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DYNAMIC ANALYSIS OF
NONCLASSICALLY DAMPED SYSTEMS

by

CARLOS ESTUARDO VENTURA

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

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HOUSTON, TEXAS

May, 1985
ABSTRACT

DYNAMIC ANALYSIS OF
OF NONCLASSICALLY DAMPED SYSTEMS

by

Carlos Estuardo Ventura Z.

The objectives of the studies reported in this dissertation are: (1) to develop improved techniques for evaluating the dynamic response of viscously damped linear systems, and (2) to contribute concepts and information which will lead to an improved insight into the dynamic response of such systems.

The dissertation consists of two major parts. The first part, reported in Chapters II through IV, deals with the analysis of the response of nonclassically damped discrete systems. A critical evaluation is first made of the generalized modal superposition method of analysis for such systems, with special emphasis on identifying the physical significance of the various terms in the solution and simplifying its implementation. Next, the response spectrum variant of the procedure is examined for base-excited systems.

The interrelationship of the spectral values of deformation and relative velocity of single-degree-of-freedom systems is identified, and simple practical rules are presented for defining the design spectra for relative velocity for such systems. These rules are similar to those available for defining the corresponding spectra for maximum deformation. Finally, a recently proposed procedure
for interrelating the steady-state and transient responses of classically damped systems is extended to nonclassically damped systems.

The second part of this dissertation, comprised of Chapters V and VI, deals with the application of the Discrete Fourier Transform (DFT) method of dynamic analysis. The limitations and principal sources of potential inaccuracies of this approach are identified, and an evaluation is made of the nature and magnitudes of the errors that may result from its indiscriminate use. Two versions of a modification are then presented which dramatically improve the efficiency of the procedure, and the relative merits of the two techniques are examined. The concepts involved are developed by reference to single-degree-of-freedom systems and are then extended to the analysis of multi-degree-of-freedom systems.
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CHAPTER I

INTRODUCTION AND ACKNOWLEDGEMENT
INTRODUCTION

The objective of the study reported herein is twofold: (1) to develop improved techniques for evaluating the dynamic response of viscously damped linear structural systems, and (2) to contribute concepts and information which will lead to an improved insight into the dynamic response of such systems.

The study consists of two parts. The first part, reported in Chapters II through IV, deals with the analysis and understanding of the response of systems of which the damping matrix is such that the governing equations of motion cannot be uncoupled by use of the undamped natural modes of vibration of the system. The classical modal superposition of analysis is not applicable in this case, and the systems are said to be nonclassically damped.

Chapter II deals with the analysis of the free and forced vibration of nonclassically damped systems by the generalized modal superposition method. Special attention is paid to simplifying the implementation of the analysis of such systems and to interpreting the physical meaning of the various terms in the expressions for the response. It is shown that the response of a nonclassically damped multi-degree-of-freedom system may be expressed as linear combinations of the displacements and velocities of similarly excited single-degree-of-freedom systems, and that the analysis may be implemented with only minor computational effort beyond that required for the analysis of a classically damped system of the same size. Numerical solutions are presented to illustrate the concepts involved, and the results are compared with those obtained by an approximate
modal superposition method that makes use of the undamped natural modes of the system.

Chapter III deals with a response spectrum method for evaluating the maximum response of nonclassically damped systems. The method involves the use of response spectra for the maximum deformation and maximum velocity of single-degree-of-freedom systems. Simple procedures are presented for evaluating the response spectra for maximum velocity, and the interrelationship of these spectra and the corresponding for maximum deformation is examined.

Chapter IV presents the extension to nonclassically damped systems of a recently proposed technique for interrelating the transient and steady-state responses of viscously damped linear systems. Making use of the information presented in Chapter II, it is shown that the transient and steady-state responses of nonclassically damped systems can be interrelated in a form similar to that developed for classically damped systems.

The second part of this dissertation, comprised of Chapters V and VI, deals with the analysis of the dynamic response of linear systems by the Discrete Fourier Transform (DFT) method. In Chapter V, the limitations of the classical DFT procedure are identified, and a critical evaluation is made of the magnitude of the errors that may result from its indiscriminate use. Two versions of a modified procedure are then presented which dramatically improve the efficiency of the DFT approach, and the relative merits of the two techniques are examined.

The basic concepts involved are developed in Chapter V by refer-
ence to single-degree-of-freedom systems, and the extension of these techniques to nonclassically damped multi-degree-of-freedom systems is discussed in Chapter VI. Both constant-parameter systems and systems with frequency-dependent parameters may be considered.

Each of the five chapters is presented in a self-contained manner, so that it may be read independently of, or with minimum reference to the other chapters. The background information for each chapter, the relevant references, notation for symbols employed, and the conclusions drawn are presented in each chapter. As a consequence, some ideas may be repeated from one chapter to another, but it is hoped that this repetition may prove helpful to the reader.

ACKNOWLEDGEMENT

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Financial support for the graduate study leading to this dissertation has also been provided by Rice University, by the Secretaria de Planificacion Economica and by the Direccion para el Fomento de Becas of Guatemala, Central America. The generosity of these institutions is gratefully acknowledged.

Chapters II and V are the same as Reports No. 28 and 31 in the series Structural Research at Rice, Department of Civil Engineering, Rice University. Reference 8 in Chapter V was also prepared in the
course of the studies reported herein, but is not included in the present document.

I wish to express my sincere appreciation to Dr. A. S. Veletsos for his excellent motivation and orientation of this work, and for making my life at Rice a learning experience that I will always cherish.

I cannot fail to thank my friends Ricardo Bordinhao, Kirk Dotson and Guillermo Hahn for their constant support and helpful suggestions.

I also want to thank my wife, Lucrecia, and my son, Estuardo, for their sacrifices, understanding and encouragement throughout this work. To both of them, my love and gratitude.
CHAPTER II

MODAL ANALYSIS OF NONCLASSICALLY DAMPED LINEAR SYSTEMS
INTRODUCTION

The modal superposition method is generally recognized as a powerful method for evaluating the dynamic response of viscously damped linear structural systems. The method enables one to express the response of a multi-degree-of-freedom system as a linear combination of its corresponding modal responses. Two versions of the procedure are in use: (1) The time history version, in which the modal responses are evaluated as a function of time and then combined to yield the response history of the system; and (2) the response spectrum version, in which first the maximum values of the modal responses are determined, usually from the response spectrum applicable to the particular excitation and damping under consideration, and the maximum response of the system is then computed by an appropriate combination of the modal maxima.

When damping is of the form specified by Caughey and O'Kelly\textsuperscript{1}, the natural modes of vibration of the system are real-valued and identical to those of the associated undamped system. Systems satisfying this condition are said to be \textbf{classically damped}, and the modal superposition method for such systems is referred to as the \textbf{classical modal method}.

The classical modal method has found widespread application in civil engineering practice because of its conceptual simplicity, ease of application and the insight it provides into the action of the system. The response spectrum variant of the method, which makes it possible to identify and consider only the dominant terms in the solution, is particularly useful for making rapid estimates of maximum
response values.

Viscously damped systems that do not satisfy the Caughey-O'Kelly condition generally have complex-valued natural modes. Such systems are said to be nonclassically damped, and their response may be evaluated by a generalization of the modal superposition method due to Foss. 2

Although well established, 2-9 the generalized modal method has found only limited application in structural engineering practice. Several factors appear to have contributed to this: (a) The generalized method is inherently more involved than the classical; (b) when used in conjunction with the response spectrum concept, it has had to rely on approximations of questionable accuracy; and (c) perhaps most important, the physical meanings of the elements of the solution for this method have not yet been identified as well as those for the classical modal method.

Considering that the specification of the nature and magnitude of damping in structures is subject to considerable uncertainty, the assumption of classical damping may be a justifiable approximation in many practical applications. There are instances, however, in which a more refined analysis is definitely warranted, and it is important that the attractiveness and reliability of the generalized procedure be improved. This chapter is intended to be responsive to this need.

Its broad objectives are: (1) To simplify the implementation of the generalized modal method for evaluating the dynamic response of nonclassically damped linear structures; and (2) by clarifying the physical significance of the elements of this method, to increase
the attractiveness of the method to, and its use by, structural engineers.

The method is described by reference to cantilever structures excited at the base, and it is then applied to force-excited systems. Following an examination of the natural frequencies and modes of vibration of such systems, the free vibrational response is formulated in several alternative forms, and the more convenient form, in combination with the concepts employed in the development of Duhamel's integral for single-degree-of-freedom systems, is used to obtain the solution for forced vibration. Novel features of the present treatment are: the physical interpretation of the solutions represented by complex conjugate pairs of characteristic vectors, the use of these solutions as building blocks in the definition of the free vibrational and forced vibrational responses of the system, and the identification of some useful relationships for the vectors in the expression for forced vibration.

It is shown that the transient displacements of a nonclassically damped multi-degree-of-freedom system may be expressed as a linear combination of the deformations and the true relative velocities of a series of similarly excited single-degree-of-freedom systems, and that the analysis may be implemented with only minor computational effort beyond that required for a classically damped system of the same size. The concepts involved are illustrated by a series of examples, and comprehensive numerical solutions are presented which elucidate the sensitivity of the response to variations in several important parameters. The exact solutions also are compared with
those obtained by an approximate modal superposition involving the use of classical modes of vibration, and the interrelationship of the two sets of solutions is discussed.

Although use is made of complex-valued algebra in the derivation of some of the expressions presented herein, all final solutions are presented in terms of real-valued quantities.

Notation

The symbols used in this chapter are defined when first introduced in the text, and those used extensively are summarized in Appendix II.

**STATEMENT OF PROBLEM**

A viscously damped, linear cantilever system with n degrees of freedom is considered. The system is presumed to be excited by a base motion the acceleration of which is $\ddot{g}(t)$. The response of this system is governed by the equations

$$[m]\{\ddot{x}\} + [c]\dot{\{x\}} + [k]\{x\} = -[m]\{1\}\ddot{g}(t)$$

(1)

in which $\{x\}$ is the column vector of the displacements of the nodes relative to the moving base; a dot superscript denotes differentiation with respect to time, t; $\{1\}$ is a column vector of ones; and $[m]$, $[c]$ and $[k]$ are the mass matrix, damping matrix and stiffness matrix of the system, respectively. The latter matrices are real and symmetric; additionally, $[c]$ and $[k]$ are positive semi-definite, and $[m]$ is positive definite. The objective is to elucidate the analysis of the response of the system when no further restriction is imposed on the form of the damping matrix.
NATURAL FREQUENCIES AND MODES

For a system in free vibration, the right-hand member of equation (1) vanishes, and the equation admits a solution of the form

\[ \{x\} = \{\psi\} e^{rt} \]  

(2)

where \( r \) is a characteristic value and \( \{\psi\} \) is the associated characteristic vector or natural mode. On substituting equation (2) into the homogeneous form of equation (1), one obtains the characteristic value problem

\[ \left( r^2[m] + r[c] + [k] \right)\{\psi\} = \{0\} \]  

(3)

in which \( \{0\} \) is the null vector.

Rather than through the solution of the system of equations (3), the desired values of \( r \) and the associated \( \{\psi\} \) may be determined more conveniently by first reducing the system of \( n \) second order differential equations (1) to a system of \( 2n \) first order differential equations, as suggested in References 3 and 4. Summarized briefly in Appendix I, this approach leads to a characteristic value problem of the form

\[ r[A]\{Z\} + [B]\{Z\} = \{0\} \]  

(4)

in which \( [A] \) and \( [B] \) are symmetric, real matrices of size \( 2n \) by \( 2n \); and \( \{Z\} \) is a vector of \( 2n \) elements, of which the lower \( n \) elements represent the desired modal displacements, \( \{\psi\} \), and the upper \( n \) elements represent the associated velocities, \( r[\psi] \). Equation (4) may be solved by well established procedures.

Provided the amount of damping in the system is not very high, the characteristic values occur in complex conjugate pairs with either
negative or zero real parts. For a system with \( n \) degrees of freedom, there are \( n \) pairs of characteristic values, and to each such pair there corresponds a complex conjugate pair of characteristic vectors.

Let \( r_j \) and \( \bar{r}_j \) be a pair of characteristic values defined by

\[
\begin{pmatrix}
  r_j \\
  \bar{r}_j
\end{pmatrix} = -q_j \pm i\bar{p}_j
\]

(5)

and

\[
\begin{pmatrix}
  \psi_j \\
  \bar{\psi}_j
\end{pmatrix} = \{\phi_j\} \pm i\{\chi_j\}
\]

(6)

be the associated pair of characteristic vectors. In these expressions, \( i = \sqrt{-1} \); \( q_j \) and \( \bar{p}_j \) are real positive scalars; and \( \{\phi_j\} \) and \( \{\chi_j\} \) are real-valued vectors of \( n \) elements each. Further, let \( p_j \) be the modulus of \( r_j \), i.e.

\[
p_j = \sqrt{q_j^2 + \bar{p}_j^2}
\]

(7)

and

\[
\zeta_j = \frac{q_j}{p_j}
\]

(8)

Then the characteristic values may be expressed as

\[
\begin{pmatrix}
  r_j \\
  \bar{r}_j
\end{pmatrix} = -\zeta_j p_j \pm i\bar{p}_j
\]

(9)

and the following relationship between \( \bar{p}_j \) and \( p_j \) is determined from equation (7)

\[
\bar{p}_j = p_j \sqrt{1 - \zeta_j^2}
\]

(10)

The values of \( p_j \) are numbered in ascending order, and the values of \( r_j \) and the associated vectors, \( \{\psi_j\} \), are numbered in the order
of the corresponding $p_j$.

Equations (9) are the same as those governing the characteristic roots of a viscously damped single-degree-of-freedom (SDF) system with an undamped circular natural frequency $p_j$ and a damping factor $\zeta_j$. Furthermore, equation (10) is the same as that relating the undamped and damped frequencies of such a system (see second section in Appendix I). In the following developments, $p_j$ will be referred to as the $j$th pseudo-undamped circular natural frequency of the system; $\bar{p}_j$ will be referred to as the corresponding damped frequency; and $\zeta_j$ as the $j$th modal damping factor. It should be noted that $p_j$ is a function of the amount of system damping present and, hence, differs from the corresponding frequency of the associated undamped system. Where confusion may arise, the latter frequency will be denoted by the symbol $p_j^0$.

**Reduction for Undamped and Classically Damped Systems**

For an undamped system, the characteristic values $r_j$ are purely imaginary and the natural modes are real-valued. Accordingly, $\{x_j\} = \{0\}$, $\{\psi_j\} = \{\phi_j\}$, and $\bar{p}_j = p_j = p_j^0$.

Caughey and O'Kelly\(^1\) have shown that if the damping matrix of the system satisfies the identity

$\begin{bmatrix} c \end{bmatrix}[m]^{-1}[k] = [k][m]^{-1}[c] \tag{11}$

the natural modes are real-valued and equal to those of the associated undamped system. The exponent in this expression denotes the inverse of a matrix. The characteristic values in this case appear in complex conjugate pairs, and the modulus of each pair, $p_j$, is the same as
the circular natural frequency of the associated undamped system, \( p_j^0 \). The Rayleigh form of damping,\(^{12} \) for which \([c]\) is proportional to either \([m]\) or \([k]\) or is a linear combination of the two, is a special case of equation (11).

**Orthogonality of Modes**

Any pair of characteristic vectors corresponding to distinct characteristic values, including a complex conjugate pair of modes, satisfies the following orthogonality relations:

\[
(r_j + r_k) \{\psi_j\}^T[m]\{\psi_k\} + \{\psi_j\}^T[c]\{\psi_k\} = 0
\]

(12)

and

\[
\{\psi_j\}^T[k]\{\psi_k\} - r_k r_j \{\psi_j\}^T[m]\{\psi_k\} = 0
\]

(13)

The derivation of these expressions is reviewed briefly in the third section of Appendix I.

For a classically damped system, for which \( \{\psi_j\} = \{\phi_j\} \) and \( p_j = p_j^0 \), it can be shown (see fourth section of Appendix I) that

\[
\{\phi_j\}^T[c]\{\phi_k\} = 2 \zeta_j p_j \{\phi_j\}^T[m]\{\phi_k\}
\]

(14)

and that equations (12) and (13) reduce to the well-known relations

\[
\{\phi_j\}^T[m]\{\phi_k\} = 0 \quad \text{for} \quad r_k \neq r_j
\]

(15)

and

\[
\{\phi_j\}^T[k]\{\phi_k\} = 0 \quad \text{for} \quad r_k \neq r_j
\]

(16)

It should be recalled that the quantities \( p_j \) and \( \{\phi_j\} \) in these expressions are the same as those for the associated undamped system.

**Examples**

The sensitivity of the free vibrational characteristics of a
non-classically damped system to the magnitude and distribution of
the damping present are examined in this section for the three-story
building frame shown in Figure 1. The structure is presumed to be
of the shear-beam type with uniform story stiffnesses, \( k \), and with
floor masses \( m, m \) and \( m/2 \), as indicated. System damping is concen-
trated either in the bottom story or the top story. The damping coeffi-
cient, \( c \), is expressed in the form

\[
c = \zeta_0 \sqrt{k m}
\]  

(17)

and several different values of the dimensionless constant, \( \zeta_0 \), are
used. The damping matrices for these systems do not satisfy equation
(11).

The characteristic values and vectors of these systems were
evaluated by use of a standard computer program\textsuperscript{13} making use of the
procedure reviewed in the first section of Appendix I. The results
for the system shown in Figure 1(a) are given in Table I for four
different values of \( \zeta_0 \). Listed in addition to the values of \( r_j \) and
\( \{\psi_j\} \) are the values of \( p_j \), \( \bar{p}_j \) and \( z_j \). The characteristic vectors
are normalized such that the real part of the displacement of the
first floor is unity and the corresponding imaginary part is zero.

The components \( \{\phi_j\} \) and \( \{x_j\} \) of the characteristic vectors for
the system with the damper in the bottom story are also plotted in
Figure 2. Note that the imaginary component, \( \{x_j\} \), may be quite
substantial for high damping values, particularly for the higher
modes of vibration. By contrast, the real component, \( \{\phi_j\} \), is not
particularly sensitive to the amount of damping present.
The circular natural frequencies of the system for each of the
two distributions of damping considered are plotted in Figure 3 as
a function of \( \zeta_0 \). As would be expected from equation (10), the damped
frequencies are lower than the pseudo-undamped. However, the former
frequencies are not necessarily lower than the true undamped natural
frequencies of the system, \( \omega_j^0 \), (i.e., those corresponding to \( \zeta_0 = 0 \));
and for a high order mode, they may well be lower than the correspon-
ding frequencies of a lower order mode. As an example, note that
\( \tilde{\omega}_2 \) in Figure 3(b) increases with increasing \( \zeta_0 \), and that the values
of \( \tilde{\omega}_3 \) for the high values of \( \zeta_0 \) are less than the corresponding values
of \( \tilde{\omega}_2 \). It should be recalled that it is the pseudo-undamped, rather
than the damped, frequencies of the system that are numbered in ascen-
ding order.

The modal damping factors for the system, \( \zeta_j \), are displayed
in Figure 4. Note that each of these factors is affected differently
by a change in the overall damping of the system, and that an increase
in system damping may increase or decrease the modal damping values.

Comparison with Results of Approximate Procedure

Included in Figure 4 for purposes of comparison are the values
of \( \zeta_j \) computed on the assumption that the transformation which in
the classical modal superposition method diagonalizes the stiffness
matrix of the system also diagonalizes the damping matrix. The dis-
placements of the system in this approximation are expressed as a
linear combination of its classical, undamped natural modes of vibra-
tion, and in evaluating the triple matricial products \( \{ \phi_k \}^T [c] \{ \phi_j \} \),
the terms corresponding to \( j \neq k \) are omitted. The superscript \( T \) in
the latter expression denotes a transposed vector. This approach, which has been the subject of numerous previous studies, \(^{11,14-18}\) will be referred to herein as the **approximate procedure**. It may be seen that while the agreement between the two sets of results is generally reasonable, there are significant differences for the system considered in Figure 3(b), particularly for the larger values of \(\zeta_0\).

Selected values of the exact and approximate modal damping factors for the system in Figure 1(a) are listed in Table II, along with the associated natural frequency values, \(p_j\) and \(\tilde{p}_j\).

**MODAL SOLUTION**

The term **modal solution** will be used to identify a solution represented by a linear combination of a complex conjugate pair of characteristic values and their associated vectors. In particular, the \(j\)th modal solution for displacements is given by

\[
\{x\} = \mathbf{C}_j \{\psi_j\} e^{r_j t} + \mathbf{\bar{C}}_j \{\bar{\psi}_j\} e^{\bar{r}_j t} \tag{18}
\]

in which \(\mathbf{C}_j\) is a complex-valued constant and \(\mathbf{\bar{C}}_j\) is its complex conjugate.

Since the second term on the right-hand member of equation (18) is the complex conjugate of the first, the sum of the imaginary terms in this expression vanishes and equation (18) reduces to

\[
\{x\} = 2 \text{Re} [\mathbf{C}_j \{\psi_j\} e^{r_j t}] \tag{19}
\]

in which \(\text{Re}\) stands for the real part of the quantity that follows.

It is instructive to express equation (19) entirely in real-valued terms, and to this end two alternative forms are considered. If
one expresses $C_j$ in terms of its modulus and a phase angle as

$$C_j = \frac{1}{2} a_j e^{i \theta_j}$$

(20)

and makes use of equations (6) and (9), and of the identity between exponential and trigonometric functions, equation (19) may be written as

$$\{x\} = a_j e^{-\zeta_j \omega_j t} [\{\phi_j\} \cos(\bar{\omega}_j t + \theta_j) - \{x_j\} \sin(\bar{\omega}_j t + \theta_j)]$$

(21)

Alternatively, if one first evaluates the product

$$2C_j \{\psi_j\} = \{\beta_j\} + i \{\gamma_j\}$$

(22)

in which $\{\beta_j\}$ and $\{\gamma_j\}$ are real-valued vectors, then substitutes equation (22) into equation (19), and makes use of the identity between exponential and trigonometric functions, one obtains

$$\{x\} = e^{-\zeta_j \omega_j t} [\{\beta_j\} \cos(\bar{\omega}_j t) - \{\gamma_j\} \sin(\bar{\omega}_j t)]$$

(23)

Equations (19), (21) and (23) represent the superposition of two exponentially decaying harmonic motions with a circular frequency, $\bar{\omega}_j$, and a damping factor, $\zeta_j$. The component motions in each case lag one another by 90 degrees or one-quarter the period $\bar{T}_j = 2\pi/\bar{\omega}_j$, and they are in different configurations. As a result, each point of the system undergoes a simple harmonic motion, but the configuration of the system does not remain constant but changes continuously, repeating itself at intervals $\bar{T}_j$. The quantity $\bar{T}_j$ is known as the jth damped natural period of the system.

It should be clear from equation (21) that the $\{\phi_j\}$ configuration is attained when $\cos(\bar{\omega}_j t + \theta_j) = \pm 1$ or $\sin(\bar{\omega}_j t + \theta_j) = 0$, whereas the $\{x_j\}$
configuration is attained when \( \sin(\tilde{p}_j t + \theta_j) = \pm 1 \) or \( \cos(\tilde{p}_j t + \theta_j) = 0 \). It can further be seen from equation (22) that the \( \{\theta_j\} \) configuration is attained when \( \cos \tilde{p}_j t = \pm 1 \) or \( \sin \tilde{p}_j t = 0 \), whereas the \( \{\gamma_j\} \) configuration is attained when \( \sin \tilde{p}_j t = \pm 1 \) or \( \cos \tilde{p}_j t = 0 \).

It is important to realize that equations (21) and (23) are only two of an infinite number of forms in which the modal solution may be expressed. Other combinations of the basic modal components \( \{\phi_j\} \) and \( \{\chi_j\} \) can be used as the reference configurations, and this fact is used to advantage in a subsequent development.

**Reduction for Classically Damped Systems**

For undamped systems, for which \( \{\psi_j\} = \{\tilde{\psi}_j\} = \{\phi_j\} \) and \( \tilde{p}_j = p_j = p_j^0 \), equation (21) reduces to

\[
\{x\} = a_j \{\phi_j\} \cos(p_j^0 t + \theta_j)
\]  

(24)

Such systems can execute simple harmonic motions in fixed, time-invariant configurations.

For damped systems satisfying equation (11), equation (21) reduces to

\[
\{x\} = a_j \{\phi_j\} e^{-\zeta_j p_j^0 t} \cos(p_j^0 t + \theta_j)
\]  

(25)

in which \( \tilde{p}_j^0 \) is related to \( p_j^0 \) by the same expression as that relating \( \tilde{p}_j \) to \( p_j \). Such systems can vibrate in time-invariant configurations but with exponentially decaying amplitudes.

**FREE VIBRATION**

The response of the system to an arbitrary initial excitation is given by the superposition of the modal solutions presented in the preceding section. In particular, the displacements \( \{x\} \) may
be expressed either in the form of equation (19) as

$$\{x\} = 2 \sum_{j=1}^{n} \Re \left[ C_j \{\psi_j\} e^{r_j^t} \right]$$  \hspace{2cm} (26)

or in the form of equations (21) and (23) as

$$\{x\} = \sum_{j=1}^{n} a_j e^{-\zeta_j p_j^t} \left[ \{\phi_j\} \cos(\tilde{p}_j t + \theta_j) - \{x_j\} \sin(\tilde{p}_j t + \theta_j) \right]$$  \hspace{2cm} (27)

or

$$\{x\} = \sum_{j=1}^{n} e^{-\zeta_j p_j^t} \left[ \{\beta_j\} \cos(\tilde{p}_j t) - \{\gamma_j\} \sin(\tilde{p}_j t) \right]$$  \hspace{2cm} (28)

The complex-valued participation factors, $C_j$, may be determined from

$$C_j = \frac{r_j \{\psi_j\}^T[m] \{x(0)\} + \{\psi_j\}^T[c] \{x(0)\} + \{\psi_j\}^T[m] \{\dot{x}(0)\}}{2 r_j \{\psi_j\}^T[m] \{\psi_j\} + \{\psi_j\}^T[c] \{\psi_j\}}$$  \hspace{2cm} (29)

in which $\{x(0)\}$ is the prescribed vector of initial displacements and $\{\dot{x}(0)\}$ is the corresponding vector of initial velocities. The derivation of this equation is given under the fifth heading in Appendix I. With the values of $C_j$ established, the constants $a_j$ and $\theta_j$ in equation (27) are determined from equation (20), and the vectors $\{\beta_j\}$ and $\{\gamma_j\}$ in equation (28) are determined from equation (22).

For a classically damped system, the following generalized version of equation (14) is valid (see fourth section in Appendix I):

$$\{\psi_j\}^T[c] \{x(0)\} = 2 \zeta_j p_j^0 \{\phi_j\}^T[m] \{x(0)\}$$  \hspace{2cm} (30)

and on making use of this result and of equations (9) and (14), equation (29) reduces to

$$C_j = \frac{1}{2} \left[ x_j^* - i \left( \frac{v_j^*}{p_j^0} + \frac{\zeta_j}{\sqrt{1 - \zeta_j^2}} x_j^* \right) \right]$$  \hspace{2cm} (31)
in which
\[
x_j^* = \frac{\{\phi_j\}^T[m]\{x(0)\}}{\{\phi_j\}^T[m]\{\phi_j\}}
\]
and
\[
v_j^* = \frac{\{\phi_j\}^T[m]\{\dot{x}(0)\}}{\{\phi_j\}^T[m]\{\phi_j\}}
\]
Similarly, the vectors \{b_j\} and \{y_j\} in equations (23) and (28) reduce to the well-known expressions\(^{10}\):
\[
\{b_j\} = x_j^* \{\phi_j\}
\]
and
\[
\{y_j\} = -\left(\frac{v_j^*}{b_j^0} + \frac{\xi_j}{\sqrt{1 - \xi_j^2}} x_j^*\right) \{\phi_j\}
\]
in which \{\phi_j\} should be interpreted as the jth real-valued mode of the associated undamped system.

