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ELASTIC AND PLASTIC STRESSES IN A POROUS MEDIUM CONTAINING SPHERICAL OR CYLINDRICAL CAVITIES

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ELASTIC AND PLASTIC STRESSES IN A POROUS MEDIUM CONTAINING SPHERICAL OR CYLINDRICAL CAVITIES

by

ERIC P. FAHRENTHOLD

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

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April, 1984
ABSTRACT

ELASTIC AND PLASTIC STRESSES IN A POROUS MEDIUM CONTAINING SPHERICAL OR CYLINDRICAL CAVITIES

by

ERIC P. FAHRENTHOLD

The influence of pore fluid flow on the elastic and plastic stresses in a porous material may be determined for a solid body containing a large spherical or cylindrical cavity.

A description of the stresses in porous media relies on a separation of the total stress into effective and neutral parts. In the elastic case, the effective stress is that part of the total stress that determines the deformation of the porous solid. The application of Drucker's postulate to a mixture of a solid and a fluid under homogeneous deformation results in a plastic flow rule written in terms of the total stress.

The elastic or plastic state of stress in the porous solid surrounding a spherical or cylindrical cavity is significantly affected by a nonuniform pore pressure distribution. For a plastic material obeying a quadratic yield condition, pore fluid flow into a spherical cavity reduces the mean compressive stress in the solid and may
lead to cavity expansion. For an elastic material, pore fluid flow into a cylindrical cavity may initiate yielding in the porous solid at an unsupported, fully supported, or partially supported cavity boundary. Application of these analyses to the rock surrounding a perforated casing in a hydrocarbon reservoir gives an estimate for the well pressures that precipitate initial yielding and the production of solid material.

Several other problems related to the behavior of rock around a perforated casing may be investigated using elasticity, plasticity, and potential flow theory. An elastic porous cylinder under uniform overburden loading represents more closely the physical boundary conditions in a reservoir than the conventional plane strain analysis. Mechanical loads required for the extrusion of a porous solid are significantly reduced by simultaneous pore fluid flow through the extruded material. Finally conformal mapping provides a simple approximate description of the pore pressure distribution around a slotted casing.
ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Professor John B. Cheatham of Rice University for his advice and assistance during this research.

The assistance of Drs. Phillip D. Pattillo and Michael B. Smith of Amoco Production Company, and the research support provided by that organization, are greatly appreciated.
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<th>Symbol</th>
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<tr>
<td>a</td>
<td>Inner radius of a sphere or cylinder; or an elastic constant</td>
</tr>
<tr>
<td>A</td>
<td>Area; or an elastic constant in Biot's model</td>
</tr>
<tr>
<td>$A_{ij}$</td>
<td>Symmetric tensor of elastic moduli for a porous solid</td>
</tr>
<tr>
<td>$\mathbf{A}_{ij}$</td>
<td>Symmetric tensor of elastic moduli</td>
</tr>
<tr>
<td>$B_i$</td>
<td>Component of a body force vector</td>
</tr>
<tr>
<td>$B_{ij}$</td>
<td>Symmetric tensor of elastic compliances</td>
</tr>
<tr>
<td>$B_{ij}$</td>
<td>Symmetric tensor of elastic compliances</td>
</tr>
<tr>
<td>b</td>
<td>Outer radius of a cylinder; or a material constant in a quadratic yield condition</td>
</tr>
<tr>
<td>b</td>
<td>Outer radius of a sphere</td>
</tr>
<tr>
<td>C</td>
<td>Cohesive strength of a rock</td>
</tr>
<tr>
<td>$C_0$</td>
<td>Constant in Darcy's Law for radial flow</td>
</tr>
<tr>
<td>c</td>
<td>Coefficient of the pore pressure in Carroll's definition of effective pressure; or a material constant in a quadratic yield condition</td>
</tr>
<tr>
<td>$c$</td>
<td>Soil cohesion in Coulomb's yield function</td>
</tr>
<tr>
<td>$C_{ijkl}$</td>
<td>Symmetric tensor of elastic moduli for a porous solid</td>
</tr>
<tr>
<td>$\mathbf{C}_{ijkl}$</td>
<td>Symmetric tensor of elastic moduli</td>
</tr>
<tr>
<td>$\mathbf{\bar{C}}_{ijkl}$</td>
<td>Symmetric tensor of elastic moduli</td>
</tr>
<tr>
<td>$\mathbf{\hat{C}}_{ijkl}$</td>
<td>Symmetric tensor of elastic moduli</td>
</tr>
<tr>
<td>$C_{ijkl}^s$</td>
<td>Tensor of elastic compliances for the material in the solid grains</td>
</tr>
<tr>
<td>$\hat{D}$</td>
<td>Displacement due to a body force</td>
</tr>
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</table>
e \quad \text{Volumetric strain of a porous solid}

e^e \quad \text{Elastic part of the volumetric strain of a porous solid}

e^p \quad \text{Plastic part of the volumetric strain of a porous solid}

e_{ij}^e \quad \text{Elastic part of a strain component in a porous solid}

e_{ij}^p \quad \text{Plastic part of a strain component in a porous solid}

e_{rr} \quad \text{Radial component of the strain in a porous solid}

e_{\theta\theta} \quad \text{Tangential component of the strain in a porous solid}

E \quad \text{Young's modulus of elasticity for a porous solid}

f \quad \text{Porosity}

F_i \quad \text{Component of a surface traction vector}

F \quad \text{Yield function for a porous solid, or the body force in a complex variable formulation}

G(z) \quad \text{A Plemelj function}

h \quad \text{Formation thickness, or extruded billet half-thickness}

i \quad \text{The imaginary number, } \sqrt{-1}

I \quad \text{First invariant of the effective stress tensor}

\hat{I} \quad \text{Nondimensional first invariant of the effective stress tensor}

J \quad \text{Square root of the second invariant of the deviatoric effective stress tensor}

\hat{J} \quad \text{Nondimensional square root of the second invariant of the deviatoric effective stress tensor}

\hat{k} \quad \text{Elastic constant for a porous solid in plane strain}

k \quad \text{Permeability; or a material constant in a quadratic yield condition}

\bar{k} \quad \text{Permeability}
K  Bulk modulus of elasticity for a porous solid
K  Coefficient of jacketed compressibility for a porous solid
K_s  Bulk modulus of elasticity for the material in the solid grains
L  Length of an extruded billet
M  An elastic constant in Biot's model
M_{ijkl}  Tensor of elastic moduli for a porous solid
n  Coefficient of the pore pressure in Johnson and Green's definition of effective stress, or porosity
N  An elastic constant in Biot's model, or a constant in Terzaghi's yield condition
P  Pore pressure
P_{\infty}  In situ reservoir pore pressure
P_e  Reservoir edge pressure
P_w  Well pressure
P_0  Radial pressure increment
P_s  Partial pressure of the solid in a mixture
P_T  Average total normal stress in a mixture
\langle P \rangle  Effective pressure
\hat{P}  Pressure due to a body force distribution
Q_i  Component of a strain vector
Q_i  Component of a stress vector
Q  Volumetric flowrate; or an elastic constant in Biot's model
r  Radial coordinate
\( \hat{r} \)  
Nondimensional radial coordinate

\( r_c \)  
Casing radius

\( r_e \)  
Reservoir radius

\( R \)  
An elastic constant in Biot's model, or a cavity radius

\( s \)  
Yield stress in shear; or a constant in a critical well pressure expression

\( \tilde{S}_{ijkl} \)  
Symmetric tensor of elastic compliances

\( \tilde{S}_{ijkl} \)  
Symmetric tensor of elastic compliances

\( S_{ijkl} \)  
Symmetric tensor of elastic compliances

\( \hat{\sigma} \)  
Tangential stress component at initial yield around an open well bore

\( \tilde{\sigma} \)  
Radial stress component at initial yield around a lined well bore

\( t \)  
Time variable, or a coordinate on a circular hole

\( u \)  
Radial displacement; or the real part of the complex number \( w \)

\( \hat{u} \)  
Displacement due to a body force distribution

\( u_i \)  
Component of a displacement vector

\( v \)  
Imaginary part of the complex number \( w \)

\( V_0 \)  
Reference volume

\( V \)  
Volume

\( V(z) \)  
Body force potential

\( w \)  
A complex number

\( X \)  
Force per unit length on a rigid liner

\( x \)  
Real part of the complex number \( z \)

\( X(z) \)  
Particular integral of the body force potential
\( y \) Imaginary part of the complex number \( z \)

\( z \) A complex number

\( \delta W_{\text{ext}} \) Work per unit volume done by an external agency over Drucker's cycle

\( \delta W_T \) Total work per unit volume done on a body subjected to Drucker's cycle

\( \delta W_T^* \) Total work per unit volume done on a body subjected to a modified Drucker's cycle

\( \delta W_0 \) Work per unit volume done on a body subjected to Drucker's cycle if the stress state is held constant

\( \alpha \) Half the slot angle; or an elastic constant in Biot's model

\( \beta \) Ratio of the bulk modulus of a porous solid to the bulk modulus of the material in the solid grains, or a nondimensional flow rate

\( \nabla \) Del operator

\( \delta \) Coefficient of unjacketed compressibility for a porous solid

\( \delta_{ij} \) Kronecker delta

\( \varepsilon \) Volumetric strain of the fluid, or a nondimensional coordinate for the radial position and time after the onset of pore fluid production

\( \bar{\varepsilon} \) The value of \( \varepsilon \) at the well bore

\( \xi \) Fluid content; or the real part of the complex number \( \Omega \)

\( \zeta \) Nondimensional coordinate in the complex plane

\( \eta \) Imaginary part of the complex number \( \Omega \)

\( \theta \) Angular coordinate

\( \lambda \) Lamé coefficient for a porous solid
Proportionality factor in the flow law for a porous solid

\( \mu \) Lamé coefficient for a porous solid; or the viscosity of a fluid

\( \mu' \) Viscosity of a fluid

\( v \) Poisson's ratio for a porous solid

\( \sigma \) Effective stress tensor

\( \sigma' \) Deviatoric effective stress tensor

\( \sigma_{\infty} \) In situ stress state in a reservoir

\( \sigma \) Stress in the fluid in a mixture, averaged over a unit area of the mixture

\( \sigma' \) Total normal stress in Coulomb's yield function

\( \sigma_{ij} \) Stress in the porous solid in a mixture, averaged over a unit area of the mixture

\( \sigma_{rr} \) Radial effective stress

\( \sigma_{\theta\theta} \) Tangential effective stress

\( \sigma_{ij} \) A part of the total stress defined by Paslay and Cheatham's decomposition of the stress tensor

\( \sigma_{rr} \) Radial component of the stress defined by Paslay and Cheatham

\( \sigma_{\theta\theta} \) Tangential component of the stress defined by Paslay and Cheatham

\( \sigma_{ij}'' \) A part of the stress in the solid defined by Lubinski's decomposition of the stress tensor

\( \langle \sigma_{ij} \rangle \) Effective stress

\( \mathbf{I} \) Total stress tensor

\( \mathbf{\tau}_e \) Total stress at the reservoir edge

\( \tau_0 \) Overburden stress
\( \tau \)  \quad \text{Time after the onset of production}

\( \hat{\tau} \)  \quad \text{Total shear stress in Coulomb's yield function}

\( \tau_{ij} \)  \quad \text{Total stress in a mixture}

\( \phi \)  \quad \text{Angle of internal friction in Coulomb's yield function, or the angular position of the edge of a slot in a cylinder}

\( \psi(z) \)  \quad \text{Power series in the variable } z

\( \omega_{\infty} \)  \quad \text{The rotation at infinity in a porous solid}

\( \omega(z) \)  \quad \text{A complex potential function}

\( \Omega(z) \)  \quad \text{A complex potential function}

Note: the placement of a dot (\( . \)) over any variable denotes the total derivative of that quantity with respect to time.
INTRODUCTION

The theories of elasticity and plasticity that describe states of stress in solid and fluid materials are well developed. Somewhat less is known about the more complex behavior of intermixed solids and fluids encountered in the study of materials such as porous media. Of particular interest in this field is the determination of the influence of a flowing pore fluid on the stresses in a porous solid experiencing elastic or inelastic deformation. The analysis of some problems in fluidization, extrusion, and subsidence requires an evaluation of the effects of nonuniform pore pressure distribution on a porous solid medium.

The objectives of this study are to extend the application of some elasticity and plasticity formulations to the analysis of porous media, and to solve some practical problems which demonstrate the effect of pore fluid flow on the state of stress in the surrounding solid structure.

First it was desired to understand fully the principle of effective stress as applied to porous materials containing a pore fluid. The work of various authors on this subject is reviewed in Chapter 1. This review indicates that the effective stress principle is best understood when studied in the context of the physical model used to describe the interacting solid and fluid. The best physical description of the problem seems to be provided by a
mixture model, applied to porous media by Biot\textsuperscript{4,10,13} and Bowen\textsuperscript{11,14,18}. In Chapter 2 Biot's model is outlined and used to establish a general definition of effective stress for elastic deformations.

A second objective was to apply general plasticity theory to a porous mixture. In Chapter 2 Drucker's postulate\textsuperscript{17} for a stable work-hardening material is applied to a porous elastic-plastic solid with pore fluid. This leads to a definition of effective stress in the plastic region and an appropriate flow law for homogeneous deformations.

A third goal was to interpret the plasticity theory of Johnson and Green\textsuperscript{9}. Their expression for plastic work and their flow law for fluid-saturated rocks are compared to the results obtained by application of Drucker's postulate to Biot's mixture model. In Chapter 2 a special case of Drucker's loading and unloading cycle is compared to the work of Johnson and Green.

The next four chapters are concerned with the analysis of fluid flow effects on several practical problems. The problems are linked by their attention to the phenomenon of the stress distribution around cavities in an elastic or plastic medium containing a flowing pore fluid. In Chapter 3 the plastic stresses around a spherical cavity are determined and the critical flowrate leading to cavity expansion is calculated. In Chapter 4 complex variable methods are
used to describe elastic stresses and initial yielding of solid material surrounding a cylindrical cavity provided with varying degrees of external support. In Chapter 5 a cylindrical cavity is again considered, with interest focused on the effect of pore pressure reduction in initiating subsidence effects. Finally in Chapter 6 the effect of pore fluid flow on the process of extrusion is investigated.

In a somewhat specialized aspect of the porous media problem, it was of interest to consider the influence of a nonsymmetric flow field on the stress distribution in a solid body. As a prerequisite it is appropriate to determine the pore pressure distribution for the extreme case of a cylinder with a single slot along its length. An expression for this flow field is determined using conformal mapping in Chapter 7.

Lastly, Chapter 8 describes some experimental observations on the effect of a pore fluid on the ultimate strength of a weak rock.

The results of this study apply basic elasticity and plasticity concepts to porous media, and provide an understanding of pore fluid flow effects in some stress analysis problems that are of practical interest.
CHAPTER 1.

THE EFFECTIVE STRESS PRINCIPLE

In the analysis of porous materials it has been found useful to introduce the concept of an effective stress. As described by Terzaghi and Peck\textsuperscript{1}, the effective stress is that portion of the stress which has a measurable influence on the mechanical properties of the material, such as void ratio or shearing resistance. Various mathematical definitions have been applied in decomposing the stress tensor into effective and neutral parts. It appears that the proper decomposition depends upon the particular method of analysis used.

In order to evaluate a particular effective stress concept, it is necessary to consider both the decomposition of the stress tensor and the accompanying stress-strain relations. In addition one must remember that a fluid-saturated porous material is composed of two constituents, a solid and a fluid, and that the stresses and deformations in the constituents are in general interdependent. From the point of view of the mixture model of Biot\textsuperscript{4,10,13}, generalized by Bowen\textsuperscript{11,14,18}, a clear comparison of the various effective stress laws can be made.

It is appropriate to specify a common notation for use in describing the work of various authors. In accordance with the mixture model referred to above, the total stress
\( \tau_{ij} \) is made up of a stress on the solid \( \sigma_{ij} \) and a stress on the fluid \( \sigma \). Since the fluid stress has only normal components,

\[
\tau_{ij} = \sigma_{ij} + \delta_{ij} \sigma
\]  

(1.1)

where \( \delta_{ij} \) is the Kronecker delta. The above stresses are all average quantities over a unit area of the mixture. The total stresses must satisfy the equations of equilibrium for the mixture. The pore pressure \( P \) is related to the fluid part of the stress by

\[
\sigma = -fP
\]  

(1.2)

where \( f \) is the porosity. Since \( P \) is a positive quantity, tension is taken as positive.

When the solid part of the mixture is linearly elastic, the effective stress \( \langle \sigma_{ij} \rangle \) will be defined as that part of the total stress which is a linear function of the strain in the solid constituent. The definition of \( \langle \sigma_{ij} \rangle \) when plastic flow occurs in the solid will be considered in Chapter 2.

In soil mechanics the effective stress is often defined as

\[
\langle \sigma_{ij} \rangle = \tau_{ij} + \delta_{ij} P.
\]  

(1.3)

Terzaghi\(^2\) arrived at equation (1.3) through experimental
observation of saturated soils. He indicates that the strain in soils and the stress conditions for failure depend only on this effective stress. For example, he writes Coulomb's yield equation as

\[
\hat{\tau} = \hat{c} + (\hat{\sigma} + P) \tan \phi 
\]

where \( \hat{\tau} \) is the total shear stress, \( \hat{\sigma} \) is the total normal stress, \( \hat{c} \) is the cohesion, and \( \phi \) is the angle of internal friction.

