdot 1/(1-\omega) + (1-1/2(1+y_1/N))\right]dL_2(B_1) = -(1-\omega)[\beta/(1-\beta)L_2(B_1)$$

$$+1/2T_t y_1]d\omega_2 + \omega_2[\beta/(1-\beta)L_2(B_1) + 1/2T_t y_1]d\theta$$

(4-43)

$$-NH_1d\omega_1 - (1-ty_0/N-y_1)dL_1(B_1) = 0$$

(4-44)

$$N(1-ty_1/N-y_1)dL_2(B_1) = -(1+i)(1-\omega)T_t y_0 d\omega_1 - \left[\omega_1 \cdot H_2 + T_t y_1 (1-\omega)\right]d\omega_2$$

$$+ [(1+i)T_t y_0 \omega_1 + T_t y_1 \omega_2]d\theta - \left[T_t \omega_1 (1-\omega) + T_\theta \right]y_0 d\theta_{LU}$$

(4-44)

In equation (4-44), we know $(1-ty_0/N-y_1)=0$ and $d\omega_1=0$, which means that the substitution of a land value tax for a wage tax does not affect the wage rate in the first period. Solving equation (4-42) for $dL_1(B_1)$ and setting $d\omega_1=0$, we obtain

$$dL_1(B_1) = a' \ d\theta$$

(4-45)

where,

$$a' = \omega_1 \left[\beta/(1-\beta)L_1(B_1) + 1/2T_t y_0 \right] / \left[\omega_1 (1-\omega) \beta/(1-\beta) + 1/2(\omega_1 + 1)\right]$$

$$a' > a > 0.$$
From equation (4-45) it follows that since $d\theta_\omega /d\theta_{LU}$ is negative $dL_1(B_1) /d\theta_{LU} \lt 0$, which means that the substitution of a land value tax for a wage tax decreases leisure in the first period. Solving equations (4-43) and (4-44) for $d\omega_2$, we obtain

$$d\omega_2 = [(b\omega_2 - e)/(b' - c')] d\theta_\omega + [f'/(b' - c')] d\theta_{LU} \tag{4-46}$$

where,

$$b' = (1-\theta_\omega)[\beta/(1-\beta) L_2(B_1) + 1/2 T_t y_1]/[\beta/(1-\beta) m_2(1-\theta_\omega) + 1 - 1/2 T_t y_1/N]$$

$$c' = [N H_2 + T_t (1-\theta_\omega) y_1]/N(1-t y_1/N - m_2)$$

$$f' = [T_t e_1 (1-\theta_\omega) + T_t] y_0$$

In equation (4-46) we know that $(b\omega_2 - e)$ is positive from Proof VI on page 78 above. Disregarding the wage tax term the denominator in equation (4-46), $(b' - c')$ is negative under the same restriction that ensures that $(b - c)$ in equation (4-17) is negative. So, $(b' - c')$ is negative if

$$[L_2(B_1) - H_2]/[L_2(B_1) - H_1 + (1-\theta_\omega)m_2 H_2] < \beta$$

As we explained in the analysis of the single tax this restriction is likely to be satisfied. So, in equation (4-46) the coefficient of the first term is positive if the following restrictions on the value of $\beta$ are satisfied.
\[
\frac{[L_2(B_1) - H_2]}{[L_2(B_1) - H_2] + H_2(1 - \theta w)} < \beta < \frac{H_2}{[H_2 + (1 - ty_1/H - \theta w)L_2(B_1)]} \quad (4-47)
\]

The restriction (4-47) has the same interpretation as the restriction (4-23) on page 79 above. The only difference between the two restrictions is that the wage tax appears in the denominator of equation (4-47). To determine the effect of substituting a land value tax for a wage tax on the wage rate, we use the budget balance condition. Solving equations (4-43) and (4-44) for \( dL_2(B_1) \), we obtain

\[
dL_2(B_1) = \left[ \frac{(b'e - bcw_2) / (b' - c'')} {d\theta} \right] - \left[ f'b' / (b' - c'') \right] d\theta_{LU} \quad (4-48)
\]

In equation (4-48) we know that \( (b' - c'') \) is negative, and that \( (e - cw_2) \) is negative when the restriction on \( \beta \), equation (4-47) is satisfied. So, the coefficient of the first term in equation (4-48) is positive. Also, the coefficient of the second term in this equation is positive since \( (b' - c'') \) is negative and \( f'b' \) is positive. But the effect of substituting a land value tax for a wage tax on leisure in the second period is ambiguous as the coefficient of the first term in equation (4-48) is positive and \( d\theta_{LU} / d\theta_{LU} \) is negative. Thus, we have to use the budget constraint to determine the change in the wage tax relative to the change in the land value tax.