Equations (26) and (28) are the more convenient of the three forms used to express the response, and will be emphasized in the material that follows.

**Initial Conditions that Excite a Single Mode**

If the initial displacements and velocities of the system are of the form
\[
\{x(0)\} = 2 \text{ Re} \left[C_k \{\psi_k\}\right]
\]
and
\[
\{\dot{x}(0)\} = 2 \text{ Re} \left[r_k C_k \{\psi_k\}\right]
\]
it can be shown (see sixth section in Appendix I) that all values of $C_j$ in equation (26), except for the $k$th, vanish, and that the response of the system is given by

$$\{x\} = 2\text{Re}[C_k^*\psi_k e^{rk^+}]$$  \hspace{2cm} (36)

To clarify the meaning of equations (35), let

$$2C_k = b_k + id_k$$  \hspace{2cm} (37)

in which $b_k$ and $d_k$ are real-valued constants. On substituting this expression into equations (36) and making use of equation (9), one obtains

$$\{x(0)\} = b_k^*\phi_k - d_k^*x_k$$  \hspace{2cm} (38a)

$$\{\dot{x}(0)\} = b_k^*\dot{\phi}_k - d_k^*\dot{x}_k$$  \hspace{2cm} (38b)

in which

$$b_k^* = -\tilde{p}_k\left(\frac{\zeta_k}{\sqrt{1 - \zeta_k^2}} b_k + d_k\right)$$  \hspace{2cm} (39a)

and

$$d_k^* = \tilde{p}_k\left(b_k - \frac{\zeta_k}{\sqrt{1 - \zeta_k^2}} d_k\right)$$  \hspace{2cm} (39b)

It should now be clear that any initial displacement configuration which is a linear combination of $\{\phi_k\}$ and $\{x_k\}$, along with an initial velocity configuration defined by equations (38b) and (39), will excite only the $k$th mode of vibration of the system.

In particular, if $\{x(0)\} = b_k^*\phi_k$, the initial velocities needed to excite only the $k$th mode of vibration are determined from equations (38b) and (39) to be
\[
\{\dot{x}(0)\} = -b_k (\zeta_k p_k \{\phi_k\} + \ddot{p}_k \{x_k\})
\] (40)

Similarly, if \(\{\ddot{x}(0)\} = d_k \{x_k\}\), the corresponding initial displacements are determined from equations (38a) by first computing the values of \(b_k\) and \(d_k\) from the system of equations (39). The result is

\[
\{x(0)\} = \frac{d_k}{p_k} \left( \sqrt{1 - \zeta_k^2} \{\phi_k\} - \zeta_k \{x_k\} \right)
\] (41)

With the proportionality factors \(b_k\) and \(d_k\) specified, the values of \(2\zeta_k\) may be determined from equation (37), and the displacements of the system at any time may be determined from equation (36). Alternatively, the displacements may be expressed directly in terms of \(b_k\) and \(d_k\) as follows

\[
\{x\} = e^{-\zeta_k p_k t} \left[ \left( b_k \{\phi_k\} - d_k \{x_k\} \right) \cos \ddot{p}_k t - \left( d_k \{\phi_k\} + b_k \{x_k\} \right) \sin \ddot{p}_k t \right]
\] (42)

Free Vibration Due to Uniform Set of Initial Velocity Changes

Before proceeding to the analysis of the forced response of the system, it is desirable to reexamine the response of the system to a uniform set of initial velocity changes with no corresponding displacement changes, i.e., \(\{x(0)\} = \{0\}\) and \(\{\dot{x}(0)\} = \{1\} v_0\).

Let \(B_j\) be the value of \(C_j\) for a unitary set of initial velocity changes. This value is determined from equation (29) to be

\[
B_j = \frac{\{\psi_j\}^T[m]\{1\}}{2 r_j \{\psi_j\}^T[m]\{\psi_j\} + \{\psi_j\}^T[c]\{\psi_j\}}
\] (43)

The value of \(C_j\) for \(\{\dot{x}(0)\} = \{1\} v_0\) is then \(C_j = B_j v_0\), and the displacements of the system may be determined from the following expression deduced from equation (26)
\[ \{x\} = 2 \sum_{j=1}^{n} \text{Re} [B_j(\psi_j) v_0 e^{r_j t}] \]  

(44)

If the product \(2B_j(\psi_j)\) is now expressed in a form analogous to equation (22) as

\[ 2B_j(\psi_j) = \{\beta_j^V\} + i\{\gamma_j^V\} \]  

(45)

in which \(\{\beta_j^V\}\) and \(\{\gamma_j^V\}\) are real-valued vectors with dimensions of time per radian, equation (44) may also be written in the form

\[ \{x\} = \sum_{j=1}^{n} e^{-\xi_j p_j t} \left[ (\beta_j^V) \cos \tilde{p}_j t - (\gamma_j^V) \sin \tilde{p}_j t \right] v_0 \]  

(46)

The significance of the superscripts \(V\) in the last two expressions is identified later.

A simple but crucial final step will now be taken. Let \(h_j(t)\) be the impulse response function for a SDF system, defined as the response of the system to a unit initial velocity change with no corresponding displacement change. For a viscously damped system with damping factor \(\xi_j\) and undamped circular natural frequency \(p_j\), this function is given by

\[ h_j(t) = \frac{1}{\tilde{p}_j} e^{-\xi_j p_j t} \sin \tilde{p}_j t \]  

(47)

and its first derivative is given by

\[ \dot{h}_j(t) = e^{-\xi_j p_j t} \left( \cos \tilde{p}_j t - \frac{\xi_j}{\sqrt{1 - \xi_j^2}} \sin \tilde{p}_j t \right) \]  

(48)

from which, on making use of equation (10), one obtains

\[ e^{-\xi_j p_j t} \cos \tilde{p}_j t = \dot{h}_j(t) + \xi_j p_j h_j(t) \]  

(49)
The time functions multiplying the vectors \( \{ \beta_j^v \} \) and \( \{ \gamma_j^v \} \) in equation (46) are now replaced by the corresponding expressions defined by equations (47) and (49) to obtain

\[
\{ x \} = \sum_{j=1}^{n} \left[ \{ \alpha_j^v \} p_j h_j(t) + \{ \beta_j^v \} \dot{h}_j(t) \right] v_0
\]  

(50)

in which

\[
\{ \alpha_j^v \} = \zeta_j \{ \beta_j^v \} - \sqrt{1 - \zeta_j^2} \{ \gamma_j^v \}
\]  

(51)

Arrived at independently in the course of this study, this transformation has also been used recently by Igusa et al\(^{20}\).

Equation (50) is analogous to equation (47) but it differs from the latter in two respects: (a) instead of the sine and cosine functions, it is expressed in terms of the functions \( h_j(t) \) and \( \dot{h}_j(t) \) which have clear physical meanings; and (b) the reference configurations are the vectors \( \{ \alpha_j^v \} \) and \( \{ \beta_j^v \} \) instead of the vectors \( \{ \beta_j \} \) and \( \{ \gamma_j \} \). In a modal solution, the configuration \( \{ \alpha_j^v \} \) is attained at the instant for which \( h_j(t) \) is an extremum, i.e., \( \dot{h}_j(t) = 0 \), whereas the configuration \( \{ \beta_j^v \} \) is attained when \( h_j(t) \) is zero. Equation (50) is fundamental to the analysis of the transient response that follows.

**FORCED VIBRATION**

The response of the system to an arbitrary excitation of the base may be evaluated from the expressions for free vibration presented in the preceding section as follows. If \( \ddot{x}_g(\tau) \) is the acceleration of the base motion at time \( \tau = \tau \), the velocity change of the base in the short time interval between \( \tau \) and \( \tau + d\tau \) is given by \( \dot{x}_g(\tau)d\tau \), and the velocity change of each mass of the structure relative to
the moving base is given by \( v(\tau) = -\ddot{x}_g(\tau) d\tau \).

The differential displacements of the system, \( \{dx\} \), at \( t > \tau \) due to these velocity changes are then given by the following expression, obtained from equation (50) by replacing \( v_0 \) with \( v(\tau) \) and \( t \) with \( t - \tau \):

\[
\{dx\} = - \sum_{j=1}^{n} \left[ \{\alpha_j^V\} p_j h_j(t-\tau) + \{\beta_j^V\} \dot{h}_j(t-\tau) \right] \ddot{x}_g(\tau) d\tau \quad (52)
\]

The displacements of the system due to the prescribed base motion, are finally obtained by integration as

\[
\{x\} = \sum_{j=1}^{n} \left[ \{\alpha_j^V\} V_j(t) + \{\beta_j^V\} \dot{D}_j(t) \right] \quad (53)
\]

in which

\[
V_j(t) = p_j D_j(t) = - p_j \int_0^t \ddot{x}_g(\tau) h_j(t-\tau) d\tau \quad (54)
\]

and

\[
\dot{D}_j(t) = - \int_0^t \ddot{x}_g(\tau) \dot{h}_j(t-\tau) d\tau \quad (55)
\]

The quantity \( V_j(t) \) in equations (53) and (54) represents the instantaneous pseudovelocity of a SDF system with circular frequency \( p_j \) and damping factor \( \zeta_j \) subjected to the prescribed excitation; and \( D_j(t) \) and \( \dot{D}_j(t) \) represent the corresponding deformation and relative velocity of the system, respectively. It follows that the response of a nonclassically damped multi-degreeof-freedom system may be expressed as a linear combination of \( n \) pairs of terms. The first member of the \( j \)th such pair represents a motion in a configuration \( \{\alpha_j^V\} \) the temporal variation of which is the same as
that of $D_j(t)$, whereas the second member represents a motion in a
correlation $\{E_j^v\}$ the temporal variation of which is the same as
that of $\dot{D}_j(t)$. The configurations $\{\alpha_j^v\}$ and $\{E_j^v\}$ are naturally
functions of the natural modes of vibration of the system and are
defined by equations (45) and (51).

In a stepwise numerical evaluation of the response of a SDF
system, the relative velocity, $\dot{D}_j(t)$, is normally computed in the
process of obtaining $D_j(t)$ or the associated pseudovelocidity value,
$V_j(t)$. Provided the natural frequencies and modes of a nonclassically
damped multi-degree-of-freedom system have been evaluated, therefore,
the analysis of such a system may be implemented with only minor
computational effort beyond that required for a classically damped
system of the same size.

Alternative Forms of Expressions for Response

Instead of the pseudovelocidity and true relative velocity func-
tions, $V_j(t)$ and $\dot{D}_j(t)$, equation (53) may also be expressed in terms
of $D_j(t)$ and $\dot{D}_j(t)$ as follows:

Let $\{\alpha_j^D\}$ and $\{E_j^D\}$ be dimensionless vectors defined by

$$\{\alpha_j^D\} = p_j \{\alpha_j^v\}$$

(56a)

and

$$\{E_j^D\} = p_j \{E_j^v\}$$

(56b)

On making use of the relationship between $V_j(t)$ and $D_j(t)$ defined
by equation (54), equation (53) may then be rewritten as

$$\{x\} = \sum_{j=1}^{n} [\{\alpha_j^D\} D_j(t) + \{E_j^D\} \frac{\dot{D}_j(t)}{p_j}]$$

(57)
Similarly, on introducing the vectors

\[ \{ \alpha_j^A \} = \frac{1}{p_j} \{ \alpha_j^V \} = \frac{1}{p_j^2} \{ \alpha_j^D \} \]

(58a)

and

\[ \{ \beta_j^A \} = \frac{1}{p_j} \{ \beta_j^V \} = \frac{1}{p_j^2} \{ \beta_j^D \} \]

(58b)

and the pseudoacceleration function, \( A_j(t) \), defined by

\[ A_j(t) = p_j V_j(t) = p_j^2 D_j(t) \]

(59)
equation (53) can also be written as

\[ \{ x \} = \sum_{j=1}^{n} \left[ \{ \alpha_j^A \} A_j(t) + \{ \beta_j^A \} p_j D_j(t) \right] \]

(60)

**Effect of Nonzero Initial Conditions**

Implicit in the foregoing development has been the assumption that the system is initially at rest. For a system with nonzero initial conditions, equation (53), or its equivalent versions defined by equations (57) and (60), should be augmented by the addition of the free vibrational solution defined by equations (26), (27) or (28).

**Reduction for Classically Damped Systems**

For classically damped systems, for which the \( p_j = p_j^0 \) and the natural modes of vibration are real-valued and satisfy equations (15) and (16), equation (43) reduces to \( B_j = -ib_j^V/(2\sqrt{1-\zeta_j^2}) \), in which

\[ b_j^V = \frac{1}{p_j^0} \frac{\{ \phi_j \}_T[m]\{1\}}{\{ \phi_j \}_T[m]\{ \phi_j \}} \]

(61)

From equations (45) and (51) it then follows that \( \{ \beta_j^V \} = \{ 0 \} \) and that \( \{ \alpha_j^V \} = b_j^V \{ \phi_j \} \). Thus, equation (53) reduces to the well known
expression
\[
\{x\} = \sum_{j=1}^{n} b_j^V \phi_j V_j(t) \tag{62}
\]

The \(v\)-superscript on the symbol \(b_j\) emphasizes the fact that the latter quantity is to be used along with the pseudovelocity function, \(V_j(t)\).

Equation (62) can also be expressed in terms of the deformation function, \(D_j(t)\), as
\[
\{x\} = \sum_{j=1}^{n} b_j^D \phi_j D_j(t) \tag{63}
\]
or in terms of the pseudoacceleration function, \(A_j(t)\), as
\[
\{x\} = \sum_{j=1}^{n} b_j^A \phi_j A_j(t) \tag{64}
\]
in which \(b_j^D\) and \(b_j^A\) are participation factors defined by
\[
b_j^D = \frac{\phi_j^T [m] \{1\}}{\phi_j^T [m] \phi_j} \tag{65a}
\]
and
\[
b_j^A = \frac{1}{(\rho_j^2)^2} \frac{\phi_j^T [m] \{1\}}{\phi_j^T [m] \phi_j} \tag{65b}
\]

Summary of Procedure

The steps involved in the analysis of the transient response of a nonclassically damped system may be summarized as follows:

1. Evaluate the characteristic values, \(r_j\), and the associated characteristic vectors, \(\{\psi_j\}\); and from equations (5), (7) and (8), determine the damped and pseudo-undamped natural frequencies of the system, \(\bar{\omega}_j\) and \(\omega_j\), and the modal damping factors, \(\zeta_j\).
2. From equation (43), compute the participation factors, $B_j$. 

3. Evaluate the complex-valued products $2B_j \psi_j = \beta_j^v + i \beta_j^y$, and by application of equation (51), compute the vectors $\alpha_j^v$. Alternatively, one may compute the vectors $\alpha_j^p$ and $\beta_j^p$ from equations (56) or the vectors $\alpha_j^A$ and $\beta_j^A$ from equations (58). 

4. From analyses of the response of single-degree-of-freedom systems to the prescribed ground motion, determine the pseudovelocity functions, $V_j(t)$, and the true relative velocities, $\dot{D}_j(t)$. 

5. Compute the displacements $\{x\}$ from equation (53), and the corresponding story deformations from 

$$ u_i = x_i - x_{i-1} $$  \hspace{1cm} (66) 

in which the subscript $i$ refers to the $i$th floor level or story. The displacements may also be computed from equation (57) or from equation (60). 

Properties of Modal Response Vectors 

The vectors $\{\alpha_j\}$ and $\{\beta_j\}$ with the various superscripts in equations (53), (57) and (60) satisfy the following relations:

$$ \sum_{j=1}^{n} \beta_j^v = \{0\} $$ \hspace{1cm} (67a) 

$$ \sum_{j=1}^{n} [\alpha_j^p - 2 \zeta_j \beta_j^p] = \{1\} $$ \hspace{1cm} (67b) 

and 

$$ -x_g \sum_{j=1}^{n} \alpha_j^A = \{x_{st}\} $$ \hspace{1cm} (67c) 

in which $\{x_{st}\}$ represents the static displacements of the structure due to the inertia forces associated with a uniform structural acceler-
ation of magnitude $\ddot{x}_g$. These expressions are of great value in check-
ing the accuracy of the solution.

Equation (67a) is deduced from equation (44) on noting that, for the conditions considered, $\{x(0)\} = \{0\}$. Similarly, equation (67b) is obtained from the first derivative of equation (44) by making use of the fact that $\{\dot{x}(0)\} = \{1\}$. Finally, equation (67c) is obtained by examining the high-frequency limiting behavior of equation (60). For very stiff systems, the maximum values of $\{x\}$ tend to $\{x_{st}\}$; the corresponding values of $A_j(t)$ and $D_j(t)$ tend to $-\ddot{x}_g$ and zero, respectively; and equation (60) reduces to equation (67c).

Illustrative Example

The response of the system shown in Figure 1(a) is evaluated in this section for two different excitations of the base: (a) the half-cycle displacement pulse shown in Figure 5, for which the acceleration trace consists of a sequence of three half-sine waves of the same peak values and durations $t_1$, $2t_1$ and $t_1$, respectively; and (b) the first 6.3 seconds of the N-S component of the El Centro, California earthquake record of May 18, 1940, as reported in Reference 19. The peak values of the acceleration, velocity and displacement of the latter motion are $\ddot{x}_g = 0.312g$, $\dot{x}_g = 14.02$ in/sec and $x_g = 8.29$ in, respectively. The dimensionless damping coefficient in equation (17) is assigned the values of $\zeta_0 = 0.5$ and $\zeta_0 = 1$. The values of $r_j$ and $\{\psi_j\}$ for these systems are listed in Table I along with the associated values of $\rho_j$, $\bar{\rho}_j$ and $\tau_j$.

The complex-valued participation factors $B_j$ are determined from equation (43), and they are listed in Table III along with the products
$2B_j \{ \psi_j \} = \{ \beta_j^V \} + i \{ \gamma_j^V \}$ and the associated vectors $\{ \alpha_j^V \}$, $\{ \alpha_j^D \}$, $\{ \beta_j^D \}$, $\{ \alpha_j^A \}$, and $\{ \beta_j^A \}$. These results, which are independent of the characteristics of the base motion, can be shown to satisfy equations (67). The common multipliers for the various quantities are identified in the extreme right-hand column of the table.

With the information presented in Table III, the displacements of the system, may be determined from either of equations (53), (57) or (60), making use of the relevant SDF response functions, $D_j(t)$ and $D_j(t)$. The interfloor deformations may then be determined from equation (66).

The histories of story deformation for systems subjected to the half-cycle displacement pulse are shown by the solid lines in Figures 6 and 7. Two different values of the frequency parameter $f_1^O t_d$ are considered, in which $f_1^O = p_1^O/2\pi$ is the fundamental undamped natural frequency of the system in cycles per unit of time, and $t_d$ is the duration of the forcing function. Similar data are presented in Figure 8 for systems with $f_1^O = 1$ cps subjected to the El Centro earthquake record. In each case, the response of the system is displayed for a period $t_o$ that exceeds the duration of the excitation by the fundamental undamped natural period of the system, $T_1^O = 1/f_1^O$; i.e., $t_o = t_d + T_1^O$, or

$$\frac{t_o}{t_d} = 1 + \frac{1}{f_1^O t_d}$$

Also shown in dashed lines are the corresponding approximate solutions, expressed in terms of the natural modes of vibration of the associated undamped system.
It can be seen from Figures 6 through 8 that the differences between the approximate and exact solutions may be substantial for systems with low natural frequencies and high damping values. The differences may be particularly large for the deformations of the upper parts of the structure, for which the contributions of the higher modes of vibration are generally more important than for the lower parts of the structure.

These trends may better be seen in Figures 9 and 10 in which response spectra are presented for the absolute maximum floor displacements, $(x_i)_{\text{max}}$, and the corresponding story deformations, $(u_i)_{\text{max}}$. The results, which are for systems with $\zeta_0 = 1$, are presented in the form of pseudovelocity values, $p_1^0(x_i)_{\text{max}}$ and $p_1^0(u_i)_{\text{max}}$, and they are non-dimensionalized with respect to the maximum value of the base velocity, $\dot{x}_g$. It may be recalled that $p_1^0$ represents the fundamental circular natural frequency of the undamped system.

Application to Systems With Real-Valued Characteristic Values

The information presented so far is applicable to systems for which all characteristic values (or roots) and the associated characteristic vectors are complex-valued. The values of $\zeta_j$ in this case are less than unity, and each modal solution is given by the sum of two exponentially decaying harmonic functions. In general, there may be an even number of real-valued negative roots, each associated with a real-valued characteristic vector. The purpose of this section is to explain how these roots and vectors should be handled in a forced vibration analysis.

Let $r_j$ and $r_k$ be a pair of such roots, with $|r_k| > |r_j|$, and
Let \( \{\psi_j\} \) and \( \{\psi_k\} \) be the associated real-valued characteristic vectors. It is convenient to express this pair of roots in a form analogous to equation (9) as
\[
\begin{align*}
    r_j &= -\zeta_j p_j + \tilde{p}_j \\
    r_k &= -\zeta_j p_j - \tilde{p}_j
\end{align*}
\] (68a)
(68b)
in which
\[
\tilde{p}_j = p_j \sqrt{\zeta^2 - 1}
\] (69)
and \( \zeta_j \) and \( p_j \) are real, positive quantities that may be determined as follows.

On multiplying equations (68a) and (68b) and making use of equation (69), one obtains
\[
p_j = \sqrt{r_j r_k}
\] (70)
Similarly, on adding equations (68a) and (68b), one obtains
\[
\zeta_j = -\frac{1}{p_j} (r_j + r_k) = -\frac{r_j + r_k}{\sqrt{r_j r_k}}
\] (71)
from which it is clear that \( \zeta_j > 1 \). Finally, the following expression for \( \tilde{p}_j \) may be obtained by subtracting equation (68b) from equation (68a):
\[
\tilde{p}_j = \frac{r_j - r_k}{2}.
\] (72)
The modal solution corresponding to such a pair of real-valued characteristic roots is given by the sum of two exponentially decaying functions and hence, the resulting motion is non-oscillatory.

For a system in forced vibration, the functions \( D_j(t) \) and \( \dot{D}_j(t) \) in equation (57) represent the deformation and relative velocity...
due to the prescribed ground motion of a SDF system with the frequency and damping factor defined by equations (70) and (71), and the vector \( \{ \alpha_j^V \} \) in equation (53) is given by the following modified version of equation (51):

\[
\{ \alpha_j^V \} = \zeta_j \{ \beta_j^V \} - \sqrt{\zeta_j^2 - 1} \{ \gamma_j^V \}
\]  
(73)

The vectors \( \{ \beta_j^V \} \) and \( \{ \gamma_j^V \} \) in the latter expression and the vector \( \{ \beta_j^V \} \) in equation (53) are given by

\[
\{ \beta_j^V \} = B_k \{ \psi_k \} + B_j \{ \psi_j \}
\]  
(74)

\[
\{ \gamma_j^V \} = B_k \{ \psi_k \} - B_j \{ \psi_j \}
\]  
(75)

in which \( B_j \) and \( B_k \) are determined from equation (43) making use of the real-valued characteristic roots, \( r_j \) and \( r_k \), and the associated real-valued characteristic vectors, \( \{ \psi_j \} \) and \( \{ \psi_k \} \). The derivation of equations (73) to (75) is given in the last section of Appendix I.

**Analysis of Force-Excited Systems**

The method of analysis for the base-excited systems presented in the preceding sections can, with minor modifications, also be applied to systems excited by a set of lateral forces, \( \{ P(t) \} \). In the following development, these forces are considered to be of the form

\[
\{ P(t) \} = \{ P \} g(t)
\]  
(76)

in which \( \{ P \} \) is an arbitrary vector with units of force and \( g(t) \) is a dimensionless time function. Note that whereas the spatial distribution of the forces considered is arbitrary, their temporal
variation is the same. The system is presumed to be initially at rest.

Let \( \{v(\tau)\} \) be the velocity changes induced at time \( \tau \) by the forces \( \{P(\tau)\} \) acting in the infinitesimal time interval between \( \tau \) and \( \tau + d\tau \). These velocities may be determined from the impulse-momentum relationship as

\[
\{v(\tau)\} = [m]^{-1}\{P(\tau)\}d\tau = [m]^{-1}\{P\}g(\tau)d\tau
\] (77)

The complex-valued participation factor \( C_j \) in the expression for the resulting free vibration (equation (26)) may then be determined from equation (29) by setting \( \{\dot{x}(0)\} = \{v(\tau)\} \) and \( \{x(0)\} = \{0\} \). If one next introduces the quantities

\[
B_j^P = \frac{1}{P_j} \frac{\{\psi_j\}^T\{P\}}{2r_j\{\psi_j\}^T[m]\{\psi_j\} + \{\psi_j\}^T[c]\{\psi_j\}}
\] (78)

and

\[
2B_j^P(\psi_j) = \{B_j^P\} + i\{\gamma_j^P\}
\] (79)

and follows the steps taken previously in the development of the corresponding solution for base-excited systems, one obtains the following expression for the displacements:

\[
\{x\} = \sum_{j=1}^{n} \left[ \{\alpha_j^P\}A_j(t) + \{B_j^P\} \frac{\dot{A}_j(t)}{p_j} \right]
\] (80)

in which

\[
\{\alpha_j^P\} = \zeta_j\{\beta_j^P\} - \sqrt{1 - \zeta_j^2}\{\gamma_j^P\}
\] (81)

\[
A_j(t) = p_j^2 \int_0^t g(\tau) h(t - \tau) d\tau
\] (82)
and \( \dot{A}_j(t) \) is the time derivative of \( A_j(t) \).

The quantity \( B_j^p \) in equation (78) and the vectors \( \{a_j^p\} \), \( \{b_j^p\} \) and \( \{y_j^p\} \) in equations (79), (80) and (81) have units of displacement, whereas the function \( A_j(t) \) is dimensionless. The latter function represents the normalized displacement of a SDF system, the natural frequency and damping factor of which are the same as those of the \( j \)th mode of vibration of the prescribed multi-degree-of-freedom system, and is excited by a force of the same temporal variation as \( g(t) \). The normalizing factor is the static displacement of the system induced by the peak value of the applied force. Thus

\[
A_j(t) = \frac{X_j(t)}{x_{st}} \tag{83}
\]

in which \( X_j(t) \) is the displacement of the SDF system and \( x_{st} \) is its maximum static value.

The vectors \( \{a_j^p\} \) and \( \{b_j^p\} \) in equation (80) are the counterparts of the vectors \( \{a_j^A\} \) and \( \{b_j^A\} \) in equation (60), and the dimensionless amplification function \( A_j(t) \) is the counterpart of the normalized pseudoacceleration function, \( A_j(t)/\ddot{x}_g \).

**Application to Harmonic Response.** The steady-state response of nonclassically damped systems to a set of harmonic forces may generally be evaluated efficiently by direct solution of the governing equations of motion. However, the modal superposition method described in this paper may be preferable for systems having a large number of degrees of freedom, and its application is described briefly in this section.