Paslay and Cheatham\(^3\) decompose the stress tensor using a stress \( \bar{\sigma}_{ij} \) defined by

\[
\tau_{ij} = \bar{\sigma}_{ij} - \delta_{ij} P. \tag{1.5}
\]

They describe the elastic stresses induced by the flow of fluids into boreholes, for the plane strain case, in terms of \( \bar{\sigma}_{ij} \) and \( P \). Their stress-strain relations are equivalent to (see Appendix A)

\[
\begin{align*}
\bar{\sigma}_{rr} - \beta P &= \lambda (e_{rr} + e_{\theta\theta}) + 2 \mu e_{rr} \tag{1.6a} \\
\bar{\sigma}_{\theta\theta} - \beta P &= \lambda (e_{rr} + e_{\theta\theta}) + 2 \mu e_{\theta\theta} \tag{1.6b} \\
\beta &= K/K_s \tag{1.6c}
\end{align*}
\]

where \( e_{ij} \) represents the strain in the porous solid, \( \mu \) and \( \lambda \) are Lamé coefficients for the porous solid, \( K \) is the bulk modulus of the porous solid, and \( K_s \) is the bulk modulus of the material in the solid grains. The subscripts
\( r \) and \( \theta \) denote radial and tangential components respectively. Equations (1.6a), (1.6b), and (1.6c) show that the effective stress is then

\[
\langle \sigma_{ij} \rangle = \bar{\sigma}_{ij} - \delta_{ij} \beta P \quad (1.7a)
\]

or using equation (1.5)

\[
\langle \sigma_{ij} \rangle = \tau_{ij} + \delta_{ij} (1 - \beta) P. \quad (1.7b)
\]

Nur and Byerlee\(^8\) derive equation (1.7b) by considering the volumetric strain of a porous material subjected to a confining pressure and a pore pressure. They superimpose the strains due to: (1) equal pore and confining pressures, and (2) confining pressure in excess of the pore pressure. In the first case the solid matrix is found to experience the same stress with either fluid or identical solid material occupying the pore volume. In the second case the strain is simply determined by the bulk modulus of the porous material. However they emphasize that their expression may not apply to inelastic processes.

Lubinski\(^5\) draws an analogy between the stresses in a porous solid due to pore pressure and the stresses in a nonporous solid due to thermal expansion. He separates the stress and strain in a porous solid into two components, one determined by the elastic constants of the porous material, the other by the elastic constants of the interpore material. The pore pressure produces a uniform strain in the
solid with a magnitude determined by the elastic constants of the interpore material. The average stress corresponding to this uniform strain is the pore pressure multiplied by the fraction of the cross sectional area which is interpore material, that is \((1-f)\). Therefore the stress in the solid is given by

\[
\sigma_{ij} = \sigma_{ij}^{\prime} - \delta_{ij}(1-f)P
\]  \hspace{1cm} (1.8)

where \(\sigma_{ij}^{\prime}\) is the part of the solid stress determined by the elastic constants of the porous material. From this decomposition of the stress in the solid Lubinski derives relations of the form

\[
\sigma_{ij} = 2\mu e_{ij} + \delta_{ij} [\lambda e - (1 - \beta - f) P]
\]  \hspace{1cm} (1.9a)

where \(e\) is the dilatation of the porous solid

\[
e = e_{ii}.
\]  \hspace{1cm} (1.9b)

Equation (1.9a) is derived by writing Hooke's law to relate the stress \(\sigma_{ij}^{\prime}\) to a corresponding strain in the porous solid. The coefficient \((1 - \beta - f)\) is analogous to a linear coefficient of thermal expansion which would be used if \(P\) were replaced by the temperature. Equation (1.9a) shows that Lubinski's effective stress is

\[
<\sigma_{ij}> = \sigma_{ij} + \delta_{ij} (1 - \beta - f) P.
\]  \hspace{1cm} (1.10)
But equations (1.1) and (1.2) give

$$\sigma_{ij} = \tau_{ij} + \delta_{ij} f P.$$  \hspace{1cm} (1.11)

Therefore equation (1.10) can be written as

$$\langle \sigma_{ij} \rangle = \tau_{ij} + \delta_{ij} (1 - \beta) P$$  \hspace{1cm} (1.12)

which is identical to equation (1.7b). Therefore the effective stress laws of Paslay and Cheatham\(^3\), Nur and Byerlee\(^8\), and Lubinski\(^5\) for an isotropic elastic solid are in agreement. If the solid grains are incompressible, $\beta = 0$ and equation (1.7b) gives the effective stress of Terzaghi, equation (1.3).

Biot and Willis\(^6\) analyze a mixture of an elastic solid and an elastic fluid. They decompose the stress tensor as shown in equations (1.1) and (1.2) and present two equivalent sets of equations describing the stresses and strains in the mixture. Each set of equations makes use of four elastic coefficients. The experimental measurements required to evaluate these coefficients are also considered. The first set of equations is

$$\sigma_{ij} = 2N e_{ij} + \delta_{ij} (Ae + Q\varepsilon)$$  \hspace{1cm} (1.13a)

$$\sigma = Qe + R\varepsilon$$  \hspace{1cm} (1.13b)

where $\varepsilon$ is the dilatation of the fluid and $A$, $N$, $Q$, and $R$ are elastic coefficients. These equations along with
equation (1.2) give the effective stress as

\[ <\sigma_{ij}> = \sigma_{ij} + \delta_{ij} \frac{Q}{R} f P \]  

(1.14a)

which with equation (1.1) becomes

\[ <\sigma_{ij}> = \tau_{ij} + \delta_{ij} \left( \frac{Q+R}{R} \right) f P. \]  

(1.14b)

The second set of elastic stress-strain relations is

\[ \tau_{ij} + \delta_{ij} \alpha P = 2 \mu e_{ij} + \delta_{ij} \lambda e \]  

(1.15a)

\[ \xi = \frac{P}{M} + \alpha e \]  

(1.15b)

where \( \alpha, M, \lambda, \) and \( \mu \) are the elastic coefficients and \( \xi \) is the fluid content, given by

\[ \xi = f(e - \varepsilon). \]  

(1.16)

In this case the effective stress is

\[ <\sigma_{ij}> = \tau_{ij} + \delta_{ij} \alpha P. \]  

(1.17)

Because of the definition of \( \alpha \) where

\[ \alpha = \left( \frac{Q+R}{R} \right) f \]  

(1.18a)

the effective stress laws of equations (1.14b) and (1.17)
are the same. Biot and Willis determine that the parameter \( \alpha \) has the limits

\[
f \leq \alpha \leq 1.
\]

(1.18b)

The value of \( \alpha \) may be calculated from

\[
\alpha = 1 - \frac{\delta}{\bar{K}}
\]

(1.18c)

where \( \bar{K} \) and \( \delta \) are the jacketed and unjacketed coefficients of compressibility for the porous material. The value of \( \delta \) is determined by measuring the dilatation of an unjacketed sample of porous solid immersed in a fluid at a fixed pressure. The value of \( \bar{K} \) is determined by measuring the dilatation of a jacketed sample subjected to constant external pressure, with the pore fluid held at atmospheric pressure. If the solid grains and the fluid are incompressible, then \( \alpha = 1 \) and equation (1.17) gives the effective stress of Terzaghi, equation (1.3).

Carroll\(^7\) considers elastic and inelastic deformations of porous media, and analyzes the various effective stress laws. He states that since the deviatoric components of the total and effective stresses are equal, it is sufficient to consider an effective pressure

\[
\langle P \rangle = P_T - cP, \; 0 \leq c \leq 1
\]

(1.19a)

with \( P_T \) being the average normal stress in the mixture and
the definition of \( c \) depending on the application. For the linear elastic response of a homogeneous isotropic solid matrix, Carroll shows that \( c \) takes on a value of

\[
\begin{align*}
  c &= 1 & \text{for the change in porosity,} & \quad (1.19b) \\
  c &= f & \text{for the solid volume change,} & \quad (1.19c) \\
  c &= 1 - \frac{K}{K_s} & \text{for the total strain, and} & \quad (1.19d) \\
  c &= 1 - \frac{fK}{K_s - K} & \text{for the pore volume strain.} & \quad (1.19e)
\end{align*}
\]

For an incompressible solid matrix, the last two definitions reduce to \( c = 1 \), and the second definition becomes irrelevant. Therefore Carroll concludes that equation (1.3) is the appropriate effective stress law for the isotropic elastic case, unless the process under study is one which is significantly affected by solid compaction.

For an anisotropic elastic material Carroll finds that the effective stress is

\[
<\sigma_{ij}> = \tau_{ij} + (\delta_{ij} - M_{ijkl} C_{klmn}^S) P \quad (1.20)
\]

where \( M_{ijkl} \) is a tensor of elastic moduli for the porous material and \( C_{klmn}^S \) is a tensor of elastic compliances for the material in the solid grains.

In discussing nonlinear volumetric deformation of a porous material, Carroll assumes that the solid volume
depends only on the partial pressure in the solid, \( P_S \), defined by the equation

\[
P_T = (1 - f) P_S + fP. \tag{1.21a}
\]

Again he uses an effective pressure defined by equation (1.19a) with \( c = 1 \), and he assumes that the porosity depends solely on this effective pressure. Since for \( c = 1 \), equation (1.21a) can be written as

\[
P_S = P + \frac{\langle P \rangle}{(1-f)} \tag{1.21b}
\]

then, in general, the total volume of the material depends on both the pore pressure and the effective pressure. But as in the linear case, if the solid is incompressible the dependence is only on the effective pressure with \( c = 1 \).

Johnson and Green\(^9\) apply plasticity theory to fluid-saturated rocks. They divide the total strain in the porous solid into elastic and plastic components, with the elastic strain proportional to an effective stress in the form of equation (1.19a). In the plastic region they again use equation (1.19a), but specify \( c = 1 \).

Johnson and Green use Drucker's definition of stability in developing their plasticity theory. Drucker\(^{17}\) states that a system is stable if the work done by an external agency applying a set of added loads to the system is
positive for all allowable added loads. For stability with respect to large variations in the stress he gives

\[ \int_{t_a}^{t_b} [\int_A \Delta F_i \Delta \dot{u}_i \, dA + \int_V \Delta B_i \Delta \dot{u}_i \, dV] \, dt \geq 0 \quad (1.22) \]

where the component of the body force increment \( \Delta B_i \) acting over the volume \( V \) and the component of the surface traction increment \( \Delta F_i \) acting over the area \( A \) produce the increment in displacement rate \( \Delta \dot{u}_i \). The values \( t_a \) and \( t_b \) are the times at the beginning and end of the load increment. Drucker as well as Johnson and Green consider time as a convenient parameter but exclude time or rate dependent behavior of the material. For homogeneous states of stress and strain Drucker's stability postulate requires

\[ \int_{t_a}^{t_b} \Delta Q_i \Delta \dot{q}_i \, dt = \int_a^b \Delta Q_i \Delta (dq_i) \geq 0 \quad (1.23) \]

where \( Q_i, q_i, \) and \( \dot{q}_i \) are respectively the stress, strain, and strain rate components in the material under loading, and \( \Delta \) denotes the increments in these quantities between state \( a \) at time \( t_a \) and state \( b \) at time \( t_b \). Malvern uses this notation for the stress and strain as vectors in a nine-dimensional space. In Chapter 2 a porous material will be represented in a ten-dimensional space.
If the total energy dissipated over the material volume is equated to the work done by the external forces the result is (for no body forces)

\[ \int_{t_a}^{t_b} \int_A \Delta F_i \Delta \dot{u}_i \, dA \, dt = \int_V \int_{t_a}^{t_b} \Delta Q_i \Delta \dot{q}_i \, dt \, dV \geq 0. \quad (1.24) \]

If states \( a \) and \( b \) are the same, then equation (1.23) shows that the second integration in equation (1.24) may be written as

\[ \int_V \phi \Delta Q_i \Delta (dq_i) \, dV = \int_V \phi \Delta Q_i \Delta \dot{q}_i \, dt \, dV \geq 0 \quad (1.25) \]

which is in the form given by Johnson and Green. Their interpretation of this inequality will be discussed in Chapter 2.

Johnson and Green state that the pore pressure affects the yield criteria for the material through the dependence of the yield surface on the effective stress. That is the yield surface has the same form with or without pore fluid present, the difference being in the definition of the stress variable in the yield function. This is consistent with Terzaghi's approach, as in equation (1.4).

This review of effective stress laws indicates that Biot's mixture model for an anisotropic solid and a fluid may be used to determine the general form of the effective stress in the elastic region. Biot's model includes both
anisotropy in the solid and the interdependence of the stress and strain in the constituents. Drucker's definition of a stable plastic material may then be used to determine the effective stress for a saturated porous solid experiencing plastic flow. This method is applied in Chapter 2.
CHAPTER 2.

PLASTICITY OF A POROUS SOLID WITH PORE FLUID

2.1 The Mixture Model of porous media.

The elastic deformation of a fluid-saturated porous material may be described using Biot's\textsuperscript{10} theory for a mixture of a fluid and an anisotropic porous solid. The total stress in the mixture is given by

\[ \tau_{ij} = \sigma_{ij} + \delta_{ij}\sigma \]  \hspace{1cm} (2.1)

where \( \tau_{ij} \) is the total stress, \( \sigma_{ij} \) is the stress in the porous solid, \( \delta_{ij} \) is the Kronecker delta, and tension is taken as positive. The stress on the fluid part of the mixture equals \( \sigma \), where

\[ \sigma = -fp \]  \hspace{1cm} (2.2)

with \( f \) being the porosity and \( P \) the pore pressure. The components of the strain in the porous solid and in the fluid are represented by \( e_{ij} \) and \( \varepsilon_{ii} \) respectively, where an index has the values

\[ i = 1,2,3. \]  \hspace{1cm} (2.3)

The dilatation of the porous solid and fluid are, respectively

\[ e = e_{ii} \]  \hspace{1cm} (2.4a)

\[ \varepsilon = \varepsilon_{ii} \]  \hspace{1cm} (2.4b)

where the repeated index indicates summation over \( i = 1,2,3 \).
For a mixture of an elastic anisotropic solid and an elastic fluid, Biot\textsuperscript{10} gives an elastic stress-strain relation of the form
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl}^{e} + A_{ij} \varepsilon^{e} \]  
(2.5a)

where \( C_{ijkl} \) and \( A_{ij} \) are symmetric tensors of constant elastic coefficients and the \( \varepsilon_{kl}^{e} \) are the components of the elastic strain in the porous solid. The stress on the fluid is for this case
\[ \sigma = A_{ij} \varepsilon_{ij}^{e} + R \varepsilon^{e} \]  
(2.5b)

where \( R \) is a constant. Solving equation (2.5b) for \( \varepsilon^{e} \) and substituting the result into equation (2.5a) gives
\[ \sigma_{ij} = (C_{ijkl} - \frac{A_{ij}}{R} A_{kl}) \varepsilon_{kl}^{e} + \frac{A_{ij}}{R} \sigma. \]  
(2.6a)

Let
\[ \bar{C}_{ijkl} = C_{ijkl} - \frac{A_{ij}}{R} A_{kl} \]  
(2.6b)

and rewrite equation (2.6a) as
\[ \sigma_{ij} - \frac{A_{ij}}{R} \sigma = \bar{C}_{ijkl} \varepsilon_{kl}^{e}. \]  
(2.6c)

If the effective stress tensor \( \langle \sigma_{ij} \rangle \) is defined as
\[ <\sigma_{ij}> = \sigma_{ij} - \frac{A_{ij}}{R} \sigma \]  

(2.7a)

then equation (6c) can be written as

\[ <\sigma_{ij}> = c_{ijkl} \epsilon_{kl}^e. \]  

(2.7b)

Inverting equation (7b) results in

\[ \epsilon_{kl}^e = \tilde{S}_{klij} <\sigma_{ij}> \]  

(2.7c)

where \( \tilde{S}_{klij} \) is a compliance tensor.

Equation (2.7a) is the general form of the effective stress in the elastic region for an anisotropic porous solid with pore fluid. Using equations (2.1) and (2.2) this becomes

\[ <\sigma_{ij}> = \tau_{ij} + (\delta_{ij} + \frac{A_{ij}}{R}) fP. \]  

(2.7d)

For the isotropic case, Biot\textsuperscript{10} gives

\[ A_{ij} = \delta_{ij} Q \]  

(2.7e)

where \( Q \) is a constant. This results in

\[ <\sigma_{ij}> = \tau_{ij} + \delta_{ij} \left( \frac{Q+R}{R} \right) fP \]  

(2.7f)

or

\[ <\sigma_{ij}> = \tau_{ij} + \delta_{ij} \alpha P \]  

(2.7g)
where
\[
\alpha = (\frac{Q+R}{R}) f. \tag{2.7h}
\]

2.2 Drucker's postulate applied to porous media.

Drucker's definition of a stable work-hardening material may be applied to porous media using the mixture model just described. The general method for an arbitrary body is described by Malvern\textsuperscript{12}.

Consider a mixture of a porous anisotropic solid and a fluid in an initial state of equilibrium with a stress state \(\tau_{1j}^s\). If an external agency slowly applies a set of self-equilibrating loads to this body and then slowly removes them, Drucker's postulate requires that: (1) the plastic work done by the external agency during this cycle be positive and (2) the net work done by the external agency during this cycle be non-negative. The cycle is defined by a return to the initial stress state \(\tau_{1j}^s\), and to the initial values of stress in the fluid and the solid, \(\sigma^s\) and \(\sigma_{1j}^s\).

Let the stress be \(\tau_{1j}^s\) at \(t = 0\), and let the external agency change the stress to state \(\tau_{1j}^{(1)}\) at \(t = t_1\), where \(\sigma_{1j}^{(1)}\) is on the yield surface for the porous solid. Then let the external loading take the body to a state \(\tau_{1j}^{(1)} + \delta\tau_{1j}\) at \(t_1 + \delta t\), followed by an unloading back to \(\tau_{1j}^s\) at \(t = t^*\). The fluid is considered to be elastic.
throughout the cycle, while the stress $\sigma_{ij}^{(1)} + \delta \sigma_{ij}$ is on or outside the initial yield surface of the porous solid. The total work per unit volume done during the cycle is given by

$$\delta W_T = \int_0^{t_1} (\sigma_{ij} \dot{e}_{ij} + \sigma \dot{e}) dt + \int_{t_1}^{t_1+\delta t} (\sigma_{ij} \dot{e}_{ij} + \sigma \dot{e}) dt$$

$$+ \int_{t_1+\delta t}^{t^*} (\sigma_{ij} \dot{e}_{ij} + \sigma \dot{e}) dt$$

(2.8a)

where the total work done during each time interval is divided into the work done on the solid and the work done on the fluid.

The bulk strains in the solid and fluid are given by the sum of the elastic and plastic parts,

$$e_{ij} = e_{ij}^e + e_{ij}^p, \quad \varepsilon = \varepsilon^e + \varepsilon^p$$

(2.8b)

where the plastic part is zero except over the interval $t_1 \leq t \leq t_1 + \delta t$. Therefore equation (2.8a) can be written as

$$\delta W_T = \int (\sigma_{ij} \dot{e}_{ij}^e + \sigma \dot{e}^e) dt + \int_{t_1}^{t_1+\delta t} (\sigma_{ij} \dot{e}_{ij}^p + \sigma \dot{e}^p) dt$$

(2.8c)

where the first integral is taken over the entire cycle from $t = 0$ to $t = t^*$.