When both a wage tax and a land value tax are imposed simultaneously, the present value of total tax revenue over the two periods is
\[ \text{TTR}_1(\theta_w, \theta_{LV}) + \text{TTR}_2(\theta_w, \theta_{LV})/(1+i) = \theta_w [U_1 + U_2/(1+i)] + \theta_{LV} [U_1 + U_2/(1+i)] \]

where,

\[ \text{TTR}_1(\theta_w, \theta_{LV}) = \text{total tax revenue in each period when both two taxes are imposed simultaneously}, \]

\[ U_i = \text{total before tax wage income in each period}, \]

\[ U_i = \text{total before tax land value in each period}. \]

From the equal yield condition, we can calculate total tax revenue over the two periods.

\[ \text{TTR}_1(\theta_w, \theta_{LV}) + \text{TTR}_2(\theta_w, \theta_{LV})/(1+i) = \theta_w [y_0 w_1 H_1 + y_1 w_2 H_2/(1+i)] + \theta_{LV} [1/2 [y_0 (1-m_1) H_1 + (T_t w_1 + T_m) y_0^2] + [(1-m_2) (y_1 - y_0)] - t (y_2 - 1/2 y_0^2) H_2/(1+i) H + (1/2 y_0^2 + y_1 - y_0 y_1) (T_t w_2 + T_m)/(1+i)] \]

We assume that the total tax revenue over the two periods is fixed exogenously. Totally differentiating this equation and setting \( \theta_{LV} = 0 \) and \( dw_1 = 0 \) to obtain

\[ d\theta_w/d\theta_{LV} = -j/h + \theta_w [y_0 w_1 dL_1(B_1)/d\theta_{LV} + 1/(1+i) y_1 w_2 dL_2(B_1)/d\theta_{LV} - y_1 H_2 d\theta_1/d\theta_{LV}] \]

Substituting equations (4-28) and (4-29) into (4-49), we obtain the following relation
\[
d\theta_e/d\theta_{LV} = -j/\theta_e (y_1w_2bf + y_1H_2f)/(1+i)(b-c) \quad (4-49)'
\]

To investigate how the tax substitution of a land value tax for a wage tax affects social welfare we totally differentiate social welfare function \((4-3)\)' and substitute equations \((4-45), (4-46)\) and \((4-48)\) into the resulting expression to obtain

\[
dSW(\theta_e, \theta_{LV}) = [\beta/(1-\beta)]^\beta \{y_1[y_1(1-\theta_e)]^{\beta-1}[a'_{w_1}(1-\theta_e) - \beta L_1(B_1)]
\]
\[
+ y_1[y_2(1-\theta_e)]^{\beta-1}/(b'-c') [w_2(b'c' - bme_2) + \beta L_2(B_1)(b'c' - e)]
\]
\[
- (b'-c')L_2(B_1)] d\theta_e + [\beta/(1-\beta)]^\beta i' y_1[y_2(1-\theta_e)]^{\beta-1}[\beta L_2(B_1)]
\]
\[
- b'w_2(1-\theta_e)/(b'-c') d\theta_{LV} \quad (4-50)
\]

Rewriting equation \((4-49)\),

\[
dSW(\theta_e, \theta_{LV}) = B d\theta_e + C d\theta_{LV} \quad (4-50)'
\]

In equation \((4-50)'\) the coefficient B has the same terms as the coefficient A in equation \((4-34)\) except for the wage tax. It follows that the coefficient B is positive when the restrictions on B, equation \((4-40)\) are satisfied. Since we know \(d\theta_e/d\theta_{LV} < 0\), the first term in equation \((4-50)'\) is negative when the land value tax is substituted for the wage tax. Thus, we conclude that if the coefficient C in the second term of equation \((4-50)'\) is negative the substitution of a land value tax for a wage tax decreases social
welfare. The proof that \( C \) is negative is as follows.

