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Numerical simulation of vortex-induced oscillation of an elastically mounted circular cylinder using body-fitted coordinates

Allen, Donald Wayne, Ph.D.

Rice University, 1987
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NUMERICAL SIMULATION OF VORTEX-INDUCED OSCILLATION
OF AN ELASTICALLY MOUNTED CIRCULAR
CYLINDER USING BODY-FITTED COORDINATES

by

DONALD W. ALLEN

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

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August, 1986
Abstract

Vortex-induced oscillation during lock-in of an elastically mounted circular cylinder is numerically modeled herein for two-dimensional flow. The model solves the incompressible Navier-Stokes equations for flow-fields containing one or more moving boundaries. A body-fitted coordinate technique is used to generate a grid that contains coordinate lines coincident with the physical boundaries. The technique maps each curvilinear line segment in the physical plane to a straight line in a computational plane by a chain-rule transformation. The model allows for time-dependent transformations so that flow-fields containing one or more arbitrarily moving boundaries may be easily transformed to the fixed computational plane.

This investigation focuses on vortex-induced vibration of a circular cylinder when the flow is laminar near a Reynolds number of 100. Both steady and unsteady flow solutions are also presented for flow over a stationary circular cylinder. The solutions for vortex-induced oscillations are performed during lock-in (synchronization of the vortex-shedding frequency and the natural frequency of the elastically mounted cylinder) for different amounts of structural damping and different ratios between the structural natural frequency and the stationary cylinder vortex shedding frequency.

ii
Special attention is given to the controversy presented by several experimental researchers regarding a discontinuity in the Strouhal-Reynolds number relationship for flow over a stationary cylinder at a Reynolds number near 100. Results of a test attempting to find two Strouhal shedding frequencies in this Reynolds number range are presented. These results indicate that the discontinuity observed in some experiments is not caused by purely fluid mechanical effects.
Acknowledgements

For his guidance throughout this research, I owe many thanks to Dr. William F. Walker, whose insight while supervising this project was invaluable.

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Finally, I would most of all like to thank God, who during my years of graduate study has gone from someone I believe in, to someone I now truly know. For His gift described in I John 4:7-11, I am forever grateful.
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Nomenclature

\( a^n \) = acceleration at time step \( n \)

\( A_{ij} \) = dummy coefficient of field variable \( z \)

\( A_r \) = amplitude ratio

\( A_y \) = transverse amplitude of cylinder motion

\( c \) = structural damping coefficient

\( c_{cr} \) = critical structural damping coefficient

\( C_D \) = drag coefficient = \( \frac{\text{drag force}}{\frac{1}{2} \varphi U_{\infty}^2 D} \)

\( C_L \) = lift coefficient = \( \frac{\text{lift force}}{\frac{1}{2} \varphi U_{\infty}^2 D} \)

\( C_p \) = pressure coefficient = \( \frac{P - P_{\infty}}{\frac{1}{2} \varphi U_{\infty}^2} \)

\( D \) = cylinder diameter

\( D_i \) = dilation

\( f_d \) = damped natural frequency

\( f_n \) = structural natural frequency

\( f_s \) = Strouhal frequency of vortex shedding

\( f_{so} \) = Strouhal frequency for a stationary cylinder

\( F_y \) = lift force on the cylinder

\( J \) = Jacobian

\( j^n \) = jerk at time step \( n \)

\( k \) = spring constant

\( m \) = mass

\( m_c \) = mass of the cylinder

\( M_r \) = mass ratio

\( n \) = time step number

\( p \) = source term for grid generation equations

\( p \) = fluid pressure
Q = source term for grid generation equations
r = radius
Re = Reynolds number
Re_{cmax} = maximum Re for central differencing of first derivatives
Re_{mesh} = mesh Reynolds number
S = Strouhal number
T = temperature
t = time
u = horizontal velocity component of the fluid
u_c = horizontal velocity of a point on the cylinder
U_\infty = free-stream velocity of the fluid
v = vertical velocity component of the fluid
v_c = vertical velocity of a point on the cylinder
v^n = velocity at time step n
v_r = radial velocity
v_\theta = circumferential velocity
x = horizontal direction in the physical plane
y = vertical direction in the physical plane
y_i = initial vertical position of the cylinder
z_{ij} = dummy field variable at node ij
a = grid transformation parameter
a_m = adjustment parameter
\beta = grid transformation parameter
\Gamma = circulation
\gamma = grid transformation parameter
\( \epsilon \) = iteration parameter for pressure adjustment
\( \xi \) = vorticity or damping ratio
\( \eta \) = grid coordinate direction in computational plane
\( \theta \) = angle measured from positive horizontal axis
\( \mu \) = fluid dynamic viscosity
\( \nu \) = fluid kinematic viscosity
\( \xi \) = grid coordinate direction in computational plane
\( \rho \) = fluid density
\( \sigma \) = grid transformation parameter
\( \tau \) = grid transformation parameter
\( \tau_n \) = natural period of structure
\( \tau_s \) = period of vortex shedding
\( \phi \) = phase angle
\( \omega \) = angular velocity of the cylinder
\( \omega_r \) = frequency ratio \((\omega_s/\omega_n)\)
Chapter 1

INTRODUCTION

1.1 Fundamentals of Vortex-Induced Oscillations

Vortex shedding can occur whenever a bluff body encounters a viscous flowing fluid. For example, consider a fluid particle in an inviscid uniform flow-field approaching a cylinder along a streamline. As the particle approaches the forward stagnation point, its velocity decreases while it experiences an increase in pressure (the Bernoulli effect). After the forward stagnation point, the pressure accelerates the particle downstream around the cylinder until it reaches the widest section of the cylinder. In this region the velocity has increased to a maximum while the pressure is at a minimum. The increasing pressure along the back side of the cylinder decelerates the fluid particle until it reaches the rear stagnation point (180 degrees from the forward stagnation point). The fluid particle arrives at this point with the same velocity and pressure that it had at the forward stagnation point. The pressure and velocity of the particle along the streamline are therefore symmetric about the axis at the widest section of the cylinder (90 degrees from the forward stagnation point), and the fluid does not separate from the cylinder surface.

When the fluid is viscous, the frictional forces on the fluid particle in the boundary layer cause the particle to
diffuse much of its momentum. When the fluid particle reaches the widest region of the cylinder, it has lost enough of its momentum such that it is unable to overcome the increasing pressure force along the backside of the cylinder surface. The particle's motion is soon arrested and the pressure forces cause it to reverse direction. The particle therefore separates from the cylinder and the boundary layer then forms a shear layer downstream from the cylinder. The particles in the inner region of the shear layer move much slower than the particles in the outer region of the shear layer (since the outermost particles are in contact with the free-stream) causing the shear layer to roll up into a viscous vortex. This process occurs on both sides of the cylinder, yielding two shear layers that surround a low pressure wake behind the cylinder.

Under conditions of near perfect symmetry, at sufficiently low Reynolds numbers (less than about 40), the two viscous vortices on each side of the dividing streamline (see Figure 1.1-1) will remain adjacent to the cylinder and lengthen until a steady flow configuration is reached. At Reynolds numbers larger than 40, this flow alignment is unstable, and disturbances in the flow-field (such as gravitational effects, pressure fluctuations, surface irregularities, etc.) cause the vortex interactions to become unbalanced. For example if a disturbance causes vortex A to draw fluid from vortex B in Figure 1.1-1, the
Figure 1.1-1 Streamlines Describing Steady Flow
vortices become unbalanced. For Reynolds numbers larger than about 40, this disturbance is not immediately balanced by a disturbance to vortex B. Instead, vortex B is moved aside as vortex A continues to draw fluid from vortex B. Eventually vortex A will "shed" from the cylinder and move downstream. Vortex B will then draw fluid from the region previously occupied by vortex A until it also sheds. This process continues, leading to an alternate periodic shedding of vortices. Note that in the "stable" configuration only two independent variables are required to describe the flow-field, whereas in the "unstable" case three independent variables are required since the flow is then time-dependent.

Each time a vortex is shed from the cylinder, it is a result of changes in the pressure and shear forces along the cylinder surface. The fluid force on the cylinder in the transverse direction (the lift force) oscillates at the frequency of the vortex shedding cycle since the cylinder experiences a net force opposite the direction of the vortex that was last shed. The fluid force on the cylinder in the in-line direction (the drag force) oscillates at a frequency twice that of the lift force.

The characteristics of the flow-field and fluid forces for flow around a circular cylinder depend on the free-stream velocity $U_\infty$: the cylinder diameter $D$: the free-stream Reynolds number, defined by
\[ \text{Re} = \rho \frac{U_\infty D}{\mu} \]  \hspace{1cm} (1.1.1)

where \( \rho \) is the fluid density and \( \mu \) is the dynamic viscosity: the length-diameter ratio (L/D): and the surface roughness. The frequency of vortex shedding, which is the Strouhal frequency \( f_s \), is defined by

\[ f_s = S \frac{U_\infty}{D} \]  \hspace{1cm} (1.1.2)

The Strouhal number, \( S \), is a proportionality constant, which for a stationary body and a given geometry depends only upon the Reynolds number. The Strouhal-Reynolds number relationship for circular cylinders (obtained from experimental data) is shown in Figure 1.1-2.

If the cylinder is oscillating transverse to the flow direction at or near the shedding frequency, the vortices will strengthen, and the correlation of the vortex shedding along the cylinder span will increase significantly. The vortex strength is measured by the circulation

\[ \Gamma = \oint_{C} \vec{V} \cdot d\vec{c} = \int_{A} \hat{n} \cdot (\nabla \times \vec{V}) dA = \int_{A} \xi_n dA \]  \hspace{1cm} (1.1.3)

where \( \xi_n \) is the component of the vorticity vector normal to \( dA \), and \( A \) is the area of the vortex bound by the curve \( C \). The vorticity vector is defined by

\[ \vec{\xi} = \nabla \times \vec{V} \]  \hspace{1cm} (1.1.4)

The spanwise correlation is a measure of how uniform the vortex shedding is along the span. If the shedding is
Figure 1.1-2 Strouhal-Reynolds Number Relationship for Circular Cylinders (Blevins [1977])
uniform along the entire span, then each point along the span sheds the vortex at precisely the same moment, and the flow may be considered two-dimensional since the vortex is parallel to the cylinder span. For low Reynolds number flows over an elastically mounted rigid circular cylinder, the flow may be considered two-dimensional for a length of about 15-20 diameters along the span (Blevins [1977]). During cylinder oscillations at a frequency near the vortex shedding frequency, the vortex shedding frequency \( f_s \) may change from the stationary vortex shedding frequency \( f_{SO} \), to the frequency of the cylinder oscillation, initiating a condition known as "lock-in". The initiation of lock-in depends upon both the cylinder and shedding frequencies, as well as the amplitude of the oscillations.

When a simple spring-mounted cylinder system with a structural natural frequency of \( f_n \) is subjected to the oscillating forces from vortex shedding, the cylinder motions are said to be vortex-induced. When lock-in is established for vortex-induced vibrations, the oscillation amplitude will increase significantly. This amplitude will grow until vortex formation at the cylinder surface is suppressed, causing the process of amplitude growth to be self-limiting. Large amplitude oscillations induced in elastic structures resulting from lock-in can cause structural failure (Blevins [1977]).
1.2 Previous Research of Vortex-Induced Oscillations

The works of Blevins [1977] and Hall [1981] contain brief historical reviews of vortex shedding including the musical applications of King David and Ktesibios of Alexandria, as well as the first systematic investigation by Strouhal [1878] who observed that the changes in Aeolian tones generated by a stretched wire were in proportion to the wind speed divided by the wire thickness. Transverse oscillations were observed and reported by Raleigh [1879]; Bernard [1908] observed the relationship between the wake period and vortex formation; and von Karman [1912] was able to discern a regular stable pattern of alternating vortices. The large amplitude oscillations during lock-in have current applications in many areas including offshore structures, heat exchangers, transmission lines and cables, pipelines, and several areas of acoustics, and therefore recent research has been undertaken in all of these areas. Marris [1964], King [1979], Sarpkaya [1979], and Bearman [1984] have all presented reviews of previous research of vortex shedding and vortex-induced oscillations since the time of von Karman.

1.3 Recent Modeling of Fluid Forces

Several mathematical models have been used to simulate the fluid forces on a cylinder during vortex shedding. These models fall into two classes: those that compute the
fluid forces by modeling the flow-field and those that do not. The models that do not solve for the flow-field itself but simply act as models for the fluid forces are best described as nonlinear oscillators. These models are highlighted by the nonlinear oscillator models of Bishop and Hassan [1964], Hartlen and Currie [1970], Skop and Griffin [1973], Iwan and Blevins [1974], and Benaroya and Lepore [1983], as well as the spanwise correlation model of Blevins and Burton [1976].

Many of the flow-field models are based upon the discrete vortex model of Abernathy and Kronauer [1962]. This model has been used for computing flow over a circular cylinder by Gerrard [1967], Diffenbaugh and Marshall [1976], Sarpkaya and Schoaff [1979], Stansby [1981], and Stansby and Dixon [1982]. Most of the recent discrete vortex models such as that of Sarpkaya and Scholaff are based upon potential flow and boundary layer interaction, rediscretization of the shear layers, and dissipation of the circulation. These models give good values for the fluid forces on an elastically mounted cylinder with linear damping, but only approximate the flow-field itself. Fluid phenomena such as the interaction of the vortices are disregarded, and the dissipation of circulation is only approximated.

The flow-field analysis during vortex-induced oscillations is best obtained by solution of the time-dependent incompressible Navier-Stokes equations for flow
over a moving cylinder, since these are the equations that govern the fluid motion. Numerical solutions can uncouple the fluid and structural interaction and update the fluid forces and cylinder motion at each time step. The pressure and shear forces obtained from the flow solution along with the structural forces determine the cylinder motion, whereas the cylinder velocity determines the fluid no-slip boundary conditions. The accuracy of numerical simulation is governed mainly by the size of the time step and the accuracy of the Navier-Stokes solution at each time step. Both Blevins [1977] and Sarpkaya [1979] outline the limitations of Navier-Stokes solutions including: Reynolds number size (usually restricted to laminar flow): number of time steps: and number of grid blocks for three dimensional simulations, all of which are related to computational cost. However, numerical modeling of flows even at low Reynolds numbers, can provide researchers with a better understanding of vortex shedding and vortex-induced oscillations. Since great progress in computational speeds of large computers has been made in recent years, it is reasonable to expect the limitations outlined above to be decreased.

