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Synthesis of optimal control systems with stable feedback

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Rice University, 1987
RICE UNIVERSITY

SYNTHESIS OF OPTIMAL CONTROL SYSTEMS
WITH STABLE FEEDBACK

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

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ABSTRACT

In this thesis, the question of optimal control system design when constrained to using a stable controller is addressed. The performance index that is minimized is $H^\infty$-norm of the sensitivity of the closed-loop system to external disturbances. It is shown that the resulting feedback associated with the minimum value of the optimality criterion is irrational. A design algorithm is developed to find an approximation whose deviation from a sub-optimal controller lies within a pre-specified error bound, thus guaranteeing internal stability. For multivariable systems, a characterization is provided of all sensitivity functions that result from a stable controller; and two important special cases are solved for optimality.

Next, the problem of finding the $H^2$-optimal stable feedback controller for a general optimality criterion is considered. It is shown that this optimization reduces to finding the "best" function in RH$^2$ (i.e. of minimum norm) that satisfies an avoidance constraint. A design procedure is proposed for obtaining an optimal stable controller that minimizes the performance index and maintains internal stability. Given an initial point, this is achieved by minimizing a nonlinear objective function subject to nonlinear inequality constraints via a sequential gradient programming method.
ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor, Dr. J. Boyd Pearson for his guidance, encouragement and support during the course of this research. I also wish to thank Dr. A. C. Antoulas and Dr. J. E. Dennis for numerous fruitful discussions and valuable suggestions, and for serving on my thesis committee.

My stay here at Rice has been enlivened by the stimulating company of fellow graduate students, most especially the 2nd floor "mafia". Thanks are especially due to my office-mates Munther Dahleh for his keen theoretical insights, and to Bob Bishop for saving me painful hours of drudgery by running MACSYMA on the NASA Space-VAX.

Finally, I would like to dedicate this thesis to my parents, Amma and Appa, for their love, support and understanding which made it all possible.
# Table of Contents

Chapter 1. INTRODUCTION ................................................................. 1  
1.1. Problem Motivation ................................................................. 1  
1.2. Historical Background .............................................................. 8  
1.3. Problem Definition ................................................................. 9  
1.4. Notation ..................................................................................... 10  
Chapter 2. \( H^\infty \)-OPTIMIZATION ............................................. 13  
2.1. Introduction .................................................................................. 13  
2.2. Internal Stability ....................................................................... 14  
2.3. Optimal Sensitivity ..................................................................... 15  
2.4. Computational Algorithm .......................................................... 25  
2.5. Sub-optimal sensitivities ............................................................. 27  
2.6. Boundary Interpolation ............................................................... 38  
Chapter 3. RATIONAL APPROXIMATION ................................... 49  
3.1. Introduction .................................................................................. 49  
3.2. Error bounds ............................................................................... 49  
3.3. Approximation technique ............................................................ 51  
3.4. Computational Algorithms .......................................................... 55  
3.5. Multivariable Systems ................................................................. 59  
3.5.1. Introduction ............................................................................ 59  
3.5.2. Disturbance Reduction ............................................................ 60  
3.5.3. Proper controllers ................................................................. 63  
Chapter 4. \( H^2 \)-OPTIMIZATION : the scalar case ......................... 65  
4.1. Introduction .................................................................................. 65  
4.2. Problem Motivation ...................................................................... 65  
4.3. Problem Formulation ................................................................. 67  
4.4. State-Space Framework ............................................................... 70  
4.5. The Optimal Controller ............................................................... 75  
4.6. Optimal Stable Controllers ......................................................... 77  
4.7. Scalar Case ................................................................................. 79  
4.8. Nonlinear Optimization ............................................................... 85
4.9. Conditional Stability ................................................................. 87
4.10. Illustrative Examples .............................................................. 88
  4.10.1. An LQG Problem .............................................................. 88
  4.10.2. Measurement-noise effect minimization ................................ 99
Chapter 5. $H^2$-OPTIMIZATION : the multivariable case .................. 104
  5.1. Introduction ........................................................................ 104
  5.2. Multivariable Systems .......................................................... 104
  5.3. Nonlinear Optimization ......................................................... 108
  5.4. Initial Condition .................................................................. 110
  5.5. Illustrative Example .............................................................. 111
Chapter 6. CONCLUSIONS ............................................................... 115
Appendix A. INTEGRAL EVALUATION ............................................ 117
Bibliography .............................................................................. 119
CHAPTER 1

INTRODUCTION

1.1. Problem Motivation

Consider the problem of stabilizing a single-input single-output open-loop system \( G(s) \) using an asymptotically stable feedback controller \( K(s) \) in the configuration shown in Figure 1.1. This question was first considered by Youla et al. [55], and is also known as the \textit{strong stabilization} problem.

![Figure 1-1. Closed loop control system](image)

The fundamental motivation for incorporating a stable controller in the system design is to avoid instabilities present in the system due to an unstable controller. For instance, in the case of a stable plant, use of an unstable controller will result in \textit{conditional stability}. A conditionally stable system is stable for values of the open-loop gain lying between the critical values, but becomes unstable if the open-loop gain is either increased or decreased sufficiently. It is advisable to avert such a situation since the gain could drop below the critical value for the following reasons (
Horowitz [30] –

(i) *Component parameter variation* – aging of electronic components such as operational amplifiers and power transistors.

(ii) *Temporary signal saturation* - a large error signal \( e(s) \) entering the controller (for example, when the system starts up from zero-state) can cause saturation of the correction signal \( u(s) \) being input to the plant. This in turn reduces the effective open-loop gain of the system.

If saturation does occur, the resulting non-linear system may go into an oscillatory limit cycle and "hang up" if the proper conditions are present. The describing-function analysis in Example 1.1 below illustrates how conditional stability together with signal saturation could lead to this possibility.

Conditional stability occurs in certain systems: in particular, in a system which has an unstable feedforward path. If the system has an unstable controller, the feedforward path will be unstable. Since a large number of real systems are inherently stable (for instance, according to Garcia and Morari [23] closed-loop stability of process control systems in the absence of plant variations is almost never an issue because the majority of chemical processes are open-loop stable), it is desirable to design for a stable controller form a practical standpoint. Typically, control system practitioners look for stable controllers as an engineering rule of thumb.

It has been shown by Vidyasagar and Vishwanadham [51] that the problem of *simultaneous stabilization* is equivalent to strong stabilization. In order to design a
single controller stabilizing two different plants simultaneously, a stable controller
must be found that stabilizes a third auxiliary system derived from the given two.
This problem is of significance in situations where there is a considerable variation in
the plant model due to changes in the operating conditions. Typically, classical con-
trol system designers look for stable controllers - one does not wish to use unstable
feedback to stabilize a closed-loop system if possible.

**Example 1.1**: A simplified form of the open-loop transfer function of an air-
plane with an auto-pilot in the longitudinal mode is given by

\[ K(s)G(s) = K \frac{(s+a)}{(s-b)} \frac{(s^2+2\rho_1\omega_1s+\omega_1^2)}{(s^2+2\rho_2\omega_2s+\omega_2^2)} \quad a>0, \ b>0 \]  
(1-1)

(this is a modified version of problem A-8-4, p.358 from Ogata [40]). Choosing
\(a=b=1\), \(\rho_1=0.0822\), \(\omega_1^2 = 37\), \(\rho_2=0.5\), \(\omega_2^2 = 16\); the root-locus plot for the closed-
loop system is shown in Figure 1-2. The system is stable for values of the gain K in
the ranges \((0.614, 1.45)\) and \((23.9, +\infty)\), and is hence conditionally stable. Gain
\(K=50\) is picked to lie in the latter, more robust region.

If the amplifier K has a saturation-type nonlinearity, the input-output charac-
teristic will be linear with a saturation limit S. Assuming that the higher harmonics
generated by the nonlinear element are sufficiently attenuated (due to the finite band-
width of the system), only the fundamental harmonic component of the output is
significant; and the stability of the system can be predicted by a *describing function
analysis*[40]. The describing function for such an element is obtained as
\[
\frac{N}{K} = \frac{2}{\pi} \left[ \sin^{-1} \left( \frac{S}{X} \right) + \frac{S}{X} \sqrt{1 - \left( \frac{S}{X} \right)^2} \right] \quad \text{for} \quad \left( \frac{S}{X} \right) < 1 \quad (1-2)
\]

\[
= 1 \quad \text{for} \quad \left( \frac{S}{X} \right) > 1 \quad (1-3)
\]

giving an effective amplifier gain of \( N \).

The Nyquist plot for the system \( G(j\omega) \) is shown in Figure 1-3 for gain \( K=50 \). The \(-1/N\) locus starts form the \(-1\) point on the negative real axis and extends to \(-\infty\), and is a function only of the input signal amplitude \( X \). The two loci intersect at points A, B and C; and a perturbation analysis shows that point B corresponds to a stable limit cycle. In the event of saturation, the output of this system at steady-state exhibits a sustained oscillation at frequency \( \omega=2.96 \ \text{rad/s} \). Simulation runs with a 4th order Runge-Kutta numerical integration scheme and a sampling frequency of 100Hz verify that this is the case, and a typical output is shown in Figure 1-4.
Figure 1.2: Root-locus plot for example 1.1 to illustrate conditional stability.
Figure 1.3: Describing function for example 1.1. "B" represents a stable limit cycle.
Figure 1.4: Simulation run for example 1.1. System oscillates in steady-state with no input.
1.2. Historical Background

It has been shown by Youla et al. [55] that a necessary and sufficient condition for the existence of a stable stabilizing controller $K(s)$ for a given plant $G(s)$ is that the real zeros and poles of $P(s)$ in the extended right half plane $\tilde{H}$ possess the parity interlacing property.

Definition: Let the total number of real poles of $G(s)$ to the right of each zero $z_i$ of $G(s)$ in $\tilde{H}$ be $\sigma_i$, $i=1, 2 \ldots n$. Then $P(s)$ is said to possess the parity interlacing property if and only if the set of numbers $\{ \sigma_i \mid i=1, 2 \ldots n \}$ is either completely even or completely odd.

If such a controller can be found, the plant is said to be strongly stabilizable.

A parametrization of all stable, stabilizing controllers for scalar systems has been given in Vidyasagar and Davidson [50], from which it is seen that the general characterization of this family is irrational (logarithmic). In Tannenbaum [46] the problem of gain margin maximization using stable controllers is solved. Sondergeld and Smith [44] consider the question of Mcmillan degree of stable controllers: it is shown that the degree could be arbitrarily large. In the context of $H^\infty$-optimization, Khargonekar and Tannenbaum [31] and Freudenberg and Looze [20] raise the issue of sensitivity minimization with stable controllers - however, a complete solution is not provided.
1.3. Problem Definition

Assuming then that the plant $G(s)$ does satisfy the parity interlacing property, it is possible to utilize a stable controller $K(s)$ to ensure the internal stability of the system. In this thesis, we consider the problem of designing a stable controller such that the overall closed-loop system has some desirable properties - specifically, be optimal in some well-defined sense. For instance, a reasonable performance requirement could be that the system reduce the effect of the disturbance $d(s)$ on the plant output $y(s)$ (see Figure 1-5); where the disturbances belong to the class of bounded energy signals (i.e. $d(s) \in L^2$).

![Diagram](image)

Figure 1-5. Output disturbance

In other words, the sensitivity of the system to the external disturbances is to be minimized when the power spectrum of the disturbance is not known a priori. In Chapters 2 and 3 it will be shown how to design a rational, stable controller $K(s)$ that minimizes the energy of the output response – this is done using the increasingly popular $H^\infty$-design methodology. In subsequent chapters we consider the more general problem of $H^2$-optimization of system performance when constrained to stable controllers. The physical significance of such a minimization procedure is well
understood, and a complete exposition can be found in many references - for instance, Stein and Athans [45] or Vidyasagar [52].

1.4. Notation

\( \mathbb{R} \) the real line

\( \mathbb{C} \) the complex plane

\( \mathbb{H} \) open right half plane, \( \text{Re}(s) > 0 \)

\( \overline{\mathbb{H}} \) closed right half plane, \( \text{Re}(s) \geq 0 \)

\( \mathring{\mathbb{H}} \) extended right half plane, \( \overline{\mathbb{H}} \cup \{\infty\} \)

\( \mathbb{D} \) open unit disc, \( |z| < 1 \)

\( \overline{\mathbb{D}} \) closed unit disc, \( |z| \leq 1 \)

\( \mathbb{C}_1 \) unit circle, \( |z| = 1 \)

\( L^\infty \) the Banach space of functions bounded on \( j\mathbb{R} \)

\[ f(s) \in L^\infty \iff \sup_{\omega \in \mathbb{R}} |f(j\omega)| < \infty \]

\[ \| f(s) \|_\infty = \sup_{\omega \in \mathbb{R}} |f(j\omega)| \]

\( H^\infty \) the Hardy space of functions analytic in \( \mathbb{H} \) and bounded in \( \mathring{\mathbb{H}} \)

\[ f(s) \in H^\infty \iff \sup_{\sigma > 0} |f(\sigma + j\omega)| < \infty \]
$\mathbb{RH}^\infty$ the subset of functions in $\mathbb{H}^\infty$ that are real-rational.

$L_\infty$ the Banach space of functions bounded on $C_1$

$$g(z) \in L_\infty \iff \sup_{\theta \in [0,2\pi]} |g(e^{i\theta})| < \infty$$

$$\|g(z)\|_\infty = \sup_{\theta \in [0,2\pi]} |g(e^{i\theta})|$$

$\mathbb{H}_\infty$ the Hardy space of functions analytic in $\mathbb{D}$ and bounded in $\overline{\mathbb{D}}$

$$g(z) \in \mathbb{H}_\infty \iff \sup_{r < 1} |g(re^{i\theta})| < \infty$$

$\mathbb{RH}_\infty$ the subset of functions in $\mathbb{H}_\infty$ that are real-rational.

$L^2$ Hilbert space of functions square-integrable on $j\mathbb{R}$

$$f(s) \in L^2 \iff \int_{-\infty}^{\infty} f^*(j\omega)f(j\omega)\,d\omega < \infty$$

where * indicates complex conjugate transpose.

$$\|f(s)\|_2 = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega)f(j\omega)\,d\omega \right]^{\frac{1}{2}}$$

$\mathbb{H}^2$ the Hardy space of functions analytic in $\mathbb{H}$ and square-integrable in $\overline{\mathbb{H}}$

$$f(s) \in \mathbb{H}^2 \iff \sup_{\sigma > 0} \int_{-\infty}^{\infty} f^*(\sigma+j\omega)f(\sigma+j\omega)d\omega < \infty$$
$\text{RH}^2$ the subset of functions in $H^2$ that are real-rational

$H_2^\perp$ orthogonal complement of $H^2$ in $L^2$

$L_2$ Hilbert space of functions square-integrable on $C_1$

\[ g(z) \in L_2 \iff \int_0^{2\pi} g^*(e^{i\theta})g(e^{i\theta}) \, d\theta < \infty \]

\[ \| g(z) \|_2 = \left( \frac{1}{2\pi} \int_0^{2\pi} g^*(e^{i\theta})g(e^{i\theta}) \, d\theta \right)^{1/2} \]

$H_2$ the Hardy space of functions analytic in $D$ and square-integrable in $\overline{D}$

\[ g(z) \in H_2 \iff \sup_{r<1} \int_0^{2\pi} g^*(re^{i\theta})g(re^{i\theta}) \, d\theta < \infty \]

$H_2^\perp$ orthogonal complement of $H_2$ in $L_2$

$G(s)$ If $G(s)$ is a LTI system with realization $(A, B, C, D)$, then

\[ G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

$\Lambda[A]$ eigenvalues of matrix $A$
CHAPTER 2

H∞-OPTIMIZATION

2.1. Introduction

In this chapter, we consider the problem of minimizing the sensitivity of the system (as measured by the energy of the output in Figure 1-5) to disturbances that are finite-energy signals. Since the input d(s) is not known precisely, such an approach represents a "worst-case" design. This is the basis of the so-called H∞-design methodology which was introduced by Zames [57]. The solution to this problem was first given in Zames and Francis [58]. The multivariable version was solved by Chang and Pearson [6] using Nevanlinna-Pick interpolation; by Francis, Helton and Zames [19] using Sarason's theory; and by Safonov and Verma [43] using Hankel approximations. A considerable volume of work has since appeared on different aspects of the general problem of H∞-optimization.

We solve the problem for the scalar case assuming that we are restricted to a stable controller. It is shown that the optimal solution is irrational (exponential) and has to be approximated to obtain a rational solution.
2.2. Internal Stability

Consider the 2x2 block transfer matrix \( T(s) \) of the system shown in Figure 2-1.

\[
\begin{bmatrix}
 x_1 \\
 x_2
\end{bmatrix} = \begin{bmatrix}
 K(1+GK)^{-1} & -GK(1+GK)^{-1} \\
 GK(1+GK)^{-1} & G(1+GK)^{-1}
\end{bmatrix} \begin{bmatrix}
 r_1 \\
 r_2
\end{bmatrix}
\]  \hspace{1cm} (2-1)

Figure 2-1. Internal Stability

The closed-loop system is said to be *internally stable* if and only if every entry in the transfer function matrix \( T(s) \) is stable (Chen and Desoer) [9]; which is equivalent to requiring that

1. \((1+GK)\) has all its' zeros in the open left half plane (OLHP).
2. there are no right half plane (RH) pole-zero cancellations between \( G \) and \( K \).

It was shown by Youla et al. [55] that the requirement of internal stability could be viewed as an interpolation problem on the sensitivity \( S(s) \) i.e. the transfer function from the disturbance \( d(s) \) to the output \( y(s) \) in Figure 1-5.

\[
S = (1+GK)^{-1}
\]  \hspace{1cm} (2-2)

The conditions on \( S(s) \) are
2.3. Optimal Sensitivity

Usually, the disturbance \( d(s) \) is not completely arbitrary and some information regarding its frequency content is present. A weighting function \( W(s) \) can be chosen to reflect this knowledge; for example, a low-pass weighting would be suitable if the disturbances affecting the system were at low frequencies. If \( S(s) \) is the transfer function from \( d(s) \) to \( x(s) \) then let

\[
\Phi(s) = W(s)S(s) = \frac{W}{1+GK}
\]  

(2-3)

where \( \Phi(s) \) is the weighted sensitivity function.

Let us assume that \( d(s) \) belongs to the class of signals with bounded energy i.e. \( d(s) \in H^2 \) and that we wish to minimize the maximum energy in the output \( y(s) \). The corresponding performance objective for the design would then be to minimize the \( H^\infty \)-norm of \( \Phi(s) \), which in the scalar case reduces to the Chebyshev norm.

\[
\|\Phi(s)\|_\infty := \sup_{\omega \in \mathbb{R}} |\Phi(j\omega)|
\]  

(2-4)

The magnitude of the optimal sensitivity is defined as

\[
\mu_0 := \inf_{\forall K(s)\text{stabilizing}} \|\Phi(s)\|_\infty
\]  

(2-5)
Finding the optimal weighted sensitivity then is equivalent to constructing a
function $\Phi(s) \in H^\infty$ such that

1. $\Phi(p_i) = 0$, $i=1,2,...,M$

2. $\Phi(s_k) = W(s_k)$, $k=1,2,...,N$

3. $\|\Phi(s)\|_\infty$ is minimum.

For simplicity, it is assumed that $[p_i]_{i=1}^{i=M}$, $[s_k]_{k=1}^{k=N}$ lie in $H$, and are distinct.

This is a purely technical assumption that enables one to avoid boundary interpo-
lation and interpolation with multiplicities - for further details and some general dis-
cussion, see Chapter 6 of Wang [54]. This is a standard interpolation problem where the
interpolating function is required to have least maximum modulus (see Walsh [53]).

Let $\Phi(s) = B_p(s)F(s)$ where $B_p(s)$ is a Blaschke product built up from the poles
$[p_i]_{i=1}^{i=M}$ of the plant $G(s)$. Let

$$B_p = \prod_{i=1}^{i=M} \frac{s-p_i}{s+p_i} \quad (2-6)$$

Then $B_p(s)$ satisfies the interpolation conditions imposed by the plant poles in $H$. To
handle the ones required by the plant zeros $[s_k]_{k=1}^{k=N}$, $F(s)$ has to satisfy

$$F(s_k) = \frac{W(s_k)}{B_p(s_k)} = w_k, \quad k=1,2,\ldots,N \quad (2-7)$$

It is well known that the solution $F(s)$ is a unique all-pass function i.e., its’ magnitude
is constant on the $j\omega$-axis, and is of the form $F(s) = \mu_0 f(s)$ The minimum norm of the
desired function $F(s)$ is $\mu_0$ and can be found by solving an eigenvalue problem set up in terms of the interpolation conditions on $F(s)$. $f(s)$ can then be constructed by the well-known Nevanlinna-Pick algorithm such that $f(s) \in H^\infty$, $\| f \|_\infty = 1$ and

$$f(s_k) = \frac{1}{\mu_0} \frac{W(s_k)}{B_p(s_k)} , \quad k=1,2,\ldots,N$$  \hspace{1cm} (2-8)

For more mathematical background and details, refer to Helton [29], Ball [4], and Delsarte et al. [14]

The optimal sensitivity is therefore

$$\Phi_{opt}(s) = \mu_0 B_p(s) f(s)$$  \hspace{1cm} (2-9)

The compensator $K(s)$ that achieves this optimal sensitivity is unique, and can be recovered through

$$K_{opt}(s) = \frac{W(s) - \Phi_{opt}(s)}{\Phi_{opt}(s)} \frac{1}{G(s)}$$  \hspace{1cm} (2-10)

Equation 2-10 shows that the unstable poles of the compensator $K(s)$ arise purely from the RHP zeros of $f(s)$. There is no contribution from either the RHP zeros of $B_p(s)$ or the RHP zeros of $G(s)$, since they are cancelled by identical poles of $G(s)$ and zeros of $[W(s) - \Phi_{opt}(s)]$ respectively. In general,

$$f(s) = \prod_{i=1}^{i=q} \frac{s-a_i}{s+a_i} \quad q \leq N-1$$  \hspace{1cm} (2-11)

For any optimal weighted sensitivity problem where the plant $G(s)$ has $N$ non-minimum phase zeros, the "optimal" compensator $K_{opt}(s)$ will usually have $N-1$
unstable poles. Hence, we observe the well-known fact that for systems with 2 or more nonminimum phase zeros, the $H^\infty$ controller is always unstable.

