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ISSUES CONCERNING SYSTEMS OF DEMAND EQUATION

by

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ISSUES CONCERNING SYSTEMS OF DEMAND EQUATIONS

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ABSTRACT

This dissertation examines two separate issues concerning systems of demand equations. The first essay deals with the econometric implementation of demand systems and the second analyzes the demand for integer-ordered goods.

In the first essay, we examine the consequences of adopting the random utility hypothesis as an approach for randomizing a system of demand equations. Random utility models are appealing since they allow the usual assumption of deterministic utility maximizing behavior by each consumer to co-exist with the apparent randomness across individuals which is exhibited by data. Our results show that the use of random utility models implies that the disturbances of the demand equations may not be homoscedastic but must be functions of prices and/or income. If the demand system is generated by random utility maximization, then empirical studies of demand which have assumed homoscedastic disturbances will suffer the usual inferential difficulties. A possible explanation for the prevalent finding of nonsymmetry, therefore, is provided by the fact that most previous demand studies have, in fact, assumed homoscedasticity. An appropriate structure for the disturbances is obtained for the specific case of the linear expenditure system. A
system of demand equations for international travel is estimated and compared using both the typical homoscedastic disturbances and the alternative specification we derive.

The second essay develops a model of consumer behavior when consumers are confronted with lumpy alternatives. Specifically, the case of a single discrete good which can be purchased only in integer quantities is modeled. The model uses a systems approach, incorporating demand for the discrete good with demand for continuous goods into a complete system of demand equations. After deriving a general form for the mixed integer/continuous model, a particular functional form for the model is derived by imposing linearity constraints on the demand equations. To obtain the stochastic specification, the framework of random utility maximization is adopted. Finally, the model is shown to be identified and an estimation procedure is developed using maximum likelihood methods.
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Issues Concerning Systems of Demand Equations

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ISSUES CONCERNING SYSTEMS OF DEMAND EQUATIONS

INTRODUCTION

In this thesis, two still unresolved problems concerning consumption analysis are addressed. The first essay deals with the link between the derivation of demand systems in theory and the empirical implementation of these models. The second essay provides a theoretic and econometric analysis of the demand for discrete goods within the framework of demand systems.

The first issue is the randomization of demand systems. The theoretical derivation of systems of demand equations is a rigorous exercise. Economists who have derived such systems do so with great care, ensuring that the equations adhere to demand theory restrictions and are sufficiently general yet do not require unrealistic consumer behavior. Before applying these demand systems to real world data it is necessary to make them stochastic. The method of specifying the stochastic component contrasts sharply with the method of specifying the systematic component - the most common procedure in applied work has been to assume an additive error term with mean zero and constant covariance.

The random utility hypothesis provides an intuitively more appealing approach to this problem. It assumes that while the consumer can completely determine her utility function, the econometric investigator
can only observe some of the relevant factors. Thus, the random element enters at the beginning rather than as an afterthought at the end. Essay I develops the implications of the random utility hypothesis on demand systems. Random utility maximization is performed for a specific demand system, the linear expenditure system. An empirical application of this theory is also provided. The demand for international travel is estimated using both the specific random utility model derived here and the usual linear expenditure system and the results are compared.

Essay II analyzes consumer demand for certain types of discrete goods. Traditional demand theory depends on the assumption that all goods can be consumed in continuously variable quantities. Since many of the choices confronting consumers involve goods which may be purchased only in discrete amounts, an extension of demand theory in this direction will be useful. The bulk of previous research into the demand for discrete goods has focused on the case of unordered alternatives, for example the choice of mode of transportation: bicycle, car, or bus. Another type of discrete good is the good which may be purchased only in integer quantities. Examples of integer ordered alternatives are certain durable goods such as televisions or air conditioners, or number of trips: how many trips should one make to the beach in a given year. While other economists and statisticians have examined the problem of ordered alternatives, this issue has not been treated in the context of consumer demand theory.

The essay develops both the theoretic and econometric implications of including an integer ordered good with other continuous goods in the consumer's choice set. A mixed integer/continuous demand system is
derived first in general terms and then the choice of a particular functional form is considered. The identification of this model is carried out and a possible maximum likelihood estimation procedure is outlined.
IMPLICATIONS OF THE RANDOM UTILITY HYPOTHESIS
FOR ESTIMATION OF DEMAND SYSTEMS

1. Introduction

Consumer demand theory is one of the most thoroughly examined areas in economics. Demand equations and systems of demand equations have been estimated for a vast array of commodity groups. Despite this abundance of empirical applications of demand theory, few economists have paid more than passing attention to the problem of correctly specifying the demand model's random component.

This lack of attention to the stochastic specification contrasts sharply with the care taken in formulating the systematic portion of the demand equation or utility function. Economists who derive systems of demand equations generally take great pains to ensure that their equations do not impose highly restrictive conditions and do not require unrealistic consumer behavior. Yet when these rigorously derived equations are applied, this same attention to rigor and generality is not extended to specifying the structure of the disturbances. As Barten [1977] remarks, "disturbances are usually tacked on to demand equations as a kind of afterthought."

In fact, the stochastic specification deserves careful consideration. An appropriate error structure is as essential to consumer demand analysis as the structure of the systematic component. Misspecifying the random component can cause errors in inference, hypothesis-testing and
assessment of goodness-of-fit. Barten [1977] suggests that one of the reasons for the continual rejection of the micro conditions of symmetry and homogeneity in applied work may be incorrect treatment of the error terms.

In this essay, we address the problem of appropriate stochastic specifications in demand analysis. In particular, we examine the consequences of adopting the random utility hypothesis as an approach for randomizing a system of demand equations. Our results show that the use of random utility models has strong implications for the error structure - specifically, the disturbances of the demand equations may not be homoscedastic but must be functions of prices and/or income. These implications have not been incorporated in previous empirical work.

In the second section of this essay, we discuss various approaches which have been taken to the problem of incorporating randomness into models - including the random utility hypothesis. The third section presents general theoretical results which derive from random utility maximization. In the fourth section, we derive the stochastic specification for the specific case of the linear expenditure system (LES) and show that the resulting model is identified. In the fifth section, we present results of some empirical tests. We compare the results from estimating a system of demand equations for international travel using both the typical "add-on" disturbance terms and the alternative specification we derive.
2. Alternative Error Specifications

Only a few economists have taken particular interest in the problem of specifying an error structure for consumer demand analysis. In this section, we discuss briefly the different methods employed by Barten [1968], Theil [1971, 1975], Pollak and Wales [1969], and McFadden [1982]. Although much of Barten's and Theil's work on this subject was performed jointly, we treat them separately primarily because Theil [1975] has substantially generalized his work in this area.

Barten [1968] speculated that the randomness existing in a demand model arises due to unobserved factors which cause changes in the individual's utility function. He proposed a stochastic utility function which has the quadratic approximation

\[ u(x) = a'x + (1/2)x'Ux \]

where \( x \) is an \( n \times 1 \) vector of quantities, \( a \) is an \( n \times 1 \) vector of random elements, and

\[ U = \begin{bmatrix} \frac{\partial^2 u(x)}{\partial x^2} \\ \frac{\partial^2 u(x)}{\partial x \partial x'} \end{bmatrix}. \]

By limiting the stochastic part to the linear portion of the approximation, Barten ensured that the disturbance terms would be additive. This is clear from the first order conditions of the utility maximization problem:

\[ \frac{\partial u(x)}{\partial x} = a + Ux = \lambda p. \]

Barten assumed that these unobserved random elements were independent of prices and income. This assumption resulted in the error terms attached to the demand equations being functions of the model's
parameters, but independent of the variables. The covariance matrix of this error specification is proportional to the matrix of cross price effects. Because the error terms are not correlated with the explanatory variables and the covariance matrix of the errors is constant, estimation problems are minimized.

Barten used this stochastic specification to estimate a system of demand equations for a group of composite goods. His results show relatively large estimated variances and covariances of the errors. He concluded that this demonstrates that the error terms are an important factor in explaining behavior.

Theil [1975] interpreted the randomness in the consumer's problem somewhat differently. He assumed that the consumer behaves statistically. This means that two stages exist in the consumer's decision-making process: the planning stage and the implementation stage. In the planning stage the individual uses a criterion function, \( f(z) \), which involves only the major factors relevant to the consumer's decision. Using the information \( f(z) \), the consumer derives a theoretically optimal decision, \( \tilde{z} \). The vector \( z \) is a theoretical optimum because it ignores various minor factors such as product quality variations or unexpected shortages which the consumer sees as random. The actual decision, \( z \), is made in the implementation stage; it reflects these minor factors. Thus, during the planning stage, the individual can only determine a decision distribution. The vector \( \tilde{z} \) is random because it depends on the realizations of the minor factors. Theil's analysis is directed toward finding criteria for the optimal decision distribution.

The optimal decision distribution must reflect both major and minor
factors. The major factors are represented by the systematic component of the actual decision, \( \bar{x} \), which is derived from the criterion function. There will always be some loss associated with the actual decision vector because \( \bar{x} \) is the vector which optimizes \( f(\ast) \). Thus, expected loss, \( E[f(\bar{x}) - f(z)] \) may be used to select the optimal decision distribution.

The consumer could achieve zero expected loss by choosing \( z = \bar{x} \), but this would exclude any consideration of the minor factors. The consumer actually prefers a distribution with a larger dispersion because this allows for more freedom of action in the implementation stage. Theil proposed the generalized variance as a dispersion measure. The generalized variance is defined as the determinant of the covariance matrix of \( \bar{x} - z \).

To derive the optimal decision distribution, the consumer optimizes a function

\[
\psi = \psi(L, G)
\]

where \( L \) is the expected loss and \( G \) is the generalized variance. The function \( \psi \) is decreasing in \( L \) and increasing in \( G \). To summarize Theil's results, he finds that the expected value of the optimal decision distribution is the theoretically optimal decision,

\[
E(z) = \bar{z}.
\]

The covariance matrix of the distribution is

\[
V = -kA^{-1}
\]

where \( k \) is any scalar and \( A \) is the Hessian matrix of the criterion function. Theil achieves these same results under less restrictive
assumptions for the dispersion measure than the generalized variance.

Theil derives the form of the disturbance terms for the linear expenditure system and the Rotterdam model. In both cases the covariance matrix of the disturbance terms does not involve any right-hand-side variables, but does involve the model's parameters. Theil also shows that his approach is equivalent to Barten's [1968] theory of random shifts in the quadratic approximation to the individual's preference ordering. The error terms derived for the Rotterdam model are the same under either approach.

Pollak and Wales [1969] focused on finding an appropriate error structure for the linear expenditure system, although their approach can easily be generalized to other models. They believed that the usual assumption of a constant covariance matrix was not plausible, that variances would tend to rise as income and prices rose. Their solution was to randomize a parameter, replacing \( \gamma_i \) in the demand equation

\[
x_i = \gamma_i + (\beta_1/p_1)(y - \sum_j \gamma_j)
\]

with \( \gamma_i = \gamma_i + \varepsilon_i \). This procedure results in a disturbance term with the form

\[
\nu_i = \varepsilon_i - (\beta_1/p_1)(\sum_j \varepsilon_j).
\]

They noted that this error term would result from the maximization of a stochastic utility function of the form

\[
U = \prod_{i=1}^{n}(x_i - \gamma_i - \varepsilon_i)^{p_i}.
\]

Even if simplifying assumptions are made about the distribution of the \( \varepsilon \)'s:
\[ E(\varepsilon) = 0 \]
\[ E(\varepsilon_i, \varepsilon_j) = 0 \quad i \neq j \]
and
\[ E(\varepsilon \varepsilon^t) = \Sigma \]

the covariance matrix of this model is complicated. Although the disturbance is independent of income, it involves prices and hence is not constant over time.