Hurlbut et. al. [1978,1982] adapted the fixed cylinder incompressible Navier-Stokes solver of Swanson and Spaulding [1978] to simulate numerically the flow over a cylinder undergoing both in-line and transverse oscillations in two dimensions. They presented results for Reynolds numbers
from 1 to 100 and amplitude ratios from 0.1 to 2.0 where the amplitude ratio is defined as

$$A_r = \frac{A_y}{D}$$  \hspace{1cm} (1.1.3)

where $A_y$ is the transverse amplitude. Lecointe and Piquet [1984] solved the Navier-Stokes equations for in-line oscillations at $Re = 200$ during lock-in. In both numerical investigations, the motion of the cylinder was specified, i.e. the motion was therefore not vortex-induced, and both used a logarithmic grid transformation that restricts the analysis to a single circular cylinder.

1.4 Current Investigation

The objective of the current investigation was to develop and implement a numerical method for modeling fluid flow over a spring-mounted cylinder experiencing vortex-induced oscillation. The method was to be able to model several bodies in the flow-field, model cylinders with multiple degrees of freedom, allow for non-circular bodies, and be easily extended to three dimensions. In the simulations performed in this investigation, the cylinder was selected to be rigid and was elastically mounted with linear damping. These criteria were chosen so as to provide a "first approximation" to the dynamic structural characteristics of a cylinder of finite span. Note that the general method does not restrict the cylinder to be rigid,
nor does it restrict the spring or damper to be linear. Chapter 2 of this study presents the numerical method developed to meet these objectives, including the body-fitted grid generation method and the attributes of the Navier-Stokes solver. Chapter 3 discusses results for flow over a stationary cylinder while Chapter 4 presents results for vortex-induced cylinder motions in the transverse direction. All computations were performed at a Reynolds number of 100 in order to compare the results with those of previous numerical investigations as well as to investigate the current controversy of a possible Strouhal-Reynolds number discontinuity near this Reynolds number. This controversy is discussed in Chapter 4. The oscillating cylinder simulations were performed with a mass ratio of 5.0, where the mass ratio is defined by

\[ M_r = \frac{m_c}{\rho D^2} \]  \hspace{1cm} (1.1.4)

\( m_c \) is the cylinder mass per unit length along the span. Simulations at this mass ratio were found to reach steady periodic conditions much more quickly than one attempted at a mass ratio of 100.0. Finally, the summary, conclusions, and recommendations for future research are discussed in the section following Chapter 4.
Chapter 2

SOLUTION PROCEDURE

2.1 General Algorithm

The general algorithm used in this project is:

1) Input flow and structural parameters (Reynold's number, free-stream velocity, fluid density, viscosity, cylinder mass, spring constant, damping constant, etc.).

2) Set initial cylinder position and generate initial grid.

3) Set initial flow boundary conditions on cylinder and compute initial flow-field such that $\nabla \cdot \vec{V} = 0$.

4) Generate body-fitted coordinate grid for the cylinder position at the current time step.

5) Solve Navier-Stokes equations at the current time step.

6) Compute fluid forces on cylinder at the current time step.

7) Update cylinder position for next time step.

8) Update cylinder flow boundary conditions for next time step.

The first step of the algorithm allows the user considerable flexibility in choosing the flow and structural problem to solve. The program should handle the Reynolds number range when the entire flow-field remains laminar (about $40 \leq Re \leq 190$).

Step 2) establishes the initial position of the cylinder with the center at the origin, and generates an
initial body-fitted coordinate grid using the local exponential interpolation technique discussed in Section 2.2. The grid points on the cylinder boundary are spaced with a constant angular increment, as is the ellipse that is used to approximate the outer free-stream boundary.

Step 3) establishes an initial flow-field that conserves mass and satisfies the no-slip boundary condition at the cylinder surface, as well as the free-stream boundary condition on the outer boundary. To conserve mass, the velocity field must satisfy the equation for mass conservation in an incompressible fluid

\[ \nabla \cdot \mathbf{V} = 0 \tag{2.1.1} \]

which in \([x,y]\) space is

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1.2} \]

and in \([r,\theta]\) space is

\[ \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0 \tag{2.1.3} \]

Steps 4) and 5) constitute most of the computational work at each time step and are discussed in Sections 2.2 and 2.3 with further detail given in Appendices A and B. Step 4) establishes the computational grid in each time step and may be skipped in simulations in which the
cylinder stays fixed. Step 5) solves the unsteady, incompressible Navier-Stokes equations in two-dimensions at each time step using a fully implicit finite-difference scheme.

Step 6) computes the fluid forces on the cylinder at each time step. These pressure and shear forces are passed to steps 7) and 8) which explicitly evaluate the new position and velocity of the cylinder from Newton’s second law of motion. For example, a cylinder experiencing vibration in the y-direction would have its position and velocity updated as:

\[
y^{n+1} = y^n + (v^n)(\Delta t) + 1/2(a^n)(\Delta t)^2 \\
+ 1/6(j^n)(\Delta t)^3 \tag{2.1.4}
\]

\[
v^{n+1} = v^n + (a^n)(\Delta t) + 1/2(j^n)(\Delta t)^2 \tag{2.1.5}
\]

where: 
\[a^n = \frac{[F_y^n - k(y^{n+1} - y_i - cv^n)]}{m_c} \tag{2.1.6}\]

\[n = \text{implies time step } n\]

\[j = \text{jerk} = (a^n - a^{n-1})/\Delta t\]

\[F_y = \text{the lift force}\]

\[k = \text{spring constant}\]

\[c = \text{structural damping coefficient}\]

\[y_i = \text{initial y-coordinate of cylinder position}\]

Equations (2.1.4) and (2.1.5) are Taylor series expansions about time step \(n\) and assume constant jerk at each time step.
For a moving cylinder, steps 4) through 8) are repeated at each time step until a constant amplitude and period of the cylinder motion is established. For a still cylinder problem with steady boundary conditions, steps 4, 7, and 8 may be omitted at each time step, while steps 5 and 6 are repeated until the oscillating fluid forces have a constant amplitude and period.

2.2 Grid Generation

The computational region was discretized by a body-fitted curvilinear coordinate system similar to the one introduced by Thompson et al. [1974]. This grid generation method produces a grid containing coordinate lines coincident with all boundary contours (regardless of the location or shape of the boundary), and eliminates interpolation when the boundary is curved. The technique has been used in several previous fluid flow calculations including those of Thompson and Shanks [1977], Haussling and Coleman [1977], and Thames et al. [1977].

The two-dimensional body-fitted coordinate technique consists of a coordinate transformation from the physical plane to a rectangular computational plane. Figure 2.2-1 shows a two-dimensional region D in the physical plane bounded by two simple closed contours of arbitrary shape Γ₁ and Γ₂, and the mapping of region D onto the computational plane, region D*. The curved boundaries Γ₁ and Γ₂ are
Figure 2.2-1 Body-Fitted Coordinate Transformation for a Single Body (Thompson et al. [1974])
mapped onto the straight lines $\Gamma_1^*$ and $\Gamma_2^*$, becoming lines of constant $\eta$ in the transformed plane. The points $A_1^*$ and $B_1^*$ in the transformed plane must be identical, as must points $A_2^*$ and $B_2^*$, so that the boundaries are continuous. This requires the branch cut of lines $\Gamma_3$ and $\Gamma_4$ which are identical in the physical plane to be mapped onto $\Gamma_3^*$ and $\Gamma_4^*$ in the transformed plane. Note that these lines are of arbitrary shape in the physical plane.

Each point in the computational plane contains a corresponding $(x,y)$ value which specifies the mapping of that point. The input boundary grid point locations determine the $(x,y)$ value of each point on $\Gamma_1^*$ and $\Gamma_2^*$. For example, if the inner boundary was a circle, the $(x,y)$ values of each boundary point on the circle would be input (in either clockwise or counter-clockwise order), and the first point would be mapped to $(\xi,\eta)=(1,1)$, the second point would be mapped to $(\xi,\eta)=(2,1)$ etc., so that the $(x,y)$ values of each point on $\Gamma_1^*$ would be known. The $(x,y)$ values of the interior points (including the points on $\Gamma_3^*$ and $\Gamma_4^*$ that do not lie on $\Gamma_1^*$ or $\Gamma_2^*$) are unknown, and determine the location of each of the interior grid points in the physical plane. These interior values are arbitrary except that the grid should not overlap and should be as smooth as possible in order to minimize the truncation errors of the finite-difference approximations (see Kalnay De Rivas [1972]). Once the interior $(x,y)$ values are
specified, the body-fitted coordinate grid is established, and the curvilinear grid in the physical plane may be graphed by plotting the \((x,y)\) values for each \(\xi\) and \(\eta\) line and connecting consecutive points on identical \(\xi\) or \(\eta\) lines.

Figure 2.2-2 shows how more than one interior body may be present in the region. Each interior body is mapped to a line of constant \(\eta\) in the transformed plane. The bodies may move independently by changing the corresponding \((x,y)\) values of the boundary points in the transformed plane, and re-computing the \((x,y)\) values of the interior points. This feature allows for modeling of vortex-induced oscillations of multiple cylinders; a feature not possible with the transformation used by Hurlbut et al. [1978,1982].

Thompson et al. [1974] computed the \((x,y)\) values of the interior points by requiring the \(\xi\) and \(\eta\) coordinates to satisfy a pair of Poisson equations:

\[
\begin{align*}
\xi_{xx} + \xi_{yy} & = P \quad (2.2.1) \\
\eta_{xx} + \eta_{yy} & = Q \quad (2.2.2)
\end{align*}
\]

where \(P=P(x,y)\) and \(Q=Q(x,y)\) are functions used to control the mesh spacing. In order to perform all computations in the computational plane, it is the \((x,y)\) values of the interior points in the computational plane that are desired. In order to compute these values from equations (2.2.1) and (2.2.2), the equations are transformed using the chain rule (see Appendix A.1) to give:
Figure 2.2-2  Body-Fitted Coordinate Transformation for Two Bodies (Thompson et al. [1974])
\[ a x_{\xi} x_{\eta} - 2\beta x_{\xi} y_{\eta} + \gamma x_{\eta} y_{\eta} + J^2(P x_{\xi} + Q x_{\eta}) = 0 \] (2.2.3)

\[ a y_{\xi} x_{\eta} - 2\beta y_{\xi} y_{\eta} + \gamma y_{\eta} y_{\eta} + J^2(P y_{\xi} + Q y_{\eta}) = 0 \] (2.2.4)

where:

\[ a = x_{\eta}^2 + y_{\eta}^2 \] (2.2.5a)

\[ \beta = x_{\xi}^2 + y_{\xi}^2 \] (2.2.5b)

\[ \gamma = x_{\xi} x_{\eta} + y_{\xi} y_{\eta} \] (2.2.5c)

\[ J = x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \] (2.2.5d)

Note that the set of partial differential equations to be solved in region D must also be transformed to \([\xi, \eta]\) space.

Solution of equations (2.2.3) and (2.2.4) with \(P=Q=0\) (using the x and y values on boundaries \(\Gamma_1^*\) and \(\Gamma_2^*\) as boundary conditions), yields a smooth grid point distribution without grid overlap for most geometries. This solution is simply the solution of Laplace's equation, an equation which, with Dirichlet boundary conditions, always yields a grid free of overlap since the equation is harmonic. Thompson used different grid spacing control functions \((P \text{ and } Q)\) to yield various grid point distributions (e.g. to resolve a boundary layer), but found that many functions produced grid overlap. Grid overlap occurs because the generating equation is no longer harmonic, and its solution may contain extrema in the interior of the field. Thompson also found that grid spacing control functions that worked well for one problem might produce grid overlap or poor mesh spacing for others.
Some researchers have overcome this problem by coupling the grid spacing control functions to the set of partial differential equations to be solved yielding an adaptive grid (see Dwyer et al. [1980] or Oujesky [1985]). Other researchers have used different types of partial differential equations to compute the interior coordinate values such as the hyperbolic partial differential equation method of Steger and Sorenson [1980].

All of the methods for computing the interior coordinate values mentioned above require the solution of a set of partial differential equations. For adaptive grids, or problems with moving boundaries (such as the motion of a cylinder due to vortex shedding), this requires several solutions of the partial differential equations used to establish the grid. This can be computationally expensive, and for most problems, an algebraic method for computing the grid point distribution will give similar results at a fraction of the computational cost of partial differential equation methods.

In this investigation an algebraic technique was used to generate the type of body-fitted coordinate grid needed for external flow over one or more moving bodies. The technique consists of a local exponential interpolation between an inner and outer boundary to yield a grid containing a high concentration of grid lines near the inner boundary. The grid lines are restricted to be orthogonal at
Figure 2.2.3(a) Body-Fitted Coordinate Grid for Flow over a Still Cylinder
Figure 2.2.3(b) Body-Fitted Coordinate Grid for Flow over a Still Cylinder
Figure 2.2-3(c) Body-Fitted Coordinate Grid for Flow over a Slit Cylinder
Figure 2.2-4: Figure 2.2-3c with the Cylinder Displaced
both the inner and outer boundaries in order to minimize truncation error at these locations due to one-sided differencing (see Thompson and Mastin [1985]). Appendix A.3 explains this algebraic grid generation technique in detail for flow over a single cylinder. Figures 2.2-3a-c show the grid generated by the above algebraic technique for flow over a single still cylinder, while Figure 2.2-4 shows the grid when the cylinder in Figure 2.2-3c has been displaced. The grid size is 40x40 with the boundary points on the cylinder and the outer boundary spaced at even angular increments. The outer elliptical boundary used to approximate the free-stream was given a major axis (the incoming flow direction) radius of 40 cylinder diameters and a radius of 25 cylinder diameters along the minor axis. These sizes were chosen to approximate the free-stream by placing the outer boundary as far from the cylinder as possible, while still retaining enough grid points near the cylinder to resolve accurately the flow detail in the boundary layer.