For exciting forces of the form
\{P(t)\} = \{P\} \sin \omega t \quad (84)

in which \(\omega\) is the circular frequency of excitation, the amplification function \(A_j(t)\) and its first derivative are given by

\[A_j(t) = A_j \sin(\omega t - \theta_j)\]  \quad (85a)

and

\[
\dot{A}_j(t) = \omega A_j \cos(\omega t - \theta_j) \quad (85b)
\]

In the latter expressions,

\[A_j = \frac{1}{\sqrt{(1 - \rho_j^2)^2 + 4 \xi_j^2 \rho_j^2}} \quad (86a)\]

\[\rho_j = \frac{\omega}{\rho_j} \quad (86b)\]

and

\[
\theta_j = \tan^{-1}\left(\frac{2 \xi_j \rho_j}{1 - \rho_j^2}\right) \quad (86c)
\]

in which \(0 \leq \theta_j \leq \pi\).

Let \(x_i(t)\) be the displacement of the \(i\)th floor of the system and \(\alpha_{ij}^p\) and \(\beta_{ij}^p\) be the corresponding values of \(\{\alpha_j^p\}\) and \(\{\beta_j^p\}\) in equation (80). On substituting equations (85a), (85b) and (86c) into equation (80), one obtains

\[
x_i(t) = \sum_{j=1}^{n} A_j [\alpha_{ij}^p \sin(\omega t - \theta_j) + \rho_j \beta_{ij}^p \cos(\omega t - \theta_j)] \quad (87)
\]

Further, on expanding the sine and cosine functions and introducing the quantities

\[
\xi_{ij} = A_j \sqrt{(\alpha_{ij}^p)^2 + (\rho_j \beta_{ij}^p)^2} \quad (88a)
\]
\[ \epsilon_{ij} = \tan^{-1} \left( \frac{\alpha_{ij}^p}{\beta_{ij}^p \rho_j} \right) \]  

(88b)

equation (86) may be rewritten as

\[ x_i(t) = \sum_{j=1}^{n} \left\{ [\epsilon_{ij} \sin(\theta_j + \epsilon_i)] \sin \omega t + [\epsilon_{ij} \cos(\theta_j + \epsilon_i)] \cos \omega t \right\} \]  

(89)

in which \( \epsilon_{ij} \) is understood to lie in the range 0 to 2\( \pi \). The maximum value of \( x_i(t) \) may finally be determined from

\[ |(x_i)_{\text{max}}| = \sqrt{\left( \sum_{j=1}^{n} \epsilon_{ij} \sin(\theta_j + \epsilon_i) \right)^2 + \left( \sum_{j=1}^{n} \epsilon_{ij} \cos(\theta_j + \epsilon_i) \right)^2} \]  

(90)

**Illustrative Example.** The steady-state response of the system shown in Figure 1(a) is evaluated for a harmonic force applied to the first floor level considering \( \zeta_0 = 1 \).

The natural frequencies and modes of the system and the associated damping factors are given in part (d) of Table I, and those quantities in the expressions for the response that are independent of the temporal characteristics of the forcing function are given in the upper part of Table IV. The latter quantities include the participation factors defined by equation (78); the vectors \( \{ \beta_j^p \} \) and \( \{ \gamma_j^p \} \) in equation (79); and the vectors \( \{ \alpha_j^p \} \) in equation (81). The quantities that do depend on the temporal variation of the exciting force are given in the lower part of Table IV assuming that the value of the exciting frequency \( \omega = \sqrt{k/m} \). The relevant quantities are the frequency ratios, \( \rho_j \); the amplification factors and phase angles defined by equations (86a) and (86c); the displacement amplitudes and phase angles defined
by equations (88); and the values of \( \sin(\theta_j + \epsilon_{ij}) \) and \( \cos(\theta_j + \epsilon_{ij}) \).

The maximum displacements of the system, \( |(x_i)_{\text{max}}| \), are then determined from equation (90) to be

\[
|\xi_{1 \text{max}}| = 0.4471 x_{st} \quad |\xi_{2 \text{max}}| = 0.4473 x_{st} \quad |\xi_{3 \text{max}}| = 0.8948 x_{st}
\]

in which \( x_{st} = P/k \) is the static displacement of the first floor due to the peak value of the applied force.

The maximum interfloor deformations, \( |(u_i)_{\text{max}}| \), are determined from equation (90) by replacing the quantities \( \xi_{ij} \) by

\[
\bar{\xi}_{ij} = \xi_{ij} - \xi_{i-1,j}
\]

(91)

The results are

\[
|\xi_{1 \text{max}}| = 0.4471 x_{st} \quad |\xi_{2 \text{max}}| = 0.8945 x_{st} \quad |\xi_{3 \text{max}}| = 0.4473 x_{st}
\]

The maximum displacements and interfloor deformations of the system are displayed in Figure 11 over a wide range of exciting frequencies, where they are also compared with the values obtained by the approximate procedure involving the use of the undamped natural modes of vibration of the system. Note that the differences in the two sets of solutions are generally not insignificant.

CONCLUSION

With the information and the physical insight contributed in this chapter, the response of a non-classically damped linear system to an arbitrary excitation may be evaluated with only minor computational effort beyond that required for the analysis of a classically damped system of the same size. The response of the system has been expressed in terms of the deformations and true relative velocities.
of a series of similarly excited single-degree-of-freedom systems.

Comprehensive numerical solutions have been presented for the maximum response of a three-degree-of-freedom system over a range of excitation and system parameters, and the results compared with those obtained by an approximate solution involving the use of classical modes of vibration. It has been shown that, depending on the characteristics of the excitation and of the system itself, the approximate solution may be substantially in error.
APPENDIX I

Reduced Form of Equation of Motion

The system of second order differential equations (1) can be reduced\(^3,4\) to the following first order system:

\[
[A] \{\dot{z}\} + [B] \{z\} = \{Y(t)\} \tag{A1}
\]

in which \([A]\) and \([B]\) are matrices of size 2n by 2n given by

\[
[A] = \begin{bmatrix}
0 & m \\
m & c
\end{bmatrix} \quad [B] = \begin{bmatrix}
-[m] & 0 \\
0 & [k]
\end{bmatrix}
\tag{A2}
\]

and \(\{z\}\) and \(\{Y(t)\}\) are vectors of 2n elements given by

\[
\{z\} = \begin{Bmatrix}
\{\dot{x}\} \\
\{x\}
\end{Bmatrix} \quad \{Y(t)\} = \begin{Bmatrix}
\{0\} \\
-[m] \{\ddot{x}_g(t)\}
\end{Bmatrix}
\tag{A3}
\]

The solution of the homogeneous form of equation (A1) may be taken as

\[
\{z\} = \{Z\} e^{rt} \tag{A4}
\]

where \(r\) is a characteristic value and \(\{Z\}\) is the associated characteristic vector of 2n elements. The lower n elements of \(\{Z\}\) represent the desired modal displacements, \(\{\psi\}\), and the upper n elements represent the corresponding modal velocities, \(r(\psi)\); i.e.

\[
\{Z\} = \begin{Bmatrix}
\{r(\psi)\} \\
\{\psi\}
\end{Bmatrix}
\tag{A5}
\]

Substituting equation (A4) into the homogeneous form of equation (A1), one obtains the characteristic value problem defined by equation (4).
Form of Characteristic Values

Substitution of \( \{x\} = \{\psi_j\} e^{r_j t} \) into the homogeneous form of equation (1) leads to

\[
[m] \{\psi_j\} r_j^2 + [c] \{\psi_j\} r_j + [k] \{\psi_j\} = \{0\} \tag{A6}
\]

and premultiplication by the transpose of the complex conjugate of \( \{\psi_j\} \) leads to

\[
\{\bar{\psi}_j\}^T [m] \{\psi_j\} r_j^2 + \{\bar{\psi}_j\}^T [c] \{\psi_j\} r_j + \{\bar{\psi}_j\}^T [k] \{\psi_j\} = 0 \tag{A7}
\]

where the superscript \( T \) denotes a transposed vector. Each of the three matricial products represents a positive real number. On letting

\[
m_j^* = \{\bar{\psi}_j\}^T [m] \{\psi_j\} \tag{A8a}
\]

\[
c_j^* = \{\bar{\psi}_j\}^T [c] \{\psi_j\} \tag{A8b}
\]

\[
k_j^* = \{\bar{\psi}_j\}^T [k] \{\psi_j\} \tag{A8c}
\]

equation (A7) can be written as

\[
m_j^* r_j^2 + c_j^* r_j + k_j^* = 0 \tag{A9}
\]

which is recognized to be the characteristic or frequency equation for a single-degree-of-freedom system with mass \( m_j^* \) damping coefficient \( c_j^* \), and stiffness \( k_j^* \). Proceeding in the usual manner and letting

\[
p_j = \sqrt{k_j^*/m_j^*} \tag{A10}
\]

and

\[
2 \xi_j p_j = c_j^*/m_j^* \tag{A11}
\]

equation (A9) can be rewritten as
\[ r_j^2 + 2 \xi_j p_j r_j + p_j^2 = 0 \]  \hspace{1cm} (A12)

the roots of which are

\[ r_j = -\xi_j p_j + i \tilde{p}_j \]  \hspace{1cm} (A13a)

and

\[ \tilde{r}_j = -\xi_j p_j - i \tilde{p}_j \]  \hspace{1cm} (A13b)

where

\[ \tilde{p}_j = p_j \sqrt{1 - \xi_j^2} \]  \hspace{1cm} (A13c)

Similar derivations have been given previously. 5,9,17,21

Orthogonality of Modes

Since [A] and [B] in equation (4) are real symmetric matrices, the characteristics vectors \{Z_j\} and \{Z_k\} corresponding to any pair of distinct characteristic values \( r_j \) and \( r_k \) satisfy the orthogonality relations

\[ \{Z_j\}^T [A] \{Z_k\} = 0 \]  \hspace{1cm} (A14)

and

\[ \{Z_j\}^T [B] \{Z_k\} = 0 \]  \hspace{1cm} (A15)

These relations also hold true for a complex conjugate pair of vectors \{Z_j\} and \{\tilde{Z}_j\} since the associated characteristic values, \( r_j \) and \( \tilde{r}_j \), are different.

On making use of equations (A2) and (A5), equation (A14) reduces to equation (12), and equation (A15) reduces to equation (13).

Damping Matricial Product for Classically Damped Systems

For classically damped systems, \[ r_j = -\xi_j p_j^o + i \tilde{p}_j^o, \{\psi_j\} = \{\phi_j\} \] and
\[ [k] \{ \phi_j \} = (p_j^0)^2 [m] \{ \phi_j \} \]  

(A16)

On making use of these facts, equation (A6) reduces to

\[ [c] \{ \phi_j \} = 2 \xi_j p_j^0 [m] \{ \phi_j \} \]  

(A17)

Finally, on multiplying both sides of equation (A17) by the transpose of \( \{ \psi_k \} \) and making use of the symmetry of \([m]\) and \([c]\) and of the fact that \( \{ \psi_k \} = \{ \phi_k \} \), one obtains

\[ \{ \psi_j \}^T [c] \{ \psi_k \} = \{ \phi_j \}^T [c] \{ \phi_k \} = 2 \xi_j p_j^0 \{ \phi_j \}^T [m] \{ \phi_k \} \]  

(A18)

A similar relationship holds for an arbitrary real-valued vector \( \{ x(0) \} \); i.e.

\[ \{ \psi_j \}^T [c] \{ x(0) \} = 2 \xi_j p_j^0 \{ \phi_j \}^T [m] \{ x(0) \} \]  

(A19)

**Complex-Valued Participation Factors for Free Vibration**

The complete solution of the homogeneous form of equation (A1) is given by

\[ \{ z \} = \sum_{j=1}^{n} C_j \{ Z_j \} e^{\lambda_j t} + \sum_{j=1}^{n} \bar{C}_j \{ \bar{Z}_j \} e^{\bar{\lambda}_j t} \]  

(A20)

in which the participation factors, \( C_j \) and \( \bar{C}_j \), may be determined from the initial conditions of the problem as follows. Let

\[ \{ z(0) \} = \left\{ \frac{\{ \ddot{x}(0) \} \{ x(0) \} \} \right\} \]  

(A21)

be the vector of the velocities and displacements of the system at \( t = 0 \). Then

\[ \{ z(0) \} = \sum_{j=1}^{n} C_j \{ Z_j \} + \sum_{j=1}^{n} \bar{C}_j \{ \bar{Z}_j \} \]  

(A22)

On premultiplying both sides of this equation by \( \{ Z_k \}^T [A] \) and making use of the orthogonality condition defined by equation (A14), it can
be shown that all terms on the right-hand member of the resulting expression, except for the \( k = j \) term, vanish. This leads to

\[
C_j = \frac{\{Z_j\}^T[A]\{z(0)\}}{\{Z_j\}^T[A]\{Z_j\}} \quad (A23)
\]

which, on making use of equations (A2), (A5) and (A21) reduces to equation (29).

**Initial Conditions that Excite a Single Mode**

When expressed in terms of the vectors \( \{z\} \) and \( \{Z_k\} \), equations (35a) and (35b) may be written as

\[
\{z(0)\} = 2 \text{Re} [C_k \{Z_k\}] = C_k \{Z_k\} + C_k \{\bar{Z}_k\} \quad (A24)
\]

The participation factors \( C_j \) may then be determined from equation (A23) as

\[
C_j = \frac{\{Z_j\}^T[A](C_k \{Z_k\} + C_k \{\bar{Z}_k\})}{\{Z_j\}^T[A]\{Z_j\}} \quad (A25)
\]

Because of the orthogonality condition defined by equation (A14) this expression vanishes, except when \( j = k \), in which case \( C_j = C_k \).

**Systems with Real-Valued Roots**

The motion represented by a linear combination of two real-valued characteristic vectors, \( \{\psi_j\} \) and \( \{\psi_k\} \), and the associated characteristic values, \( r_j \) and \( r_k \), is given by

\[
\{x\} = C_j \{\psi_j\} e^{r_j t} + C_k \{\psi_k\} e^{r_k t} \quad (A26)
\]

in which the quantities \( C_j \) and \( C_k \) in this case are real-valued constants than can be evaluated from equation (29). On making use of
equations (68) and of the relationship between exponential and hyperbolic functions, equation (A26) may be rewritten as

\[ \{x\} = e^{-\zeta_j p_j t} \left[ \{\beta_j\} \cosh \bar{p}_j t - \{\gamma_j\} \sinh \bar{p}_j t \right] \tag{A27} \]

where

\[ \{\beta_j\} = C_k \{\psi_k\} + C_j \{\psi_j\} \tag{A28} \]

\[ \{\gamma_j\} = C_k \{\psi_k\} - C_j \{\psi_j\} \tag{A29} \]

For a system subjected to a set of unit initial velocity changes, \{\dot{x}(0)\} = \{1\}, the participation factors \(C_j\) and \(C_k\) must be replaced by \(B_j\) and \(B_k\), and the vectors \{\beta_j\} and \{\gamma_j\} must be replaced by \{\beta_j^\prime\} and \{\gamma_j^\prime\}, respectively. Equations (A28) and (A29) then reduce to equations (74) and (75). Further, on recalling that the impulse response function for an overdamped single-degree-of-freedom system is given by

\[ h_j(t) = \frac{1}{\bar{p}_j} e^{-\zeta_j p_j t} \sinh \bar{p}_j t \tag{A30} \]

it can be shown that equation (A26) leads to

\[ \{x\} = \{\alpha_j^\prime\} p_j h_j(t) + \{\beta_j^\prime\} h_j(t) \tag{A31} \]

in which \(\bar{p}_j\) is defined by equation (69) and \(\alpha_j^\prime\) is defined by equation (73).
APPENDIX II

Notation

$A_j(t)$  pseudoacceleration of a SDF system
$B_j$  participation factor for base-excited system
$B_j^p$  participation factor for force-excited system
$c$  damping coefficient
$[c]$  damping matrix of system
$C_j$  participation factor for system in free-vibration
$D_j(t)$  deformation of a base-excited SDF system
$\dot{D}_j(t)$  relative velocity of a base-excited SDF system
$h_j(t)$  impulse response function for a SDF system
$i$  $\sqrt{-1}$; when used as a subscript, it indicates level of floor or story
$j$  integer number indicating order of mode under consideration
$k$  stiffness coefficient; when used as a subscript, it indicates order of mode
$[k]$  stiffness matrix of system
$m$  mass coefficient
$[m]$  mass matrix of system
$n$  number of degrees of freedom in a system
$P_j, \tilde{P}_j$  pseudo-undamped and damped circular frequency of jth mode, respectively
$P_j^0$  $2\pi f_j^0$ = circular natural frequency of jth mode of associated undamped system
$r$  characteristic value
$SDF$  single-degree-of-freedom system
$\tau_j^0$  natural period of fundamental mode of associated undamped system
\( t \) \hspace{1cm} \text{time}
\( t_d \) \hspace{1cm} \text{duration of excitation}
\( t_0 \) \hspace{1cm} \( t_d + T_i^0 \)
\( \{u\} \) \hspace{1cm} \text{vector of interfloor deformations}
\( u_i \) \hspace{1cm} \text{interfloor deformation of } i^{th} \text{ story}
\( |u_i|_{\text{max}} \) \hspace{1cm} \text{absolute maximum value of } u_i
\( V_j(t) \) \hspace{1cm} \text{pseudovelocity of a SDF system}
\( \{x\} \) \hspace{1cm} \text{vector of displacements relative to moving base for a base-excited system, and of absolute displacements for a force-excited system}
\( \{x(0), \dot{x}(0)\} \) \hspace{1cm} \text{initial displacements and initial velocities of system, respectively}
\( x_i, x_i(t) \) \hspace{1cm} \text{displacement of } i^{th} \text{ floor or level}
\( |x_i|_{\text{max}} \) \hspace{1cm} \text{absolute maximum value of } x_i
\( \ddot{x}_g(t) \) \hspace{1cm} \text{acceleration of the moving base}
\( \ddot{x}_g \) \hspace{1cm} \text{peak value of } \ddot{x}_g(t)
\( \{Z\} \) \hspace{1cm} \text{characteristic vector of size } 2n
\( A_j(t) \) \hspace{1cm} \text{amplification function in a force-excited system}
\( \{\alpha_j^y\} \) \hspace{1cm} \text{modal configuration defined by equation (51)}
\( \{\alpha_j^p\} \) \hspace{1cm} \text{modal configuration defined by equation (81)}
\( \{\beta_j, \gamma_j\} \) \hspace{1cm} \text{modal configurations defined by equation (22)}
\( \{\beta_j^y, \gamma_j^y\} \) \hspace{1cm} \text{modal configurations defined by equation (45)}
\( \{\beta_j^p, \gamma_j^p\} \) \hspace{1cm} \text{modal configurations defined by equation (79)}
\( \zeta_j \) \hspace{1cm} \text{modal damping factor}
\( \zeta_0 \) \hspace{1cm} \text{dimensionless damping coefficient in equation (17)}
\( \{\phi_j, x_j^\prime\} \) \hspace{1cm} \text{real part and imaginary part of } j^{th} \text{ complex-valued natural mode, } \{\psi_j\}, \text{ respectively}
\( \{\psi\} \) \hspace{1cm} \text{characteristic vector or natural mode of system}
REFERENCES


12. NATS Project, 'Eigenystem subroutine package (EISPACK)', A control program for the eigensystem package, Subroutines F269 to F298 and F220 to F247 (1975).


Table I. Free vibrational characteristics of system shown in Figure 1(a)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Floor Level</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_j / \sqrt{k/m}$</td>
<td></td>
<td>(a) For $\zeta_o = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5176i</td>
<td>1.4142i</td>
<td>1.9318i</td>
</tr>
<tr>
<td>${\psi_j}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.7321</td>
<td>0</td>
<td>-1.7321</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>$p_j^0 / \sqrt{k/m}$</td>
<td></td>
<td>0.5176</td>
<td>1.4142</td>
<td>1.9318</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) For $\zeta_o = 0.1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_j / \sqrt{k/m}$</td>
<td></td>
<td>-0.0083 + 0.5178i</td>
<td>-0.0334 + 1.4138i</td>
<td>-0.0083 + 1.9311i</td>
</tr>
<tr>
<td>${\psi_j}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.7312 + 0.0431i</td>
<td>-0.0011 + 0.0470i</td>
<td>-1.7300 + 0.1611i</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.9987 + 0.0598i</td>
<td>-0.9956 + 0.0002i</td>
<td>1.9968 - 0.2233i</td>
</tr>
<tr>
<td>$p_j / \sqrt{k/m}$</td>
<td></td>
<td>0.5179</td>
<td>1.4142</td>
<td>1.9312</td>
</tr>
<tr>
<td>$\bar{p}_j / \sqrt{k/m}$</td>
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<td>0.5178</td>
<td>1.4138</td>
<td>1.9311</td>
</tr>
<tr>
<td>$\zeta_j$</td>
<td></td>
<td>0.0161</td>
<td>0.0236</td>
<td>0.0043</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c) For $\zeta_o = 0.5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_j / \sqrt{k/m}$</td>
<td></td>
<td>-0.0420 + 0.5207i</td>
<td>-0.1724 + 1.4022i</td>
<td>-0.0356 + 1.9159i</td>
</tr>
<tr>
<td>${\psi_j}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.7096 + 0.2166i</td>
<td>-0.0225 + 0.2175i</td>
<td>-1.6871 + 0.8217i</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.9681 + 0.3001i</td>
<td>-0.8963 + 0.0248i</td>
<td>1.9281 - 1.1419i</td>
</tr>
<tr>
<td>$p_j / \sqrt{k/m}$</td>
<td></td>
<td>0.5224</td>
<td>1.4127</td>
<td>1.9162</td>
</tr>
<tr>
<td>$\bar{p}_j / \sqrt{k/m}$</td>
<td></td>
<td>0.5207</td>
<td>1.4022</td>
<td>1.9159</td>
</tr>
<tr>
<td>$\zeta_j$</td>
<td></td>
<td>0.0804</td>
<td>0.1221</td>
<td>0.0186</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(d) For $\zeta_o = 1.0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_j / \sqrt{k/m}$</td>
<td></td>
<td>-0.0861 + 0.5313i</td>
<td>-0.3865 + 1.3425i</td>
<td>-0.0454 + 1.8870i</td>
</tr>
<tr>
<td>${\psi_j}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.6391 + 0.4398i</td>
<td>-0.0350 + 0.3531i</td>
<td>-1.6041 + 1.7155i</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.8680 + 0.6089i</td>
<td>-0.6623 + 0.1524i</td>
<td>1.7944 - 2.3987i</td>
</tr>
<tr>
<td>$p_j / \sqrt{k/m}$</td>
<td></td>
<td>0.5382</td>
<td>1.3922</td>
<td>1.8875</td>
</tr>
<tr>
<td>$\bar{p}_j / \sqrt{k/m}$</td>
<td></td>
<td>0.5313</td>
<td>1.3425</td>
<td>1.8870</td>
</tr>
<tr>
<td>$\zeta_j$</td>
<td></td>
<td>0.1599</td>
<td>0.2647</td>
<td>0.0241</td>
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Table II. Comparison of exact and approximate natural frequencies and damping factors for system considered in Figure 1(a)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>Approx.</td>
<td>Exact</td>
</tr>
<tr>
<td>( p_j/\sqrt{k/m} )</td>
<td>0.5179</td>
<td>0.5176</td>
<td>1.4142</td>
</tr>
<tr>
<td>( \bar{p}_j/\sqrt{k/m} )</td>
<td>0.5178</td>
<td>0.5175</td>
<td>1.4138</td>
</tr>
<tr>
<td>( \zeta_j )</td>
<td>0.0161</td>
<td>0.0161</td>
<td>0.0236</td>
</tr>
<tr>
<td>( p_j/\sqrt{k/m} )</td>
<td>0.5224</td>
<td>0.5176</td>
<td>1.4127</td>
</tr>
<tr>
<td>( \bar{p}_j/\sqrt{k/m} )</td>
<td>0.5207</td>
<td>0.5159</td>
<td>1.4022</td>
</tr>
<tr>
<td>( \zeta_j )</td>
<td>0.0804</td>
<td>0.0805</td>
<td>0.1221</td>
</tr>
<tr>
<td>( p_j/\sqrt{k/m} )</td>
<td>0.5382</td>
<td>0.5176</td>
<td>1.3922</td>
</tr>
<tr>
<td>( \bar{p}_j/\sqrt{k/m} )</td>
<td>0.5313</td>
<td>0.5108</td>
<td>1.3425</td>
</tr>
<tr>
<td>( \zeta_j )</td>
<td>0.1599</td>
<td>0.1610</td>
<td>0.2647</td>
</tr>
</tbody>
</table>

(a) \( \zeta_0 = 0.1 \)

(b) \( \zeta_0 = 0.5 \)

(c) \( \zeta_0 = 1.0 \)
Table III. Response of system considered in Figure 1(a) due to a base excitation

<table>
<thead>
<tr>
<th>Floor Level</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
<th>Common Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(a) For $\zeta_0 = 0.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Values of $B_j$</td>
<td>Values of $2B_j{\psi_j^Y} = {\beta_j^Y} + i{\gamma_j^Y}$</td>
<td>$\sqrt{m/k}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.1585 - 1.2163i</td>
<td>0.1295 - 0.2737i</td>
<td>0.0290 + 0.0002i</td>
<td>$\sqrt{m/k}$</td>
</tr>
<tr>
<td>2</td>
<td>-0.0075 - 2.1138i</td>
<td>0.0566 + 0.0343i</td>
<td>-0.0491 + 0.0234i</td>
<td>$\sqrt{m/k}$</td>
</tr>
<tr>
<td>3</td>
<td>0.0531 - 2.4415i</td>
<td>-0.1093 + 0.2485i</td>
<td>0.0562 - 0.0326i</td>
<td>$\sqrt{m/k}$</td>
</tr>
<tr>
<td></td>
<td>Values of ${\alpha_j^Y}$ and ${\beta_j^Y}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.1997 - 0.1585</td>
<td>0.2875</td>
<td>0.1295</td>
<td>0.0003</td>
</tr>
<tr>
<td>2</td>
<td>2.1063 - 0.0075</td>
<td>-0.0272</td>
<td>0.0566</td>
<td>-0.0243</td>
</tr>
<tr>
<td>3</td>
<td>2.4378</td>
<td>0.0531</td>
<td>-0.2600</td>
<td>-0.1093</td>
</tr>
<tr>
<td></td>
<td>Values of ${\alpha_j^D}$ and ${\beta_j^D}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.6267 - 0.0828</td>
<td>0.4061</td>
<td>0.1829</td>
<td>0.0006</td>
</tr>
<tr>
<td>2</td>
<td>1.1004</td>
<td>-0.0039</td>
<td>-0.0384</td>
<td>0.0800</td>
</tr>
<tr>
<td>3</td>
<td>1.2736</td>
<td>0.0277</td>
<td>-0.3673</td>
<td>-0.1544</td>
</tr>
<tr>
<td></td>
<td>Values of ${\alpha_j^A}$ and ${\beta_j^A}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.2963 - 0.3034</td>
<td>0.2035</td>
<td>0.0917</td>
<td>0.0002</td>
</tr>
<tr>
<td>2</td>
<td>4.0319</td>
<td>-0.0143</td>
<td>-0.0192</td>
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</tr>
<tr>
<td>3</td>
<td>4.6664</td>
<td>0.1016</td>
<td>-0.1840</td>
<td>-0.0774</td>
</tr>
<tr>
<td></td>
<td>(b) For $\zeta_0 = 1.0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Values of $B_j$</td>
<td>Values of $2B_j{\psi_j^Y} = {\beta_j^Y} + i{\gamma_j^Y}$</td>
<td>$\sqrt{m/k}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.1822 - 0.6290i</td>
<td>0.1751 - 0.1796i</td>
<td>0.0071 + 0.0138i</td>
<td>$\sqrt{m/k}$</td>
</tr>
<tr>
<td>2</td>
<td>0.3645 - 1.2580i</td>
<td>0.3502 - 0.3592i</td>
<td>0.0143 + 0.0277i</td>
<td>$\sqrt{m/k}$</td>
</tr>
<tr>
<td>3</td>
<td>0.0851 - 2.5718i</td>
<td>-0.1772 + 0.2913i</td>
<td>0.0921 + 0.0154i</td>
<td>$\sqrt{m/k}$</td>
</tr>
<tr>
<td></td>
<td>Values of ${\alpha_j^Y}$ and ${\beta_j^Y}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.1835</td>
<td>-0.3645</td>
<td>0.4391</td>
<td>0.3502</td>
</tr>
<tr>
<td>2</td>
<td>2.1866</td>
<td>-0.0441</td>
<td>-0.1010</td>
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<td>3</td>
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<td>-0.1772</td>
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<td>Values of ${\alpha_j^D}$ and ${\beta_j^D}$</td>
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<td></td>
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</tr>
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<td>-0.1962</td>
<td>0.6113</td>
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<tr>
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<td>1.1768</td>
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<td>-0.1407</td>
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</tr>
<tr>
<td>3</td>
<td>1.3736</td>
<td>0.0458</td>
<td>-0.4564</td>
<td>-0.2467</td>
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<td></td>
<td>Values of ${\alpha_j^A}$ and ${\beta_j^A}$</td>
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<td></td>
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<tr>
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<td>-0.0726</td>
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<tr>
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<td>4.7425</td>
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<td>-0.1273</td>
</tr>
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</table>
Table IV. Harmonic response of force-excited systems considered in example.