Equation (2.8c) can be written as
\[ \delta W_T = \int \left( \sigma_{ij} - \frac{\dot{\sigma}_{ij}}{R} \right) \epsilon_{ij}^e \, dt + \int \sigma \left( \frac{\dot{\sigma}_{ij}}{R} \right) \epsilon_{ij}^p + \epsilon^e \, dt \]
\[ + \int_{t_1}^{t_1 + \delta t} (\sigma_{ij} \epsilon_{ij}^p + \sigma \epsilon^p) \, dt. \]

(2.8d)

Using equations (2.5b), (2.7a), and (2.7b) this becomes

\[ \delta W_T = \int \left\langle \sigma_{ij} \right\rangle \left\langle \ddot{\epsilon}_{ijkl} \right\rangle \, dt + \int \sigma \dot{\sigma} \, dt \]
\[ + \int_{t_1}^{t_1 + \delta t} (\sigma_{ij} \epsilon_{ij}^p + \sigma \epsilon^p) \, dt. \]

(2.8e)

Since \( \ddot{\epsilon}_{ijkl} \) is symmetric, and since at the end of the cycle both \( \sigma \) and \( \sigma_{ij} \) return to their original values, the first two integrals are zero, and therefore for homogeneous deformations \( (\epsilon = \epsilon) \)

\[ \delta W_T = \int_{t_1}^{t_1 + \delta t} \tau_{ij} \epsilon_{ij}^p \, dt. \]

(2.8f)

The net work done by the external agency over the cycle is represented by \( \delta W_{\text{ext}} \) where

\[ \delta W_{\text{ext}} = \delta W_T - \delta W_0 \]

(2.9a)

and \( \delta W_0 \) is the work that would be done by the stress \( \tau_{ij}^{*} \) if held constant over the cycle, 

\[ \delta W_0 = \int_{t_1}^{t_1 + \delta t} \tau_{ij}^{*} \epsilon_{ij}^p \, dt. \]

(2.9b)
Therefore the net work per unit volume done by the external agency over the cycle is

\[ \delta W_{\text{ext}} = \int_{t_1}^{t_1 + \delta t} (\tau_{ij} - \tau_{ij}^*) \dot{\epsilon}_{ij}^P \, dt. \]  \hspace{1cm} (2.9c)

Drucker's postulate requires that this work be non-negative for an arbitrarily small \( \delta t \), therefore

\[ (\tau_{ij} - \tau_{ij}^*) \dot{\epsilon}_{ij}^P \geq 0. \]  \hspace{1cm} (2.10a)

The first part of Drucker's postulate requires for \( \tau_{ij}^{(1)} = \tau_{ij}^* \) that

\[ (\tau_{ij} - \tau_{ij}^*) \, d\epsilon_{ij}^P = d\tau_{ij} \, d\epsilon_{ij}^P > 0 \]  \hspace{1cm} (2.10b)

which implies

\[ \tau_{ij} \dot{\epsilon}_{ij}^P > 0. \]  \hspace{1cm} (2.10c)

Equation (2.10a) requires that the yield surface in stress space for the porous solid be convex, and that

\[ \dot{\epsilon}_{ij}^P = \lambda \frac{\partial F}{\partial \tau_{ij}} \]  \hspace{1cm} (2.10d)

where \( F = 0 \) is the equation describing the yield surface in stress space and \( \lambda \) is a non-negative scalar function.

Thus application of Drucker's postulate to the mixture leads to plasticity equations (2.10 a-d) of a conventional
form but written in terms of the total stress in the porous mixture

$$\tau_{ij} = \sigma_{ij} - \delta_{ij} fP.$$  \hspace{1cm} (2.10e)

This indicates that in the plastic region the effective stress should be defined by

$$\langle \sigma_{ij} \rangle = \tau_{ij}.$$ \hspace{1cm} (2.10f)

2.3 A discussion of the plasticity theory of Johnson and Green.

The preceding formulation using a mixture model may be compared to the plasticity theory of Johnson and Green\textsuperscript{9}. Equation (2.8c) may be written as

$$\delta W_T = \oint (\sigma_{ij} \dot{e}_{ij} + \sigma \dot{e}) dt.$$ \hspace{1cm} (2.11a)

Substituting equation (2.1) into the above gives

$$\delta W_T = \oint \tau_{ij} \dot{e}_{ij} dt - \oint \sigma (\dot{e} - \dot{\varepsilon}) dt.$$ \hspace{1cm} (2.11b)

Biot\textsuperscript{13} has introduced a variable $\xi$ defined by

$$\xi = f(e - \varepsilon)$$ \hspace{1cm} (2.11c)

which is the change in fluid content of the mixture in volume per unit volume. If the fluid content of the mixture is constant during the cycle, then $\xi = 0$ and equation (2.11b) becomes
\[ \delta W_i^* = \oint \tau_{ij} \dot{e}_{ij} dt \]  

which corresponds to Johnson and Green's equation (2.21). The prime indicates that this cycle is different from the one previously considered, since the fluid content is now held constant. For the condition \( \xi = 0 \), equation (2.11c) gives

\[ \varepsilon = e \]  

where in general the porous solid strain has elastic and plastic parts, that is

\[ e = e^e + e^P. \]  

In the elastic range equation (2.5a) is then

\[ \sigma_{ij} = (C_{ijkl} + A_{ij} \delta_{kl}) e^e_{kl} \]  

or

\[ \sigma_{ij} = \bar{C}_{ijkl} e^e_{kl} \]  

where \( \bar{C}_{ijkl} = C_{ijkl} + A_{ij} \delta_{kl} \). Similarly equation (2.5b) is then

\[ \sigma = (A_{ij} + \delta_{ij} R) e^e_{ij}. \]  

or

\[ \sigma = \bar{A}_{ij} e^e_{ij} \]  

where \( \bar{A}_{ij} = A_{ij} + \delta_{ij} R \).

Comparison of Johnson and Green's equation (2.19) with equation (2.1) shows that if their parameter \( n \) is taken as
the porosity, then their effective stress is the same as the stress in the porous solid $\sigma_{ij}$. The difference in sign is accounted for by their use of compression as positive. Then if the fluid content is taken as constant during the loading and unloading cycle, Johnson and Green's equation (2.17) corresponds to equation (2.13b) above. Equation (2.11d) for this cycle can be written as

$$\delta W_T = \int \tau_{ij} e_{ij}^e dt + \int_{t_1}^{t_1 + \delta t} \tau_{ij} e_{ij}^p dt. \tag{2.14a}$$

Equations (2.1), (2.13b), and (2.13d) give

$$\tau_{ij} = [\bar{\tau}_{ijkl} + \delta_{ij} \bar{\alpha}_{kl}] e_{kl}^e \tag{2.14b}$$

or

$$\tau_{ij} = \hat{C}_{ijkl} e_{kl}^e \tag{2.14c}$$

where $\hat{C}_{ijkl} = \bar{\tau}_{ijkl} + \delta_{ij} \bar{\alpha}_{kl}$.

If equation (2.14c) is inverted, the elastic strain is

$$e_{kl}^e = \hat{S}_{klij} \tau_{ij} \tag{2.14d}$$

where $\hat{S}_{klij}$ is a compliance tensor. Substituting equation (2.14d) into equation (2.14a) results in

$$\delta W_T = \int \tau_{ij} \hat{S}_{ijkl} \tau_{kl}^e dt + \int_{t_1}^{t_1 + \delta t} \tau_{ij} e_{ij}^p dt. \tag{2.14e}$$

Since $\hat{S}_{ijkl}$ is symmetric and the total stress is the same
at the beginning and end of the cycle, the first integral is zero and

\[ \delta W_T = \int_{t_1}^{t_1 + \delta t} \tau_{ij} e_{ij}^p \, dt \]  
(2.14f)

which may be compared to Johnson and Green's equation (2.25). The latter equation contains an additional term of the form

\[ \Delta \delta W_T = \int \tau_{ij} n \tilde{S}_{ijkl} \hat{P} \delta_{kl} \, dt \]  
(2.15a)

where equation (2.13b) has been inverted to give

\[ e_{ij}^e = \tilde{S}_{ijkl} \sigma_{kl} \]  
(2.15b)

and \( \tilde{S}_{ijkl} \) is a compliance tensor. Similarly equation (2.13d) can be inverted to give

\[ e_{ij}^e = \sigma \tilde{B}_{ij} \]  
(2.15c)

where \( \tilde{B}_{ij} \) is a compliance tensor. Equating the elastic strain from the relations (2.14d) and (2.15c) results in

\[ \sigma \tilde{B}_{ij} = \hat{S}_{ijkl} \tau_{kl} \]  
(2.15d)

or

\[ \tau_{kl} = \sigma \hat{B}_{kl} \]  
(2.15e)

where

\[ \hat{B}_{kl} = \hat{C}_{klij} \tilde{B}_{ij} \]  
(2.15f)

Since in the Johnson and Green formulation \( n \) was taken as constant, let
\[ \dot{\sigma} = -n \dot{p}. \]  

(2.15g)

Now equation (2.15a) becomes

\[ \Delta \delta W_T' = - \int_{t_0}^{t_1} \dot{\bar{S}}_{ijkl} \dot{\sigma} \delta_{kl} \, dt \]  

(2.16a)

which with equation (2.15e) can be written as

\[ \Delta \delta W_T' = - \int \dot{\sigma} \hat{B}_{ij} \bar{S}_{ijkl} \dot{\delta}_{kl} \, dt. \]  

(2.16b)

Since the tensors \( \hat{B}_{ij}, \bar{S}_{ijkl}, \) and \( \dot{\delta}_{kl} \) are symmetric and \( \sigma \) returns to its original value at the end of the cycle, then

\[ \Delta \delta W_T' = 0 \]  

(2.16c)

and the second term in Johnson and Green's equation (2.25) is zero. The latter expression is equivalent to equation (2.14f) for a cycle at constant fluid content, if Biot's elastic mixture model is applied.

2.4 Drucker's stability postulate.

In its general isothermal form Drucker's stability postulate is written in terms of body forces, surface tractions, and displacement rates for a body occupying a known volume in space. For the special case of a body with homogeneous stress and strain rate fields this postulate takes its familiar form as a time integral.

In applying Drucker's postulate to porous media, a requirement for homogeneity has particular significance,
since the body is now composed of two or more immiscible constituents.

Porous media may satisfy the requirement for homogeneity, allowing use of the special formulation of Drucker's postulate, if the constituents undergo the same deformations from a common reference configuration. This special case leads to an expression for the total work over Drucker's cycle that depends on the total stress in the mixture.

A more general method that allows Drucker's postulate to be written as a time integral consists of transforming the volume integrals for each constituent to integrals over a common reference configuration. For example, let Drucker's postulate be given by

\[ \int_t \left\{ \int_{\Omega_s} \text{tr}(T_s^T L_s) \, dV_s + \int_{\Omega_f} \text{tr}(T_f^T L_f) \, dV_f \right\} \, dt \geq 0 \quad (2.17a) \]

where \( \text{tr} \) is the trace operation, \( T \) and \( L \) are the stress and the velocity gradient tensors, \( V \) is a spatial volume, \( t \) is the time, and the subscripts \( s \) and \( f \) refer to the solid and fluid respectively. The superscript \( T \) denotes the transpose. These integrals can be transformed to integrals over a common volume in the reference configuration, \( V_0 \), using the identities\(^{12} \)
\[
\begin{align*}
\text{d}V &= | \det \mathbf{F} | \text{d}V_O \quad (2.17b) \\
\mathbf{T}_R &= | \det \mathbf{F} | \mathbf{T} \mathbf{F}^{-1T} \quad (2.17c) \\
\dot{\mathbf{F}} &= \mathbf{L} \mathbf{F} \quad (2.17d)
\end{align*}
\]

where \( \mathbf{F} \) is the deformation gradient, \( \mathbf{F}^{-1T} \) is the transpose of its inverse, \( \dot{\mathbf{F}} \) is its material derivative, \( \mathbf{T}_R \) is the first Piola-Kirchoff stress tensor, and det is the determinant operation. Equations (2.17b) through (2.17d) allow equation (2.17a) to be written

\[
\int_t \text{tr}(\mathbf{T}_R^T \dot{\mathbf{F}}_S + \mathbf{T}_R^T \dot{\mathbf{F}}_f) \, dt \geq 0
\]

where the integrand has been taken to be homogeneous over the reference configuration.

Drucker\textsuperscript{21} extended his stability postulate to include the effects of temperature changes on the work done on a body which is acted upon by some external agency. A comparison of the extended stability postulate with Bowen's\textsuperscript{18} entropy inequality for a mixture indicates that the latter more general formulation may be the best one to use in understanding and defining the stability of a solid material containing pore fluid.
CHAPTER 3.
THE PLASTIC STATE OF STRESS AROUND A HEMISPHERICAL CAVITY

3.1 Introduction.

The stress distribution in a porous solid subjected to external loads may be significantly modified by the existence of a pore pressure and its associated flow field in the porous solid material. In this chapter an example of the general problem discussed in Chapters 1 and 2 is considered. A sphere or hemisphere of porous material with a void at its center is subjected to a radial flow field and the resulting plastic stresses are determined. This analysis describes conditions required for the stability of the porous material surrounding a casing perforation (see Figure A).

The plastic stresses in the porous solid around a perforated casing are determined by the material properties of the rock and the rate at which the reservoir is being depleted. In the work that follows a quadratic yield condition appropriate for Dania chalk is used to estimate the well pressure that avoids the formation of cavities around casing perforations. A similar analysis has been performed by Bratli and Risnes$^{57}$ to describe the minimum allowable well pressure for a material obeying the Coulomb yield condition, while the corresponding elastic problem has been solved by Nordgren$^{58}$. 

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3.2 The pore pressure distribution.

Consider a flat circular plate of radius \( b \) with a circular hole of radius \( a \) at its center. Neglecting the curvature of a well casing, the flat plate represents a casing surface and the circular hole a single perforation. Above the plate lies a hemisphere of porous rock containing a hemispherical void adjacent to the perforation. If the flow of pore fluid into the perforation is radial, the pore pressure distribution \( (P) \) is given by Darcy's law as

\[
P = P_w + \frac{Q \mu}{4 \pi k} \left( \frac{1}{a} - \frac{1}{r} \right)
\] (3.1a)

where \( Q \) is the volumetric flow rate of pore fluid, \( \mu \) is the fluid viscosity, \( k \) is the rock permeability, \( P_w \) is the pore pressure at the perforation, and \( r \) is a radial coordinate measured from the center of the hole in the plate (see Figure B).

If \( P_e \) is the pore pressure at \( r = b \), then

\[
\frac{Q \mu}{4 \pi k a} = \frac{P_e - P_w}{1 - \frac{a}{b}}.
\] (3.1b)

3.3 The equations of equilibrium.

The total stress \( \tau_{ij} \) on the hemispherical body of rock must satisfy the equations of equilibrium. If the casing surface is taken to be frictionless, then the
equations of equilibrium reduce to

\[ \frac{d\tau_{rr}}{dr} + \frac{2}{r} (\tau_{rr} - \tau_{\theta\theta}) = 0 \]  \hspace{1cm} (3.2a)

where \( \tau_{rr} \) and \( \tau_{\theta\theta} \) are the radial and tangential components of the total stress in spherical coordinates.

Introducing the effective stress \( \sigma_{ij} \), with tension taken as positive, gives

\[ \sigma_{ij} = \tau_{ij} + \delta_{ij} \alpha \]  \hspace{1cm} (3.2b)

where \( \delta_{ij} \) is the Kronecker delta and \( \alpha \) is an experimentally determined constant. Substituting (3.2b) and equations (3.1) into (3.2a), the equilibrium equation becomes

\[ \frac{d\sigma_{rr}}{dr} + \frac{2}{r} (\sigma_{rr} - \sigma_{\theta\theta}) - \frac{\alpha \sigma_0}{r^2} = 0 \]  \hspace{1cm} (3.2c)

where

\[ \sigma_0 = \frac{Q \mu}{4\pi \kappa a} = \frac{P_e - P_w}{1 - \frac{a}{b}}. \]  \hspace{1cm} (3.2d)

3.4 The quadratic yield condition.

If the porous rock is assumed to obey a quadratic yield condition then\(^\text{16}\)

\[ J^2 = k - bI - cI^2 \]  \hspace{1cm} (3.3a)
where $J^2$ is the second invariant of the deviatoric effective stress tensor ($\sigma'$, defined by (3.3e))

$$J^2 = \frac{1}{2} \text{tr} [(\sigma')^T(\sigma')]$$  \hspace{1cm} (3.3b)

and $I$ is the first invariant of the effective stress tensor ($\sigma$, defined by (3.3d))

$$I = \text{tr}(\sigma).$$  \hspace{1cm} (3.3c)

The coefficients $k$, $b$, and $c$ are material constants, "tr" denotes the trace operation, and the superscript "T" denotes the transpose. The effective stress and deviatoric effective stress are given by

$$\sigma = \tilde{\tau} + \alpha PI \tilde{\tau}$$  \hspace{1cm} (3.3d)

$$\sigma' = \sigma - \frac{1}{3} \text{tr}(\sigma)I$$  \hspace{1cm} (3.3e)

where $\tilde{\tau}$ is the total stress tensor and $I$ is the identity tensor.

For the case of radial symmetry, equations (3.3b) and (3.3c) reduce to

$$J^2 = \frac{1}{3} (\sigma_{rr} - \sigma_{\theta\theta})^2$$  \hspace{1cm} (3.4a)

$$I = \sigma_{rr} + 2\sigma_{\theta\theta}. $$  \hspace{1cm} (3.4b)
Inverting these expressions, the effective stresses are

\[ \sigma_{rr} = \frac{1}{3} (I + 2\sqrt{3} \ J) \]  

\[ \sigma_{\theta\theta} = \frac{1}{3} (I - \sqrt{3} \ J) \]  

where

\[ \sqrt{3} \ J = \sigma_{rr} - \sigma_{\theta\theta}. \]  

The quadratic yield surface differs from the Coulomb yield condition in that it allows plastic deformation under hydrostatic loading and more realistically describes the behavior of porous rock.\textsuperscript{16}

3.5 The state of stress in a plastic region around a perforation.

Using the equilibrium equation and the yield condition it is possible to determine the state of stress in a plastic region around the perforation.