2.3 Navier-Stokes Solution Procedure

The Navier-Stokes equations for unsteady incompressible flow of a Newtonian fluid in \([x,y]\) space are:

\[
\begin{align*}
u_t + uu_x + vv_y &= -p_x + (u_{xx} + u_{yy})/Re \quad (2.3.1) \\
v_t + uv_x + vv_y &= -p_y + (v_{xx} + v_{yy})/Re \quad (2.3.2)
\end{align*}
\]
where:

\[ u = \frac{u^*}{U_\infty^*}, \quad v = \frac{v^*}{U_\infty^*}, \quad p = \frac{p^*}{\rho^*(U_\infty^*)^2} \]

\[ x = \frac{x^*}{D^*}, \quad y = \frac{y^*}{D^*}, \quad t = \frac{t^* U_\infty^*}{D^*} \]

\[ \text{Re} = \frac{\rho^* U_\infty^* D^*/\mu^*}{} \tag{2.3.3} \]

These equations together with the equation for mass conservation

\[ u_x + v_y \equiv \text{Di} = 0 \tag{2.3.4} \]

and the appropriate boundary conditions (discussed below) govern the motion of the fluid.

The partial differential equation for mass conservation (2.3.4) may be replaced by a second order elliptic partial differential equation for pressure by taking the divergence of the momentum equations and making use of equation (2.3.4) to obtain

\[ P_{xx} + P_{yy} = -(u_x)^2 - 2u_y v_x - (v_y)^2 - \text{Di} t \tag{2.3.5} \]

Hirt and Harlow [1967] have shown that the convergence of the above system of equations for unsteady flows can be improved by retaining the last term in equation (2.3.5). The improvement is based upon the fact that the finite difference approximation to Di is non-zero even though it is
zero in the continuum. The inclusion of $D_i$ corrects for the error from the previous time step and keeps the error from accumulating.

Equations (2.3.1), (2.3.2), and (2.3.5) represent a coupled set of non-linear partial differential equations to be solved for $u$, $v$, and $p$ at each time step. If the unsteady term in each of equations (2.3.1) and (2.3.2) is neglected (steady flow), then all of the equations are second-order elliptic partial differential equations and therefore each equation requires a single boundary condition at each grid point on the boundaries. For unsteady flows, equations (2.3.1) and (2.3.2) require initial conditions for $u$ and $v$ in addition to the boundary conditions. The only initial condition needed in the pressure equation (2.3.5) is for $D_i$ (since it is the only unsteady term in this equation), which can be computed from the initial velocities. Equations (2.3.1) and (2.3.2) are often called "mixed parabolic-elliptic" since they are elliptic for steady flows and both parabolic and elliptic for unsteady flows. The velocity boundary conditions for external flow over a moving cylinder are:

$$u = U_\infty, \ v = 0$$  \hspace{1cm} (2.3.6a)

on the free-stream boundary, and are:

$$u = u_c(x,y,t), \ v = v_c(x,y,t)$$  \hspace{1cm} (2.3.6b)
at the cylinder boundary, where $u_c$ and $v_c$ are the $x$ and $y$ components of the local velocity of the cylinder surface at each grid point.

Since each of equations (2.3.1), (2.3.2) and (2.3.5) contain only derivatives of pressure, the pressure solution is unique up to a constant. The pressure level was selected here by setting the free-stream pressure $p_\infty=0$ at the free-stream boundary. The pressure boundary condition at the cylinder surface was selected to be:

$$p = p(Di) \text{ such that } Di=0$$  \hspace{1cm} (2.3.7)

This pressure boundary condition requires that the pressure insure mass conservation across the cylinder boundary (introduced by Viecelli [1969]). Note that the pressure equation (2.3.5) has replaced the equation for mass conservation (2.3.4) and therefore must impose this constraint on the flow-field.

The initial conditions for $u$ and $v$ should be selected such that boundary conditions (2.3.6a-b) are satisfied as well as equation (2.3.1). If the initial condition does not conserve mass, several time steps are required before the corrective term in equation (2.3.5) can produce a pressure field that imposes mass conservation. No initial conditions are required for pressure, since the pressure equation does not contain any unsteady terms.

The above equations and boundary conditions must be
transformed to \([\xi, \eta]\) space in order to perform all computations in the transformed plane. Using the derivative relations from Appendix A.2, the transformed equations are:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= x_t(y_\eta u_\xi - y_\xi u_\eta) / J - y_t(x_\eta u_\xi - x_\xi u_\eta) / J \\
+ u(y_\eta u_\xi - y_\xi u_\eta) / J + v(x_\xi u_\eta - x_\eta u_\xi) / J \\
&= -(y_\eta P_\xi - y_\xi P_\eta) / J + (au_\xi - 2\beta u_\xi + \gamma u_\eta \\
+ \sigma u_\eta + \tau u_\xi) / Re J^2
\end{align*}
\] (2.3.8)

\[
\begin{align*}
\frac{\partial v}{\partial t} &= x_t(y_\eta v_\xi - y_\xi v_\eta) / J - y_t(x_\xi v_\eta - x_\eta v_\xi) / J \\
+ u(x_\eta v_\xi - x_\xi v_\eta) / J + v(x_\xi v_\eta - x_\eta v_\xi) / J \\
&= -(x_\eta P_\eta - x_\xi P_\xi) / J + (av_\xi - 2\beta v_\xi + \gamma v_\eta \\
+ \sigma v_\eta + \tau v_\xi) / Re J^2
\end{align*}
\] (2.3.9)

\[
\begin{align*}
\frac{\partial p}{\partial t} &= 2\beta p_\eta + \gamma p_\xi + \sigma p_\eta + \tau p_\xi = -(y_\eta u_\xi - y_\xi u_\eta)^2 \\
- 2(x_\xi u_\eta - x_\eta u_\xi)(y_\eta v_\xi - y_\xi v_\eta) \\
- (x_\xi v_\eta - x_\eta v_\xi)^2 - J^2 D_t \\
+ x_t(y_\eta D_\xi - y_\xi D_\eta) + y_t(x_\xi D_\eta - x_\eta D_\xi) J
\end{align*}
\] (2.3.10)

where:

\[
\begin{align*}
a &= (x_\eta)^2 + (y_\eta)^2
\end{align*}
\] (2.3.11a)

\[
\begin{align*}
\beta &= x_\xi x_\eta + y_\xi y_\eta
\end{align*}
\] (2.3.11b)

\[
\begin{align*}
\gamma &= (x_\xi)^2 + (y_\xi)^2
\end{align*}
\] (2.3.11c)

\[
\begin{align*}
J &= x_\xi y_\eta - x_\eta y_\xi
\end{align*}
\] (2.3.11d)

\[
\begin{align*}
P &= ax_\xi - 2\beta x_\xi + \gamma x_\eta
\end{align*}
\] (2.3.11e)

\[
\begin{align*}
Q &= ay_\eta - 2\beta y_\xi + \gamma y_\eta
\end{align*}
\] (2.3.11f)
\[ \sigma = \left( y_\xi P - x_\xi Q \right) / J \]  \hspace{1cm} (2.3.11g)

\[ \tau = \left( x_\eta Q - y_\eta P \right) / J. \]  \hspace{1cm} (2.3.11h)

The boundary conditions (2.3.6) and (2.3.7) remain the same when transformed to \([\xi, \eta]\) space, except that now the free-stream boundary is mapped to the grid line \(\eta = \eta_{\text{max}}\) (see Figure 2.2-1), and the cylinder boundary is mapped to the grid line \(\eta = 1\). Note that the quantities \(x_t, y_t, x_\eta, y_\eta, x_\xi, \) and \(y_\xi\) together with quantities (2.3.11a-h) are all constants once the grid has been established at each time step, yielding a system of equations that are only slightly more complex than the system in \([x,y]\) space.

The above system of equations was solved at each time step by a fully implicit finite-difference method. The equations were uncoupled by lagging the coupling terms (e.g. the pressure term as well as the terms containing \(v\) in equation 2.3.8) at each iteration. Lagging was performed by evaluating the quantities from the previous iteration and treating them as constants during the current iteration. The finite-difference equations were linearized by lagging the nonlinear convection terms as well, yielding three sets of uncoupled finite-difference equations to be solved at each time step. All partial derivatives were differenced with second-order central differences, with the following exceptions:

1. The temporal derivatives in the momentum equations
were differenced with first-order backward differences.

2. The pressure derivative in the \( \eta \) direction was differenced with a first-order backward difference to eliminate "wiggles" (spacial oscillations that occur when the mesh Reynolds number exceeds a value of 2 - see Roache [1972]).

3. The convective terms were differenced with a hybrid upwind difference scheme to eliminate "wiggles" when the mesh spacings were large. This hybrid method consists of a weighted average between a second-order central difference and a third order upwind difference developed by Leonard [1981]. If the mesh Reynolds number exceeds \( \text{Rec}_{\text{max}} \), then the convective terms were evaluated as

\[
\psi(\partial \phi / \partial z) = (\text{Rec}_{\text{max}} / \text{Re}_{\text{mesh}}) \psi(\partial \phi / \partial z)_c \\
+ (1 - \text{Rec}_{\text{max}} / \text{Re}_{\text{mesh}}) \psi(\partial \phi / \partial z)_u
\] (2.3.12)

where the central difference \( \psi(\partial \phi / \partial z)_c \) is

\[
\psi(\partial \phi / \partial z)_c = \psi[(\phi_{i+1} - \phi_{i-1}) / (2\Delta z)]
\] (2.3.13a)

and the upwind difference \( \psi(\partial \phi / \partial z)_u \) is

\[
\psi(\partial \phi / \partial z)_u = \psi[-\phi_{i+2} + 8\phi_{i+1} - 8\phi_{i-1} + \phi_{i-2}] \\
+ |\psi|[(\phi_{i+2} - 4\phi_{i+1} + 6\phi_i - 4\phi_{i-1} \\
+ \phi_{i-2})] / [12\Delta z]
\] (2.3.13b)
Note that $\psi$ and $\phi$ represent either $u$ or $v$ while $z$ represents either $\xi$, or $\eta$.

$\text{Re}_{\text{cmax}}$ is the maximum Reynolds number for which central differences are used for the convective terms, therefore when $\text{Re}_{\text{mesh}}$ is less than $\text{Re}_{\text{cmax}}$ central differences are used, and when $\text{Re}_{\text{mesh}}$ exceeds $\text{Re}_{\text{cmax}}$ equation (2.3.12) is used. $\text{Re}_{\text{cmax}}$ may range anywhere from 0.0 to 2.0; a value of 1.0 was used in this investigation. This hybrid method has the advantage of being able to use central differences to evaluate these terms near the cylinder wake and boundary layer (and thereby eliminating the added diffusion of upwind differencing), while also eliminating "wiggles" by smoothly switching to upwind differences when the mesh spacing gets large (away from the cylinder).

The three sets of finite difference equations were solved using the modified strongly implicit procedure (MSIP) of Schneider and Zedan [1981], which consists of an incomplete [L]-[U] factorization. The method has been shown to be faster than most other iterative methods for the grid size used here (40x40), including most preconditioned conjugate-gradient algorithms (see Jacobs [1983]). Appendix B discusses the theory of MSIP, its extension to body-fitted coordinates, and the tests used to select the method's adjustment parameter (a value of 0.9 was used) which
significantly influences the method's performance.

The pressure at each point on the cylinder boundary was adjusted by the equation (taken from Thompson and Shanks [1977])

\[ p^{k+1} = p^k - \frac{2\epsilon J^2 \Delta \xi}{(a + \gamma) \Delta t} \]  \hspace{1cm} (2.3.14)

where \( \epsilon \) is an iteration parameter which was chosen to equal 0.95, and \( \Delta t \) is the time step which was 0.05. Equation (2.3.14) was used at each iteration to adjust the pressure until the flow conserved mass at each point on the cylinder boundary.

The above procedure was tested on the classical viscometer problem which consists of parallel flow between concentric cylinders. The inner cylinder was given a dimensionless angular velocity of 0.1 (radius = 1) and the outer cylinder was given a dimensionless angular velocity of 0.2 (radius = 5). The analytical solution of this problem with these boundary conditions and a density \( \rho = 1 \) is

\[ v_\theta(r) = 0.204167r - 0.104167/r \]  \hspace{1cm} (2.3.15)

\[ p(r) = 0.02084r^2 - 0.042741n(r) - 0.00543/r^2 \]  \hspace{1cm} (2.3.16)

where \( v_\theta \) is the circumferential velocity. Note that the solution is independent of viscosity. Figure 2.3-1 presents a plot of the analytical and computational solutions for velocity as a function of radius, while Figure 2.3-2 is a
Figure 2.3.1 Circumferential Velocity vs. Radius for Analytical and Numerical Solutions of the Viscosimeter Problem.
Figure 2.3-2 Pressure vs Radius for Analytical and Numerical Solutions of the Viscometer Problem
similar plot for pressure. Both figures show excellent agreement with the analytical solution and indicated the consistency of the finite-difference approximations to equations (2.3.8)-(2.3.10). The number of time-steps to reach convergence was a function of the initial conditions, while the number of iterations to reach convergence for each time step decreased as the solution approached the final steady-state solution. The iteration of the pressure on the cylinder boundary was the slowest value to converge at each time step, since the velocities were somewhat insensitive to the pressure. Several values of $\epsilon$ were tried for the first several time steps, with the value of 0.95 being near the optimal value. These observations were found to be valid for the steady solution of the still cylinder problem as well.
Chapter 3
FLOW OVER A STATIONARY CYLINDER

3.1 Steady Flow

Steady flow configurations for laminar viscous flow over a circular cylinder consist of a pair of symmetric standing vortices attached to the cylinder's downstream surface. These flow configurations have been observed experimentally for Reynolds numbers up to about 40, and have been obtained computationally for Reynolds numbers as high as 600 by Fornberg [1985]. Above these Reynolds number limits, the symmetric flow pattern becomes unstable, and the vortices begin shedding from the cylinder.

The differences for the limiting stable Reynolds number between experimental and computational results, is due to the difficulty in experimentally maintaining a symmetrical flow-field. Real flow experiments are never perfectly symmetric, and disturbances in the flow-field such as gravitational effects, pressure fluctuations, and surface irregularities cause the symmetric flow configurations to become unstable. Numerical simulations have the advantage of being able to produce a flow-field that is free from disturbances and stable symmetric flow configurations can be obtained for higher Reynolds numbers than those observed experimentally.