<table>
<thead>
<tr>
<th>Floor Level</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
<th>Factor or Units</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B^p_j$</td>
<td></td>
<td></td>
<td>$x_{st}$</td>
</tr>
<tr>
<td></td>
<td>$-0.1561 - 0.2946i$</td>
<td>$0.0415 - 0.2317i$</td>
<td>$0.0139 + 0.0006i$</td>
<td></td>
</tr>
<tr>
<td>Values of $2B^p_j {\psi^p_j} = {\beta^p_j} + i{\gamma^p_j}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$-0.3123 - 0.5891i$</td>
<td>$0.0830 - 0.4634i$</td>
<td>$0.0278 + 0.0011i$</td>
<td>$x_{st}$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.2527 - 1.030i$</td>
<td>$0.1607 + 0.0456i$</td>
<td>$-0.0465 + 0.0459i$</td>
<td>$x_{st}$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.2246 - 1.2907i$</td>
<td>$0.0156 + 0.3196i$</td>
<td>$0.0527 - 0.0647i$</td>
<td>$x_{st}$</td>
</tr>
<tr>
<td>Values of ${\alpha^p_j}$ and ${\beta^p_j}$</td>
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<td></td>
<td></td>
<td></td>
</tr>
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</tr>
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<tr>
<td>Values of $\theta_j$</td>
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<td>$0.6659$</td>
<td>$0.0355$</td>
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</tr>
<tr>
<td>Values of $\varepsilon_{ij}$</td>
<td>$1.4443$</td>
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<td>$2.3998$</td>
<td>$rad$</td>
</tr>
<tr>
<td>1</td>
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<td>$6.2493$</td>
<td>$2.3998$</td>
<td>$rad$</td>
</tr>
<tr>
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<td>$1.9919$</td>
<td>$6.2711$</td>
<td>$4.2296$</td>
<td>$rad$</td>
</tr>
<tr>
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<td>$1.8959$</td>
<td>$4.7492$</td>
<td>$1.1695$</td>
<td>$rad$</td>
</tr>
<tr>
<td>Values of $\xi_{ij}$</td>
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</tr>
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<td>$0.0205$</td>
<td>$x_{st}$</td>
</tr>
<tr>
<td>2</td>
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<td>$0.0737$</td>
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<tr>
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<td>$0.0993$</td>
<td>$x_{st}$</td>
</tr>
<tr>
<td>Values of $\sin(\theta_j + \varepsilon_{ij})$ and $\cos(\theta_j + \varepsilon_{ij})$</td>
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<td></td>
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<td></td>
</tr>
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</tr>
<tr>
<td>3</td>
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<td>$0.0873$</td>
<td>$-0.7631$</td>
<td>$0.6463$</td>
</tr>
</tbody>
</table>
Figure 2. Effect of damping on natural modes of system considered in Figure 1(a).
Figure 3. Effect of damping on natural frequencies of systems considered in Figure 1.

(a) Damper in Bottom Story
(b) Damper in Top Story

Values of $\tilde{\rho}_j/\sqrt{k/m}$

Normalized Frequencies
Figure 4. Effect of damping on modal damping factors for systems considered in Figure 1.
Figure 5. Simple base motion considered
Figure 6. Interfloor deformations for system shown in Figure 1(a); system with $f_{td} = 1$ subjected to simple base motion.
Figure 7. Interfloor deformations for system shown in Figure 1(a); system with $f_1^0 t_d = 2$ subjected to simple base motion.
Figure 8. Interfloor deformations for system shown in Figure 1(a); system with $f_1^0 = 1$ cps subjected to El Centro earthquake record.
Figure 9. Response spectra for maximum floor displacements and story deformations for system shown in Figure 1(a); system with $\zeta_0 = 1$ subjected to simple base motion.
Figure 10. Response spectra for maximum floor displacements and story deformations for system shown in Figure 1(a); system with $\zeta_0 = 1$ subjected to El Centro earthquake record.
Figure 11. Response spectra for maximum floor displacements and story deformations for system shown in Figure 1(a): system with $\zeta_0 = 1$ subjected to a harmonic force at the first floor level.
CHAPTER III

RESPONSE SPECTRUM METHOD FOR EVALUATING THE DYNAMIC RESPONSE OF
NONCLASSICALLY DAMPED LINEAR SYSTEMS
INTRODUCTION

This chapter deals with the response spectrum version of the modal superposition method for evaluating the maximum response of nonclassically damped linear systems. The modal maxima in this approach are first determined from the response spectrum or spectra applicable to the particular excitation and damping under consideration, and the maximum response of the system is then approximated by an appropriate combination of the modal maxima.

When system damping is of the classical type and the desired responses are the deformations of the system, the response spectra needed are those for maximum deformation. For nonclassically damped systems, however, it is also necessary to have the associated spectra for maximum relative velocity, and this complicates the implementation of the method.

In previous applications of the method to nonclassically damped systems\textsuperscript{1,2}, the spectral values of relative velocity were approximated by the corresponding pseudovelocity values. While it may lead to results of reasonable accuracy under certain conditions, this approximation is clearly not generally valid, and there is a need to reexamine the issues involved in the application of the method to nonclassically damped systems.

The objective of this chapter is to present information and concepts with which the response spectrum version of the modal superposition method may be applied to the analysis of nonclassically damped systems with essentially the same accuracy and ease as it is possible to do for classically damped systems.
After reviewing briefly the method, the interrelationship of the spectral values of deformation and relative velocity of single-degree-of-freedom systems is identified, and simple practical rules are presented for defining the design spectra for relative velocity in terms of certain cardinal characteristics of the ground motion. The approach is essentially a generalization of available techniques for approximating the corresponding spectra for maximum deformation.

A basic step in the application of the modal superposition method to nonclassically damped systems is the combination, for each mode of vibration, of the terms that are proportional to the deformation and the relative velocity of that mode. It is proposed that the peak value of the combination be determined by application of the square root of the sum of the squares rule, and the accuracy of this proposal is evaluated through comprehensive parametric studies of the response of a three-degree-of-freedom system.

**STATEMENT OF PROBLEM AND BACKGROUND**

The system considered is a viscously damped, linear cantilever structure with n degrees of freedom excited at the base. Using the response spectrum version of the modal superposition method, it is desired to evaluate the maximum response of the system without imposing any restriction on the form of its damping matrix.

The displacements of the system relative to the moving base are given by equation (53) of Chapter II, which is reproduced below:

\[
\{x(t)\} = \sum_{j=1}^{n} \left[ (a_j^v V_j(t) + (\beta_j^v) \dot{D}_j(t)) \right]
\]

(1)
The quantity $D_j(t)$ in this expression represents the instantaneous value of the deformation of a single-degree-of-freedom (SDF) oscillator that has the natural frequency and damping of the $j^{th}$ mode of vibration of the actual system and is subjected to the same base motion as the prescribed motion; $\dot{D}_j(t)$ is the relative velocity of the oscillator; and $V_j(t)$ is its pseudovelocity, which is related to $D_j(t)$ by

$$V_j(t) = p_j D_j(t) \quad (2)$$

For the meaning of the remaining quantities reference should be made to Chapter II.

In the response spectrum approach, $V_j(t)$ and $\dot{D}_j(t)$ are replaced by their maximum or spectral values, and the maximum values of the elements of $\{x(t)\}$ are approximated by an appropriate combination of their corresponding modal maxima. The spectral values of $D_j(t)$, $V_j(t)$ and $\dot{D}_j(t)$ will be denoted by $D_j$, $V_j$ and $\dot{D}_j$, respectively. For classically damped systems, $\{\beta_j^V\} = \{0\}$ and only $D_j$ is needed to evaluate the maximum contribution of the $j^{th}$ mode. By contrast, for a nonclassically damped system, it is necessary to know both $D_j$ and $\dot{D}_j$.

In a time-domain analysis of the system, in which the responses of the SDF oscillators are evaluated by numerical integration, the values of $\dot{D}_j(t)$ are computed in the process of evaluating $D_j(t)$, and there is no special difficulty in determining the maximum value of equation (1) once the vectors $\{\alpha_j^V\}$ and $\{\beta_j^V\}$ have been computed. However, if the excitation is specified indirectly in terms of response spectra, as is often the case in design applications, it is necessary to specify the spectra for both maximum deformation and maximum relative velocity.
In previous studies of the response of structures to earthquakes\cite{1,2}, \( \dot{D}_j \) has been considered to equal to \( V_j \). This approximation which obviates the need of specifying the relative velocity spectra, is justified by the studies reported in References 3 and 4, which have shown that, for systems with moderate natural frequencies and small damping, \( \dot{D}_j \) and \( V_j \) may be used interchangeably. However, as indicated in the following section, this approximation is valid only under restrictive conditions and should be used with care.

PSEUDO VELOCITY AND TRUE RELATIVE VELOCITY

Comparison of Response Spectra

In Figure 1 are shown response spectra for \( V \) and \( \dot{D} \) for undamped SDF systems subjected to the base motion shown in Figure 2. The acceleration trace of the motion is represented by a sequence of three half-sine waves of the same peak values and durations \( t_1 \), \( 2t_1 \) and \( t_1 \). Included in the figure are the velocity and displacement histories of the ground motion, as well as that of the derivative of the acceleration, the so-called jerk. Similar spectra are presented in Figure 3 for systems with 2 percent of critical damping, \( \zeta = 0.02 \), subjected to the N-S component of the El Centro, California earthquake record of May 18, 1940. For simplicity, the subscript \( j \) has been deleted from the symbols \( V_j \) and \( \dot{D}_j \) in these plots.

Figures 1 and 3 clearly show that, within the middle-frequency region of the spectra for which \( V \) is nearly constant, \( \dot{D} \) and \( V \) are approximately the same. By contrast, the two quantities are significantly different outside this region. Specifically, \( \dot{D} \) is much greater
than \( V \) in the low-frequency spectral region, and in the high-frequency region it may be substantially less than \( V \), the difference increasing with increasing frequency. As the natural frequency of the system tends to zero, \( D \) tends to the maximum ground displacement, \( x_g \), and hence \( V \) tends to zero. By contrast, \( \dot{D} \) tends to the maximum ground velocity, \( \dot{x}_g \).

The width of the middle frequency region of the response spectrum depends on the frequency content of the velocity trace of the ground motion. The broader the band of frequencies involved, the greater is the width of this region, and hence the broader is the range of natural frequencies for which \( \dot{D} \) and \( V \) are the same.

**Interrelationship of Pseudovelocity and Relative Velocity**

The interrelationship between \( V(t) \) and \( \dot{D}(t) \) may be determined from Duhamel's integral,

\[
D(t) = - \int_{0}^{t} \ddot{x}_g(\tau) h(t-\tau) \, d\tau \tag{3}
\]

in which \( \ddot{x}_g(\tau) \) is the ground acceleration at time \( \tau \); \( h(t) \) is the impulse response function of the SDF oscillator under consideration, given by

\[
h(t) = \frac{1}{p} e^{-\xi \phi t} \sin \varphi t \tag{4}
\]

\( \xi \) is the damping factor of the oscillator; and \( p \) and \( \varphi \) are its undamped and damped circular natural frequencies, respectively. Differentiation of equation (3) with respect to \( t \) yields

\[
\dot{D}(t) = - \int_{0}^{t} \ddot{x}_g(\tau) \dot{h}(t-\tau) \, d\tau \tag{5a}
\]

and on integrating the right hand member by parts, one obtains
\[
\dot{D}(t) = - \int_{0}^{t} \dddot{x}_g(\tau) h(t-\tau) \, d\tau - \dddot{x}_g(0) h(t)
\]  
(5b)

in which \(\dddot{x}_g(0)\) is the initial or starting value of the ground acceleration. This result also follows from the derivative theorem of the convolution integral (see page 118 of Reference 5).

For \(\dddot{x}_g(0) = 0\), a condition that should be satisfied for any physically realizable ground motion, equation (5b) reduces to

\[
\dot{D}(t) = - \int_{0}^{t} \dddot{x}_g(\tau) h(t-\tau) \, d\tau
\]  
(6)

Note that, except for the fact that \(\dddot{x}_g(t)\) appears in lieu of \(\dddot{x}_g(t)\), equation (6) for \(\dot{D}(t)\) is identical to equation (3) for \(D(t)\). It follows that the relative velocity, \(\dot{D}(t)\), induced in a SDF system by a ground motion of a specified acceleration trace is the same as the deformation induced in the same system by a substitute motion the acceleration trace of which is equal to the jerk trace of the actual motion. The peak values of the two traces are considered to be the same in this analogy. More specifically, let the subscripts 1 and 2 denote two different ground motions such that the jerk trace of the first and the acceleration trace of the second have the same shape, i.e.,

\[
\frac{\dddot{x}_{g1}(t)}{\dddot{x}_{g1}} = \frac{\dddot{x}_{g2}(t)}{\dddot{x}_{g2}}
\]  
(7)

in which \(\dddot{x}_{g1}\) and \(\dddot{x}_{g2}\) are the peak values of the respective traces.

If \(D_1(t)\) and \(D_2(t)\) are the deformations of the system due to motions 1 and 2, respectively, then

\[
\frac{\dot{D}_1(t)}{\dot{x}_{g1}} = \frac{D_2(t)}{x_{g2}}
\]  
(8a)
\[
\frac{pD_1(t)}{\ddot{x}_{g1}} = \frac{V_2(t)}{\ddot{x}_{g2}} \quad (8b)
\]

and
\[
\frac{p^2D_1(t)}{\dddot{x}_{g1}} = \frac{A_2(t)}{\dddot{x}_{g2}} \quad (8c)
\]

The terms in the denominators of these expressions represent the peak values of the relevant traces of the ground motions, and in the latter expression \( A_2(t) = pV_2(t) = p^2D_2(t) \) is the pseudoacceleration of the system due to the second motion. It follows that the maximum values of the two sets of normalized quantities will also be the same, and that the normalized \( \dddot{D} \)-spectrum for the first motion will be identical to the normalized \( \dot{V} \)-spectrum for the second motion.

**Tripartite Plot of Deformation Spectra**

The deformation spectra for ground excited systems can most conveniently be displayed on a tripartite logarithmic plot with the abscissa representing the natural frequency of the system and the ordinate representing the pseudovelocity, \( V \). On such a plot diagonal lines extending upward from left to right represent lines of constant maximum deformation, \( D \), and diagonal lines extending downward from left to right represent lines of constant pseudoacceleration, \( A = pV = p^2D \).

When plotted in this form, the salient features of the deformation spectrum can be defined by reference to the trapezoid shown by the dashed lines in Figure 4(a), in which the diagonal line on the left corresponds to a deformation value equal to the maximum ground displacement, \( x_g \), the horizontal line corresponds to a pseudovelocity
value equal to the maximum ground velocity, \( \dot{x}_g \), and the diagonal line on the right corresponds to a pseudoacceleration value equal to the maximum ground acceleration, \( \ddot{x}_g \). The response spectrum then approaches the limiting value \( D = x_g \) on the left, the limiting value \( A = \ddot{x}_g \) on the right; it is nearly horizontal in the central region; and it exhibits a hump on either side of the central region.

The detailed characteristics of the spectrum for the right inclined portion depends on the characteristics of the acceleration trace of the ground motion; those for the middle region depend on the characteristics of the associated velocity trace; and those for the left inclined portion depend on the characteristics of the displacement trace.

Let \( C_D \) be a dimensionless amplification factor such that \( C_D x_g \) represents the absolute maximum value of \( D \), as shown in Figure 4(a), and let \( C_V \) and \( C_A \) be the corresponding factors for the peak values of \( V \) and \( A \), respectively. In addition to the magnitude of system damping, these factors depend on the degree of periodicity of the controlling trace of the ground motion. Specifically, \( C_A \) depends on the degree of periodicity of the acceleration trace of the ground motion, and \( C_V \) and \( C_D \) depend on the degrees of periodicity of the corresponding velocity and displacement traces, respectively. The term 'degree of periodicity' is defined approximately as the number of halfwaves of nearly equal amplitude and duration that characterize the dominant or intense portion of the controlling trace of the ground motion. For lightly damped systems (with damping coefficients of less than 1 percent of the critical value) the value of \( C_A \) is approximately equal to 1.5
times the number of nearly identical waves in the acceleration trace of the ground motion, and the values of \( C_V \) and \( C_D \) are defined by the same rule provided it is applied to the corresponding velocity and displacement traces, respectively. Since the degree of periodicity for a earthquake ground record typically decreases with integration (i.e. as one moves from the acceleration trace to the velocity and displacement traces), the value of \( C_A \) is generally the largest of the three and \( C_D \) is the smallest.

The peak value of \( A \) depends on the duration of the dominant pulses in the acceleration trace of the ground motion, and the frequency beyond which \( A \) may be considered to be approximately equal to \( \ddot{x}_g \) depends on the rise time of these dominant pulses. Similarly, the boundaries of the transition region between the peak value of \( D \) and its limiting value of \( D = x_g \) on the left depend on the corresponding quantities of the displacement trace. The detailed relationships are available in Reference 6, but are not needed in the discussion that follows.

In design applications for earthquake ground motions, the pseudovelocity spectrum is typically represented by a piecewise linear diagram, and the values of \( C_D \), \( C_V \) and \( C_A \), and the boundaries of the various transition segments in the low and high frequency regions of the spectrum are defined approximately based on statistical studies of the results obtained for a reasonably large number of earthquake motions of the same family\(^3\),\(^7\). For lightly damped systems with damping values of one percent of the critical value or less, the magnitudes of \( C_D \), \( C_V \) and \( C_A \) are of the order of 2.5, 3.5 and 4.5, respectively, and the boundaries of the transition segment between the peak value
of A and its limiting value of $A = \dddot{x}_g$ are about 8 cps and about 25 cps, respectively.

A word of caution is in order at this stage. The information summarized above assumes that the acceleration and velocity traces of the ground motion are continuous diagrams. The response of systems to base motions with discontinuous acceleration and velocity traces may be significantly larger than those intimated above, and the rules presented should not be used without the appropriate modifications. This matter is considered further later.

**Tripartite Plots for Relative Velocity Spectra**

The relationship between $\dot{D}$ and $V$ referred to earlier suggests that it would also be desirable to display the response spectra for $\dot{D}$ on a tripartite logarithmic plot, as shown in part (b) of Figure 4. The vertical scale in this case must represent the product $p\dot{D}$, and the diagonal scales on the left and the right must represent the values of $\dot{D}$ and $p^2\dot{D}$, respectively. Similarly, the diagonal dashed line on the left must represent the peak value of the ground velocity, $\dot{x}_g$; the horizontal dashed line must represent the peak value of the ground acceleration, $\dddot{x}_g$; and the diagonal dashed line on the right must represent the maximum value of the jerk, $\dddot{x}_g$.

The characteristics of the $\dot{D}$ spectra displayed in this form may then be defined by the same rules as those used to interrelate the characteristics of the $V$ spectra to those of the ground motion. However, instead of the characteristics of the displacement, velocity and acceleration traces of the ground motion, one must now work with those of the velocity, acceleration and jerk. Specifically, the
characteristics of the left inclined portion of the $\ddot{D}$ spectrum will depend on the characteristics of the ground velocity; those of the central region will depend on the characteristics of the ground acceleration; and those of the right inclined portion will depend on the characteristics of the ground jerk. As before, the relevant characteristics for a given trace are the degree of periodicity of its dominant pulses, their durations and rise times.

Let $C_{D}$, $C_{pD}$ and $C_{p2D}$ be the amplification factors for the peak values of $\ddot{D}$, $pD$ and $p^2D$, as shown in Figure 4(b). Considering that the ground velocity diagram for the relative velocity spectrum in Figure 2 is the same as the acceleration trace for the pseudovelocity spectrum in Figure 4(a) it follows that

$$C_{\ddot{D}} = C_{V} \quad (9)$$

and

$$C_{p\ddot{D}} = C_{A} \quad (10)$$

Further, considering that the jerk diagram of the ground motion is likely to be more nearly periodic and to be associated with higher frequency oscillations than the associated velocity diagram, the amplification factor $C_{p2\ddot{D}}$ can be expected to be larger than $C_{p\ddot{D}}$, and the boundaries of the transition segment of the spectrum for the right inclined portion can be expected to correspond to higher frequency values than those of the corresponding segment in the $V$ spectrum. Finally, provided that the jerk diagram of the ground motion has no discontinuities, the $\ddot{D}$ spectrum will approach the limiting value of $p^2D = \ddot{x}_g$ at high frequencies and the limiting value of $\ddot{D} = \ddot{x}_g$ at lower frequencies.
With one major exception identified below, these predictions are confirmed by the deformation and relative velocity spectra presented in Figures 5 and 6. These data refer to systems subjected to the two ground motions referred to earlier. A very wide range of damping values is considered.

The exception refers to the right-hand inclined portions of the \( D \)-spectra and their high frequency limits. These differences are due to the fact that whereas the approximate design rules presume the jerk trace of the ground motion to be a continuous diagram, the ground motions considered in Figures 5 and 6 have discontinuous jerk diagrams. The discontinuities are particularly numerous for the earthquake record, for which the acceleration trace is represented by a piecewise linear diagram and hence the associated jerk diagram consists of a series of rectangular steps. For the effects of such discontinuities on the response of high frequency systems, the reader is referred to Reference 6.

Since sudden changes in jerk are unlikely to occur in actual ground motions, the high-frequency regions of the \( D \)-spectra presented in Figures 5 and 6 cannot be considered to represent correctly the effects of such motions. It follows that the common practice of representing the acceleration traces of earthquake ground motions by piecewise linear diagrams may not be appropriate for the computation of the spectral values of relative velocity of high-frequency systems, and that continuous representations of the jerk diagrams are required in this case. These issues clearly deserve additional studies. Pending the results of such studies, it is recommended that the characteristics
of the $\dot{D}$-spectra in the high-frequency region be determined in the manner suggested in Figure 4(b) making due provision for the fact that, compared to the acceleration diagram, the jerk diagram is more nearly periodic and has a higher frequency content. The effect of these differences is twofold:

1. It leads to a higher amplification factor $C_{p2D}^\dot{D}$ for the right, amplified diagonal segment of the $\dot{D}$-spectrum than the factor $C_A$ for the corresponding segment of the $V$-spectrum; and

2. The boundaries of the transition region between the peak value of $\dot{D}$ and its limiting value of $\ddot{x}_g$ will correspond to higher frequency values than those of the corresponding region in the $V$-spectrum.

A more precise, quantitative definition of these changes is not possible at present.

**Alternative Tripartite Plot of Velocity Spectra.** For ease in application and comparing the relative magnitudes of $\dot{D}$ and $V$, it would be desirable to display the $\dot{D}$-spectra in a logarithmic scale with $\dot{D}$ displayed on the vertical rather than the left diagonal scale. This may be accomplished simply by rotating the spectra displayed in Figures 5(b) and 6(b) by 45 degrees clockwise, as shown in Figure 7(b). On such a plot diagonal lines extending downward to the right represent constant values of $p\dot{D}$, and lines extending downward with a slope of 2 to 1 represent constant values of $p^2\dot{D}$. The values of $\dot{x}_g$, $\ddot{x}_g$ and $\dddot{x}_g$ are then represented by the dashed lines shown in Figure 7(b). This format is the same as that employed in Figures 1 and 3.
COMPUTATION OF MODAL MAXIMA

With the values of $V$ and $\dot{D}$ determined, the question that remains is how the two terms in the expression for a modal component should be combined.

For harmonically excited systems, the steady-state values of $V_j(t)$ and $\dot{D}_j(t)$ are orthogonal to each other, with the result that when one term is maximum the other is zero. The maximum value of the $j^{th}$ modal response for the $i^{th}$ coordinate of the system can then be determined by taking the square root of the sum of the squares of the individual maxima. More specifically, if $\xi_{ij}$ denotes the maximum modal displacement of the $i^{th}$ coordinate for the $j^{th}$ mode of vibration, then

$$\xi_{ij} = \sqrt{\left[\alpha_{ij} V_j\right]^2 + \left[\beta_{ij} \dot{D}_j\right]^2}$$

(11)

in which $\alpha_{ij}^V$ and $\beta_{ij}^V$ are the corresponding values of $\{\alpha_j^V\}$ and $\{\beta_j^V\}$, respectively. For transient excitations, $\dot{D}_j(t)$ vanishes when $V_j(t)$ is an extremum. Although the opposite is not true, it is likely that the square root of the sum of the squares rule will still yield results of sufficient accuracy for practical purposes, and it is recommended that it be used generally.

The effectiveness and degree of accuracy of the proposed approximation depends on the characteristics of the system and of the exciting motion, as these factors affect the relative magnitudes of $\alpha_{ij}^V V_j$ and $\beta_{ij}^V \dot{D}_j$. Very accurate solutions can be expected if one of the terms is small in comparison to the other. The solution is, naturally, exact if one of the terms is zero. For classically damped systems, $\beta_j^V = 0$ and the proposed approximation can be expected to
be of high accuracy if the departure from the classical condition of damping is not very great.

Illustrative Example

The system considered is the three-degree-of-freedom frame of the shear-beam type presented in Figure 1(a) in the previous chapter. The damping coefficient for the dashpot between the first floor and the base is considered to be such that \( \varepsilon_0 = 1 \) (See equation (17) in Chapter II). The relevant values of \( p_j \), \( \xi_j \) and of the vectors \( \{ \alpha_j \} \) and \( \{ \beta_j \} \) are summarized in Tables I and III of the preceding chapter. The maximum modal responses of the system to the two base motions considered in this study are evaluated both exactly and by application of the proposed square root of the sum of the squares rule. A very wide range of frequencies is considered for the fundamental mode of vibration of the structure.