Substituting (3.4c) and (3.4e) into (3.2c) gives

\[ \frac{1}{3} \frac{d}{dr} (I + 2\sqrt{3} \ J) + \frac{2\sqrt{3}}{r} J - \frac{\alpha a P_0}{r^2} = 0. \]  

(3.5a)

Next equation (3.3a) may be rearranged to the nondimensional form

\[ \hat{r}^2 + \hat{\theta}^2 = 1 \]  

(3.5b)
where

\[ \hat{t} = \frac{\sqrt{c}}{A} (1 + \frac{b}{2c}) \]  
(3.5c)

\[ \hat{J} = \frac{J}{A} \]  
(3.5d)

\[ A = \sqrt{k + \frac{b^2}{4c}} \]  
(3.5e)

In terms of these nondimensional variables equation (3.5a) becomes

\[ \frac{1}{3} \frac{d}{dr} \left[ \frac{A}{\sqrt{c}} \hat{t} - \frac{b}{2c} + 2\sqrt{3} A \hat{J} \right] + \frac{2\sqrt{3} A}{r} \hat{J} - \frac{\alpha a P_0}{r^2} = 0. \]  
(3.5f)

Defining the nondimensional radial coordinate

\[ \hat{r} = \frac{r}{a} \]  
(3.5g)

and differentiating equation (3.5b)

\[ d \hat{J} = - \frac{\hat{t}}{\sqrt{1 - \hat{t}^2}} d\hat{t} \]  
(3.5h)

allows (3.5f) to be written as follows:

\[ \frac{d\hat{t}}{d\hat{r}} = \frac{3 \left[ \frac{\alpha}{r^2} - \frac{2\sqrt{3} c}{r} \sqrt{1 - \hat{t}^2} \right]}{1 - \frac{2\sqrt{3} \hat{t}}{\sqrt{1 - \hat{t}^2}}} \]  
(3.5i)
where
\[ \beta = \alpha \sqrt{\frac{a}{c}} \frac{P_0}{A}. \]  \hfill (3.5j) 

The solution of equation (3.5i) determines approximately the plastic state of stress in the rock around a single casing perforation. Since equations (3.2d), (3.5e), and (3.5j) show that

\[ \beta = \frac{\alpha Q u}{4 \pi \kappa a} = \frac{\alpha (P_e - P_w)}{\sqrt{\frac{k}{c} + \frac{b^2}{4c^2}} (1 - \frac{a}{b}) \sqrt{\frac{k}{c} + \frac{b^2}{4c^2}}} \]  \hfill (3.5k) 

it can be seen that the rock stresses are dependent upon the pore fluid flow field and the rock and fluid material properties.

3.6 The plastic state of stress with no fluid flow.

For the case of no fluid flow, \( \beta = 0 \), equation (3.5i) may be integrated analytically. Separating variables in that expression yields

\[ -6 \sqrt{3} \frac{d \hat{r}}{\hat{r}} = \left[ \frac{1}{\sqrt{1 - \hat{t}^2}} - \frac{2 \sqrt{3} \hat{t}}{(1 - \hat{t}^2)} \right] d \hat{t} \]  \hfill (3.6a) 

with the result
\[ \hat{r} = \frac{c_1 \sin^{-1} \hat{r}}{(1 - \hat{r}^2)^{1/6} \exp\left(\frac{-1}{6\sqrt{3c}}\right)} \]  

(3.6b)

where \( c_1 \) is a constant of integration. The value of \( c_1 \) is obtained from the boundary condition

\[ \sigma_{rr} = 0 \text{ at } r = a. \]  

(3.6c)

With the value of \( \hat{I} \) known, \( \hat{J} \) is calculated from equation (3.5b) and the plastic state of stress is specified.

As an illustration consider the special case

\[ c = \frac{1}{12} \]  

(3.7a)

\[ k = 0. \]  

(3.7b)

This value of \( k \) describes a "cohesionless" material. The boundary condition (3.6c) now becomes

\[ \hat{I} = 0 \text{ at } \hat{r} = 1 \]  

(3.7c)

so that

\[ c_1 = 1. \]  

(3.7d)

The variation of the nondimensional trace of the effective stress with the nondimensional radius of the plastic zone, as given by equation (3.6b), is shown in Figure 1 (\( \beta = 0 \)).
3.7 The plastic state of stress with fluid flow.

When the rock is subjected to a nonuniform pore pressure distribution, the nondimensional flow rate must be specified and equation (3.5i) integrated numerically.

To illustrate the effect of fluid flow on the rock stresses, consider again the special case described by equations (3.7a) through (3.7c). The relation (3.5i) is now

\[
\frac{d\hat{f}}{dr} = \frac{3\left(\frac{\beta}{\hat{f}^2} - \frac{1}{\hat{f}} \frac{1}{\sqrt{1 - \hat{f}^2}}\right)}{1 - \frac{1}{\sqrt{1 - \hat{f}^2}}}. \tag{3.8a}
\]

With the initial condition (3.7c) this expression was integrated using a fourth order Runge-Kutta method for two values of $\beta$. The results are shown in Figure 1. Again equation (3.5b) can be used to determine the plastic state of stress at a particular radial position.

The boundary condition (3.7c) along with (3.8a) shows that

\[
\frac{1}{3} \frac{d\hat{f}}{dr} = \beta - 1 \text{ at } \hat{f} = 1. \tag{3.8b}
\]

The limiting case $\beta = 1$ is shown in Figure 1. To interpret this limiting case, note that equations (3.4c) and (3.5d) through (3.5e) give the radial effective stress as
\[
\frac{\sigma_{rr}}{2b} = \hat{I} - 1 + \sqrt{1 - \hat{I}^2}.
\] (3.8c)

Hence tensile radial effective stresses are developed for \( \hat{I} \) positive. For \( \beta \) values larger than one, \( \hat{I} \) takes on positive values near \( \hat{I} = 1 \) and tensile radial stresses are developed in the vicinity of the perforation. Since these tensile stresses may be conducive to the creation of voids near the casing surface, the condition \( \beta = 1 \) may be used to define a limit to the allowable well drawdown. For this example equation (3.5k) reduces to

\[
\beta = \frac{\alpha(P_e - P_w)}{(1 - \frac{\beta}{D})(6b)}
\] (3.8d)

and with

\[
\beta = 1
\] (3.8e)

\[
\alpha = 1
\] (3.8f)

\[
P_e = 6500 \text{ psi}
\] (3.8g)

\[
b = 624 \text{ psi}
\] (3.8h)

\[
a \ll b
\] (3.8i)

the minimum allowable well pressure is

\[
P_w = 2576 \text{ psi}.
\] (3.8j)

3.8 Conclusion

The plastic stress distribution around a spherical cavity derived in this chapter describes approximately the
state of stress around hemispherical rock surfaces adjacent to casing perforations. The influence of fluid flow rate, rock cohesive strength, and other parameters on the stability of a hemispherical interface has been determined.

In the absence of pore fluid flow, the effective stresses are compressive and the material rapidly approaches a state of hydrostatic compression with increasing distance from the perforation surface. The introduction of a fluid flow field causes a reduction in the magnitude of the mean compressive stress, until tensile radial stresses develop in the immediate vicinity of the perforation. Since tensile radial stresses tend to draw material from the rock surface into the void around the perforation, the critical flow rate creates conditions tending to enlarge the existing hemispherical cavity. The desire to avoid the development of cavities adjacent to the casing surface thus defines a limit to the fluid flow rate, that is a limit for the well pressure. The variation of the critical well pressure with rock strength parameters for a cohesionless material is shown by the "cavity formation" curve in Figure 14. The results shown are consistent with those obtained previously through the use of finite element analysis\textsuperscript{16,59}. 
CHAPTER 4.

ASYMMETRIC LOADING ON A CYLINDER IN A POROELASTIC MEDIUM

4.1 Introduction

The distribution of rock stresses around a perforated casing may be studied through the application of complex variable methods to poroelasticity. Rice and Cleary\textsuperscript{22} formulated the coupled poroelasticity problem using complex variables, and applied it to suddenly pressurized cylindrical and spherical cavities. They present solutions for hydraulic fracturing problems, in which the pore pressure in the cavity is raised suddenly to a very large value, and the pore fluid diffuses outward. An extension of their work relevant to this study would be the solution of the coupled problem for a conventional producing reservoir model of the type applied in this chapter.

In this chapter England's\textsuperscript{23} two-dimensional analysis of elasticity problems with body forces is employed to solve several uncoupled porous media problems with complex variables. Specifically the rock stresses around open, fully supported, and partially supported well bores are studied to determine the boundary loads and the flow rates for which plastic yielding of the solid is initiated. The case of a well bore supported by a liner with a single slot may be considered a two-dimensional analog to the perforated casing problem (see Figure C ).

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4.2 Problem statement.

The objective of the elastic analysis in this chapter is to determine the stress distribution in an infinite body of porous elastic solid containing a circular hole, with the well bore located along the circular hole. The boundary conditions are a specified constant state of stress at infinity and zero displacement or radial stress on the boundary of the hole. The latter condition implies a casing which is permeable but absent along a slot (see Figure D).

4.3 Method of solution

Two-dimensional elasticity problems with body forces, like the one described in Section 4.2, may be solved using the method of complex variables. Mushkhelishvili\textsuperscript{25} and Green and Zerna\textsuperscript{26} describe the development and application of this method. Timoshenko and Goodier\textsuperscript{27} employ it in the solution of thermoelastic problems, which are similar in many respects to poroelastic problems of the type considered here. England's\textsuperscript{23} treatment of the complex variable approach will be followed below, since he shows clearly how body force effects are accounted for, and since he works various examples involving the geometry of the problem in Section 4.2.
4.4 The equations of equilibrium and the stress-strain relations.

The total stress \( \tau_{ij} \) on the solid-fluid mixture must satisfy the equations of equilibrium for a plane problem without body forces

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (4.1a)
\]

\[
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0 \quad (4.1b)
\]

where \( x \) and \( y \) are rectangular coordinates, related to the complex variable \( z \) by

\[
z = x + iy, \quad i = \sqrt{-1}. \quad (4.1c)
\]

In terms of the polar coordinates \( r \) and \( \theta \),

\[
z = re^{i\theta}. \quad (4.1d)
\]

The total stress on the mixture is a function of the effective stress \( \sigma_{ij} \), the pore pressure \( P \), and an experimentally determined constant \( \alpha \), and can be written

\[
\tau_{ij} = \sigma_{ij} - \delta_{ij} \alpha P \quad (4.1e)
\]

where \( \delta_{ij} \) is the Kronecker delta and tension is taken as positive. If (4.1e) is substituted into (4.1a) and (4.1b) the equilibrium equations become
\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \alpha \frac{\partial P}{\partial x} = 0 \tag{4.2a}
\]
\[
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \alpha \frac{\partial P}{\partial y} = 0. \tag{4.2b}
\]

The effective stress determines the strain in the porous solid, so that the pertinent stress-strain relations are
\[
\sigma_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} \lambda \varepsilon \tag{4.2c}
\]
where \(\mu\) and \(\lambda\) are Lamé coefficients for the porous solid, the \(\varepsilon_{ij}\) are components of the strain in the porous solid, and
\[
\varepsilon = \varepsilon_{kk} \tag{4.2d}
\]
with the repeated index indicating a sum over
\[
i = x, y, z. \tag{4.2e}
\]

With the assumption of plane strain
\[
\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{4.2f}
\]
equations (4.2c) reduce to
\[
\sigma_{xx} = (\lambda + 2\mu) \varepsilon_{xx} + \lambda \varepsilon_{yy} \tag{4.2g}
\]
\[
\sigma_{yy} = (\lambda + 2\mu) \varepsilon_{yy} + \lambda \varepsilon_{xx} \tag{4.2g}
\]
\[
\sigma_{xy} = 2\mu \varepsilon_{xy} \tag{4.2i}
\]
\[
\sigma_{zz} = \lambda (\varepsilon_{xx} + \varepsilon_{yy}) = \frac{\lambda}{2(\mu + \lambda)} (\sigma_{xx} + \sigma_{yy}). \tag{4.2j}
\]
Equations (4.2a), (4.2b), and (4.2g) through (4.2j) show that the plane strain poroelastic problem takes the form of a conventional elasticity problem with body forces when written in terms of the effective stress $\sigma_{ij}$. Thus the established methods of complex variables may be applied without modification to the problem of interest. In the subsequent discussion, England's\textsuperscript{23} general formulas will be repeated as needed with the stress variable in his equations replaced by the effective stress $\sigma_{ij}$.

4.5 Stresses and displacements expressed in terms of complex potential functions.

The strains in the porous solid $e_{ij}$ are determined from the displacements in the $x$ and $y$ directions ($u$ and $v$) using

\begin{align*}
e_{xx} &= \frac{\partial u}{\partial x} \quad (4.3a) \\
e_{yy} &= \frac{\partial v}{\partial y} \quad (4.3b) \\
e_{xy} &= \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \quad (4.3c)
\end{align*}

If these equations are substituted into the constitutive relations (4.2g) through (4.2i), the resulting expressions allow the equilibrium equations (4.2a) and (4.2b) to be written in the form
\[(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{F} = 0 \quad (4.3d)\]

where

\[\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (4.3e)\]

and \(\nabla\), \(\mathbf{u}\), and \(\mathbf{F}\) are vectors with \(x\) and \(y\) components

\[\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] \quad (4.3f)\]

\[\mathbf{u} = [u, v] \quad (4.3g)\]

\[\mathbf{F} = [-\alpha \frac{\partial P}{\partial x}, -\alpha \frac{\partial P}{\partial y}] \quad (4.3h)\]

If Navier's displacement equations of equilibrium \((4.3d)\) are written in terms of the complex variable \(z\) and its conjugate

\[\overline{z} = x - iy = re^{-i\theta} \quad (4.3i)\]

it can be shown that there exists a general solution for the displacements that depends on three functions \(\Omega(z)\), \(\omega(z)\), and \(X(z)\). The functions \(\Omega(z)\) and \(\omega(z)\) are complex potentials whose form must be determined for the particular boundary value problem under consideration, while \(X(z)\) is derived from the given body force distribution. The general solution for the displacement is

\[2\mu(u + iv) = \hat{\kappa} \Omega(z) - z \overline{\Omega'(z)} - \overline{\omega(z)} - \frac{\mu}{2(\lambda + 2\mu)} X(z) \quad (4.4a)\]
where a horizontal bar denotes the complex conjugate, the prime denotes differentiation with respect to \( z \), and

\[
\hat{k} = \frac{\lambda + 3\mu}{\lambda + \mu}.
\]  

(4.4b)

Using the strain-displacement and constitutive relations the stresses produced by the displacements (4.4a) can be determined, the result being

\[
\sigma_{xx} + \sigma_{yy} = 2[\Omega'(z) + \overline{\Omega'(z)}] - \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) V(z)
\]  

(4.4c)

\[
\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2[z \overline{\Omega'(z)} + \overline{\omega'(z)}] - \frac{\mu}{(\lambda + 2\mu)} \frac{\partial X(z)}{\partial z}
\]  

(4.4d)

where the double prime indicates the second derivative with respect to \( z \) and \( V(z) \) is the body force potential

\[
V(z) = \frac{\partial X(z)}{\partial z}.
\]  

(4.4e)

In equations (4.4a), (4.4c), and (4.4d) the independent variables \( z \) and \( \overline{z} \) have taken the place of \( x \) and \( y \). These equivalent pairs of variables are related by the transformations (4.1c) and (4.3i). The notation used to indicate functional dependence

\[
f = f(z)
\]  

(4.4f)
is understood to be shorthand for

\[ f = f(z, \bar{z}). \]  \hfill (4.4g)

With stresses and displacements expressed in terms of
the complex potentials \( \Omega(z) \), \( \omega(z) \), \( V(z) \) and the particular
integral \( X(z) \), it is necessary to determine these func-
tions for the problem in Section 4.2. The body force poten-
tial \( (V(z)) \) and the particular integral \( (X(z)) \) are cal-
culated from the pore pressure distribution.

In polar coordinates the stresses and displacements
are\footnote{28}

\[ \sigma_{rr} + \sigma_{\theta\theta} = \sigma_{xx} + \sigma_{yy} \] \hfill (4.5a)
\[ \sigma_{rr} - \sigma_{\theta\theta} + 2i\sigma_{r\theta} = (\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy})e^{-2i\theta} \] \hfill (4.5b)
\[ u_r + iu_\theta = (u + iv)e^{-i\theta} \] \hfill (4.5c)

which with equations (4.4) give

\[ \sigma_{rr} + i\sigma_{r\theta} = \Omega'(z) + \overline{\Omega'(z)} - z\Omega''(z) - \frac{\omega'}{\mathcal{Z}} - \frac{1}{2}(\frac{\lambda+\mu}{\lambda+2\mu})V(z) - \frac{1}{2}(\frac{\mu}{\lambda+2\mu})\overline{\omega'} \] \hfill (4.5d)
\[ \frac{\partial X(z)}{\partial z}. \] \hfill (4.5e)

In the following sections England's\footnote{23} general solutions
will be used to determine the functions \( \Omega(z) \) and \( \omega(z) \) for
the particular boundary value problems under consideration.
4.6 The pore pressure distribution

If the well bore is taken to be a circular hole of radius \( R \) in an infinite medium, then the pore fluid flow field can be described by the constant terminal rate solution\(^{29}\) for the pore pressure \( P \)

\[
P = P_\infty - \hat{Q} \int_0^\infty \frac{e^{-s}}{s} \, ds
\]  \hspace{1cm} (4.6a)

\[
\hat{Q} = \frac{Q}{4\pi kh}
\]  \hspace{1cm} (4.6b)

\[
\varepsilon = \frac{n\mu_cr^2}{4k\tau}
\]  \hspace{1cm} (4.6c)

where \( P_\infty \) is the initial reservoir pressure, \( Q \) is the fluid volumetric flow rate, \( k \) is the formation permeability, \( h \) is the formation thickness, \( n \) is the formation porosity, \( c \) is the fluid compressibility (assumed small and constant), and \( \tau \) is the time after the onset of production from the well. The quantity \( \hat{Q} \) represents the rate at which fluid is being produced from the given reservoir, while \( \varepsilon \) is determined by the particular combination of radial position \( (r) \) and time \( (\tau) \) for which the pressure is desired.