Figure 3.1-1 shows the streamline plot for the steady
flow solution obtained in this investigation. The Reynolds number was 100 with \( p=1, \ D=1, \ U_{\infty}=1, \) and \( \mu=0.01. \) The unsteady algorithm presented in Chapter 2 was used to march forward in time towards the steady solution. The solution converged (at about \( t=50, \) corresponding to 1000 time steps) to the steady-state flow pattern with vortices that were symmetric up to round-off error. The final drag coefficient was 0.98 which is slightly lower than the recent values of 1.06 obtained by Fornberg [1980,1985] who used very fine meshes to solve the Navier-Stokes equations in steady form. Dawson and Marcus [1970] solved the unsteady equations and obtained a value of 0.98, which at the final time of \( t=9.6, \) was still slowly decreasing. Swanson and Spaulding [1978] obtained a larger value of 1.16 using a MAC method.

The pressure coefficient around the cylinder surface is presented in Figure 3.1-2. The pressure at the forward stagnation point (\( \theta=180 \) degrees from the positive x-axis) was 0.542 and the pressure at the wake centerline (\( \theta=0 \) degrees) was -0.193. Comparable values obtained by Fornberg [1980] were 0.53 and -0.17 respectively, indicating that the differences in computed drag coefficient were not due to the pressure solution in these locations. Brooks and Hughes [1982] obtained values of 0.62 and -0.25 using a Petrov-Galerkin finite element method, however they do not state a value for their overall drag coefficient. The pressure drag coefficient in the present solution was about 72 percent of
the overall drag coefficient, a value also obtained by Dawson and Marcus [1970].

The separation angle and the separation distance were 62.5 degrees and 5.96 cylinder diameters, respectively. The separation angle was measured relative to the positive x-axis and the separation distance was measured from the cylinder center to the downstream stagnation point (the point along the x-axis with zero velocity). Swanson and Spaulding [1978] obtained values of 66.1 degrees and 6.54 cylinder diameters, while Fornberg computed values of about 67 degrees (approximated from a graph) and 6.85 cylinder diameters. Dawson and Marcus obtained values of 60 degrees and 2.7 diameters, however the fact that their drag coefficient was still decreasing indicates that their vortices were still growing in size.

3.2 Unsteady Flow

In all computational methods which obtain a stable symmetric pair of standing vortices, it is necessary to introduce a disturbance into the flow-field in order to obtain an alternate periodic shedding of vortices. This was performed by imposing a rotational boundary condition on the cylinder of

\[ \omega = 2.0 \text{ (ccw)} \text{ for } 10.0 \leq t \leq 10.5 \]
\[ \omega = -2.0 \text{ (cw)} \text{ for } 10.5 \leq t \leq 11.0 \]
which disturbed the flow-field enough to induce vortex-shedding fairly rapidly.

The flow solution was continued after the perturbation until the lift force from the vortex-shedding reached a constant period and amplitude as shown in Figure 3.2-1. The final maximum amplitude of the lift coefficient was 0.23 which is somewhat lower than the value of 0.345 obtained by Swanson and Spaulding [1978] (note that their solution is subject to "wiggles"). The Strouhal number was 0.154, a value well within the range of experimental data (see Figure 1.1-2). In fact some researchers such as Tritton [1959,1971] and Gerrard [1978] report indications of a discontinuity in the Strouhal-Reynolds number relationship near this Reynolds number (Re = 90-110).

Figure 3.2-2 shows a similar plot of the drag coefficient which oscillates at a frequency twice that of the lift coefficient. It also contains a second peak during each of its periods. A view of the streamline plots during a drag period (Figure 3.2-3a-e) reveals that this phenomenon is accompanied by a temporary shrinking of the weaker (lower) vortex, shortly after the stronger (upper) vortex has been shed. This effect was also observed in the computational results of Gresho [1985] at Re = 200. To understand the cause of this phenomenon, consider the generalized circulation theorem for the change of circulation in an area S bounded by a curve c in an
Figure 3.2-1 Lift Coefficient Time History for the Unsteady Flow Solution
Figure 3.2-2 Drag Coefficient Time History for the Unsteady Flow Solution
Figure 3.2-3(b) Streamline Plots for the Unsteady Flow Solution
Figure 3.2-3(d) Streamline Plots for the Unsteady Flow Solution
Figure 3.2-3(e) Streamline Plots for the Unsteady Flow Solution
incompressible fluid (free of gravitational effects)

\[ \frac{D\Gamma}{Dt} = \nu \int_\mathcal{C} \nabla^2 \mathbf{V} \cdot d\mathbf{c} \quad (3.2.1) \]

where \( d\mathbf{c} \) is a differential length tangent to the path of line integration and \( \mathbf{V} \) is the velocity vector. From vector analysis

\[ \nabla^2 \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V}) \quad (3.2.2) \]

which for an incompressible fluid reduces to

\[ \nabla^2 \mathbf{V} = -\nabla \times \mathbf{\xi} \quad (3.2.3) \]

where \( \mathbf{\xi} = \nabla \times \mathbf{V} \) is the vorticity. If the flow is irrotational the vorticity is zero and the circulation remains constant in time.

Substitution of equation (3.2.3) into equation (3.2.1) gives

\[ \frac{D\Gamma}{Dt} = -\nu \int_\mathcal{C} (\nabla \times \mathbf{\xi}) \cdot d\mathbf{c} \quad (3.2.4) \]

When Stokes theorem is applied to equation (3.2.4) the equation may be expressed in terms of surface integrals as

\[ \frac{D\Gamma}{Dt} = -\nu \int_A [\nabla \times (\nabla \times \mathbf{\xi})] \cdot \mathbf{n} \quad (3.2.5) \]

where \( \mathbf{n} \) is the unit vector normal to the surface bounded by the previous path of line integration \( \mathcal{C} \). Equation (3.2.5) must hold for any point in the flow-field, therefore it can be written as
\[\lim_{A \to \infty} \left\{ D \left( \int_A \nabla \cdot \mathbf{n} \right) \right\}/Dt = -\lim_{A \to \infty} \int_A \left[ \nabla \times (\nabla \times \bar{\xi}) \right] \cdot \mathbf{n} \]  \hspace{1cm} (3.2.6)

which reduces to

\[D \left( \lim_{A \to \infty} \int_A \nabla \cdot \mathbf{n} \right)/Dt = -\lim_{A \to \infty} \int_A \left[ \nabla \times (\nabla \times \bar{\xi}) \right] \cdot \mathbf{n} \]  \hspace{1cm} (3.2.7)

and finally to

\[D\bar{\xi}/Dt = -\nu [\nabla \times (\nabla \times \bar{\xi})] \]
\[= -\nu [\nabla(\nabla \cdot \bar{\xi}) + \nabla^2 \bar{\xi}] \]  \hspace{1cm} (3.2.8)

and since

\[\nabla \cdot \bar{\xi} = \nabla \cdot (\nabla \times \bar{V}) = 0, \]  \hspace{1cm} (3.2.9)

then

\[D\bar{\xi}/Dt = \nu \nabla^2 \bar{\xi} \]  \hspace{1cm} (3.2.10)

which is the expression for the diffusion of vorticity of every fluid particle in a viscous incompressible two dimensional flow-field. Equation (3.2.10) is known as the vorticity transport equation, and may also be derived directly from the Navier-Stokes equations. The left-hand side of equation (3.2.10) is the substantial derivative of the vorticity which is defined by

\[D\bar{\xi}/Dt = \partial \bar{\xi}/\partial t + (\bar{V} \cdot \nabla) \bar{\xi} \]  \hspace{1cm} (3.2.11)

Substitution of equation (3.2.11) into equation (3.2.10) gives
\[ \dot{\zeta}/\dot{t} = - (\nabla \cdot \nabla) \zeta + \nu \nabla^2 \zeta \]  (3.2.12)

Equation (3.2.12) is an expression for the change in vorticity at each point in the flow-field, both by convection and diffusion. Note that a fluid particle may experience a change in its vorticity only through the action of viscosity as expressed in equation (3.2.10), while the vorticity at each point in the flow-field may change both by diffusion and by the movement of the fluid particles by convection, as expressed in equation (3.2.12).

Consider what occurs if a vortex of finite area and initially uniform vorticity (say positive) is present in a viscous incompressible fluid which is initially at rest (and therefore irrotational) outside the vortex, as shown in Figure 3.2-4. Neglecting boundary effects, the vortex will diffuse its vorticity uniformly around its perimeter, a process governed by equation (3.2.10). The change of vorticity at each point in the fluid may be examined by expressing equation (3.2.12) in \([r, \theta]\) coordinates which gives

\[
\dot{\zeta}/\dot{t} = - v_r (\partial \zeta/\partial r) - (v_\theta/r) (\partial \zeta/\partial \theta) \\
+ \nu [\partial^2 \zeta/\partial r^2 + (1/r)(\partial \zeta/\partial r) \\
+ (1/r^2)(\partial^2 \zeta/\partial \theta^2)] 
\]  (3.2.13)

where \(\zeta\) is the component of the vorticity vector normal to the \([r, \theta]\) plane. For this single vortex, it is assumed that
Figure 3.2-4 Single Vortex Placed in a Fluid Initially at Rest
there are no circumferential changes, and therefore the radial velocity component is zero since mass must be conserved in the region initially occupied by the vortex, called region Q. Therefore, the convective terms may be neglected, and equation (3.2.13) reduces to

$$\frac{\partial \xi}{\partial t} = \nu \left[ \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \left( \frac{\partial \xi}{\partial r} \right) \right]$$

(3.2.14)

Consider point P in Figure 3.2-4. Each term in the right-hand side of equation (3.2.14) is negative at this point, so that vorticity is diffused outward from the vortex. Since it has been assumed that there are no changes in the \( \theta \) direction, the entire region Q will diffuse vorticity outward, and therefore the circulation of this area will decrease. Each fluid particle in region Q will also diffuse its vorticity, however it will remain in region Q and continue to rotate in the \( \theta \) direction, with the viscous forces causing it to slow down.

If a vortex of negative uniform vorticity is placed near the first vortex as shown in Figure 3.2-5, both vortices will diffuse more of their vorticity in the region of the other vortex than in the region where the outer fluid is irrotational. This is because the region near the opposite vortex has the largest potential for vorticity diffusion. In addition, vorticity will be convected into the regions with the largest amount of vorticity diffusion. For example, in the positive vortex the gradient \( \partial \xi / \partial \theta \) is
Figure 3.2-5 Two Vortices with Vorticity of Opposite Sign in a Fluid Initially at Rest
negative near the opposite vortex (since the vortex has lost more of its vorticity at this angle due to diffusion) so that the term \(-v_\theta/r\)(\partial \zeta/\partial \theta)\) is positive, meaning that vorticity is convected into this area. Likewise, in the negative vortex this gradient is negative near the opposite vortex, so that the term \(-v_\theta/r\)(\partial \zeta/\partial \theta)\) is negative \(v_\theta\) is negative in this case), and therefore vorticity is also convected into the area of maximum diffusion. The proximity of a vortex of opposite sign therefore not only weakens the vortex by diffusion, but by convection as well. The above situation somewhat resembles that of the cylinder steady flow solution where the two attached vortices with vorticity of opposite sign diffuse vorticity across the wake, however in this case they are being fed vorticity by their shear layers so that the net circulation of each vortex is constant.

During unsteady vortex shedding from a cylinder, the convection and diffusion of vorticity by the vorticies is complicated by the presence of the cylinder boundary and the shear layers. Shortly before a vortex is shed from the cylinder (here shedding is defined by movement of the vortex downstream) another vortex begins to form on the opposite side of the wake. As this second vortex begins to form, its shear layer is entrained by the fluid of the stronger vortex across the wake, which soon causes the vortex to shed. It is during the formation of the weaker vortex that the drag
coefficient begins to decrease. As more of the shear layer feeding the weaker vortex is drawn across the wake by the stronger vortex, the vortices begin to diffuse their vorticity more rapidly. This is due to a large area of the stronger vortex being exposed to the shear layer containing fluid with vorticity of opposite sign. The weaker vortex experiences a two-fold effect; its vorticity is diffused by the proximity of the stronger vortex, and its shear layer is drawn across the wake so that it is not fed as much vorticity. The result of this two-fold effect is that the weaker vortex shrinks in size and weakens until the stronger vortex moves downstream. The movement of the stronger vortex downstream allows the weaker vortex to grow again, and the process of vortex shedding repeats itself at this side of the cylinder.

If the vortices did not interact, the drag would decrease smoothly after a vortex is shed, and the time series for the drag coefficient in Figure 3.2-2 would be sinusoidal like that of the lift coefficient, with a frequency twice that of the lift coefficient. The vortex interaction causes the process of drag reduction to reverse itself temporarily, and produces a second peak in each period of the drag record. The vortex interaction does not seem to affect the sinusoidal appearance of the lift record in Figure 3.2-1. However the peak to peak values of the drag coefficient are many times smaller than that of the lift
record, and therefore may reveal fluctuations of the fluid forces on the cylinder more than the lift record.

This vortex interaction lowers the Strouhal frequency since it delays the process of vortex growth. These vortex interactions may explain the decrease in the Strouhal number as the Reynolds number is decreased through the laminar flow range, since changes in the Reynolds number affects the vortex strength and the ratio of convective to diffusive forces in the wake. Section four discusses the effect of cylinder motion on this phenomenon, and addresses the discontinuity observations of Tritton [1959,1971] and Gerrard [1978].

The motion of the separation angles is shown in Figure 3.2-6. The mean separation angle has decreased to 56.5 degrees from the symmetric solution value of 62.5 degrees. Each angle reaches a minimum several time steps before a vortex on its side is shed.

Figure 3.2-7 shows the pressure coefficients for maximum and minimum lift along the cylinder surface. The maximum lift occurs just before a vortex is shed from the upper portion of the cylinder, while the minimum lift occurs just before a vortex is shed from the lower portion.