Since Pollak and Wales rejected the independence of the error term and income, they added a further complication to the above specification. They assumed that
\[ E(\varepsilon_i^2) = \sigma_i^2 x_i^2 \]
where
\[ x_i = E(x_i). \]

A higher level of income will increase \( E(x_i) \) and thus increase the variance of the \( i \)th equation.

Pollak and Wales estimated several versions of the linear expenditure system using this error structure. They demonstrated that even though the nonconstant covariance matrix complicated maximum likelihood estimation procedures, estimation was feasible.

The random utility hypothesis originated in the psychology literature on individual choice. Thurstone [1927] devised the concept of a random decision rule. McFadden [1982] has applied this hypothesis to consumer demand theory, assuming that utility is a random function in modelling the choice among discrete alternatives. While the use of random utility models by McFadden [1982, 1984] and Hanemann [1984] has, to our knowledge, been limited to discrete dependent variable models, the hypothesis applies equally well to the traditional case of all
continuous variables. In fact, Barten [1968] and Pollak and Wales [1969] were actually proposing random utility models although they did not link it to the earlier literature in psychology.

In a random utility model, the stochastic component directly enters the utility function. McFadden [1982] suggested that the individual may be viewed as selecting a utility function from a random distribution each time she makes a decision. Hanemann [1984] gave an appealing interpretation of these models saying that although the individual knows what affects her utility, the "econometric investigator" cannot observe all the relevant factors. These unobserved factors may be characteristics of the individual or of the goods and may change over time.

The random utility hypothesis is appealing for several reasons. Primarily, it motivates the randomness existing in an applied demand model. While we prefer to retain the assumption that individuals follow rational utility maximizing behavior, it is clear that some randomness exists across individuals. The random utility hypothesis resolves this conflict. Furthermore, the use of random utility models provides a structure for the stochastic specification. The specification of the random utility function dictates the form taken by the disturbance terms for the demand equations. However, as our theoretical results will show, some simple specifications imply serious restrictions on the error terms.
3. Some General Results

We first rewrite the general utility maximization problem as a random utility model:

\[
\max_{x^S} U^S(x^S) = U(x^S, \xi^S) \quad \text{subject to} \quad p'x^S = y^S
\]

where \( x \) is an \( n \times 1 \) vector of goods, \( p \) is the \( n \times 1 \) vector of prices of the goods, \( y \) is income, \( s \) denotes individual \( s \), and \( \xi^S \) is a vector of disturbance terms representing unobserved factors which influence the individual's utility. The vector \( \xi \) is unconstrained. The solution to this optimization problem is the vector of demand equations

\[
x^S = f^S(p, y^S, \xi^S).
\]

Demand equations may either be derived by constrained utility maximization as in (3.2) or they may be postulated without reference to a specific utility function. If the latter procedure is followed, we must ensure that the demand equations reflect the axioms of consumer choice. The assumption that consumers follow rational utility-maximizing behavior places three constraints on demand equations. They are:

1. Homogeneity of degree zero in prices and income - multiplying all prices and income by a scalar will leave demand unchanged.

2. Adding-up - expenditures on all goods must sum to income,

\[
\sum_{i} p_i x_i = y.
\]

3. Slutsky symmetry - cross-derivatives of the Hicksian demand functions
must be equal. In terms of Marshallian demand functions, this means

$$\frac{\partial x_i}{\partial p_j} + x_j(\frac{\partial x_i}{\partial y}) = \frac{\partial x_j}{\partial p_i} + x_i(\frac{\partial x_j}{\partial y}).$$

Demand equations obtained by differentiating a utility function subject to the budget constraint will satisfy these restrictions automatically. If we begin with the demand function rather than the utility function, these restrictions must be imposed. We examine the effects of these constraints on demand equations generated by random utility maximization and obtain some theoretical results on the form of the stochastic term.

The most common error specification in empirical demand studies is the addition of an error term to each demand equation, or to each expenditure equation. Simplifying assumptions are usually made on the distribution of these disturbances; in particular, they are assumed to have mean zero and a constant variance/covariance matrix. We demonstrate that assuming the errors are homoscedastic for either functional form violates the general restrictions on demand equations.

We assume expenditure functions of the form

$$(3.3) \quad p_i x_i = g_i(p, y) + u_i(p, y)$$

or equivalently, demand functions

$$(3.4) \quad x_i = f_i(p, y) + v_i(p, y)$$

with

$$E(u_i \mid p, y) = 0 \quad \text{and} \quad E(v_i \mid p, y) = 0.$$  

Additionally, the functions $f_i$, $g_i$, $u_i$, and $v_i$ are assumed to be differentiable with respect to $p$ and $y$.

**Theorem 1**: Suppose homogeneity is satisfied for the expenditure equation (3.3), then
\[ \frac{\partial u_i(p, y)}{\partial p} \neq 0 \]
and/or
\[ \frac{\partial u_i(p, y)}{\partial y} \neq 0. \]

**Proof:** By homogeneity (of degree one for the expenditure equation) for any \( k, \)

\[
k p_i x_i = g_i(k p, ky) + u_i(k p, ky) = k g_i(p, y) + k u_i(p, y).
\]

If to the contrary, \( \frac{\partial u_i}{\partial p} = 0 \) and \( \frac{\partial u_i}{\partial y} = 0, \) then differentiating with respect to \( k \) yields:

\[
g_i(p, y) + u_i = \frac{\partial g_i(k p, ky)}{\partial p} p + [\frac{\partial g_i(k p, ky)}{\partial y}] y
\]
and

\[
u_i = [\frac{\partial g_i(k p, ky)}{\partial p}] p + [\frac{\partial g_i(k p, ky)}{\partial y}] y - g_i(p, y)
\]
for all \( p, y, \) and \( u_i. \) But since \( u_i \) is independent of \( p \) and \( y \) and is nonconstant, this is impossible.

Q.E.D.

This formulation, an expenditure equation with an additive, homoscedastic error term, had previously been rejected by Pollak and Wales [1969] in the context of the linear expenditure system because its interpretation is particularly unappealing. It implies that a proportional increase in all prices and income would leave the variance of the expenditure equation unchanged, but would decrease the variance of the demand equation. We have now shown that in the general case this specification cannot satisfy the homogeneity restriction.

The obvious solution appears to be to append the error to the demand equation. This formulation satisfies the homogeneity restriction if the systematic portion is homogeneous. However, the assumption of an
additive, homoscedastic disturbance term is incorrect here as well; this specification violates Slutsky symmetry. The argument proceeds in two parts - we first demonstrate that these assumptions result in proportional disturbance terms; second, we show that proportional error terms cannot exist in a system of demand equations.

Lemma 1: Suppose $\partial \xi / \partial p = 0$, $\partial \xi / \partial y = 0$ and that symmetry is satisfied, then

$$v_i = [(\partial f_i(p, y)/\partial y)/(\partial f_j(p, y)/\partial y)] v_j.$$

Proof: The symmetry restriction requires that

$$\frac{\partial f_i}{\partial p} + \frac{\partial v_i}{\partial p} + (f_j + v_j)(\frac{\partial f_i}{\partial y} + \frac{\partial v_i}{\partial y}) = \frac{\partial f_j}{\partial p} + \frac{\partial v_j}{\partial p} + (f_i + v_i)(\frac{\partial f_j}{\partial y} + \frac{\partial v_j}{\partial y}).$$

Because we assumed that the $v$'s are not functions of $p$ or $y$, all partial derivatives of the $v_i$ and $v_j$ terms will equal zero. The symmetry condition must hold if we take expected values of both sides. This implies

$$\frac{\partial f_i}{\partial p} + f_j(\frac{\partial f_i}{\partial y}) = \frac{\partial f_j}{\partial p} + f_i(\frac{\partial f_j}{\partial y})$$

for all $p, y$. The systematic portion of (3.5) then drops out and we have

$$(\frac{\partial f_j}{\partial y})v_i = (\frac{\partial f_i}{\partial y})v_j$$

or

$$v_i = [(\frac{\partial f_i}{\partial y})/(\frac{\partial f_j}{\partial y})] v_j.$$

Q.E.D.

Lemma 2: Suppose $v_i = [(\partial f_i/\partial y)/(\partial f_j/\partial y)] v_j$ and that adding-up is satisfied, then $v_i = v_j = 0$. 
Proof: The adding up constraint requires that expenditures on all goods must sum to income.

\[ y = \sum_i x_i = \sum_i f_i + \sum_i v_i. \]

Taking expected values of both sides,

\[ E(y) = y = \sum_i f_i \]

since \( E(v_i) = 0 \). This implies

\[ 0 = \sum_i v_i. \]

Substituting in the proportionality result from above

\[ 0 = \sum_i [(\partial f_i/\partial y)/(\partial f_j/\partial y)]v_j \]

(3.10) \[ = \sum_i [(\partial f_i/\partial y)/(\partial f_j/\partial y)]v_j \]

\[ = v_j/(\partial f_j/\partial y) [\sum_i (\partial f_i/\partial y)]. \]

Because we have assumed that \( \partial v_i/\partial y = 0 \),

\[ \sum_i (\partial f_i/\partial y) = 1 \]

and equation (3.10) simplifies to

(3.11) \[ 0 = v_j/(\partial f_j/\partial y). \]

Since \( \partial f_j/\partial y \neq 0 \), equation (3.11) implies that \( v_j = 0 \). This is turn implies that all the disturbance terms will be 0.

Q.E.D.

Combining Lemmas 1 and 2, we see that the disturbance terms in a system of demand equations cannot be independent of both \( p \) and \( y \). Homos-
Hedasticity of the disturbances may only be achieved if all error terms are zero, which is an empirically unpalatable assumption.

**Theorem 2:** Suppose that symmetry and adding-up are satisfied and that \( v_i \) is nonconstant, then

\[
\frac{\partial v_i}{\partial p} \neq 0
\]

and/or

\[
\frac{\partial v_i}{\partial y} \neq 0.
\]

The assumption of an additive disturbance results in heteroscedasticity—a nonconstant variance/covariance matrix. Therefore, the disturbance term cannot be independent of both prices and income on either the demand equation or the expenditure equation.

This result has serious implications. First and most obvious, inferences based on the assumption of homoscedastic errors will not be accurate. The estimated covariances will be biased, although the direction of the bias is not known. Second, we must re-examine existing demand systems. Demand systems have typically been derived by imposing the theoretical constraints on the systematic portion of the demand equation. The addition of a homoscedastic error term complicates the adding-up constraint, but otherwise does not affect the functional form. However, our results have now established that the error terms cannot be independent of prices and income. Since we prefer to retain the previously derived functional forms, it is necessary to find the restrictions on the disturbances implied by a demand equation with a systematic component which separately satisfies homogeneity, adding-up, and Slutsky symmetry.

Homogeneity of degree zero in prices and income for the demand
equation as a whole combined with homogeneity for the systematic portion requires that the error term itself be homogeneous. Assume a demand equation with an additive, heteroscedastic error term:

\[ x_i = f_i(p, y) + v_i(p, y). \]

Suppose

\[ f_i(p, y) = f_i(kp, ky) \]

and by homogeneity

\[ x_i = f_i(kp, ky) + v_i(kp, ky). \]

Then

\[ x_i - f_i(p, y) = v_i(kp, ky) \]

(3.12)

\[ v_i(p, y) = v_i(kp, ky). \]

Thus, to retain a functional form with a homogeneous systematic component, the disturbance term must be homogeneous as well.

Adding-up requires that

\[
\sum_{i} x_i = y.
\]

\[
\sum_{i} x_i = \sum_{i} f_i(*) + \sum_{i} v_i(*).
\]

Taking expected values on each side, we have

\[
(3.13) \quad E y = E(\sum_{i} f_i(*)) + E(\sum_{i} v_i(*)).
\]

Since \( Ey = y \) and \( Ev_i(*) = 0 \), the above yields

\[
(3.14) \quad y = \sum_{i} f_i(*).
\]

This, in turn, implies a restriction on the stochastic component:

\[
(3.15) \quad \sum_{i} v_i(*) = 0.
\]

Condition (3.15) is a well-known result in applied demand analysis. It is generally dealt with by dropping one equation from an \( n \)-good system.
and estimating the remaining \( n - 1 \) equations. The \( n^{th} \) equation takes the part of a residual.