**Example 2.1**: Let the system model and its' weighting function be

\[
G(s) = \frac{(s-1)(s-2)}{(s+3)(s+4)}, \quad W(s) = \frac{1}{100} \frac{s+10}{s+0.1}
\]  
(2-12)

The optimal weighted sensitivity is found to be

\[
\Phi_{\text{opt}}(s) = -0.1635 \frac{s-4.1487}{s+4.1487},
\]  
(2-13)

and the corresponding compensator is

\[
K_{\text{opt}}(s) = -1.061 \frac{(s+3)(s+4)}{(s-4.1487)(s+0.1)};
\]  
(2-14)

illustrating the fact that excluding the RHP poles of $G(s)$ (if any), the RHP zeros of the sensitivity function appear as the unstable poles of the compensator.

If the controller $K(s)$ that stabilizes the system is required to be stable, i.e. $K(s)$ has no poles in $H$; the optimization problem now becomes more restrictive. Instead of minimizing the weighted sensitivity over all possible controllers, consideration is limited to a smaller class of controllers. The least magnitude of the weighted sensitivity over all stable, stabilizing controllers is defined to be

\[
\mu_T := \inf_{\forall \text{ K(s) stable, stabilizing}} \| \Phi(s) \|_\infty
\]  
(2-15)

and is called the **threshold sensitivity**.
Let the sensitivity function $\Phi(s)$ be further restricted such that

$$\Phi(s) \neq 0 \quad \forall \, s \in \overline{H} \text{ except } [p_1], \, i=1,2, \ldots, M$$

(2.16)

It is easily seen from equation (2.10) and the discussion above that the controller $K(s)$ will be stable. The resulting modified interpolation problem on $F(s)$ will then be

$$F(s_k) = w_k, \, j=1,2, \ldots, N$$

(2.17)

where $F(s) \in H^\infty$ and

$$F(s) \neq 0 \quad \forall \, s \in \overline{H}$$

(2.18)

This so-called interpolation with an outer function (i.e. a function which has no RHP poles or zeros) was first considered by Ball and Helton [5], and was used by Tannenbaum [46] for stabilizing feedback systems with uncertainty in the gain factor.

At this point, it is convenient to introduce a one-to-one conformal mapping $\Psi$ of the extended right half plane $\overline{H}$ in the $s$-domain onto the closed unit disc $\overline{D}$ in the $z$-domain (Ahlfors[1]).

Let $z = \Psi(s)$ and $\Psi: s_k \rightarrow z_k, \, k=1,2, \ldots, N$

(2.19)

The modified interpolation problem on $f(z)$ then becomes

$$f(z_k) = v_k = \frac{1}{\mu} \, w_k \quad k=1,2, \ldots, N$$

(2.20)

(note that $\mu$ is now an unknown parameter!) where $f(z) \in H_\infty$, norm $\|f\|_\infty \leq 1$ and $f(z) \neq 0 \quad \forall \, z \in \mathbb{D}$. Consider the function
\[ g(z) = -\ln f(z) \]  
(2-21)

Note that \( g(z) \) is well-defined because of the presence of the zero avoidance condition on \( f(z) \). The interpolation problem for \( g(z) \) is then

\[ g(z_k) = -\ln v_k - j2\pi n_k, \quad k=1,2,\ldots,N \]  
(2-22)

where \( g(z) \in H_\infty \), and

\[ g : \overline{D} \rightarrow \overline{H} \]  
(2-23)

The integer set \( \{n_k\} \) arises due to the non-uniqueness of the complex logarithm. Such a function \( g(z) \) exists if and only if the Pick matrix \( Q(\mu, \{n_k\}) \geq 0 \), where

\[ Q(\mu, \{n_k\}) = \begin{bmatrix}
-\ln v_1 - \ln \bar{v}_k + j2\pi(n_k - n_i) \\
1 - z_k \bar{z}_k
\end{bmatrix}_{i,k=1}^N \]  
(2-24)

Since the matrix \( Q \) depends only on the integer differences \( (n_k - n_i) \), one can normalize \( \{n_k\} \) by setting \( n_1 = 0 \). In all subsequent discussion, integer set \( \{n_k\} \) is assumed to be normalized in this fashion. It has been shown by Ball and Helton [5] that the existence of these integer sets is intrinsic to the problem, and not a superficial consequence of the method.

**Proposition 2.1**

Consider the interpolation problem of equation (2-25) with \( \{n_k\} = 0 \) for all \( k=1,2,\ldots,N \). Then the *reference sensitivity* \( \mu_R \) is the least value of the parameter \( \mu \) such that a solution \( g_R(z) \) exists, and
\[ \mu_R = \exp \left( \frac{\beta_R}{2} \right) \]  

(2-25)

where \( \beta_R \) is the largest eigenvalue of \( A^{-1}B \). Matrices \( A \) and \( B \) are

\[ A = \begin{bmatrix} \frac{1}{1 - z_i \overline{z}_k} \end{bmatrix}_{i,k=1}^N \]  

(2-26)

\[ B = \begin{bmatrix} \ln w_i + \ln \overline{w}_k \end{bmatrix}_{i,k=1}^N \]  

(2-27)

and \( \{ w_i \} \) are as defined in equation (2-7).

Proof

By equation (2.25) above, \( g_R(z) \) exists

\[ \Leftrightarrow \begin{bmatrix} -\ln w_i - \ln \overline{w}_k + 2 \ln \mu \end{bmatrix}_{i,k=1}^N \geq 0 \]  

(2-28)

\[ \Leftrightarrow \beta A \geq B \] , where \( \beta = 2 \ln \mu \)  

(2-29)

Since \( A \) is always positive-definite, equation (2-29) holds

\[ \Leftrightarrow \beta \geq \lambda_{\text{max}}(A^{-1}B) \]  

(2-30)

Hence, \( \beta_R = \min \beta = \lambda_{\text{max}}(A^{-1}B) \). "

The total number of choices that are possible for integer set \( \{ n_k \} \) appear to be
infinite in number, but this is not really so. As the integers become larger in magnitude,
the off-diagonal terms of the Piek matrix in equation (2-24) also get larger, but
the diagonal elements remain the same - hence $Q(\mu, \{n_k\})$ cannot stay positive-definite for very many choices of $\{n_k\}$.

**THEOREM 2.2**

The possible choices of integer set $\{n_k\}$ to be considered in finding the threshold sensitivity $\mu_T$ are finite in number. For every choice $\{n_k\}_p, p=1,2,\cdots, P$; there exists a minimum value of $\mu > 0$, say $\mu_p$, such that $Q(\mu_p, \{n_k\}_p) \geq 0$. Then

$$\mu_T = \text{minimum} \{ \mu_p \mid p=1,2,\cdots, P \} \quad (2-31)$$

**PROOF**

Suppose there exists a value of $\mu$, say $\mu_1 < \mu_R$ and some choice of integers $\{n_k\} \neq 0$ such that the Pick matrix $Q(\mu_1, \{n_k\}) \geq 0$.

$\implies Q(\mu_R, \{n_k\}) > 0$ (by definition of $\mu$)

$\iff$ Every leading principal minor of $Q(\mu_R, \{n_k\}) > 0$

Hence, every 2x2 principal minor of $Q(\mu_R, \{n_k\}) > 0$ (see Gantmacher [22]).

Consider the 2x2 principal minor of $Q(\mu_R, \{n_k\})$ built up from the $(i,i)$, $(i,k)$, $(k,i)$ and the $(k,k)$th elements -

$$
\hat{Q} = 
\begin{bmatrix}
-\ln w_i - \ln w_i + 2\ln \mu_R & -\ln w_i - \ln w_k + 2\ln \mu_R + j2\pi nd(i,k) \\
1 - z_i z_i & 1 - z_i z_k
\end{bmatrix}
\begin{bmatrix}
-\ln w_k - \ln w_i + 2\ln \mu_R - j2\pi nd(i,k) \\
1 - z_k z_i & 1 - z_k z_k
\end{bmatrix}
$$

where $nd(i,k) = n_k - n_i$. 

Now $\hat{Q} > 0$ implies that

$$- 4\pi^2 nd^2(i,k) + 4\pi nd(i,k) \Re[ j(-\ln w_i - \ln \bar{w}_k)] + (ab - c\bar{c}) \cdot |1 - z_i \bar{z}_k|^2 > 0 \quad (2-32)$$

where

$$a = \frac{-\ln w_i - \ln \bar{w}_i + 2\ln \mu_R}{1 - z_i \bar{z}_i} \quad (2-33)$$

$$b = \frac{-\ln w_k - \ln \bar{w}_k + 2\ln \mu_R}{1 - z_k \bar{z}_k} \quad (2-34)$$

$$c = \frac{-\ln w_i - \ln \bar{w}_i + 2\ln \mu_R}{1 - z_i \bar{z}_k} \quad (2-35)$$

Equation (2-32) is an inequality $A nd^2(i,k) + B nd(i,k) + C > 0$ with $A < 0$.

Let $X_L(i,k)$, $X_U(i,k)$ be the zeros of this polynomial (note they are always real).

$$nd(i,k) \in [X_L(i,k), X_U(i,k)] \text{ and is purely integer.} \quad (2-36)$$

for $i=1,2 \cdots N-1$ and $k=i+1, i+2 \cdots N$.

There are $N(N-1)/2$ differences between the variables $\{n_k\}, \; k=1,2 \ldots N$; however in general only $N-1$ of these are independent. This can be easily seen by considering the following system of linear equations -
\[
\begin{bmatrix}
  -1 & 1 \\
  -1 & 1 & \ddots & \ddots \\
  & \ddots & \ddots & \ddots \\
  0 & -1 & 1 & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots 
\end{bmatrix}
\begin{bmatrix}
  n_1 \\
  n_2 \\
  n_3 \\
  \vdots \\
  n_N 
\end{bmatrix}
= 
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_N \\
  b_{N+1} \\
  b_{N(N-1)/2} 
\end{bmatrix}
\]  
(2-38)

\[
H_n = b
\]  
(2-39)

The components of the right-hand vector \( b \) are integers

\( b_1 \in [X_L(2,1), X_U(2,1)] \)

\( b_2 \in [X_L(3,1), X_U(3,1)] \) and so on ... .

The row dependencies of matrix \( H \) are seen to be

Row \( N = \) Row 2 - Row 1

Row \( N+1 = \) Row 3 - Row 1 and so on ... .

Hence, the components of \( b \) are further restricted such that

\( b_N = b_2 - b_1 \)

\( b_{N+1} = b_3 - b_1 \) and so on ... .

Thus, there exist only a finite number of vectors \( b \) such that an exact solution \( n \) to equation (2-38) can be found.

Hence, the possible choices of integer set \( \{n_k\} \) which may permit parameter \( \mu < \mu_R \) are finite in number, and can be found by solving equation (2-38) for allowable choices of \( b \).
Let there be P admissible choices for the integer set \( \{n_k\}_p \), \( p=1,2 \ldots P \) (including the zero set). Then there exists, for each choice \( \{n_k\}_p \), a minimum value of the parameter \( \mu > 0 \), say \( \mu_p \), such that a solution to the interpolation problem

\[
g_p(z_k) = - \ln \left( \frac{w_k}{\mu} \right) + j2\pi n_k \mu_p
\]  

(2-40)

can be found, where \( g_p(z) \in \mathcal{H}_\infty \), and \( g_p : \overline{D} \rightarrow \mathcal{H} \). The optimal choice of \( \mu \) is then

\[
\mu_T = \text{minimum} \left\{ \mu_p \mid p=1,2 \ldots P \right\}
\]  

(2-41)

- the threshold sensitivity defined earlier. ■

2.4. Computational Algorithm

The algorithm for finding the threshold sensitivity \( \mu_T \) works in the following way:

1. Generate all possible integer sets of the first \( (N-1) \) differences using the discrete sets of possible values for \( \text{nd}(i,k) \). Since \( \{n_k\} \) is assumed to be normalized, these are actually all possible integer sets \( \{n_k\} \).

2. Check whether each of the possible integer sets is admissible or not by verifying the consistency of the remaining \( (N-1)(N-2)/2 \) differences. This is equivalent to constructing all vectors \( \mathbf{b} \) such that equation (2-38) has an integer solution \( \mathbf{n} \).

3. Since \( \mu_T \leq \mu_R \), a further substantial reduction in computational effort is achieved by checking the positive definiteness of \( Q(\mu_R, \{n_k\}_p) \) for \( p=1,2 \ldots \).
P - this follows from

\[ \mu_p \leq \mu_R \iff Q(\mu_R, \{n_k\}_p) \geq 0 \]  \hspace{1cm} (2-42)

Suppose there are \( U \) such choices \( \{n_k\}_u, u=1,2 \ldots U \).

(4) For each choice \( \{n_k\}_u \), find the minimum value of \( \mu = \mu_u \) such that 
\[ Q(\mu_u, \{n_k\}_u) \geq 0. \] Then \( \mu_T = \min \{ \mu_u | u=1,2 \ldots U \} \).

The optimal interpolating function \( g_T(z) \) corresponding to the threshold sensitivity \( \mu_T \) can be then used to recover the optimal sensitivity function \( \Phi_T(z) \) over all stable controllers.

\[ \Phi_T(z) = \mu_T B_p(z) \exp[-g_T(z)] \]  \hspace{1cm} (2-43)

**Example 2.2**: Consider the same problem as in Example 2.1 -

the only possible choice for integer set \( \{n_k\} = 0 \); and threshold sensitivity

\[ \mu_T = 0.1750 > \mu_0 = 0.1635 \]  \hspace{1cm} (2-44)

which is the magnitude of optimal sensitivity over all stable controllers.

**Example 2.3**: Consider the interpolation problem of finding \( g(z) \in H_\infty \)

\[ g: (0.2+j0.4, 0.2-j0.4, 0.333\overline{3}) \longrightarrow (-1+j1, -1-j1, -1) \]  \hspace{1cm} (2-45)

The minimum norm over all possible functions is \( \mu_0 = 3.8951 \).

If \( g(z) \) is constrained to be an outer function, \( \mu_R = 9961.52 \) with integer set \( \{n_k\} = 0 \)

The bounds on \( \{n_j\} \) are found to be:

\[ nd(1,2) = n_2 - n_1 \in [-2.07, 3.57] \]
\( \text{nd}(1,3) = n_3 - n_1 \in [-1.56, 1.31] \)

\( \text{nd}(2,3) = n_3 - n_2 \in [-2.31, 0.56] \)

There are 9 admissible combinations of integers for \( \{n_k\} \); however \( Q(\mu_R, \{n_k\}) > 0 \) only for the sets (0, 1, 0) and (0, -1, -1). The optimal choice is integer set (0, 1, 0) and the threshold sensitivity \( \mu_T = 4.276 \).

2.5. Sub-optimal sensitivities

**THEOREM 2.3**

The optimal sensitivity over all stable controllers \( \Phi_T(z) \) is unique and is a *singular* inner function - that is, it has at least one *essential singularity* (p.129, Ahlfors [2]) on the unit circle \( C_1 \).

**PROOF**

Let \( \{n_i\}_T \) be the integer set associated with \( \mu_T \). The optimum interpolating function \( g_T(z) \) can be constructed using the composite scheme shown in Figure 2-2. \( \overline{g} \) is a conformal mapping from \( D \) to \( \tilde{D} \) - hence \( \overline{g}(z) \in H_\infty \). Let \( \Psi: \tilde{D} \rightarrow \tilde{H} \) be the one-to-one conformal mapping described earlier.

Hence, there exists a unique point \( z_\infty \) on the unit circle \( |z|=1 \) such that

\[
\Psi: z_\infty \rightarrow \infty
\]  

(2-46)

Since the Pick matrix \( Q(\mu_T) \) is positive semi-definite, \( \overline{g}_T(z) \in H_\infty \) such that

\[
\overline{g}_T(z_i) = \hat{w}_i, \quad i=1,2,...N
\]  

(2-47)

is a unique, all-pass function with \( ||\overline{g}_T(z)|| = 1 \).
Since $g_T(z)$ is onto, there exist points $z_{01}, z_{02}, \ldots, z_{0n}$ on the unit circle $|z|=1$ such that

$$g_T(z_{0j}) = z_{\infty}, j=1,2,\ldots,n$$

(2-48)

$g_T(z) = \Psi \circ g_T(z)$. $\therefore g_T(z)$ is also unique, and has poles $\{ z_{0j} \} j=1,2,\ldots,n$ on $|z|=1$. Hence $\Phi_T(z)$ is unique, and has at least one essential singularity on the unit circle $C_1$.

The optimal sensitivity over all stable controllers, $\Phi_T(z)$ is unique but irrational. The construction scheme outlined above shows that $\Phi_T(z)$ will have at least one essential singularity on the unit circle $|z|=1$. Hence, $\Phi_T(z)$ comes arbitrarily close to every complex number $\lambda$ in any $\varepsilon$-neighbourhood of the essential singularity $z_0$. $\Phi_T(z)$ is a singular inner function for $|z|=1$; but in the vicinity of $z_0$ the phase of the complex function $\Phi_T(e^{j\phi})$ changes very rapidly and goes through an infinite number
of oscillations. By p.37 of Duren [17], \( \Phi_T(z) \) cannot be approximated by a stable rational function.

If a sub-optimal value of the performance index \( \mu > \mu_T \) is chosen a whole new realm of possibilities opens up. The interpolation problem that is required to be solved for \( \tilde{g}(z) \) now has an infinite number of solutions. The strategy adopted is to construct a particular solution \( \tilde{g}(z) \) that moves the essential singularities of the corresponding sub-optimal sensitivity \( \Phi(z) \) away from \( |z|=1 \). This gives rise to an approximation problem on the irrational controller \( C(z) \) that can be solved to within a pre-specified error bound.

**Example 2.4:** Consider the problem of examples 2.1 and 2.2 -

Let \( \Psi : s=4 \frac{1+z}{1-z} \) be the conformal mapping from \( D \) to \( H \). To construct the optimal sensitivity over all stable controllers \( \Phi_T(z) \), solve for

\[
\tilde{g}_T : (-0.6, -0.3333) \longrightarrow \Psi^{-1} (\ln 0.1/\mu_T, -\ln 0.05714/\mu_T) = (-0.7545, -0.5627)
\]

\[
\tilde{g}_T(z) = \frac{-3.5415z + 1}{z - 3.5415} \quad \text{hence}
\]

\[
\Phi_T(z) = 0.175 \exp \left[ 2.2385 \frac{z+1}{z-1} \right]
\]  

(2-49)  

(2-50)

This optimal sensitivity has an essential singularity at \( z_0 = 1 \).

Let \( \mu > \mu_T \), then there exist an infinite number of solutions to the interpolation problem on \( \tilde{g}(z) \).

Let \( \tilde{g}(z) \) be constructed using the Nevanlinna-Pick algorithm - this is called the
minimal degree solution $g_{MDS}(z)$. Figure 2-3 shows the location of the essential singularity $z_0$ of the corresponding sensitivity $\Phi(z)$ (which arises from the pole of $g(z)$) versus $\alpha$, where $\alpha := 1/\mu$. As $\alpha$ is decreased from $\alpha_T = 1/\mu_T$, $z_0$ moves away from the unit circle $|z|=1$.

A "good" choice of $\alpha$ that moves the singularity $z_0$ away is $\alpha = 3.25$, giving

$$g_{MDS}(z) = \frac{0.9881z - 0.7661}{z + 3.0211}$$

(2-51)

with a corresponding sensitivity function

$$\Phi(z) = 0.3077 \exp\left[-\frac{7.9526z + 9.02}{0.0119z + 3.7871}\right]$$

(2-52)

The singularity $z_0 = -319.3$, which is distant enough from $C_1$ to give a reasonable approximation problem for $C(z)$.

Another solution $\bar{g}(z)$ that was constructed is the least maximum modulus solution $\bar{g}_{LMM}(z)$. This is so called because it has the minimum $H_\infty$-norm among all possible solutions $\bar{g}(z)$ to the (sub optimal) interpolation problem. Figure 2.4 shows the location of the singularity $z_0$ versus $\alpha$ where the sensitivity $\Phi(z)$ is constructed using $\bar{g}_{LMM}(z)$. It is observed that at $\alpha = 0.35$ the singularity is fairly distant from $C_1$, giving another possible set of design choices $[\mu, \Phi(z)]$.

Since $\Phi(z) = \mu B_p(z)f(z)$, $\|B_p\| = 1$ and $\|f\| \leq 1$

$\Rightarrow \|\Phi(z)\| \leq \mu$

Figure 2.5 shows the profile of $\|\Phi\|_\infty$ versus $\alpha$ for the classes of solutions $g_{MDS}(z)$
and $\mathbf{g}_{LM}\mathbf{M}(z)$, which are seen to be bounded by the curve $1/\alpha$. 

It is possible, depending on the problem being solved, that the essential singularity $z_0$ closest to the unit circle $|z|=1$ does not move sufficiently away from it for either of the two classes of solutions considered above. A more general approach would then be to parametrize all solutions $\mathbf{g}(z)$ to the interpolation problem for a particular choice of $\alpha$. This is of the form

$$
\mathbf{g}_\alpha(z) = \frac{\mathbf{P}(z)h(z) + \mathbf{Q}(z)}{\mathbf{P}(z) + \mathbf{Q}(z)h(z)}
$$

(2-53)

where $\mathbf{P}(z), \mathbf{Q}(z), \mathbf{P}(z), \mathbf{Q}(z) \in \mathbf{RH}_\infty$. $h(z) \in \mathbf{H}_\infty$ is an arbitrary function with $||h||_\infty \leq 1$ (i.e. $h(z)$ belongs to the unit ball $\mathbf{BH}_\infty$). Equation (2-53) is a special case of the Adamjan, Arov and Krein [1] characterization of all solutions to the Nevanlinna-Pick interpolation problem (see Krein and Nudelman [32]). Explicit formulas for the functions $\mathbf{P}(z), \mathbf{Q}(z), \mathbf{P}(z), \mathbf{Q}(z)$ have been given in Khargonekar and Tannenbaum [31].