Proof (VIII)

\[
C = \left[ \frac{\beta}{(1-\beta)} \right] y' y \left[ \omega_2 (1-\theta_y) \right] \beta^{-1} \left[ \beta L_2(B_1) - b' \omega_2 (1-\theta_y) \right] / (b'-c')
\]

Since we know that \((b'-c')\) is negative, if \( \beta L_2(B_1) \) is greater than \( b' \omega_2 (1-\theta_y) \) then \( C \) is negative.

\[
\beta L_2(B_1) - b' \omega_2 (1-\theta_y) = \beta L_2(B_1) - \frac{(1-\theta_y)^2 \beta}{(1-\beta) L_2(B_1) + 1/2 t y_1} \omega_2 \frac{\beta}{(1-\beta) \omega_2 (1-\theta_y) + 1 - 1/2 t y_1 / N}
\]

We know the denominator in the second term of this equation is greater than 1 and \((1-\theta_y)^2 \omega_2 \) is less than 1. So, \( C \) is negative if

\[
\beta L_2(B_1) > \frac{\beta}{(1-\beta) L_2(B_1)} \text{, i.e., } \beta < 1.
\]

This restriction is always satisfied with a Cobb-Douglas utility function.

For a simultaneous tax system we have two restrictions for a wage tax to be relatively more efficient than a land value tax at the level of a system of cities.

\[
0 < \beta < \frac{H_2}{H_2 + (1-ty_1/N-\omega_2) L_2(B_1)} \quad (4-51)
\]
\[
\frac{[\beta/(1-\beta)L_{2}(1-\theta_{w})+1-1/2t_{y}\gamma_{1}/N]H_{2}}{\beta[T_{1}(1-\theta_{w})+T_{2}]y_{0}[\beta/(1-\beta)L_{2}(B_{1})+1/2T_{y}]}
\]

\[
h/j < \frac{[NH_{2}y_{2}(1-\theta_{w})-(1+i)T_{1}y_{0}y_{1}(1-\theta_{w})]}{N(1-t_{y}/N-w_{2})[T_{1}(1-\theta_{w})+T_{y}]y_{0}}
\]  (4-52)

Except for the wage tax term the two restrictions, equations (4-51) and (4-52) that ensure that a wage tax is more efficient when a wage tax and a land value tax are imposed simultaneously are exactly the same restrictions which ensured that the wage tax is more efficient for the single tax case. The likelihood of these restrictions being satisfied were analysed above on pages 88-89.
Chapter V
Conclusions

In this dissertation we have investigated the differential effects of a wage tax and a land value tax on urban development and economic efficiency. A two-period model has been used to confirm that a land value tax is non-neutral and will accelerate urban development in the first period. The basic object of this essay has been to determine the practical importance of this non-neutrality. Is a land value tax efficient relative to a wage tax of equal yield? To my knowledge this research is the first attempt to study the comparative efficiency of two distortionary taxes imposed at the local level. Earlier analysis was restricted to comparing a lump-sum tax or a rental land-rent tax to a distortionary property tax or wage tax.

The general answer to this question is that at the level of a single city where the supply of labor is perfectly elastic a land value tax is more efficient than a wage tax of equal yield under very weak restrictions. These restrictions are that the size of the industrial zone in the second period is greater than \((1+i)\) times the size of the industrial zone in the first period and the price of the industrial good is \((1+i)\) times greater in the second period than in the first period, where \(i\) is the discount rate, and the size of the residential zone is more than two times larger than the industrial zone in each period.

However, when we investigate the issue for a system of identical cities, we are able to develop quite plausible
restrictions under which a wage tax, which distorts the work-leisure choice, is more efficient relative to a land value tax which distorts the pattern of urban development. The reason why this result is obtained is that at the national level the supply of labor is relatively inelastic, whereas for a single city the supply of labor is very elastic.

These results establish a basic policy problem. If the choice of taxes is decided by individual governments, they will choose land value taxes rather than wage taxes. But we have shown that at the national level a wage tax system is likely to be more efficient relative to a system of land value taxes. This establishes the danger of making national policy prescriptions in the basis of results obtained for a single city.

There is a remaining complication that has not been addressed by this thesis: the problem of tax rate differentials between cities. In analysing the case of a system of cities we have considered the imposition of a uniform set of taxes across cities and have not allowed for differential tax rates in different cities. On the basis of our analysis we conjecture that the introduction of tax rate differences would undermine the case for wage taxes, but this is a question for further research.
Appendix 4-1

The non-linear land-rent functions are derived by making use of utility function and income equations. We solve the utility maximization problem for a worker who lives on the boundary between the industrial and the residential zones, $B_1$ and for a worker who resides on the outer edge of the residential zone in the first period, $B_2$. They have the same level of utility at each location.