While homogeneity and adding-up may be handled easily, Slutsky symmetry proves to be more complicated. The full symmetry condition, involving both the systematic and stochastic components, is

\[
\frac{\partial f_i}{\partial p_j} + \frac{\partial v_i}{\partial p_j} + (f_j + v_j)(\frac{\partial f_i}{\partial y} + \frac{\partial v_i}{\partial y}) = \frac{\partial f_j}{\partial p_i} + \frac{\partial v_j}{\partial p_i} + (f_i + v_i)(\frac{\partial f_j}{\partial y} + \frac{\partial v_j}{\partial y}).
\]

However, with the result that the error terms must be functions of prices and/or income, the partial derivatives of the \( v_i \) terms do not drop out. Thus the usual symmetry constraint given in (3.6) which involves only the systematic component no longer holds. Since existing demand systems have generally had condition (3.6) imposed on them, we now examine the conditions under which it will still be valid.

**Theorem 3:** Suppose symmetry is satisfied for the demand equation, then

\[
\frac{\partial f_i}{\partial p_j} + f_j(\frac{\partial f_i}{\partial y}) = \frac{\partial f_j}{\partial p_i} + f_i(\frac{\partial f_j}{\partial y})
\]

if and only if

\[
(\frac{\partial v_i}{\partial p_j}) + f_j(\frac{\partial v_i}{\partial y}) + v_j[(\frac{\partial f_i}{\partial y}) + (\frac{\partial v_i}{\partial y})] = (\frac{\partial v_j}{\partial p_i}) + f_i(\frac{\partial v_j}{\partial y}) + v_i[(\frac{\partial f_j}{\partial y}) + (\frac{\partial v_j}{\partial y})].
\]  
(3.16)

Condition (3.16) will not be true automatically. Therefore, to achieve the usual symmetry result on the systematic component, we must impose condition (3.16) on the stochastic component.

It is possible to make some assumptions to simplify the above result. One alternative is to assume that the \( v_i \)'s are functions of the price vector, but not of income. This reduces equation (3.16) to
\[
\frac{\partial v_i}{\partial p_j} + v_j(\frac{\partial f_i}{\partial y}) = \frac{\partial v_j}{\partial p_i} + v_i(\frac{\partial f_j}{\partial y}).
\]

Condition (3.17) is satisfied if the \( p_i \)'s are treated symmetrically in the disturbance terms.
4. Specification and Identification of a Random Utility Model

In this section, the feasibility of implementing a random utility model is examined. The purpose of defining a theoretically consistent stochastic specification is, after all, to improve applied analysis. We show that although a theoretically correct error specification involves substantially more complex disturbance terms than have generally been used, these complications are not intractable.

We begin by deriving a specific demand system from the general random utility model derived in section 3. We obtain a functional form with an error structure consistent with random utility maximization, and illustrate that the resulting demand system is identified.

4.1 The LES as a Random Utility Model

We have chosen to work with the straightforward example of the linear expenditure system. Since the LES was first derived by Klein-Rubin [1947-48], it has been used extensively in applied demand analysis despite its somewhat restrictive properties. Because of its prominence in empirical analysis, the LES provides an appropriate and interesting starting point for this exercise. Clearly, analysis of the implications of the random utility model for other more flexible demand systems is a natural extension.

The derivation begins with our result from the previous section, a general demand equation with an additive, heteroscedastic error term:

\[ x_i = f_i(p, y) + v_i(p, y). \]  

(4.1)

Two sets of restrictions are imposed on equation (4.1). First, we assume specific functional forms for \( f_i \) and \( v_i \). Since we want a linear expendi-
ture system, $f_i$ is assumed to be linear in prices and income. A bilinear form is assumed for $v_i$. That is, $v_i$ will be linear in $p$ and $y$ given $\xi$ and linear in $\xi$ given $p$ and $y$, where $\xi$ is a vector of random disturbances with mean zero and unconstrained covariance. This functional form for $v_i$ seems appropriate for a linear expenditure system. Linearity in prices and income corresponds to the treatment of these variables in the systematic component and linearity in $\xi$ is a simple way of incorporating the random element. Thus,

(4.2) \[ f_i = \alpha p_i / \rho_i + \beta_i y / \rho_i. \]

(4.3) \[ v_i = (p_i / \rho_i)Q_i \xi + (y_i / \rho_i)r_i \xi. \]

where $\alpha$, $\beta$, $Q_i$, (an $n \times n$ matrix), and $r_i$ (an $n \times 1$ vector) are unknown parameters.

The second set of restrictions are the three demand theory constraints of homogeneity, adding-up, and Slutsky symmetry. Note that equations (4.2) and (4.3) are written so that homogeneity of degree zero in prices and income is satisfied automatically. In order to obtain the usual LES function for the systematic component, we assume that $f_i$ as well as the demand equation as a whole satisfies these restrictions. Thus the stochastic term must satisfy the conditions derived in section 3.

Following Klein-Rubin [1947-48], we see that imposing symmetry on an equation linear in prices and income yields the functional form

(4.4) \[ f_i = \gamma_i + (\beta_i / \rho_i)[y - \Sigma \gamma_j y_j]. \]

The adding up constraint requires
\[ y = \sum_{j} p_{j} x_{j} \]

so we must restrict

\[ \sum_{i} B_{i} = 1. \]

We now derive the appropriate forms for \( Q_{i} \) and \( \Gamma_{i} \) in the disturbance term. The application of the symmetry and adding-up restrictions results in the \( \Gamma_{i} \) term dropping out of the disturbance altogether. We demonstrate this before examining the correct form for \( Q_{i} \). As we showed in section 3, the adding-up constraint requires that the disturbances sum to zero. Thus

\[ \sum_{i}[p' Q_{i} \xi + y \Gamma'_{i} \xi] = 0 \]

or

\[ p' \sum_{i} Q_{i} \xi + y \sum_{i} \Gamma'_{i} \xi = 0 \quad \text{for all } p, y \]

Hence,

\[ \sum_{i} Q_{i} = 0 \]

and

\[ \sum_{i} \Gamma'_{i} = 0 \]

Theorem 3 derived in the previous section states that symmetry requires:

\[ (\alpha v_{j}/\alpha p_{i}) + f_{j}(\alpha v_{j}/\alpha y) + v_{j}[(\alpha f_{i}/\alpha y) + (\alpha v_{j}/\alpha y)] \]

\[ = (\alpha v_{j}/\alpha p_{i}) + f_{i}(\alpha v_{j}/\alpha y) + v_{i}[(\alpha f_{j}/\alpha y) + (\alpha v_{j}/\alpha y)] \]

or

\[ e'_{j} Q_{i} \xi / p_{i} + f_{j} \xi'_{j} / p_{i} + [(p/p_{j})' Q_{i} \xi + (y/p_{j}) \xi'_{j}] [(\beta_{i}/p_{i}) + \Gamma'_{i} \xi / p_{i}] \]

\[ e'_{i} Q_{j} \xi / p_{j} + f_{j} \xi'_{j} / p_{j} + [(p/p_{i})' Q_{j} \xi + (y/p_{i}) \xi'_{j}] [(\beta_{j}/p_{j}) + \Gamma'_{j} \xi / p_{j}] \]

where \( e_{i} \) is a vector of zeros with 1 in the \( i \)th position, and \( \beta_{i}/p_{i} = \alpha f_{i}/\alpha y \). Multiply both sides by \( p_{i} \) and \( p_{j} \) to obtain

\[ p' E_{j} Q_{i} \xi + p_{j} f_{j} \xi'_{j} + p' Q_{j} \xi'_{j} + (\alpha f_{j}/\alpha y) \xi'_{j} + (\alpha v_{j}/\alpha y) \xi'_{j} \]

\[ e'_{i} Q_{j} \xi / p_{j} + p_{i} f_{i} \xi'_{i} + p' Q_{i} \xi'_{i} + (\alpha f_{i}/\alpha y) \xi'_{i} + (\alpha v_{i}/\alpha y) \xi'_{i} \]
where $E_{i,i}$ is a square matrix with 1 in the $(i,i)$ position and zeros elsewhere. This equation must hold for all $p$ vectors and for all values of $y$. Equation (4.6) reduces to

$$\beta_i r^i \xi = \beta_j r^j \xi \quad \text{for all } \xi$$

$$\beta_i \Gamma_j = \beta_j \Gamma_i.$$  

Summing across $i$ yields

$$\Sigma \beta_i \Gamma_j = \Sigma \beta_j \Gamma_i.$$  

Since $\Sigma \beta = 1$,

$$\Gamma_j = \beta_j \Sigma \Gamma_i.$$  

From our result on adding-up, we know $\Sigma \Gamma_i = 0$. Therefore $\Gamma_j = 0$ and terms involving $y$ drop out of the disturbance term.

The symmetry constraint then becomes

$$p'(E_{jj}Q_j)\xi + \beta_j p'Q_j \xi = p'(E_{ii}Q_i)\xi + \beta_i p'Q_i \xi.$$  

(4.7)

Since (4.7) must hold for every possible $p$ and for every possible $\xi$, the condition reduces to:

$$E_{jj}Q_j + \beta_j Q_j = E_{ii}Q_i + \beta_i Q_i$$  

(4.8)

or

$$(\beta_j I_n - E_{jj})Q_j = (\beta_i I_n - E_{ii})Q_i.$$  

Define

$$B_i = (\beta_i I_n - E_{ii})$$

$$B_j = (\beta_j I_n - E_{jj}).$$

Both are diagonal matrices, so we can write
(4.9) \[ B_jQ_i = B_iQ_j \]
\[ Q_i = B_j^{-1}B_iQ_j \]
or
\[ Q_i = B_iB_j^{-1}O_j \]
or
\[ Q_i = B_iF \]

which says that \( Q_i \) is a linear transformation of \( F \), where \( F = B_j^{-1}Q_j \). The adding-up constraint requires that \( \Sigma Q_i = 0 \). For this to be true, \( \Sigma B_i = 0 \), which is easily verified.

Thus the demand equations become
\[ x_i = f_i(p/p_i, y/p_i) + p'B_iF\varepsilon. \]

Since we did not constrain \( \varepsilon \), and \( F \) is an arbitrary matrix, we may replace \( \varepsilon \) with \( \varepsilon^* = F\varepsilon \) and write
\[ (4.10) \quad x_i = f_i(p/p_i, y/p_i) + p'B_i\varepsilon^*. \]

Or, to write the full demand equation, we have
\[ (4.11) \quad x_i = \gamma_i + (B_i/p_i)(y - \Xi p_i\gamma_i + \varepsilon_i^* - \beta_i(\Xi p_i\varepsilon_i)/p_i. \]

The stochastic specification given in equation (4.11) is the only bilinear error structure which is compatible with the LES and consistent with random utility maximization. It is interesting to note that this error structure is identical to the one first suggested by Pollak and Wales [1969], who achieved it by positing a distribution on the \( \gamma_i \)'s. Pollak and Wales proposed this error structure as a convenient method for incorporating their intuitive notion that changes in prices will affect the variance of demand. We have demonstrated that their specification is theoretically correct as well as convenient. It is, in fact, the only bilinear error structure which transforms the LES into a
random utility model and maintains similarity in functional form for the systematic and stochastic components.

4.2 Identification of a RUM

Despite the complexity added by the non-constant covariance matrix, this random utility model is fully identified. We divide the model's parameters into two sets and consider the identification of the $\beta_i$'s and $\gamma_i$'s separately from the identification of the covariances. The demand equation in (4.11) written in expenditure form is

\[(4.12) \quad p_i x_i = p_i \gamma_i + \beta_i (y - \sum_j p_j y_j) + p_i u_i\]

where $u_i$ is the heteroscedastic error term derived above.