Consider the conformal mapping $\Psi$ from $\overline{\mathbb{D}}$ to $\overline{\mathbb{H}}$ described earlier, and let

$$
\Psi : z_\infty | \longrightarrow \infty
$$

(2-54)

Recall that $g = \Psi \circ \mathbf{g}$, then all solutions $\{z_{0i}\}, i=1,2, ..., n$ to the equation $\mathbf{g}(z) = z_\infty$ constitute poles of $g(z)$, or the essential singularities of the sensitivity $\mathbf{F}(z)$. Thus, moving these singularities away from $C_1$ is equivalent to solving the following optimization problem -
\[
\max_{\forall h \in \mathbb{H}_L} \min_{i=1,2,\ldots,N} \left\{ |z_{0i}| \mid h(z_{0i})A(z_{0i}) + B(z_{0i}) = 0 \right\}
\] (2-55)

where \( A(z) = \tilde{P}(z) - z_{\infty}Q(z) \) and \( B(z) = \tilde{Q}(z) - z_{\infty}P(z) \).

This is, in general, not a viable problem because of the very large class of functions \( h(z) \) that are possible. Let \( h(z) \) be restricted to a real number \(-1, 1\).

Then the optimization problem reduces to a classical root-locus of the type

\[ 1 + G(z)h = 0 \quad \text{where} \quad G(z) = \frac{A(z)}{B(z)} \] (2-56)

The region of stability is now outside the unit disc \( \overline{D} \) - the roots are to be pushed out as far as possible under the constraint \(|h| \leq 1\).

**Example 2.5**: Consider the same problem as in example 2.4 - since \( \Psi : z = \frac{s-4}{s+4} \) is the conformal mapping used, \( z_{\infty} = 1 \). Let \( \alpha = 3.25 \) as before, then

\[ G(z) = \frac{9.086z^2 + 3.39z + 1}{z^2 + 3.39z + 9.086} \] (2-57)

Figure 2-6 illustrates \( \min |z_{0i}| \quad i=1,2 \) versus \( h \), where \( h \in [-1, 1] \). The best choice is \( h = 0.08 \), giving the solution

\[ \tilde{g}(z) = \frac{0.5729z^2 + 1.6028z - 1.845}{0.846z^2 + 4.7216z + 7.1611} \] (2-58)

and the corresponding sensitivity

\[ \Phi(z) = \exp \left[ -\frac{5.6756z^2 + 25.2978z + 21.2649}{0.2731z^2 + 3.1188z + 9.006} \right] \] (2-59)
The essential singularities are now at $-5.7097 \pm j 0.6115$. 
Figure 2.3: Singularity location $z_0$ vs. $\alpha$ for minimal degree solution (example 2-4).
Figure 2.4: Singularity location $z_0$ vs. $\alpha$ for least maximum solution (example 2-4).
Figure 2.5: Sensitivity vs. $\alpha$ for $\bar{g}_{MDS}(z)$, $\bar{g}_{LMM}(z)$ (example 2-4).
Figure 2.6: \( \min_h |z_{01}| \), singularity locations vs. \( h \) (example 2-5).
2.6. Boundary Interpolation

In the theoretical development of stable, stabilizing controllers so far, cases involving -

(a) interpolation on the boundary of the disc i.e. unit circle $C_1$

(b) interpolation with multiplicities (or repeated points)

have been avoided for the sake of simplicity and clarity of exposition.

However, in many practical situations the systems to be considered are

(i) strictly proper, which is equivalent to having one or more zeros at $\infty$;

and/or

(ii) have poles and zeros on the imaginary axis (for instance, an integrator would introduce a pole at $s=0$).

In this section, the current framework of interpolation with outer functions is extended to incorporate case(a) i.e. points on the boundary of $C_1$.

Let us first prove the following modified version of Theorem 1.5 from Kargonekar and Tannenbaum [31].

THEOREM 2.4

Consider the interpolation problem on $f(z) \in H_\infty$

\[ f(z_i) = w_i \quad \text{for} \quad |z_i| < 1 \quad , \quad i=1, 2 \cdots k \quad \text{(2-60)} \]

\[ f(z_i) = w_i \quad \text{for} \quad |z_i| = 1 \quad , \quad i=k+1, k+2 \cdots l \quad \text{(2-61)} \]
where the $\left[ z_i \right]_{i=1}^1$ are distinct.

Let $\mu_0$ be the minimum norm of the function $f_{in}(z) \in H_\infty$ that interpolates the interior points: that is

$$\mu_0 = \min_{f \in H_\infty} \| f(z) \|_{\infty} \quad \ni \quad f_{in}(z_i) = w_i \quad \text{for } i=1, 2 \cdots k$$

(2-62)

and define $\mu_c := \max \left[ \mu_0, \ |w_{k+1}|, \ |w_{k+2}|, \cdots, |w_1| \right]$

Then, there exists a function $f(z) \in H_\infty$ such that

$$f(z_i) = w_i \quad i = 1, 2 \cdots k, k+1 \cdots 1$$

(2-63)

with $\|f(z)\|_{\infty} \leq \mu_c + \epsilon \ \forall \ \epsilon > 0$

(2-64)

\textbf{PROOF}

Consider the normalized interpolation problem

$$\tilde{f}(z_i) = \tilde{w}_i \quad i = 1, 2, \cdots 1$$

(2-65)

where $\tilde{w}_i = \frac{w}{\mu}$ for $i = 1, 2, \cdots 1$ and $\mu = \mu_c + \epsilon$ for some $\epsilon > 0$. Let $G$ be the Pick matrix for the interior interpolation points $(z_i, \tilde{w}_i)$; $|z_i| < 1$ for $i = 1, 2, \cdots k$

$$G = \begin{bmatrix} 1 - \tilde{w}_i \tilde{w}_j^* \end{bmatrix}_{i,j=1}^k$$

(2-66)

Since $\mu > \mu_0$, matrix $G > 0$. Suppose we move the points on the boundary $C_1$:

$|z_i| = 1$; $i = k+1, k+2 \cdots 1$ by an infinitesimally small amount $\delta > 0$ inside $D$. Consider the Pick matrix $H_\delta$ for this new problem $(z_i, \tilde{w}_i)$; $i = 1, 2 \cdots 1$
\[ H_\delta = \left[ \frac{1 - \hat{w}_i \hat{w}_j^*}{1 - z_i z_j^*} \right]_{i,j=1}^1 \]  
\hspace{1cm} \text{(2-67)}

For \( \delta \) sufficiently small, rows \( k+1, k+2 \cdots 1 \) of \( H_\delta \) will be *diagonally dominant* i.e.

\[ h(i,i) \gg h(i,j) \quad \text{for } i=k+1, k+2 \cdots 1 ; j=1, 2, \cdots 1 ; i \neq j \]  
\hspace{1cm} \text{(2-68)}

Also, \( h(i,i) > 0 \) since \( |\hat{w}_i| < 1 \) for all \( i=1, 2 \cdots 1 \).

\[
\text{Let } H_\delta^{(g)} = \begin{bmatrix}
I_k & 0 & 0 \\
p_{k+1} & 0 & 0 \\
0 & p_{k+2} & 0 \\
& & \ddots \\
0 & 0 & p_1
\end{bmatrix} . H_\delta \]  
\hspace{1cm} \text{(2-70)}

where \( p_i = \frac{1}{h(i,i)} \) for \( i=k+1, k+2 \ldots 1 \).

Then \( H_\delta \) is scaled such that rows \( k+1, k+2 \cdots 1 \) are normalized with respect to their diagonal elements, or

\[ H_\delta^{(g)}(i,i) = 1 \quad \text{for } i=k+1, k+2 \cdots 1 \]  
\hspace{1cm} \text{(2-71)}

Since \( \lim \limits_{\delta \to 0} h(i,i) = \infty \) for \( i=k+1, k+2 \cdots 1 \)

\[
\lim \limits_{\delta \to 0} H_\delta^{(g)} = \begin{bmatrix} G^* \\
0 & I_{t-k} \end{bmatrix} > 0
\]  
\hspace{1cm} \text{(2-72)}

Since this is the transformed Pick matrix for the function \( \hat{f}(z) \), we have that \( \hat{f}(z) \in H_\infty \) exists and \( ||\hat{f}(z)|| \leq 1 \). Let \( f(z) = \mu \hat{f}(z) \).

Then \( f(z) \in H_\infty \) exists such that \( f(z_i) = w_i \) for \( i=1, 2 \cdots 1 \) and \( ||f(z)|| \leq \mu \).
Let \( \Phi(z) \) be the sub-optimal weighted sensitivity of the desired closed-loop system. Then the corresponding controller \( K(z) \) that achieves this sensitivity is given by

\[
K(z) = \frac{1}{G(z)} \frac{W(z) - \Phi(z)}{\Phi(z)}
\]

\( K(z) \) is irrational and must be approximated by a stable rational approximation \( K_d(z) \) as explained in Chapter 3. However, the approximation procedure requires that \( K(z) \) be analytic in \( D \) and continuous on \( \overline{D} \). This is guaranteed by imposing the interpolation conditions \( \Phi(z_i) = W(z_i) \) at \( \left[ z_i \right]_{i=1}^{k} \) the zeros of \( G(z) \) in \( D \); and at \( \left[ z_i \right]_{i=k+1}^{1} \) the zeros of \( G(z) \) on the boundary \( C_1 \). Here, the complete set of zeros \( \left[ z_i \right]_{i=1}^{1} \) is assumed to be distinct as before.

Recall from Section 2.3 that by setting \( \Phi(z) = B_p(z)F(z) \) (where \( B_p(z) \) is a Blaschke product constructed from the poles of \( G(z) \) inside \( D \)) we obtain an equivalent interpolation problem on \( F(z) \) and that \( \|\Phi(z)\| = \|F(z)\| \). However, if \( G(z) \) has poles on the boundary \( C_1 \) it is not possible to find an all-pass \( B_p(z) \). Let \( \hat{B}_p(z) = B_p(z) h(z) \) where \( h(z) \in H_\infty \) interpolates the poles on \( |z|=1 \) to 0 and \( \|h(z)\| = 1 \) (this is always possible because of Theorem 2.4). Now on setting \( \Phi(z) = \hat{B}_p(z)F(z) \) the equivalent interpolation problem on \( F(z) \) is such that \( \|\Phi(z)\| \leq \|F(z)\| \).

Let \( \mu_T(z) \) be the threshold sensitivity as defined previously for the partial system with zeros \( \left[ z_i \right]_{i=1}^{k} \) in \( D \). That is, \( \mu_T(z) \) is the minimum norm over all outer functions \( F(z) \in H_\infty \) such that
(i) \( F(z_i) = w_i \) for \( i = 1, 2 \cdots k \)

where \( w_i = \frac{\hat{W}(z_i)}{\hat{B}_p(z_i)} \)

(ii) \( F(z) \neq 0 \quad \forall z \in D. \)

We can then prove the following theorem for the complete system \( G(z) \) -

**THEOREM 2.5**

There exists a stable controller \( K(z) \) such that the weighted sensitivity \( \Phi(z) \leq \mu_d + \varepsilon \) for any \( \varepsilon > 0 \), where

\[
\mu_d := \max \left[ \mu_T, \left| w_{k+1} \right|, \left| w_{k+2} \right| \cdots \left| w_1 \right| \right].
\]

**PROOF**

This proof is along the same lines as the one in Theorem 2.3.

Let \( \mu = \mu_d + \varepsilon \) for some \( \varepsilon > 0 \). The discussion in Section 2.4 shows that defining

\[
g(z) := -\ln \frac{F(z)}{\mu}; \text{ an integer set } \{n_i\}_{i=1}^k \text{ can be found such that the interpolation problem}
\]

\[
g(z_i) = -\ln \frac{w_i}{\mu} + j2\pi n_i \quad \text{for } i = 1, 2 \cdots k \quad (2-74)
\]

for \( g: D \rightarrow H \) has at least one solution. That is, the Pick matrix

\[
Q(\mu, \{n_i\}) = \begin{bmatrix}
-\ln v_i -\ln \bar{v}_m + j2\pi(n_m-n_i) \\
1 - z_i \bar{v}_m
\end{bmatrix}^k
\]

is positive definite, where \( v_i = \frac{w_i}{\mu}. \)
Suppose the boundary points \( \{ z_i \}_{i=k+1} \) are moved inside the disk \( D \) by an infinitesimally small amount \( \delta > 0 \): the Pick matrix for the new interpolation problem becomes

\[
H_\delta = \begin{bmatrix} Q(\mu, \{ n_i \}) & * \\ * & G_\delta \end{bmatrix}
\]

(2-76)

where

\[
G_\delta = \begin{bmatrix} -\ln v_i - \ln v_m \\ 1 - z_i z_m \end{bmatrix}_{i,m=k+1}^{1}
\]

(2-77)

and is diagonally dominant. Since \( |v_i| < 1 \) and \( |z_i| < 1 \); \( g_\delta(i,i) > 0 \) and \( \lim_{\delta \to 0} g_\delta(i,i) = \infty \). Scaling \( H_\delta \) such that rows \( k+1, k+2 \ldots 1 \) are normalized with respect to their diagonal elements, it can be shown that

\[
\lim_{\delta \to 0} H_\delta^{(s)} = \begin{bmatrix} Q(\mu, \{ n_i \}) & * \\ 0 & I \end{bmatrix} > 0
\]

(2-78)

Hence, there exists a function \( \hat{g}: D \longrightarrow H \) analytic in \( D \) such that

\[
\hat{g}(z_i) = -\ln \frac{w_i}{\mu} + j2\pi n_i \quad \text{for } i=1, 2 \ldots k
\]

(2-79)

\[
= -\ln \frac{w_i}{\mu} \quad \text{for } i=k+1, k+2 \ldots 1
\]

(2-80)

Let \( F(z) = \mu \exp[ -\hat{g}(z)] \). Then \( F(z) \in H_\infty(z) \) is such that \( F(z_i) = w_i \) for \( i=1, 2 \ldots 1 \), \( ||F(z)|| \leq \mu \) and \( F(z) \neq 0 \ \forall \ z \in D \)

Since \( \Phi(z) = \hat{B}_p(z)F(z) \), \( ||\Phi(z)|| \leq \mu \) and the corresponding controller \( K(z) \) is stable. \( \blacksquare \)
**Example 2.6**: Consider the following system model

\[ G(s) = \frac{(s-1)(s-2)}{(s+3)(s+4)(s+5)} \]  \hspace{1cm} (2-81)

and let the disturbances entering the system be attenuated by the low-pass filter

\[ W(s) = \frac{1}{100} \frac{s+10}{s+0.1} \]  \hspace{1cm} (2-82)

Since the plant is strictly proper, the additional interpolation condition

\[ \Phi(\infty) = W(\infty) \]  \hspace{1cm} (2-83)

must be included to ensure the properness of the controller \( K(s) \).

Using the transformation \( \Psi : s = 4 \frac{1+z}{1-z} \), the interpolation problem on the sensitivity is

\[ \Phi : (-0.6, -0.3333, 1) \longrightarrow (0.1, 0.05714, 0.01) \]  \hspace{1cm} (2-84)

Since for the "interior" problem \( \mu_0 = 0.1635 \), we choose

\[ \mu = 0.164 > \mu_c = \max \left[ \mu_0, 0.01 \right] \]  \hspace{1cm} (2-85)

and obtain the normalized interpolation problem

\[ \hat{\Phi}_\mu : (-0.6, -0.3333, 1) \longrightarrow (0.1/\mu, 0.05714/\mu, 0.01/\mu) \]  \hspace{1cm} (2-86)

where \( \Phi(z) = \mu \hat{\Phi}_\mu(z) \).

Now, all solutions \( \hat{\Phi}_\mu(z) \in H_\infty \) with \( \|\hat{\Phi}_\mu(z)\| \leq 1 \) to the interpolation problem at the points interior to \( D \) (for this choice of \( \mu = 0.164 \)) are given by
\[ \Phi_\mu(z) = \frac{P(z)g(z) + Q(z)}{R(z)g(z) + S(z)} \]  

(2-87)

This is the so-called AAK parametrization of all interpolating functions: \( P(z), Q(z), R(z) \) and \( S(z) \in \mathbb{H}_\infty \) are uniquely determined by the interpolation data and \( g(z) \) is the free function parameter. On simplifying, we get

\[ \Phi_\mu(z) = \frac{A \cdot z^2 + B \cdot z + C}{D \cdot z^2 + E \cdot z + F} \]  

(2-88)

\[ A = 317.004 \cdot g(z) \quad B = -0.8467 \cdot g(z) - 312.123 \quad C = g(z) + 6.5502 \]  

(2-89)

\[ D = 1 + 6.5502 \cdot g(z) \quad E = -0.8467 - 312.123 \cdot g(z) \quad F = 317.004 \]  

(2-90)

where function \( g(z) \in \mathbb{H}_\infty \) and \( \|g(z)\| \leq 1 \).

The boundary condition imposes \( g(1) = 0.9676 \) - choosing \( g(z) \) identically equal to this value gives

\[ \Phi(z) = 0.164 \cdot \frac{306.735z^2 - 312.942z + 7.5178}{7.338z^2 - 302.86z + 317.004} \]  

(2-91)

Recovering the optimal compensator \( K(z) \) and transforming back to the s-plane

\[ K(s) = 1952.63 \cdot \frac{(s + 3)(s + 4)(s + 5)}{(s + 0.1)(s^2 - 1827.41s + 7660.45)} \]  

(2-92)

Note that this "\( \mathbb{H}_\infty \) - optimal" compensator has 2 unstable poles at 4.2016 and 1823.2 : the resulting closed-loop system is conditionally stable for values of gain in the range [0.937, 1.28]. From the Nyquist plot, the system is seen to have poor robustness properties with very low gain and phase margins (0.937 and 20.9°).
Example 2.7: Consider the same plant model G(s) and weighting function W(s) as in Example 2.6 - however, the disturbance rejection is to be achieved using a stable compensator. Under the conformal mapping $\Psi : s = \frac{1+z}{1-z}$ the normalized interpolation problem on the sensitivity is

$$
\Phi_\mu : ( -0.6, -0.3333, 1 ) \rightarrow ( 0.1/\mu, 0.05714/\mu, 0.01/\mu ) \quad (2-93)
$$

where $\Phi_\mu(z) \neq 0 \forall z \in \mathbb{D}$ (i.e. $\Phi_\mu(z)$ is an outer function).

By Theorem 2.5, existence of a stable controller is guaranteed for $\mu > 0.175 = \max \left[ \mu_T, 0.01 \right]$. Using the logarithmic transformation as described in Section 2.4 we obtain the following disc-to-disc interpolation problem -

$$
\bar{g}_\alpha : ( -0.6, -0.3333, 1 ) \rightarrow \Psi^{-1}( -\ln 0.1\alpha, -\ln 0.05714\alpha, -\ln 0.01\alpha ) \quad (2-94)
$$

where $\alpha := 1/\mu < 5.714$ and $g_\alpha(z) = \Psi \left[ \bar{g}_\alpha(z) \right]$.

The poles of $g_\alpha(z)$ will be the essential singularities of the sensitivity $\Phi(z)$ - it is desirable to move them as far away from the unit circle $C_1$ as possible. This would result in a reasonable approximation problem for $K(z)$. To achieve this, we utilize the free design parameter $\alpha$. Figure 2.6 shows the magnitude of the closest pole, $|z_{oi}| \forall i$, versus $\alpha$ for minimal degree solutions $g_\alpha(z)$. A judicious choice is $\alpha_e = 2$ giving

$$
\bar{g}_\alpha(z) = \frac{0.2233z^2 + 0.7522z - 1.1048}{0.9571z^2 + 4.7213z + 5.9509} \quad (2-95)
$$
with a corresponding sensitivity of

\[ \Phi(z) = 0.5 \exp \left[ \frac{4.7217z^2 + 21.894z + 19.385}{0.7339z^2 + 3.9692z + 7.0556} \right] \] (2-96)

The essential singularities are seen to be at \(-2.7043 \pm j1.517\).
Figure 2-7: \( \min |z_{01}| \), singularity locations vs. \( \alpha \) (example 2-7).
CHAPTER 3

RATIONAL APPROXIMATION

3.1. Introduction

In this chapter, the problem of approximating the irrational controller $K(z)$ corresponding to the sub-optimal sensitivity $\Phi(z)$ is considered. To ensure that the approximation $K_a(z)$ still maintains internal stability of the closed-loop, an a priori error bound on the difference is proved. This in turn leads to a lower bound on the degree of the rational approximation. We conclude this chapter with some observations on the multivariable problem.

3.2. Error bounds

The following result from Vidyasagar and Davidson [50] is utilized to prove an error bound on the difference $K - K_a$.

**LEMMA 3.1**

Let the plant $G(z) = \frac{n(z)}{d(z)}$ and let $K(z)$ be stable and internally stabilizing.

If $K_a(z)$ is a stable, rational approximation to $K(z)$ such that

$$\| K - K_a \|_{\infty} < \delta$$  \hspace{1cm} (3-1)

where $\delta = \frac{1}{\| n \| \| u^{-1} \|}$ and $u^{-1} = \frac{S}{d}$, then the closed-loop system with $K_a(z)$ is inter-
nally stable.

**PROOF**

The compensator $K(z)$ is stable and internally stabilizing $\Leftrightarrow u = d + nK$ is a unit in $H_\infty$ (i.e. it has a stable inverse $u^{-1}$ such that $u.u^{-1} = 1$.) Let $u_a = d + nK_a$. Then by the Schwarz inequality

$$
\| u - u_a \| \leq \| n \| \| K - K_a \| < \frac{1}{\| u^{-1} \|} \quad \text{or} \quad (3-2)
$$

$$
\| 1 - u^{-1}u_a \| < 1 \quad (3-3)
$$

$u_a$ is thus a unit in $H_\infty$.

Hence the closed-loop system with stable compensator $K_a(z)$ is internally stable.

$\delta$ is then a prespecified error bound on the approximation $K_a(z)$ to guarantee internal stability of the system. The better the approximation is to the compensator $K(z)$, the closer the actual sensitivity $\Phi_a(z)$ is to the desired sensitivity; which the following result illustrates.