Simultaneous examination of both Figure 3.2-6 and Figure 3.2-7 reveals some additional insight into the process of vortex shedding. Consider the separation angle for the top portion of the cylinder. The separation angle
Figure 3.2-6 Separation Angles Time Histories for the Unsteady Flow Solution
Figure 3.2-7  Pressure Coefficients around the Cylinder Surface for Maximum and Minimum Lift
here reaches a maximum just before a vortex is shed from the bottom portion of the cylinder surface. This is due to the high pressure on this part of the surface when the lower vortex is large, which causes the boundary layer to separate at a larger angle, since it is unable to overcome the increased pressure force. Once the lower vortex is shed, the separation angle decreases sharply, reaching a minimum shortly after the upper vortex begins to form. As the upper vortex grows larger, the upper separation angle increases, however the fluid particles in the boundary layer are able to traverse farther along the cylinder surface, since the vortex growth has lowered the pressure in this area. The result of these two opposing phenomena is a thinning of the shear layer at the separation point, which eventually causes the shear layer to detach from the cylinder and the vortex then sheds. As noted above, this is accompanied by entrainment of the shear layer from across the wake. The separation angle slowly increases during the vortex formation and continues to increase after the vortex is shed. The net effect of vortex shedding on the separation angle is seen in Figure 3.2-6 where the time of the half cycle from the minimum separation angle to the maximum separation angle is about 10-15 percent longer than the other half cycle.
Chapter 4

FLOW OVER A SPRING-MOUNTED CYLINDER DURING LOCK-IN

4.1 Effect of Structural Damping

The oscillating lift force on a spring-mounted cylinder during vortex shedding can induce large transverse oscillations. The largest amplitudes will occur when the vortex shedding frequency is near the structural resonant frequency, however there is a range of frequencies near the resonant frequency in which the cylinder motion and the lift will oscillate at the same frequency; a condition known as lock-in. Lock-in is accompanied by an increase in the vortex strength which thereby increases the lift force, resulting in growth of the cylinder oscillation amplitude. As the amplitude grows larger, the vortex formation is suppressed, so that the vortex-induced forces on the cylinder tend to be self-limiting.

The presence of structural damping causes both the structural frequency and the cylinder amplitude to decrease. This section presents results for flow-induced transverse oscillations of a spring-mounted cylinder with various amounts of damping present. In each simulation the cylinder mass was selected to produce a mass ratio of 5.0, and the spring constant was selected so that the undamped natural frequency coincided with the Strouhal frequency of the stationary cylinder simulation \( f_{SO} = 0.154 \).
Consider the single degree of freedom system consisting of a spring, mass, and damper shown in Figure 4.1-1. The equation of motion for this system is

$$my_{tt} + cy_t + ky = 0 \quad (4.1.1)$$

where $m$ is the mass of the body, $c$ is the damping constant, and $k$ is the spring constant. The undamped natural frequency ($c=0$) for this system is

$$f_n = (k/m)^{1/2} / 2\pi = \omega_n / 2\pi \quad (4.1.2)$$

and the damped natural frequency is

$$f_d = [(k/m) - (c^2/4m^2)]^{1/2} / 2\pi = \omega_d / 2\pi \quad (4.1.3)$$

Equation (4.1.3) can also be expressed in terms of the undamped natural frequency as

$$\omega_d = \omega_n(1 - \xi^2)^{1/2} \quad (4.1.4)$$

The damping ratio $\xi$ is defined as

$$\xi = c / c_{cr} \quad (4.1.5)$$

where $c_{cr}$ is the critical damping constant defined as

$$c_{cr} = (4mk)^{1/2} \quad (4.1.6)$$

If the simple spring-mass-damper system is subjected to a harmonic forcing function $F \sin \omega t$, then the equation of motion becomes
Figure 4.1-1 Single Degree of Freedom Spring, Mass and Damper System (Beachley and Harrison [1978])
\[ m_y + c_y + k_y = F \sin \omega t. \quad (4.1.7) \]

The steady state solution (i.e. the transient terms are neglected) of equation (4.1.7) is

\[ y = A \cos \omega t + B \sin \omega t \quad (4.1.8) \]

where

\[ A = -\frac{F(\omega)}{m^2[(\omega_n^2 - \omega^2)^2 + (c \omega/m)^2]} \quad (4.1.9a) \]

and

\[ B = \frac{F(\omega_n - \omega)}{m[(\omega_n - \omega^2)^2 + (c \omega/m)^2]}. \quad (4.1.9b) \]

The magnitude of the motion \( y \) is

\[ y = \frac{F}{k[(1 - \omega_r^2)^2 + (2\zeta \omega_r)^2]^{1/2}} \quad (4.1.10) \]

and the phase angle \( \phi \) is

\[ \tan \phi = \frac{-A}{B} = \frac{(2\zeta \omega_r)}{(1 - \omega_r^2)} \quad (4.1.11) \]

where

\[ \omega_r = \frac{\omega}{\omega_n} \quad (4.1.12) \]

If \( \zeta = 0 \), then the phase angle is 0 degrees when \( \omega < \omega_n \) and 180 degrees when \( \omega > \omega_n \). If \( \omega = \omega_n \) the phase angle is 90 degrees. Figures 4.1-2 and 4.1-3 show the amplitude ratio \((y/k)/F\) and the phase angle \( \phi \) for various frequency ratios \( \omega_r \). Note that increasing the damping ratio has both the effect of decreasing the amplitude ratio for all frequency
Figure 4.1-2 Amplitude Ratio ($Y_k/F$) versus Frequency Ratio (Beachley and Harrison [1978])
Figure 4.1-3  Phase Angle versus Frequency Ratio
(Beachley and Harrison [1978])
ratios, and the effect of shifting the phase angle towards $\pi/2$. The maximum amplitude ratio (found from equation 4.1.10) occurs when

$$\omega_r = (1 - 2\zeta^2)^{1/2} \quad (4.1.13)$$

The maximum amplitude ratio occurs at frequencies slightly lower than the damped natural frequency $\omega_d$ defined by equation (4.1.3). Note that the application of the above equations to the vortex-induced vibration of a spring-mounted cylinder is valid as long as the lift force obeys the assumptions of equation (4.1.7) such that its amplitude is constant and the force is harmonic.

The above set of equations are simplified when $\omega = \omega_s = \omega_n$ which was set initially here, however changes in the vortex interactions caused the Strouhal frequency to increase sharply when the cylinder was allowed to oscillate transversely. The vortex interactions inhibited vortex growth of the attached vortex for the still cylinder case discussed in Chapter 3, but this effect diminished as the amplitude of the cylinder motion increased; allowing the vortices to grow faster and resulting in a substantial increase in the Strouhal frequency. The resulting transverse vibrations were not at resonance, however steady amplitudes at lock-in were obtained for each damping ratio.

Figure 4.1-4 is a plot of the cylinder amplitude as a
Figure 4.1-4 Cylinder Amplitude versus Damping Ratio
function of the damping ratio, and Figure 4.1-5 is a plot of the actual frequency ratio $\omega_s/\omega_n$ as a function of the cylinder amplitude. The amplitudes were output for each simulation and used with the simulation parameters to determine $\omega_s/\omega_n = \omega_r$ in equation (4.1.10). The frequency ratio was also verified with the time series for cylinder amplitude and lift coefficient, but equation (4.1.10) was used instead of linear interpolation between time steps in order to provide greater accuracy.

Both Figure 4.1-4 and Figure 4.1-5 indicate limiting values. The limiting cylinder amplitude for large damping ratios is of course zero, while the limiting frequency ratio for large cylinder amplitudes is about 1.24. The corresponding maximum Strouhal frequency $f_s$ is about 0.19; a value near the upper range of the experimental values.

An important feature of Figure 4.1-5 is that it indicates a significant increase in the frequency ratio even at low cylinder amplitudes. This result has important consequences for researchers who have observed a discontinuity in the Strouhal-Reynolds number relationship near Re=100, since their observations may be influenced by small cylinder vibrations in their experiments.

A test attempting to obtain two different Strouhal shedding frequencies at this Reynolds number was performed for flow over a stationary cylinder by approaching the unsteady Re=100 flow solution from two different Reynolds
Figure 4.1-5 Actual Frequency Ratio versus Cylinder Amplitude
numbers. In the first case, the flow solution for $Re=60$ was obtained and then the Reynolds number was slowly increased to $Re=100$ by decreasing the viscosity throughout the flowfield. In the second case, the flow solution for $Re=150$ was obtained and then the Reynolds number was slowly decreased to $Re=100$ by increasing the viscosity throughout the flowfield. Both cases produced Strouhal shedding frequencies identical to the results presented in Chapter 3 for unsteady flow over a still cylinder when $Re=100$ throughout the simulation ($f_{so}=0.154$).

The results of this test indicate that either the discontinuity is not present, or that it is a three-dimensional (spanwise) effect. If it is a spanwise phenomena, then a three-dimensional numerical simulation would be required in order to obtain two different Strouhal frequencies at this Reynolds number. The cause could pertain to the spanwise correlations of the vortex shedding, or to the end conditions on the cylinder in the laboratory.

If the discontinuity does not exist, then Figure 4.1-5 indicates that small cylinder vibrations are a likely cause of the observed differences in the Strouhal frequency near $Re=100$. For tests involving a mounted cylinder, the experimentalist must insure that the entire length of the cylinder is stationary. Even small deflections (or sections of the cylinder span deflecting more than others) might cause enough spanwise changes to induce a different
Figure 4.1-6  Mean Drag Coefficient versus Cylinder Amplitude
Figure 4.1-7  Peak-to-Peak Drag Coefficient versus Cylinder Amplitude
frequency of vortex shedding.

Figures 4.1-6 and 4.1-7 are plots of the mean and peak-to-peak drag coefficients as a function of cylinder amplitude. The mean drag coefficient appears to increase linearly with cylinder amplitude while the peak-to-peak drag coefficient appears to increase exponentially with cylinder amplitude. Specific time histories of the drag coefficient for each damping ratio are in Appendix C.

4.2 Effect of Structural Frequency

In the vortex-induced oscillation tests presented in Section 4.1, the change in the Strouhal frequency due to cylinder movement caused the Strouhal frequency to deviate substantially from resonance. This section reports the results of an additional test which was performed in order to investigate the effect of approaching resonance on the cylinder motion and lift force. The resonance condition was approached by changing the structural natural frequency through modification of the spring constant. The objective was to increase the structural natural frequency so that it approached the Strouhal frequency for the undamped oscillating cylinder case reported in Section 4.1 ($f_s=0.189$).

Table 4.2-1 lists the results of the test described above. The quantity $f_{nb}/f_{no}$ is the ratio between the structural natural frequency used in this test and the structural natural frequency used in Section 4.1.
<table>
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</table>
(f_s = f_{so} = 0.154). The actual ratios used in this test were 1.06 and 1.12; attempts to increase this ratio to 1.15 (and obtain results closer to resonance) failed to achieve lock-in during the first few cycles of vortex shedding.

Despite the limited range of ratios used in this test, the results listed in Table 4.2-1 indicate the trends for the lift coefficient, the Strouhal frequency, and the cylinder amplitude as resonance is approached. When any linear spring-mass system approaches resonance during forced harmonic vibration, the amplitude ratio (Y_k/F) approaches infinity. As resonance was approached in this test, the lift force decreased sharply, so that even though the amplitude ratio increased sharply, the actual cylinder amplitude increased only slightly. This self-limiting effect of vortex-induced vibration of a cylinder is well known by experimentalists, although the exact cause of this self-limiting effect is unknown.

Figures 4.2-1a-h are streamline plots of the first case in Table 4.2-1 (f_{nb}/f_{no} = 1.0) plotted one-eighth of a cycle apart. These figures show that near the maximum cylinder amplitude the formation of vortices becomes suppressed. Figures 4.2-2a-h are streamline plots of the last case (f_{nb}/f_{no} = 1.10) in Table 4.2-1 plotted one-eighth of a cycle apart. The latter set of figures are nearly identical with the set of figures for the first case in that they show suppression of the vortex formation, although in the latter
Figure 4.2-1(c) Streamlines for $f_{nb}/f_{no}=1.0$
Figure 4.2-1(d) Streamlines for \( \frac{f_{nb}}{f_{no}} = 1.0 \)
Figure 4.2-1(e) Streamlines for $f_{nb}/f_{inc}=1.0$
Figure 4.2-1(g) Streamlines for $f_{nb}/f_{no}=1.0$
Figure 4.2-1(h) Streamlines for $f_{nb}/f_{no}=1.0$
STREAMLINES AT TIME - 167.30

Figure 4.2-2(a): Streamlines for $f_{nb}/f_{po}=1.2$
Figure 4.2-2(b) Streamlines for $f_{nb}/f_{no}=1.2$
STREAMLINES AT TIME = 168.60

Figure 4.2-2(c) Streamlines for $f_{nb}/f_{no}=1.2$
Figure 4.2-2(d) Streamlines for $f_{nb}/f_{no}=1.2$
Figure 4.2-2(f) Streamlines for $f_{hb}/f_{no}=1.2$
STREAMLINES AT TIME = 171.85

Figure 4.2-2(h) Streamlines for $f_{nb}/f_{no}=1.2$
set of figures the vortex suppression is slightly more pronounced. This indicates that as resonance is approached, the decrease in the lift force is accompanied by suppression of vortex formation which places a limit on the amplitude of the cylinder vibration. The vortex suppression itself is influenced by the velocity of the cylinder through the no-slip boundary condition. The motion of the cylinder induces motion on the fluid particles near the cylinder such that the boundary layer remains near or attached to the cylinder surface. The conclusion is that for a given lock-in frequency and Reynolds number, the maximum cylinder amplitude during vortex-induced vibration is the amplitude at which the vortex formation is entirely suppressed, resulting in a decrease in the lift force on the cylinder as the maximum cylinder amplitude is approached.
Summary, Conclusions, and Recommendations

A model was developed for solving the two-dimensional incompressible Navier-Stokes equations for flow-fields containing one or more moving boundaries. The model was used to simulate flow over a stationary cylinder at Re=100, as well as vortex-induced oscillation of an elastically mounted cylinder during lock-in. A body-fitted coordinate technique was used in order to transform all curved boundaries into a computational plane that remained fixed regardless of the motion of the boundaries in the physical plane.

For flow over a stationary cylinder, results were presented for both steady and unsteady flows. The steady flow solution consisted of two stationary vortices adjacent to the cylinder in the wake, and was in excellent agreement with other recent computational solutions. The unsteady flow solution at Re=100 consisted of periodic vortex shedding at a Strouhal frequency of $f_{50}=0.154$. Particular attention was given to the current controversy over a possible discontinuity in the Strouhal-Reynolds number relationship at this Reynolds number. A computational experiment was performed in an attempt to obtain two different shedding frequencies at this Reynolds number. The experiment consisted of two simulations which obtained flow solutions at Re=60 and Re=150 and then slowly modified the viscosity until a solution at Re=100 was obtained. Both
solutions matched the initial result at $\text{Re}=100$ ($f_{so} = 0.154$), and therefore no discontinuity was found.