The identification of the $\beta_i$'s and $\gamma_i$'s proceeds very simply. Rewrite equation (4.12) as

\[(4.13) \quad p_i x_i = \gamma_i (1 - \beta_i) p_i + \beta_i y - \sum_{j \neq i} (\beta_i \gamma_j) p_j + p_i u_i\]

\[= \delta_i p_i + \delta_0 y - \sum_{j \neq i} \delta_j p_j + p_i u_i.\]

Estimates may be obtained of $\delta_i$, $\delta_0$, and $\delta_j$, for $j \neq i$. Because $\beta_i = \delta_0$, $\gamma_i$ is obtained from $\delta_i$. Similarly, because $\beta_i$ is known, each of the $\gamma_j$'s can be extracted.

Identification of the covariance matrix is more complicated due to the heteroscedasticity of the $u_i$'s. The variance of $p_i x_i$ is

\[(4.14) \quad \text{Var}(p_i x_i) = p' B_i \Sigma B_i p\]

where $\Sigma$ is the unconstrained $n \times n$ covariance matrix of $x$, and $B_i$ is defined as

\[B_i = \beta_i I_n - E_{ii}\]
where $I_n$ is an $n \times n$ identity matrix and $E_{ij}$ is a square matrix with 1 in the $(i,i)$ position and zeros elsewhere. Note that $\Sigma$ is the covariance matrix for the full system. An off-diagonal element of the covariance is:

$$\omega_{ij} = (p/p_i)'(\beta_iI_n - E_{ii})\Sigma(\beta_jI_n - E_{jj})(p/p_j).$$

Multiplying through by $p_i$ and $p_j$ and simplifying, we have

$$(4.15) \quad p_i p_j \omega_{ij} = \beta_i \beta_j p^' \Sigma p - \beta_i e_k p_j - \beta_j e_k p_i + p_i p_j \sigma_{ij}$$

where $\Sigma_j$ is the $j^{th}$ column of the $\Sigma$ matrix and $\sigma_{ij}$ is the $(i,j)$ element of this matrix.

By allowing the price vector to change, we show that $\Sigma$ is determined completely. We change $p_k$, where $k \neq i \neq j$. The covariance between $x_i$ and $x_j$ with the new $p_k$ is written

$$(4.16) \quad p_i p_j \omega_{ij}^k = \beta_i \beta_j (p + e_k)' \Sigma (p + e_k) - \beta_i (p + e_k)' \Sigma_p p_j$$

$$- \beta_j (p + e_k)' \Sigma_p p_i + p_i p_j \sigma_{ij}$$

where $e_k$ is a vector with 1 in the $k^{th}$ position and 0's elsewhere.

Subtracting $\omega_{ij}^k$ from $\omega_{ij}$ gives

$$(4.17) \quad p_i p_j (\omega_{ij} - \omega_{ij}^k) = \beta_i \beta_j \sigma_{kk} + 2 \beta_i \beta_j p^' \Sigma e_k$$

$$- \beta_i e_k p_j - \beta_j e_k p_i.$$

This equation may be rewritten to extract $\Sigma_k$.

$$(4.18) \quad p_i p_j (\omega_{ij} - \omega_{ij}^k) = [\beta_i \beta_j e_k + p^'(2 \beta_i I_n - \beta_i E_{jj} - \beta_j E_{ii})] \Sigma_k$$

for all $p$. Since $\beta_i$ and $\beta_j$ are identified, the matrix

$$(2 \beta_i I_n - \beta_i E_{jj} - \beta_j E_{ii})$$

is known and is non-singular. By varying $p$, we may obtain $n$ linearly independent vectors and hence $\Sigma_k$ is determined uniquely. By choosing
different values for \( k \), every column of the \( \Sigma \) matrix may be identified in like fashion.

Identification of all parameters, including the covariance matrix, has now been completed.
5. Estimation of a Random Utility Model

In this section, we describe the estimation of the random utility model derived above. We apply our theoretical model to aggregate data on international travel examining the demand for travel by U.S. residents to various Western European countries. The purpose of this analysis is two-fold. First, we illustrate that although the stochastic specification consistent with random utility maximization is complex, estimation of the RUM is feasible. Second, we compare the results from estimating the usual LES with homoscedastic errors to the results from the RUM. This will determine whether the error specification is critical to the parameter estimates.

While we have interpreted the random utility model as explaining differences in individuals' behavior, we have applied the model to aggregate time series data. The specific model we employ aggregates in a straightforward fashion because the equations are linear in income. For individual $i$ in time period $t$, the demand for good $j$ can be written as follows:

$$x_{jt}^i = f(p_{jt}, y_{jt}^i) + p_{jt}^i B_j \varepsilon_{jt}^i.$$ 

To aggregate across all individuals, we sum across $i$.

$$\sum_{i} x_{jt}^i = f(p_{jt}, \sum_{i} y_{jt}^i) + p_{jt}^i B_j \sum_{i} \varepsilon_{jt}^i.$$ 

or

$$\tilde{x}_{jt} = f(p_{jt}, \tilde{y}_{jt}) + p_{jt}^i B_j \tilde{\varepsilon}.$$ 

where $\tilde{}$ represents the summation of the variable.

$$\tilde{\varepsilon} \sim N(0, n_t \Sigma)$$

where $n_t$ represents the number of individuals at time $t$. Thus, aggrega-
tion presents no problem for this model. Note, however, that if the model were applied to a cross-section, panel data would be required. Observations across time are needed to get variations in prices.

As in any demand system, this model explains the relative effects of price and income variables on expenditures on a set of commodities. In this example, we assume that consumers have previously allocated a certain proportion of their income to international travel to Western Europe. Our model analyzes the allocation of that predetermined amount among the various destinations and transportation. This assumption implies that this commodity subset is separable from other goods in the consumers' choice set. This assumption is not tested.

The model covers travel to the destinations most popular with U. S. tourists: the United Kingdom, France, Italy, Switzerland, West Germany, Austria, and Spain as well as transportation to and from the U. S. and Western Europe. The data is grouped into four destinations. We treat the U. K. and France as separate destinations, but group Austria, Germany, and Switzerland together on the grounds that an American tourist visiting one of these countries is likely to visit the others in the same trip. Italy and Spain, both being southern European countries, are lumped together as well.

The model consists of five demand equations. The first four represent travel to the four destinations named above. Aggregate tourism expenditures in these areas are explained by the relevant price variables and total travel (tourism plus transportation) expenditures. The last equation explains aggregate expenditures on transportation from the U. S. to Europe in terms of the price variables and total travel expen-
diture.

Transportation is treated in a separate equation despite the fact that one cannot consume tourism without also consuming transportation. This is done because the data do not allow division of transportation expenditures into transport to each of the destinations separately. However, for a U.S. tourist, a trip to Europe generally means flying into a major airport such as London or Frankfurt, then visiting a number of countries, so a separate equation for transportation is reasonable. We also assume that the transportation variable is continuous even though for an individual, it is a discrete variable—a consumer may take only a discrete number of trips to Europe. Ideally, our estimation technique should comprehend the discrete nature of transportation. However, the lack of disaggregated data prevents us from taking this approach.

5.1 Data

The data may be divided into two categories: expenditure data and price data. The expenditure data make up the payments side of the travel account and part of the passenger fare accounts in the U.S. balance of payments accounts. These data are collected and published by the Department of Commerce’s Bureau of Economic Analysis. Tourism payments include expenditures on lodging, food, and entertainment purchased abroad. U.S. military and other government personnel expenditures are not included, nor are the expenditures of U.S. citizens living or working abroad.

Transportation expenditures cover the payments made by Americans to U.S. and foreign carriers for transport to and from the U.S. Transportation between two foreign points is included under tourism expenditures.
Because the transportation data provided by the Bureau of Economic Analysis includes all international travel rather than simply transport to Western Europe, this variable was constructed by multiplying the average U.S.-European airfare by the number of visitors from the U.S. to the Western Europe-Mediterranean area.

It is quite probable that this data contains measurement error. White and Walker [1982] demonstrated that very little agreement exists among tourism expenditure data collected by various countries. For example, the data on U.S. expenditures in West Germany and West German receipts from U.S. tourists showed differences of up to 100 percent. However, we have not dealt with the measurement error problem in this study. Since we use expenditure data from only one source, it is hoped that the upward or downward bias in the data is consistent.

The prices confronting international tourists have two components: the internal prices of the goods tourists buy and the exchange rates between the domestic and foreign currencies. This study uses tourism price indexes compiled by the Austrian Institute for Economic Research. These indexes cover the prices of goods which tourists are likely to buy, such as lodging, restaurant meals, and entertainment. The exchange rates were taken from the International Monetary Fund's International Financial Statistics. The tourism price indexes were multiplied by the appropriate exchange rates in order to include both price variations across countries and exchange rate fluctuations in the price variables.

Because in two cases, tourism expenditures in two or more countries were aggregated, we formed composite price indexes to represent the cost of tourism in these areas. These tourism price indexes are weighted
averages of the prices in the individual countries, where the weights
are the share each country has in total tourism expenditures for the
group.

5.2 Empirical Results

We present the results from estimating two models, the usual LES
and the RUM. Both models were estimated with the equations in expendi-
ture form. The systematic component of each model is the same:

\[(5.1) \quad p_i x_i = p_i y_i + \beta_i (y - \Sigma_{j} p_j y_j)\]

where \(p_i x_i\) represents tourism expenditures in country \(i\). In the last
equation, \(p_g x_g\) represents transportation expenditures. The \(p_k\)'s are the
price variables described above. The variable \(y\) represents total travel
expenditure, or total income allocated to this branch. The \(y\) parameters
are often interpreted as subsistence quantities of the various com-
modities, although it is entirely possible for the estimated parameters
to turn out negative. A negative \(y_i\) means that the demand for good \(i\) is
elastic with respect to its own price. (See Pollak and Wales [1969]).
Since we are dealing with European travel, negative \(y_i\)'s are possible.
The \(\beta_i\) parameter measures the proportion of supernumerary income allo-
cated to good \(i\).

We assume that the additive homoscedastic disturbances for the LES
have a multivariate normal distribution, or

\[y \sim N(0, \Sigma_0)\]

where \(\Sigma_0\) is the constant covariance matrix, with \(E(\sigma_{i, t}, \sigma_{i, t+1}) = 0\). For
the RUM, the disturbance term is \(u_i\) where
\[ u_i = \varepsilon_i - (B_i / p_i)(\Sigma_j \varepsilon_j). \]

The random component, \( \varepsilon \), is also assumed to have a multivariate normal distribution
\[ \varepsilon \sim N(0, \Sigma_1) \]
with \( E(\sigma_i, t, \sigma_i, t+1) = 0 \). This implies that the density of \( u \) is
\[ u \sim N(0, \Omega(\rho; \Sigma_1)). \]

where the \((i, j)\) element of \( \Omega \) is
\[ \omega_{ij} = \Sigma_i B_i \Sigma B_j. \]

Both models were estimated using maximum likelihood techniques. In both cases, one equation was dropped to avoid having an over-determined system. In estimating any demand system, the adding-up constraint means that the parameters of the last equation may be recovered from the estimates of the other equations in the system. In our example, the transportation equation was dropped from each model. Since \( \Sigma B_i = 1 \), the estimated \( B \) for this equation was easily calculated, although no standard error is available.

Estimation of the LES was quite simple, it is a non-linear system of seemingly unrelated regressions. Table 5.1 presents these results.