The pore pressure gradient acts as a pseudo-body force \( (F) \) loading the solid
\[ F = -a \left( \frac{\partial P}{\partial x} + i \frac{\partial P}{\partial y} \right) = -2a \frac{\partial P}{\partial z}. \] 

(4.7a)

Since the body force is related to its potential by\(^3\)

\[ F = \frac{\partial V(z)}{\partial z} \] 

(4.7b)

then

\[ V(z) = -2aP. \] 

(4.7c)

The particular integral is related to the body force potential by

\[ V(z) = \frac{\partial X(z)}{\partial z}. \] 

(4.8a)

Integration of this expression yields

\[ X(z) = z(V(z) - 2a \hat{Q} \frac{e^{-\xi}}{\xi}). \] 

(4.8b)

Differentiating this relation to evaluate the stresses in (4.4) results in

\[ \frac{\partial X(z)}{\partial z} = 2a \hat{Q} \frac{e^{-\xi}}{\xi} \frac{z}{z}. \] 

(4.8c)

4.7 The rock stresses around an open well bore.

Consider the stress distribution in the porous solid around an open well bore of radius \( R \). For an infinite reservoir, the pore pressure is given by equation (4.6a).

Given a uniform stress state and zero rotation at infinity
\[ \sigma_{xx} = \sigma_{yy} = -\sigma_w \]  
(4.9a)  
\[ \sigma_{xy} = 0 \]  
(4.9b)  

the form of the functions \( \Omega(z) \) and \( \omega(z) \) may be determined.

For this radially symmetric problem, the resultant force over the hole is zero and the general forms for the complex potentials are\(^{32}\)

\[ \Omega(z) = (A + iB)Rz + \Omega_0(z) \]  
(4.10a)  
\[ \omega(z) = (C + iD)Rz + \omega_0(z) \]  
(4.10b)  

where \( A, B, C, \) and \( D \) are constants, and

\[ \zeta = \frac{z}{R} \]  
(4.11)  

is a dimensionless coordinate.

The complex potentials are determined by the Cauchy integrals\(^{33}\)

\[ \Omega_0(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{F_0(t)}{(t - z)} \, dt \]  
(4.12b)  
\[ \omega_0(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{F_0(t)}{(t - z)} \, dt - \frac{m(1/\zeta)}{m'(\zeta)} \Omega_0'(z) \]  
(4.12b)  

where

\[ t = e^{i\Theta} \]  
(4.12c)  

is the variable of integration around the circular hole in the infinite medium.
The function $m(\zeta)$ is the conformal transformation

$$m(\zeta) = R\zeta = z$$  \hspace{1cm} (4.13)

which, along with the constants in (4.10), determines the integrands in (4.12)

$$F_0(t) = iR(t) - (A + iB)Rt - (A - iB)R \frac{m(t)}{m'(\zeta)} - \frac{(C - iD)}{t}.\hspace{1cm} (4.14)$$

The function $R(t)$ in the last equation has the form

$$R(\zeta) = i\hat{R}m(\zeta)\hspace{1cm} (4.15)$$

where $\hat{R}$ is calculated below for the given body forces.

For the stress boundary value problem, the constant $\gamma$ in (4.12a) has the value

$$\gamma = 1.\hspace{1cm} (4.16)$$

Finally the prime in (4.12b) and (4.14) denotes differentiation with respect to $\zeta$ (or $t$) and the path of integration ($\Gamma$) is the contour

$$|t| = 1\hspace{1cm} (4.17)$$

described in an anticlockwise sense.

The constant $\hat{R}$ is determined by the boundary condition along $\Gamma$, which is the well bore. Equations (4.5d),
(4.7c), and (4.8c) give the radial and shearing stresses due to the body forces as

\[
\sigma_{rr} = \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha P - \left( \frac{\mu}{\lambda + 2\mu} \right) \alpha Q \frac{e^{-\varepsilon}}{\varepsilon} \quad (4.18a)
\]

\[
\sigma_{r\theta} = 0
\]

which is along the well bore

\[
\sigma_{rr} = \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha P_w - \left( \frac{\mu}{\lambda + 2\mu} \right) \alpha Q \frac{e^{-\varepsilon}}{\varepsilon} \quad (4.18b)
\]

From equation (4.6c), the value of \( \varepsilon \) along \( r = R \) is defined as

\[
\varepsilon = \frac{n \bar{u} c R^2}{4kt} \quad (4.18c)
\]

while \( P_w \) is the well pressure.

With a boundary condition of zero radial and shearing stresses along the well bore, the complex potentials must cancel the stress in (4.18b) due to the body force.\(^{34}\) This requires\(^{35}\)

\[
\hat{P} = \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha P_w - \left( \frac{\mu}{\lambda + 2\mu} \right) \alpha \hat{Q} \frac{e^{-\varepsilon}}{\varepsilon} \quad (4.19)
\]

Next the constants in (4.10) must be found from the specified state of stress at infinity, equations (4.9). The complex potentials (4.10) and the body forces in (4.5) give
a general state of stress at infinity\(^{36}\)

\[
\sigma_{xx} + \sigma_{yy} = 4A + 2\left(\frac{\lambda}{\lambda + 2\mu}\right)\alpha P_\infty
\]  \hspace{1cm} (4.20a)

\[
\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2(C - iD)
\]  \hspace{1cm} (4.20b)

where from (4.8c)

\[
\frac{\partial X(z)}{\partial z} \to 0 \text{ as } z \to \infty.
\]  \hspace{1cm} (4.20c)

With (4.9) the last three equations imply

\[
C = D = 0
\]  \hspace{1cm} (4.21a)

\[
A = \frac{1}{2}\left[\sigma_\infty + \left(\frac{\lambda}{\lambda + 2\mu}\right)\alpha P_\infty\right].
\]  \hspace{1cm} (4.21b)

Since the rotation at infinity \((\omega_\infty)\) is zero and\(^{36}\)

\[
\omega_\infty = \text{Im}\left[\frac{\partial}{\partial z}(u + iv)\right] = (1 + \hat{k})\frac{B}{2\mu}
\]  \hspace{1cm} (4.22a)

where \text{Im} denotes the imaginary part of the function in brackets, the last constant in (4.10) is

\[
B = 0.
\]  \hspace{1cm} (4.22b)

As implied by the form of (4.22a), the body force potential is real-valued.\(^{37}\)

Equation (4.14) is now

\[
F_0(t) = -(\hat{\beta} + 2A)Rt
\]  \hspace{1cm} (4.23)
so that

$$\Omega_0(\zeta) = \frac{\hat{\rho} + 2A}{2\pi i} \int_{\Gamma \setminus (t - \zeta)} \frac{tdt}{t(t - \zeta)}.$$  (4.24a)

This Cauchy integral has the value\textsuperscript{38}

$$\Omega_0(\zeta) = 0$$  (4.24b)

since $\zeta$ is holomorphic in the region

$$|\zeta| < 1.$$  (4.24c)

With

$$F_0(\zeta) = -(\hat{\rho} + 2A) \frac{R}{\zeta}$$  (4.25a)

and

$$\Omega_0^1(\zeta) = 0$$

equation (4.12b) is

$$\omega_0(\zeta) = \frac{\hat{\rho} + 2A}{2\pi i} \int_{\Gamma \setminus (t - \zeta)} \frac{dt}{t(t - \zeta)}.$$  (4.26a)

This Cauchy integral has the value\textsuperscript{38}

$$\omega_0(\zeta) = -(\hat{\rho} + 2A) \frac{R}{\zeta}$$  (4.26b)

since $\zeta^{-1}$ is holomorphic in the region

$$|\zeta| > 1.$$  (4.26c)

The previous development, with the conformal mapping (4.13), gives the complex potentials for the stress boundary value problem as
\( \Omega(z) = Az \)  \hspace{1cm} (4.27a)

\[ \omega(z) = -(\hat{p} + 2A) \frac{R^2}{z} \]  \hspace{1cm} (4.27b)

Equations (4.4) and (4.5) show that the stresses and displacements are therefore

\[ \sigma_{rr} + \sigma_{\theta\theta} = 4A + 2(\frac{\lambda + \mu}{\lambda + 2\mu})\alpha p \]  \hspace{1cm} (4.28a)

\[ \sigma_{rr} + i\sigma_{r\theta} = 2A - (\hat{p} + 2A) \frac{R^2}{z^2} + (\frac{\lambda + \mu}{\lambda + 2\mu})\alpha p \]

\[ - (\frac{\mu}{\lambda + 2\mu})\alpha \hat{Q} \frac{\Theta}{\xi}^{-\epsilon} \]  \hspace{1cm} (4.28b)

\[ 2\mu(u_r + iu_\theta)(\frac{z}{z})^{1/2} = (k - 1)\alpha z + (\hat{p} + 2A) \frac{R^2}{z} \]

\[ + (\frac{\mu}{\lambda + 2\mu})\alpha z(\hat{p} + \hat{Q} \frac{\Theta}{\xi}^{-\epsilon}) \]  \hspace{1cm} (4.28c)

These expressions reduce to

\[ \sigma_{rr} + \sigma_{\theta\theta} = -2[\sigma_\infty + (\frac{\lambda + \mu}{\lambda + 2\mu})\alpha(p_\infty - p)] \]  \hspace{1cm} (4.29a)
\[ \sigma_{rr} = -\sigma_\infty (1 - \frac{R^2}{r^2}) + \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha (P_\infty - P_w) \frac{R^2}{r^2} \]

\[- \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha (P_\infty - P) - \left( \frac{\mu}{\lambda + 2\mu} \right) \frac{(e^{-\varepsilon} - e^{-\varepsilon})}{\varepsilon} \alpha \hat{Q} \tag{4.29b} \]

\[ \sigma_{r\theta} = 0 \tag{4.29c} \]

\[ 2\mu u_r = -\sigma_\infty \left[ \frac{\mu}{(\lambda + \mu)} + \frac{R^2}{r^2} \right] r \]

\[- \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \left[ \frac{\mu}{(\lambda + \mu)} \alpha P_\infty + \alpha (P_\infty - P_w) \frac{R^2}{r^2} \right] r \]

\[ + \left( \frac{\mu}{\lambda + 2\mu} \right) \alpha P + \alpha \hat{Q} \left( \frac{e^{-\varepsilon} - e^{-\varepsilon}}{\varepsilon} \right) r \tag{4.29d} \]

\[ u_\theta = 0. \tag{4.29e} \]

At the well bore, where

\[ r = R \tag{4.30} \]

the stresses and displacements are

\[ \sigma_{rr} = u_\theta = 0 \tag{4.31a} \]

\[ \sigma_{\theta\theta} = -2[\sigma_\infty + \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha (P_\infty - P_w)] \tag{4.31b} \]

\[ u_r = -\frac{R}{2\mu} \left[ \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \sigma_\infty + \alpha (P_\infty - P_w) \right]. \tag{4.31c} \]
4.8 The rock stresses around a fully supported well bore.

In this section the stress distribution around a cased well bore will be estimated by taking the boundary condition on the rock around the casing to be that of zero displacement. The method of solution is very similar to that employed in the last section.

For the displacement boundary value problem equations (4.14) and (4.16) are replaced by

\[
F_0(t) = -2\mu D(t) + \hat{k}(A + iB)Rt - (A - iB)R \frac{m(t)}{m'(t)}
\]

\[-(C - iD) \frac{R}{t} \]

\[
\gamma = -\hat{k}. \quad (4.32)
\]

The displacement due to the body force must be negated by the complex potentials in order to realize a net zero displacement on the boundary \( \Gamma \). With (4.4a) and (4.8b) this requires

\[
2\mu D(t) = \frac{\mu}{2(\lambda + 2\mu)}(-2\alpha Rt)(P_w + \hat{Q} \frac{\phi}{\epsilon}) \quad (4.34a)
\]

or

\[
2\mu D(t) = -2\mu \hat{D}t \quad (4.34b)
\]

where

\[
\hat{D} = \frac{\alpha R}{2(\lambda + 2\mu)}(P_w + \hat{Q} \frac{e^{-\epsilon}}{\epsilon}). \quad (4.34c)
\]
With the boundary conditions at infinity the same as in the last section, the constants $A$, $B$, $C$, and $D$ retain their previous values. Hence

$$F_0(t) = [2\mu \hat{D} + (\hat{k} - 1)AR] t$$

(4.35a)

and (4.12a) yields

$$\Omega_0(\xi) = 0.$$  

(4.35b)

Since

$$\overline{F_0}(t) = [2\mu \hat{D} + (\hat{k} - 1)AR](1/t)$$

(4.36a)

the integral in (4.12b) is

$$\omega_0(\xi) = [2\mu \hat{D} + (\hat{k} - 1)AR](1/\xi).$$

(4.36b)

The complex potentials (4.10) are now

$$\Omega(z) = Az$$

(4.37a)

$$\omega(z) = [2\mu \hat{D} + (\hat{k} - 1)AR](R/z).$$

(4.37b)

With these potential functions the stresses and displacements are

$$\sigma_{rr} + \sigma_{\theta\theta} = 4A + 2(\frac{\lambda}{\lambda + 2\mu})\alpha P$$

(4.38a)

$$\sigma_{rr} + i\sigma_{r\theta} = 2A + [2\mu \hat{D} + (\hat{k} - 1)AR] \frac{R}{zz}$$

$$+ \left(\frac{\lambda}{\lambda + 2\mu}\right)\alpha P - \left(\frac{\mu}{\lambda + 2\mu}\right)\alpha \frac{\phi}{\epsilon} e^{-\epsilon}$$

(4.38b)
\[ 2\mu(u_r + i u_\theta)(\frac{z}{z})^{1/2} = (\hat{k} - 1)Az - [2\mu \hat{D} + (\hat{k} - 1)AR] \frac{R}{z} \]
\[ + \left( \frac{\mu}{\lambda + 2\mu} \right) \alpha z(P + \hat{Q} \frac{e^{-\epsilon}}{\epsilon}) \quad (4.38c) \]

These expressions reduce to

\[ \sigma_{rr} + \sigma_{\theta\theta} = -2[\sigma_\infty + \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha(P_\infty - P)] \quad (4.39a) \]

\[ \sigma_{rr} = - [\sigma_\infty + \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha P_\infty] \left[ 1 + \frac{\mu}{(\lambda + \mu)} \frac{R^2}{r^2} \right] + \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha P \]
\[ + \left( \frac{\mu}{\lambda + 2\mu} \right) \left[ \alpha P w \frac{R^2}{r^2} - \alpha \hat{Q} \left( \frac{e^{-\epsilon} - e^{-\xi}}{\epsilon} \right) \right] \quad (4.39b) \]

\[ \sigma_{r\theta} = 0 \quad (4.39c) \]

\[ 2\mu u_r = \frac{-\mu}{(\lambda + \mu)} \left[ \sigma_\infty + \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) \alpha P_\infty \right] (1 - \frac{R^2}{r^2}) r \]
\[ + \left( \frac{\mu}{\lambda + 2\mu} \right) [\alpha P - \alpha P w \frac{R^2}{r^2} + \alpha \hat{Q} \left( \frac{e^{-\epsilon} - e^{-\xi}}{\epsilon} \right)] r \quad (4.39d) \]

\[ u_\theta = 0. \quad (4.39e) \]

At the casing surface, where

\[ r = R \quad (4.40) \]

the stresses and displacements are
\[ \sigma_{r\theta} = u_\theta = u_r = 0 \]  
\[ \sigma_{rr} = -\left(\frac{\lambda + 2\mu}{\lambda + \mu}\right)\sigma_\infty - \alpha(P_\infty - P_w) \]  
\[ \sigma_{\theta\theta} = -\left(\frac{\lambda}{\lambda + \mu}\right)\sigma_\infty - \left(\frac{\lambda}{\lambda + 2\mu}\right)\alpha(P_\infty - P_w). \]

4.9 The rock stresses around a partially supported well bore.

A principal obstacle in the analytical study of rock stresses around casing perforations is the difficulty in describing simultaneously the unstressed rock surface adjacent to the perforation and the supported rock surface cemented to the casing. However for the two-dimensional analog of the perforation problem, complex variable methods allow the study of the mixed boundary value problem of a well bore supported along only part of its circumference.

Again take the well bore to lie in an infinite porous solid with the pore pressure distribution given in Section 4.6. The effect of partially supporting the wall of the well can be obtained by negating the displacement obtained in Section 4.7 over all but a single arc of the contour \( \Gamma \). Physically this approximates the stresses around a well bore supported by a rigid permeable liner with a single slot.
The solution to the stated problem is obtained by superimposing on the stresses in Section 4.7 the stresses obtained for the following mixed boundary problem.