Two sets of simulations were performed which allowed the cylinder to oscillate transverse to the flow direction. Both sets were performed at $\text{Re}=100$ and a mass ratio of 5 during lock-in. The first set contained solutions for different amounts of structural damping on the mounted cylinder while the second set contained solutions for different structural frequencies.

The principal effect of increased damping was to lower the amplitude of cylinder vibration. Note that each simulation was started with the structural undamped natural frequency identical to the vortex-shedding frequency for the still cylinder case, however the cylinder oscillations had the effect of increasing the shedding frequency even at small cylinder amplitudes (corresponding to large amounts of damping). This result, together with the results of the still cylinder simulations, indicates that the discontinuity in the Strouhal-Reynolds number relationship observed by some experimentalists may be a result of small vibrations of the cylinder. The mean drag coefficient increased linearly with cylinder amplitude, while the peak-to-peak drag coefficient appeared to increase rapidly with increasing cylinder amplitudes.

The structural natural frequency was varied in an attempt to approach a resonance condition during lock-in; a
difficult task since the amplitude of the cylinder motion effected the Strouhal frequency and whether lock-in was established. It was concluded that for a given lock-in frequency and Reynolds number, the maximum cylinder amplitude during vortex-induced oscillation is the amplitude at which the vortex formation is entirely suppressed, resulting in a decrease in the lift force as resonance is approached.

The results and conclusions described above leave many questions about vortex-induced oscillation of a cylinder in the Re=100 range unanswered. The failure to find two Strouhal frequencies at this Reynolds number in two-dimensions implies that some three-dimensional numerical simulations should give more insight into whether the discontinuity exists at all. The experimentalist should be aware of these results when attempting tests in this Reynolds number range in the future, and should pay careful attention to possible problems resulting from vibrations of the test cylinder.

The numerical model developed in this investigation should be useful in solving several types of problems involving vortex-induced vibration including: non-circular cylinders; different Reynolds number flows in the laminar flow range; multiple cylinders in the flow-field; various structural problems (e.g. inelastically mounted cylinders); shear flows; flow over a cylinder near a boundary such as a
wall or the ocean floor; and as mentioned above, three-dimensional flow problems. With the rapid advances in computational hardware, it is reasonable to expect numerical solutions to the Navier-Stokes equations to play an increasingly important role in vortex-induced vibration problems.
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Appendix A

BODY FITTED COORDINATE GRID GENERATION

A.1 Mathematical Development of Body-Fitted Coordinate Transformations

The general transformation from the \([x,y]\) plane to the \([\xi,\eta]\) plane (see Figure 2.2-1) is given by

\[
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
= 
\begin{bmatrix}
\xi(x,y) \\
\eta(x,y)
\end{bmatrix}
\]  
(A.1.1)

whose Jacobian (based on the chain rule) is

\[
J_1 = \begin{bmatrix}
\xi_x & \xi_y \\
\eta_x & \eta_y
\end{bmatrix}
\]  
(A.1.2)

The inverse transformation is:

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
x(\xi,\eta) \\
y(\xi,\eta)
\end{bmatrix}
\]  
(A.1.3)

whose Jacobian is

\[
J_2 = \begin{bmatrix}
x_\xi & x_\eta \\
y_\xi & y_\eta
\end{bmatrix}
\]  
(A.1.4)

Since transformation (A.1.3) is the inverse of transformation (A.1.1), then \(J_1\) and \(J_2\) are related by

\[
J_1 = [J_2]^{-1}
\]  
(A.1.5)
which implies the relations

\[ \xi_x = y_\eta / J \quad (A.1.6a) \]
\[ \xi_y = -x_\eta / J \quad (A.1.6b) \]
\[ \eta_x = -y_\xi / J \quad (A.1.6c) \]
\[ \eta_y = x_\xi / J \quad (A.1.6d) \]

where

\[ J = \det[J_2] = x_\xi y_\eta - x_\eta y_\xi \quad (A.1.6e) \]

These relations can be used in conjunction with the chain rule to yield partial derivatives of a sufficiently differentiable function of \( x, y, \) and \( t \). For example

\[ f_x = f_\xi \xi_x + f_\eta \eta_x \quad (A.1.7a) \]
\[ = (f_\xi y_\eta - f_\eta y_\xi) / J. \quad (A.1.7b) \]

Other derivatives are transformed by similar applications of relations (A.1.6) and the chain rule. For a list of all derivative transformations used in this project see Appendix A.2.
A.2 List of Derivative Transformations

This section lists the derivative transformations used in this project. Most are taken from Thompson et al. [1974] which contains a more comprehensive list.

Definitions:

- \( f(x,y,t) \) - a twice continuously differentiable scalar function of \( x, y, \) and \( t. \)

- \( F(x,y) = iF_1(x,y) + jF_2(x,y) \) - a continuously differentiable vector valued function of \( x \) and \( y. \)

- \( a = x_\eta^2 + y_\eta^2 \)

- \( \beta = x_\xi x_\eta + y_\xi y_\eta \)

- \( \gamma = x_\xi^2 + y_\xi^2 \)

- \( J = x_\xi y_\eta - y_\xi x_\eta \)

- \( P = a x_\xi - 2\beta x_\xi \eta + \gamma x_\eta \eta \)

- \( Q = a y_\xi - 2\beta y_\xi \eta + \gamma y_\eta \eta \)

- \( \tau = (x_\eta Q - y_\eta P)/J \)

- \( \sigma = (y_\xi P - y_\eta Q)/J \)

First Derivative Transformations

\[
f_x \equiv (\partial f/\partial x)_y, t = (y_\eta f_\xi - y_\xi f_\eta)/J \tag{A.2.1a}
\]

\[
f_y \equiv (\partial f/\partial y)_x, t = (x_\xi f_\eta - x_\eta f_\xi)/J \tag{A.2.1b}
\]
\[ f_t = (\partial f/\partial t)_x, y = (\partial f/\partial t)_x, y \]
\[ - (y_\eta f_x - y_x f_\eta) (\partial x/\partial t)_x, y \]
\[ - (x_\xi f_\eta - x_\eta f_\xi) (\partial y/\partial t)_x, y \]  
(A.2.1c)

**Second Derivative Transformations**

\[ f_{xx} = (\partial^2 f/\partial x^2)_y, t = \frac{(y_\eta f_{xx}^x - 2y_x f_\eta f_\xi + y_\xi f_{\eta\eta})}{J^2} + \left[ (y_\eta f_{xx}^x - 2y_x f_\eta f_\xi + y_\xi f_{\eta\eta}) (x_\eta f_x - x_\xi f_\eta) \right] / J^3 \]  
(A.2.2a)

\[ f_{yy} = (\partial^2 f/\partial y^2)_x, t = \frac{(x_\eta f_{yy}^x - 2x_x f_\eta f_\xi + x_\xi f_{\eta\eta})}{J^2} + \left[ (x_\eta f_{yy}^x - 2x_x f_\eta f_\xi + x_\xi f_{\eta\eta}) (y_\xi f_x - y_\eta f_\xi) \right] / J^3 \]  
(A.2.2b)

**Del Operator Transformations and Relations**

\[ \nabla^2 f = \frac{(a f_{xx}^x + y f_{yy}^x + y f_{yy}^y + x f_{xx}^x + x f_{xx}^y)}{J^2} \]  
(A.2.3a)

\[ \nabla f = \left[ (y_\eta f_x^x - y_x f_\eta) i + (x_\xi f_\eta - x_\eta f_x) j \right] / J \]  
(A.2.3b)

\[ \nabla \cdot F = \left[ y_\eta (F_1)_x^x - y_x (F_1)_\eta + x_\xi (F_2)_\eta - x_\eta (F_2)_x^x \right] / J \]  
(A.2.3c)

**Derivatives of \( \xi(x, y) \) and \( \eta(x, y) \)**

\[ \xi_x = y_\eta / J \]  
(A.2.4a)

\[ \xi_y = -x_\eta / J \]  
(A.2.4b)

\[ \eta_x = -y_\xi / J \]  
(A.2.4c)

\[ \eta_y = x_\xi / J \]  
(A.2.4d)

\[ \xi_{xx} = (\xi x y_\xi + \eta y y_\eta) / J - (\xi x^2 y_\xi - \xi x y_\eta) / J \]  
(A.2.5a)
\[ \xi_{yy} = -\left( \eta_x \eta \eta + \xi_y \xi \eta \right)/J - \left( \xi_y \eta \eta J \eta + \xi_y^2 J \xi \right)/J \]  
(A.2.5b)

\[ \xi_{xy} = \left( \eta_y \eta \eta + \xi_y \xi \eta \right)/J - \left( \xi_x \xi \eta J \xi + \xi_x \eta J \eta \right)/J \]  
(A.2.5c)

\[ \eta_{xx} = -\left( \xi_x \xi \xi \xi + \eta_x \eta \xi \eta \right)/J - \left( \xi_x \eta \xi \eta J \xi + \eta_x \xi J \eta \right)/J \]  
(A.2.5d)

\[ \eta_{yy} = \left( \eta_x \xi \xi \eta + \xi_x \xi \xi \eta \right)/J - \left( \xi_y \eta \eta J \xi + \eta_y \eta J \eta \right)/J \]  
(A.2.5e)

\[ \eta_{xy} = -\left( \eta_y \xi \xi \eta + \xi_y \xi \xi \eta \right)/J - \left( \xi_x \eta \eta J \xi + \eta_x \xi J \eta \right)/J \]  
(A.2.5f)
A.3 Algebraic Grid Generation Technique

The computational region for external flow over a moving cylinder in this project consists of the area between the cylinder surface and the boundary used to approximate the free-stream. The computational grid must place as many grid points as possible near the cylinder, while extending far enough away from the cylinder so that the location of the free-stream boundary affects the flow solution as little as possible. These two criteria imply using as many grid points as possible, however the number of grid points is proportional to the computational cost. The grid generation technique used here attempts to at least partially overcome this problem by using the grid points economically.

This project uses an algebraic grid generation technique to generate a body-fitted coordinate grid. The technique consists of an exponential interpolation between an inner and an outer boundary, with a high concentration of grid lines being placed near the inner boundary. The grid lines are restricted to be orthogonal at both the inner and outer boundaries in order to minimize error (see Thompson and Mastin [1985]).

Figure A.3-1 is useful in illustrating the sample calculation of a grid point location in the physical plane. Note that in the physical plane each \( \eta \)-line is a closed contour around the inner boundary, and each \( \xi \)-line consists of a curvilinear line segment from the inner boundary to the
outer boundary. The \((x,y)\) coordinates of each grid point on the boundaries is assumed to have been input and therefore known. The \((x,y)\) coordinates of each interior grid point is still unknown and are to be determined. This is equivalent to determining the \((x,y)\) values of each interior point in the \((\xi,\eta)\) plane of Figure 2.2-1, where the \((x,y)\) values of the boundary points (the points on \(\Gamma_1^*\) and \(\Gamma_2^*\)) are assumed to have been input.

In this grid generation technique, the \((x,y)\) values of the interior points are based upon only the \((x,y)\) values of the two endpoints in its \(\xi\)-line. This allows the grid point locations of each \(\xi\)-line to be computed independently.

Consider a \(\xi\)-line running from point \((a_1, b_1)\) on the inner boundary to point \((a_2, b_2)\) on the outer boundary as shown in Figure A.3-1. As discussed above, these coordinates are assumed to have been input and are therefore known. The distance from the inner boundary's center \((c_1, d_1)\) to point \((a_1, b_1)\) is denoted \(r_1\), and is essentially the "local radius" (note that the body is not restricted to be circular) of that point. This center can be either computed or input, in either case it is assumed to be known. Similarly, the distance from the outer boundary's center \((c_2, d_2)\) to point \((a_2, b_2)\) is denoted \(r_2\).

The radius (distance) from the inner boundary's center to a sample grid point \((x_i, y_j)\) is computed as

\[
R = r_1 \exp[Q(\eta_j/\eta_{\text{max}})] \quad \text{(A.3.1)}
\]
where \[ Q = \ln \left( \frac{r_2}{r_1} \right). \] (A.3.2)

Equation A.3.1 determines only the actual distance of the grid point being computed to the inner boundary center but does not determine its actual location. The angle \( \theta \) between the radial line joining the inner boundary center to the grid point being computed and the positive \( x \)-axis can be computed in order to uniquely determine the location of the grid point. This is accomplished by first requiring that the two closest grid points to each boundary on every \( \xi \)-line insure that the \( \xi \)-line is orthogonal to that boundary. This requires knowledge of the coordinates of the outer boundary center which may either be input or computed. The above orthogonality condition establishes \( \theta \) for \( \eta=1,2 \) (inner boundary is \( \eta=0 \)) and for \( \eta=\eta_{\text{max}}-1,\eta_{\text{max}}-2 \) (outer boundary is \( \eta=\eta_{\text{max}} \)) since the distances of these grid points from the inner boundary center can be computed independent of one another using equation A.3.1. The other angles are calculated based upon a weighted interpolation between \( \theta \) at \( \eta=2 \) and \( \theta \) at \( \eta=\eta_{\text{max}}-2 \). The weights are simply based upon \( r_1, r_2, \) and \( R \) such that \( \theta \) varies smoothly along each \( \xi \)-line. The grid can be established by calculating the grid point locations for all values of \( \eta \) on a given \( \xi \)-line and then sweeping through the grid for each \( \xi \)-line. Note that each \( \eta \)-line consists of simply connecting each point of constant \( \eta \) for all of the \( \xi \)-lines.

The actual coordinates \( (x_i, y_j) \) for each grid point are
computed from the corresponding $R$ and $\theta$ values by the trigonometric relations:

$$x_i = R\cos(\theta) + c_1 \quad (A.3.3a)$$

$$y_j = R\sin(\theta) + d_1. \quad (A.3.3b)$$

The above values $(x_i, y_j)$ are computed and stored for each grid point in the $[\xi, \eta]$ plane, and are used for all subsequent calculations. At each time step in which one of the boundaries moves (in this problem it is the cylinder), the grid point locations must be re-computed.

The general algorithm for computing the $(x,y)$ coordinates of each grid point in the physical plane for a field consisting of inner and outer closed boundaries is

1. Input the $(x,y)$ coordinates of each point on the boundaries. This is identical to inputting the $(x,y)$ values of each boundary point in the computational plane.

2. Compute or input the $(x,y)$ coordinates of the centers of both boundaries.

3. For the line $\xi=1$, compute the distances from the inner boundary center to the grid points corresponding to $\eta=1,2$ and $\eta=\eta_{\text{max}}-1, \eta_{\text{max}}-2$ using equation (A.3.1).