Estimating the RUM required a general nonlinear optimization algorithm because the heteroscedastic \( \Omega \) matrix precluded the use of any least squares approximation to estimating the covariances. The major difficulty in estimating the RUM was maintaining the positive definite-ness of the covariance matrix. Leaving all elements of the \( \Omega \) matrix unconstrained meant that the estimation algorithm could stray into areas where the function was undefined. This problem was surmounted by using
the fact that the Ω matrix may be written as the product of two triangular matrices:

Ω = PP'

Since Ω is the product of P and P', Ω will always be positive definite regardless of the values of P. Thus we estimated the elements of P, rather than of Ω.

Other than this problem, estimation of the RUM was straightforward. The number of parameters in the model was, of course, much larger than the number in the LES. We had to estimate not only the β's and γ's, which in this example were nine, but also the elements of P. In a five equation system, there are (6×5)/2 = 15 elements. Thus, our example contained 24 parameters. We used numeric rather than analytic derivatives. Convergence probably could have been obtained faster with more accurate derivatives. The RUM results are given in Table 5.2.
Table 5.1

Model 1: LES
(Standard errors in parentheses.)

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. U. K.</td>
<td>0.0809</td>
<td>0.5835</td>
</tr>
<tr>
<td></td>
<td>(.0164)</td>
<td>(.1114)</td>
</tr>
<tr>
<td>2. France</td>
<td>0.0611</td>
<td>2.1040</td>
</tr>
<tr>
<td></td>
<td>(.0046)</td>
<td>(.6682)</td>
</tr>
<tr>
<td>3. Ger.-Aus.-Swit.</td>
<td>0.0948</td>
<td>2.4623</td>
</tr>
<tr>
<td></td>
<td>(.0085)</td>
<td>(.8825)</td>
</tr>
<tr>
<td>4. Italy-Spain</td>
<td>0.0692</td>
<td>116.9800</td>
</tr>
<tr>
<td></td>
<td>(.0095)</td>
<td>(27.8570)</td>
</tr>
<tr>
<td>5. Transportation</td>
<td>0.6940</td>
<td>-0.0239</td>
</tr>
<tr>
<td></td>
<td>(.2764)</td>
<td></td>
</tr>
</tbody>
</table>

Log-likelihood function = -506.2586

Table 5.2

Model 2: RUM
(Standard errors in parentheses.)

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. U. K.</td>
<td>0.1170</td>
<td>0.2193</td>
</tr>
<tr>
<td></td>
<td>(.0126)</td>
<td>(.1103)</td>
</tr>
<tr>
<td>2. France</td>
<td>0.0620</td>
<td>2.3025</td>
</tr>
<tr>
<td></td>
<td>(.0067)</td>
<td>(.6709)</td>
</tr>
<tr>
<td>3. Ger.-Aus.-Swit.</td>
<td>0.1327</td>
<td>0.4310</td>
</tr>
<tr>
<td></td>
<td>(.0132)</td>
<td>(.1239)</td>
</tr>
<tr>
<td>4. Italy-Spain</td>
<td>0.0876</td>
<td>127.1900</td>
</tr>
<tr>
<td></td>
<td>(.0128)</td>
<td>(40.2370)</td>
</tr>
<tr>
<td>5. Transportation</td>
<td>0.6007</td>
<td>0.2037</td>
</tr>
<tr>
<td></td>
<td>(.1128)</td>
<td></td>
</tr>
</tbody>
</table>

Log-likelihood function = -490.6338
A comparison of Tables 5.1 and 5.2 shows that the structure of the error terms makes a clear difference in the empirical results. We would expect there to be more variation in the estimated standard errors than in the actual parameter estimates. The presence of heteroscedasticity leaves parameter estimates consistent but biases estimated variances and covariances. Thus, if the RUM is the true model, the estimated $\beta$'s and $\gamma$'s from the LES are inefficient but the standard error estimates are actually biased. However, an examination of the estimated parameters shows that some changes in the estimates did occur, especially among the $\gamma$'s. It is possible that with a larger sample less variation across estimated parameters would occur.

For the estimated $\beta$'s, it was true that the parameter estimates changed less under the alternate specification than the estimated standard errors. The $\beta$ for the U. K. for example changed from .08 in the LES to .12 in the RUM. However, the calculated t-ratio changed from 4.9 for the LES estimate to 9.3 for the RUM estimate. Among the $\gamma$ estimates, two particular values are interesting. The $\gamma$ for the U. K. in the LES is .58 with a standard error of .11. The calculated t-ratio is 5.24, leading one to believe that the parameter estimate is accurate. However, under the RUM the same $\gamma$ is estimated at .22 with a t-ratio of 1.99 — still significantly different from zero at the 95% confidence level, but not by much. The estimates of the transportation $\gamma$ are also interesting. In the LES, $\gamma = -.024$ and it is very far from being significantly different from zero. In the RUM, however, the estimate of this parameter improves. The $\gamma = .204$ and the t-ratio is 1.81, barely under the critical value for 95%.
A final comparison worth noting is the value of the log-likelihood function for each model. The RUM specification substantially lowered the value of likelihood function. Although this is not a formal test of goodness-of-fit, we would conclude that the lower value of the function for the RUM implies that this model fits this data better than the simple LES.
6. Concluding Remarks

In this essay, we have explored the problem of appropriate stochastic specifications for demand analysis. Despite the obvious importance of the error specification to applied work, few economists have dealt with this issue in any detail. We have concluded that the random utility hypothesis introduced into the economic literature by McFadden [1982] is an intuitively appealing method for incorporating randomness into utility theory. The random utility hypothesis reconciles the usual assumption of deterministic utility maximizing behavior by each consumer with the apparent randomness existing across individuals.

However, as our theoretical results have shown, adopting the random utility hypothesis implies strong restrictions on the disturbances. In particular, additive error terms on a complete system of demand equations derived by random utility maximization cannot be homoscedastic if we wish to retain the usual form for the systematic component. The error terms must be functions of prices and/or income. This result derives from imposing the symmetry restriction on a general demand equation with an additive error.

In addition to the general theoretical results, we have obtained an appropriate error structure for the specific case of the LES. The stochastic specification we derive is identical to the error structure suggested by Pollak and Wales [1969], who proposed it as a convenient way to incorporate their intuitive notion that changes in prices will affect the variance of demand. We have shown that this stochastic specification is the only bilinear error structure which is compatible with the LES and consistent with random utility maximization.
We then estimated a five equation demand system using both the typical homoscedastic disturbances and the alternative random utility model. Although the heteroscedastic covariance matrix in the RUM adds considerable complexity to the model, estimation is feasible. Our empirical results demonstrate that changing the error structure affects the parameter estimates to some extent and estimated standard errors more substantially. These results are what we would expect from correcting for heteroscedasticity - the parameter estimates become more efficient and the estimated standard errors become unbiased. Our results also indicate that for this example, the RUM specification fits the data better.

This analysis contains several limitations. First, our theoretical results apply only to the case where additive errors are assumed. We chose this case because the majority of applied demand analysis has also made this assumption. It is possible that for a non-linear demand system, non-additive errors could be assumed. The implications of random utility maximization could be different in such a case. Second, we should note that as the number of goods in the model increases, the number of parameters to be estimated grows very quickly. Our five good model contained 24 parameters, an eight good model would contain 51 parameters, and so on. Estimation costs will increase very rapidly as the number of parameters increases.

Several interesting extensions of this work are possible. Most obviously, random utility maximization could be extended to other demand systems with more flexible properties than the LES. A formal hypothesis test to accommodate the non-nested RUM and LES could strengthen our
empirical results, although the generality of such a test is questionable. Another possibility, although complicated, would be to combine a correct stochastic specification with Stapleton's [1984] measurement error correction. This would attack two widespread problems simultaneously.
Footnotes

1. As noted in section 2, Pollak and Wales [1969] added a further complication to their stochastic specification. Rather than assuming that
   \[ \text{E}(\varepsilon_i^2) = \sigma_i^2 \]
   they assumed
   \[ \text{E}(\varepsilon_i^2) = \sigma_i^2 f_i^2. \]
   They did this to allow changes in income (which would affect \( f_i \)) to influence the disturbance terms. We have determined that this formulation does not satisfy the symmetry constraint for a random utility model.

2. The use of tourism price indexes marks a significant improvement over most previous travel demand studies. Most earlier studies use consumer price indexes, which because they include the cost of housing and other consumer durables, cover many prices which are irrelevant to tourists.

3. The LES results were obtained using the econometric software package SHAZAM.

4. The RUM results were obtained using the GRADX method from the GQOPT program.

5. Stapleton [1984] examined the case of measurement error in demand systems. He concluded that these errors could affect the acceptance or rejection of the symmetry constraint, but did not seem to be the sole cause for rejection.
Appendix

Data

Table A.1

U. S. Tourism Expenditures, 1958-1983
(Thousands of Dollars)

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Source: Personal communications with S. Schulmeister, Austrian Institute of Economic Research.
Table A.3

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($/Unit of Foreign Currency)

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1 This observation was taken from K. J. White [1983].

2 This observation was computed by adjusting the observation from the previous year by the change in the U.S. airfare index compiled by the Bureau of Labor Statistics.

1. Introduction

Traditional consumer choice theory depends on the assumption that all goods are infinitely divisible. The usual utility-maximizing solution yields a tangency point on a continuous indifference curve. While this approach applies to a wide range of problems, in many cases, assuming goods are continuous is inappropriate. Many of the choices which actually confront the consumer involve discrete alternatives.

Previous work linking discrete choice with utility theory has centered on the problem where the consumer confronts a number of mutually exclusive unordered alternatives. The classic example of this is the choice of mode of transportation—drive, ride a bus, or walk. McFadden [1973, 1974, 1976, 1978, 1979, 1982] has developed models to handle this type of discrete choice problem within the context of consumer demand theory. His extensions to the multinomial logit model combine utility maximizing behavior with probabilistic choice.

In many situations, the alternatives have a natural ordering, in addition to being mutually exclusive. Level of education attained—high school, college, graduate school—illustrates ordered alternatives. For the general class of ordered response models, the multinomial logit specification is inappropriate because it fails to exploit the order inherent in the alternatives. The natural ordering of the discrete good
provides additional information which should not be ignored. Ordered response models have been estimated (see McKelvey and Zavoina [1975], Aitchison and Silvey [1957], or Gurland, et. al., [1960]), but the link with utility theory has not been developed.

A refinement that has not been dealt with in the ordered response literature is the possibility that the ordered alternatives may be assigned cardinal (typically integer) values. Consider, for example, the decision on how many tickets to purchase for the coming opera season or the purchase of consumer durables such as air conditioners or televisions. In these cases, the consumer must attempt to maximize utility subject to the constraint that one of the goods may be purchased only in integer quantities. The integer constraint will affect not only the demand for the discrete good but also the demand for the continuous goods in the choice set.

In this essay, we explore the theoretic and econometric implications of a mixed integer/continuous model from the standpoint of utility maximization. The second section of this paper deals with utility maximizing behavior when the choice set includes ordered discrete alternatives. The derivation of the demand system is carried out in terms of a general utility function. The third section considers the choice of a specific functional form. The fourth section outlines a possible estimation procedure based on the method of maximum likelihood.

The model derived here has elements in common with previous work. Specifically, McFadden’s development of the conditional indirect utility function, although used differently, is necessary to derive the demand for the discrete good in a mixed integer/continuous model. However, we
extend the model significantly by incorporating the demand for the
discrete good into a complete system of demand equations. McFadden
[1974] made reference to this approach but did not develop it ex-
licitly. The system of demand equations which we derive includes demand
for continuous goods as well as the discrete good. This procedure allows
closer adherence to demand theory by allowing for cross-effects between
goods.