Consider a circular hole of radius $R$ in an infinite medium with zero stresses and rotation at infinity. Along the circular hole the boundary conditions are

\[ \sigma_{rr} + i\sigma_{r\theta} = 0 \quad \text{for } t = Re^{i\theta} \text{ on } L', \quad (4.42a) \]
\[ u + iv = f(t) \quad \text{for } t = Re^{i\theta} \text{ on } L \quad (4.42b) \]

where the arcs $L$ and $L'$ are defined by

$L : r = R, \ |\theta| < \phi$
$L' : r = R, \ |\theta| > \phi, -\pi \leq \theta \leq \pi.$

The slot in the liner is along $L'$ where the effective stress will be zero. From (4.5c) and (4.31c) the specified displacement on $L'$ must be

\[ f(t) = \hat{u}t \quad (4.43a) \]

\[ \hat{u} = \frac{1}{2\mu} \left[ \left( \frac{\lambda + 2\mu}{\lambda} \right) \sigma_{\infty} + \alpha(P_{\infty} - P_w) \right]. \quad (4.43b) \]

England\textsuperscript{41} shows that the solution for the posed problem is given by

\[ \sigma_{rr} + \sigma_{r\theta} = 2[\Omega'(z) + \overline{\Omega'(z)}] \quad (4.44a) \]
\[ \sigma_{rr} + i \sigma_{r\theta} = \Omega'(z) - \frac{R^2}{zz} \Omega'(\frac{R^2}{z}) \]
\[ + (1 - \frac{R^2}{zz}) [\Omega'(\bar{z}) - \bar{z} \Omega''(\bar{z})] \quad (4.44b) \]

where

\[ \Omega'(z) = \frac{\mu G(z)}{k \pi i} \int_{L} \frac{f'(t)dt}{G^+(t)(t-z)} + \psi(z)G(z) \quad (4.45) \]

\[ G(z) = (z - Re^{i\phi})^{-\gamma(z - Re^{-i\phi})^{-1}} \gamma - 1 \quad (4.46) \]

\[ \gamma = \frac{1}{2} + i\beta, \quad \beta = \frac{1}{2\pi} \ln(k) \quad (4.47) \]

\[ \psi(z) = \frac{D_{-2}}{z^2} + \frac{D_{-1}}{z} + D_0 + D_1z \quad (4.48) \]

and the \( D_i \) are constants. The function \( G^+(t) \) is the limit approached by \( G(t) \) as \( z \) approaches a point on \( L \) from the region outside the circular hole.\(^{43}\)

The integral in (4.45)

\[ I(z) = \int_{L} \frac{f'(t)dt}{G^+(t)(t-z)} \quad (4.49a) \]

can be written

\[ I(z) = \frac{k}{(1+k)} [2\pi i \frac{\hat{u}}{G(z)} - \lim_{a \to \infty} \int_{0}^{2\pi} \frac{\hat{u} e^{i\theta} i d\theta}{G(\hat{u} e^{i\theta})(\hat{u} e^{i\theta} - z)}]. \quad (4.49b) \]
The value of $I(z)$ depends on the form of $G(z)^{-1}$ for large $z$, which is

$$\frac{1}{G(z)} = z[l - \frac{1}{z}(\gamma \text{Re}^i\phi + (1 - \gamma)\text{Re}^{-i\phi}) + ...]. \quad (4.50)$$

With this expansion, (4.49b) becomes

$$I(z) = \frac{2\pi i \hat{\mu}}{(1 + \hat{k})} \left[ \frac{1}{G(z)} - z + \gamma \text{Re}^i\phi + (1 - \gamma)\text{Re}^{-i\phi} \right] \quad (4.51)$$

and therefore the derivative of the potential function in (4.45) is

$$\Omega'(z) = \frac{2\mu \hat{\mu}}{(1 + \hat{k})} \left[ l - (z - \gamma \text{Re}^i\phi - (1 - \gamma)\text{Re}^{-i\phi})G(z) \right]$$

$$+ \psi(z)G(z). \quad (4.52)$$

The constants in $\psi(z)$ are determined from the form of $\Omega'(z)$ at infinity and the origin.\(^{47}\)

For large $z$, $\Omega'(z)$ takes the form\(^{48}\)

$$\Omega'(z) = \frac{-\hat{x}}{2\pi(1 + \hat{k})} \frac{1}{z} + 0(\frac{1}{z^2}) \quad (4.53)$$

where the second term on the right hand side denotes a quantity of order $\frac{1}{z^2}$ and $\hat{x}$ is the resultant force over the arc $L$, a real number by symmetry.

For small $z$, $\Omega'(z)$ has the form\(^{49}\)
\[ \Omega'(z) = \frac{\hat{k} \hat{x}}{2\pi (1 + \hat{k})} \frac{1}{z} + \text{constant.} \quad (4.54) \]

For large \( z \), (4.46) and (4.52) become

\[ G(z) = \frac{1}{z} + \frac{\gamma \text{Re} e^{i\phi} + (1 - \gamma) \text{Re} - e^{i\phi}}{z^2} + O \left( \frac{1}{z^3} \right) \quad (4.55) \]

\[ \Omega'(z) = \frac{2\mu \hat{u}}{(1 + \hat{k})} \left\{ \frac{1}{z} + \gamma \text{Re} e^{i\phi} + (1 - \gamma) \text{Re} - e^{i\phi} \right. \]
\[ - \frac{\gamma \text{Re} - e^{i\phi} + (1 - \gamma) \text{Re} e^{i\phi}}{z} + O \left( \frac{1}{z^2} \right) \}
\]
\[ + \frac{D_0}{z} + D_1 \left[ 1 + \frac{\gamma \text{Re} e^{i\phi} + (1 - \gamma) \text{Re} - e^{i\phi}}{z} \right] + O \left( \frac{1}{z^2} \right) \]
\[ (4.56a) \]

\[ = \frac{D_0}{z} + D_1 \left[ 1 + \frac{\gamma \text{Re} e^{i\phi} + (1 - \gamma) \text{Re} - e^{i\phi}}{z} \right] + O \left( \frac{1}{z^2} \right). \]
\[ (4.56b) \]

Comparing this result with (4.53)

\[ D_1 = 0 \quad (4.57a) \]
\[ D_0 = \frac{-\hat{x}}{2\pi (1 + \hat{k})}. \quad (4.57b) \]

For small values of \( z \), (4.46) and (4.52) become
\[ G(z) = \frac{e^{2\phi \beta}}{R} [1 + \frac{z}{R} (\gamma e^{-i\phi} + (1 - \gamma)e^{i\phi}) + \ldots] \quad (4.58) \]

\[ \Omega'(z) = \frac{e^{2\phi \beta}}{z^2} \left( \frac{D_{-2}}{Rz} (\gamma e^{-i\phi} + (1 - \gamma)e^{i\phi}) + \frac{D_{-1}}{z} \right) + \frac{D_{-2}}{z^2} + \text{constant} \]. \quad (4.59) \]

Comparing this result with (4.54),

\[ D_{-2} = 0 \quad (4.60a) \]

\[ D_{-1} = \frac{\hat{k} R e^{-2\phi \beta}}{2 \pi (1 + \hat{k})}. \quad (4.60b) \]

With equations (4.57) and (4.60) the final form for \( \Omega'(z) \) is then

\[ \Omega'(z) = \frac{2 \mu \hat{u}}{(1 + \hat{k})} - \left[ \frac{2 \mu \hat{u}}{(1 + \hat{k})} (z - \gamma Re^{i\phi} - (1 - \gamma)Re^{-i\phi}) \right] \quad (4.61) \]

\[ + \frac{\hat{k}}{2 \pi (1 + \hat{k})} \left( 1 + \frac{R}{z} \hat{k} e^{-2\phi \beta} \right) (z - Re^{i\phi} - (z - Re^{-i\phi})^\gamma - 1. \]

Differentiating this expression,
\[ \Omega''(z) = \left\{ \frac{\hat{k}^2 \text{Re}^{-2\phi \beta}}{2\pi(1 + \hat{k})} \frac{1}{z^2} - \frac{2\mu \hat{\nu}}{(1 + \hat{k})} \right. \\
+ \left[ \frac{2\mu \hat{\nu}}{(1 + \hat{k})} (z - \gamma \text{Re}^i\phi -(1 - \gamma) \text{Re}^{-i\phi}) + \frac{\hat{X}}{2\pi(1 + \hat{k})} (1 + \text{Re}^{-2\phi \beta}) \right] \\
\left. \left\{ \frac{\gamma}{(z - \text{Re}^i\phi)} + \frac{(1 - \gamma)}{(z - \text{Re}^{-i\phi})} \right\} (z - \text{Re}^i\phi)^{-\gamma} (z - \text{Re}^{-i\phi})^{\gamma-1} \right\} \\
\text{(4.62)} \]

As noted by England\textsuperscript{50}, the form of \( \Omega''(z) \) produced by this solution leads to singularities of the stresses at the ends of the arc \( L' \), that is the edges of the slot. This phenomenon is not uncommon at points where the boundary conditions change in type.\textsuperscript{51}

Substituting (4.61) and (4.62) into equations (4.44) gives a stress distribution which, superimposed on the results of Section 4.7, describes the rock stress around a partially supported well bore under nonsteady flow conditions.

4.10 Numerical examples of the elastic stress distribution around open and fully supported well bores.

To illustrate the solutions in Sections 4.7 and 4.8, consider the following example:

\[ \frac{\sigma}{P_\infty} = \frac{1}{10} \quad \text{(4.63a)} \]

\[ \frac{\hat{Q}}{P_\infty} = \frac{1}{50} \quad \text{(4.63b)} \]
\[ \alpha = 1 \quad (4.63c) \]
\[ \nu = .25 \quad (4.63d) \]

where \( \nu \) is Poisson's ratio for the porous solid. Poisson's ratio determines the value of the coefficients \(^{52}\)

\[ \frac{\lambda + \mu}{\lambda + 2\mu} = \frac{1}{2(1 - \nu)} \quad (4.64a) \]

\[ \frac{\mu}{\lambda + 2\mu} = \frac{(1 - 2\nu)}{2(1 - \nu)}. \quad (4.64b) \]

Calculation of the pore pressure from equations (4.6) is simplified when

\[ \varepsilon < .01 \quad (4.65a) \]

allowing (4.6a) to be written \(^{53}\)

\[ \frac{P}{P_\infty} = 1 + \frac{\hat{Q}}{P_\infty} (\ln \varepsilon + .5772). \quad (4.65b) \]

Defining the nondimensional radius \( (\rho) \)

\[ \rho = \frac{r}{R} \quad (4.66) \]

makes the pore pressure distribution

\[ \frac{P}{P_\infty} = 1 + \frac{\hat{Q}}{P_\infty} [\ln (\rho^2 \varepsilon) + .5772] \quad (4.67) \]
where (4.18c) was used. With this expression the well pressure is

\[
\frac{P_w}{P_\infty} = 1 + \frac{\hat{Q}}{P_\infty} [\ln \bar{\varepsilon} + .5772]. \tag{4.67}
\]

The nondimensional effective stresses

\[
\frac{\sigma_{rr}}{P_\infty}, \frac{\sigma_{\theta\theta}}{P_\infty}, \frac{\sigma_{zz}}{P_\infty}
\]

were calculated for the region

\[
1 \leq \rho \leq 5
\]

(4.69)

for times corresponding to

\[
\bar{\varepsilon} = 4 \times 10^{-4}, 1 \times 10^{-6}, 1 \times 10^{-9}
\]

(4.70a)

which represent well pressures

\[
\frac{P_w}{P_\infty} = 0.855, 0.735, 0.597.
\]

(4.70b)

With the rock in plane strain,

\[
\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}).
\]

(4.71)

The effective stress distribution for an open well bore is shown in Figures 2 through 4. The maximum principal stress difference occurs at the well bore and increases in
magnitude as the drawdown increases. The order of the principal effective stresses at the well bore is

\[ \sigma_{rr} > \sigma_{zz} > \sigma_{\theta \theta}. \] (4.72)

The effective stress distribution for a fully supported well bore is shown in Figures 5 through 7. Again the maximum principal stress difference occurs at the well bore and increases as the drawdown increases. The order of the principal effective stresses at the well bore is

\[ \sigma_{\theta \theta} = \sigma_{zz} > \sigma_{rr}. \] (4.73)

This elastic analysis makes it possible to establish a criterion for initial yielding of the reservoir rock.

4.11 Conditions for initial yielding of the rock around open and fully supported well bores.

Initial yielding is predicted when the elastic stresses first satisfy an appropriate yield condition. Consider for example the Coulomb yield condition of Terzaghi\(^2\)

\[ \sigma_3 = \sigma_1 N^2 - 2CN \] (4.74a)

where \( \sigma_1 \) and \( \sigma_3 \) are the maximum and minimum effective stresses, \( C \) is the rock cohesive strength, and

\[ N = \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \] (4.74b)
with \( \phi \) the angle of internal friction for the porous solid.

For the unsupported case, equation (4.72) shows that the yield condition is
\[
\sigma_{\theta\theta} = \sigma_{rr} N^2 - 2CN. \tag{4.75}
\]

With the stresses at the well bore given by (4.31), this becomes
\[
P_w = P_\infty - \frac{2}{\alpha} (1 - \nu)(CN - \sigma_\infty). \tag{4.76a}
\]

Equation (4.76a) gives the critical well pressure which initiates yielding around an open well bore. For the special case defined by equations (4.63a), (4.63c), and (4.63d),
\[
\frac{P_w}{P_\infty} = 1 - \frac{3}{2} \left( \frac{CN}{P_\infty} - \frac{1}{10} \right). \tag{4.76b}
\]

Figure 8 shows the variation of the non-dimensional critical well pressure of (4.76b) with the rock properties \( \frac{C}{P_\infty} \) and \( \phi \). The figure shows that the critical well pressure increases as the cohesive strength and angle of internal friction decrease. That is the weaker the rock, the smaller the allowable drawdown.
For a fully supported well bore, the yield condition is

$$
\sigma_{rr} = 0\epsilon N^2 - 2CN. \tag{4.77}
$$

With the stresses at the well bore given by (4.41), this relation becomes

$$
\frac{P_w}{P_\infty} = 1 + \frac{2}{\alpha} (1 - \nu) \left[ \frac{\sigma_\infty}{P_\infty} - \frac{N}{(1 - \nu - \nu N^2) C} \right]. \tag{4.78a}
$$

Equation (4.78a) gives the critical well pressure which initiates yielding around a fully supported well bore. For the special case defined by equations (4.63a), (4.63c), and (4.63d),

$$
\frac{P_w}{P_\infty} = 1 + \frac{3}{2} \left[ \frac{1}{10} - \left( \frac{4N}{3 - N^2} \right) \frac{C}{P_\infty} \right]. \tag{4.78b}
$$

Figure 9 shows the variation of the non-dimensional critical well pressure of (4.78b) with the rock properties \( \frac{C}{P_\infty} \) and \( \phi \). The results are similar to those obtained for the open hole case, except that lower well pressures are required to yield the rock around a fully supported well bore.

The allowable drawdown for a perforated casing would be expected to lie somewhere between the extreme cases of an open and a fully supported well bore. That is equations (4.76a) and (4.78a) may be used to estimate upper and lower
bounds for the critical well pressure for a perforated casing.

For the special case defined by equations (4.63a), (4.63c), and (4.63d), Figure 10 shows bounds for the value of \( \frac{P_w}{P_\infty} \) that initiates yielding around a perforated casing with \( \frac{C}{P_\infty} = 0.2 \) and low angles of internal friction.

4.12 Initial yielding of Dania chalk around a perforated casing.

The allowable drawdown for perforated casing located in Dania chalk formations is a principal objective in this study. Upper and lower bounds for the critical well pressure in this type of completion may be estimated using the method outlined in Section 4.11.

Dania chalk obeys a quadratic yield condition\(^{16}\)

\[
J^2 = k - b\mathbf{I} - c\mathbf{I}^2 \tag{4.79}
\]

where

\[
J^2 = \frac{1}{2} \text{tr} \left[ (g')^T (g') \right] \tag{4.80a}
\]

\[
\mathbf{I} = \text{tr}(g) \tag{4.80b}
\]

\[
g' = g - \frac{1}{3} \text{tr}(g)\mathbf{I} \tag{4.80c}
\]

with \( g \) the effective stress tensor, \( \text{tr} \) denoting the trace operation, and \( k, b, \) and \( c \) material constants.
The symbols $I$ and $J^2$ denote the first invariant of the effective stress tensor and the second invariant of the deviatoric effective stress tensor respectively. For an axisymmetric plane strain problem,

$$I = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} \quad (4.81a)$$

$$2J^2 = \sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{zz}^2 \quad (4.81b)$$

Consider first the case of an open well bore. At the well bore, from Section 4.7,

$$\sigma_{rr} = 0 \quad (4.82a)$$

$$\sigma_{zz} = \nu\sigma_{\theta\theta} \quad (4.82b)$$

so that

$$I = (1 + \nu)\sigma_{\theta\theta} \quad (4.82c)$$

and the deviatoric effective stresses are

$$\sigma_{rr}' = -(1 + \nu)\frac{\sigma_{\theta\theta}}{3} \quad (4.83a)$$

$$\sigma_{\theta\theta}' = (2 - \nu)\frac{\sigma_{\theta\theta}}{3} \quad (4.83b)$$

$$\sigma_{zz}' = -(1 - 2\nu)\frac{\sigma_{\theta\theta}}{3}. \quad (4.83c)$$

The second stress invariant above is then

$$J^2 = (1 - \nu + \nu^2)\frac{\sigma_{\theta\theta}^2}{3}. \quad (4.83d)$$
The quadratic yield condition (4.79) is now

$$0 = k - \hat{b}\sigma_{\theta\theta} - \hat{c}r^2$$

(4.84a)

where

$$\hat{b} = (1 + \nu)b$$

(4.84b)

$$\hat{c} = (1 + \nu)^2c + \frac{1}{3}(1 - \nu + \nu^2)$$

(4.84c)

Initial yielding of the rock around an open hole occurs when the tangential effective stress satisfies the condition (4.84a). This occurs when

$$-2[\sigma_{\infty} + \frac{\lambda + \mu}{\lambda + 2\mu}a(P_{\infty} - P_w)] = \frac{b}{2\hat{c}} + \frac{(\hat{b}^2)}{4\hat{c}^2} + \frac{k}{\hat{c}}^{1/2}$$

(4.85)

Since the left hand side of this equation is less than zero, the negative square root is taken and therefore

$$\frac{P_w}{P_{\infty}} = 1 - \frac{2}{a}(1 - \nu)[\frac{\hat{s}}{P_{\infty}} - \frac{\sigma_{\infty}}{P_{\infty}}]$$

(4.86a)

where

$$\hat{s} = \frac{b}{4\hat{c}} + \frac{(\hat{b}^2)}{16\hat{c}^2} + \frac{k}{4\hat{c}}^{1/2}$$

(4.86b)

and $\hat{s}$ depends on $\nu$ and the rock strength. Physically $\hat{s}$ represents the magnitude of the tangential stress on the open hole at initial yield. Equation (4.86a) gives the critical pressure that initiates yielding around an open
well bore in Dania chalk. It estimates an upper bound for the critical well pressure around a perforated casing.