For the residential location $B_1$

$$\max U_1(B_1) = [C_1(B_1)]^{\beta} [L_1(B_1)]^{1-\beta}$$

s.t. \( w_1 [T_0 - L_1(B_1) - T_L y_1/N] + TR_1/y_0 = C_1(B_1) + R_1 H(B_1) + T_m y_1/N \) \hspace{1cm} (1)

For the residential location $B_2$

$$\max U_1(B_2) = [C_1(B_2)]^{\beta} [L_1(B_2)]^{1-\beta}$$

s.t. \( w_1 [T_0 - L_1(B_2) - T_L (y_1/N+y_0)] + TR_1/y_0 = C_1(B_2) + T_m (y_1/N+y_0) \) \hspace{1cm} (2)

They supply the same work hours:

$$[T_0 - L_1(B_1) - T_L y_1/N] = [T_0 - L_1(B_2) - T_L (y_1/N+y_0)] \hspace{1cm} (3)$$

Solving equation (3) for $L_1(B_2)$ to obtain
\[ L_1(B_2) = L_1(B_1) - T_l y_0 \]  \hspace{2cm} (4)

Since the workers have the same level of utility, we substitute equation (4) into the utility function for \( B_2 \) and equate two utility functions to obtain

\[ \left[ 1 - T_l y_0 / L_1(B_1) \right] (1 - \beta) / \beta = C_1(B_1) / C_2(B_1) \]  \hspace{2cm} (5)

Solving equation (2) for \( C_1(B_2) \), we have

\[ C_1(B_2) = Y - T_m (y_1 / H + y_0) \]  \hspace{2cm} (6)

where \( Y \) is worker's income. Since they have the same income (as we substitute equation (4) into (2) the left hand sides of equations (1) and (2) exactly equal), we equate the right hand sides of equations (1) and (2) to calculate land rents.

\[ R_1^H(B_1) = C_1(B_2) - C_1(B_1) + T_m y_0 \]  \hspace{2cm} (7)

Solving equations (5) and (6) for \( C_1(B_1) \) we have

\[ C_1(B_1) = \left[ 1 - T_l y_0 / L_1(B_1) \right] (1 - \beta) / \beta \left[ Y - T_m (y_1 / H + y_0) \right] \]  \hspace{2cm} (8)

Substituting equations (6) and (8) into (7), we obtain the
non-linear land-rent function.

\[ R_1^H(B_1) = Y - T_B y_1 / N - [1 - T_t \bar{y}_0 / L_1(B_1)]^{(1 - \beta) / \beta} [Y - T_B (y_1 / N + y_0)] \]  \( (9) \)

For analytical simplicity let \( \beta = .5 \). Then the non-linear land-rent function is shown by

\[ R_1^H(B_1)' = T_t y_0 [Y - T_B (y_1 / N + y_0)] / L_1(B_1) + T_B y_0 \]  \( (9)' \)

On the other hand, the linear land-rent function which we assumed is shown by

\[ R_1^L(B_1) = (T_t + T_B) y_0 \]  \( (10) \)

From equations \( (9)' \) and \( (10) \) we know that two land rents exactly equal if the following expression is satisfied.

\[ w_1 L_1(B_1) = Y - T_B (y_1 / N + y_0) = w_1 + T_t y_0 - T_B (y_1 / N + y_0) \]  \( (11) \)

From the non-distorted equilibrium we can calculate the value of each endogenous variables. From equation \( (4-10) \) we obtain

\[ w_1 = 1 - T_t y_0 / N \]  \( (12) \)

Solving equation \( (4-8) \) for \( L_1(B_1) \), we obtain
\[ L_t (B_t) = \frac{(1+w_t)(T_0 T_t y_t/H) - (T_t w_1 + T_0) y_0 - 2T_t y_t/H}{1+3w_t} \]  

For the numerical analysis we assume that

\[ t = 1, \quad 1/H = 0.25, \quad y_0 = 1, \quad y_t = 1.5, \quad T_t = 0.025, \quad T_0 = 0.08, \quad T_0 = 24 \]

Substituting these estimated values for the exogenous variables into equation (11), we find that the left hand side of equation (11) equals 11.729 and the right hand side equals 11.753. The difference between these two values is less than 0.03. Therefore, the approximated linear land-rents we have assumed are very close to the value of the actual non-linear land-rents.
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