An additional insight is gained for some applications of the in-
teger response model by using a systems approach. For some goods, the
consumer may choose not only how many units of the good to buy, but also
the intensity of use of each unit. For example, consider an individual's
decision on how many ski trips to take. The individual decides the
number of trips to make and the length of stay on each trip. By incor-
porating demand for the discrete good into a system of demand equations,
we allow simultaneous decisions on usage and the quantity purchased.
2. Utility Maximizing Behavior

2.1 An Ordered Discrete Good

The starting point for the mixed integer/continuous system is the assumption that the individual's preferences for all goods, including the discrete good, can be characterized by continuous indifference curves. This implies that a consumer might actually prefer to consume a non-integer quantity. A latent continuous variable underlies the discrete variable. Clearly, this assumption exploits the ordered nature of the discrete good. Making the usual assumptions about preferences, the indifference curve representing preferences between the discrete good and a continuous good would be as follows:

![Figure 1](image)

D = a discrete good

x = a continuous good

Actual consumption of the discrete good is restricted to integer points on the indifference curve. While a tangency may occur on an integer
value for \( D \), the indifference curve actually achieved generally will be below the one tangent to the budget line.

Characterizing the indifference curve as continuous gives rise to an underlying continuous demand curve for \( D \). Define \( D^* \) as the latent continuous variable.

![Figure 2](image)

Here, constraining actual consumption to integer values breaks the continuous demand curve into a series of steps. The prices at either end of each segment are "switchpoint" prices, meaning that as the price of a discrete good falls below a certain level, the number of units purchased by the individual jumps from \( k \) to \( k+1 \). For example, in the case of three discrete alternatives

\[
D = \begin{cases} 
0, & \text{if } p_d > p(0) \\
1, & \text{if } p(0) > p_d > p(1) \\
2, & \text{if } p(1) > p_d 
\end{cases}
\]
where \( p(\ell) \) denotes the \( p_d \) at which the individual switches from \( \ell \) to \( \ell + 1 \) units of \( D \).

A point on the underlying continuous demand curve corresponds to each switchpoint price. This point gives the number of units the individual would choose if the variable could have non-integer values. These are labelled \( D(\ell) \) in the graph. Writing the above in terms of units of \( D \) rather than prices, we have

\[
D = \begin{cases} 
0, & \text{if } D^* < D(0) \\
1, & \text{if } D(0) < D^* < D(1) \\
2, & \text{if } D(1) < D^* 
\end{cases}
\]

2.2 Deriving the System

The next step incorporates the demand for the discrete good with that for continuous goods in the framework of a complete system of demand equations. We postulate a utility function for the individual, maximize utility subject to the budget constraint and the constraint that consumption of \( D \) be integer values, and, thereby, derive the system of demand equations.

Assume a strictly quasi-concave utility function of the form

\[ U^S = U^S(D^S, x^S) \]

where \( D \) is the discrete good, \( x^S \) is an \( n \times 1 \) vector of continuous goods, and the \( s \) superscript denotes individual \( s \). If the demand equations are derived from (2.1), substituting \( D^* \) for \( D \), the system is given by

\[
\begin{align*}
D^*_s &= f_d^s(p_d'D, y^S) \\
x^S &= \int f^S(p_d'D, y^S)
\end{align*}
\]
where \( p_d \) is the price of a unit of \( D \), \( p_x \) is an \( n \times 1 \) vector of prices for the continuous goods, and \( y \) is income. The \( D^* \) equation represents latent demand for the discrete good—the quantity the consumer would buy if \( D \) were continuous.

Deriving the system of demand equations using \( D \) rather than \( D^* \) requires a different procedure. The demand for the continuous goods depends on the level of \( D \) consumed while the demand for \( D \) is given by the equation for the step function shown in Figure 2. Deriving the demands for the continuous goods, \( x \), requires maximizing utility while holding \( D \) constant.

\[
\max_x U^S(D^S, x) \quad \text{subject to} \quad y^S = p_d D^S + p_x x^S
\]

The conditional demand equations are

\[(2.3) \quad x_i^S = f_i^S(p_x, y^S, D^S) \]

where \( y^* = y^S - p_d D^S \).

To derive the demand for the discrete good, we must find a switchpoint price of \( D \), meaning that we must find a \( p_d \) such that the individual is indifferent between \( i \) and \( i + 1 \) units of \( D \).

**Theorem:** Existence. Suppose that the conditional indirect utility function is a continuous function in \( p_d \), then a price of \( D \) exists such that the consumer is indifferent between \( i \) and \( i + 1 \) units of \( D \).

**Proof:** See appendix.

Figure 3 illustrates a switchpoint price for \( D \). The budget line
labelled B implies a price of D such that the consumer is indifferent between 1 and 2 units of D.

Figure 3

Figure 3 also demonstrates that \( p_d(l) \) lies between the prices which would allow the consumer to achieve a tangency solution at \( D=l \) or \( D=l+1 \). The price of D implied by budget line A would allow the consumer to maximize utility by consuming 1 unit of D. Budget line C gives the \( p_d \) which would allow utility maximization at \( D=2 \). Budget line B, which implies the switchpoint price \( p_d(1) \), lies between A and C.

Mathematically, the switchpoint equation is derived from

\[
U^S(l, x) = U^S(l+1, x)
\]

Substituting (2.3) for \( x \) into (2.4) gives the conditional indirect utility function.

\[
V^S(l, \text{end}^S(p_x, y^S, l)) = V^S(l+1, \text{end}^S(p_x, y^S, l+1))
\]

where \( p_x \), and \( y \) are constant. Using (2.5) to solve for \( p_d \) gives
\[(2.6) \quad p^S_d(\lambda) = g^S(\lambda, P, Y).\]

**Theorem:** Let $D$ be a normal good, then $p^S_d(\lambda)$, the switchpoint price between $\lambda$ and $\lambda + 1$ units of $D$, is unique.

**Proof:** See appendix.

To derive $D(\lambda)$, the switchpoint quantity, (2.6) is substituted into the latent demand equation to obtain

\[(2.7) \quad D(\lambda) = h^S(\lambda, P, Y).\]

The demand for the discrete good is given by

\[(2.8) \quad D = \begin{cases} 0 & \text{if } D^S(0) \leq D^S < D^S(1) \\ 1 & \text{if } D^S(0) \leq D^S < D^S(1) \\ 2 & \text{if } D^S(1) \leq D^S < D^S(2) \\ \vdots & \end{cases}\]

The complete system consists of (2.3) and (2.8). The conditional demand system in (2.3) will satisfy the demand theory restrictions of homogeneity, symmetry, and adding-up. Adding-up for the complete system is guaranteed by the way \( y^* \) is defined. Unlike demand systems for continuous goods, however, the conditional demand equations are simultaneous. The presence of $D$ in the equations for the continuous goods complicates the model relative to the usual case.

2.3 **An Application**

In order to illuminate the problem, we provide a possible applica-
tion of the mixed integer/continuous model. This example illustrates the way in which consumption behavior must adjust to accommodate the "lumpiness" of some commodities.

Consider an individual's consumption of ski trips. The number of trips to make in a year is an integer variable. The individual may choose not only the number of trips to make, but also how intensively to "use" each trip. That is, one may vary the length of stay and by doing so, vary the consumption of such services as nights spent at the ski lodge and meals eaten in restaurants. Intensity of use is a continuous variable.

Because one cannot consume these skiing services without incurring the cost of making the trip, the price of the trip may be viewed as a set-up cost. An important insight here is that very high set-up costs will imply totally different behavior from that resulting from insignificant set-up costs. If the price of a round-trip ticket to Aspen is quite high, we would expect the individual to choose one lengthy trip. If the trip is quite inexpensive, we would more likely observe a number of weekend trips being made.

Our model refines this general observation; first, by allowing concurrent decisions on the number of trips to make and the usage of each trip and, second, by showing explicitly the effects on consumption of both trips and services of changes in the prices of either good. The indifference curve diagram in Figure 1 could be used to represent the consumer's preferences over trips and services. As we allow the relative prices of trips and services to change, we may observe the combination of the two goods the consumer will choose.
The consumer will achieve the utility level represented by this indifference curve only in the case where the tangency between the budget line and the indifference curve occurs at an integer value for trips. Since this is likely to be infrequent, the consumer generally achieves a lower level of utility than that which would be possible if trips were a continuous variable. That is, there is some loss of utility resulting from the lumpiness of the good. A measure of this loss of utility may be obtained by comparing the effective demand for trips with the demand curve which would exist if trips were continuous. Figure 4 makes this comparison. When the individual must consume an amount which is away from the continuous demand curve, consumer surplus is lost. The combined area of these triangles represents the total loss of consumer surplus due to the lumpiness of the good.

Figure 4

This measure of loss of consumer surplus may be taken as a measure
of the importance of the problem of lumpy goods. If the total area of the triangles is small relative to the total area under the demand curve, the problem is insignificant and little will be lost by simply treating the good as continuous. A relatively large loss of consumer surplus indicates that the good should not be modelled as continuous.
3. Functional Form

Specifying a particular functional form for the mixed integer/continuous system can be done from two different standpoints. A specific utility function with desirable properties may be chosen and the equations derived from it. While this approach seems straightforward, it risks deriving intractable equations for both the conditional demand equations and the switchpoint equation. An alternate approach specifies demand equations with desirable properties and recovers the implied utility function by solving the demand system as a set of partial differential equations. We choose to adopt the second "bottom-up" approach and specify the conditional demand equation as the starting point for deriving a particular functional form.

3.1 A Random Utility Model

We adopt the random utility hypothesis in our specification of a particular functional form, which means that the stochastic elements will enter the utility function directly. Rewriting the utility function as a random utility model, we have

\[ U^S(D^S, X^S) = U(D^S, X^S, \xi^S) \]

where \( \xi^S \) is an unconstrained vector of disturbance terms representing unobserved factors which influence the individual's utility. The conditional demand equations derived from this utility function are written

\[ X^S = f^S(D^S, P^S, Y^S, \xi^S) \]

The latent demand equation becomes

\[ D^* = f_d^*(P, Y^S, \xi^S) \]
where $p' = (p, p'x')$. The switchpoint between $l$ and $l+1$ is

$$p_d^s(\xi) = s^*(l, p_x, y, \xi)$$

or

$$D^s(\xi) = h^*(l, p_x, y, \xi)$$

The sorting function is written

$$D^s =
\begin{cases}
0 & \text{if } D^s < D^s(0) \\
1 & \text{if } D^s(0) < D^s < D^s(1) \\
2 & \text{if } D^s(1) < D^s < D^s(2) \\
\vdots
\end{cases}$$

In the sequel, we will dispense with the superscript $s$ as it will be understood that all functions refer to individual $s$.

In Essay I, we demonstrated that an additive stochastic term of a demand equation derived from a random utility function cannot be independent of both prices and income. In view of this result we write the conditional demand equation as

$$x_i = f_i(D, p/p_i', y/p_i') + u_i(D, p/p_i, y/p_i')$$

We assume a simple form for both $f_i(*)$ and $u_i(*)$. Specifically, we assume that the systematic portion of the equation is linear in $p$ and $y^*$, so that

$$f_i = \sum_k \alpha_{ik}(D)p_k/p_i + \beta_i(D)y^*/p_i$$

where $\alpha_{ik}(D)$ and $\beta_i(D)$ are unknown parameters. For $u_i$ we assume a bilinear form, that is, $u_i$ will be linear in $p$ given $\xi$ and linear in $\xi$ given $p$, where $\xi$ is a vector of disturbances with mean zero and uncon-
strained covariance.

\[(3.3) \quad u_i = (e/p_i)^T Q_i(D) \epsilon_i\]

where \( Q_i \) is a matrix of unknown parameters. Writing the parameters of (3.2) and (3.3) as functions of \( D \) reflects that \( D \) enters these equations directly. This implies that the preference ordering over the continuous goods may change with \( D \).

The equation as a whole must satisfy the demand theory restrictions of homogeneity of degree zero, symmetry, and adding up. Note that homogeneity is assured by the division by the appropriate price. Since we have assumed that the disturbance term has the bilinear form in (3.3), the systematic and stochastic parts of the demand equations must separately satisfy these restrictions. Thus we derive the specific functional form by imposing the restrictions on each part in turn.