Selected solutions are listed in Tables I and II in which \( f_1 \) is the fundamental pseudo-undamped natural frequency of the system in cycles per second; \( \varepsilon_{ij} \) is the maximum displacement relative to the moving base of the \( i^{th} \) floor for response in the \( j^{th} \) mode; and \( \xi_{ij} \) is the corresponding maximum deformation of the \( i^{th} \) story. All responses are expressed in dimensionless form, as indicated. The solutions for interfloor deformations of systems subjected to the first 6.3 seconds of the El Centro record are all shown in Figures 8(a) through 8(c). These data show clearly that the proposed approximation is indeed of high accuracy.
MAXIMUM RESPONSE OF SYSTEM

With the maximum values of the modal responses established, the maximum response of the system can be obtained by an appropriate combination of modal maxima.

Treatment of Special Cases

There are two special cases for which the evaluation of the maximum response of nonclassically damped systems can be simplified significantly. The first concerns systems for which all natural frequency values fall in the left part of the response spectrum for which \( D_j(t) = -x_g(t) \) and \( \dot{D}_j(t) = -\dot{x}_g(t) \). On making use of equation (67a) of the previous chapter, the second term on the right hand member of equation (1) vanishes and the response may be expressed independently of \( \dot{D}_j(t) \) as

\[
\{x(t)\} = \sum_{j=1}^{n} \{\alpha_j v_j\} \nu_j(t) \quad (12)
\]

The second case concerns systems for which all natural frequency values correspond to the high frequency region of the response spectrum for which \( p_j^2 D_j(t) = A_j(t) \approx -\ddot{x}_g(t) \) and \( \dot{D}_j(t) = -\dddot{x}_g(t)/p_j^2 \approx 0 \). The second term on the right hand member of equation (1) may be neglected and the response of the system is given by

\[
\{x(t)\} = -\dddot{x}_g(t) \sum_{j=1}^{n} \frac{1}{p_j} \{\alpha_j v_j\} \quad (13)
\]

CONCLUSIONS

The interrelationship between the relative velocity and pseudovelocity of viscously damped linear systems excited at the base
has been identified, and simple practical rules have been presented for defining the design spectra for relative velocity for such systems. These rules are similar to those available for defining the corresponding spectra for maximum deformation.

With this information, the response spectrum variant of the modal superposition method of analysis can be applied to nonclassically damped systems with almost the same ease and accuracy as for classically damped systems. The square root of the sum of the squares rule has been proposed for combining the two terms in the expression for a modal response, and the adequacy of this approximation has been demonstrated by numerical examples.
### APPENDIX I

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$p^2D = pV = \text{pseudoacceleration of a base-excited SDF system}$</td>
</tr>
<tr>
<td>$D(t)$</td>
<td>instantaneous deformation of a base-excited SDF system</td>
</tr>
<tr>
<td>$\dot{D}(t)$</td>
<td>instantaneous relative velocity of a base-excited SDF system</td>
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<td>$D, \dot{D}$</td>
<td>absolute maximum values of $D_j(t)$ and $\dot{D}_j(t)$, respectively</td>
</tr>
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<td>$f_j$</td>
<td>$p_j/2\pi = \text{pseudo-natural frequency of } j^{th} \text{ mode of system}$</td>
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<td>$h(t)$</td>
<td>impulse response function for a SDF system</td>
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<tr>
<td>$i$</td>
<td>level of floor or story</td>
</tr>
<tr>
<td>$j$</td>
<td>integer number indicating order of mode under consideration</td>
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<td>$p_j$</td>
<td>pseudo-undamped circular natural frequency of $j^{th}$ mode</td>
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<td>time</td>
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<td>modal configurations of system</td>
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<tr>
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<td>elements of vectors ${\alpha_j^V}$ and ${\beta_j^V}$ at $i^{th}$ floor or story</td>
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<tr>
<td>$\zeta$</td>
<td>damping factor</td>
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<tr>
<td>$\bar{\xi}_{ij}$</td>
<td>maximum value of story deformation of the $i^{th}$ floor for response in the $j^{th}$ mode</td>
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REFERENCES


2) R. Villaverde, 'Earthquake response of systems with nonproportional damping by the conventional response spectrum method', 7th World Conference on Earthquake Engineering, Istambul, Turkey (1980)


<table>
<thead>
<tr>
<th>(f_{1d}^*)</th>
<th>(\frac{v_1}{x_g})</th>
<th>(\frac{D_1}{x_g})</th>
<th>(\xi_{1j}/\hat{x}_g)</th>
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(a) First Mode

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(b) Second Mode

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<th>(\xi_{1j}/\hat{x}_g)</th>
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(c) Third Mode
Table II. Exact and approximate modal maxima of three-story building subjected to first 6.3 sec of El Centro Earthquake

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<th>$\frac{V_d}{x_g}$</th>
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Figure 1. Comparison of spectra for pseudovelocity and true relative velocity; undamped systems subjected to simple base motion.
Figure 2. Simple base motion considered
Figure 3. Comparison of spectra for pseudovelocity and true relative velocity; systems with $z = 0.02$ subjected to El Centro Record
Figure 4. General forms of spectra for pseudovelocity and relative velocity.
Figure 5. Pseudovelocity and relative velocity spectra for systems subjected to simple base motion.
Figure 6. Pseudovelocity and relative velocity spectra for systems subjected to El Centro record
Figure 7. Direct comparison of spectra for pseudovelocity and relative velocity
Figure 8. Comparison of approximate and exact modal story deformations for illustrative example
Figure 8. (Continued)
Figure 8. (Concluded)
CHAPTER IV

STEADY-STATE AND TRANSIENT RESPONSES OF NONCLASSICALLY DAMPED LINEAR SYSTEMS
INTRODUCTION

In a recent paper\(^1\), a simple, computationally efficient procedure was developed for interrelating the steady-state and transient responses of discrete linear systems. The procedure enables one to evaluate the transient response of the system from knowledge of its steady-state response. Conversely, the steady-state response of the system to a periodic extension of the excitation can be evaluated from knowledge of its transient response over a single cycle of the excitation.

The fundamental concepts of the procedure were introduced by reference to viscously damped single-degree-of-freedom systems, and they were then extended to the analysis of multi-degree-of-freedom systems for which the classical modal superposition method is applicable.

In this section, the procedure is extended to nonclassically damped systems using the generalized modal superposition method described in Chapter II. The procedure is developed for base-excited systems and it is illustrated by simple examples.

**Notation**

The symbols used in this chapter are defined when first introduced in the text, and those used extensively are summarized in Appendix II for this chapter.

**SYSTEM AND BACKGROUND**

*System*

The system investigated is a viscously damped, cantilever, linear system with \( n \) degrees of freedom excited at the base. The equations
of motion of the system are

\[ [m]\ddot{\mathbf{x}}(t) + [c]\dot{\mathbf{x}}(t) + [k]\mathbf{x}(t) = -[m]\mathbf{1}\ddot{\mathbf{g}}(t) \quad (1) \]

where \([m]\), \([c]\) and \([k]\) are its mass, damping and stiffness matrices, respectively; \(\{x(t)\}\) is the vector of the displacements at time \(t\) relative to the moving base; \(\{1\}\) is a vector of ones; and \(\ddot{\mathbf{g}}(t)\) represents the acceleration of the moving base. A dot superscript denotes differentiation with respect to time.

**Statement of Problem**

Given the transient response of the system to a prescribed excitation, it is desired to evaluate the corresponding steady-state response to a periodic extension of the excitation. Similarly, given the steady-state response of the system to a periodic excitation, it is desired to evaluate the corresponding transient response for any duration of the excitation.

The procedure for interrelating the two responses involves the superposition on the prescribed response of an appropriate free vibrational response. The basic concepts involved are described in Reference 1 and are reviewed briefly in the next chapter (pages 142 and 143). The detailed implementation of the procedure for classically damped systems has been presented in Reference 1. For the nonclassically damped systems considered herein, the component responses may be evaluated conveniently by the modal superposition method discussed in Chapter II. The details are presented in the following sections.
Computation of Forced Vibration

The response of the system to the prescribed excitation is defined by the following expression, presented previously as equation (53) in Chapter II.

\[ \{x(t)\} = \sum_{j=1}^{n} \left[ \{\alpha_j^y\} p_j D_j(t) + \{\beta_j^y\} \dot{D}_j(t) \right] \quad (2) \]

The vectors \( \{\alpha_j^y\} \) and \( \{\beta_j^y\} \) in this expression depend on the characteristic values of the system, \( r_j \), the associated natural modes, \( \{\psi_j\} \), and the participation factors, \( B_j \). The values of \( B_j \) and of the vectors \( \{\beta_j^y\} \) and \( \{\alpha_j^y\} \) are determined from equations (43), (45) and (51) of Chapter II, respectively. The functions \( D_j(t) \) and \( \dot{D}_j(t) \) represent the deformation and relative velocity of a single-degree-of-freedom system (SDF) with circular frequency \( p_j \) and damping factor \( \zeta_j \) subjected to the prescribed excitation. The values of \( p_j \) and \( \zeta_j \) are determined from the characteristic values, \( r_j \), according to equations (7) and (8) of Chapter II.

Computation of Free Vibrational Response

The response of the system to a prescribed set of initial displacements, \( \{x(0)\} \), and initial velocities, \( \{\dot{x}(0)\} \), may be evaluated from equation (28) of Chapter II. However, because it is expressed in terms of the reference vectors \( \{\beta_j^y\} \) and \( \{\gamma_j^y\} \) which are different from the vectors \( \{\alpha_j^y\} \) and \( \{\beta_j^y\} \) employed in equation (2) for forced vibration, equation (28) is not particularly convenient for the intended purpose here. An alternative formulation, involving the same reference vectors as those employed in the solution for forced vibration, may be obtained from equation (26) of Chapter II by letting...
\[ C_j = B_j (a_j + i b_j) \bar{p}_j \]  

(3)

in which \( B_j \) are complex valued participation factors defined by equation (43) of Chapter II; \( a_j \) and \( b_j \) are real valued constants that remain to be determined; \( i = \sqrt{-1} \); and \( \bar{p}_j \) is the \( j \)th damped circular natural frequency of the system, defined by equation (10) of Chapter II.

On substituting equation (3) into equation (26), making use of equations (9) and (45) of Chapter II and of the relationship between exponential and trigonometric functions, expanding the resulting expression and using equation (51) of Chapter II, the following equation is obtained after rearrangement of terms:

\[ \{x(t)\} = \sum_{j=1}^{n} \left[ \{\alpha_j^y\} p_j \xi_j(t) + \{\beta_j^y\} \dot{\xi}_j(t) \right] \]  

(4)

where

\[ \xi_j(t) = e^{-\xi_j p_j t} \left[ a_j \sin \bar{p}_j t + b_j \cos \bar{p}_j t \right] \]  

(5)

The functions \( \xi_j(t) \) and \( \dot{\xi}_j(t) \) in equations (4) and (5) are now recognized as the free vibrational deformation and relative velocity, respectively, of a SDF system with circular frequency \( p_j \) and damping factor \( \zeta_j \). If \( \xi_j(0) \) and \( \dot{\xi}_j(0) \) represent the initial values of these functions, then the constants \( a_j \) and \( b_j \) are given by

\[ a_j = \frac{\dot{\xi}_j(0)}{\bar{p}_j} + \frac{\zeta_j}{\sqrt{1 - \zeta_j^2}} \xi_j(0) \]  

(6a)

\[ b_j = \xi_j(0) \]  

(6b)
Equations (2) and (4) are applicable to nonclassically as well to classically damped systems. For classically damped systems, \( \{\beta_j^V\} = \{0\} \).

EXTENSION OF PROCEDURE

Following the approach described in Reference 1, the steady-state and transient displacements of the system are interrelated by the equation

\[
\{x(t)\} = \{y(t)\} + \{z(t)\}
\]

(7)
in which \( \{x(t)\} \) are the transient displacements; \( \{y(t)\} \) are the steady-state displacements; and \( \{z(t)\} \) are the displacements for the corrective, free vibrational solution. The computation of the corrective solution differs depending on which is the prescribed initial solution.

Computation of Transient Response from Steady-State

The desired transient response is expressed in slightly different forms depending on whether the steady-state response is expressed directly or in terms of its modal components.

Steady-State Response Specified by Modal Components. The starting solution in this case is defined by

\[
\{y(t)\} = \sum_{j=1}^{n} \left[ \{a_j^V\} p_j S_j(t) + \{\beta_j^V\} \dot{S}_j(t) \right]
\]

(8)
in which \( S_j(t) \) represents the steady-state deformation of a SDF system with undamped natural frequency \( p_j \) and damping factor \( \zeta_j \), and \( \dot{S}_j(t) \) represents the corresponding relative velocity. The responses \( S_j(t) \)
and $\dot{S}_j(t)$ may be evaluated either by a frequency-domain analysis making use of Fourier Series, or by a time-domain analysis, such as the method described in Appendix I.

On substituting equation (8) into equation (7), and replacing \{z(t)\} in the latter equation by the right hand side of equation (4), one obtains

$$\{x(t)\} = \sum_{j=1}^{n} \left\{ \{a_j \} p_j [S_j(t) + \xi_j(t)] + \{b_j \} [\dot{S}_j(t) + \dot{\xi}_j(t)] \right\}$$  \hspace{1cm} (9)

in which $\xi_j(t)$ is defined by equation (5). From a comparison of this equation with equation (2) it should be clear that the modal components of the desired transient response may be obtained from the corresponding components of the prescribed steady-state response by correcting each component separately.

The constants $a_j$ and $b_j$ in equation (5) must be determined so that equation (9) satisfies the initial conditions of the desired transient motion. If these conditions are specified in terms of the initial values of $D_j$ and $\dot{D}_j$ in equation (2), then the initial values of $\xi_j$ and $\dot{\xi}_j$ are given by

$$\xi_j(0) = D_j(0) - S_j(0)$$  \hspace{1cm} (10a)

$$\dot{\xi}_j(0) = \dot{D}_j(0) - \dot{S}_j(0)$$  \hspace{1cm} (10b)

and the values of $a_j$ and $b_j$ are determined from equations (6a) and (6b). For the practically important case in which the initial conditions of the desired transient response are zero, $D_j = \dot{D}_j = 0$ and the procedure may be implemented readily.
If the initial conditions of the desired transient response are specified directly in terms of \( \{x(0)\} \) and \( \dot{x}(0) \), to determine the constants \( a_j \) and \( b_j \) it is first necessary to decompose the prescribed vectors into their modal components. In this case, it is convenient to think of \( a_j + ib_j \) as being made up of two parts: the component \( a_j^T + ib_j^T \) which is contributed by the initial conditions of the desired transient response, and the component \( a_j^S + ib_j^S \) which is contributed by the initial conditions of the prescribed steady-state response. It then follows that

\[
a_j + ib_j = a_j^T - a_j^S + i(b_j^T - b_j^S)
\]

in which \( a_j^S \) and \( b_j^S \) are determined from the right hand member of equations (6a) and (6b) by taking \( \xi_j(0) = s_j(0) \) and \( \dot{\xi}_j(0) = \dot{s}_j(0) \), and \( a_j^T \) and \( b_j^T \) are determined from

\[
a_j^T + ib_j^T = \frac{r_j\{\psi_j\}^T[m]\{x(0)\} + \{\psi_j\}^T[c]\{x(0)\} + \{\psi_j\}^T[m]\{\dot{x}(0)\}}{\bar{p}_j\{\psi_j\}^T[m]\{1\}}
\]

The latter expression is obtained from equation (3) by interpreting \( a_j + ib_j \) to be \( a_j^T + ib_j^T \) and replacing \( C_j \) and \( B_j \) by the right hand members of equations (29) and (43) in Chapter II.

**Steady-State Response Specified Directly.** The desired transient displacements in this case are expressed in the form

\[
\{x(t)\} = \{y(t)\} + \sum_{j=1}^{n} \left( \{\alpha_j^y\} p_j \xi_j(t) + \{\beta_j^y\} \dot{\xi}_j(t) \right)
\]

in which only the corrective solution is expressed by its modal
components. The constants $a_j$ and $b_j$ in the expressions for $\xi_j(t)$ and $\dot{\xi}_j(t)$ are determined from equation (12) by interpreting its left hand member to be $a_j + ib_j$ and replacing the vectors $\{x(0)\}$ and $\{\dot{x}(0)\}$ by $\{x(0) - y(0)\}$ and $\{\dot{x}(0) - \dot{y}(0)\}$, respectively.

**Steady-State Response from Transient**

The corrective solution in this case must be determined so that the desired steady-state solution satisfies the periodicity conditions

\[
\{y(t_o)\} = \{y(0)\} \quad (14a)
\]

\[
\{\dot{y}(t_o)\} = \{\dot{y}(0)\} \quad (14b)
\]

which, on making use of equation (7), can also be written as

\[
\{x(t_o)\} - \{z(t_o)\} = \{x(0)\} - \{z(0)\} \quad (15a)
\]

\[
\{\dot{x}(t_o)\} - \{\dot{z}(t_o)\} = \{\dot{x}(0)\} - \{\dot{z}(0)\} \quad (15b)
\]

The symbol $t_o$ in these expressions represents the period of the excitation and of the associated steady-state response.

As in the previous section, the evaluation of the corrective solution, $\{z(t)\}$, depends on whether the prescribed transient solution is defined directly or in terms of its modal components.

**Transient Response Specified by Modal Components.** The desired steady-state displacements in this case are given by

\[
\{y(t)\} = \sum_{j=1}^{n} \left\{ \left\{ \alpha_j \right\} p_j \left[ D_j(t) - \xi_j(t) \right] + \left\{ \beta_j \right\} \left[ \dot{D}_j(t) - \dot{\xi}_j(t) \right] \right\} \quad (16)
\]

This expression is obtained from equation (7) by making use of equation (2) and replacing $\{z(t)\}$ by the right hand member of equation (4). From a comparison of equation (16) with equation (8) it is clear that the
modal components of the desired steady-state response are obtained from the corresponding components of the prescribed transient response by correcting each component separately.

The constants \(a_j\) and \(b_j\) in the expressions for \(\xi_j(t)\) and \(\dot{\xi}_j(t)\) are determined by insuring that each term of equation (16) satisfies the periodicity conditions defined by equations (15). These conditions are expressed as

\[
D_j(t_0) - \xi_j(t_0) = D_j(0) - \xi_j(0) \tag{17a}
\]

\[
\dot{D}_j(t_0) - \dot{\xi}_j(t_0) = \dot{D}_j(0) - \dot{\xi}_j(0) \tag{17b}
\]

which, on making use of equation (5) and its corresponding derivative, lead to a system of two simultaneous equations in \(a_j\) and \(b_j\), the solution of which yields

\[
a_j = \frac{1}{\Delta_j} \left\{ \left[ \frac{\zeta_j}{\sqrt{1 - \zeta_j^2}} \left( 1 - e^{-\zeta_j p_j t_o \cos \tilde{p}_j t_o} \right) - e^{-\zeta_j p_j t_o \sin \tilde{p}_j t_o} \right] \Delta D_j \right. \\
+ \left. \left( 1 - e^{-\zeta_j p_j t_o \cos \tilde{p}_j t_o} \right) \frac{\Delta \dot{D}_j}{\dot{p}_j} \right\} \tag{18a}
\]

and

\[
b_j = \frac{1}{\Delta_j} \left\{ \left[ 1 - e^{-\zeta_j p_j t_o \left( \cos \tilde{p}_j t_o - \frac{\zeta_j}{\sqrt{1 - \zeta_j^2}} \sin \tilde{p}_j t_o \right)} \right] \Delta D_j \\
+ \left( e^{-\zeta_j p_j t_o \sin \tilde{p}_j t_o} \right) \frac{\Delta \dot{D}_j}{\dot{p}_j} \right\} \tag{18b}
\]

In these expressions,
\[ \Delta D_j = D_j(0) - D_j(t_o) \] 
\[ \Delta \dot{D}_j = \dot{D}_j(0) - \dot{D}_j(t_o) \]
and
\[ \Delta_j = 1 + e^{-2\xi_j p_j t_o} - 2e^{-\xi_j p_j t_o} \cos \bar{\theta}_j t_o \]

**Transient Response Specified Directly.** The desired steady-state displacements in this case may be expressed in the form of equation (13) as

\[ \{y(t)\} = \{x(t)\} - \sum_{j=1}^{n} \left[ \{a_j^T\} p_j \varepsilon_j(t) + \{b_j^T\} \dot{\varepsilon}_j(t) \right] \]

(20)

The constants \( a_j \) and \( b_j \) in the expressions for \( \varepsilon_j(t) \) and \( \dot{\varepsilon}_j(t) \) can then be determined from equations (18) provided \( \Delta D_j \) and \( \Delta \dot{D}_j \) are determined from

\[ \Delta D_j = \Delta b_j^T \]
\[ \Delta \dot{D}_j = \bar{p}_j \Delta a_j^T - \xi_j p_j \Delta b_j^T \]

(21a)
(21b)

and the quantities \( \Delta a_j^T \) and \( \Delta b_j^T \) are evaluated from equation (12) by interpreting \( a_j^T + ib_j^T \) to be \( \Delta a_j^T + i\Delta b_j^T \) and replacing the vectors \( \{x(0)\} \) and \( \{\dot{x}(0)\} \) by \( \{x(0) - x(t_o)\} \) and \( \{\dot{x}(0) - \dot{x}(t_o)\} \), respectively. The quantities \( \Delta D_j \) and \( \Delta \dot{D}_j \) in equations (21) are the counterparts of \( \varepsilon_j(0) \) and \( \dot{\varepsilon}_j(0) \) in equations (6), and the quantities \( \Delta a_j^T \) and \( \Delta b_j^T \) are the counterparts of \( a_j \) and \( b_j \) in the same equations.

**ILLUSTRATIVE EXAMPLES**

The following examples refer to the three-story structure of the shear-beam type shown in Figure 1 of Chapter II, with the damping
factor, \( \zeta_0 \), taken as unity. The characteristic values of this system, \( r_j \), the associated natural frequencies and damping factors, \( p_j, \tilde{p}_j \) and \( \zeta_j \), and the complex-valued natural modes, \( \{\psi_j\} \), have been presented in Table I of Chapter II, while the participation factors, \( B_j \), and the vectors \( \{\alpha_j^V\} \) and \( \{\beta_j^V\} \) are listed in part (b) of Table IV in the same chapter.

**Steady-State Response from Transient**

The steady-state response of the system to a periodic extension of the base excitation shown in Figure 1(a) is computed in this example from its transient response to a single cycle of the excitation. With \( t_d \) denoting the duration of the excitation, the period of its periodic extension and of the associated steady-state response is \( t_0 = t_d \).

The system is considered to be initially at rest and its dimensions are considered to be such that

\[
f_0 t_0 = f_0 t_d = 1
\]

in which \( f_0 \) is a reference frequency defined by

\[
f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}}
\]

The values of \( p_j t_0 = p_j t_d \) for the various modes are then \( p_1 t_0 = 3.382 \), \( p_2 t_0 = 8.747 \) and \( p_3 t_0 = 11.860 \). The starting transient response is presumed to be specified by its modal components. The deformations, \( D_j(t) \), and relative velocities, \( \dot{D}_j(t) \), for the SDF systems in the modal solutions are shown by the dashed lines in Figures 2 and 3. The functions \( D_j(t) \) on these plots are normalized with respect to the peak displacement of the ground motion, \( x_g \), whereas the functions \( \dot{D}_j(t) \) are normalized with respect to the product \( p_j x_g \). The maximum ground
displacement is related to the maximum ground acceleration, $\ddot{x}_g$, by the expression

$$x_g = \frac{4 + \pi}{16\pi^2} \ddot{x}_g t_d^2 = 0.0452 \ddot{x}_g t_d^2$$  \hspace{1cm} (22)

Since the system is initially at rest, the values $\Delta D_j$ and $\Delta \dot{D}_j$ in equations (19a) and (19b) are the negative of those for $D_j$ and $\dot{D}_j$ at $t = t_d$; these values are listed in Table I for each of the modes, with the common multipliers identified in the extreme right hand column of the table. Also listed in the table are the values of $a_j$ and $b_j$, determined from equations (18).

With the constants $a_j$ and $b_j$ established, the corrective solutions $\xi_j(t)$ and $\dot{\xi}_j(t)$ for any time $0 \leq t \leq t_d$ are determined from equation (5) and its derivative, and the steady-state response functions $S_j(t)$ and $\dot{S}_j(t)$ are determined by subtracting $\xi_j(t)$ from $D_j(t)$ and $\dot{\xi}_j(t)$ from $\dot{D}_j(t)$, respectively. The results are shown in Figures 2 and 3. Finally, the steady-state response of the system is evaluated from equation (16). The resulting interfloor deformations are shown in Figure 4, where they are compared with those of the starting transient solutions.

**Transient Response from Steady-State**

In this example, the transient response of the system to the excitation shown in Figure 1(a) is to be evaluated from its corresponding steady-state response. The desired transient response is to be computed over a time interval that exceeds the duration of the excitation, $t_d$, by the undamped natural period of the system, $T_1^0$.

The period, $t_0$, of the periodic extension of the excitation and
of the associated steady-state response must then be equal to or greater than $t_d + T_1^0$, as shown in Figure 1(b). In the material presented herein, the starting steady-state solution is presumed to have been obtained by the Discrete Fourier Transform (DFT) method, taking $t_o = t_d + T_1^0$. The duration of the excitation is considered to equal $T_1^0$. Accordingly, $t_o = 2t_d$, and the values of $p_j t_o$ for the various modes are $p_1 t_o = 13.065$, $p_2 t_o = 33.797$ and $p_3 t_o = 45.821$.

The initial values of the steady-state displacements and velocities of the system obtained by the DFT method are:

\[
\begin{align*}
    y_1(0) &= -0.0337 x_g \\
    y_2(0) &= -0.1734 x_g \\
    y_3(0) &= -0.2347 x_g \\
    \dot{y}_1(0) &= -0.2638 p_1 x_g \\
    \dot{y}_2(0) &= -0.4116 p_1 x_g \\
    \dot{y}_3(0) &= -0.4440 p_1 x_g
\end{align*}
\]

in which the maximum ground displacement, $x_g$, is defined by equation (22). The transient response is computed considering that $\{x(0)\} = \{\dot{x}(0)\} = \{0\}$. The constants $a_j$ and $b_j$ are evaluated from the right hand member of equation (12) by replacing the vectors $\{x(0)\}$ and $\{\dot{x}(0)\}$ by $\{x(0) - y(0)\}$ and $\{\dot{x}(0) - \dot{y}(0)\}$, respectively. The results are listed in part (b) of Table 1, and the resulting histories of $\xi_j(t)/x_g$ and $\dot{\xi}_j(t)/p_j x_g$ in the interval $0 \leq t \leq 2t_d$ are shown in Figure 5. Note that the values of $a_2$ and $b_2$ are very small compared with the rest, and, therefore, the contribution of the second mode to the corrective solution is insignificant. The transient displacements of the system are finally evaluated from equation (13). The response histories for interfloor deformation are shown in Figure 6, where they are compared with the prescribed steady-state deformations.
CONCLUSION

The recently proposed procedures for interrelating the steady-state and transient responses of classically damped discrete linear systems have been extended to nonclassically damped systems. The procedures have been implemented by use of the generalized modal superposition method considered in Chapter II of this dissertation. Although described for base excited systems, the procedures can readily be extended to force excited systems.