**LEMMA 3.2**

Let $\| K - K_a \| = \delta_1$ be the error in the approximation of $K(z)$, and let $\delta_2 = \| \frac{P \Phi}{W} \|$. If $\delta_1 \delta_2 < 1$, then

$$
\| \Phi_a \| \leq \frac{\| \Phi \|}{1 - \delta_1 \delta_2} \quad (3-4)
$$
PROOF

The difference between the approximate and the desired sensitivity function

\[ \Phi_a - \Phi = \frac{W}{1 + GK_a} - \frac{W}{1 + GK_a} \]

\[ = \frac{W}{1 + GK_a} \frac{G}{1 + GK} \left[ K_a - K_a \right] \tag{3-6} \]

Suppose \( \delta_1 \delta_2 = \| \frac{G \Phi}{W} \| \| K_a - K_a \| < 1 \). Then by the Schwarz inequality

\[ \| \Phi_a \| - \| \Phi \| \leq \| \Phi_a \| \| \frac{G}{1 + GK} \| \| K_a - K_a \| \quad \text{giving} \tag{3-7} \]

\[ \| \Phi_a \| \leq \frac{\| \Phi \|}{1 - K \delta_1} \quad \blacksquare \tag{3-8} \]

3.3. Approximation technique

Let \( R_{mn} \) be the space of rational functions of type \( (m,n) \) - i.e. with at most \( m \) finite zeros and at most \( n \) finite poles, with no poles in the unit disk \( D \). Given a function \( f(z) \) analytic in \( D \) and continuous on \( \overline{D} \) the rational Chebyshev approximation problem is to find \( r^*(z) \in R_{mn} \) such that

\[ \| f - r^* \| = \inf_{r \in R_{mn}} \| f - r \| \quad \tag{3-9} \]

The subsequent development follows closely the so-called Caratheodory-Fejer method from Trefethen [49]. It has been shown by Gutknecht and Trefethen [27] that for any given \( f(z) \), \( m \) and \( n \); a best approximation exists, but is not unique for
Given $f(z)$ and $r(z)$, the curve in the complex plane described by $[f(e^{i\theta}) - r(e^{i\theta})]$ for $0 < \theta \leq 2\pi$ is called the error curve $e$ corresponding to $r(z)$. If the error curve can be contained in a circle of minimal radius about the origin, $r$ is the best approximation $r^*$.

**LEMMA 3.3**

Let $\eta(e,0)$ be the winding number of the error curve $e$ about the origin 0.

If $\eta(e,0) \geq m+n+1$

$$\min_{\forall z \in C_1} |f - r| \leq \|f - r^*\| \leq \|f - r\| \quad (3-10)$$

If the error curve is a perfect circle, then $r = r^*$.

**PROOF**: See p.301 of Trefethen [49]

Let $\tilde{R}_{mn} \supseteq R_{mn}$ be the extended approximation space of all functions that are bounded on $C_1$ and can be written in the form

$$f(z) = \sum_{j=m}^{\infty} d_j z^j \sum_{k=n}^{\infty} e_k z^k \quad (3-11)$$

The numerator converges in $1 < |z| \leq \infty$ and is bounded there except near $\infty$, and $\tilde{f}(z)$ is meromorphic in $1 < |z| \leq \infty$. The extended best approximation $\tilde{r}^* \in \tilde{R}_{mn}$ is such that

$$\|f - \tilde{r}^*\| = \inf_{\forall \tilde{r} \in \tilde{R}_{mn}} \|f - \tilde{r}\| \quad (3-12)$$
As before, if the error curve \( e \) for some \( \tilde{r} \) has \( \eta(e,0) \geq m+n+1 \) then

\[
\min_{\forall z \in C_1} |f - \tilde{r}| \leq \|f - \tilde{r}^*\| \leq \|f - \tilde{r}\|
\] (3-13)

The Chebyshev approximation problem in \( R_{mn} \) has no closed form solution, but the same problem in \( \tilde{R}_{mn} \) does. (Adamjan et al.) [1]

For simplicity it is assumed that

\[
f(z) = c_0 + c_1 z + \cdots + c_K z^K
\] (3-14)

(this represents a truncation of the infinite Taylor series expansion of \( f(z) \) around \( z=0 \)). Let the Hankel matrix of the coefficients be \( H_f \), and it is shifted left or right depending on the value of \( \nu = n-m \).

\[
H_f^{(\nu)} = \begin{bmatrix}
0 & \cdots & c_0 & c_1 & \cdots & c_K \\
c_0 & c_1 & \cdots & c_K \\
c_0 & c_1 & \cdots \\
\cdot & \cdot & \cdot \\
c_K \\
\end{bmatrix}_{(K+\nu \times K+\nu)} \quad (\nu > 0) \quad (3-16)
\]

If \( \nu \leq 0 \), the corresponding Hankel matrix is

\[
H_f^{(\nu)} = \begin{bmatrix}
c_{1-\nu} & c_{2-\nu} & \cdots & c_K \\
c_{2-\nu} & \cdots & c_K \\
\cdot & \cdot & \cdot \\
c_K \\
\end{bmatrix}_{(K+\nu \times K+\nu)} \quad (3-17)
\]
THEOREM 3.4

The polynomial $f(z) = c_0 + c_1 z + \cdots + c_K z^K$ has a unique approximation $\tilde{f}^*$ out of $\tilde{R}_{mn}$. The error is

$$
\| f - \tilde{f}^* \| = \sigma_{n+1} \begin{bmatrix} H_f^{(\nu)} \end{bmatrix}
$$

(3-18)

and the error curve is a perfect circle about the origin with $v(e,0) = m+n+1$ if $\sigma_{m+n+1}$ is simple. $\tilde{f}^*$ is given by

$$
f(z) - \tilde{f}^*(z) = \sigma_{n+1} z^K B_f(z)
$$

(3-19)

where $B_f(z)$ is an all-pass function constructed using the $(n+1)$st left and right singular vectors from the singular value decomposition (SVD) $H_f^{(\nu)} = U\Sigma V^*$. 

Proof: See Trefethen [49]

Remark: In the reduced-order model reduction problem of Glover [25], results of a similar flavour are presented. For example, in Theorem 7.2(i) -

Let $G(s)$ be a stable rational transfer matrix i.e. $G \in \mathcal{H}_+^{ss}$. Then

$$
\sigma_{k+1} \begin{bmatrix} H_G \end{bmatrix} = \inf \begin{bmatrix} F \in \mathcal{H}_+ \end{bmatrix}, \g \in \mathcal{H}_+^{\infty} \| G - \hat{G} - F \|_{\infty}
$$

(3-20)

where $\hat{G}(s)$ has Mcmillan degree $\leq k$ and is the reduced order approximation to $G(s)$. 

Remark: The rational CF method consists of constructing $\tilde{f}^*$ and then truncating it to obtain an approximation $r_{cf} \in R_{mn}$ that is near best. This truncation is only one of several reasonable methods for obtaining $r_{cf}$; some others are described in Gutknecht [28].
3.4. Computational Algorithms

The following algorithm was implemented to compute the rational approximation \( r_{cf}(z) \) to the function \( f(z) \).

1. **The first K Taylor series coefficients of \( f(z) \) i.e. \{ \( c_0, c_1, \ldots, c_K \) \} are found using the Fast Fourier Transform (FFT). In our implementation \( K \) is chosen to be a power of 2 to minimize computation time, and sufficiently large that all subsequent coefficients are negligible.**

2. **The matrix \( H_f^{(y)} \) is set up corresponding to a specified values of \( v = n - m \), and its' SVD is computed. (This corresponds to choosing the relative order of the controller.) The minimum order \( k \) of the desired approximation is then such that**

\[
\sigma_{k+1} \left[ H_f^{(y)} \right] < \delta \tag{3-21}
\]

where \( \delta \) is the prespecified bound on the approximation.

3. **Pick \( n \geq k \). Then \( \sigma_{n+1} \) and the corresponding \((n+1)\)st left and right singular vectors \( u_{n+1}, v_{n+1} \) are used to find the optimal error in equation (2-1).**

4. **The denominator of \( B_f(z) \) is factored to determine its' zeros outside \( \overline{D} \). Let**

\[
\text{denominator}(B_f(z)) = q_{in}(z) \cdot q_{out}(z) \tag{3-22}
\]

where \( q_{in}(z) \) and \( q_{out}(z) \) are polynomials with all zeros inside and outside \( \overline{D} \) respectively. Then \( q_{out}(z) \) is the denominator of \( r_{cf}(z) \).
Let the extended best approximation to \( f(z) \) be

\[
\tilde{r}^*(z) = \frac{\sum_{j=0}^{j=m} d_j z^j}{\sum_{k=n}^{j=\infty} e_k z^k}
\]

(3-23)

Then the coefficients \( \{ ..., d_{-1}, d_0, d_1, ..., d_m \} \) are found by doing an FFT on

\[
h(z) = \left[ f(z) - \sigma_{n+1} B_f(z) \right] q_{out}(z).
\]

(3-24)

The terms of negative degree in the numerator of \( \tilde{r}^*(z) \) are dropped to obtain the approximation

\[
r_{cf}(z) = \frac{\sum_{j=0}^{j=m} d_j z^j}{\sum_{k=n}^{j=\infty} e_k z^k}
\]

(3-25)

Mathematical software for performing all the stages in the compensator design has been developed and tested extensively on the VAX 11/750. The routines have been designed to be computationally efficient, robust and general-purpose in nature; and have been found to be numerically sound.

**Example 3.1:** Consider the problem of example 2.4 -

It is desired to approximate the irrational compensator \( K(z) \) corresponding to sensitivity \( \Phi(z) \) such that

\[
\| K - K_d \| < \delta = 3.2340
\]

(3-26)
Choosing m=n, is is seen that

\[ \sigma_2 \left[ H_0^{(0)} \right] = 0.3805 \]  (3-27)

Hence, a first order approximation will suffice.

\[ K_a(z) = \frac{1.0683z + 1.9894}{z + 1.0471} \]  (3-28)

Since \( \delta_1 \delta_2 = \|P \frac{\Phi}{W} \| \| K_a - K \| = 0.13434 \), \( \| \Phi_a \| \leq 1.1552 \| \Phi \| \). The variation in sensitivity due to approximation is expected to be at most 15%. It can be easily verified that the closed-loop system is internally stable, and has a weighted sensitivity of magnitude \( \| WS_a \|_\infty = 0.2424 \). In comparison, \( \| WS \|_\infty = 0.2319 \), so that the actual variation due to the approximation is less than 5%.

On transforming back to the s-domain, the compensator obtained is

\[ K_a(s) = \frac{3.0578s + 3.6844}{2.0471s + 0.1885} \]  (3-29)

From the Nyquist plot, the system is seen to have the improved gain and phase margins of 1.63 and 85°.

**Example 3.2**: Consider the problem of example 2.5 -

the error bound on the approximation is now \( \delta = 0.8018 \), and the Hankel matrix has

\[ \sigma_3 \left[ H_c^{(0)} \right] = 0.0845 \]. The second order approximation to \( K(z) \) is found to be

\[ K_a(z) = \frac{0.9507z^2 + 3.2133z + 2.9962}{(z + 1.7226)(z + 1.0524)} \]  (3-30)

Computing \( \delta_1 \delta_2 = 0.074 \), the maximum variation in sensitivity due to the
approximation can be no more than 8% . The actual system sensitivity has magnitude $\| WS_a \|_\infty = 0.2367$, which is within 1% of the desired $\| WS \|_\infty = 0.2357$.

In the s-domain we get

$$K_a(s) = \frac{7.1602s^2 + 16.3638s + 11.7387}{(2.7226s + 2.8904)(2.0525s + 0.2099)} \quad (3-31)$$

Example 3.3 : Consider the problem of Example 2.7 -

the irrational compensator $K(z)$ must be approximated by a stable, rational function $K_a(z)$ such that

$$\| K - K_a \| < \delta = 0.5 \quad (3-32)$$

for the closed-loop system to be internally stable. Choosing $m=n$ (i.e. $\nu=0$, or relative order of compensator=0) the Hankel matrix has $\sigma_3 \left[ H_c^{(0)} \right] = 0.1302$ - a second order approximation will do.

$$K_a(z) = \frac{4.8651z^2 + 0.1662z - 27.087}{(z - 4.6823)(z + 1.0511)} \quad (3-33)$$

The maximum predicted error in the sensitivity due to approximation is 1.2% since $\delta_1 \delta_2 = 0.01138$. The weighted sensitivity for the "approximated" system has magnitude $\| WS_a \|_\infty = 0.28024$, which is within 0.1% of the ideal $\| WS \| = 0.28021$.

Transforming back to the s-domain, the compensator is

$$K_a(s) = \frac{22.056s^2 + 255.62s + 358.21}{(3.6823s + 22.729)(2.0511s + 0.2045)} \quad (3-34)$$

From the Nyquist plot the closed-loop system is seen to be robust with gain and phase
margins of 2.41 and 87.6°.

Remark: The upper bound for the error in the approximate sensitivity found in Lemma 3.2 appears to be very conservative. This is a consequence of working with the ||·||∞: a better estimate would be of the form

$$| \Phi_a(j\omega) | \leq \frac{| \Phi(j\omega) |}{1 - |P \frac{\Phi}{W}| |K-K_a| (j\omega)} \quad \forall \omega$$  \hspace{1cm} (3-35)

3.5. Multivariable Systems

3.5.1. Introduction

In the multivariable case, let plant $G(s)$ and controller $K(s)$ be two real rational matrices of compatible sizes $pxm$ and $mxp$ respectively. To state the conditions a general multivariable plant must satisfy so that it can be stabilized by an asymptotically stable controller, it is necessary to recall the following definitions.

The point $s=s_0$ is called a blocking zero of $G(s)$ if it is a zero of every entry of $G(s)$; i.e. $G(s_0) = 0$.

If $G = ND^{-1}$ is a right coprime factorization, then the zeros of $\det D(s)$ are the poles of $G(s)$.

Let the real blocking zeros of $G(s)$ in $H$ be $\{ z_1, z_2, \cdots, z_n \}$; and let the total number of real poles of $G(s)$ to the right of $z_i$ be denoted by $\sigma_i$. Then $G(s)$ is strongly stabilizable if and only if the integer set $\{ \sigma_i \}_{i=1}^n$ is either completely even or com-
pletely odd. ( Youla et al. [55] ).

Generically, most multivariable plants do not possess any blocking zeros unless they are inherent in the structure of the problem. Hence, in practice it is always possible to stabilize a multivariable plant with a stable controller.

3.5.2. Disturbance Reduction

Let \( G(s) \) have left and right coprime factorizations \( G = A_2^{-1}B_2 = B_1A_1^{-1} \), with corresponding Bezout identity matrices \( X_2, Y_2, X_1, Y_1 \). All stabilizing controllers can then be expressed by the YJBK parameterization

\[
K(s) = \left( Y_2 + A_1Q \right) \left( X_2 - B_1Q \right)^{-1}
\]

where \( Q \in \text{RH}^\infty \). Since this is a coprime factorization, stability of the controller is equivalent to \( |X_2 - B_1Q| \neq 0 \ \forall \ s \in \tilde{H} \).

The optimal sensitivity problem (restricted to stable controllers)

\[
\text{minimum } \| (I + GK)^{-1} W \|_\infty \quad \forall \ K(s) \text{ stabilizing and stable }
\]

is then equivalent to

\[
\text{minimum }_{K \in \text{RH}^\infty} \| (X_2 - B_1Q) AW \|_\infty \quad \exists \ |X_2 - B_1Q| \neq 0 \quad \forall \ s \in \tilde{H}.
\]

Let the inner-outer factorization of \( A_2W = A_0A_1 \). Then (3-38) is equivalent to

\[
\text{minimum }_{Q \in \text{RH}^\infty} \| \Phi \| \quad \exists \ |\Phi| = h(s) \in \left[ \mathcal{Z} \mid \text{set of units in RH}^\infty \right]
\]

where \( \Phi = (X_2 - B_1Q) A_0 = \bar{X}_2 - B_1\bar{Q} \).
Let the inner-outer factorization of $B_1 = B_1 B_0$. If the unstable zeros $\{z_i\}_{i=1}^{N}$ of $B_i$ are distinct, we have as an all-pass function

$$\phi = | B_1 | = \prod_{i=1}^{N} \frac{s - z_i}{s + z_i} \quad (3-40)$$

Let $\hat{\Phi} = \Phi B_i^* \Phi = \hat{X}_2 - \hat{Q}$ \quad (3-41)

where $\hat{X}_2 = \phi B_i^* X_2 A_0$ and $\hat{Q} = B_0 QA_0$. We then have the following result -

**LEMMA 3.5**

The sensitivity minimization problem over all stable controllers (3-37) is equivalent to the modified interpolation problem

$$\min_{\Phi \in RH^\infty} || \hat{\Phi} ||_\infty \quad \exists \quad | \hat{\Phi} | = \phi^{p-1} h(s), \quad h(s) \in \mathbb{S} \quad (3-42)$$

and $\hat{\Phi}(z_i) = \hat{X}_2(z_i) \quad i=1, 2 \cdots N$. \quad (3-43)

**PROOF**

Since the zeros $\{z_i\}_{i=1}^{N}$ of $B_i(s)$ are distinct, by Chang and Pearson [6]

$\hat{Q}(z) \in RH^\infty \iff \hat{\Phi}(s) \in RH^\infty$ and satisfies the interpolation conditions $\hat{\Phi}(z_i) = \hat{X}_2(z_i) \quad i=1, 2 \cdots N$. Also, since $B_i^* = B_i^{-1}$; $|\phi B_i^*| = \phi^{p-1}$ and the determinant condition follows from equation (3-39). ■

Consider the interpolation values $\hat{X}_2(z_i) = W_i = \phi B_i^{-1} X_2(z_i), \quad i=1, 2 \cdots N$

The Smith-McMllan form for $B_i^{-1}(s) = U(s)M(s)V^*(s)$ where $U(s), V(s)$ are unimodular matrices and
\[
M(s) = \begin{bmatrix}
\frac{1}{N} \\
\prod_{i=1}^{N} (s - z_i) \\
\vdots \\
\vdots \\
\frac{1}{N} \\
\prod_{i=1}^{N} (s + z_i)
\end{bmatrix}
\] (3-44)

The above structure for \( M(s) \) is a consequence of the divisibility properties of the numerator and denominator polynomials in the Smith-McMillan matrix, and the fact that the zeros of \( B_i(s) \) are distinct. Since adjoint \( \left[ B_i(s) \right] = \phi(s) B_i^{-1}(s) \), it is clear that rank adjoint \( [ B_i(z_i) ] = 1 \).

\[ \therefore \text{ rank of } W_1 \leq 1 \quad \forall \quad i = 1, 2 \cdots N \quad \text{(by application of Sylvester's inequality)}. \]

Utilizing this property of the matrix interpolation values, we now solve for the important practical cases of systems with either no rhp zeros or 1 zero in \( \overline{\mathcal{H}} \). Suppose the plant \( P(s) \) has 1 zero in the rhp - i.e. \( z_1 \in \overline{\mathcal{H}} \). Consider the singular value decomposition (SVD) of the interpolation value \( W_1 \)

\[
W_1 = U_1 \begin{bmatrix}
\sigma_1 \\
0
\end{bmatrix} V_1^* \quad \text{(3-45)}
\]

Then \( \hat{\Phi}(s) \in \mathcal{RH}_{\infty} \Rightarrow \hat{\Phi}(z_1) = W_1 \), \( |\hat{\Phi}| = \phi^{p-1} h(s) \) and \( ||\hat{\Phi}||_{\infty} \) is minimum has the general, optimal solution.
\[ \hat{\Phi}_{opt}(s) = U_1 \begin{bmatrix} \sigma_1 \\ \Omega(s) \end{bmatrix} V_1^* \quad \Omega(z_1) = 0, \quad ||\Omega|| < \sigma_1 \]  

(3-46)

Also \( |\Omega(s)| = \phi^{p-1} h(s), h(s) \in \mathcal{S} \) A simple choice for \( \Omega(s) = \rho \phi(s) I_{p-1} \). Then

\[ ||\hat{\Phi}|| = \sigma_1, \text{ and } \hat{\Phi}_{opt}(s) = \frac{\hat{\Phi}_{opt}(s) - \hat{X}_2(s)}{\phi(s)} \]

leads to a \( H^\infty \)-optimal controller \( K_{opt}(s) \) that is stable.

Suppose the plant \( G(s) \) has no zeros in the \( rhp \). Then it is well-known [57] that the optimal sensitivity function is zero - however, it would require a controller \( K(s) \) with infinite gain to achieve this optimum. The additional requirement that \( K(s) \) be stable is satisfied by choosing the sensitivity function of the form

\[ \Phi(s) = \gamma I \quad \gamma \in \mathbb{R} \text{ arbitrary} \]

(3-47)

Then \( ||\Phi|| = ||\gamma||, \quad |\Phi| \in \mathcal{S} \) and \( \hat{Q}_{so} = \gamma I - \hat{X}_2 \) leads to a stable controller \( K_{so}(s) \).