4. For the line $\xi=1$, compute the angles between a line joining the inner boundary center to the grid points of step 3, by requiring the two grid points closest to each boundary insure that the $\xi$-line
is orthogonal to that boundary.

5. For the line $\xi=1$, compute the distances and angles of the remaining grid points on that line using equation A.3.1 to compute the distances, and using a weighted interpolation between $\theta$ at $\eta=2$ and $\theta$ at $\eta=\eta_{\text{max}}-2$ to compute the angles. The exact interpolation method used should insure that $\theta$ varies smoothly along each $\xi$-line.

6. Repeat steps 3-5 for each $\xi$-line.

7. Compute and store the actual $(x,y)$ coordinates for each grid point using equations (A.3.3a) and (A.3.3b).

Note that the location of the re-entrant boundary is determined by specifying which grid points on the boundaries correspond to the line $\xi=1$. 
Appendix B

MODIFIED STRONGLY IMPLICIT PROCEDURE (MSIP)

B.1 General MSIP Theory

The modified strongly implicit procedure (MSIP) is an efficient method for solving a nine diagonal coefficient matrix resulting from a nine-point finite difference scheme. The original strongly implicit procedure (SIP) was introduced by Stone [1968] for solving pentadiagonal matrices resulting from five-point finite difference schemes. The method was developed for solving parabolic and elliptic equations in reservoir simulation and gave good results for simultaneous solution of multiphase flow equations (Weinstein et al. [1969a]) and three-dimensional equations (Weinstein et al. [1969b]). The method was extended to nine diagonal coefficient matrices by Schneider and Zedan [1981] for heat conduction problems, and yields superior results in comparison to the original SIP.

The modified strongly implicit procedure is developed to solve a system of nine point finite difference equations for the variable \( z \) of the form (see Figure B.1-1)

\[
A_{ij}^1 z_{i-1,j+1} + A_{ij}^2 z_{i,j+1} + A_{ij}^3 z_{i+1,j+1} + A_{ij}^4 z_{i-1,j} + A_{ij}^5 z_{i,j} + A_{ij}^6 z_{i+1,j} + A_{ij}^7 z_{i-1,j-1} + A_{ij}^8 z_{i,j-1} + A_{ij}^9 z_{i+1,j-1} = q_{ij}
\]  

(B.1.1)

where the \( i,j \) subscript denotes grid point location rather than matrix location. The system of equations can be
Figure B.1-1  Nine Point Finite-Difference Molecule
written in matrix form as

\[
[A] \{z\} = \{q\} \tag{B.1.2}
\]

where the coefficient matrix \([A]\) has the form

\[
[A] = \begin{bmatrix}
A_{ij}^5 & A_{ij}^6 & A_{ij}^1 & A_{ij}^2 & A_{ij}^3 \\
A_{ij}^4 & A_{ij} & A_{ij}^2 & A_{ij} & * \\
& A_{ij} & A_{ij}^2 & A_{ij} & * \\
& & A_{ij} & A_{ij}^2 & A_{ij} & * \\
& & & A_{ij} & A_{ij}^2 & A_{ij} & * \\
& & & & A_{ij} & A_{ij} & * \\
& & & & & A_{ij} & * \\
& & & & & & A_{ij}
\end{bmatrix}
\]

Note that the diagonals marked with an asterisk correspond to grid points having the same value for the \(i\) index (same grid column).

The matrix \([A]\) is now preconditioned such that when the preconditioned matrix

\[
[B] = [A] + [P] \tag{B.1.3}
\]

is factored into an \([L][U]\) product (where \([L]\) is a lower
triangular matrix and \([U]\) us an upper triangular matrix), \([L]\) and \([U]\) have nonzero diagonals only where the matrix \([A]\) has nonzero diagonals. The \([L]\) and \([U]\) matrices then have the form

\[
[L] = \\
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

\[
[U] = \\
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

Multiplication of \([L]\) and \([U]\) produces the equations used to determine the coefficients of \([L]\) and \([U]\):

\[
a_{i,j} = A_{i,j}^7 \quad (B.1.4a)
\]

\[
a_{i,j}f_{i-j-1} + b_{i,j} = A_{i,j}^8 \quad (B.1.4b)
\]

\[
b_{i,j}f_{i,j-1} + c_{i,j} = A_{i,j}^9 \quad (B.1.4c)
\]
\[
a_i, j^{h_i-1}, j-1 + b_i, j^{g_i}, j-1 + d_i, j = A_i, j^4 \\
a_i, j^{s_i-1}, j-1 + b_i, j^{h_i}, j-1 + c_i, j^{g_i+1}, j-1 + d_i, j^{f_i-1}, j + e_i, j = A_i, j^5 \\
b_i, j^{s_i}, j-1 + c_i, j^{h_i+1}, j-1 + e_i, j^{f_i}, j = A_i, j^6 \\
d_i, j^{h_i-1}, j + e_i, j^{g_i}, j = A_i, j^1 \\
d_i, j^{s_i-1}, j + e_i, j^{h_i}, j = A_i, j^2 \\
e_i, j^{s_i}, j = A_i, j^3
\]

and the matrix \([B]\)

\[
[B] = \begin{bmatrix}
\phi_{ij} & A_{ij}^1 & A_{ij}^2 & A_{ij}^3 \\
\phi_{ij}^2 & A_{ij}^4 & A_{ij}^5 & A_{ij}^6 & \phi_{ij}^3 \\
\phi_{ij} & A_{ij}^7 & A_{ij}^8 & A_{ij}^9 & \phi_{ij}^1 \\
\end{bmatrix}
\]
where: 

\[ \phi_{ij}^1 = c_{ij}^{f_{i+1,j-1}} \]  
(B.1.5a) 

\[ \phi_{ij}^2 = a_{ij}^{g_{i-1,j-1}} \]  
(B.1.5b) 

\[ \phi_{ij}^3 = c_{ij}^{g_{i+1,j-1}} \]  
(B.1.5c) 

\[ \phi_{ij}^4 = d_{ij}^{g_{i-1,j}} \]  
(B.1.5d) 

The diagonals in [B] defined by (B.1.5) compose the pre-conditioning matrix [P] and result from the incomplete L-U factorization (see Figure B.1-2). Both Stone [1968] and Schneider and Zedan [1981] use Taylor series expansions to obtain values for \( z_{i-2,j} \), \( z_{i+2,j} \), \( z_{i+2,j-1} \), and \( z_{i-2,j+1} \) which are:

\[ z_{i-2,j} = -z_{i,j} + 2z_{i-1,j} \]  
(B.1.6a) 

\[ z_{i+2,j} = -z_{i,j} + 2z_{i+1,j} \]  
(B.1.6b) 

\[ z_{i+2,j-1} = -2z_{i,j} + 2z_{i+1,j} + z_{i,j-1} \]  
(B.1.6c) 

\[ z_{i-2,j+1} = -2z_{i,j} + 2z_{i-1,j} + z_{i,j+1} \]  
(B.1.6d) 

These are used in conjunction with an adjustment parameter \( a_m \) to partially cancel the influence of the \( \phi_{ij} \) terms in matrix [B]. The system of finite-difference equations (B.1.1) are now of the form:

\[ A_{ij}^{1} z_{i-1,j+1} + A_{ij}^{2} z_{i,j+1} + A_{ij}^{3} z_{i+1,j+1} + A_{ij}^{4} z_{i-1,j} + A_{ij}^{5} z_{i,j} + A_{ij}^{6} z_{i+1,j} + A_{ij}^{7} z_{i-1,j-1} + A_{ij}^{8} z_{i,j-1} + A_{ij}^{9} z_{i+1,j-1} + \phi_{i,j}^{1} [z_{i+2,j-1} - a_m (-2z_{i,j} + 2z_{i+1,j} + z_{i,j-1})] + \phi_{i,j}^{2} [z_{i-2,j} - a_m (-2z_{i,j} + 2z_{i-1,j})] + \phi_{i,j}^{3} [z_{i+2,j} - a_m (-2z_{i,j} + 2z_{i+1,j})] \]
Figure B.1-2 Finite-Difference Molecule Resulting from Incomplete (L)-[0] Factorization
\[ + \phi_{i,j}^4 \left[ z_{i+2,j+1} - a_m(-2z_{i,j} + 2z_{i-1,j} + z_{i,j+1}) \right] = q_{i,j} \quad (B.1.7) \]

The effect of the adjustment parameter \( a_m \) is to modify the pre-conditioning matrix \([P]\) (and therefore \([B]\)) through the terms in equation (B.1.7) (note that this adds more diagonals to \([P]\)), in an attempt to speed up the convergence of the MSIP algorithm presented later in this section. Equations (B.1.4) can be modified to include the additional terms of equation (B.1.7) and rearranged to evaluate the components of the factorized matrices explicitly as:

\[
\begin{align*}
a_{i,j} &= A_{i,j}^7 \\
b_{i,j} &= A_{i,j}^8 - a_{i,j}f_{i-1,j-1} - a_m A_{i,j}^9 f_{i+1,j-1} / [1 - a_m(f_{i,j+1}f_{i+1,j-1})] \\
c_{i,j} &= A_{i,j}^9 - b_{i,j}f_{i,j-1} \\
d_{i,j} &= A_{i,j}^4 - a_{i,j}g_{i-1,j-1} - b_{i,j}g_{i+1,j-1} - 2a_m a_{i,j}g_{i-1,j-1} / (1 + 2a_m g_{i-1,j-1}) \\
e_{i,j} &= A_{i,j}^5 - a_{i,j}h_{i-1,j-1} - b_{i,j}h_{i+1,j-1} - c_{i,j}g_{i+1,j-1} - d_{i,j}f_{i-1,j} + a_m (2\phi_{i,j}^1 + \phi_{i,j}^2 + \phi_{i,j}^3 + 2\phi_{i,j}^4) \\
f_{i,j} &= A_{i,j}^6 - b_{i,j}s_{i-1,j-1} - c_{i,j}h_{i+1,j-1} - 2\alpha(\phi_{i,j}^1 + \phi_{i,j}^3) / e_{i,j} \\
g_{i,j} &= A_{i,j}^1 - d_{i,j}h_{i-1,j} / e_{i,j} \\
h_{i,j} &= A_{i,j}^2 - d_{i,j}s_{i-1,j} - a\phi_{i,j}^4 / e_{i,j} \\
s_{i,j} &= A_{i,j}^3 / e_{i,j} \\
\end{align*}
\]

and the \( \phi_{i,j} \) terms are evaluated as in (B.1.5) using the
quantities defined in equations (B.1.8). Note that the original SIP derivation follows exactly the one for MSIP presented above, with the exception that only the diagonals associated with a five point finite difference scheme are used in matrices [A], [L], and [U], giving $\phi_{ij}^2=\phi_{ij}^3=0$. (Note also that this gives different equations for the components of [B] than those in equations (B.1.8).

The general iterative algorithm for MSIP is as follows:
1. Add the vector $[P]{z}$ to both sides of the equation (B.1.2) to obtain
   \[ [A+P]{z} = {q} + [P]{z} \]  \hspace{1cm} (B.1.9)
2. Solve the iteration equation
   \[ [B]{z}^{k+1} = {q} + [P]{z}^k \]  \hspace{1cm} (B.1.10)
   where $k$ = iteration number. Equation (B.1.10) is equivalent to:
   \[ [L]{w}^{k+1} = {q} + [P]{z}^k \]  \hspace{1cm} (B.1.11)
   where:
   \[ {w}^{k+1} = [U]{z}^{k+1} \]  \hspace{1cm} (B.1.12)
   therefore, step 2 may be divided into the substeps:
   a) Solve for ${w}^{k+1}$ by forward substitution
   b) Solve for ${z}^{k+1}$ by backward substitution

The above two steps are repeated until ${z}^{k+1}$ is sufficiently close to ${z}^k$. Note that computation of the left-hand side of equation of (B.1.9) need only be performed on the first iteration (by storing [L] and [U]) if the coefficients in matrix [A] are not a function of the solution vector {z}. 
B.2. Use of MSIP with Body-Fitted Coordinate Systems

The procedure discussed in the previous section must be modified for use in body-fitted coordinate systems. This modification is required because the grid node numbering at the re-entrant boundary (see Figure 2.2-1) changes the form of the system of finite differences in equation (B.1.1). This section explains how to implement the modified strongly implicit procedure at this re-entrant boundary, while retaining the nine diagonal banded structure of the coefficient matrix.

Figure (B.2-1) displays the finite difference molecules for both $\xi=1$ and $\xi=\xi_{\text{max}}$ in the computational $[\xi, \eta]$ plane. Grid points in column $i-1$ for $\xi=1$ are those that lie on the line $\xi=\xi_{\text{max}}$ since the re-entrant line is double-valued in the computational plane (see Figure 2.2-1). Similarly, grid points in column $i+1$ for $\xi=\xi_{\text{max}}$ are those that lie on the line $\xi=1$. Note that the grid is numbered by $\eta$-line from left to right as in Figure B.2-2. For the line $\xi=1$, the finite difference equations will have the form

$$A_i,j^{n w z_{\text{max}}},j+1 + A_i,j^{n z_i},j+1 + A_i,j^{n e z_i+1},j+1$$
$$+ A_i,j^{w z_{\text{max}}},j + A_i,j^{0 z_i},j + A_i,j^{e z_i+1},j$$
$$+ A_i,j^{s w z_{\text{max}}},j-1 + A_i,j^{s z_i},j-1 + A_i,j^{s e z_i+1},j-1$$
$$= q_i,j \quad (B.2.1)$$

and for the line $\xi=\xi_{\text{max}}$, the finite difference equations will have the form
Figure B.2-1  Finite-Difference Molecules for $\xi = 1$ and $\xi = \xi_{\text{max}}$
where \( \text{imax}=\xi_{\text{max}} \) (number of \( \xi \)-lines). If the coefficients are numbered in equations (B.2.1) and (B.2.2) as they are for the other \( \xi \)-lines (i.e. as in the previous section: \( A^{nw}=A^1 \), \( A^n=A^2 \), etc.), then the coefficient matrix \([A]\) (see equation (B.1.2)) loses its nine diagonal banded structure. It will then have 11 diagonals and several of the coefficients will lie in the wrong diagonals (i.e. coefficient \( A_{i,j} 4=A_{i,j} W \) in equation B.2.1 will now lie in the diagonal that is supposed to be composed of \( A_{i,j} 1 \)).