In addition, a Duhamel type integral has been presented which in combination with the modal superposition method may be used to evaluate the steady-state response of the system by integration in the time domain.
APPENDIX I

Computation of Steady-State Response by Integration in Time Domain

Consider a SDF system with a circular frequency \( \omega \) and damping factor \( \zeta \) subjected to a periodic base acceleration, \( \ddot{x}_g(t) \), as shown in Figure 7. Let \( t_0 \) be the period of the excitation, and let it be desired to evaluate the steady-state response of the system in the interval \( 0 \leq t \leq t_0 \).

The desired response may be evaluated by integration in the time domain in a manner similar to determine the transient response of the system by use of Duhamel's integral. To this end, let \( H(t) \) be the periodic impulse response function for the system, defined as the steady-state displacement at time \( t \) in the interval between zero and \( t_0 \) due to a periodic set of unit velocity changes applied at the ends of the interval. This function is given by

\[
H(t) = \frac{e^{-\zeta \omega t}}{\Delta} \left[ (e^{-\zeta \omega t_0} \sin \omega t_0) \cos \omega t + (1 - e^{-\zeta \omega t_0} \cos \omega t_0) \sin \omega t \right] \quad (A1)
\]

in which

\[
\Delta = 1 + e^{-2\zeta \omega t_0} - 2 e^{-\zeta \omega t_0} \cos \omega t_0 \quad (A2)
\]

Further, let \( v(\tau) = -\ddot{x}_g(\tau)d\tau \) be the periodic set of velocity changes induced by the acceleration \( \ddot{x}_g(\tau) \) considered to act over the infinitesimal interval \( d\tau \). The resulting steady-state displacement at any time \( 0 \leq t \leq t_0 \) is then given by

\[
dS(t) = \begin{cases} 
  v(\tau) H(t-\tau) & \text{for } t > \tau \\
  v(\tau) H(t+t_0-\tau) & \text{for } t < \tau
\end{cases} \quad (A3)
\]
and the corresponding steady-state displacement due to the prescribed acceleration is obtained by integration as follows

\[ S(t) = - \int_0^t \ddot{x}_g(\tau) h^*(t-\tau) \, d\tau + \int_t^{t_0} \ddot{x}_g(\tau) h^*(t + t_0 - \tau) \, d\tau \]  \hspace{1cm} (A4)

This expression is a specialized form of the so-called cyclic convolution integral\(^3\).

As an example, consider an undamped system subjected to a sinusoidal base acceleration of amplitude $\ddot{x}_g$ and circular frequency $\omega = 2\pi/t_0$. The steady-state response of the system due to this excitation may be evaluated from equation (A4) making use of the following specialized form of equation (A1):

\[ h^*(t) = \frac{1}{2p} \left[ \cot \frac{pt_0}{2} \cos pt + \sin pt \right] \]  \hspace{1cm} (A5)

On making use of this equation, it can be shown that

\[ \int_0^t \ddot{x}_g(\tau) h^*(t-\tau) \, d\tau = \frac{\ddot{x}_g}{2p^2[1-(\omega/p)^2]} \left[ \sin \omega t + \frac{\omega}{p} \cot \frac{pt_0}{2} \cos \omega t \right. \\
\left. - \frac{\omega}{p} (\sin pt + \cot \frac{pt_0}{2} \cos pt) \right] \]  \hspace{1cm} (A6a)

and that

\[ \int_t^{t_0} \ddot{x}_g(\tau) h^*(t + t_0 - \tau) \, d\tau = \frac{\ddot{x}_g}{2p^2[1-(\omega/p)^2]} \left[ \sin \omega t - \frac{\omega}{p} \cot \frac{pt_0}{2} \cos \omega t \right. \\
\left. + \frac{\omega}{p} (\sin pt + \cot \frac{pt_0}{2} \cos pt) \right] \]  \hspace{1cm} (A6b)

Finally, substituting equations (A6) into equation (A4) one obtains
the following well known expression for the steady-state (harmonic) response of the system

\[ S(t) = - \frac{\ddot{x}_g}{p^2} \frac{1}{1 - (\omega/p)^2} \sin \omega t \]  

(A7)
APPENDIX II

Notation

- \( a_j, b_j \): constants in equation (5)
- \( a_j^S, b_j^S \): steady-state components of constants \( a_j \) and \( b_j \)
- \( a_j^T, b_j^T \): transient components of constants \( a_j \) and \( b_j \)
- \( B_j \): complex-valued participation factor
- \( D_j(t) \): transient deformation of a SDF system
- \( \dot{D}_j(t) \): transient relative velocity of a SDF system
- \( i \sqrt{-1} \): integer variable indicating mode number
- \( n \): number of degrees of freedom of a system
- \( p_j, \tilde{p}_j \): pseudo-undamped and damped circular natural frequency of \( j \)th mode
- \( \text{SDF} \): single-degree-of-freedom system
- \( S_j(t) \): steady-state deformation of a SDF system
- \( \dot{S}_j(t) \): steady-state relative velocity of a SDF system
- \( T_1^0 \): fundamental period of undamped system
- \( t \): time
- \( t_d \): duration of excitation
- \( t_0 \): period during which the response is to be evaluated; also period of periodic extension of excitation
- \( \{x(t)\} \): transient displacements of the system relative to the moving base
- \( \{y(t)\} \): steady-state displacements of the system relative to the moving base
- \( \{z(t)\} \): corrective free vibrational displacements
- \( \{\alpha_j^v\}, \{\beta_j^v\} \): real-valued vectors
\[ \Delta D_j = D_j(0) - D_j(t_o) \]
\[ \dot{D}_j = \dot{D}_j(0) - \dot{D}_j(t_o) \]
\[ \xi_j \quad \text{modal damping factor} \]
\[ \xi_{j}(t) \quad \text{free vibrational displacement of a SDF system} \]
\[ \dot{\xi}_j(t) \quad \text{free vibrational velocity of a SDF system} \]
REFERENCES


Table I. Steady-state and transient responses of system considered in examples.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
<th>Common Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta D_j)</td>
<td>(-0.8077)</td>
<td>0.5420</td>
<td>0.3009</td>
<td>(x_g)</td>
</tr>
<tr>
<td>(\Delta D_j)</td>
<td>1.5144</td>
<td>0.4743</td>
<td>(-0.0439)</td>
<td>(p_j x_g)</td>
</tr>
<tr>
<td>(a_j)</td>
<td>0.8515</td>
<td>0.5639</td>
<td>0.3097</td>
<td>(x_g)</td>
</tr>
<tr>
<td>(b_j)</td>
<td>(-0.5757)</td>
<td>0.5583</td>
<td>0.3467</td>
<td>(x_g)</td>
</tr>
</tbody>
</table>

(a) Steady-State Response from Transient

(b) Transient Response from Steady-State

| \(a_j\) | 0.3683 | 0.0008 | 0.0252 | \(x_g\) |
| \(b_j\) | 0.1565 | -0.0006 | 0.0079 | \(x_g\) |
Figure 1. Acceleration of base motion considered in examples
Figure 2. Deformation functions for computation of response of first example.
Figure 3. Relative velocity functions for computation of response of first example.
Figure 4. Story deformations of system considered in first example
Figure 5. Response functions of corrective solution for second example
Figure 6. Steady-state and transient story deformations of system considered in second example.
Figure 7. Representative acceleration trace of periodic ground motion
CHAPTER V

PITFALLS AND IMPROVEMENTS OF DFT METHOD OF DYNAMIC ANALYSIS
INTRODUCTION

The Discrete Fourier Transform (DFT) method in combination with any of the available Fast Fourier Transform (FFT) algorithms\textsuperscript{1,6} provides a powerful means for evaluating the dynamic response of linear structural systems. This approach may be used for classically damped systems, for which the natural modes of vibration are real-valued and identical to those of the associated undamped systems, as well as for nonclassically damped systems, for which the modes are complex-valued. Both constant-parameter systems and systems with frequency-dependent parameters of the type encountered in studies of soil-structure interaction may be considered.

A discretized version of the Fourier Transform method of analysis, the DFT approach is a generalization of the Fourier series method in which the forcing function is expressed as a linear combination of sinusoids of different frequencies, amplitudes and phase angles, and the response is computed by the superposition of the effects of the component functions. This approach is particularly efficient for evaluating the steady-state response of a system to a periodic excitation. When applied to the computation of the response to an arbitrary transient excitation, however, the DFT procedure may still entail considerable computational effort, and it may lead to significant inaccuracies if it is not properly implemented.

The first objective of this chapter is to identify the principal sources of potential inaccuracies in the classical DFT procedure for evaluating the transient response of linear systems and to examine the nature and magnitudes of the errors that may result. The second
objective is to present two versions of a recently proposed technique with which the efficiency of the DFT approach may be improved dramatically.

In the proposed improvement, the steady-state response of the system is first computed by application of the standard DFT method in combination with one of the highly efficient FFT algorithms, and a simple corrective solution is then applied which transforms the steady-state response to the desired transient response. The first version of the proposed procedure can efficiently be applied to classically damped systems only, whereas the second can be used efficiently for a much broader class of systems.

The concepts involved are developed by reference to single-degree-of-freedom systems, and they are illustrated with the aid of judiciously selected numerical solutions. While the extension of these concepts to more involved systems is developed in the next chapter, the basic usefulness of the proposed method in the analysis of multi-degree-of-freedom systems is identified here.

STATEMENT OF PROBLEM

A viscously damped, single-degree-of-freedom (SDF) linear system of mass, \( m \), stiffness, \( k \), and coefficient of viscous damping, \( c \), is considered. The system is presumed to be excited by a force, \( P(t) \), shown by the solid line in Figure 1, in which \( t \) = time and \( t_d \) = the duration of the excitation.

Using the DFT method of analysis or the improved versions proposed herein, it is desired to evaluate the resulting displacement of the system, \( x(t) \). Of special interest is the numerically largest
value of the displacement, \( |x_{\text{max}}| \). Unless otherwise indicated, the system will be presumed to be initially at rest.

Since the maximum response of the system may be attained after cessation of the excitation during the first half-cycle of free vibration, the analysis should be carried out over a time interval, \( t_o \), which exceeds \( t_d \) by at least one-half the natural period of the system, \( T \), i.e.

\[
t_o \geq t_d + \frac{1}{2} T
\]

(1)

**REVIEW AND ANALYSIS OF FUNDAMENTALS**

Let the forcing function in the time interval \( t_o \) be sampled at \( N \) equally spaced points, numbered from zero to \( N - 1 \), and let \( \Delta t \) be the sampling interval, as shown in Figure 1; thus, \( t_o = N \Delta t \). Further, let \( P_n = P(t_n) = \) the value of \( P(t) \) at \( t_n = n \Delta t \), and \( x_n = x(t_n) = \) the corresponding displacement of the system. The discretized forcing function is represented by the dots in Figure 1.

The sampling interval, \( \Delta t \), should be small compared both to the periods of the significant oscillations in the forcing function and to the natural period of the system itself. The first requirement ensures adequate representation of the forcing function and of the forced vibrational component of the response, and the second requirement ensures adequate definition of the free vibrational response component. The first condition generally controls the analysis of low-frequency systems or systems subjected to short-duration excitations, whereas the second controls the analysis of high-frequency systems or systems subjected to long-duration excitations.
Fourier Series Representation of Excitation

The elements of the discretized forcing function, $P_n$, may be expressed as a linear combination of $N$ harmonic functions, in the form

$$P_n = \sum_{r=0}^{N-1} F_r e^{i\omega_r t_n} = \sum_{r=0}^{N-1} F_r e^{i(2\pi nr/N)}$$

(2)

in which $i = \sqrt{-1}$; $r$ = a positive integer; $\omega_r = r\Delta \omega$ = the circular frequency of the $r$th harmonic;

$$\Delta \omega = \frac{2\pi}{t_0} = \frac{2\pi}{N \Delta t}$$

(3)

is the corresponding frequency of the fundamental or first harmonic; and $F_r$ is a complex-valued coefficient that defines the amplitude and phase of the $r$th harmonic. The latter quantity is given by

$$F_r = \frac{1}{t_0} \sum_{n=0}^{N-1} P_n e^{-i\omega_r t_n \Delta t} = \frac{1}{N} \sum_{n=0}^{N-1} P_n e^{-i(2\pi nr/N)}$$

(4)

The sequence of $F_r$ values represents the DFT of the forcing function.

Note that only positive frequencies are considered in equations (2) and (4). In such a one-sided expansion, the values of $F_r$ on either side of $\omega_{N/2}$ are the complex conjugates of each other. Specifically,

$$F_r = F_{N-r}^* \quad \text{for} \quad \frac{N}{2} < r \leq N-1$$

(5)

in which the star superscript denotes a complex conjugate. The frequencies corresponding to $N/2 < r \leq N-1$ have no physical significance; they are merely the counterparts of the negative frequencies in a two-sided Fourier expansion in which $\omega$ extends from $-\omega_{N/2}$ to $\omega_{N/2}$. Thus, $\omega_{N/2}$ defines the frequency of the highest participating
harmonic. Denoted also by the symbol $\omega_{\text{max}}$, this frequency is known as the Nyquist or folding frequency and is given by

$$\omega_{\text{max}} = \frac{N}{2} \Delta \omega = \frac{\pi}{\Delta t}$$  \hspace{1cm} (6)

The maximum and minimum periods of the participating harmonics are determined from equations (3) and (6) to be $t_o$ and $2\Delta t$, respectively.

**Frequency Response Function**

Let $H(\omega)$ be the complex-valued function that defines the amplitude and phase angle of the steady-state, harmonic response of the system to a harmonic force of unit amplitude. For the SDF system examined here,

$$H(\omega) = \frac{1}{k - m_\omega^2 + i \omega} = \frac{1}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_p}\right)^2 + i 2\varsigma \left(\frac{\omega}{\omega_p}\right)}$$  \hspace{1cm} (7)

in which $p = 2\pi/T = \sqrt{k/m}$ = the undamped circular natural frequency of the system, and $\varsigma = c/(2mp)$ = the fraction of critical damping.

Further, let $H_r$ be the value of $H(\omega)$ for $\omega = \omega_r$. For the one-sided Fourier series representation of the forcing function presented in equation (2), the values of $H_r$ for $r > N/2$ must be the folded images, or complex conjugates, of those for $r < N/2$. These values may be determined from equation (7) giving the following interpretation to $\omega_r$:

$$\omega_r = \begin{cases} 
\Delta \omega & \text{for } 0 \leq r \leq N/2 \\
-(N-r)\Delta \omega & \text{for } N/2 < r \leq N-1
\end{cases}$$  \hspace{1cm} (8)
Computation of Response

In the DFT approach, one first computes the products

\[ Y_r = H_r F_r \quad \text{for} \quad 0 \leq r \leq N - 1 \]  

(9)

and then evaluates the discretized response of the system, \( y_n = y(t_n) \), from

\[ y_n = \sum_{r=0}^{N-1} Y_r e^{i \omega_n t_n} = \sum_{r=0}^{N-1} Y_r e^{i (2\pi nr/N)} \]  

(10)

which is the counterpart of equation (2). The symbol \( y_n \) has been used in equation (10) to emphasize the fact that the response computed in this manner may differ from the desired, actual transient response, \( x_n \). The sequence of \( y_n \) values represents the inverse DFT of the \( Y \)-sequence.

Implicit in the foregoing development has been the assumption that the system is initially at rest. For a system with an arbitrary initial displacement, \( x(0) \), and arbitrary initial velocity, \( \dot{x}(0) \), equation (9) should be generalized as follows:

\[ Y_r = H_r \left\{ F_r + m [2\zeta p + i \omega_n] x(0) + m \dot{x}(0) \right\} \]  

(11)

POSSIBLE ERRORS IN CLASSICAL SOLUTION

As already intimated, the response defined by equation (10) does not generally represent the desired transient response of the system; rather it represents its steady-state response to a periodic extension of the excitation. This may be appreciated from the following considerations.

Fundamental to equation (2) is the assumption that the discretized force sequence is periodic, repeating with a period \( t_0 \). Accord-
ingly, the elements $F_n e^{i\omega t}$ represent the harmonic components of the periodic force sequence; the elements $Y_n e^{i\omega t}$ in equation (10) represent the harmonic components of the resulting response; and the combination of terms in equation (10) represents the steady-state response of the system to a periodic extension of the forcing function.

The interrelationship between $y(t)$ and $x(t)$ depends on the properties of the system under study and the characteristics of the excitation, particularly the duration of free vibration, $t_f$, employed. If, as indicated in Figure 2, $t_f$ is sufficiently long such that the steady-state response of the system at the end of each cycle of a periodic extension of the forcing function is negligible, then the response of the system within a given cycle is independent of its response in the preceding cycle, and there is no practical difference between $y(t)$ and $x(t)$. To ensure that this is indeed the case, the period of definition of the forcing function must be augmented by the addition of a sufficiently long band of zeros at the end, and the response must be evaluated for the augmented force sequence. With $t_f$ interpreted to be the duration of the appended band of zeros, the period of the extended excitation, $t_o^*$, is given by

$$t_o^* = t_d + t_f \quad (12)$$

It is important that there be no misunderstanding regarding the durations defined by equations (1) and (12). The former represents the duration over which the response of the system is actually
desired, whereas the latter represents the duration that must be considered in the DFT analysis to ensure that the computed steady-state response, \( y(t) \), is a reasonable approximation to the desired transient response, \( x(t) \).

The longer the natural period of the system or the smaller its damping, the longer will, in general, be the time needed for the free vibrational response to subside, and hence the longer will be the required values of \( t_f \) and \( t_0^* \). For the limiting case of a truly undamped system, the required values of \( t_f \) and \( t_0^* \) tend to infinity and, as is well known, the method cannot yield the desired transient response. An inappropriate choice of \( t_f \) is the principal source of inaccuracies in analyses of transient response by this method.

The value of \( t_0 \) defined by the equality in equation (1) represents the minimum or optimum value of \( t_0^* \) that may be employed. Use of a larger value increases the number of points, \( N \); decreases the frequency increment, \( \Delta \omega \) (see equation (3)); and increases the number of frequencies and the associated values of \( H_r \) and \( Y_r \) that must be considered. These changes, in turn, increase the number of terms in the DFT sequence and the required computational effort. The increased effort may be particularly significant for multi-degree-of-freedom systems, for which a family of such functions must be computed.
Sensitivity of Solution

The sensitivity of the DFT solution to the choice of the free vibrational period, \( t_f \), was studied considering the effect of the sinusoidal force pulse shown in the inset diagram of Figure 3.

The forcing function in these solutions was sampled at intervals \( \Delta t = t_d/40 \), and the response was evaluated at the same intervals. Thus, the circular frequency of the highest harmonic in the Fourier series representation of the forcing function was \( \omega_{\text{max}} = 40\pi/t_d \) (see equation (6)), and the associated period was \( t_d/20 \). All DFTs in these and all other solutions that follow were evaluated by use of the FFT algorithm of the Fortran IMSL subroutine package\(^5\). Unlike the original Cooley-Tukey algorithm\(^3\) which requires that the number of points, \( N \), be a power of 2, this algorithm permits consideration of an arbitrary value of \( N \).

In Figure 3 are shown the displacement histories obtained for systems with \( \zeta = 0.05 \) using six different values of \( t_f \) in the range between \( t_d \) and \( 10t_d \). The undamped natural frequency of the system in cycles per second, \( f = 1/T \), is taken such that \( ft_d = 0.5 \). The time scale in these plots is normalized with respect to the duration of the sinusoidal pulse, \( t_d \); and the displacement of the system is normalized with respect to \( x_{st} = P/k \), the static displacement induced by the peak value of the applied force, \( P \). Although the response in each case was evaluated for the period of the extended forcing function, \( t^*_0 = t_d + t_f \), only its initial part, corresponding to \( t_0 = 2t_d = t_d + 0.5T \), is displayed.

The results clearly show that, unless \( t_f \) is quite long, the
DFT solution may differ significantly from the exact solution. For the example considered, the errors are substantial even when \( t_f = 10t_d \).

The DFTs of the extended forcing functions are shown in Figure 4(a), and dimensionless plots of the associated discretized frequency response functions are shown in Figure 4(b). The frequency axes in these plots are normalized with respect to the Nyquist frequency, \( \omega_{\text{max}} \), and the values of \( F_r \) in Figure 4(a) are normalized with respect to the peak value of the forcing function, \( \mathcal{P} \). As \( t_f \) tends to infinity, \( \Delta \omega \) tends to zero; the DFT of the extended force tends to the continuous Fourier transform of the prescribed force; the discretized sequence of \( H_r \) values tends to the continuous frequency response function, \( H(\omega) \); the sequence of \( Y_r = H_r F_r \) values tends to the continuous inverse Fourier transform of the response; and the computed response tends to the desired transient response.

**Sensitivity of Peak Response**

The values of \( |x_{\text{max}}|/x_{\text{st}} \) for systems with \( \zeta = 0.01 \) and \( \zeta = 0.05 \) are plotted in Figure 5(a) as a function of the frequency parameter, \( ft_d \). In addition to the exact solutions, the DFT solutions corresponding to \( t_f = 10t_d \) are displayed. The sampling interval in these solutions was taken equal to or less than \( \Delta t = t_d/40 \) or \( \Delta t = T/20 \), whichever is smaller.

It is clear from Figure 5(a) that the accuracy of the DFT solution corresponding to a fixed value of \( t_f/t_d \) depends importantly on the values of the frequency parameter and damping involved. In the high-frequency region of the response spectrum, for which
the duration of the excitation is long compared to the natural period of the system, the peak response computed by the DFT procedure is in excellent agreement with the exact solution almost irrespective of the amount of damping involved. The response of the system to each cycle of the excitation in this case is essentially independent of its response to the preceding cycles, and there is practically no difference between its steady-state and transient responses.

By contrast, the two solutions may differ significantly in the medium-frequency and low-frequency regions of the response spectrum, particularly when system damping is low. The lower the value of the frequency parameter or system damping, the slower is the rate of decay of the motion in the region of the appended zeros, and hence the less accurate is the DFT solution.

In addition to explaining the trends displayed in Figure 5(a), the foregoing considerations suggest that in assessing the accuracy of the DFT solution it would be preferable to consider $t_f$ to be a fixed multiple of the natural period of the system, $T$, rather than a fixed multiple of the duration of the forcing function, $t_d$. To this end, the exact response spectra presented in Figure 5(a) are compared in Figure 5(b) with those computed by the DFT approach taking $t_f = 10T$. All other parameters are the same as before. For systems with $\zeta = 0.05$, the DFT solution in this case is in reasonable agreement with the exact solution over the entire range of frequencies. The errors are still quite large, however, for systems with $\zeta = 0.01$, particularly at selected frequencies in the low-frequency region of the response spectrum.
The effect on peak response of the duration of the appended band of zeros, $t_f$, is demonstrated in Tables I and II. Listed are the values of $|x_{\text{max}}|/x_{\text{st}}$ computed by the DFT procedure using several different values of $t_f/T$ for systems subjected to the sinusoidal force pulse considered previously. Also listed are the corresponding values of $|\ddot{x}_{\text{max}}|/x_{\text{st}}$, the significance of which is identified in the next section. Note that increasing $t_f$ in increments of $T/2$ changes drastically the peak values of the response, the answers oscillating about the exact values. Note further that extremely large values of $t_f$, on the order of 25 to about 200 times the natural period of the system, are required to compute the peak displacement accurately to within four significant figures; the larger values correspond to low-frequency, lightly damped systems.

The sensitivity of $|x_{\text{max}}|/x_{\text{st}}$ to variations in the parameter $t_f/T$, just like that due to variations in the frequency parameter, $f_{t_d}$, stem from the fact that the terminal values of the response computed by the DFT procedure are quite sensitive to changes in either of these parameters.

**Index of Accuracy of Solution**

The initial values of the displacement and velocity of the system in the DFT solution, $y(0)$ and $\dot{y}(0)$, will in general be different from the prescribed values, $x(0)$ and $\dot{x}(0)$. The differences

$$
\Delta y(0) = y(0) - x(0)
$$

(13)

$$
\Delta \dot{y}(0) = \dot{y}(0) - \dot{x}(0)
$$

could then be used as indices of the accuracy of the DFT solution.
A preferable single index would be the quantity $|\tilde{x}_{\text{max}}|$, which represents the numerical value of the maximum transient displacement of the system induced by the unsatisfied initial conditions. The smaller this displacement is in comparison to $|x_{\text{max}}|$, the better is the accuracy of the DFT solution, the value of $|\tilde{x}_{\text{max}}|$ tending to zero for the exact solution. These statements are confirmed by the solutions presented in Tables I and II.

The displacement $|\tilde{x}_{\text{max}}|$ may be determined from

$$|\tilde{x}_{\text{max}}| = e^{-\zeta \alpha} \sqrt{[\Delta y(0)]^2 + 2\zeta \Delta y(0) \frac{\Delta y(0)}{p} + \left[\frac{\Delta y(0)}{p}\right]^2} \quad (14a)$$

in which

$$\alpha = \frac{1}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta + \frac{p \Delta y(0)}{\Delta y(0)}}\right) + n\pi \quad (14b)$$

and $n = 0$ if the first term of the right-hand member of equation (14b) is positive; otherwise $n = 1$. Further, the value of $\dot{y}(0)$ needed to compute $\Delta \dot{y}(0)$ may conveniently be determined from

$$\dot{y}(0) = -\frac{4\pi}{t_0} \sum_{r=0}^{N/2} r \text{Im}(Y_r) \quad (15)$$

in which $\text{Im}$ denotes the imaginary part of the quantity that follows. Equation (15) is obtained from equation (10) by differentiation, making use of the facts that $\omega_r = r \omega = 2\pi r/t_0$ and that the values of $Y_r$ for $r > N/2$ are the complex conjugates of those for $0 < r < N/2$. 
PROPOSED IMPROVEMENT

In the proposed procedure, the period of the periodic extension of the forcing function is taken equal to the duration, $t_0$, over which the response of the system is actually desired; the steady-state response of the system is computed only within this period; and the desired transient response is obtained from the steady-state response by the superposition of a simple, corrective solution.

Fundamental Concept

Consider the forcing function shown by the solid line in Figure 6(a), and let $t_0$ be the time interval over which the displacement of the system, $x(t)$, is to be evaluated. Consider next its periodic extension, shown by the dashed line, and let $y(t)$ be the steady-state periodic displacement of the system. The period of the latter displacement is naturally $t_0$. Possible histories of $x(t)$ and $y(t)$ are shown in Figures 6(b) and 6(c), and they are compared in Figure 6(d) with the origin of $y(t)$ taken at $t=0$. The initial values of $x(t)$ in these plots are considered to be different from zero in the interest of generality.

Since the forcing function over the period $t_0$ for both the transient and steady-state responses is the same, the difference in the two solutions must stem from differences in the initial states of the two motions. Accordingly, if one response history is known, the other may be determined by the superposition of a corrective, free vibrational solution which ensures that the initial state of the desired motion conforms to the desired state.$^7$ In particular,
if the steady-state displacement, \( y(t) \), has been determined by the DFT approach — which can be done quite efficiently — then the transient displacement, \( x(t) \), may be determined from

\[
x(t) = y(t) + \xi(t)
\]

(16)
in which \( \xi(t) \) is the corrective displacement representing the effect of the unsatisfied initial conditions.