3.5.3. Proper controllers

If the plant \( G(s) \) is strictly proper, \( Q_{opt}(s) \) as obtained above is improper leading to an improper controller \( K_{opt}(s) \). To ensure a proper stable controller that makes the system internally stable, we modify the optimal controller to

\[ K_e(s) = K_{opt}(s) \frac{1}{(1 + \varepsilon s)^p} = (Y_2 + A_1 Q_{opt}) \frac{1}{(1 + \varepsilon s)^p} : (X_2 - B_1 Q_{opt})^{-1} \]

(3-48)

For \( \varepsilon = 0 \), \( K_0(s) = K_{opt}(s) \) and by continuity we expect for small values of \( \varepsilon > 0 \) to be able to find \( K_e(s) \) to be internally stabilizing. This idea is formalized in the following lemma.
LEMMA 3.6

There exists $\varepsilon > 0$ such that $K_s(s)$ as defined above is proper, stable and internally stabilizing if

$$
\overline{\sigma} \left[ I - A_2 \Phi A_0^{-1} \right]_{s=\infty} < 1
$$

(3-49)

PROOF

$K_s(s)$ is internally stabilizing if and only if

$$
A_2(X_2 - B_1 Q_{\text{opt}}) + B_2(Y_2 + A_1 Q_{\text{opt}}) \frac{1}{(1 + \varepsilon s)^p} = U_s \text{ is a unit in RH}^{\infty}.
$$

Since

$$
A_2(X_2 - B_1 Q_{\text{opt}}) + B_2(Y_2 + A_1 Q_{\text{opt}}) = I \text{ is a unit, we have}
$$

$$
I - U_s = B_2(Y_2 + A_1 Q_{\text{opt}}) \left[ 1 - \frac{1}{(1 + \varepsilon s)^p} \right]
$$

(3-50)

Suppose $s = j\omega_0$ for some $\omega_0 < \infty$. Then by choosing $\varepsilon > 0$ sufficiently small we can force

$$
\overline{\sigma} \left[ I - U_s(j\omega_0) \right] < 1
$$

(3-51)

Now $\overline{\sigma} \left[ I - U_s(j\omega_0) \right]$ is a continuous function of $\omega$, and since by assumption

$$
\overline{\sigma} \left[ I - U_s \right]_{s=\infty} = \overline{\sigma} \left[ B_2(Y_2 + A_1 Q) \right]_{s=\infty}
$$

(3-52)

$$
= \overline{\sigma} \left[ I - A_2(X_2 - B_1 Q) \right]_{s=\infty} < 1
$$

(3-53)

we have that $\| I - U_s \|_{\infty} < 1$. $\therefore U_s$ is a unit in RH$^{\infty}$
CHAPTER 4

H$^2$-OPTIMIZATION: the scalar case

4.1. Introduction

In this chapter we address the question of designing an asymptotically stable feedback controller that optimizes a general system performance criterion in the H$^2$-sense. The H$^2$-optimal system design methods based on Weiner-Hopf techniques were developed in Youla et al. [56] Since then, a considerable volume of work on the subject has appeared in the literature dealing with issues such as stability margins [42] and robustness [36], especially in the context of the LQG/LTR design methodology. We present a design procedure for finding a stable controller in the event that the H$^2$-optimal controller (which is unique) is unstable. The consequent degradation in overall performance is minimized while maintaining the internal stability of the closed loop. The advantage of this approach is that any general 2 or 4-block problem (such as the mixed sensitivity of Kwakernaak [35]) can be handled with the same ease as a simple disturbance reduction type problem.

4.2. Problem Motivation

There are two important practical situations for which H$^2$-optimization is appropriate. These are as follows -
(1) Consider the standard feedback configuration with output disturbance shown in Figure 1-5. Let the disturbance \( d(s) \) at the output of the plant be a finite energy signal. This would be the case when transient noise or impulsive type perturbations enter the system (as opposed to persistent disturbances, which are handled in the \( l^1 \)-framework by Dahleh and Pearson [13]). In the time domain, \( \bar{d}(t) \) (inverse Laplace transform of \( d(s) \)) can then be modelled as an \( L^2(\mathbb{R}+) \) function.

Suppose we wish to minimize the maximum amplitude of the system response \( \bar{x}(t) \) over all time \( t \) to these disturbances. For instance, the peak deviation in the output due to the incoming transients is to be made as small as possible. Let \( \bar{x}(t) = \Phi(t)^* \bar{d}(t) \), and assume that \( \Phi(t) \) is a causal, finite-energy impulse response i.e. \( \Phi(t) \in L^2(\mathbb{R}+) \).

\[
\therefore \quad \bar{x}(t) = \int_0^t \Phi(t-\tau) \bar{d}(\tau) d\tau
\]

(4-1)

By application of Holder's inequality [41] it follows that \( |\bar{x}(t)| \leq \| \Phi \|_2 \| \bar{d} \|_2 \).

\[
\therefore \quad \| \bar{x}(t) \|_{\infty} \leq \| \Phi \|_2 \quad \forall \bar{d}(t) \in L^2(\mathbb{R}^+) , \| \bar{d} \|_2 \leq 1
\]

(4-2)

Suppose \( \bar{d}(\tau) = \frac{\Phi(t-\tau)}{\| \Phi \|_2} \) for \( t > \tau \geq 0 \) and zero otherwise. Then in the limit as \( t \to \infty \), we have

\[
\sup_{d \in L^2 , \| d \|_2 \leq 1} \| \bar{x}(t) \|_{\infty} = \| \Phi \|_2
\]

(4-3)

This is in fact the exact condition under which equality holds in Holder's inequality.
Hence, uniformly good disturbance rejection for all time is achieved by

\[ \min \| \Phi(s) \|_2 \quad \forall \ C(s) \text{ stabilizing} \quad (4-4) \]

(2) Consider the standard probabilistic LQG framework of the form [34]

\[ \dot{x}(t) = Ax(t) + Bu(t) + w_1(t) \quad (4-5) \]

\[ y(t) = Cx(t) + w_2(t) \quad (4-6) \]

The state disturbances \( w_1(t) \) and measurement errors \( w_2(t) \) are assumed to be uncorrelated white Gaussian noise with intensities \( V_1(t) \) and \( V_2(t) \) respectively.

Suppose we desire to minimize a quadratic performance criterion of the type

\[ J(x,u) = \lim_{\tau \to \infty} \frac{1}{\tau} \mathbb{E} \left[ \int_0^\tau x^T(t)Qx(t) + u^T(t)Ru(t) \, dt \right] \quad (4-7) \]

i.e. the cost functional is a minimum mean square error. Since the inputs are stochastic processes with known means and covariances, this can be formulated as an \( H^2 \) optimization problem (Stein and Athans [45]).

4.3. Problem Formulation

Consider the general system configuration shown below in Figure 4-1.
where $e(s)$, $u(s)$ are the exogenous and control inputs, and $z(s)$, $y(s)$ are the regulated and measured outputs respectively. Let

$$
\begin{bmatrix}
  z(s) \\
  y(s)
\end{bmatrix} =
\begin{bmatrix}
P_{11}(s) & P_{12}(s) \\
P_{21}(s) & P_{22}(s)
\end{bmatrix}
\begin{bmatrix}
e(s) \\
u(s)
\end{bmatrix}
$$

(4-8)

then the closed-loop transfer function from $e(s)$ to $z(s)$ is

$$
z(s) = P_{11} - P_{12} K (I + P_{22} K)^{-1} P_{21} u(s).
$$

(4-9)

The transfer function matrix $P(s)$ depends on the nominal plant model $G(s)$ and the particular optimization problem under consideration. Most optimal filtering and control problems with different modelling assumptions on the inputs and corresponding performance specifications can be described in this framework [3], [16].

Assuming that $P(s)$ is admissible [10], the controller $K(s)$ stabilizes the closed-loop system if and only if it stabilizes $P_{22}(s)$. Let $P_{22}(s) = B_1(s) A_1^{-1}(s) A_2^{-1}(s) B_2(s)$ be a stable rational coprime factorization, and let $X_1(s), Y_1(s), X_2(s), Y_2(s)$ be the corresponding Bezout identity matrices such that
\[
\begin{bmatrix}
X_1 & Y_1 \\
-B_2 & A_2
\end{bmatrix}
\begin{bmatrix}
A_1 & -Y_2 \\
B_1 & X_2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] (4-10)

The YJBK parametrization \([56], [33]\) of all stabilizing controllers is

\[
K(s) = \begin{bmatrix}
Y_2 + A_1 Q \\
B_2
\end{bmatrix}
\begin{bmatrix}
X_2 - B_1 Q
\end{bmatrix}^{-1}(s)
\] (4-11)

\[
= \begin{bmatrix}
X_1 - QB_2
\end{bmatrix}^{-1}
\begin{bmatrix}
Y_1 + QA_2
\end{bmatrix}(s)
\] (4-12)

where \(Q(s) \in \mathbb{RH}^\infty\). This leads to an affine characterization of the input-output transfer function

\[
z(s) = \left[ (P_{11} - P_{12} A_2 P_{21}) - (P_{12} A_1) Q (A_2 P_{21}) \right] u(s)
\] (4-13)

\[
:= [ T(s) - U(s)Q(s)V(s) ] u(s) \quad \text{where} \quad T(s), U(s), V(s) \in \mathbb{RH}^\infty
\] (4-14)

Let \(\Phi := T - UQV\) and let inner-outer factorizations of \(U(s)\) and \(V(s)\) (which must have full rank) be \(U = U_i U_o\) and \(V = V_o V_i\). In the so-called 4-block problem \([11]\), \(U(s)\) and \(V(s)\) have column rank and row rank respectively, and we find complementary inner factors \(U_{ci}(s)\) and \(V_{ci}(s)\) such that \(\begin{bmatrix}
U_i & U_{ci}
\end{bmatrix}\) and \(\begin{bmatrix}
V_i \\
V_{ci}
\end{bmatrix}\) are square and inner. Algorithms based on polynomial methods for performing all the above computations are given in \([7], [8]\) and \([21]\). Other methods based on the development in Glover \([25]\) are given by a number of authors.

\[
\Phi = T - \begin{bmatrix}
U_i & U_{ci}
\end{bmatrix}
\begin{bmatrix}
U_o \\
0
\end{bmatrix} Q \begin{bmatrix}
V_o & 0
\end{bmatrix}
\begin{bmatrix}
V_i \\
V_{ci}
\end{bmatrix}
\] (4-15)
Premultiplying by $\begin{bmatrix} U_i^* & 0 \\ 0 & U_{ci}^* \end{bmatrix}$ and postmultiplying by $\begin{bmatrix} V_i^* & V_{ci}^* \end{bmatrix}$ we obtain

$$\| \Phi \|_2^2 = \| \begin{bmatrix} R_{11} - \tilde{Q} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \|_2^2$$

(4-16)

$$= \| R_{11} - \tilde{Q} \|_2^2 + \| R_{12} \|_2^2 + \| R_{21} \|_2^2 + \| R_{22} \|_2^2$$

(4-17)

where $\tilde{Q} = U_0 Q V_0$.

Consider the optimization problem

$$\text{minimize} \quad \| \Phi \|_2$$

$$\text{subject to} \quad Q(s) \in RH^\infty$$

(4-18)

The physical significance of such a minimization procedure is well understood, and a complete exposition can be found in many references - for instance [45] or [52]. By the Projection Theorem [38], the solution to the optimization problem in equation (4-18) is

$$\tilde{Q}_{\text{opt}}(s) = P_{H_2}(R_{11}) + \tilde{Q}_0$$

(4-19)

where $P_{H_2}(R_{11})$ is the projection of $R_{11}$ onto $H_2$ and $\tilde{Q}_0 = R_{11}(\infty)$.

4.4. State-Space Framework

In Doyle [16], an elegant state-space framework for performing all the computations is described. Suppose that
\[ P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \] (4-20)

is a realization for the system \( P(s) \) such that

\[ P_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij} \] (4-21)

For the general problem, \( P_{12}(s) \) has more rows than columns and \( P_{21}(s) \) has more columns than rows. By assumption, \( D_{12} \) has full column rank and \( D_{21} \) has full row rank; hence we can factorize

\[ D_{12}^T D_{12} = R_D = R_1^T R_1 > 0 \quad \text{and} \quad D_{21}^T D_{21} = \bar{R}_D = R_2 R_2^T > 0 \] (4-22)

The Cholesky factors \( R_1 \) and \( R_2 \) are square and positive definite.

We introduce scaling matrices \( R_1^{-1} \) and \( R_2^{-1} \) at the control input \( u(s) \) and measured output \( y(s) \) as shown in Figure 4-2. The plant \( P(s) \) then becomes

\[ P(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} R_1^{-1} \\ R_2^{-1} C_2 & R_2^{-1} D_{21} & R_2^{-1} D_{22} R_1^{-1} \end{bmatrix} \] (4-23)

As proved in Limebeer and Hung [37], this transformation does not result in any loss of generality since the effect of the scaling can be reversed by replacing the controller \( K(s) \) with \( R_1^{-1} K(s) R_2^{-1} \) at the end of the design process. Henceforth, it is assumed that \( P(s) \) has already been scaled such that \( D_{12}^T D_{12} = I \) and \( D_{21} D_{21}^T = I \). The motiva-
tion for such a scaling is two-fold:

(i) it simplifies later calculations

(ii) together with a specific choice of stabilizing matrices $F$ and $H$ (to be described later), this leads to convenient forms for matrices $U(s)$ and $V(s)$ in equation (4-14) - namely that $U^*(s)U(s) = I$ and $V(s)V^*(s) = I$.

![Diagram of generalized system configuration after scaling](image)

Figure 4-2. Generalized system configuration after scaling

Let $(C_2, A, B_2)$ be stabilizable and detectable (this implies admissibility). Then there exists a state feedback matrix $F$ and an output injection matrix $H$ such that $A+B_2F$ and $A+HC_2$ are stable. By Nett et al. [39] we have that the coprime factorization and Bezout identity matrices are given by
\[
\begin{bmatrix}
A_1(s) & -Y_2(s) \\
B_1(s) & X_2(s)
\end{bmatrix} =
\begin{bmatrix}
A + B_2 F & B_2 & -H \\
F & I & 0 \\
C_2 + D_{22} F & D_{22} & I
\end{bmatrix}
\]  \hspace{1cm} (4-24)

\[
\begin{bmatrix}
X_1(s) & Y_1(s) \\
-B_2(s) & A_2(s)
\end{bmatrix} =
\begin{bmatrix}
A + H C_2 & -B_2 - H D_{22} & H \\
F & I & 0 \\
C_2 & -D_{22} & I
\end{bmatrix}
\]  \hspace{1cm} (4-25)

A realization for the matrices T, U, V in equation (4-14) problem is given in [12]

\[
\begin{bmatrix}
T(s) & U(s) \\
V(s) & 0
\end{bmatrix} =
\begin{bmatrix}
A + B_2 F & -H C_2 & -H D_{21} & B_2 \\
0 & A + H C_2 & B_1 + H D_{21} & 0 \\
C_1 + D_{12} F & C_1 & D_{11} & D_{12} \\
0 & C_2 & D_{21} & 0
\end{bmatrix}
\]  \hspace{1cm} (4-26)

The following particular choice of stabilizing matrices F and H results in matrices U(s) and V(s) being inner, or that outer factors U_0(s) and V_0(s) in equation (4-15) are identity.

Consider the Algebraic Riccati equation (ARE) ( using notation in [16] )

\[
E^T X + X E - X W X + Q = 0
\]  \hspace{1cm} (4-27)

where \( E, W, Q \in \mathbb{R}^{n \times n}, W = W^T \geq 0 \) and \( Q \geq 0 \).

Let A, B, P, S and R be matrices of compatible dimensions for the ARE such that

(i) \( P = P^T, R = R^T > 0 \)
(ii) \((A, B)\) is stabilizable and \((P, A)\) is detectable.

For \(E = A - BR^{-1}S^T\), \(W = BR^{-1}B^T\) and \(Q = P - SR^{-1}S^T\) there exists a unique real symmetric solution \(X\) to the ARE such that \(E - BR^{-1}B^TX\) has stable eigenvalues. The solution \(X\) is denoted by

\[
X = \text{Ric} \begin{bmatrix} E & -W \\ -Q & -B^T \end{bmatrix}
\]

(4-28)

Since \(D_{12}\) and \(D_{21}\) have full column and row rank, there exist matrices \(D_1\) and \(D_2\) such that \(\begin{bmatrix} D_{12} & D_1 \end{bmatrix}\) and \(\begin{bmatrix} D_{21} \\ D_2 \end{bmatrix}\) are square and inner.

The stabilizing state feedback matrix \(F\) is given by

\[
F = - \begin{bmatrix} D_{12}^T C_1 + B_2^T X \end{bmatrix}
\]

where

\[
X = \text{Ric} \begin{bmatrix} A - B_2 D_{12}^T C_1 & -B_2 B_2^T \\ -C_1^T D_{12}^T C_1 & -(A - B_2 D_{12}^T C_1)^T \end{bmatrix}
\]

(4-30)

The stabilizing output injection matrix \(H\) is given by

\[
H = - \begin{bmatrix} Y C_2^T + B_1 D_{21}^T \end{bmatrix}
\]

where

\[
Y = \text{Ric} \begin{bmatrix} (A - B_1 D_{21}^T C_2)^T & -C_2^T C_2 \\ -B_1 D_{21}^T D_2 B_1^T & -(A - B_1 D_{21}^T C_2) \end{bmatrix}
\]

(4-32)

With this choice of stabilizing matrices \(F\) and \(H\) we have that \(U^*(s)U(s) = I\) and \(V(s)V^*(s) = I\). \(\Phi(s)\) can be then put in the form of equation (4-15) as
\[ \Phi(s) = T(s) - \begin{bmatrix} U(s) & U_{ci}(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} Q \begin{bmatrix} V(s) \\ V_{ci}(s) \end{bmatrix} \] 

where

\[ U_{ci} = \begin{bmatrix} A+B_2F & -X^T C_1^T D_1 \\ C_1+D_12F & D_1 \end{bmatrix}, \quad V_{ci} = \begin{bmatrix} A+HC_2 & B_1+HD_{21} \\ -D_2B_1^TY^T & D_2 \end{bmatrix} \] 

The optimization problem (4-18) can be then reduced to minimizing

\[ \| \Phi(s) \|_2 = \| R(s) - \bar{Q}(s) \|_2 = \| \begin{bmatrix} R_{11}(s)-Q(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix} \|_2 \] 

Using state-space considerations it can be shown [12] that matrix R(s) is completely unstable i.e. all the poles are in \( \overline{\mathbb{H}} \), giving

\[ Q_{opt}(s) = Q_0 = R_{11}(\infty) = D_{12}^T D_{11} D_{21}^T \] 

4.5. The Optimal Controller

All stabilizing controllers can be represented as

\[ K(s) = (Y_2 + A_1Q) (X_2 - B_1Q)^{-1} \] 

\[ = Y_2X_2^{-1} + X_1^{-1} Q (I - X_2^{-1} B_1 Q)^{-1} X_2^{-1} \] 

which can be configured as shown below.
Figure 4-3. All stabilizing controllers

where

\[
J(s) = \begin{bmatrix}
Y_2X_2^{-1} & X_1^{-1} \\
X_2^{-1} & X_2^{-1}B_1
\end{bmatrix} (s) = \begin{bmatrix}
J_{11}(s) & J_{12}(s) \\
J_{21}(s) & J_{22}(s)
\end{bmatrix}
\] (4-39)

\(Q(s) \in \mathbb{R}^{m}\) is the free parameter. A state-space realization for \(J(s)\) is given by [12]

\[
J(s) = \begin{bmatrix}
A + B_2F + HC_2 + HD_{22}F & H & B_2 + HD_{22} \\
F & 0 & I \\
C_2 + D_{22}F & I & D_{22}
\end{bmatrix}
\] (4-40)

From equation (4-36) the \(H^2\)-optimal solution for \(Q(s)\) is a constant; and hence has the minimal realization

\[
Q_{\text{opt}}(s) = \begin{bmatrix}
0 & 0 \\
0 & Q_0
\end{bmatrix}
\] (4-41)

The \(H^2\)-optimal controller \(K_{\text{opt}}(s) = F_1(J, Q_{\text{opt}})\) has the realization

\[
K_{\text{opt}}(s) := \begin{bmatrix}
\hat{A} & \hat{B} \\
\hat{C} & \hat{D}
\end{bmatrix} = \begin{bmatrix}
\hat{A} + \hat{B}_2Q_0\Delta \hat{C}_2 & H + \hat{B}_2Q_0\Delta \\
F + Q_0\hat{C}_2 & Q_0\Delta
\end{bmatrix}
\] (4-42)
where \( \tilde{A} = \hat{A} + B_2 F + H C_2 + H D_{22} F \)

(4-43)

\[ \tilde{B}_2 = B_2 + H D_{22}, \quad \tilde{C}_2 = C_2 + D_{22} F \]  

(4-44)

\[ \Lambda = (I - D_{22} Q_0)^{-1} \]  

(4-45)

**LEMMA 4.1**

The "\( H^2 \)-optimal" controller \( K_{opt}(s) \) for the system configuration in Figure 4-1 has its' unstable poles at the eigenvalues of \( \tilde{A} \) in the closed rhp - i.e.

\[ \Lambda \left[ (\tilde{A} + B_2 Q_0 \Delta \tilde{C}_2) \right] \cap \overline{H} \]  

(4-46)

**PROOF**

From equation (4-42), poles of \( K_{opt}(s) \) \( \subseteq \Lambda \left[ \hat{A} \right] \).

By Theorem 4.4 of [37], we have that all unobservable and uncontrollable modes of this realization of \( K_{opt}(s) \) are contained in the system zeros of \( J_{12}(s) \) and \( J_{21}(s) \). Since the zeros of a system \( G(s) \) with realization \((A, B, C, D)\) and \( D \) invertible are given by \( \Lambda \left[ A - B D^{-1} C \right] \), from equation (4-40) it follows that -

Zeros of \( J_{12}(s) \) \( \subseteq \Lambda \left[ A + H C_2 \right] \) and zeros of \( J_{21}(s) \) \( \subseteq \Lambda \left[ A + B_2 F \right] \), which are stable.

The result then follows.

4.6. Optimal Stable Controllers

In the event the \( H^2 \)-optimal controller (which is unique) is unstable the question that naturally arises is - what does the designer have to give up in terms of
performance if constrained to use a stable controller? The resulting optimization problem is over the restricted class of stable controllers.

From the previous discussion (equation (4-36)), it is evident that since

\[ R_{11} - Q_0 = P_{H^2}(R_{11}) \]

\[ \| \Phi_0 \|_2^2 = \| P_{H^2}(R_{11}) \|_2^2 + \left[ \| R_{12} \|_2^2 + \| R_{21} \|_2^2 + \| R_{22} \|_2^2 \right] \quad (4-47) \]

Let us introduce \( S(s) \in RH^2 \) as an additional "add-on" component to \( Q_0 \) such that

\[ Q_1(s) = Q_0 + S(s) \quad \text{where} \quad S(s) \in RH^2 \quad (4-48) \]

The system will still remain internally stable since \( Q_1 \in RH^{in} \).

Then \( R_{11} - Q_1 = P_{H^2}(R_{11}) + S \) giving

\[ \| \Phi_1 \|_2^2 = \| P_{H^2}(R_{11}) \|_2^2 + \| S \|_2^2 + \left[ \| R_{12} \|_2^2 + \| R_{21} \|_2^2 + \| R_{22} \|_2^2 \right] \quad (4-49) \]

\[ = \| \Phi_0 \|_2^2 + \| S \|_2^2 \quad (4-50) \]

The question of finding the "best" system utilizing a stable controller can be formulated as

\[ \min_{K(s) \in RH^{in}} \| \Phi \|_2 \quad \exists \quad K(s) \text{ is internally stabilizing and stable} \quad (4-51) \]

\[ \Leftrightarrow \min_{S \in RH^2} \| S \|_2 \quad \exists \quad \| X_2 - B_1 Q_1 \|_2 \neq 0 \quad \forall \ s \in \bar{H} \quad (4-52) \]

The optimization problem of finding the stable controller that minimizes the \( H^2 \)-norm of the transfer function \( \Phi(s) \) has thus been reduced to the much simpler problem.
described in equation (4-52).