Both of the problems mentioned above can be overcome by adjusting the numbering convention for the coefficients in order to keep the diagonals consistent and then moving the terms that lie outside the nine diagonal banded matrix structure to the right-hand side of the equation. The finite difference equations for line \( \xi=1 \) become

\[
A_{i,j}^{1z}_{\text{imax}},j + A_{i,j}^{2z_{i},j+1} + A_{i,j}^{3z_{i}+1,j+1} + A_{i,j}^{4z_{\text{imax}},j-1} + A_{i,j}^{5z_{i},j} + A_{i,j}^{6z_{i}+1,j} + A_{i,j}^{7z_{\text{i}},j-1} + A_{i,j}^{8z_{i}+1,j-1} = q_{i,j} - A_{i,j}^{\text{nw}z_{\text{imax}},j+1}
\]  

(B.2.3)

and the finite difference equations for line \( \xi=\xi_{\text{max}} \) become

\[
A_{i,j}^{1z_{i-1},j+1} + A_{i,j}^{2z_{i},j+1} + A_{i,j}^{4z_{i-1},j} + A_{i,j}^{5z_{i},j}
\]
\[+ A_{i,j}^6 z_{1,j+1} + A_{i,j}^7 z_{i-1,j-1} + A_{i,j}^8 z_{i,j-1} + A_{i,j}^9 z_{i,j} = q_{i,j} - A_{i,j}^{se} z_{1,j-1}\]  
(B.2.4)

Use of equations (B.2.3) and (B.2.4) will give consistent diagonals for matrices [A], [L], [U], and [B], however the diagonals \(\phi_{i,j}^2\) and \(\phi_{i,j}^4\) are now coefficients of \(z_{\text{max}-1,j-1}\) and \(z_{\text{max}-1,j}\) (instead of \(z_{i-2,j}\) and \(z_{i-2,j+1}\) respectively, in equation (B.2.3). Similarly, the diagonals \(\phi_{i,j}^1\) and \(\phi_{i,j}^3\) are coefficients of \(z_{2,j}\) and \(z_{2,j+1}\) (instead of \(z_{i+2,j-1}\) and \(z_{i+2,j}\) respectively, in equation (B.2.4). Equations (B.1.6a-d) are still used to partially cancel the influence of these terms so that for line \(\xi=1\), equation (B.1.7) is modified to be

\[A_{i,j}^1 z_{\text{max},j} + A_{i,j}^2 z_{i,j+1} + A_{i,j}^3 z_{i+1,j+1} + A_{i,j}^4 z_{\text{max},j-1} + A_{i,j}^5 z_{i,j} + A_{i,j}^6 z_{i+1,j} + A_{i,j}^7 z_{i-1,j-1} + A_{i,j}^8 z_{i,j-1} + A_{i,j}^9 z_{i+1,j}\]

\[+ \phi_{i,j}^1 [z_{i+2,j-1} - a_m(-2z_{i,j} + 2z_{i+1,j} + z_{i,j-1})]\]

\[+ \phi_{i,j}^2 [z_{\text{max}-1,j-1} - a_m(-2z_{i,j} + 2z_{\text{max},j} + z_{i,j-1})]\]

\[+ \phi_{i,j}^3 [z_{i+2,j} - a_m(-2z_{i,j} + 2z_{i+1,j})] + \phi_{i,j}^4 [z_{\text{max}-1,j} - a_m(-2z_{i,j} + 2z_{\text{max},j})] = q_{i,j} - A_{i,j}^{nw} z_{\text{max},j+1}\]

(B.2.5)

and for line \(\xi=\xi_{\text{max}}\), equation (B.1.7) is modified to be

\[A_{i,j}^1 z_{i-1,j+1} + A_{i,j}^2 z_{1,j+1} + A_{i,j}^3 z_{i-1,j} + A_{i,j}^4 z_{i,j} + A_{i,j}^5 z_{i,j} + A_{i,j}^6 z_{1,j} + A_{i,j}^7 z_{i-1,j-1} + A_{i,j}^8 z_{i,j-1} + A_{i,j}^9 z_{i,j}\]

\[+ \phi_{i,j}^1 [z_{2,j} - a_m(-z_{i,j} + 2z_{i,j})]\]

\[+ \phi_{i,j}^2 [z_{i-2,j} - a_m(-z_{i,j} + 2z_{i-1,j})]\]
\[ + \phi_{i,j}^3[z_{2,j+1} - a_m(-2z_{i,j} + 2z_{i,j} + z_{i,j+1})] \\
+ \phi_{i,j}^4[z_{i-2,j+1} - a_m(-2z_{i,j} + 2z_{i-1,j} + z_{i,j+1})] \]
\[ = z_{i,j} - A_{i,j} \delta z_{i,j-1} \]  

(B.2.6)

Use of equations (B.2.5) and (B.2.6) for equation (B.1.7) along the re-entrant boundary yields a system of finite difference equations that can be easily solved for \( z_{i,j} \) using the MSIP algorithm in the previous section. The effect of these equations on the performance of the iteration parameter \( a_m \) for heat conduction between concentric circles is discussed in the following section.
B.3. Performance of MSIP in Body-Fitted Coordinates for Heat Conduction between Concentric Infinite Cylinders

The extension of MSIP to body-fitted coordinate systems was tested for steady two-dimensional heat conduction between concentric infinite cylinders to determine its performance. This test was also used to determine the best adjustment parameter ($a_m$) for use in solving the Navier-Stokes equations.

The problem of steady two-dimensional heat conduction between concentric infinite cylinders consists of an inner cylinder of radius $r_1$ and an outer cylinder of radius $r_2$, with heat being conducted between the two cylinders. The governing partial differential equation (Laplace's equation) for steady heat conduction in [x,y] space is

$$T_{xx} + T_{yy} = 0. \quad (B.3.1)$$

The boundary conditions for the test problem were selected to be:

$$T = T_1 \quad \text{at} \quad x^2 + y^2 = r_1 \quad (B.3.2a)$$
$$T = T_2 \quad \text{at} \quad x^2 + y^2 = r_2. \quad (B.3.2b)$$

Note that the solution to this problem is

$$T = T_1 - (T_1 - T_2) \ln(r/r_1)/\ln(r_2/r_1). \quad (B.3.3)$$

This problem was chosen to test the MSIP extensions to body-fitted coordinate systems for the following reasons:
1. The problem geometry resembles that of fluid flow over a circular cylinder.

2. The governing Laplace equation closely resembles that of the Poisson equation for pressure, which is known from many previous investigations (e.g. Lecointe and Piquet [1984]) to constitute most of the computational cost.

3. The equation is relatively simple and has an available closed form solution.

The transformation of equations (B.3.1) and (B.3.2) to \([\xi, \eta]\) space gives:

\[\alpha T_{\xi\xi} - 2\beta T_{\xi\eta} + \gamma T_{\eta\eta} + \sigma T_{\eta} + \tau T_{\xi} = 0 \quad (B.3.4)\]

with boundary conditions

\[T = T_1 \quad \text{at} \quad \eta = 1 \quad (B.3.5a)\]

\[T = T_2 \quad \text{at} \quad \eta = \eta_{\text{max}} \quad (B.3.5b)\]

The boundary conditions used for the test problem were:

\[T = T_1 \equiv 1 \quad \text{at} \quad \eta = 1 \quad (r=r_{1}=1.0)\]

\[T = T_2 \equiv 5 \quad \text{at} \quad \eta = \eta_{\text{max}} \quad (r=r_{2}=5.0)\]

The MSIP extensions of Appendix B.2 were utilized, and second-order central derivatives were used to approximate all derivatives (including the grid metrics). The tolerance for convergence was defined as the sum of the absolute value of all of the residuals of every grid point, and the value
used was $1.0 \times 10^{-6}$.

Figure B.3-1 shows a plot of the number of iterations required to reach convergence as a function of the adjustment parameter for different grid sizes when $\xi_{\text{max}} = \eta_{\text{max}}$. As the number of grid points is increased, the optimum $a_m$ decreases slightly. The curves also become more convex, increasing the importance of choosing a good value of $a_m$ for fine meshes. The optimum $a_m$ occurs at $a_m > 0.9$ for all of the grid sizes in Figure B.3-1, while at $a_m = 1.0$ the number of iterations increase dramatically, indicating that values above the optimum should be used with caution.

Figure B.3-2 shows a similar plot when $\xi_{\text{max}} > \eta_{\text{max}}$ (this increases the grid aspect ratio). As in the previous figure, the optimum $a_m$ decreases as the number of grid points is increased. The curves also become more convex, and although the optimal $a_m$ for 80x10 and 160x10 are slightly less than 0.9, the method diverges for values close to 1.0 (in fact, for the 160x10 grid, the method diverges for $a_m > 0.95$). Note that for a given number of grid points, the increase in aspect ratio has the effect of substantially increasing the number of iterations required for convergence (by several times) for all values of $a_m$. 
Figure B.3-1 MSIP Adjustment Parameter Characteristics for $\xi_{\max} = \eta_{\max}$
ADJUSTMENT PARAMETER CHARACTERISTICS
HEAT CONDUCTION BETWEEN CIRCLES

Figure B.3-2 MSIP Adjustment Parameter Characteristics for $\xi_{\text{max}}$, $\eta_{\text{max}}$
Appendix C

COLLECTION OF FIGURES DESCRIBING STRUCTURAL DAMPING EFFECTS IN MOVING CYLINDER OSCILLATIONS
Figure C.1-3 Cylinder Amplitude for $\xi = 0.0$
PRESSURE COEFFICIENTS
DAMPING RATIO = 0.0

ANGULAR (DEGREES)

Figure C.1-4 Pressure Coefficients for $\zeta = 0.0$
STREAMLINES AT TIME = 119.15
DAMPING RATIO = 0.0

Figure C.1-5(a) Streamlines for $\zeta = 0.0$
STREAMLINES AT TIME = 119.80
DAMPING RATIO = 0.0

Figure C.1-5(b) Streamlines for $\zeta=0.0$
STREAMLINES AT TIME * 120.50
DAMPING RATIO * 0.0

Figure C.1-5(c) Streamlines for $\xi = 0.0$
Figure C.1.5(f) Streamlines for $\gamma = 0.0$
Figure C.1.5(g) Streamlines for $\xi=0.0$

STREAMLINES AT TIME $t=123.15$
DAMPING RATIO $\zeta=0.0$
Figure C.2-1 Lift Coefficient for $S=0.1$
Figure C.2-3 Cylinder Amplitude for $\zeta = 0.1$
PRESSURE COEFFICIENTS
DAMPING RATIO = 0.1

Figure C.2-4 Pressure Coefficients for \( \delta = 0.1 \)
Figure C.2-5(b) Streamlines for $\xi = 0.1$
Figure C.2-5(c) Streamlines for $\zeta = 0.1$
Figure C.2-5(d)  Streamlines for $\zeta = 0.1$
Figure C.2-5(e) Streamlines for $\zeta = 0.1$
STREAMLINES AT TIME = 144.60
DAMPING RATIO = 0.1

Figure C.2-5(f) Streamlines for $\zeta = 0.1$
STREAMLINES AT TIME: 145.25
DAMPING RATIO: 0.1

Figure C.2-5(g) Streamlines for $\xi=0.1$
STREAMLINES AT TIME = 145.95
DAMPING RATIO = 0.1

Figure C.2-5(h) Streamlines for $\zeta = 0.1$
Figure C.3-1 Lift Coefficient for $\zeta = 0.2$
Figure C.3-2 Drag Coefficient for $\zeta = 0.2$
CYLINDER AMPLITUDE
DAMPING COEFFICIENT = 0.2

Figure C.3-3 Cylinder Amplitude for $\zeta = 0.2$
PRESSURE COEFFICIENTS
DAMPING RATIO = 0.2

Figure C.3-4  Pressure Coefficients for $\zeta = 0.2$
STREAMLINES AT TIME = 165.15
DAMPING = 0.2

Figure C.3-5(a) Streamlines for $\xi = 0.2$
Figure C.3-5(b) Streamlines for $\tilde{\tau} = 0.2$

STREAMLINES AT TIME $\tau = 165.85$
DAMPING RATIO $\zeta = 0.2$
Figure C.3-5(c) Streamlines for $\xi=0.2$
Figure C.3-5(d) Streamlines for $\zeta = 0.2$
STREAMLINES AT TIME = 187.90
DAMPING RATIO = 0.2

Figure C.3-5(e) Streamlines for $\xi = 0.2$
STREAMLINES AT TIME = 168.60
DAMPING RATIO = 0.2

Figure C.3-5(f) Streamlines for $\zeta=0.2$
Figure C.3-5(h) Streamlines for $\xi=0.2$
Figure C.4-1 Lift Coefficient for $\xi = 0.3$
Figure C.4-2  Drag Coefficient for $\zeta=0.3$
Figure C.4-3 Cylinder Amplitude for $\beta = 0.3$
Figure C.4-4 Pressure Coefficients for $\zeta=0.3$
STREAMLINES AT TIME = 140.70
DAMPING RATIO = 0.3

Figure C.4-5(c) Streamlines for $\delta = 0.3$
Figure C.4-5(e) Streamlines for $S=0.3$
STREAMLINES AT TIME: 142.85
DAMPING RATIO: 0.3

Figure C.4-5(f) Streamlines for $\xi = 0.3$
Figure C.4-5(g) Streamlines for $\xi = 0.3$
Figure C.5.1 Lift Coefficient for $\zeta = 0.4$
Figure C.5-2 Drag Coefficient for $\zeta = 0.4$
Figure C.5-4  Pressure Coefficients for $\zeta = 0.4$
STREAMLINES AT TIME = 140.55
DAMPING RATIO = 0.4

Figure C.5-5(a) Streamlines for $\zeta = 0.4$
Figure C.5-5(c) Streamlines for $\gamma = 0.4$
Streamlines at time: 142.80
Damping ratio: 0.4

Figure C.5-5(d) Streamlines for $\gamma = 0.4$
STREAMLINES AT TIME = 144.20
DAMPING RATIO = 0.4

Figure C.5.5(f) Streamlines for $\zeta = 0.4$
STREAMLINES AT TIME = 144.95
DAMPING RATIO = 0.4

Figure C.5-5(g) Streamlines for $\zeta = 0.4$
Figure C.5-5(h) Streamlines for $\xi = 0.4$