Now consider a well bore supported by a rigid permeable liner. At the well bore, from Section 4.8

$$\sigma_{\theta \theta} = \sigma_{zz} = \frac{\nu}{(1 - \nu)} \sigma_{rr}$$  \hspace{1cm} (4.87a)

so that

$$I = \left( \frac{1 + \nu}{1 - \nu} \right) \sigma_{rr}$$  \hspace{1cm} (4.87b)

and the deviatoric effective stresses are

$$\sigma'_{rr} = 2 \frac{(1 - 2\nu)}{(1 - \nu)} \frac{\sigma_{rr}}{3}$$  \hspace{1cm} (4.88a)

$$\sigma'_{\theta \theta} = \sigma'_{zz} = \frac{(1 - 2\nu)}{(1 - \nu)} \frac{\sigma_{rr}}{3}.$$  \hspace{1cm} (4.88b)

The second stress invariant is then

$$J^2 = \frac{(1 - 2\nu)^2}{(1 - \nu)} \frac{\sigma_{rr}^2}{3}.$$  \hspace{1cm} (4.88c)

The quadratic yield condition (4.79) is now

$$0 = k - \tilde{b} \sigma_{rr} - \tilde{c} \sigma_{rr}^2$$  \hspace{1cm} (4.89a)

where

$$\tilde{b} = \left( \frac{1 + \nu}{1 - \nu} \right) b$$  \hspace{1cm} (4.89b)
\[ \tilde{c} = \left( \frac{1 + \nu}{1 - \nu} \right)^2 c + \frac{1}{3} \left( \frac{1 - 2\nu}{1 - \nu} \right)^2. \quad (4.89c) \]

Initial yielding of the rock around a fully supported hole occurs when the radial effective stress satisfies the condition (4.89a). This occurs when

\[- \left( \frac{\lambda + 2\mu}{\lambda + \mu} \right) \sigma_0 - \alpha (P_\infty - P_w) = - \frac{b}{2\tilde{c}} + \left( \frac{\tilde{b}^2}{4\tilde{c}^2} + \frac{k}{\tilde{c}} \right)^{1/2}. \quad (4.90)\]

Since the left hand side of this equation is less than zero, the negative square root is taken and therefore

\[ \frac{P_w}{P_\infty} = 1 - \frac{2}{\alpha (1 - \nu)} \left[ \frac{1}{(1 - \nu)} \tilde{s} \right] \sigma_m \left( \frac{P_m}{P_\infty} \right) \quad (4.91a) \]

where

\[ \tilde{s} = \frac{\tilde{b}}{4\tilde{c}} + \left( \frac{\tilde{b}^2}{16\tilde{c}^2} + \frac{k}{4\tilde{c}} \right)^{1/2} \quad (4.91b) \]

and \( \tilde{s} \) depends on \( \nu \) and the rock strength. Physically \( \tilde{s} \) represents the magnitude of the radial compressive stress on the cased hole at initial yield. Equation (4.91a) gives the critical well pressure that initiates yielding around a fully supported well bore in Dania chalk. It estimates a lower bound for the critical well pressure around a perforated casing.

Using the definitions in (4.84) and (4.89), the expressions (4.86a) and (4.91a) may be written
\[
\frac{P_w}{P_\infty} = 1 - \frac{2}{a} (1 - \nu) \left[ \frac{1}{(1 + \nu)} \frac{s}{P_\infty} - \frac{\sigma_\infty}{P_\infty} \right] 
\] (4.92a)

where

\[
s = \frac{b}{4(c + a)} + \left[ \frac{b}{4(c+a)} \right]^2 + \frac{k}{4(c + a)} \right]^{1/2} \] (4.92b)

and the elastic parameter \( a \) is defined by

\[
\frac{(1 - \nu + \nu^2)}{(1 + \nu)^2} \text{ for an open hole} \] (4.92c)

\[
\frac{(1 - 2\nu)^2}{(1 + \nu)^2} \text{ for a fully supported hole.} \] (4.92d)

Equations (4.92) give approximate upper and lower bounds on the critical well pressure for a perforated casing in a material obeying a quadratic yield condition. The critical well pressure depends on Poisson's ratio and the dimensionless rock strength parameters \( c, b/P_\infty, \) and \( \sqrt{\kappa}/P_\infty. \)

To illustrate the previous solution, consider the special case

\[
\alpha = 1 \] (4.93a)

\[
\nu = 0.25 \] (4.93b)

\[
c = \frac{1}{12} \] (4.93c)

\[
\frac{\sigma_\infty}{P_\infty} = \frac{1}{10}. \] (4.93d)
Figures 11 and 12 show the variation of the critical well pressure of equation (4.92) around open and fully supported boreholes as a function of the material constants $b/P_\infty$ and $\sqrt{K}/P_\infty$ in the quadratic yield condition. These curves may be used to bound the critical pressure that initiates yield around a perforated casing. For the particular example,

\begin{align*}
\alpha &= 1 & (4.94a) \\
\nu &= 0.25 & (4.94b) \\
c &= 0.0845 & (4.94c) \\
k &= 50,000 \text{ psi}^2 & (4.94d) \\
\sigma_\infty &= 500 \text{ psi} & (4.94e) \\
P_\infty &= 6500 \text{ psi} & (4.94f)
\end{align*}

the limiting curves are shown in Figure 13 as a function of the material constant $b/P_\infty$. With the material constant appropriate for Dania chalk

\begin{equation}
\mathbf{b} = 624 \text{ psi} \quad (4.94g)
\end{equation}

initial yielding occurs for

\begin{equation}
4486 < P_w < 5751 \text{ psi}. \quad (4.95)
\end{equation}

This result may be compared with the finite element analysis which indicated that severe losses in near well bore permeability occur for

\begin{equation}
P_w < 4000 \text{ psi}. \quad (4.96)
\end{equation}
4.13 Numerical example of the elastic stress distribution around a partially supported well bore.

The solution derived in Section 4.9 for the elastic stresses in rock around a slotted casing may be illustrated with the particular example

\[
\frac{\sigma_\infty}{P_\infty} = \frac{1}{10} \quad (4.97a)
\]

\[
\frac{\hat{\phi}}{P_\infty} = \frac{1}{50} \quad (4.97b)
\]

\[\alpha = 1 \quad (4.97c)\]

\[\nu = .25 \quad (4.97d)\]

\[\phi = \frac{3\pi}{4} \quad (4.97e)\]

\[\hat{\chi} = 2P_w R \sin(\pi - \phi) \quad (4.97f)\]

at well pressures

\[
\frac{P_w}{P_\infty} = .597 \text{ and } .855.
\]

Radial, tangential, and shearing stresses are shown as a function of the nondimensional radius

\[
\rho = \frac{r}{R} \quad (4.98)
\]

along the radial lines

\[
\theta = 0, \frac{\pi}{2}, \text{ and } \pi. \quad (4.99)
\]
Figures 15 and 16 show the stresses along a radial line directly opposite of the casing slot \((\theta = 0)\). Radial effective stresses are large and compressive as in the case of a fully supported well bore.

Figures 17 and 18 show the stresses along a radial line closer to the slot \((\theta = \pi/2)\). With the casing support still present in this area, the form of the stress distribution is again similar to the fully supported case. However, note the introduction of large shearing stresses near the casing due to the asymmetry of the problem and the bonding of the rock to the casing surface.

Figures 19 and 20 show the stresses along a radial line through the center of the slot \((\theta = \pi)\). The form of the stress distribution is similar to the open well bore case, with zero radial effective stress at the slot.

Figures 21 and 22 show the variation of the stresses as a function of the angular coordinate \(\theta\) for a nondimensional radius of two. The ordering of the normal stresses changes, as does the mean stress, depending upon proximity to the slot. It is not obvious where initial yielding will occur. A possible method of analysis is to determine the rock strength required to avoid yield, as a function of position, as has been done by Paslay and Cheatham for the symmetric open hole case.
Qualitatively the results of the mixed boundary value problem are as expected. It remains to apply an appropriate yield condition to determine the location and well pressure drawdown at which yielding is initiated. It is anticipated that an initial yield curve between the upper and lower bounds shown in Figure 13 will be developed. That is, the analysis of the slotted casing problem should provide a close estimate of the well pressure at initial yield for a perforated casing.

4.14 Suggestions for further work.

Suggestions for further study are as follows:

(a) The strain-hardening behavior of Dania chalk may be included in the analysis of the plastic stresses around a single perforation, to better understand the effect on the casing of plastic flow after initial yield.

(b) The extrusion of Dania chalk into a slotted casing may be described theoretically using the method of characteristics.

(c) Compressible and transient flow effects may be included in the analysis of the plastic stresses around a single perforation, in order to aid in the interpretation of existing experimental data.
(d) Complex variable methods may be used to determine the effects of a nonuniform flow field on the elastic stresses in the porous solid around a casing, using stress, displacement, or mixed boundary conditions.

(e) Extrusion or fluidization experiments may be conducted to evaluate the theoretical analyses suggested in (a) through (d) above.

Study of the casing collapse problem has defined the critical well pressures for initial yield and cavity formation around perforated casings in Dania chalk. The work suggested above is expected to provide additional understanding of the problem of casing collapse in weak rock formations.

4.15 Conclusions.

The principal results of this study are exemplified in Figure 14, which shows the critical well pressure calculated in Chapter 3 as a dotted line. These results are consistent with a previous finite element analysis\(^{16}\), which predicted severe losses in near well permeability at drawdown values between the initial yield and cavity formation values predicted here. The elastic analysis provides upper and lower bounds for the well pressure at initial yield around a perforated casing. The plastic analysis provides the well
pressure that initiates cavity formation around casing perforations. Although the particular example of a cohesionless material has been shown, any material obeying a quadratic yield condition may be considered. Since the critical well pressures are obtained from analytical analyses, the results are easily applied to various rock formations, pore fluids, and reservoir characteristics.

In summary, given the material constants in the yield condition, Poisson's ratio for the bulk porous solid, and the in situ reservoir stress state and pore pressure, the drawdown values that precipitate plastic flow and later cavity formation are predicted from this analysis. The results of this study are intended to estimate the allowable well pressures for the testing and production of completions in weak chalk formations.
CHAPTER 5.

THE APPLICATION OF LUBINSKI'S POROELASTICITY THEORY

TO SEVERAL PRACTICAL PROBLEMS

5.1 Introduction

Lubinski\textsuperscript{5} analyzed the uncoupled problem in porous media by drawing an analogy between poroelasticity and thermoelasticity. He showed that the stress in a porous material due to a pore fluid flow field and imposed boundary loads may be calculated from Timoshenko's\textsuperscript{27} solution for thermal stress problems. If the product of the temperature and the thermal expansion coefficient is replaced by the product of the pore pressure and an experimentally determined poroelasticity constant, stresses and displacements for the bulk porous solid may be determined for the particular boundary value problem of interest. In this chapter Lubinski's method is employed to solve several practical problems for the cylindrical geometry shown in Figure E.

5.2 Rock stresses induced by the flow of a pore fluid into a lined borehole.

Rock stresses produced by steady radial flow of an incompressible pore fluid into a borehole have been analyzed for the case of an open hole.\textsuperscript{3} In this section a similar problem is studied in which the wall of the hole is supported by a rigid but permeable casing. The objectives are
to determine the elastic stress distribution in the porous rock and the well pressure at which yielding of the solid material begins.

Consider a hollow circular cylinder of porous rock of inner radius \( a \) and outer radius \( b \) in a state of plane strain. With a steel casing cemented in place along the borehole, the boundary condition on the inner surface of the cylinder is taken to be that of zero displacement. The external surface of the cylinder, which represents the reservoir edge, is subjected to a total radial stress of magnitude \( \tau_e \). The total stress \( \tau_{ij} \) is related to the effective stress \( \sigma_{ij} \) and the pore pressure \( P \) by

\[
\tau_{ij} = \sigma_{ij} - \delta_{ij} \alpha P
\]

(5.1a)

where \( \delta_{ij} \) is the Kronecker delta, \( \alpha \) is an experimentally determined constant, and the indices range over

\[ i \text{ or } j = 1, 2, 3. \]

(5.1b)

Tension is taken as positive.

The pore pressure distribution is from Darcy's law

\[
P = P_w + C_0 \ln \frac{R}{a}
\]

(5.1c)

\[
C_0 = \frac{Q_u}{2 \pi k h} = \frac{P_e - P_w}{\ln \frac{d}{a}}
\]

(5.1d)
where $P_w$ and $P_e$ are the pore pressures at the well and reservoir edge respectively, $Q$ is the volumetric flow rate, $\bar{\mu}$ is the viscosity, $\bar{k}$ is the permeability, and $h$ is the formation thickness.

The equation of equilibrium in plane polar coordinates $(r, \theta)$ is

$$\frac{d\tau_{rr}}{dr} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$$  \hspace{1cm} (5.2a)

where the shear stress and strain are zero by symmetry. The strain in the porous solid $e_{ij}$ is related to the effective stress by Hooke's law

$$\langle \sigma_{ij} \rangle = 2\mu e_{ij} + \lambda e_{ii}$$ \hspace{1cm} (5.2b)\[ e = e_{ii} \hspace{1cm} (5.2c)

where $\mu$ and $\lambda$ are Lamé coefficients for the porous solid and the repeated index indicates a sum over

$$i = 1, 2, 3.$$ \hspace{1cm} (5.2d)

If the Lamé coefficients are replaced by Young's modulus, $E$, and Poisson's ratio, $\nu$, using

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}$$ \hspace{1cm} (5.2e)

$$\mu = \frac{E}{2(1 + \nu)}$$ \hspace{1cm} (5.2f)
then for the problem under study the stress-strain relations (5.2b) reduce to

\[ \varepsilon_{rr} = \frac{(1 - \nu^2)}{E} \left[ \langle \sigma_{rr} \rangle - \frac{\nu}{(1 - \nu)} \langle \sigma_{\theta \theta} \rangle \right] \quad (5.2g) \]

\[ \varepsilon_{\theta \theta} = \frac{(1 - \nu^2)}{E} \left[ \langle \sigma_{\theta \theta} \rangle - \frac{\nu}{(1 - \nu)} \langle \sigma_{rr} \rangle \right]. \quad (5.2h) \]

With equation (5.1a) the last two expressions may be written in the form

\[ \varepsilon_{rr} - (1 + \nu)(1 - 2\nu) \frac{\alpha P}{E} = \frac{(1 - \nu^2)}{E} \left[ \tau_{rr} - \frac{\nu}{(1 - \nu)} \tau_{\theta \theta} \right] \quad (5.2i) \]

\[ \varepsilon_{\theta \theta} - (1 + \nu)(1 - 2\nu) \frac{\alpha P}{E} = \frac{(1 - \nu^2)}{E} \left[ \tau_{\theta \theta} - \frac{\nu}{(1 - \nu)} \tau_{rr} \right]. \quad (5.2j) \]

Equations (5.2a), (5.2i), and (5.2j) are identical in form to Timoshenko and Goodier's thermoelasticity equations for a hollow cylinder in plane strain, with the product of the temperature and thermal expansion coefficient replaced by the quantity \((1 - 2\nu) \frac{\alpha}{E} P\). Timoshenko and Goodier's general solution gives the radial displacement \((u)\) and total stresses in the present problem as

\[ u = \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)} \frac{\alpha}{E} \frac{1}{r} \int_a^r P \, dr + \frac{C_1}{r} + \frac{C_2}{r} \quad (5.3a) \]

\[ \tau_{rr} = \frac{(1 - 2\nu)}{(1 - \nu)} \frac{\alpha}{r^2} \int_a^r P \, dr + \frac{E}{(1 + \nu)} \left[ \frac{C_1}{1 - 2\nu} - \frac{C_2}{r^2} \right] \quad (5.3b) \]
\[ \tau_{\theta\theta} = \frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{r^2} \int_a^r P r \, dr - \frac{(1-2\nu)}{(1-\nu)} \frac{E}{\alpha P + (1+\nu)} \left[ \frac{C_1}{1-2\nu} + \frac{C_2}{r^2} \right]. \quad (5.3b) \]

Substituting (5.1c) into the above and integrating yields

\[ u = \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \frac{\alpha}{2E} \left[ (P_w - \frac{C_0}{2}) r - \frac{a^2}{r} \right] + C_0 r \ln \frac{r}{a} + C_1 r + \frac{C_2}{r} \quad (5.3d) \]

\[ \tau_{rr} = -\frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{2} \left[ (P_w - \frac{C_0}{2}) (1 - \frac{a^2}{r^2}) + C_0 \ln \frac{r}{a} \right] + \frac{E}{(1+\nu)} \left[ \frac{C_1}{1-2\nu} - \frac{C_2}{r^2} \right]. \quad (5.3e) \]

\[ \tau_{\theta\theta} = \frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{2} \left[ (P_w - \frac{C_0}{2}) (1 - \frac{a^2}{r^2}) + C_0 \ln \frac{r}{a} \right] + \frac{E}{(1+\nu)} \left[ \frac{C_1}{1-2\nu} + \frac{C_2}{r^2} \right] - \frac{(1-2\nu)}{(1-\nu)} \alpha P. \quad (5.3f) \]

The boundary condition

\[ u = 0 \quad \text{for} \quad r = a \quad (5.3g) \]

requires

\[ C_2 = -a^2 C_1. \quad (5.3h) \]

Then the boundary condition

\[ \tau_{rr} = -\tau_e \quad \text{for} \quad r = b \quad (5.3i) \]

requires

\[ C_1 = \frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{2} \left[ (P_w - \frac{C_0}{2}) (1 - \frac{a^2}{b^2}) + C_0 \ln \frac{b}{a} \right] - \tau_e \]

\[ C_1 = \frac{E}{(1+\nu)} \left[ \frac{1}{1-2\nu} + \frac{a^2}{b^2} \right]. \quad (5.3j) \]
The final expressions for the displacement and total stresses are therefore

\[ u = \frac{[aP_w - \tau_e - \frac{ac_0}{2} [1 - \frac{(1-2\nu)}{(1-\nu)} \ln \frac{b}{a}]]}{[\frac{1}{1-2\nu} + \frac{a^2}{b^2}]} \frac{(1+\nu)}{E} \frac{r}{[r - \frac{a^2}{r}]} \]

+ \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \frac{ac_0}{2E} \frac{r}{\ln \frac{r}{a}} \]

(5.4a)

\[ \tau_{rr} = -\frac{(1-2\nu)}{(1-\nu)} \frac{a}{2} \left[ (P_w - \frac{c_0}{2})(1 - \frac{a^2}{r^2}) + c_0 \ln \frac{r}{a} \right] + \frac{c_0}{2} \left[ (P_w - \frac{c_0}{2})(1 - \frac{b^2}{a^2}) + c_0 \ln \frac{b}{a} \right] \]

- \tau_e \frac{[\frac{1}{1-2\nu} + \frac{a^2}{r^2}]}{[\frac{1}{1-2\nu} + \frac{a^2}{b^2}]} \]

(5.4b)

\[ \tau_{\theta \theta} = \frac{(1-2\nu)}{(1-\nu)} \frac{a}{2} \left[ (P_w - \frac{c_0}{2})(1 - \frac{a^2}{r^2}) - P_w - \frac{c_0}{2} \ln \frac{r}{a} \right] + \frac{c_0}{2} \left[ (P_w - \frac{c_0}{2})(1 - \frac{a^2}{b^2}) + c_0 \ln \frac{b}{a} \right] - \tau_e \frac{[\frac{1}{1-2\nu} + \frac{a^2}{r^2}]}{[\frac{1}{1-2\nu} + \frac{a^2}{b^2}]} \]

(5.4c)
Using equations (5.1a), (5.1c), (5.4b), and (5.4c) the effective stresses can now be expressed as

\[
< \sigma_{rr} > = \frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{2} \left[ \left( P_w - \frac{C_0}{2} \right) + \frac{C_0}{r^2} \right] + \frac{\alpha}{2(1-\nu)} \left( P_w + C_0 \ln \frac{r}{a} \right) \\
+ \left\{ \frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{2} \left[ \left( P_w - \frac{C_0}{2} \right) \left( 1 - \frac{a^2}{b^2} \right) + C_0 \ln \frac{b}{a} \right] - \tau_e \right\} \\
\times \frac{\frac{1}{1-2\nu} + \frac{a^2}{r^2}}{\frac{1}{1-2\nu} + \frac{a^2}{b^2}}. 
\]

\[
< \sigma_{ee} > = -\frac{\alpha}{2} (1-2\nu) \left[ \left( P_w - \frac{C_0}{2} \right) + \frac{C_0}{r^2} \right] + \frac{\alpha}{2(1-\nu)} \left( P_w + C_0 \ln \frac{r}{a} \right) \\
+ \left\{ \frac{(1-2\nu)}{(1-\nu)} \frac{\alpha}{2} \left[ \left( P_w - \frac{C_0}{2} \right) \left( 1 - \frac{a^2}{b^2} \right) + C_0 \ln \frac{b}{a} \right] - \tau_e \right\} \\
\times \frac{\frac{1}{1-2\nu} - \frac{a^2}{r^2}}{\frac{1}{1-2\nu} + \frac{a^2}{b^2}}. 
\]  

These equations describe the stress distribution for the elastic case. For a given yield condition, they can be used to determine the drawdown that initiates plastic flow.