Notice that equation (4-48) implies that we are restricted to $Q_1(s) \in RH^\infty \ni Q_1(\infty) = Q_0$, enabling $\| \Phi_1 \|$ in equation (4-49) to have a finite value.

Let the resulting family of controllers be denoted by class $\chi$ -

$$\chi = \left\{ K(s) \in RL^\infty \mid K(\infty) = [Y_2(\infty) + A_1(\infty)Q_0]\left[ X_2(\infty) - B_1(\infty)Q_0 \right]^{-1} \right\}$$ (4-53)

4.7. Scalar Case

Consider the situation where $G(s)$ is a single-input single-output system: $K(s)$ and $Q(s)$ are then scalar rational functions. Equation (4-52) is then equivalent to

$$\min_{S \in RH^2} \| S(s) \|_2 \ni \Theta := [X_2 - B_1Q_0 - B_1S] \neq 0 \ \forall s \in \bar{H}$$ (4-54)

Let $\Psi: \bar{D} \longrightarrow \bar{H}$ be the conformal mapping from the closed unit disc to the extended right half plane. Denote the transformed function $\hat{F}(z) = F(s)|_{s=\Psi(z)}$ and define $z_{\infty} := \Psi^{-1}(\infty)$ to be the image of $s=\infty$. Then equation (4-54) transforms to

$$\min_{\hat{S} \in RH^2} \| \hat{S} \|_2 \ni \hat{\Theta}(z) = [\hat{X}_2 - \hat{B}_1Q_0 - \hat{B}_1\hat{S}] \neq 0 \ \forall z \in \bar{D}$$ (4-55)

$$\hat{S}(z) \in RH_2 \iff (i) \hat{\Theta}^{(i)}(z_i) = \hat{X}_2^{(i)}(z_i) \quad i=1,2 \cdots M \ ; \ j=0,1,2 \cdots m_i-1$$ (4-56)

where $\left[ z_i \right]_{j=1}^M$ are zeros of $\hat{B}_1(z)$ in $\bar{D}$ of multiplicity $m_i$, and

$$(ii) \ \hat{S}(z_{\infty}) = 0$$ (4-57)

Condition (ii) gives rise to an additional interpolation condition on $\hat{\Theta}(z)$ in the
following manner:

suppose \( z_{z_0} \) is a zero of \( \hat{B}_1(z) \) of multiplicity \( m_{z_0} \), then \( Q_0 = \text{constant} \) and (ii) imply

\[
(iii) \quad \hat{\Theta}^{(j)}(z_{z_0}) = \hat{X}_2^{(j)}(z_{z_0}) - \hat{B}_1^{(j)}(z_{z_0})Q_0 \quad j=m_{z_0} \quad (4-58)
\]

( multiplicity \( m_{z_0} = 0 \Rightarrow z = z_{z_0} \) is not a zero of \( \hat{B}_1(z) \).)

Let \( z_i, i=1, 2 \cdots M_r \) be the set of \textbf{real} zeros of \( \hat{B}_1(z) \) in \( \overline{D} \) ( \( M_r \leq M \)).

We now show the following result -

**THEOREM 4.2**

There exists a stable controller in class \( \chi \) for the system configuration in Figure 4-1 if and only if the numbers \( \{ \hat{\Theta}(z_i), \hat{\Theta}(z_{z_0}) \mid i=1, 2 \cdots M_r \} \) have the same sign.

**PROOF**

Let \( \hat{\Theta}(z) \) be the denominator in a coprime factorization of the stable controller \( \hat{K}(z) \).

Since \( \hat{\Theta}(z) \neq 0 \quad \forall \; z \in \overline{D} \), it is evident that the numbers \( \{ \hat{\Theta}(z_i), \hat{\Theta}(z_{z_0}) \mid i=1, 2 \cdots M_r \} \)

must have the same sign, and the necessity of the condition follows.

To prove sufficiency, first assume without loss of generality that the numbers

\( \{ \hat{\Theta}(z_i), \hat{\Theta}(z_{z_0}) \mid i=1, 2 \cdots M_r \} \) are all positive. To construct a solution \( \hat{\Theta}(z) \) such that

(i) and (iii) are satisfied and \( \hat{\Theta}(z) \neq 0 \quad \forall \; z \in \overline{D} \), the following recursive scheme is utilized ( this is based on the construction given in Theorem 1 of Youla et al 55 ).

**Case (a) : Distinct real zero -**

Let \( \hat{\Theta}_K(z) \) be the solution at the Kth stage of the recursion such that

\[
\hat{\Theta}_K^{(j)}(z_i) = \hat{X}_2^{(j)}(z_i) = W_{ij} \quad i=1, 2 \cdots f-1 & \quad j=0, 1 \cdots m_i-1 ; \quad \hat{\Theta}_K(z) \neq 0 \quad \forall \; z \in \overline{D} \quad (4-59)
\]
\[
\sum_{i=1}^{f-1} m_i = K. \text{ Suppose that } \hat{\Theta}_K(z_t) = \hat{X}_2(z_t) = W_{f0} \text{ - define }
\]
\[
\hat{\Theta}_{K+1}(z) := \left[ 1 + \mu_K B_K(z) \right]^p \hat{\Theta}_K(z)
\]
(4-60)

where \( B_K(z) = \prod_{i=1}^{f-1} \left( \frac{z-z_i}{1-z_i z} \right)^{m_i} \) is the all-pass function constructed from the zeros \( z_i, i=1, 2 \cdots f-1 \) with multiplicity \( m_i \).

To force \( \hat{\Theta}_{K+1}(z_t) = W_{f0} \) and be a unit, let
\[
\mu_K = \frac{\left\{ W_{f0}/\hat{\Theta}_K(z_t) \right\}^{1/p} - 1}{B_K(z_t)}
\]
(4-61)

\( W_{f0} > 0, \hat{\Theta}_K(z_t) > 0, \hat{\Theta}_K(z_t) \neq W_{f0} \) and \( B_K(z_t) \neq 0 \) imply that

\( |\mu_K| \neq 0 \) and is well-defined. Choose \( p \) integer large enough \( \exists |\mu_K| < 1. \)

\[
|\mu_K B_K(z)| < 1 \quad \forall \, z \in \mathbb{D} \Rightarrow \hat{\Theta}_{K+1}(z) \text{ is a unit, and}
\]
\[
\hat{\Theta}_{K+1}(z_i) = \hat{\Theta}_K^{(j)}(z_i) \quad i=1, 2 \cdots f-1 \quad \& \quad j=0, 1 \cdots m_i-1
\]
(4-62)

**Case (b): Multiple real zero**

Let \( \hat{\Theta}_K(z) \) be the solution at the \( K \)th iteration such that equation (4-59) is satisfied.

Let us also assume that \( \hat{\Theta}_K^{(j)}(z_t) = W_{fj} \quad j=0, 1 \cdots q-1 \) (where \( 1 \leq q < m_f \)). Define
\[
\hat{\Theta}_{K+1}(z) := \left[ 1 + \mu_K B_K(z) \right]^p \hat{\Theta}_K(z)
\]
(4-63)

where \( B_K(z) = \prod_{i=1}^{f-1} \left( \frac{z-z_i}{1-z_i z} \right)^{m_i} \left( \frac{z-z_f}{1-z_f z} \right)^q \) is the all-pass function constructed from the zeros \( z_i, i=1, 2 \cdots f-1 \) with multiplicity \( m_i \) and with \( z_f \) as a zero repeated \( q \) times.
To force $\hat{\Theta}_{k+1}(z_t) = W_{j}$ for $j=0,1 \cdots q$ (i.e. we have the same q conditions as before, plus one additional derivative condition) and satisfy equation (4-59) - consider the binomial expansion

$$\frac{d^{(j)}}{dz^{(j)}} \left[ 1 + \mu_k B_k(z) \right]^p = \frac{d^{(j)}}{dz^{(j)}} \left[ 1 + \sum_{r=1}^{p} C_r (\mu_k B_k) r \right] \tag{4-64}$$

For $z = z_i, i=1,2 \cdots f-1$ & $j=0,1 \cdots m_i-1$ this derivative is 1, ensuring equation (4-59) is satisfied. For $j=0,1 \cdots q-1$ the derivative above evaluated at $z_t$ will always be 1, giving $\hat{\Theta}_{k+1}(z_t) = W_{j}$. However, for $j=q$

$$\hat{\Theta}_{k+1}(z_t) = \hat{\Theta}_{k}(z_t) + p \mu_k \hat{\Theta}_{k}(z_t) B_k(z_t) \tag{4-65}$$

$$\therefore p \mu_k = \frac{W_{q} - \hat{\Theta}_{k}(z_t)}{W_{q} B_k(z_t)} \tag{4-66}$$

Choosing p integer large $\varepsilon |\mu_k| < 1$ guarantees that $\hat{\Theta}_{k+1}(z)$ is a unit.

**Remark**: If $z_t = z_{oo}$ then the 2nd component of $B_k(z)$ as an all-pass function reduces to a constant. However, we can always choose $\varepsilon \in \mathbb{R}$ and

$$B_{oo}(z) = \left[ \frac{z - z_{oo}}{1 - (z_{oo} - \varepsilon)z} \right]^q \varepsilon |z_{oo} - \varepsilon| < 1 \tag{4-67}$$

and p correspondingly $\varepsilon |\mu_k| < 1$.

**Case (c): Complex zeros (assumed to be in conjugate pairs)** -

Let $\hat{\Theta}_K(z)$ be the solution at the Kth stage of the iteration such that equation (4-59) is satisfied. Define
\[ \hat{\Theta}_{K+1}(z) := \left[ 1 + \mu_K \frac{z - \beta}{1 - \beta z} B_K(z) \right]^p \hat{\Theta}_K(z) \] (4-68)

where \( B_K(z) = \prod_{i=1}^{e-1} \left[ \frac{z - z_i}{1 - z_i z} \right]^{m_i} \), \( \text{Im}(z_i) > 0 \) and \( \mu, \beta, p \in \mathfrak{R} \) are to be found such that \( \hat{\Theta}_{K+1}(z_f) = W_{f0} \) and \( \hat{\Theta}_{K+1}(\bar{z}_f) = \bar{W}_{f0} \).

Then

\[ \mu_K \frac{z_f - \beta}{1 - \beta z_f} = \frac{\{W_{f0}/\hat{\Theta}_K(z_f)\}_{1/p} - 1}{B_K(z_f)} \] (4-69)

Let \( \gamma = \min \left| \frac{z_f - \beta}{1 - \beta z_f} \right| \quad \forall \beta \in (-1, 1). \) Choose \( p \) integer \( \varphi \)

\[ |\mu_K| \leq \frac{1}{\gamma} \left| \frac{\{W_{f0}/\hat{\Theta}_K(z_f)\}_{1/p} - 1}{B_K(z_f)} \right| < 1 \] (4-70)

and find \( \beta \in (-1, 1) \) such that \( \mu_K \in \mathfrak{R} \). This is always possible - if

\[ \alpha = \arg \left( \frac{\{W_{f0}/\hat{\Theta}_K(z_f)\}_{1/p} - 1}{B_K(z_f)} \right) \] (4-71)

\( \exists \beta_0 \in (-1, 1) \) such that \( \arg \left[ \frac{z_f - \beta_0}{1 - \beta_0 z_f} \right] = \alpha \) (see Proposition 4.2a).

Hence \( \hat{\Theta}_{K+1}(z_f) = \bar{W}_{f0} \), and since \( |\mu_K| < 1 \), \( \hat{\Theta}_{K+1}(z) \) is a unit.

For the situation involving repeated complex zeros, a similar construction can be utilized - except that \( B_K(z) \) contains zeros \( z_f \) with multiplicity \( q < m_f \). We have unit \( \hat{\Theta}(z) \) under conditions (i) and (iii) are satisfied.

\( \hat{\Theta}(z) \) will then be the denominator of a controller \( \bar{K}(z) \) in class \( \chi \) that is stable. \( \blacksquare \)
Proposition 4.2a

Suppose \( z_f \in D \) such that \( \text{Im}(z_f) > 0 \). Let \( \alpha = \arg \left( \frac{z_f - \beta}{1 - \beta z_f} \right) \).

Then for \( \beta : -1 \rightarrow 1 \), \( \alpha \) ranges from 0 to \( \pi \) radians.

**Proof**

Let \( z_f = a + jb \) where \( a > 0 \), \( b > 0 \) and let \( \beta : 0 \rightarrow 1 \) (i.e. \( 0 < \beta < 1 \)).

\[
\alpha_1 = \arg (z_f - \beta) = \tan^{-1} \left( \frac{b}{a - \beta} \right) > \alpha_0 = \tan^{-1} \left( \frac{b}{a} \right)
\]

\[
\alpha_2 = \arg (1 - \beta z_f) = \tan^{-1} \left( \frac{-\beta b}{1 - \beta a} \right) < 0 \text{  (assuming that } \tan^{-1}(.) \in (-\pi, \pi]\)
\]

\( \therefore \alpha = \alpha_1 - \alpha_2 > \alpha_0 \). As \( \beta \rightarrow 1 \) we have \( \alpha \rightarrow \pi \).

Hence for \( \beta : 0 \rightarrow 1 \), \( \alpha \) ranges from \( \alpha_0 \) to \( \pi \) radians.

Suppose \( \beta : 0 \rightarrow -1 \) or \( -1 < \beta < 0 \). Then \( \alpha_1 < \alpha_0 \) and \( \alpha_2 > 0 \).

\( \therefore \alpha = \alpha_1 - \alpha_2 < \alpha_0 \). As \( \beta \rightarrow -1 \) we have \( \alpha \rightarrow 0 \).

Hence for \( \beta : 0 \rightarrow -1 \), \( \alpha \) ranges from \( \alpha_0 \) to \( 0 \) radians.

A similar argument holds for the case \( a \leq 0 \), and hence the proposition follows.

**Remark**: A new procedure for interpolation with a unit in \( RH^\infty \) has been recently proposed in Ting et al. [48] which yields low order rational functions. This technique is computationally attractive as it appears to be easy to implement as a computer algorithm.
4.8. Nonlinear Optimization

The constructive procedure outlined above builds \( Q_1 = Q_0 + S \) such that \( K(s) \) is stable, and guarantees internal stability of the closed-loop system. To maintain the system performance level (in terms of \( \| \Phi \|_2 \)) we formulate and solve the following nonlinear optimization problem.

\[
S_x(s) = \sum_{i=1}^{N} \frac{x_{1i}s + x_{2i}}{x_{3i}s^2 + x_{4i}s + x_{5i}} \quad \text{where} \quad x_{3i} \geq 0, \ x_{4i}, \ x_{5i} > 0 \quad (4-72)
\]

and let \( x' = \left[ x_{11} \ldots x_{1N}, \ldots, x_{51} \ldots x_{5N} \right] \) be a \( 5N \) parameter vector. Let the anticlockwise semicircular contour \( \Omega \) be defined on the imaginary axis with radius \( R \to \infty \) enclosing \( \overline{H} \). By integrating along \( \Omega \) and applying Cauchy’s Residue Theorem [2] we get (the computation is given in the Appendix)

\[
\| S_x \|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x^*(j\omega)S_x(j\omega) \, d\omega \quad (4-73)
\]

\[
= \frac{1}{2\pi j} \int_{\Omega} S_x(-s)S_x(s) \, ds \quad (4-74)
\]

\[
= \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{x_{1i}x_{1j}\rho_3 + x_{2i}x_{2j}\rho_2 + \rho_1\rho_4}{\rho_2\rho_3 + \rho_4^2} := f(x) \quad (4-75)
\]

\[
\rho_1 = x_{1i}x_{2j} - x_{2i}x_{1j}, \quad \rho_2 = x_{3i}x_{4j} + x_{4i}x_{3j} \quad (4-76)
\]

\[
\rho_3 = x_{4i}x_{5j} + x_{5i}x_{4j}, \quad \rho_4 = x_{3i}x_{5j} - x_{5i}x_{3j} \quad (4-77)
\]
To ensure stability of the controller $K_x(s) = F_I(J, Q_x)$, denominator $\Theta_x(s)$ must be a Hurwitz polynomial -

$$\Theta_x(s) = X_2(s) - B_1(s)Q_0 - B_1(s)S_x(s)$$

$$= C_M(x)s^M + C_{M-1}(x)s^{M-1} + \cdots + C_1(x)s + C_0(x)$$

(4-78) (4-79)

\[ \therefore \text{coefficients } \begin{bmatrix} C_j(x) \end{bmatrix}^M_{j=1} \text{ must satisfy the well-known Routh-Hurwitz constraints - } \]

$$C_j(x) > 0 \quad j = 0, 1 \cdots M \quad \& \quad g_k \begin{bmatrix} C_j(x) \end{bmatrix} > 0 \quad k = 1, 2 \cdots M-1$$

(4-80)

Recall from equation (4-50) that $\| \Phi_x \|^2 = \| \Phi_0 \|^2 + \| S_x \|^2$. Then a constrained optimization problem to minimize the performance index can be formulated as

$$\min f(x) \quad \exists \quad x_{ji} \geq 0, x_{ii}, x_{i1} > 0, i = 1, 2 \cdots N; \quad x \in \mathbb{R}^{5N}$$

(4-81)

$$C_j(x) > 0; \quad g_k \begin{bmatrix} C_j(x) \end{bmatrix} > 0 \quad j = 0, 1 \cdots M, k = 1, 2 \cdots M-1$$

(4-82)

$f(x)$ is a smooth, scalar nonlinear objective function with nonlinear constraints. A feasible point $x_0$ can be generated utilizing the constructive method of Theorem 4.2 such that $S_{x_0}(s)$ is of the desired form. This is always possible since the expression developed for $\| S_x(s) \|$ is of sufficient generality to cover all finite pole combinations.

The sequential quadratic programming algorithm NPSOL [24] was used to search for optimal solutions to (4-81) in the constrained parameter space. As with most nonlinear optimization algorithms, the rate of convergence and the solution converged to depends on the initial starting point i.e. a local minimum is obtained. Our
experience indicates that the solutions obtained will always lie on the constraint boundary, which corresponds to placing the controller poles on the imaginary axis. This is intuitively very appealing, as we are trying to move away from the unique unstable optimal controller to optimal stable controllers that are possibly non-unique. One would naturally expect therefore to reach a point where the controller poles "move" across the stability boundary - in this case the jω-axis.

4.9. Conditional Stability

Suppose the plant is minimum phase, and the optimal stable controller being designed has poles on or close to the imaginary axis. The branches of the root-locus starting at these controller poles could easily cross over into the right half plane, resulting in a system that is conditionally stable. This conditional stability can be eliminated by moving the controller poles further back into the left half plane - away from the stability boundary. In terms of the Nyquist plot, this would be effectively adding a phase lead to the system such that the open-loop frequency response does not intersect the negative real axis (or the system has an infinite gain margin).

To push the controller poles back, we adopt the standard approach of shifting the imaginary axis (see p.257 of Ogata [40]). Let

\[ s = w - d := \xi(w) \quad \text{then} \quad \text{Re}(s) \geq -d \iff \text{Re}(w) \geq 0 \quad (4-83) \]

Transforming the denominator polynomial of the controller \( \Theta_x(s) \big|_{s=\xi(w)} \) the new stability polynomial is
\[ \tilde{\Theta}_\lambda(w) = D_M(x)w^M + D_{M-1}(x)w^{M-1} + \cdots + D_1(x)w + D_0(x) \]  \hspace{1cm} (4-84)

Applying the Routh-Hurwitz conditions

\[ D_j(x) > 0 \hspace{1cm} j=0, 1 \cdots M \hspace{1cm} \text{and} \hspace{1cm} g_k \left[ D_j(x) \right] > 0 \hspace{1cm} k=1, 2 \cdots M-1 \]  \hspace{1cm} (4-85)

are the new nonlinear constraints to be satisfied.

To generate an initial feasible point \( x_0 \) that satisfies equation (4-85), the constructive method of Theorem 4-2 can be utilized with the following difference. Define

\[ \Gamma = \{ s \in C \mid \text{Re}(s) \geq -d \} \]  \hspace{1cm} (4-86)

Since the region of instability for the controller is now \( \Gamma \) (and not \( \tilde{H} \)), the unit disc \( \overline{D} \) must be mapped into \( \Gamma \).

Let \( \Psi_1 = \xi \circ \Psi \) be the composition of conformal maps \( \Psi : \overline{D} \to \tilde{H} \) and \( \xi : \tilde{H} \to \Gamma \), then \( \Psi_1 : \overline{D} \to \Gamma \) is the conformal map desired. \( \hat{\Theta}_1(z) = \Theta(s) \mid_{s=\Psi_1(z)} \) is the transformed function to be constructed such that \( \hat{\Theta}_1(z) \) is a unit in \( \overline{D} \).

### 4.10. Illustrative Examples

#### 4.10.1. An LQG Problem

Consider the observer-based controller obtained in the following example taken from Doyle and Stein [15]. The plant \( G(s) = \frac{s+2}{(s+1)(s+3)} \) has the realization
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} d = Ax + Bu + Md \\
(4-87)
\]

\[
y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + n = Cx + Nn \\
(4-88)
\]

where \(d(t), n(t)\) are state excitation and measurement noise inputs. Suppose \(d(t), n(t)\)
are modelled as white-noise stochastic processes with unit intensity -

\[
\mathbb{E}\{d(t)\} = \mathbb{E}\{n(t)\} = 0 ; \ \mathbb{E}\{d(t_1) d(t_2)\} = \mathbb{E}\{n(t_1) n(t_2)\} = \delta(t_1 - t_2) \\
(4-89)
\]

and the measure of performance is the square integrable criterion

\[
\min J(x, u) = \operatorname{Lim}_{\tau \to \infty} \frac{1}{\tau} \mathbb{E}\left[ \int_0^\tau x^T(t)q^T qx(t) + u^T(t)R^T R u(t) \, dt \right] \\
(4-90)
\]

\[
= \operatorname{Lim}_{t \to \infty} \mathbb{E}\{ r^T(t)r(t) + v^T(t)v(t) \} \quad (\text{see p. 264 of [34]})) \quad (4-91)
\]

where \(r = Qx, v = Ru, Q = 4\sqrt{5} \begin{bmatrix} \sqrt{35} & 1 \end{bmatrix}, R = 1 \).