**Computation of Corrective Solution**

Two different schemes may be used\(^7,8\) to compute \( \xi(t) \). The first involves the use of transient response concepts, whereas the second utilizes steady-state response concepts. In the first scheme, \( \xi(t) \) is expressed as

\[
\xi(t) = a g(t) + b h(t)
\]

(17)
in which \( g(t) \) and \( h(t) \) are the well-known unit response functions\(^2\) representing the displacement at time \( t \) induced by a unit initial displacement and a unit initial velocity, respectively; and \( a \) and \( b \) are constants that must be determined from the prescribed initial conditions so that

\[
\begin{align*}
x(0) &= y(0) + \xi(0) \\
\dot{x}(0) &= \dot{y}(0) + \dot{\xi}(0)
\end{align*}
\]

(18)

For completeness, the expressions for \( g(t) \) and \( h(t) \) are presented in Appendix I.

On substituting into equation (18) the values of \( \xi(0) \) and \( \dot{\xi}(0) \) determined from equations (17) and its first derivative, and making use of the fact that \( g(0) = \dot{h}(0) = 1 \) and \( g(0) = h(0) = 0 \), one obtains
a = x(0) - y(0)

b = \dot{x}(0) - \dot{y}(0)

(19)

The corrective solution may also be expressed as

\[ \xi(t) = \tilde{a} \tilde{g}(t) + \tilde{b} \tilde{h}(t) \]  

(20)

in which \( \tilde{g}(t) \) represents the steady-state displacement of the system at any time \( 0 < t < t_0 \) due to a periodic set of unit displacement changes applied at intervals \( t_0 \), and \( \tilde{h}(t) \) represents the corresponding displacement due to a periodic set of similarly spaced unit velocity changes.

The expressions for \( \tilde{g}(t) \) and \( \tilde{h}(t) \) are given in Appendix I. Note that by contrast to the transient functions, \( g(t) \) and \( h(t) \), which depend only on the dimensionless measure of time, \( pt \), and the damping of the system, \( \zeta \), the functions \( \tilde{g}(t) \) and \( \tilde{h}(t) \) depend, in addition, on the excitation period, \( t_0 \). A representative set of such functions is shown in Figure 7. These plots are for systems with \( \zeta = 0.05 \) and \( ft_0 = 1.9 \).

The initial values of \( \tilde{g}(t) \) and \( \tilde{h}(t) \), like those of \( g(t) \) and \( h(t) \), involve unit discontinuities. However, unlike the discontinuities of \( g(0) \) and \( h(0) \), those of \( \tilde{g}(0) \) and \( \tilde{h}(0) \) do not start from a zero position but satisfy the expressions

\[ \tilde{g}(0) - \tilde{g}(t_0) = 1 \]  

(21)

\[ \tilde{h}(0) - \tilde{h}(t_0) = 1 \]

Furthermore, the initial values of \( \dot{\tilde{g}}(t) \) and \( \dot{\tilde{h}}(t) \), unlike those of
\( \dot{g}(t) \) and \( h(t) \), are generally different from zero. The value \( t = 0 \) in the above expressions and those that follow refers to a time slightly in excess of zero, and \( t = t_0 \) refers to a time slightly less than \( t_0 \).

The quantities \( \tilde{a} \) and \( \tilde{b} \) in equation (20) are constants similar to those in equation (17), and must be determined from the solution of the following system of algebraic equations, obtained from equation (18) by making use of equation (20) and its first derivative:

\[
\begin{bmatrix}
\dot{g}(0) & \dot{h}(0) \\
\ddot{g}(0) & \ddot{h}(0)
\end{bmatrix}
\begin{bmatrix}
\tilde{a} \\
\tilde{b}
\end{bmatrix}
= 
\begin{bmatrix}
x(0) - y(0) \\
\dot{x}(0) - \dot{y}(0)
\end{bmatrix}
\]  
(22)

The values of \( \dot{g}(0) \) and \( \dot{h}(0) \) in the latter expression may be determined from

\[
\dot{g}(0) = -p^2 \ddot{h}(0)
\]  
(23)

and

\[
\dot{h}(0) = \ddot{g}(0) - 2\zeta p \ddot{h}(0)
\]  
(24)

and the value of \( \dot{y}(0) \) may be determined from equation (15). Equations (23) and (24) are special cases of more general expressions presented in Reference 8.

APPLICATION OF PROCEDURE

The two versions of the proposed procedure are illustrated by reference to a system with \( \zeta = 0.05 \) subjected to the sinusoidal force pulse considered previously. The system is presumed to be initially at rest, and its natural frequency is considered to be such that \( ft_d = 1.4 \). The response is evaluated for a period, \( t_o \), that exceeds
the duration of the forcing function by one-half the natural period of the system. Thus \( t_0 = t_d + 0.5T \) and \( ft_0 = ft_d + 0.5 = 1.9 \), the value for which the functions \( \ddot{g}(t) \) and \( \ddot{h}(t) \) are presented in Figure 7. The sampling interval is taken as \( \Delta t = 0.02 t_d \).

Obtained by the classical DFT procedure, the steady-state displacement of the system is shown by the dashed line in Figure 8. The initial value of this displacement is \( y(0) = 2.483 x_{st} \), and the value of the corresponding velocity, determined from equation (15), is \( \dot{y}(0) = -1.758 px_{st} \). The values of \( a \) and \( b \) in equation (17) are then determined from equation (19) to be

\[
\begin{align*}
a &= -2.483 x_{st} \\
b &= 1.758 px_{st}
\end{align*}
\]

Similarly, the following values are determined from equation (22) for the constants \( \tilde{a} \) and \( \tilde{b} \) in equation (20), making use of the fact that \( \ddot{g}(0) = 1.286 \) and \( \ddot{h}(0) = -0.784 \), and determining the values of \( \ddot{g}(0) \) and \( \ddot{h}(0) \) from equations (23) and (24):

\[
\begin{align*}
\tilde{a} &= -0.849 x_{st} \\
\tilde{b} &= 1.775 px_{st}
\end{align*}
\]

The corrective displacement, \( \xi(t) \), which may be determined either from equation (17) or from equation (20), is shown by the dotted line in Figure 8, and the desired transient displacement, \( x(t) \), which is determined from equation (16), is shown by the solid line.

Similar data are presented in Figure 9 for a system with \( f = 1.05 \) cps and \( \zeta = 0.02 \) subjected to the first 6.3 sec. of the N-S component of the El Centro, California earthquake record of May, 1940, as reported in Reference 9. The sampling interval in
this solution was taken as $\Delta t = 0.01$ sec., and the response was evaluated for a total period of $t_0 = 6.78$ sec., which exceeds the duration of the excitation by one-half the natural period of the system. The deformation of the system, $x(t)$, is expressed in terms of the pseudovelocity function, $px(t)$, normalized with respect to the maximum ground velocity, $\dot{x}_g = 14.02$ in/sec(35.61 cm/sec). The function $y(t)$ in Figure 9 represents the deformation computed classical DFT procedure (i.e., the steady-state deformation of the system to a periodic extension of the base motion), and $\xi(t)$ represents the corrective deformation that transforms $y(t)$ to the desired $x(t)$.

EFFICIENCY OF PROCEDURE

Response spectra for the maximum deformation of systems subjected to the two inputs considered previously were evaluated both by the proposed procedure and the classical DFT approach using a value of $t_f = T/2$, and the results are compared in Figures 10 and 11.

As would be expected from the material already presented, the differences between the two sets of solutions are quite large, particularly for systems with low frequencies and small amounts of damping. It is particularly noteworthy that the errors for the earthquake input are significant not only in the low-frequency region of the response spectrum but in the medium-frequency region as well. The results obtained by the proposed procedure are exact for the discretized forcing functions considered.

Further insight into the problem may be gained from Table III, in which the exact solution for systems with $\zeta = 0.02$ and
f = 1.05 cps is compared with the results obtained by the classical DFT procedure using several different values of $t_f$ in the range between one-half and thirty times the natural period of the system. Also listed are dimensionless measures of the maximum free vibrational deformation, $|\ddot{x}_{\text{max}}|$, defined by equation (14). Note that to compute $|x_{\text{max}}|$ correctly to within three significant figures by the classical DFT procedure, the duration of the appended band of zeros in this case must be $t_f = 30T = 30(1/1.05) = 28.57$ sec., or 4.5 times the duration of the earthquake record considered. Note further that the values of $|\ddot{x}_{\text{max}}|$ relative to those of $|x_{\text{max}}|$ provide a reliable simple indicator of the accuracy of the classical DFT solution.

EXTENSION OF PROCEDURE

In extending the proposed procedure to multi-degree-of-freedom, viscously damped systems, a distinction must be made between systems for which the classical modal superposition method is applicable, and nonclassically damped systems with either constant or frequency-dependent parameters.

For classically damped systems, the modal components of the steady-state response may first be computed by application of the classical DFT procedure; each modal component of the steady-state response may then be transformed to its corresponding transient response by the superposition of a corrective solution; and the desired transient response may be obtained as a linear combination of the component transient responses.
The corrective solution for each modal response component may be determined either from equation (17) by use of the transient unit response functions, \( g(t) \) and \( h(t) \), or from equation (20) by use of their steady-state counterparts, \( \bar{g}(t) \) and \( \bar{h}(t) \). As is true of SDF systems, the first option is preferable for classically damped systems because the expressions for the functions \( g(t) \) and \( h(t) \) are simpler than those for \( \bar{g}(t) \) and \( \bar{h}(t) \). The details of the procedure involving the use of the functions \( g(t) \) and \( h(t) \) have been reported in Reference 7.

The procedure involving the use of the steady-state response functions is advantageous for the analysis of nonclassically damped systems, particularly systems with frequency-dependent parameters of the type encountered in studies of soil-structure interaction. The family of unit response functions needed to transform the steady-state response of such a system to the desired transient response may be evaluated efficiently in this case by the classical DFT approach, using as excitation period the time interval, \( t_0^* \), over which the response of the system is actually desired. By contrast, the computation of the corresponding family of transient response functions is fraught with the same difficulties as those involved in the application of the classical DFT procedure to the analysis of the transient response of force-excited systems. Specifically, to compute these functions accurately it would generally be necessary to use as the excitation period a value \( t_0^* \) which is much greater than \( t_0 \).
CONCLUSIONS

The nature and magnitude of the errors that may result in analyses of the dynamic transient response of linear systems by the Discrete Fourier Transform method have been identified, and two versions of a procedure have been presented which dramatically improve the efficiency of this approach. The first version is ideally suited to the analysis of classically damped systems, whereas the second is particularly efficient for the analysis of nonclassically damped systems, with either constant or frequency-dependent parameters. The basic concepts involved have been introduced by reference to single-degree-of-freedom systems, and comprehensive numerical solutions have been presented which illustrate the basic concepts involved and demonstrate the superiority of the proposed procedure over the classical approach.

It has been shown that, unless the duration of the band of zeros appended to the end of the forcing function is a large multiple of the natural period of the system under study, the classical DFT procedure may lead to significant errors, particularly for lightly damped systems. In the proposed modifications, the excitation period considered is the same as the period over which the response of the system is actually desired.
APPENDIX I.- RESPONSE FUNCTIONS TO UNIT EXCITATIONS

The transient response functions, $g(t)$ and $h(t)$, are given by

$$g(t) = e^{-\zeta \omega t} \left( \cos \tilde{\omega} t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \tilde{\omega} t \right)$$

(A1)

and

$$h(t) = \frac{1}{p} e^{-\zeta \omega t} \sin \tilde{\omega} t$$

(A2)

in which $\tilde{\omega} = \text{the damped circular natural frequency of the system, defined by}$

$$\tilde{\omega} = p \sqrt{1-\zeta^2}$$

(A3)

Similarly, the steady-state response functions, $\tilde{g}(t)$ and $\tilde{h}(t)$, are given by

$$\tilde{g}(t) = \frac{e^{-\zeta \omega t}}{\Delta} \left\{ \left[ 1 - e^{-\zeta \omega t} \left( \cos \tilde{\omega} t_0 - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \tilde{\omega} t_0 \right) \right] \cos \tilde{\omega} t \right. \\
+ \left[ \frac{\zeta}{\sqrt{1-\zeta^2}} - e^{-\zeta \omega t} \left( \frac{\zeta}{\sqrt{1-\zeta^2}} \cos \tilde{\omega} t_0 + \sin \tilde{\omega} t_0 \right) \right] \sin \tilde{\omega} t \right\}$$

(A4)

and

$$\tilde{h}(t) = \frac{e^{-\zeta \omega t}}{\tilde{p} \Delta} \left[ \left( e^{-\zeta \omega t} \sin \tilde{\omega} t_0 \right) \cos \tilde{\omega} t \right. \\
+ \left( 1 - e^{-\zeta \omega t} \cos \tilde{\omega} t_0 \right) \sin \tilde{\omega} t \right]$$

(A5)

in which $\Delta = 1 + e^{-2\zeta \omega t_0} - 2 e^{-\zeta \omega t_0} \cos \tilde{\omega} t_0$

(A6)

Equations A4 and A5 are valid for $0 \leq t \leq t_0$. 


APPENDIX II - NOTATION

The following symbols are used in this chapter:

\( a, b \) = constants in Eq. 17, defined by Eq. 19;

\( \bar{a}, \bar{b} \) = constants in Eq. 20, defined by Eq. 22;

\( c \) = coefficient of viscous damping for system;

\( \text{DFT} \) = Discrete Fourier Transform;

\( F_r \) = complex-valued amplitude of rth harmonic in a Fourier series representation of discretized forcing functions;

\( f \) = \( 1/T \) = natural frequency of system;

\( \text{FFT} \) = Fast Fourier Transform;

\( g(t) \) = displacement of system due to a unit displacement change;

\( \bar{g}(t) \) = steady-state displacement of system due to a periodic set of unit displacement changes applied at intervals \( t_0 \);

\( h(t) \) = displacement of system due to a unit initial velocity change;

\( \bar{h}(t) \) = steady-state displacement of system due to a periodic set of unit velocity changes applied at intervals \( t_0 \);

\( H(\omega) \) = complex-valued frequency response function of system;

\( H_r \) = \( H(\omega_r) \) = value of \( H(\omega) \) for \( \omega = \omega_r \);

\( i \) = \( \sqrt{-1} \);

\( k \) = stiffness of system;

\( m \) = mass of system;

\( n \) = variable integer;

\( N \) = number of time or frequency intervals considered;

\( p, \bar{p} \) = circular natural frequency of system without and with damping, respectively;

\( P(t) \) = instantaneous value of exciting force;

\( P_n \) = \( P(t_n) \) = value of \( P(t) \) at \( t_n \);

\( P \) = peak value of \( P(t) \);

\( r \) = variable integer;

\( \text{SDF} \) = single-degree-of-freedom system;
\( T \) = natural period of system;
\( t \) = time;
\( t_d \) = duration of excitation;
\( t_f \) = duration of free vibration or of appended band of zeros;
\( t_n \) = \( nA \bar{t} \) discretized time;
\( t_o \) = period for which the response of the system is desired;
\( t_o^* \) = \( t_d + t_f \) augmented period of excitation;
\( x(t) \) = transient displacement or deformation of system;
\( |x_{\text{max}}| \) = absolute maximum value of \( x(t) \);
\( |\ddot{x}_{\text{max}}| \) = absolute maximum displacement of a system subjected to an initial displacement, \( \Delta y(0) \), and an initial velocity, \( \Delta \dot{y}(0) \);
\( x_n \) = \( x(t_n) \) value of \( x(t) \) at \( t_n \);
\( x_{st} \) = \( P/k \) static displacement induced by \( P \);
\( Y_r \) = \( H_r F_r \) complex-valued amplitude of \( r \)th harmonic component of response;
\( y(t) \) = steady-state displacement of system;
\( y_n \) = \( y(t_n) \) value of \( y(t) \) at \( t_n \);
\( \Delta t \) = \( t_o/N \) time increment used to discretize forcing function and response;
\( \Delta y(0) \) = unsatisfied initial displacement, defined by Eq. 13;
\( \Delta \dot{y}(0) \) = unsatisfied initial velocity, defined by Eq. 13;
\( \Delta \omega \) = \( 2\pi/t_o = 2\pi/(N\Delta t) \) frequency increment;
\( \zeta \) = \( c/(2mp) \) fraction of critical damping;
\( \xi(t) \) = corrective displacement or deformation;
\( \omega \) = circular frequency of a harmonic component of excitation or response;
\( \omega_{\text{max}} \) = \( N\Delta \omega/2 = \pi/\Delta t \) Nyquist or folding frequency;
\( \omega_r \) = \( r\Delta \omega \) sample of \( \omega \).
REFERENCES


TABLE I. Effect of $t_f$ on Maximum Displacements of Systems with $\zeta = 0.01$ Subjected to Sinusoidal Force Pulse

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</tr>
<tr>
<td>20</td>
<td>1.013</td>
<td>0.288</td>
<td>4.156</td>
<td>1.183</td>
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<tr>
<td>50</td>
<td>1.240</td>
<td>0.054</td>
<td>3.174</td>
<td>0.137</td>
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<tr>
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<td>1.290</td>
<td>0.002</td>
<td>3.050</td>
<td>0.006</td>
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<tr>
<td>200</td>
<td>1.292</td>
<td>0.000</td>
<td>3.045</td>
<td>0.000</td>
</tr>
</tbody>
</table>

(a) DFT Solution

(b) Exact Solution

0.5  1.292  3.045  2.769  1.707
TABLE II. Effect of $t_f$ on Maximum Displacements of Systems with $\zeta = 0.05$ Subjected to Sinusoidal Force Pulse

<table>
<thead>
<tr>
<th>$t_f$</th>
<th>$f_{d} = 0.5$</th>
<th>$f_{d} = 1$</th>
<th>$f_{d} = 1.5$</th>
<th>$f_{d} = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{</td>
<td>x_{\max}</td>
<td>}{x_{st}}$</td>
<td>$\frac{</td>
</tr>
<tr>
<td>0.5</td>
<td>4.267</td>
<td>3.624</td>
<td>1.662</td>
<td>1.214</td>
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<td>0.704</td>
<td>0.515</td>
<td>5.780</td>
<td>4.220</td>
</tr>
<tr>
<td>1.5</td>
<td>2.447</td>
<td>1.529</td>
<td>1.854</td>
<td>0.989</td>
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<td>0.786</td>
<td>0.419</td>
<td>4.419</td>
<td>2.355</td>
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<td>1.871</td>
<td>0.854</td>
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<td>0.202</td>
<td>3.181</td>
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<td>0.239</td>
<td>2.385</td>
<td>0.424</td>
</tr>
<tr>
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<td>0.012</td>
<td>0.154</td>
<td>3.028</td>
<td>0.460</td>
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<td>6.5</td>
<td>1.284</td>
<td>0.167</td>
<td>2.461</td>
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<td>2.930</td>
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<td>2.700</td>
<td>0.001</td>
</tr>
<tr>
<td>30</td>
<td>1.143</td>
<td>0.000</td>
<td>2.699</td>
<td>0.000</td>
</tr>
</tbody>
</table>

(a) DFT Solution

(b) Exact Solution

0.5  1.143  2.699  2.471  1.620
TABLE III. Comparison of Solutions Obtained by Classical DFT Approach and Proposed Improvement for Systems
with $f = 1.05$ cps. and $\zeta = 0.02$ Subjected to Earthquake Motion; $t_d = 6.3$ sec., $\Delta t = 0.01$

| $t_f$ \(T\) | $t^*_{0}$ \(\text{sec}\) | $p|x_{\text{max}}| \dot{x}_g$ | $p|\ddot{x}_{\text{max}}| \ddot{x}_g$ |
|-----------|-----------------|------------------|------------------|
| (1)       | (2)             | (3)              | (4)              |
| 0.5       | 6.78            | 3.047            | 2.655            |
| 1         | 7.25            | 2.543            | 1.389            |
| 1.5       | 7.73            | 2.955            | 2.267            |
| 2         | 8.20            | 2.563            | 1.263            |
| 2.5       | 8.68            | 2.883            | 1.940            |
| 5         | 11.06           | 2.624            | 0.932            |
| 5.5       | 11.54           | 2.858            | 1.235            |
| 6         | 12.01           | 2.640            | 0.839            |
| 6.5       | 12.49           | 2.845            | 1.068            |
| 7         | 12.97           | 2.652            | 0.753            |
| 10        | 15.82           | 2.680            | 0.538            |
| 15        | 20.59           | 2.713            | 0.300            |
| 20        | 25.35           | 2.733            | 0.164            |
| 25        | 30.11           | 2.743            | 0.088            |
| 30        | 34.87           | 2.748            | 0.048            |

(b) Proposed and Exact Solutions

| 0.5 | 6.78 | 2.754 |
FIGURE 1. - Forcing Function and Its Discretized Version
FIGURE 2 - Actual and Augmented Forcing Functions and Desired Response
FIGURE 3.1 - Responses Computed by DFT Method using Different Values of $t_r/t_d$; Systems with $\zeta = 0.05$
FIGURE 4. - DFTs of Sinusoidal Force Pulse with Different Values of $t_r / t_d$ and Corresponding Values of $H(\omega_r)$ for Systems with $\zeta = 0.05$ and $ft_d = 0.5$.
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FIGURE 7. - Periodic Response Functions, $\tilde{g}(t)$ and $\tilde{p}(t)$, for Systems with $\zeta = 0.05$ and $f t_0 = 1.9$
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FIGURE 11. - Comparison of Response Spectra Computed by Standard DFT Method and Proposed Method Using $t_f = T/2$; Systems with $\zeta = 0.02$ Subjected to Earthquake Record.
CHAPTER VI

IMPROVED DFT METHOD OF DYNAMIC ANALYSIS OF NONCLASSICALLY DAMPED LINEAR SYSTEMS
INTRODUCTION

In the preceding chapter two versions of a procedure for improving the effectiveness of the classical Discrete Fourier Transform (DFT) method of dynamic analysis of single-degree-of-freedom (SDF) systems were discussed. It was also stated that the procedure can be extended to the analysis of nonclassically damped multi-degree-of-freedom (MDF) systems, including systems with frequency-dependent parameters of the type encountered in studies of soil-structure and fluid-structure interaction. In such cases, neither the standard modal superposition method, nor its generalized extension described in Chapter II can be used efficiently, and the DFT method becomes a desirable alternative.

For the evaluation of the transient response of systems to nonperiodic excitations, however, the classical DFT method may entail considerable amount of computational effort, and it may lead to significant inaccuracies if it is not properly implemented. The purpose of this chapter is to extend the applicability of the techniques described in the preceding chapter to the analysis of nonclassically damped systems. Special attention is given to the version of the procedure that makes use of the steady-state concepts. This version is effective both for constant-parameter systems and systems with frequency-dependent parameters.

The method is developed and illustrated by reference to constant-parameter systems excited at the base, and its extension to systems with frequency-dependent parameters is also indicated. The superiority of the proposed method over the classical DFT approach is illustrated by selected numerical solutions.
STATEMENT OF PROBLEM

Consider a viscously damped, linear system of \( n \) degrees of freedom subjected to the arbitrary base acceleration, \( \ddot{x}_g(t) \), and let \( t_d \) be the duration of the excitation and \( t_o \) be the interval of time during which the transient response of the system is to be evaluated. The system is governed by the differential equations:

\[
[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = -[m]\{1\}\ddot{x}_g(t)
\]  \hspace{1cm} (1)

in which \([m]\), \([c]\) and \([k]\) are the mass matrix, damping matrix and stiffness matrix of the system, respectively; \(\{x(t)\}\) is a vector of the displacements of the nodes relative to the moving base, a dot superscript denotes differentiation respect to time, \(t\), and \([1]\) is a vector of ones.

Using the classical DFT method of analysis, or the improved version proposed herein, it is desired to evaluate the resulting transient displacements of the system, \(\{x(t)\}\). Of special interest are the numerically largest values of the displacements that may occur during that interval. Since the maximum displacements may occur either during the application of the excitation or after cessation of it, a value of \(t_o > t_d\) must be considered. In this study, the response of the system will be determined for a period \(t_o\) that exceeds the duration of the excitation by at least the fundamental undamped natural period of the system, \(T^0_1\); that is, \(t_o \geq t_d + T^0_1\).

RESPONSE BY CLASSICAL DFT METHOD

Let the exciting base motion defined in the interval zero to
\( t_0 \) be sampled at \( N \) equally spaced points, and let \( \Delta t \) be the sampling interval, thus \( t_0 = N \Delta t \). Further, let \( \ddot{x}_{gr} = \ddot{x}_g(t_r) \) be the value of \( \ddot{x}_g(t) \) at \( t = r \Delta t \), and \( \{x(t_r)\} \) be the corresponding displacements of the system.

Making use of the sequence of values \( \ddot{x}_{gr} \), one can evaluate the DFT of the excitation. If \( G_s \) defines the \( s \)th complex-valued harmonic component of a Discrete Fourier Series expansion of the discretized excitation, then

\[
G_s = \frac{1}{N} \sum_{r=0}^{N-1} \ddot{x}_{gr} e^{-i \omega_s t_r \Delta t} = \frac{1}{N} \sum_{r=0}^{N-1} \ddot{x}_{gr} e^{-i (2\pi rs/N)} \quad \text{for } 0 \leq s \leq N-1 \tag{2}
\]

in which \( i = \sqrt{-1} \), \( r \) and \( s \) are positive integers, and \( \omega_s = s \Delta \omega \) is the circular frequency of the \( s \)th harmonic such that

\[
\Delta \omega = \frac{2\pi}{t_0} = \frac{2\pi}{N} \frac{1}{\Delta t} \tag{3}
\]

The sequence of \( G_s \) values represents the DFT of the discretized base acceleration, \( \ddot{x}_{gr} \). Since only positive frequencies are considered in equation (2), the values of \( G_s \) on either side of \( \omega_{N/2} \) are the complex conjugates of each other, and \( \omega_{N/2} \) is the Nyquist or folding frequency that defines the value of the frequency of the highest participating harmonic (see also Chapter V, pages 131 to 133).

The steady-state displacements of the system to the \( s \)th harmonic of the excitation, \( \{y_s(t)\} \), are given by

\[
\{y_s(t)\} = \{Y_s\} e^{i \omega_s t} \tag{4}
\]

in which \( \{Y_s\} \) is a complex-valued vector that can be determined from the
solution of equation (1). On taking \( \ddot{x}_g(t) = G_s e^{i\omega_s t} \) and interpreting \( \{x(t)\} \) as \( \{y_s(t)\} \), one obtains

\[
\{Y_s\} = -[H(\omega_s)] [m] [I] G_s \tag{5}
\]

where

\[
[H(\omega_s)] = \left[ [k] - \omega_s^2 [m] + i\omega_s [c] \right]^{-1} \tag{6}
\]

and the \(-1\) superscript denotes the inverse of the matrix. The matrix \([H(\omega_s)]\) will be referred to as the complex-valued dynamic flexibility matrix of the system. The element on the \(i\)th row and \(j\)th column of this matrix defines the amplitude and phase angle of the steady-state response of the \(i\)th node due to a harmonic force of unit amplitude and circular frequency \(\omega_s\) applied at the \(j\)th node.