3.2 Specifying the Conditional System

The particular functional form for the systematic portion of the conditional demand equation falls out very quickly. For (3.2) to be a demand equation, it must satisfy demand theory restrictions of symmetry and adding-up. Imposing symmetry on an equation linear in prices and income yields the linear expenditure system first derived by Klein and Rubin [1947-48]. Thus the functional form for (3.2) is

\[(3.4) \quad f_i = \gamma_i(D) + (\beta_i(D)/p_i)[y^* - \sum_j p_j y_j(D)]\]

The adding up constraint requires

\[ y^* = \sum_j p_j x_j \]

so we must restrict \( \Sigma \beta = 1 \).
The particular form of a bilinear disturbance term for the linear expenditure system was derived in Essay I. The specific form for \( u_i \) is

\[
(3.5) \quad u_i = \varepsilon_i - \beta_i(D)(\sum_k \varepsilon_k)/p_i
\]

At this point, we simplify further and assume that the \( \beta_i(D) \) and \( \gamma_i(D) \) are constant across all values of \( D \). This implies that the preference ordering over \( x \) does not shift for different levels of \( D \). In other words, the preference ordering is separable with respect to \( D \). With this assumption, the full conditional demand equation becomes

\[
(3.6) \quad x_i = \gamma_i + (\beta_i/p_i)[\gamma^* - \sum_j \gamma_j] + \varepsilon_i - \beta_i(\sum_j \varepsilon_j)/p_i
\]

3.3 Discrete Demand

In order to derive demand for the discrete good, we must obtain the utility function implied by (3.6). Samuelson [1947-48] has shown that the preference ordering for the linear expenditure system is

\[
(3.7) \quad U = \sum_j \log(x_j - \gamma_j - \varepsilon_j) + C
\]

or

\[
U = C(\Pi_j(x_j - \gamma_j - \varepsilon_j)^{B_j}
\]

where \( C \) is the constant of integration and can be considered the function involving \( D \).

Next, we define the preference ordering over \( D \). We would not expect the switchpoint equation to have the same form as the demand equations for \( x \), yet a simple equation is desirable for ease of estimation. The general form of the switchpoint equation can be derived from (3.7). The indirect utility function for the continuous goods is
(3.8) \[ V = \log[y^* - \sum_i (y_i + \varepsilon_i)] + \sum_i \beta_i \log \beta_i - \sum_i \beta_i \log p_i. \]

Including the preference ordering for \( D \) gives

(3.9) \[ V = U_0(D) + \log[y^* - \sum_i (y_i + \varepsilon_i)] + \sum_i \beta_i \log \beta_i - \sum_i \beta_i \log p_i. \]

Since at the switchpoint between \( D = k \) and \( D = k+1 \), \( V(k) = V(k+1) \), we have

(3.10) \[ U_0(k) + \log[y - p_d k - \sum_i (y_i + \varepsilon_i)] \]
\[ = U_0(k+1) + \log[y - p_d (k+1) - \sum_i (y_i + \varepsilon_i)]. \]

Solving this for \( p_d \) yields

(3.11) \[ p_d(k) = \frac{\left[ e^{U_0(k)} - e^{U_0(k+1)} \right] \left[ e^{U_0(k)} - e^{U_0(k+1)} \right]}{\left[ e^{U_0(k)} - e^{U_0(k+1)} \right]} \left( y - \sum_i (y_i + \varepsilon_i) \right). \]

Equation (3.11) indicates that no choice of \( U_0 \) will allow for significant simplification of the expression on the right-hand side. Because no particular choice of \( U_0 \) is compelling, we choose to specify \( D \) in the same manner as \( x \). Additionally, our initial assumption that the consumer's preferences are based on \( D^* \), the underlying continuous variable, makes analogous treatment of the variables more appealing.

Treating the discrete good in a similar fashion as the continuous goods, the functional form for \( U_0 \) is

(3.12) \[ U_0(D) = B_d \log(D - y_d). \]

The latent demand equation becomes

(3.13) \[ D^* = y_d + \left( \frac{\beta_d}{p_d} \right) (y - p_d y_d - \sum_i y_i) + \left( \frac{\beta_d}{p_d} \right) (\sum_i y_i), \]

and the sorting function is
\[ D = \begin{cases} 
0 & \text{if } D^* < D(0) \\
1 & \text{if } D(0) < D^* < D(1) \\
2 & \text{if } D(1) < D^* < D(2) \\
\vdots & \text{\ldots} 
\end{cases} \]

\[ (3.14) \]

where

\[ D(\xi) = (1 - \beta_d) \gamma_d + \beta_d h(\xi) \]
\[ h(\xi) = \frac{\xi - k(\xi)(\xi+1)}{1 - k(\xi)} \]
\[ k(\xi) = \frac{\xi + 1 - \gamma_d}{\xi - \gamma_d} \frac{\beta_d / (\beta_x + \beta_z)}{\beta_d} \]

Note that we have not allowed for a distribution on \( \gamma_d \). This implies that the switchpoint equations are fixed across individuals.
4. Estimation

In this section, we outline an estimation procedure for the mixed integer/continuous demand system using the method of maximum likelihood. The estimation of this model differs from the usual estimation of a system of demand equations due to two factors. First, the presence of $D$ in the conditional demand equations makes the system simultaneous, and second, the heteroscedasticity of the error term complicates the variance/covariance matrix. The primary effect of these factors is to make identification of the model more problematic; however, we show that the model is fully identified.

We begin by making some distributional assumptions on the model given below.

\begin{align}
D^* &= f_d(p_d, p_x^*, y) + u_d^* \\
X^* &= f_x(p_x^*, y^*) + u_x^*
\end{align}

(4.1)

where $u_d^*$ and $u_x^*$ represent the stochastic terms given in the previous section. The distribution of the random disturbance terms, $\varepsilon$, is

\begin{equation}
\varepsilon \sim N(0, \Sigma).
\end{equation}

(4.2)

This implies that the density of $y$ is

\begin{equation}
y = \begin{bmatrix} u_d^* \\ u_x^* \end{bmatrix} \sim N(0, \Omega(p, \Theta, \Sigma))
\end{equation}

(4.3)

where

\begin{align}
\omega_{dd} &= (\Theta_d / p_d)^{\prime} \Sigma p \\
\omega_{ij} &= (1 / p_i p_j)^{\prime} \Theta_i \Sigma \Theta_j p \\
\omega_{dj} &= (\Theta_d / p_d p_j)^{\prime} \Sigma \Theta_j p \\
\end{align}

and $\Theta_i$ is defined as
\[ B_i = B_i I_n - E_{ii} \]

where \( I_n \) is an \( n \times n \) identity matrix and \( E_{ii} \) is a square matrix with 1 in the \((i,i)\) position and zeros elsewhere.

### 4.1 Identification

We first consider the identification of the conditional demand system. The only parameters not included in the conditional system are those specific to the discrete demand equation, \( \beta_d \) and \( \gamma_d \), which we deal with separately. The conditional demand equation written in expenditure form is

\[(4.4) \quad p_i x_i = p_i y_i + \beta_i (y - p_d D - \sum_j p_j y_j) + p_i u_i \]

where \( u_i \) is the heteroscedastic error term derived in the previous section.

The identification of the \( \beta_i \)'s and \( \gamma_i \)'s proceeds very simply. Rewrite equation (4.4) as

\[(4.5) \quad p_i x_i = \gamma_i (1 - \beta_i) p_i + \beta_i (y - p_d D) - \sum_{j \neq i} \delta_j p_j + u_i \]

The only complication arises from the \( p_d D \) term. \( D \) is endogenous and hence correlated with the disturbance term. However, \( p_d \), which is correlated with \( D \), may be used as an instrument. Thus estimates may be obtained of \( \delta_i \), \( \delta_o \), and \( \delta_j \), for \( j \neq i \). Because \( \beta_i = \delta_o \), \( \gamma_i \) is obtained from \( \delta_i \). Similarly, because \( \beta_j \) is known, each of the \( \gamma_j \)'s can be extracted.

Identification of the covariance matrix is more complicated due to the heteroscedasticity of the \( u_i \)'s. The variance of \( p_i x_i \) is
(4.6) \( \text{Var}(p_i x_i) = p_i B_i \Sigma B_i p \)

where \( \Sigma \) is the unconstrained \( n \times n \) covariance matrix of \( \tilde{e} \). Note that \( \Sigma \) is the covariance matrix for the full system. Since we did not posit a distribution on \( \gamma_d \), there is no \( \tilde{e}_d \) element of \( \tilde{e} \). An off-diagonal element of \( \Omega \) is:

\[
\omega_{ij} = (p/p_i)'(B_i I_n - E_{ii})\Sigma(B_j I_n - E_{jj})(p/p_j).
\]

Multiplying through by \( p_i \) and \( p_j \) and simplifying, we have

(4.7) \[ p_i p_j \omega_{ij} = B_i B_j p' \Sigma p - B_i p' \Sigma p_j - B_j p' \Sigma p_i + p_i p_j \sigma_{ij} \]

where \( \Sigma_j \) is the \( j^{th} \) column of the \( \Sigma \) matrix and \( \sigma_{ij} \) is the \( (i,j) \) element of this matrix.

By allowing the price vector to change, we show that \( \Sigma \) is determined completely. We change \( p_k \), where \( k \neq i \neq j \). The covariance between \( p_i x_i \) and \( p_j x_j \) with the new \( p_k \) is written

(4.8) \[ p_i p_j \omega_{ij}^k = B_i B_j (p + e_k)' \Sigma (p + e_k) - B_i (p + e_k)' \Sigma p_j - B_j (p + e_k)' \Sigma p_i + p_i p_j \sigma_{ij} \]

where \( e_k \) is a vector with 1 in the \( k^{th} \) position and 0's elsewhere.

Subtracting \( \omega_{ij}^k \) from \( \omega_{ij} \) gives

(4.9) \[ p_i p_j (\omega_{ij} - \omega_{ij}^k) = B_i B_j \sigma_{kk} + 2B_i B_j p' \Sigma e_k - B_i e_k' \Sigma p_j - B_j e_k' \Sigma p_i \]

This equation may be rewritten to extract \( \Sigma_k \):

(4.10) \[ p_i p_j (\omega_{ij} - \omega_{ij}^k) = [B_i B_j e_k + p'(2B_i B_j I_n - B_i E_{jj} - B_j E_{ii})] \Sigma_k \]

for all \( p \). Since \( B_i \) and \( B_j \) are identified, the matrix

\[ (2B_i B_j I_n - B_i E_{jj} - B_j E_{ii}) \]

is known and is non-singular. By varying \( p \), we may obtain \( n \) linearly
independent vectors and hence $\Sigma_k$ is determined uniquely. By choosing different values for $k$, every column of the $\Sigma$ matrix may be identified in like fashion.