Let \(e = \begin{bmatrix} d \\ n \end{bmatrix}\) be the exogenous inputs, and \(z = \begin{bmatrix} r \\ v \end{bmatrix}\) be the regulated outputs. Then the problem can be represented in the configuration of Figure 4-1 (with \(\phi = (sI-A)^{-1}\)) as

\[
\begin{bmatrix} r \\ v \\ y \end{bmatrix} = \begin{bmatrix} Q\phi M & 0 & Q\phi B \\ 0 & 0 & R \\ C\phi M & N & C\phi B \end{bmatrix} \begin{bmatrix} d \\ n \\ u \end{bmatrix} \\
(4-92)
\]

Recall that \(P_{12}(s)\) and \(P_{21}(s)\) must have full column and row rank - this is satisfied by
having \(R > 0\) and \(N\) being full rank. This assumption means in essence that the input
cost weighting matrix \(W_u = R^T R\) must be positive definite and that all components of
the observed variable $y(t)$ are corrupted by measurement noise. Then $P(s)$ has the realization

$$P(s) = \begin{bmatrix} A & M & 0 & B \\ Q & 0 & 0 & 0 \\ 0 & 0 & 0 & R \\ C & 0 & N & 0 \end{bmatrix}$$

(4-93)

The particular choice of state feedback matrix $F = [ -50 \ -10 ]$ and output injection matrix $H = [ -30 \ 50 ]$ such that $A+BF$ and $A+HC$ are stable are obtained by solving the ARE's (4-30) and (4-32). The minimization problem $\min J(x,u)$ is equivalent to the $H^2$-optimization problem $\min_{Q \in \mathbb{R}^{2x2}} \| T(s) - U(s)Q(s)V(s) \|_2$ and has the solution

$$Q_{opt}(s) = Q_0 = 0$$

(4-94)

with $\| . \|_2 = 493.76$. The resulting $H^2$-optimal controller (which is shown to be a combination of state feedback/Kalman filter in [12])

$$K_{opt}(s) = F_1(J_0) = \frac{1000(s + 2.6)}{(s^2 + 24s - 797)}$$

(4-95)

is unstable with a pole at 18.68. The closed-loop system is conditionally stable with a gain margin of 0.8 @ $\omega=4.57$ and a phase margin of 14.8° @ $\omega=12.6$ (from the Nyquist plot).

To find the stable optimal controller - let $Q_1(s) = Q_0 + S(s)$.

Then $\Theta = X_2 - B_1S$ where $X_2 = \frac{s^2 + 24s - 797}{s^2 + 14s + 53}$ and $B_1 = \frac{s+2}{s^2 + 14s + 53}$.

Let $\Psi : s = \frac{1-z}{1+z}$ be the mapping from $\bar{D}$ to $\bar{H}$. 
\[ S \in \mathbb{RH}^2 \iff \hat{\Theta}(-1) = 1, \hat{\Theta}'(-1) = 5. \] Utilizing the construction procedure of Theorem 4.2, we get

\[ \hat{\Theta}_{01}(z) = \frac{13z + 15}{3z + 5} \quad (4-96) \]

which is a unit in \( \mathbb{RH}_w \). Consequently

\[ S_{x_{01}} = \frac{-950s - 3930}{s^2 + 6s + 8}, \quad \| S_{x_{01}} \|_2 = 485.89 \quad (4-97) \]

This gives an initial, feasible solution \( x'_{01} = \begin{bmatrix} -950 & -3930 & 1 & 6 & 8 \end{bmatrix} \) for the constrained optimization problem

\[ \min_{x \in \mathbb{R}^5} f(x) = \frac{x_1^2 x_5 + x_2^2 x_3}{2x_3 x_4 x_5} \quad \text{\ s.t. } x_3 \geq 0, x_4, x_5 > 0 \quad (4-98) \]

\[ C_0(x), C_1(x), C_2(x), C_3(x), C_4(x) > 0 \quad (4-99) \]

\[ C_2(x)C_3(x) - C_1(x)C_4(x) > 0 \quad (4-100) \]

\[ C_1(x)C_2(x)C_3(x) - C_1^2(x)C_4(x) - C_0(x)C_3^2(x) > 0 \quad (4-101) \]

where \( C_0(x) = -2x_2 - 797x_5 \quad (4-102) \]

\[ C_1(x) = -2x_1 - x_2 - 797x_4 + 24x_5 \quad (4-103) \]

\[ C_2(x) = -x_1 - 797x_3 + 24x_4 + x_5 \quad (4-104) \]

\[ C_3(x) = 24x_3 + x_4 \quad (4-105) \]
\[ C_4(x) = x_3 \] (4-106)

The nonlinear programming algorithm NPSOL converges to the solution

\[ x' = \begin{bmatrix} -555.12 & -3774.2 & 0.89840 & 6.4138 & 9.4711 \end{bmatrix} \] (4-107)

which gives \( \| S \|_2 = 379.46 < \| S_{x_0} \| \). The controller is found to be

\[ K_{01}(s) = \frac{382.10}{s^2} \frac{s^3 + 8.0253s^2 + 27.339s + 38.751}{s^2 + 31.139s + 2.7841} \] (4-108)

Notice the double integrator - 2 poles are located exactly on the stability boundary to give a stable controller. The closed loop system has \( \| \Phi \|_2 = 874.22 \) and improved stability margins: gain margin = 0.292 @ \( \omega = 4.46 \) and phase margin = 39.8° @ \( \omega = 11.7 \).

Various feasible solutions \( \hat{\Theta}_0(z) \) were used to provide an initial starting point for the algorithm. For instance, \( \hat{\Theta}_{02}(z) = 5z + 6 \) leads to

\[ S_{x_0} = \frac{-980s - 1380}{s^2 + 3s + 2} \text{, } \| S_{x_0} \| = 564.66 \] (4-109)

NPSOL converges to the solution \( x' = \begin{bmatrix} -1233.3 & -1529.9 & 1.6959 & 5.1301 & 5.1301 \end{bmatrix} \)

with \( \| S \|_2 = 383.19 < \| S_{x_0} \|_2 \). The corresponding controller is

\[ K_{02}(s) = \frac{272.75}{s^2} \frac{s^3 + 6.6505s^2 + 15.907s + 11.657}{s^2 + 27.025s + 5.1108} \] (4-110)

Once again, there is a double integrator: very similar in characteristics to the system obtained with the first controller. The resulting system has \( \| \Phi \|_2 = 876.95 \), gain margin = 0.167 @ \( \omega = 2.81 \) and phase margin = 43.8° @ \( \omega = 9.87 \).
In the above designs, even though both the plant and the controller are minimum phase, the system is conditionally stable. An examination of the root-locus plots given in Figures 4-4 and 4-5 (generated using the program CC [47]) shows that the branches originating at the poles at the origin go out into the right half plane, then come back in. To push these poles into the left half plane the stability boundary is moved back to $s=-0.5$. The new denominator polynomial for the controller gives rise to the nonlinear constraints:

\[ D_0(x) , D_1(x) , D_2(x) , D_3(x) , D_4(x) > 0 \]  
\[ D_2(x)D_3(x) - D_1(x)D_4(x) > 0 \]  
\[ D_1(x)D_2(x)D_3(x) - D_1^2(x)D_4(x) - D_0(x)D_3^2(x) > 0 \]  

where \( D_0(x) = 0.75x_1 - 1.5x_2 - 202.19x_3 + 404.38x_4 - 808.75x_5 \)  
\[ D_1(x) = -x_1 - x_2 + 814.5x_3 - 820.25x_4 + 23x_5 \]  
\[ D_2(x) = -x_1 - 831.5x_3 + 22.5x_4 + x_5 \]  
\[ D_3(x) = 22x_3 + x_4 \]  
\[ D_4(x) = x_3 \]  

With initial condition \( x'_{01} \) the algorithm NPSOL converges to

\[ x' = [-1101.2 \  -3972.4 \  1.4901 \  7.9440 \  9.9460] \]  

(4-119)
with $\| S \|_2 = 388.70 > 379.46$ (the optimum when the poles were free to move to the $j\omega$-axis).

\[
K_{s01} = \frac{260.96}{(s + 0.5)^2} \frac{s^3 + 8.8489s^2 + 29.336s + 35.856}{s^2 + 28.331s + 48.087}
\]  

(4-120)

has 2 poles at $s = -0.5$, the new stability boundary. From a root-locus/Nyquist plot analysis, it is seen that the conditional stability has been eliminated: gain margin is infinite, phase margin=46.1° @ $\omega = 9.76$ (Figure 4-6).

With initial condition $x'_{02} = [-980 -1380 1 3 2]$ we obtain

\[
x' = [-1291.5 -1559.3 1.6818 5.2550 3.9015]
\]  

(4-121)

giving $\| S \|_2 = 392.00 > 383.19$ (the $j\omega$-axis optimum)

\[
K_{s02} = \frac{232.07}{(s + 0.5)^2} \frac{s^3 + 7.4362s^2 + 19.095s + 14.084}{s^2 + 26.125s + 21.866}
\]  

(4-122)

The Nyquist plot (Figure 4-7) shows there is no conditional stability: gain margin is infinite, phase margin=48° @ $\omega = 9.18$. 
Figure 4.4: Root-locus for example 4.10.1 with controller $K_{01}$. 

REAL S
Figure 4.5: Root-locus for example 4.10.1 with controller $K_{02}$. 
Figure 4.6: Nyquist plot for example 4.10.1 with controller $K_{s01}$. 
Figure 4.7: Nyquist plot for example 4.10.1 with controller $K_{o2}$. 
4.10.2. Measurement-noise effect minimization

Consider the feedback configuration shown below in Figure 4-8, where \( n(s) \) is the measurement noise present at the output.

\[
\begin{align*}
\text{Figure 4-8. Measurement noise}
\end{align*}
\]

Assume that \( n(s) \) is a transient noise input: it can be regarded as a finite-energy signal and thus modelled as an \( L^2 \) function. Let the signal at the plant input due to this measurement noise be \( u(s) \), then

\[
u(s) = K(1+GK)^{-1} n(s) := \psi(s) n(s) \tag{4-123}\]

Suppose we wish to minimize the peak amplitude of this signal \( \bar{u}(t) \) for all time over all possible noise inputs -

\[
\min_{\forall K(s) \text{ stabilizing}} \sup_{\| \bar{u}(t) \|_{\infty} \in L^2} \| \bar{u}(t) \|_{\infty} \tag{4-124}\]

This is equivalent to the \( H^2 \)-optimization problem

\[
\min_{\forall K(s) \text{ stabilizing}} \| \psi(s) \|_{2} \tag{4-125}\]

Let \( n(s) \) and \( u(s) \) be the exogenous and control inputs, and let \( u(s) \) and \( y(s) \) be the regulated and measured outputs respectively. Then the problem can be expressed in
the configuration of Figure 4-1 as

\[
\begin{bmatrix}
  u(s) \\
  Y(s)
\end{bmatrix} = \begin{bmatrix} 0 & I \\ I & G(s) \end{bmatrix} \begin{bmatrix} n(s) \\
  u(s)
\end{bmatrix} = P(s) \begin{bmatrix} n(s) \\
  u(s)
\end{bmatrix}
\]

(4-126)

Let \( G(s) = \frac{0.8s + 0.2}{s - 1} \), then \( P(s) \) has the minimal realization

\[
P = \begin{bmatrix} 1 & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 1 & 0.8
\end{bmatrix}
\]

(4-127)

Solving for the state-feedback matrix and output-injection matrix from the ARE's in Section 4.4, we get the particular solutions \( F = -2 \) and \( H = -2 \). Realizations for the coprime factorizations and Bezout identity matrices are computed as

\[
\begin{bmatrix} X_1 & Y_1 \\
  -B_2 & A_2
\end{bmatrix} = \begin{bmatrix} -1 & 0.6 & -2 \\
  -2 & 1 & 0 \\
  1 & 0.8 & 1
\end{bmatrix}
\]

(4-128)

\[
\begin{bmatrix} A_1 & -Y_2 \\
  B_1 & X_2
\end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\
  -2 & 1 & 0 \\
  -0.6 & 0.8 & 1
\end{bmatrix}
\]

(4-129)

Utilizing the YJBK parameterization of all stabilizing controllers, the optimization problem is equivalent to

\[
\min_{Q(s) \in \mathcal{RH}^+} \| -4 \frac{s-1}{(s+1)^2} - \frac{s-1}{s+1} Q \frac{s-1}{s+1} \|_2
\]

(4-130)

This has the solution \( Q_{opt}(s) = Q_0 = 0 \) and \( || \psi_{opt} || = 8 \).
The $H^2$-optimal controller can be represented in the configuration of Figure 4-3 with

$$J = \begin{bmatrix} 0.2 & -2 & -0.6 \\ -2 & 0 & 1 \\ -0.6 & 1 & 0.8 \end{bmatrix}$$

Then $K_{\text{opt}} = F_1(J, 0) = \frac{4}{s-0.2}$ is unstable with a pole at $s=0.2$. The closed-loop system is conditionally stable with gain margin $= 0.375$ @ $\omega = 0.707$ and phase margin $= 63.4^\circ$ @ $\omega = 3.05$.

Proceeding as in Section 4-5 to find a stable controller that maintains performance, let $Q_1(s) = Q_0 + S(s)$. Then $\Theta = X_2 - B_1 S = \frac{s-0.2}{s+1} - \frac{0.8s+0.2}{s+1} S$.

Let $\Psi : z = \frac{1-s}{1+s}$ be the mapping from $H$ to $D$.

$\therefore S \in RH^2 \iff \hat{\Theta}(-1)=1$. Let $\hat{\Theta}(z) = 1$, we get

$$S_0 = \frac{-1.5}{s+0.25} \quad , \quad \| S_0 \|_2 = 2.1213$$

Let $x'_0 = \begin{bmatrix} -1.5 \\ 0.25 \end{bmatrix}$ be the initial feasible solution to the nonlinear optimization problem

$$\min_{x \in \mathbb{R}^2} f(x) = \frac{x_1^2}{2x_2} \quad \exists \ x_2 > 0 \ ; \ C_0(x) , C_1(x) > 0$$

where $C_0(x) = -0.2x_1 - 0.2x_2$ \quad , \quad $C_1(x) = x_2 - 0.8x_1 - 0.2$.

Since the constraints are linear, the solution is unique -
\[ x' = \begin{bmatrix} -0.1111 \ 0.1111 \end{bmatrix} \]  \hspace{1cm} (4-134)

giving \( \| S \|_2 = 0.2357 \). The corresponding controller

\[ K_{s0}(s) = \frac{5}{9} \frac{7s+1}{s^2} \]  \hspace{1cm} (4-135)

has poles exactly on the stability boundary, and would result in a system with an \( H^2 \)-performance index of 8.2357. The closed-loop system has the improved stability margins of \( GM=0.351 \) @ \( \omega=0.655 \) and \( PM=63.7^\circ \) @ \( \omega=2.96 \).

Suppose it is desired to move the controller poles further back into the left half plane, enabling it to have a degree of relative stability. Adopting the standard approach of shifting the \( j\omega \)-axis, we let \( s=w-d \) and apply the Routh-Hurwitz criterion to the new stability polynomial in \( w \). Choosing \( d=0.2 \), we obtain

\[ \Theta(w) = w^2 + (x_2 - 0.8x_1 - 0.6)w + (-0.4x_2 - 0.04x_1 + 0.08) \]

The optimization problem

\[ \min_{x \in \mathbb{R}^2} f(x) = \frac{x_1^2}{2x_2} \] \( \exists \ x_2 > 0 \) ; \( D_0(x) \), \( D_1(x) > 0 \)  \hspace{1cm} (4-136)

has the solution \( x' = \begin{bmatrix} -0.4444 \ 0.2444 \end{bmatrix} \) which gives \( \| S \|_2 = 0.6356 \). The controller

\[ K_{s0}(s) = \frac{160}{45} \frac{s+0.4}{(s+0.2)^2} \]  \hspace{1cm} (4-137)

has poles at \( s=-0.2 \) and results in a sub-optimal system with a performance index of 8.6356. The stability margins are \( GM=0.294 \) @ \( \omega=0.429 \) and \( PM=64.3^\circ \) @ \( \omega=2.69 \).
Suppose that the order of \( S(s) \) is increased -

\[
S_x = \frac{x_1s + x_2}{x_3s^2 + x_4s + x_5} \quad \forall x_3 > 0, \quad x_4, x_5 > 0
\]  

This leads to the nonlinear optimization problem

\[
\min_{x \in \mathbb{R}} \quad f(x) = \frac{x_1^2x_5 + x_2^2x_3}{2x_3x_4x_5} \quad \Rightarrow \quad C_1(x), C_2(x), C_3(x), C_4(x), C_5(x) > 0
\]

where \( C_1(x) = -0.2x_2 - 0.2x_5 \)  

\[
C_2(x) = -0.2x_1 - 0.8x_2 - 0.2x_4 + x_5
\]

\[
C_3(x) = -0.8x_1 - 0.2x_3 + x_4
\]

\[
C_4(x) = x_3
\]

\[
C_5(x) = C_2(x)C_3(x) - C_1(x)C_4(x)
\]

NPSOL converges to \( x' = [ -1.7588 \quad -0.26218 \quad 27.627 \quad 4.1184 \quad 0.26818 ] \) giving \( \| S \|_2 = 0.21313 \). The corresponding stable controller

\[
K_{s0}(s) = \frac{3.9363s^2 + 0.65046s + 0.04745}{s^3}
\]

results in an overall performance index of 8.21313.
CHAPTER 5

$H^2$-OPTIMIZATION: the multivariable case

5.1. Introduction

In this chapter, we solve the problem of designing a stable controller that optimizes the $H^2$-performance of a multi-input multi-output system in the closed-loop configuration of figure 4-1. The problem formulation is as given in Sections 4.1 through 4.6, and a strategy similar to that employed in the scalar case is utilized. Necessary and sufficient conditions are derived for the existence of a stable controller in class $\chi$. The general optimization problem over all such controllers (equation 4-51) is shown to be equivalent to minimization of a nonlinear objective functional subject to nonlinear constraints. An algorithm is developed to find an initial, feasible solution for this nonlinear optimization problem, which is then tackled using NPSOL [24].

5.2. Multivariable Systems

Let the plant model $G(s)$ be a pxm matrix of rational functions i.e. $G(s) \in \mathbb{R}(s)^{p \times m}$ and let $[s_i]_{i=1, 2 \cdots M_r}$ be the real blocking zeros [18] of $G(s)$ in $\hat{H}$ i.e. $G(s_i) = 0$ for $i=1, 2 \cdots M_r$. It is well-known [55] that there exists a stable controller $K(s)$ that stabilizes the system $G(s)$ if and only if the real blocking zeros and poles of

\[ 104 \]
G(s) in $\tilde{H}$ exhibit the parity interlacing property (p.i.p.) - i.e. between any two adjacent zeros $s_i, s_{i+1}$ there exist an even number of poles.

Since $K(s) = (Y_2 + A_1Q)(X_2 - B_1Q)^{-1} = N_K D_K^{-1}$ is a minimal MFD (matrix fraction description) of the controller, stability of $K(s)$ is equivalent to having $D_K(s)$ be a unit in $RH^\infty$ or $\Theta(s) := |X_2 - B_1Q| \neq 0 \ \forall \ s \in \tilde{H}$. The following theorem gives necessary and sufficient conditions for the existence of a stable controller in class $\chi$, and provides an equivalent characterization of the p.i.p. in terms of the denominator determinant $\Theta(s)$. The proof of the sufficiency requires an algorithm for constructing one particular $Q(s) \in RH^\infty$ that makes $\Theta(s)$ be a unit: we utilize the one from Theorem 2 of Youla et al. [55]

As usual, let $\Psi$ be the conformal mapping from $\tilde{D}$ to $\tilde{H}$. Then transforming from the s-plane to the z-domain for convenience, we have $\hat{\chi}(z) = \chi(s)|_{s=\Psi(z)}$, $z_i = \Psi^{-1}(s_i)$ and $z_\infty = \Psi^{-1}(\infty)$ is the inverse image of $s=\infty$.

**THEOREM 5.1**

There exists a stable controller $\hat{K}(z)$ in class $\chi$ that internally stabilizes the closed-loop system if and only if the numbers $\{ \hat{\Theta}(z_i), \hat{\Theta}(z_\infty) \ i=1, 2 \cdots M_r \}$ have the same sign.

**PROOF**

*Necessity* - Let controller $\hat{K}(z)$ in class $\chi$ be stable. Since $\hat{\Theta}(z) \neq 0 \ \forall \ z \in \tilde{D}$, $\hat{\Theta}(z_i) = |\hat{X}_2(z_i)| \ i=1, 2 \cdots M_r$ and $\hat{\Theta}(z_\infty)$ have the same sign.

**Remark:** To show that this condition implies the parity interlacing property, recall
\[ \hat{A}_2 \hat{X}_2 + \hat{B}_2 \hat{Y}_2 = I. \] At the blocking zero \( z_i \), \( |\hat{X}_2(z_i)| = \frac{1}{|\hat{A}_2(z_i)|} \).

\[ \therefore |\hat{A}_2(z_i)| \quad i=1, 2 \cdots M_r \text{ have the same sign. Also, } |\hat{A}_2(z)| = 0 \text{ at the poles of } \hat{G}(z). \]

\[ \therefore \text{The blocking zeros and poles of plant } \hat{G}(z) \text{ in } [-1, 1] \text{ possess the p.i.p.} \]

**Sufficiency** - Let \{ \( \hat{\Theta}(z_i) \), \( \hat{\Theta}(z_{\infty}) \) \( i=1, 2 \cdots M_r \) \} be nonzero and have the same sign.