In the classical DFT approach, one evaluates equations (6) and (5) for each different value of \(\omega_s\). However, since the values of \(\omega_s\) for \(N/2 < s \leq N-1\) are the folded images of those of \(\omega_s\) for \(0 < s \leq N/2\); i.e.,

\[
\omega_s = \begin{cases} 
s \Delta \omega & \text{for } 0 < s \leq N/2 \\
-(N-s) \Delta \omega & \text{for } N/2 < s \leq N-1 \end{cases} \tag{7}
\]

it is necessary to evaluate equations (6) and (5) only for values of \(s\) in the range zero to \(N/2\). The values of \(\{Y_s\}\) for \(N/2 < s \leq N-1\) are the associated complex conjugates, i.e.,

\[
\{Y_s\} = \{Y_{N-s}^*\} \quad \text{for } N/2 < s \leq N-1 \tag{8}
\]

In this expression the \(*\) superscript denotes complex conjugate. The
complete response of the system is then obtained by superposition
of the component responses, i.e.,

\[
\{y(t_r)\} = \sum_{s=0}^{N-1} \{y_s(t_r)\}
\]  

(9)

On substituting equation (4) in this equation one obtains

\[
\{y(t_r)\} = \sum_{s=0}^{N-1} \{Y_s\} e^{i\omega_s t_r} = \sum_{s=0}^{N-1} \{Y_s\} e^{i(2\pi rs/N)}
\]  

(10)

The sequence of values \(\{y(t_r)\}\) is defined as the Inverse DFT of \(\{Y_s\}\).

As indicated in Chapter V, the displacements determined from
equation (10) define the steady-state displacements of the system
to a periodic extension of the excitation rather than the desired
transient displacements, \(\{x(t_r)\}\). To ensure that the results obtained
from equation (10) are reasonable approximations of the desired
transient response values, the period of definition of the excitation,
\(t_o\), should be augmented by the addition of a significantly long band
of zeros at the end, such that the response of the system at the end
of the extended period is negligible and there is no substantial
difference between the transient and steady-state responses.

Increasing the period \(t_o\) naturally increases the number of points
that must be considered to define the response of the system, and
this, in turn, increases the number of operations required to evaluate
equations (2) and (10). Inasmuch as equations (2) and (10) can be
efficiently evaluated by use of the FFT algorithm, an increase in
the number of terms is not the cause of a major increase in the
computational work. The major increase is related to the evaluation
of equation (6), particularly for systems having many degrees of
freedom. Recall that to evaluate equation (6) it is necessary to invert a complex-valued matrix of size \( n \) by \( n \), and that this operation has to be performed for each frequency component considered.

With the technique presented in the following section the transient response of the system is determined from the corresponding steady-state response without having to increase the period \( t_0 \).

PROPOSED IMPROVEMENT

The fundamental concept for interrelating the steady-state and the transient responses has been presented in Reference 1 and discussed in the preceding chapter.

The transient displacements of the system, \( \{x(t)\} \), may be related to their steady-state counterparts, \( \{y(t)\} \) by the equations

\[
\{x(t)\} = \{y(t)\} + \{z(t)\} \tag{11}
\]

in which \( \{z(t)\} \) is a corrective free vibrational solution which ensures that the initial conditions of the transient response conform to the prescribed conditions.

It has been shown in the preceding chapter that the corrective displacements evaluated by the DFT method can be computed more effectively using steady-state concepts rather than transient response concepts. In extending the same idea to MDF systems it is necessary to define the counterparts of the functions \( \tilde{g}(t) \) and \( \tilde{h}(t) \) developed for SDF systems in Reference 2 and given in the preceding chapter. To this end, let the matrices \( [\tilde{g}(t)] \) and \( [\tilde{h}(t)] \) represent families of periodic functions such that the element \( \tilde{g}_{ij}(t) \) is the element of the \( i \)th row and \( j \)th
column of \([\ddot{g}(t)]\) defines the steady-state response at any time \(0 < t < t_o\) of the \(i\)th node due to unit displacement changes applied periodically at intervals \(t_o\) to the \(j\)th node. Similarly, the element of the \(i\)th row and \(j\)th column of the matrix \([\ddot{h}(t)]\) defines the steady-state response of the \(i\)th node due to unit velocity changes applied periodically at the \(j\)th node. It should be clear that the functions along the main diagonal of \([\ddot{g}(t)]\) and the first derivatives of the functions along the main diagonal of \([\ddot{h}(t)]\) are discontinuous at \(t = 0\) and at \(t = t_o\), the total discontinuity being unity in each case. Specifically,

\[
[\ddot{g}(0)] - [\ddot{g}(t_o)] = [I] \tag{12a}
\]

and

\[
[\ddot{h}(0)] - [\ddot{h}(t_o)] = [I] \tag{12b}
\]

in which \([I]\) is the unit matrix. The values \(t = 0\) and \(t = t_o\) in the above expressions and those that follow refer to times slightly greater than zero and smaller than \(t_o\), respectively. The evaluation of the matrices \([\ddot{g}(t)]\) and \([\ddot{h}(t)]\) is considered later.

The corrective solution, \(\{z(t)\}\), in the interval zero to \(t_o\) may be expressed as a linear combination of \([\ddot{g}(t)]\) and \([\ddot{h}(t)]\) as

\[
\{z(t)\} = [\ddot{g}(t)]\{a\} + [\ddot{h}(t)]\{b\} \tag{13b}
\]

in which \(\{a\}\) and \(\{b\}\) are vectors of constants with units of displacement and velocity, respectively. These vectors may be determined from equation (11) by requiring that
\[
\{x(0)\} = \{y(0)\} + \{z(0)\} \quad (14a)
\]
\[
\{\hat{x}(0)\} = \{\hat{y}(0)\} + \{\hat{z}(0)\} \quad (14b)
\]

and determining \{z(0)\} and \{\hat{z}(0)\} from equation (13) and its derivative. This leads to the following system of simultaneous equations:

\[
\begin{bmatrix}
[\bar{g}(0)] & [\bar{h}(0)] \\
[\hat{g}(0)] & [\hat{h}(0)]
\end{bmatrix}
\begin{bmatrix}
\{a\} \\
\{b\}
\end{bmatrix}
= 
\begin{bmatrix}
\{x(0)\} - \{y(0)\} \\
\{\hat{x}(0)\} - \{\hat{y}(0)\}
\end{bmatrix}
\quad (15)
\]

With the values of \{a\} and \{b\} established, the corrective displacements, \{z(t)\}, may be determined from equation (13), and the desired \{x(t)\} may be determined from equation (11).

**Evaluation of \([\bar{g}(t)]\) and \([\bar{h}(t)]\)**

It can be shown (see Appendix I) that the families of the periodic response functions \([\bar{g}(t)]\) and \([\bar{h}(t)]\) at any time \(0 < t < t_0\) can be evaluated from

\[
[\bar{g}(t)] = \left(\frac{1}{2} - \frac{t}{t_0}\right)[I] + \frac{1}{t_0} \left([H(0)][c] + \sum_{j=-M}^{M} \frac{i}{\omega_j} [H(\omega_j)][k] e^{i\omega_j t}\right) \quad (16)
\]

and

\[
[\bar{h}(t)] = \frac{1}{t_0} \sum_{j=-M}^{M} [H(\omega_j)][m] e^{i\omega_j t} \quad (17)
\]

in which \(\omega_j = 2\pi j/t_0\); \([H(0)]\) is the value of \([H(\omega_j)]\) at \(j = 0\); and \(M\) is the largest value of \(j\) that must be considered to account for the contribution of all important frequencies.

The series in equation (16) converges quite rapidly by virtue
of the term \( \omega_j \) in the denominator while the series in equation (17) converges less rapidly, especially near the ends of the interval, i.e., \( t = 0 \) and \( t = t_0 \). Since the vectors \{a\} and \{b\} in equation (15) depend on the values of \( \tilde{g}(0) \), \( \tilde{h}(0) \) and their corresponding derivatives, it is desirable to improve the accuracy of computation of \( \tilde{h}(0) \) and \( \dot{\tilde{h}}(0) \). The convergence of equation (17) can greatly be improved by rewriting this equation in the form (see Appendix I for derivation)

\[
\tilde{h}(t) = \frac{t}{12} \left\{ \left( 1 - \frac{t}{t_0} \right)^{5t - 1} \right\} \left[ I + \frac{1}{t_0} \left\{ [H(0)][m] \right. \right.
\]

\[
+ \frac{1}{\omega_j^2} \sum_{j=M}^{M} \hat{H}(\omega_j) \left[ k + \frac{i}{\omega_j} [c] e^{i\omega_j t} \right] \right\} \quad (18)
\]

Because of the presence of the terms \( \omega_j \) and \( \omega_j^2 \) in the denominator of the series, this expression converges much more rapidly than equation (17).

It should be emphasized that equations (16) and (17) or (18) are valid for systems with constant parameters as well as for systems with frequency-dependent parameters. In problems of soil-structure interaction for which the elements of the stiffness matrix, \([k]\), and damping matrix, \([c]\), are frequency-dependent, the matrices \([k]\) and \([c]\) in equations (16) and (18) must be replaced by their frequency-dependent counterparts, \([k(\omega_j)]\) and \([c(\omega_j)]\). The same replacement has to be made in evaluating the complex-valued dynamic flexibility matrix, \([H(\omega_j)]\), from equation (6).
Derivatives of $[\ddot{g}(t)]$ and $[\ddot{h}(t)]$

The derivatives of $[\ddot{g}(t)]$ and $[\ddot{h}(t)]$ can be evaluated from (see Appendix I)

$$[\dot{\ddot{g}}(t)] = -\frac{1}{t_0} \sum_{j=-M}^{M} [H(\omega_j)] [k] e^{i\omega_j t}$$

(19)

and

$$[\dot{\ddot{h}}(t)] = \left(\frac{1}{2} - \frac{t}{t_0}\right) [I] + \frac{1}{t_0} \sum_{j=-M, j\neq 0}^{M} [H(\omega_j)] \left( \frac{1}{\omega_j} [k] - [c] \right) e^{i\omega_j t}$$

(20)

For systems with constant parameters, $[\ddot{g}(t)]$ and $[\ddot{h}(t)]$ can be evaluated more readily from:

$$[\dot{\ddot{g}}(t)] = - [\ddot{h}(t)] [m]^{-1} [k]$$

(21)

and

$$[\dot{\ddot{h}}(t)] = [\ddot{g}(t)] - [\ddot{h}(t)] [m]^{-1} [c]$$

(22)

The derivation of these expressions is also given in Appendix I.

Equations (16) and (18) may also be used to evaluate the families of impulse response functions of the system needed in an extension of the procedure involving the use of transient response concepts. However, to evaluate these functions accurately, a reasonably long period must be considered, which may be several times greater than the period over which the response is desired.

EFFICIENCY OF PROPOSED PROCEDURE

As a measure of the computational efficiency of the proposed procedure, the maximum response of the three-story structure considered
in Figure 1(a) of Chapter II is evaluated for the simple base motion shown in Figure 5 of the same chapter. A wide range of natural frequencies is considered and the results are compared with those obtained by the classical DFT approach as well as with the "exact" solutions obtained by the generalized modal superposition method described in Chapter II.

The sampling interval in these solutions was taken equal to or less than $\Delta t = t_d/10$ or $\Delta t = T_1^0/73$, whichever was smaller. The first condition ensures adequate representation of the excitation, while the second condition ensures adequate definition of the free vibrational components. All DFTs were evaluated using the FFT algorithm of the Fortran IMSL subroutine package$^3$ that permits consideration of an arbitrary value of $N$.

Solutions for maximum floor displacements, $|x_i|_{\text{max}}$, and corresponding story deformations, $|u_i|_{\text{max}}$, of systems with damping factor $\xi = 1$ and three values of the frequency parameter $f_1^0 t_d$ are presented in Table I. The quantity $f_1^0 = 1/T_1^0 = p_1^0/2\pi$ is the fundamental natural frequency of the associated undamped system. The results are presented in the form of pseudovelocity values, $p_1^0 |x_i|_{\text{max}}$ and $p_1^0 |u_i|_{\text{max}}$, and they are nondimensionalized with respect to the maximum value of the base velocity, $\dot{x}_g$. The exact solutions and those computed by the proposed procedure were obtained for a duration of free vibration, $t_f$, equal to the fundamental period of the undamped system; i.e., $t_f = T_1^0$. The solutions by the classical DFT approach were obtained for five different durations of $t_f$ in the range between $T^0_1$ and $5T^0_1$.

The data presented in Table I clearly show the effectiveness
of the proposed procedure. For the number of significant figures used in this table, the exact solutions and those obtained by the proposed procedure are the same. The accuracy of the solutions by the classical DFT procedure depends importantly on the value of $t_f$. The lower the value of the frequency parameter, $f_1^0 t_d$, the greater is generally the value of $t_f$ required to achieve a desired degree of accuracy. Note also that the errors are generally the greatest for the story deformations of the upper part of the structure.

Further insight into the problem may be gained from Figure 1 in which a comparison of the solutions obtained both by the proposed procedure and the classical DFT approach using a value of $t_f = T_1^0$ is presented. The results in this figure are presented in the form of response spectra with the ordinates representing pseudvelocities normalized respect to $\ddot{x}_g$. The maximum differences between the two sets of solutions are obtained for systems with low values of $f_1^0 t_d$, particularly for the story deformations displayed in part (b) of the figure.

CONCLUSIONS

A procedure has been presented for improving the efficiency with which the transient response of multi-degree-of-freedom linear systems may be evaluated by the Discrete Fourier Transform method of analysis.

The concepts have been developed by reference to viscously damped systems with constant parameters, but can also be applied to systems with frequency-dependent parameters.

It has been shown that the responses obtained by the classical
Discrete Fourier Transform method are highly dependent on the duration of the band of zeros appended at the end of the excitation. In the proposed procedure, the period to be considered is the same as the time interval over which the response of the system is actually desired.
APPENDIX I

Derivation of Expressions for $[\ddot{q}(t)]$ and $[\ddot{h}(t)]$

Consider a viscously damped multi-degree-of-freedom system excited by periodic sets of displacement changes $\{\Delta y\}$, and velocity changes $\{\Delta \dot{y}\}$, applied concurrently at intervals $t_0$. Let $\{y\} = \{y(t)\}$ be the resulting steady-state displacements of the system at time $0 < t < t_0$. The homogeneous form of equation (1) can be written in this case as

$$[m]\{\ddot{y}\} + [c]\{\dot{y}\} + [k]\{y\} = \{0\} \tag{A1}$$

in which $\{0\}$ is a null vector.

Consider now a family of periodic functions $\{s\} = \{s(t)\}$ defined in the interval $0 \leq t < t_0$ such that

$$\{s(t)\} = \begin{cases} \frac{1}{2} \left[ (y(0^+)) + (y(0^-)) \right] & \text{for } t = 0 \\ \{y(t)\} & \text{for } 0 < t < t_0 \end{cases} \tag{A2a}$$

in which the plus and minus sign superscripts denote values slightly greater than and less than zero, respectively. The derivatives of $\{s(t)\}$ at any time $0 \leq t < t_0$ may then be expressed as

$$\{\dot{s}(t)\} = \{\Delta y\} \delta(t) + \begin{cases} \frac{1}{2} \left[ (\dot{y}(0^+)) + (\dot{y}(0^-)) \right] & \text{for } t = 0 \\ \{\dot{y}(t)\} & \text{for } 0 < t < t_0 \end{cases} \tag{A2b}$$

and

$$\{\ddot{s}(t)\} = \{\Delta y\} \ddot{\delta}(t) + \{\Delta \dot{y}\} \delta(t) + \{\ddot{y}(t)\} \tag{A2c}$$

in which $\delta(t)$ is the unit impulse function, and $\ddot{\delta}(t)$ is its derivative.
or unit doublet. Upon making use of equations (A2) one can rewrite equation (A1) in the following form

\[ [m]\{\ddot{s}\} + [c]\{\dot{s}\} + [k]\{s\} = \{Q(t)\} \]  

(A3)

in which the vector \(\{Q(t)\}\) represents an equivalent set of periodic lateral forces given by

\[ \{Q(t)\} = ([c]\delta(t) + [m]\dot{\delta}(t))\{\Delta y\} + [m]\{\Delta \dot{y}\}\delta(t) \]  

(A4)

In equation (A3) the effects of the displacement and velocity changes have effectively been represented by a set of impulsive periodic forces. The steady-state response of the system \(\{s(t)\} = \{y(t)\}\) for \(0 < t < t_0\) may then be expressed in the form

\[ \{s(t)\} = \{y(t)\} = \sum_{j=-M}^{M} [H(\omega_j)]\{Q(\omega_j)\}e^{i\omega_j t} \]  

(A5)

in which \([H(\omega_j)]\) is defined by equation (6); \(\omega_j = \frac{2\pi j}{t_0}\);

\[ \{Q(\omega_j)\} = \frac{1}{t_0} \int_{0}^{t_0} \{Q(t)\}e^{-i\omega_j t} dt \]  

(A6)

and \(M\) is an integer representing the largest value of \(j\) that must be considered to account for the contribution of all important frequencies.

On substituting equation (A4) into equation (A6), integrating and substituting the result into equation (A5), one obtains the following expression for \(\{y(t)\}\) within \(0 < t < t_0\):
\[ \{y(t)\} = \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} [H(\omega_j)](c) + i\omega_j[m] e^{i\omega_j t} \right\} \{\Delta y\} \]

\[ + \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} [H(\omega_j)][m] e^{i\omega_j t} \right\} \{\Delta \dot{y}\} \quad (A7) \]

The families of response functions [\(\tilde{g}(t)\)] may be obtained from this equation by letting \(\{\Delta \dot{y}\} = \{0\}\) and each one of the elements of the vector \(\{\Delta y\}\) be equal to one while the rest are equal to zero. Inasmuch as the elements of the main diagonal of \([\tilde{g}(t)]\) involve discontinuities at the ends of the period, the series converges to the mean values of the total discontinuities at the ends rather than to the desired values on either side. Furthermore, the elements of the main diagonal of \([\tilde{g}(t)]\) near the ends of the period are affected by the Gibb's phenomenon, and the overall convergence of the series is slow because of the presence of the factor \(\omega_j\) multiplying the \([m]\) and \([c]\) matrices. These difficulties may be overcome by the following transformation in equation (A7). If one takes \(\{\Delta \dot{y}\} = \{0\}\), and if the term corresponding to \(j = 0\) is extracted from the first series of this equation, and the term \([k]/i\omega_j\) is added and subtracted from the remainder of the series, and use is made of equation (6), one obtains

\[ \{y(t)\} = \left\{ \frac{1}{t_0} \left( [H(0)][c] + \sum_{j=-M}^{M} \frac{i}{\omega_j} [H(\omega_j)][k] e^{i\omega_j t} \right) \right\} \{\Delta y\} \quad (A8) \]

in which \([I]\) is the identity matrix. Noticing that the second series on the right-hand member of this expression can be expressed as
(1/2 - t/t_o), and letting each one of the elements of \{\Delta y\} to be unity while the rest are zero, one obtains the expression for \(\tilde{g}(t)\) given by equation (16).

Finally, the families of response functions \(\tilde{h}(t)\) may be obtained from equation (A7) by letting \(\{\Delta y\} = \{0\}\) and letting each one of the elements of \(\{\Delta \dot{y}\} \) be equal to one while the others are zero. The results for \(\tilde{h}(t)\) are presented as equation (17).

**Improved Expression for \(\tilde{h}(t)\)**

If one extracts the term corresponding to \(j = 0\) from equation (17), and the term \(\left(\frac{k}{w_j^2} + \frac{1}{w_j} [c]\right)\) is added and subtracted from the remainder of the series, and use is made of equation (6), one obtains:

\[
\tilde{h}(t) = \frac{1}{t_o} \left\{ [H(0)][m] + \sum_{j=-M}^{M} [H(w_j)] \left( \frac{1}{2} [k] + \frac{1}{w_j} [c] \right) e^{i w_j t} \right\} \\
- \frac{1}{t_o} [I] \sum_{j=-M}^{M} \frac{1}{w_j^2} e^{i w_j t} \tag{A9}
\]

Finally, the second series of this expression can be expressed as \([1 - t/t_o] 6t/t_o - 1\) \(t_o/12\) to yield equation (18).

**Expressions for \(\tilde{g}(t)\) and \(\tilde{h}(t)\)**

The derivative of \(\{y(t)\}\) at any time \(0 < t < t_o\) is obtained from equation (A2b) as

\[
\{\dot{y}(t)\} = \{\dot{s}(t)\} - \{\Delta y\} \delta(t) \tag{A10}
\]

in which \(\{\dot{s}(t)\}\) can be obtained by differentiation of equation (A7). On substituting the resulting expression into equation (A10) and
expressing the second term on the right-hand side of equation (A10) 
y by its corresponding Fourier series, one obtains 

\[ \{ \dot{y}(t) \} = \frac{1}{t_0} \left\{ \sum_{j=-M}^{M} \left[ H(\omega_j) \right] (i\omega_j[c] - \omega_j^2[m]) e^{i\omega_j t} - \left[ I \right] \sum_{j=-M}^{M} e^{i\omega_j t} \right\} \{ \Delta y \} \]

\[ + \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} i\omega_j[H(\omega_j)][m] e^{i\omega_j t} \right\} \{ \Delta \dot{y} \} \]  

(A11)

This expression may be simplified to the following by: a) adding and 
subtracting \([k]\) inside the first series; b) extracting the term corre-
responding to \( j = 0 \) in the third series, and adding and subtracting 
\([k + i\omega_j[c]]/i\omega_j \) to the remainder of the series; and c) making use 
of equation (6) and cancelling common terms:

\[ \{ \dot{y}(t) \} = - \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} \left[ H(\omega_j) \right][k] e^{i\omega_j t} \right\} \{ \Delta y \} + \left\{ \frac{1}{t_0} \left[ I \right] \sum_{j=-M}^{M} \frac{1}{i\omega_j} e^{i\omega_j t} \right. \]

\[ + \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} \left[ H(\omega_j) \right]( - \frac{i}{\omega_j}[k] - [c]) e^{i\omega_j t} \right\} \{ \Delta \dot{y} \} \]  

(A12)

Finally, on noticing that the second series in this expression 
can be expressed as \((1/2 - t/t_0)\), one can show that the first part 
of equation (A10) leads to equation (19) and that the second part 
leads to equation (20).

Values of \([\ddot{g}(t)]\) and \([\ddot{h}(t)]\) for systems with constant-parameters

For systems in which \([k]\) and \([c]\) are constant, one can write 
equation (19) as

\[ [\ddot{g}(t)] = - \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} \left[ H(\omega_j) \right][m] e^{i\omega_j t} \right\} [m]^{-1}[k] \]  

(A13)
On comparing the term inside the braces with equation (17) one establishes the relationship between \( \tilde{g}(t) \) and \( \tilde{h}(t) \) given in equation (21). Similarly, on adding and subtracting \([H(0)][c]\) in equation (20) leads to

\[
\begin{align*}
\dot{\tilde{h}}(t) & = \left( \frac{1}{2} - \frac{t}{t_0} \right)[I] + \frac{1}{t_0} \left\{ [H(0)][c] + \sum_{j=-M}^{M} \sum_{j \neq 0}^{M} i \omega_j [H(\omega_j)][k] e^{i\omega_j t} \right\} \\
& \quad - \left\{ \frac{1}{t_0} \sum_{j=-M}^{M} [H(\omega_j)][m] e^{i\omega_j t} \right\} [m]^{-1}[c] \quad (A14)
\end{align*}
\]

which on making use of equations (16) and (17) leads to equation (22).
### APPENDIX II

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
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<tbody>
<tr>
<td>{a}, {b}</td>
<td>constants in equation (13)</td>
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<td>[c]</td>
<td>damping matrix of system</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
</tr>
<tr>
<td>$f_1^0$</td>
<td>fundamental natural frequency of undamped system</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>$G_s$</td>
<td>complex-valued amplitude of sth harmonic in a Fourier series representation of discretized base accelerations</td>
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<tr>
<td>$[\tilde{g}(t)]$</td>
<td>matrix of periodic functions in which the function of the ith row and jth column defines the steady state response at time $t$ of the ith node due to periodic unit displacement changes applied at intervals $t_0$ at jth node</td>
</tr>
<tr>
<td>$[\tilde{h}(t)]$</td>
<td>matrix of periodic function in which the function of the ith row and jth column defines the steady-state response at time $t$ of the ith node due to periodic unit velocity changes applied at intervals $t_0$ at the jth node</td>
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<td>$[H(\omega_s)]$</td>
<td>complex valued dynamic flexibility matrix of system at $\omega_s$</td>
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<tr>
<td>$i$</td>
<td>$\sqrt{-1}$; when used as a subscript indicates level of floor or story</td>
</tr>
<tr>
<td>$j$</td>
<td>variable integer</td>
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<td>[k]</td>
<td>stiffness matrix of the system</td>
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<td>[m]</td>
<td>mass matrix of the system</td>
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<td>$M$</td>
<td>largest value of $j$ in a Fourier series</td>
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<tr>
<td>MDF</td>
<td>multi-degree-of-freedom system</td>
</tr>
<tr>
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<td>number of degrees of freedom in a system</td>
</tr>
<tr>
<td>$N$</td>
<td>number of time or frequency intervals</td>
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<tr>
<td>$p_1^0$</td>
<td>$2\pi f_1^0 = $ fundamental natural circular frequency of undamped system</td>
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</table>
\( r \) \: \text{variable integer} \\
SDF \: \text{single-degree-of-freedom system} \\
\{s(t)\} \: \text{vector of periodic responses defined for } 0 \leq t < t_0 \\
t \: \text{time} \\
t_d \: \text{duration of excitation} \\
t_f \: \text{duration of free vibration or of appended band of zeroes} \\
t_r \: r \Delta t = \text{discretized time} \\
t_0 \: \text{period during which the response is to be evaluated; also period of periodic extension of excitation} \\
T_1^0 \: 1/t_1^0 = \text{natural period of fundamental mode of undamped system} \\
\{x(t)\} \: \text{vector of transient displacements relative to moving base} \\
\{y(t)\} \: \text{vector of steady-state displacements relative to moving base} \\
\{z(t)\} \: \text{corrective displacements} \\
\ddot{x}_g(t) \: \text{acceleration of base motion} \\
\ddot{x}_{gr} \: \ddot{x}_g(t_r) = \text{value of } \ddot{x}_g(t) \text{ at } t_r \\
\{Y_s\} \: \text{complex-valued amplitudes of } s\text{th harmonic component of response} \\
\delta(t) \: \text{unit impulse function} \\
\dot{\delta} \: \text{derivative of } \delta(t) \text{ or unit doublet} \\
\Delta t \: t_0/N = \text{time increment} \\
\Delta \omega \: 2\pi/t_0 = \text{frequency increment} \\
\{\Delta y\}, \{\Delta \dot{y}\} \: \text{periodic displacements and velocity changes, respectively} \\
\omega \: \text{circular frequency of harmonic component of excitation or response} \\
\omega_j \: j\Delta \omega = \text{sample of } \omega \\
\omega_s \: s\Delta \omega = \text{sample of } \omega
REFERENCES


Table I. Comparison of solutions obtained by classical DFT method and proposed improvement for system considered in example.

<p>| $t_f$ | Values of $p_i^0| (x_i)_{\text{max}}| / \dot{x}<em>g$ | Values of $p_i^0| (u_i)</em>{\text{max}}| / \dot{x}_g$ |
|-------|-----------------------------------------------|-----------------------------------------------|</p>
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Figure 1. Comparison of response of system considered computed by classical DFT method and proposed method using $t_f = T_1^0$. 

\[ \frac{|\langle x_1 \rangle_{\text{max}}| p_i^0}{\dot{x}_g} \] 

\[ \frac{|\langle x_2 \rangle_{\text{max}}| p_i^0}{\dot{x}_g} \] 

\[ \frac{|\langle u_1 \rangle_{\text{max}}| p_i^0}{\dot{x}_g} \] 

\[ \frac{|\langle u_2 \rangle_{\text{max}}| p_i^0}{\dot{x}_g} \]