The only remaining parameters to be identified are $\beta_d$ and $\gamma_d$. The difficulty here is that $E(D(\xi))$ is an intractable expression. We define a new binary variable.

$$\tilde{D}(\xi) = \begin{cases} 1 & \text{if } D > \xi \\ 0 & \text{otherwise} \end{cases}$$

or

$$(4.11) \quad \tilde{D}(\xi) = \begin{cases} 1 & \text{if } \xi > D(\xi) \\ 0 & \text{otherwise} \end{cases}$$

Substituting in for $D^*$ and $D(\xi)$, we obtain

$$(4.12) \quad \tilde{D}(\xi) = \begin{cases} 1 & \text{if } (y - \Sigma_p \gamma_i) - p_d h(\xi) \geq -p' \xi \\ 0 & \text{otherwise} \end{cases}$$

where $h(\xi)$ is as defined in (3.24). Let

$$\tilde{y} = y - \Sigma_{i \neq d} \gamma_i.$$ 

Thus

$$(4.13) \quad \tilde{D}(\xi) = \begin{cases} 1 & \text{if } \tilde{y} - p_d h(\xi) \geq -p' \xi \\ 0 & \text{otherwise} \end{cases}$$

The expected value of $D(\xi)$ is now very simple.

$$(4.14) \quad E(\tilde{D}(\xi)) = \Pr(\tilde{D}(\xi) = 1) \quad \frac{\tilde{y} - p_d h(\xi)}{p_d \Sigma_p} = 1 - \Phi\left(\frac{\tilde{y} - p_d h(\xi)}{p_d \Sigma_p}\right)$$

where $\Phi$ is the cdf for the standard normal. The income term is known because $\gamma_i$ for $i = 1, 2, \ldots, n$ is known, $p$ is exogenous, and $\Sigma$ is determined. Since the function $\tilde{y} - p_d h(\xi)$ is monotonic, $h(\xi)$ is identified. By computing $h(\xi)$ for two different values of $\xi$, $\beta_d$ and $\gamma_d$ are
identified.

Identification of all parameters, including the covariance matrix, has now been completed.

4.2 Specifying the Likelihood Function

To specify the likelihood function, we must find the joint density function over both discrete and continuous variables. Heckman [1978] outlined this procedure in a two equation model with a binary discrete variable. By extending his procedure slightly, we derive the joint density function for our model.

Because $D$, rather than $D^*$, is observed we must use the sorting function

$$
D = \begin{cases} 
0 & \text{if } u_d < q(0) \\
1 & \text{if } q(0) \leq u_d < q(1) \\
2 & \text{if } q(1) \leq u_d < q(2) \\
\vdots & \\
\end{cases}
$$

where $q(\xi)$ is obtained by subtracting the systematic part of the latent demand equation from $D(\xi)$.

$$
q(\xi) = D(\xi) - f_d(\xi)
$$

The joint density function over $u_x$ and $D$ is

$$
\begin{align*}
\psi(u_x, D) &= \begin{cases} 
\int_{q(0)}^{q(1)} \phi(u_x, u_d) \, du_d & \text{if } D = 0 \\
\int_{q(1)}^{q(2)} \phi(u_x, u_d) \, du_d & \text{if } D = 1 \\
\int_{q(2)}^{q(3)} \phi(u_x, u_d) \, du_d & \text{if } D = 2 \\
\vdots & \\
\end{cases}
\end{align*}
$$

where $\phi(\cdot)$ is the probability density function over $u_d$ and $u_x$. 
At this point, we must perform a change of variables so that the pdf covers the variables of interest, D and \( x \). In this case, the change of variables is quite simple because the system is triangular. Thus the determinant of the Jacobian transformation is 1 and

\[
(4.17) \quad \text{pdf}(x,D) = \psi(D,x - f_x(\cdot)).
\]

The joint density function over \( x \) and \( D \) becomes

\[
(4.18) \quad \psi(x,D) = \begin{cases} 
\int_{-\infty}^{q(0)} \phi[x - f_x(\cdot), u_d] \; du_d & \text{if } D = 0 \\
\int_{q(0)}^{q(1)} \phi[x - f_x(\cdot), u_d] \; du_d & \text{if } D = 1 \\
\int_{q(1)}^{q(2)} \phi[x - f_x(\cdot), u_d] \; du_d & \text{if } D = 2 \\
\ldots 
\end{cases}
\]

In addition to data on individuals, the data sample necessary to estimate this model would include observations over time so that there would be variation in prices. Given a pooled cross-section time series, the likelihood function is

\[
(4.19) \quad L = \prod_{t=1}^{T} \psi(p_{x_t}, p_{d_t}, y_{i_t}, D_{i_t})
\]

where the \( t \) subscript is across time periods and the \( i \) subscript is across individuals.

The maximum likelihood estimators obtained for this model will be consistent, asymptotically normal, and efficient. Heckman [1978] has verified that the LeCam [1953] conditions are met in his two equation model with a binary discrete variable. His results generalize in a straightforward fashion so that these estimators will have the usual properties.
5. Concluding Remarks

In this essay, we have devised a model to describe consumer behavior when consumers are confronted with lumpy alternatives. In particular, we deal with the case of a single discrete good which can be purchased only in integer quantities. The model uses a systems approach, incorporating demand for the discrete good with demand for continuous goods into a complete system of demand equations. After deriving a general form for the mixed integer/continuous model, we derived a particular functional form for the model by imposing linearity constraints on the demand equations. Finally, we have shown that the model is identified and have outlined an estimation procedure using maximum likelihood methods.

This mixed integer/continuous model extends previous work in consumer demand theory in two key ways. First, we specify the discrete alternatives as an ordered discrete good. The existing literature, while extensive, has concentrated on discrete alternatives which are mutually exclusive and unordered. Second, we analyze this good jointly with the remaining continuous goods in the consumer’s choice set. Formulating the demand for the discrete good in the context of a system of demand equations, a technique which had not previously been developed explicitly, appears useful, particularly in applications where the individual chooses the intensity of use of the discrete good as well as the number of units to buy.

It is possible that some problems where the mixed integer/continuous model could be applied could also be dealt with in a sequential choice model. However, whether or not a sequential model will be practical to
implement depends on the time aggregation of the data. Given sufficiently disaggregated data, a sequential choice model may be suitable. For data collected on, say, an annual basis, the integer/continuous model may be preferred.

One limitation of our model is that the derivation of the discrete demand equation requires the direct utility function. This precludes the use of certain functional forms for the system of demand equations where the direct utility function cannot be written explicitly. Eliminating this restriction is a goal for further research. A second limitation is that we have specified nonrandom switchpoints, which implies that the switchpoints are fixed across individuals. While this assumption is restrictive, it reduced substantially the complexity of the joint distributions. Current efforts are directed towards relaxing this assumption.

The mixed integer/continuous model may be extended in several ways. One desirable extension would be to expand the model to include more than one discrete good. Another possibility is to utilize splines for direct estimation of the switchpoints. This procedure would allow us to adopt a more flexible form for the demand equation by eliminating many of the demand theory restrictions.

The application of this model to such problems as symphony tickets or tickets to baseball games is worth exploring. In these situations, the number of tickets one may buy is an integer variable. However, the "producer" generally offers tickets in a variety of packages and the price of a single ticket varies according to the package. This complicates the consumer's problem because the pricing scheme is typically
non-linear. Additionally, it is interesting to consider this from the producer's viewpoint - what packages should be offered? There are a number of applications of this type.
Appendix

In this appendix, we provide proofs of the existence and the uniqueness of the switchpoint price of the discrete good D.

Existence

Theorem: If the conditional indirect utility function is a continuous function in $p_d$, then a price of D exists such that the consumer is indifferent between $x$ and $x+1$ units of D.

Proof: The conditional indirect utility function is obtained by solving the following constrained optimization problem.

(A.1) \[ \max_{x} U(D,x) \text{ subject to } y - p_d D = p' x \]

gives

(A.2) \[ x^* = h(D, p_x, y^*) \]

where \[ y^* = y - p_d D. \]

Substituting (A.2) back into the direct utility function gives

(A.3) \[ V(D, h(D, p_x, y^*)). \]

Let $p_d^x$ be the price of D which would allow utility maximization at a tangency at the point where $D = x$. Let $p_d^{x+1}$ be the similar price for $D = x+1$, with $p_d^x > p_d^{x+1}$. To shorten the notation, define
\[ V(\ell, p_d^\ell) = V[\ell, h(\ell, p_d^\ell, y - p_d^\ell(\ell))] \]
\[ V(\ell+1, p_{d+1}^{\ell+1}) = V[\ell+1, h(\ell+1, p_{d+1}^{\ell+1}, y - p_{d+1}^{\ell+1}(\ell))] \]

It will be the case that

\[ (A.4) \quad V(\ell+1, p_{d+1}^{\ell+1}) > V(\ell, p_d^{\ell+1}) > V(\ell, p_d^{\ell}) > V(\ell+1, p_d^{\ell}). \]

As \( p_d \) rises from \( p_d^{\ell+1} \) to \( p_d^{\ell} \),

\[ (A.5) \quad V(\ell, p_d^{\ell+1}) > V(\ell, p_d^{\ell}) \]
\[ V(\ell+1, p_{d+1}^{\ell+1}) > V(\ell+1, p_d^{\ell}). \]

Thus there is some point at which \( V(\ell, p_d) = V(\ell+1, p_d) \), and a switchpoint price exists.

Q.E.D.

Uniqueness of the switchpoint price means that only one price of \( D \) will exist such that the individual is indifferent between \( \ell \) and \( \ell + 1 \) units of \( D \).

**Theorem:** A sufficient condition to ensure uniqueness of the switchpoint price is that \( D \) be a normal good.

**Proof:** The conditional indirect utility function can be written as

\[ (A.6) \quad V(\ell, p, p_d, y) \]

The conditional expenditure function is derived by solving the optimization problem

\[ \min_{\ell} p_d D + p'x \quad \text{subject to} \quad u = \bar{u}, \]

and can be written as
Let $p^*_d$ be a switchpoint price. Then it will be true that

(A.8) \[ V(\ell, p, p^*_d, y) = V(\ell + 1, p, p^*_d, y) \]

and

(A.9) \[ e[\ell, p, p^*_d, V(\ell, p, p^*_d, y)] = e[\ell + 1, p, p^*_d, V(\ell + 1, p, p^*_d, y)]. \]

For $p_d > p^*_d$, there is some income, $\phi$, which will compensate for the higher price of D and allow

(A.10) \[ V(1, p, p_d, y) = V(\ell + 1, p, p_d, \phi). \]

Now suppose that another switchpoint price exists, with

\[ p^*_{d+1} > p^*_d \]

where reswitching occurs. At that price, it must be true that $\phi = y$. But

(A.11) \[ \partial y / \partial p_d = \partial e[\ell, p, p_d, V(\ell, p, p_d, y)] / \partial p_d = 0 \]

and

(A.12) \[ \partial \phi / \partial p_d = \partial e[\ell + 1, p, p_d, V(\ell + 1, p, p_d, \phi)] / \partial p_d \]
\[ + \{ \partial e[\ell + 1, p, p_d, V(\ell + 1, p, p_d, \phi)] / \partial V \} \partial V(\ell, p, p_d, y) / \partial p_d \].

Since

\[ \partial V(\ell, p, p_d, y) / \partial p_d = \partial V(\ell, p, p_d, y) / \partial y \cdot (-\ell) \]

and

\[ \partial e[\ell + 1, p, p_d, V(\ell + 1, p, p_d, \phi)] / \partial p_d = \ell + 1 \]

(A.12) simplifies to

(A.13) \[ \partial \phi / \partial p_d = \ell + 1 - \ell \{ \partial e[\ell + 1, p, p_d, V(\ell + 1, p, p_d, \phi)] / \partial V \} \partial V(\ell, p, p_d, y) / \partial y \}

This, in turn, is simply

\[ \partial \phi / \partial p_d = \ell + 1 - \ell (\partial \phi / \partial y).
\]

or

(A.14) \[ \partial \phi / \partial p_d = [(\ell + 1)/\ell - \partial \phi / \partial y] \ell. \]
To rule out reswitching, $\frac{\partial p}{\partial p_d}$ must be greater than 0. Since $(\lambda + 1)/\lambda > 1$, a sufficient condition is that $0 \leq \frac{\partial p}{\partial y} < 1$. This condition, $0 \leq \frac{\partial p}{\partial y} < 1$, will hold if $D$ is a normal good. Thus $D$ a normal good is a sufficient condition to ensure uniqueness of the switchpoint price.

In the limiting case, as $D$ gets very large, the ratio $(\lambda + 1)/\lambda + 1$ and $D$ a normal good becomes the necessary condition as well.

Q.E.D.
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