Let \( \hat{Q}(z) = \hat{Q}_1(z) = \hat{Q}_0 + (z - \infty) \hat{Q}_1(z) \) where \( \hat{Q}_1(z) \in \text{RH}_{\infty} \).

\[ \therefore D_K = (\hat{X}_2 - \hat{B}_1 \hat{Q}_0) - (z - \infty) \hat{B}_1 \hat{Q}_1 = \hat{X}_2 - \hat{B}_1 \hat{Q}_1 \] (5-1)

and \( \hat{\Theta} = |\hat{X}_2| \ |I - \hat{X}_2^{-1} \hat{B}_1 \hat{Q}_1| = |I + \hat{HQ}_1| \). Since

\[
\begin{bmatrix}
\hat{A}_2 & \hat{B}_2 \\
-\hat{\hat{Y}}_1 & \hat{\hat{X}}_1
\end{bmatrix}
\begin{bmatrix}
\hat{X}_2 - \hat{B}_1 \hat{Q}_0 & -\hat{B}_1 \\
\hat{Y}_2 + \hat{A}_1 \hat{Q}_0 & \hat{A}_1
\end{bmatrix}
= 
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\] (5-2)

matrices \( \hat{X}_2 - \hat{B}_1 \hat{Q}_0 \), \( \hat{B}_1 \) are left coprime. Together with the hypothesis on \( \hat{\Theta}(z_{\infty}) \), we have that \( \overline{X}_2 \), \( \overline{B}_1 \) are also left coprime.

Hence \( \hat{H} = -\overline{X}_2^{-1} \overline{B}_1 \) is a minimal MFD and \( |\overline{X}_2| = 0 \) at the poles of \( \hat{H}(z) \).

Now \{ \( z_i \), \( z_{\infty} \) \( i=1, 2 \cdots M_r \) \} are the blocking zeros of \( \hat{H}(z) \), and \( |\overline{X}_2(z_i)| = \hat{\Theta}(z_i) \) and \( |\overline{X}_2(z_{\infty})| = \hat{\Theta}(z_{\infty}) \) have the same sign. It follows that \( \hat{H}(z) \) has the parity interlacing property.

Let the Smith-McMillan form for \( \hat{H}(z) \) be

\[
\hat{H}(z) = U(z) \begin{bmatrix} \Omega_1(z) & 0 \\ 0 & 0 \end{bmatrix}_{mn} V(z) = U(z) \Omega(z) V(z)
\] (5-3)

where \( \Omega_1(z) = \text{diag} \begin{bmatrix} \varepsilon_k(z) \\ \psi_k(z) \end{bmatrix} \) \( k=1, 2 \cdots r \); \( \varepsilon_k \) and \( \psi_k \) are relatively prime.
From the divisibility properties of the numerator and denominator polynomials

\[ \varepsilon_k(z)/\varepsilon_{k+1}(z), \quad \psi_{k+1}(z)/\psi_k(z) \quad k=1, 2 \cdots r-1 \]  \hspace{1cm} (5-4)

we see that \( \phi(z) = \prod_{k=1}^{r} \psi_k(z) \) consists of the poles and \( \varepsilon_1(z) \) consists of the blocking zeros of \( \hat{H}(z) \). Hence, the system \( h(z) := \frac{\varepsilon_1(z)}{\phi(z)} \) possesses the parity interlacing property.

It is well-known [55] that there exists a stable, stabilizing controller \( q(z) \) for plant \( h(z) \) such that \( 1 + h(z)q(z) = \frac{u_n(z)}{\phi(z)} \) where \( u_n(z) \) is a unit in \( \mathbb{R}H_\infty \). This can be done by constructing unit \( u_n(z) \) such that \( u_n(z_i) = \phi(z_i) \) for \( i=1, 2 \cdots M_r \) and \( u_n(z_\infty) = \phi(z_\infty) \).

We now utilize the constructive procedure in Theorem 2, Youla et al. [55] for finding \( \hat{Q}_1(z) \in \mathbb{R}H_\infty \) such that \( |I + \hat{H}(z)\hat{Q}_1(z)| = 1 + h(z)q(z) \). The following polynomials can be shown to be coprime by the divisibility properties of \( \varepsilon_k(z) \) and \( \psi_k(z) \):

Let \( \theta_1(z) = \psi_2(z)\psi_3(z)\psi_4(z) \cdots \psi_r(z) \)  \hspace{1cm} (5-5)

\[ \theta_2(z) = \varepsilon_2(z)\psi_3(z)\psi_4(z) \cdots \psi_r(z) \]  \hspace{1cm} (5-6)

\[ \theta_3(z) = \varepsilon_2(z)\varepsilon_3(z)\psi_4(z) \cdots \psi_r(z) \] until \hspace{1cm} (5-7)

\[ \theta_r(z) = \varepsilon_2(z)\varepsilon_3(z)\varepsilon_4(z) \cdots \varepsilon_r(z) \]  \hspace{1cm} (5-8)

Hence, there exist polynomials \( b_k(z) \) \( k=1, 2 \cdots r \) such that \( \sum_{k=1}^{r} b_k(z)\theta_k(z) = 1 \). Sup-
pose \( M(z) = 
\begin{bmatrix}
M_1(z) & 0 \\
0 & 0
\end{bmatrix} \in \mathbb{R}^{m \times n} \) where

\[
M_1(z) = 
\begin{bmatrix}
q(z)b_1(z) & q(z)b_2(z) & \cdots & q(z)b_r(z) \\
-1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & -1
\end{bmatrix} \in \mathbb{R}^{r \times r}
\] (5-9)

Let \( Q_1(z) = V^{-1}(z)M(z)U^{-1}(z) \), then \( Q_1(z) \) is analytic and such that

\[ |I_m + \hat{H}(z)Q_1(z)| = |I_r + \Omega_1(z)M_1(z)| = 1 + h(z)q(z). \]

\[ \therefore |\tilde{X}_2| |I + \hat{H}Q_1| \text{ is a unit in } RH_\infty, \text{ and this choice of } \hat{Q}(z) = Q_0 + (z - z_\omega)Q_1(z) \]
gives a stable, stabilizing controller in class \( \chi \). ■

5.3. Nonlinear Optimization

The optimization problem of equation (4-52) in Section 4.6 is equivalent to

\[
\min_{\tilde{S} \in RH_{\text{exp}}} \| \tilde{S} \|_2 \quad \exists \quad |X_3 - B_1S| \neq 0 \quad \forall s \in \overline{H}
\] (5-10)

where \( X_3(s) := X_2(s) - B_1(s)Q_0 \) is known.

Let

\[
f_{ij}(s) = \sum_{k=1}^{N_d} \frac{a_k(i,j)s + b_k(i,j)}{c_k(i,j)s^2 + d_k(i,j)s + e_k(i,j)}
\] (5-11)

be the \((i, j)\) th element of \( \tilde{S}(s) \) for \( i = 1, 2 \ldots m; j = 1, 2 \ldots p \).

Then

\[
\| \tilde{S}(s) \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left[ \tilde{S}^*(j\omega)\tilde{S}(j\omega) \right] d\omega
\] (5-12)
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{p} |f_{ij}(j\omega)|^2 \, d\omega \]

\[ \sum_{i=1}^{m} \sum_{j=1}^{p} \|f_{ij}(s)\|^2 := f(x) \]

Dropping the parameter location pointer \((i, j)\) for convenience, we have that

\[ \|f_{ij}(s)\|^2 = \sum_{i=1}^{N_{ij}} \sum_{k=1}^{N_{ij}} \frac{a_k a_i \rho_3 + b_k b_i \rho_2 + \rho_1 \rho_4}{\rho_2 \rho_3 + \rho_4^2} \]

\[ \rho_1 = a_k b_l - a_l b_k, \quad \rho_2 = c_k d_l - c_l d_k \]

\[ \rho_3 = d_k e_l + d_l e_k, \quad \rho_4 = c_k e_l - c_l e_k \]

\[ \Theta(x) = |X_3(s) - B_1(s)X(s)| \] is a function of coefficients \(x := (a_k, b_k, c_k, d_k, e_k)\)

for \(i=1, 2 \ldots m\); \(j=1, 2 \ldots p\); \(k=1, 2 \ldots N_{ij}\) and is computed with the aid of a symbolic manipulator program such as MACSYMA [26]. Suppose

\[ \Theta(x) = C_M(x)s^M + C_{M-1}(x)s^{M-1} + \cdots + C_1(x)s + C_0(x) \]

The Routh-Hurwitz conditions for the stability of \(\Theta(x)\) give rise to nonlinear constraints on the coefficients \((a_k, b_k, c_k, d_k, e_k)\) of the form

\[ C_j(x) > 0 \quad j=0, 1 \cdots M \quad \text{&} \quad g_k \left[ C_j \right] > 0 \quad k=1, 2 \cdots M-1 \]

The resulting optimization problem then is to minimize the performance index \(f(x)\) such that equation (5-19) is satisfied. This was done using the sequential quadratic programming algorithm NPSOL [24] to search in the constrained parameter space.
5.4. Initial Condition

To find an initial feasible point for $S(s)$ to start the nonlinear programming algorithm, we utilize the following convenient choice derived from the state-space framework of the problem. Recall from equation (4-24) that

$$
\begin{bmatrix}
B_1(s) & X_2(s)
\end{bmatrix}
= \begin{bmatrix}
A + B_2F & B_2 & -H \\
C_2 + D_{22}F & D_{22} & I
\end{bmatrix}
$$

(5-20)

It is easily shown that $X_2^{-1}B_1(s)$ has the representation

$$
X_2^{-1}B_1(s) = \begin{bmatrix}
A + B_2F + HC_2 + HD_{22}F & B_2 + HD_{22}F \\
C_2 + D_{22}F & D_{22}
\end{bmatrix}
$$

(5-21)

Suppose we wish to stabilize the system $T := -X_2^{-1}B_1(s)$ with a stable, strictly proper "controller" $S(s)$. Since $X_2(s), B_1(s)$ are left coprime, the closed-loop characteristic polynomial of this fictitious system is given by

$$
\phi(s) = |X_2(s)| |I - X_2^{-1}B_1(s)S(s)|
$$

(5-22)

Consider the general $H^2$-optimization problem in equation (4-34). Generically, the $H^2$-optimal solution $Q_{opt}(s) = Q_0$ is always zero since matrix $R$ is strictly proper - if not, $\| R_{12} \|_2, \| R_{21} \|_2$ and $\| R_{22} \|_2$ would be infinite. The constraint of stability of the controller $K(s)$ as given in equation (5-10) $|X_3 - B_1S| \neq 0 \forall s \in \hat{H}$ is then equivalent to the above stabilization problem in equation (5-22).
Let us choose "controller" $S(s)$ with the realization

$$S(s) = \begin{bmatrix} A & H \\ -F & 0 \end{bmatrix} \tag{5-23}$$

The closed-loop system matrix $A(T, S)$

$$A(T, S) = \begin{bmatrix} A - HD_{22}F & HC_2 + HD_{22}F \\ -B_2F + HD_{22}F & A + B_2F + HC_2 + HD_{22}F \end{bmatrix} \tag{5-24}$$

is seen (under a similarity transformation) to have its eigenvalues at $A + B_2F$ and $A + HC_2$, which are stable. For this choice of $S(s)$ to be stable, the original system matrix $A$ must be stable.

5.5. Illustrative Example

We solve the following LQG example to illustrate the mechanics of solving a MIMO problem - in fact, this is a "multivariable" version of the example in Section 4.10.1. The plant $G(s)$ has the realization

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} d = Ax + Bu + Md \tag{5-25}$$

$$y = \begin{bmatrix} 2 & 1 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} n = Cx + Nn \tag{5-26}$$

where $d(t), n(t)$ are state excitation and measurement noise inputs. Suppose $d(t), n(t)$ are modelled as white-noise stochastic processes with unit intensity -
\[ E\{d(t)\} = E\{n(t)\} = 0 \quad ; \quad E\{d(t_1) \, d(t_2)\} = E\{n(t_1) \, n(t_2)\} = \delta(t_1 - t_2) \]  

(5-27)

and the measure of performance is the square integrable criterion

\[ \min J(x, u) = \lim_{\tau \to \infty} \frac{1}{\tau} \mathbb{E} \left[ \int_0^\tau r^T(t) r(t) + v^T(t) v(t) \, dt \right] \]  

(5-28)

where \( r = Qx, v = Ru, Q = 4\sqrt{5} \left[ \begin{array}{cc} \sqrt{35} & 1 \end{array} \right], R = I_2 \).

Let \( e = \begin{bmatrix} d \\ n \end{bmatrix} \) be the exogenous inputs, and \( z = \begin{bmatrix} r \\ v \end{bmatrix} \) be the regulated outputs. Then the problem can be represented in the standard configuration of Figure 4-2 as described in Section 4.10.1. The particular choice of state feedback matrix \( F \) and output injection matrix \( H \) such that \( A+BF \) and \( A+HC \) are stable are obtained by solving the ARE's (4-30) and (4-32).

\[ F = \begin{bmatrix} -17.714 & -3.2789 \\ -48.785 & -8.8569 \end{bmatrix} \quad H = \begin{bmatrix} -16.322 & -27.737 \\ 27.500 & 47.313 \end{bmatrix} \]  

(5-29)

The minimization problem \( \min J(x,u) \) is equivalent to the \( H^2 \)-optimization problem \( \min_{Q \in \mathbb{R}^{n \times n}} \| T - UQV \|_2 \) and has the solution \( Q_{\text{opt}}(s) = Q_0 = 0 \) with \( \| . \|_2 = 356.622 \).

The resulting \( H^2 \)-optimal controller, which is a combination of state feedback/Kalman filter, is

\[ K_{\text{opt}}(s) = \frac{1}{s^2 + 50.684s - 289.38} \begin{bmatrix} 198.96s+482.15 & 552.71s+1501.9 \\ 336.19s+887.51 & 934.09s-2747.1 \end{bmatrix} \]  

(5-30)

is unstable with a pole at 5.1800.
To find the *stable* optimal controller - let \( Q_1(s) = Q_0 + S(s) \). An initial choice for \( S(s) \) is given by the realization in equation (5-23) which leads to

\[
S_1(s) = \frac{1}{s^2 + 4s + 3} \begin{bmatrix}
-198.96s & -508.83 & -336.19s & -854.38 \\
-552.70s & -1409.8 & -934.09s & -2367.4
\end{bmatrix}
\] (5-31)

This gives an initial, feasible solution \( x \) for the constrained optimization problem

\[
\min_{x \in \mathbb{R}^2} f(x) = \frac{x_9 \Delta_1 + x_{11} \Delta_2}{2x_9 x_{10} x_{11}} \quad \exists \quad x_9, x_{10}, x_{11} > 0
\] (5-33)

where \( \Delta_1 = x_2^2 + x_4^2 + x_6^2 + x_8^2 \) and \( \Delta_2 = x_1^2 + x_3^2 + x_5^2 + x_7^2 \) (5-34)

\[\exists C_1(x) = -z_1 x_2 - z_3 x_4 - z_2 x_5 - z_4 x_8 + \beta x_{11} + 0.25 x_{12} > 0\] (5-35)

\[C_2(x) = -z_1 x_1 - x_2 - z_3 x_3 - z_2 x_5 - x_6 - z_4 x_7 - 0.25 x_8 + \beta x_{10} + \alpha x_{11} > 0\] (5-36)

\[C_3(x) = -x_1 - x_5 - 0.25 x_7 + \beta x_9 + \alpha x_{10} + x_{11} > 0\] (5-37)

\[C_4(x) = \alpha x_9 + x_{10} > 0\] (5-38)

\[C_5(x) = x_9 > 0\] (5-39)

\[C_6(x) = C_3(x) C_4(x) - C_2(x) C_5(x) > 0\] (5-40)

\[C_7(x) = C_2(x) C_3(x) C_4(x) - C_2^2(x) C_5(x) - C_1(x) C_4^2(x) > 0\] (5-41)

\[C_8(x) = x_1 x_7 - x_3 x_5 - x_9 x_{12} = 0\] (5-42)
\[ C_9(x) = x_1 x_8 + x_2 x_7 - x_3 x_6 - x_4 x_5 - x_{10} x_{12} = 0 \]  
(5-43)

\[ C_{10}(x) = x_2 x_8 - x_4 x_6 - x_{11} x_{12} = 0 \]  
(5-44)

\[ z_1 = 31.404, z_2 = 8.7502, z_3 = -9.8753, z_4 = -5.0553, \alpha = 50.684, \beta = -289.38 \]  
(5-45)

The nonlinear algorithm NPSOL gives one possible (suboptimal) solution

\[ S(s) = \frac{1}{3.3433 s^2 + 15.118 s + 6.4308} \begin{bmatrix} -255.44 s - 537.90 & -358.93 s - 873.69 \\ -691.13 s - 1461.9 & -972.12 s - 2375.5 \end{bmatrix} \]  
(5-46)

which gives \( \| S \|_2 = 247.83 \ll \| S_i \|_2 = 723.79 \). The resulting controller is

\[ K(s) = N_K(s)D_K(s)^{-1} \]  
where

\[ N_K = \begin{bmatrix} 409.74 s^3 + 2898.8 s^2 + 4800.2 s + 653.85 & 765.06 s^3 + 5241.4 s^2 + 8543.1 s + 1242.1 \\ 1156.7 s^3 + 9006.4 s^2 + 17893 s + 5347.1 & 2150.8 s^3 + 16048 s^2 + 30906 s + 9189.6 \end{bmatrix} \]

\[ D_K = \begin{bmatrix} d_{k11}(s) & 27.285 s^3 - 1890.20 s^2 - 8520.3 s + 1575.4 \\ 27.285 s^3 + 652.37 s^2 + 2848.5 s + 2423.4 & d_{k22}(s) \end{bmatrix} \]

where \( d_{k11}(s) = 3.3433 s^4 + 138.20 s^3 - 82.415 s^2 - 2610.2 s + 1870.2 \)

and \( d_{k22}(s) = 3.3433 s^4 + 167.37 s^3 + 1902.7 s^2 + 6403.2 s + 4680.8 \).

This controller is stable, with poles at

\[ -0.47535, -2.6141, -3.8259, -4.0465, -27.845, -49.705, -1.4433 \pm j3.8083 \].
CHAPTER 6

CONCLUSIONS

In this thesis, the question of optimal control system design when constrained to using a stable controller is addressed.

In Chapter 2, the problem of finding the stable controller which minimizes the $H^\infty$-norm of the weighted sensitivity function is solved in the scalar case. The optimum value of the performance index is found by solving a finite number of eigenvalue problems. It is shown that the optimal sensitivity is a singular inner function, and hence the controller that achieves this minimum is irrational.

In Chapter 3, the problem of approximating the irrational controller associated with a sub-optimal value of the optimality criterion is considered. A design algorithm has been developed to find a stable approximation whose deviation from this controller lies within a pre-specified error bound and consequently guarantees internal stability of the system. For multivariable systems, a characterization is given of all the desired sensitivity functions that would result in a stable controller. The important special cases of plants with zero or one nonminimum phase zeros are solved for optimality.

In Chapters 4 and 5, the problem of obtaining the $H^2$-optimal stable feedback controller for a general optimality criterion and system configuration is considered.
The optimization problem reduces to one of finding the "best" function in RH^2 (i.e. of minimum norm) that satisfies an avoidance constraint.

A design procedure is presented for finding such an optimal stable controller. Given an initial point, this is achieved by minimizing a nonlinear objective function subject to nonlinear inequality constraints. The algorithm used is NPSOL, which is based on a sequential gradient programming method.

An interesting question for future research is to incorporate robustness in the above design procedures. In H^∞-problems, a natural optimization criterion that is suggested is the so-called "mixed-sensitivity" problem. With the recently developed equivalence relationship between H^2 and H^∞-optimization, it may be possible to include robustness in the H^2-framework. Also, it would be desirable if the state-space framework were employed to do all the computations.

Another open problem is the issue of stability of 1^1_controllers; especially in the practically important case of discrete-time systems where the optimal sensitivity function has a finite impulse response, resulting in a "deadbeat" controller.
APPENDIX A

INTEGRAL EVALUATION

To evaluate the integral in equation (4-74) we proceed as follows.

Let
\[ S_x(s) = \sum_{i=1}^{N} F_i(s) = \sum_{i=1}^{N} \frac{x_{2i} + x_{1i} s}{x_{3i} s^2 + x_{4i} s + x_{5i}} \in RH^2 \]  
(A-1)

\[ S_x(-s) = \sum_{j=1}^{N} H_j(s) = \sum_{j=1}^{N} \frac{x_{2j} - x_{1j} s}{x_{3j} s^2 - x_{4j} s + x_{5j}} \in RH^{2*} \]  
(A-2)

Then the integral of the \((i,j)\)th term of the product \(S_x(s)S_x(-s)\) is given by

\[ I(i,j) = \frac{1}{2\pi j} \int_{\Omega} F_i(s)H_j(s) \, ds \]  
(A-3)

Assuming that \(x_{3i}, x_{3j} > 0\) (if either \(x_{3i}=0\) or \(x_{3j}=0\) this integral is trivial), and dividing through to simplify we have

\[ I(i,j) = \frac{1}{2\pi j} \int_{\Omega} \frac{\beta + \alpha s}{s^2 + \gamma s + \delta} \frac{B - As}{s^2 - Cs + D} \, ds \]  
(A-4)

where \(\alpha = x_{1i}/x_{3i}, \beta = x_{2i}/x_{3i}, \gamma = x_{4i}/x_{3i}, \delta = x_{5i}/x_{3i}\);

and \(A = x_{1j}/x_{3j}, B = x_{2j}/x_{3j}, C = x_{4j}/x_{3j}, D = x_{5j}/x_{3j}\).

Let \(s^2 + \gamma s + \delta = (s+\xi_1)(s+\xi_2)\); \(\gamma = \xi_1 + \xi_2, \delta = \xi_1\xi_2; -\xi_1, -\xi_2\) are poles of \(F_i(s)\).

Let \(s^2 - Cs + D = (s-r_1)(s-r_2)\); \(C = r_1 + r_2, D = r_1r_2; r_1, r_2\) are poles of \(H_j(s)\) in \(H\).

By Cauchy’s Residue Theorem.
\[ I(i,j) = \frac{1}{2\pi j} \int_{\Omega} S_x(s)S_x(-s) \, ds = \sum_{j=1}^{N} \sum_{i=1}^{N} I(i,j) \]
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