If the solid material satisfies the Coulomb yield condition written in terms of the effective stress, then Terzaghi's equation

\[ \text{equation}^2 \]
\[ <\sigma_{rr}> = \sigma_{\theta\theta} N^2 - 2CN \] (5.5a)

\[ N = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \] (5.5b)

is applicable where \( C \) is the cohesion, \( \phi \) is the angle of internal friction, and \( <\sigma_{\theta\theta}> \) has been taken as the maximum principal stress. In terms of the total stresses equation (5.5a) is

\[ \tau_{rr} = \tau_{\theta\theta} N^2 + (N^2 - 1) \alpha P - 2CN \] (5.5c)

where equation (5.1a) was used.

At the casing surface, the total stresses are

\[ \tau_{rr} = \alpha (P_w - \frac{C_0}{2})(1 - \frac{a^2}{b^2}) + \frac{\alpha C_0 \ln \frac{b}{a} - 2(1-v)\tau_e}{1-2v} \]

\[ \frac{1}{1-2v} + \frac{a^2}{b^2} \] (5.5d)

\[ \tau_{\theta\theta} = -\frac{(1-2v)}{(1-v)} \frac{v\alpha}{P_w} + \frac{(P_w - \frac{C_0}{2})(1 - \frac{a^2}{b^2}) + C_0 \ln \frac{b}{a} - \frac{2v\tau_e}{1-2v}}{1-2v} \]

\[ \frac{1}{1-2v} + \frac{a^2}{b^2} \] (5.5e)

Substituting (5.5d) and (5.5e) into (5.5c) shows that the condition for initial yielding of the porous solid at the casing surface is
\[
(1-\nu-\nu N^2) \left[ \frac{2}{1-2\nu} (\alpha \tau_e - \tau_e) + \alpha C_0 \frac{2\ln b}{a} \frac{b^2}{1+\frac{a^2}{b^2}} \right] + 2CN = 0
\]

where from equation (5.1d)

\[
C_0 = \frac{(P_e - P_w)}{\ln \frac{b}{a}}. \tag{5.5g}
\]

To illustrate the problem solution just derived, consider the following numerical example:

\[
\nu = 0.25 \tag{5.6a}
\]

\[
\frac{b}{a} = 500 \tag{5.6b}
\]

\[
\tau_e = P_e \tag{5.6c}
\]

\[
\alpha = 1 \tag{5.6d}
\]

\[
\phi = 20^\circ \tag{5.6e}
\]

\[
C = \frac{P_e}{25}. \tag{5.6f}
\]

Equations (5.6c) and 5.6d) demand that the radial effective stress be zero at the reservoir edge. The total stresses and the effective stresses may be nondimensionalized by dividing equations (5.4b), (5.4c), 5.4d), and (5.4e)
by \( P_e \). The total stresses \( \frac{\tau_{rr}}{P_e} \) and \( \frac{\tau_{\theta\theta}}{P_e} \) are shown in Figures 23 through 25 for \( \frac{P_w}{P_e} = .9, .5, \) and 0. The corresponding stresses \( \frac{\langle \sigma_{rr} \rangle}{P_e} \) and \( \frac{\langle \sigma_{\theta\theta} \rangle}{P_e} \) are shown in Figures 26 through 28. The stresses are plotted as a function of the nondimensional radius \( \frac{r}{a} \). Yielding at the casing, equation (5.5f), when combined with the definition (5.5g), gives the well pressure that initiates plastic flow in this example as \( \frac{P_w}{P_e} = .66 \).

The sample calculations demonstrate that the principal stress difference \( \tau_{\theta\theta} - \tau_{rr} \) (or equivalently \( \langle \sigma_{\theta\theta} \rangle - \langle \sigma_{rr} \rangle \)) is greater near the casing surface, and that this principal stress difference increases with increasing drawdown (defined as \( P_e - P_w \)). This indicates that well pressure drawdown can provide a mechanism for yielding the porous rock around a lined borehole.

For the Coulomb failure criterion, equation (5.5f) defines the well pressure that initiates plastic flow in the solid. By limiting the well pressure to a value above that which yields the rock, the creation of voids and hence the collapse of the casing should be precluded. Equation (5.5f) then determines an approximate upper bound for the well pressure that precipitates casing collapse. That is if the
well pressure is not lowered below the critical value found from (5.5f), then the integrity of the production liner should be maintained.

The vertical effective stress \( \langle \sigma_{zz} \rangle \), given by

\[
\langle \sigma_{zz} \rangle = \nu(\langle \sigma_{rr} \rangle + \langle \sigma_{\theta\theta} \rangle)
\]

is the maximum principal stress. Since \( \langle \sigma_{zz} \rangle \) and \( \langle \sigma_{\theta\theta} \rangle \) are equal at the casing surface, \( \langle \sigma_{\theta\theta} \rangle \) has been used in the yield condition.

It is of interest to study the effect of rock properties such as cohesive strength and angle of internal friction on the critical value of the well pressure. Equations (5.5f) and (5.5g) can be combined to give the critical well pressure that precipitates Coulomb failure in the porous solid as

\[
\frac{P_w}{P_e} = \frac{2(1-\nu) \tau_e}{(1-2\nu) \alpha P_e} - 1 + \frac{(1-\frac{b^2}{a^2})}{2 \ln \frac{b}{a}} - \frac{2CN(1-\nu)(1-2\nu) + \frac{a^2}{b^2}}{\alpha P_e(1-\nu-\nu N^2)}
\]

Figure 29 shows the nondimensional well pressure \( \frac{P_w}{P_e} \) as a function of the angle of internal friction \( \phi \) for various values of the nondimensional cohesive strength \( \frac{C}{P_e} \), where
\[ N = \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \quad (5.7b) \]

\[ \nu = 0.25 \quad (5.7c) \]

\[ \alpha = 1 \quad (5.7d) \]

\[ \frac{b}{a} = 500 \quad (5.7e) \]

\[ \tau_e = P_e. \quad (5.7f) \]

Note that the critical well pressure increases as the cohesion and angle of internal friction decrease. That is the larger the cohesive strength and the larger the angle of internal friction, the greater the allowable well drawdown.

A limiting case for equation (5.7a) occurs when

\[ 1 - \nu - \nu N^2 = 0 \quad (5.7g) \]

so that for the stated problem and boundary conditions yielding will not be induced by well pressure drawdown.

Equations (5.7b) and (5.7g) require

\[ \phi = 2 \tan^{-1} \sqrt{\frac{1 - \nu}{\nu}} - \frac{\pi}{2}. \quad (5.7h) \]

This equation defines a limiting value of the angle of internal friction for a given value of Poisson's ratio (see Figure 30). The solid must have an angle of internal friction less than this limiting value for Coulomb yielding to take place. For \( \nu = 0.25 \), the last expression gives
\[ \phi = 30^\circ. \] \tag{5.7i}

In other words, for a porous solid with Poisson's ratio of .25, Coulomb yielding will not occur in the problem under study unless the angle of internal friction is less than 30°.

For the condition

\[ \frac{b}{a} \rightarrow \infty \] \tag{5.7j}

equation (5.7a) simplifies to

\[ \frac{P_w}{P_e} = 2(1 - \nu) \frac{\tau_e}{\alpha P_e} - 1 + 2\nu - \frac{2CN(1 - \nu)}{\alpha P_e(1 - \nu - \nu N^2)}. \] \tag{5.7k}

Equation (5.7k) determines approximately the minimum well pressure that preserves casing integrity.

5.3 The rock stresses around open and cased well bores under uniform overburden load.

a. Introduction

The rock stresses around a producing well are determined by three primary factors: (1) the pore pressure distribution in the reservoir, (2) the overburden loading, and (3) the degree of support provided for the formation along the borehole. In this section the distribution of stresses in a hollow cylinder of reservoir rock is found for the case of steady radial pore fluid flow and uniform
overburden loading, with either an open or a cased wellbore. This formulation differs from previous work\textsuperscript{3,54} in that here the total vertical (overburden) stress on the rock is a constant. In the previous work just cited, plane strain was assumed. The stresses calculated from a poroelasticity analysis are combined with a Coulomb yield condition to obtain a value for the well pressure that precipitates yielding of the rock around an open or a cased hole. The behavior of the porous solid near a perforated casing can be inferred if the limiting cases of an open hole and a rigid, permeable casing are analyzed, since these conditions should provide upper and lower bounds, respectively, for the critical well pressure.

b. The rock stresses around an open borehole.

Consider a circular cylinder of porous solid with a concentric hole, the outer and inner radii of the cylinder located at \( r = a \) and \( r = b \) in a cylindrical \((r, \theta, z)\) coordinate system. For steady radial flow of an incompressible pore fluid through the rock, the pore pressure \( (P) \) is given by Darcy's law\textsuperscript{24} as

\[
P = P_w + C_0 \ln \frac{r}{a}
\]  

\[
C_0 = \frac{P_e - P_w}{\ln \frac{b}{a}}
\]  

(5.8a)  

(5.8b)
where $P_e$ and $P_w$ are the values of the pore pressure at
the reservoir edge ($r = b$) and the wellbore ($r = a$).

The porous solid is subjected to the following boundary
loads, written in terms of the total stress ($\tau_{ij}$): (1) The
upper and lower surfaces of the cylinder are subjected to a
stress

$$\tau_{zz} = - \tau_0$$  \hspace{1cm} (5.8c)

where $\tau_0$ is the magnitude of the overburden stress and
tension is taken as positive. (2) The surface of the
wellbore experiences a uniform radial stress

$$\tau_{rr} = - \tau_w$$  \hspace{1cm} (5.8d)

(3) The outer surface of the cylinder is loaded by

$$\tau_{rr} = - \tau_e$$  \hspace{1cm} (5.8e)

where $\tau_e$ is chosen to match the state of stress prevailing
a long distance from the well.

To find the state of stress in the reservoir, two
separate poroelasticity problems will be solved and their
results superimposed. In the first problem the cylinder is
loaded only by the pore pressure of equation (5.8a). In the
second problem the effects of the boundary loads $\tau_0$, $\tau_w$,
and $\tau_e$ are found.
In the first poroelasticity problem the boundary conditions and fluid pressure in the porous cylinder are

\[ \tau_{rr} = 0 \quad \text{at} \quad r = a \]  
(5.9a)

\[ \tau_{rr} = 0 \quad \text{at} \quad r = b \]  
(5.9b)

\[ \tau_{zz} = 0 \quad \text{for all} \quad r \]  
(5.9c)

\[ P = P_w + C_0 \ln \frac{r}{a}. \]  
(5.9d)

The rock is in a state of plane stress. When written in terms of the effective stress

\[ <\sigma_{ij}> = \tau_{ij} + \delta_{ij} \alpha P \]  
(5.9e)

Hooke's law for the porous solid takes the conventional form for plane stress\(^{27}\)

\[ e_{rr} = \frac{1}{E} (<\sigma_{rr}> - \nu<\sigma_{\theta\theta}>) \]  
(5.9f)

\[ e_{\theta\theta} = \frac{1}{E} (<\sigma_{\theta\theta}> - \nu<\sigma_{rr}>) \]  
(5.9g)

where the \( e_{ij} \) are the strains in the porous solid. The quantities \( E \) and \( \nu \) are Young's modulus and Poisson's ratio for the porous solid, while \( \alpha \) is an experimentally determined constant that depends on the rock and fluid compressibilities\(^6\). Using the definition (5.9e) equations (5.9f) and (5.9g) can be written
\[ e_{rr} - (1 - \nu) \frac{a}{E} P = \frac{1}{E} (\tau_{rr} - \nu \tau_{\theta\theta}) \quad (5.9h) \]

\[ e_{\theta\theta} - (1 - \nu) \frac{a}{E} P = \frac{1}{E} (\tau_{\theta\theta} - \nu \tau_{rr}) \quad (5.9i) \]

The equation of equilibrium for the cylinder is

\[ \frac{d\tau_{rr}}{dr} + \frac{(\tau_{rr} - \tau_{\theta\theta})}{r} = 0. \quad (5.9j) \]

The solution for the problem just described is given by Timoshenko and Goodier's\textsuperscript{27} thermoelasticity theory if the thermal strain is replaced by the strain

\[ (1 - \nu) \frac{a}{E} P. \]

The general solution is

\[ \tau_{rr} = -(1 - \nu) \frac{a}{r^2} \int_a^r \frac{P \, dr}{r^2} \]

\[ + \frac{E}{(1 - \nu^2)} \left[ C_1 (1 + \nu) - C_2 (1 - \nu) \frac{1}{r^2} \right] \quad (5.10a) \]

\[ \tau_{\theta\theta} = (1 - \nu) \frac{a}{r^2} \int_a^r \frac{P \, dr}{r^2} - (1 - \nu) \frac{a}{E} P \]

\[ + \frac{E}{(1 - \nu^2)} \left[ C_1 (1 + \nu) + C_2 (1 - \nu) \frac{1}{r^2} \right] \quad (5.10b) \]
\[ u = \frac{(1 - \nu^2)}{E} \frac{a}{r} \int_a^r \frac{Prdr}{r} + C_1 \frac{r}{a} + \frac{C_2}{r} \] 

\[ (5.10c) \]

where \( u \) is the radial displacement of the porous solid.

Evaluating the constants \( C_1 \) and \( C_2 \) with the boundary conditions \((5.9a)\) and \((5.9b)\), the solution to the first poroelasticity problem is

\[ \tau_{rr} = -(1 - \nu) \frac{a}{r^2} \int_a^r \frac{Prdr}{a} + \left[ (1 - \nu) \frac{a}{\mu^2} \int_a^b \frac{Prdr}{a} \right] \frac{(1 - \frac{a^2}{b^2})}{(1 - \frac{a^2}{b^2})} r^2 
\]

\[ (5.10d) \]

\[ \tau_{\theta\theta} = (1 - \nu) \frac{a}{r^2} \int_a^r \frac{Prdr}{a} - (1 - \nu) \frac{a}{\mu^2} \int_a^b \frac{Prdr}{a} \left( \frac{1 + \frac{a^2}{b^2}}{(1 - \frac{a^2}{b^2})} \right) r 
\]

\[ (5.10e) \]

\[ u = \frac{(1 - \nu^2)}{E} \frac{a}{r} \int_a^r \frac{Prdr}{r} + (1 - \nu^2) \frac{a}{E} \left[ \int_a^b \frac{Prdr}{a} \right] \left( \frac{1 + \frac{(1 - \nu)}{E}}{(1 + \nu) \frac{a^2}{b^2}} \right) \frac{r}{a^2} \left( \frac{b^2}{a^2} - 1 \right) 
\]

\[ (5.10f) \]

In the second poroelasticity problem the boundary conditions on the cylinder are

\[ \tau_{rr} = -\tau_w \quad \text{at} \quad r = a \] 

\[ (5.11a) \]

\[ \tau_{rr} = -\tau_e \quad \text{at} \quad r = b \] 

\[ (5.11b) \]

\[ \tau_{zz} = -\tau_0 \quad \text{for all} \quad r \] 

\[ (5.11c) \]
and the pore pressure is zero. The rock is in a state of generalized plane strain. Hooke's law takes the conventional form written in terms of the total stress

\[
e_{rr} = \frac{(1 - \nu^2)}{E} \left[ \tau_{rr} - \frac{\nu}{(1 - \nu)} \tau_{\theta\theta} \right] \quad (5.11d)
\]

\[
e_{\theta\theta} = \frac{(1 - \nu^2)}{E} \left[ \tau_{\theta\theta} - \frac{\nu}{(1 - \nu)} \tau_{rr} \right]. \quad (5.11e)
\]

The equation of equilibrium is again equation (5.9j). The general solution to this problem is given by Timoshenko and Goodier as

\[
\tau_{rr} = \frac{E}{(1 + \nu)} \left[ \frac{C_1}{(1 - 2\nu)} - \frac{C_2}{r^2} \right] \quad (5.11f)
\]

\[
\tau_{\theta\theta} = \frac{E}{(1 + \nu)} \left[ \frac{C_1}{(1 - 2\nu)} + \frac{C_2}{r^2} \right] \quad (5.11g)
\]

\[
\tau_{zz} = \frac{2\nu E C_1}{(1 + \nu)(1 - 2\nu)} + C_3 \quad (5.11h)
\]

\[
u = C_1 r + \frac{C_2}{r} - \frac{\nu}{E} C_3 r. \quad (5.11i)
\]

Evaluating the constants \( C_1, C_2, \) and \( C_3 \) with the boundary conditions (5.11a) through (5.11c), the solution to the second poroelasticity problem is
\[
\tau_{rr} = \left[ \frac{a^2}{b^2} - \tau_e \right] \frac{(1 - \frac{a^2}{b^2})}{\frac{r^2}{b^2}} - \tau_e \frac{a^2}{r^2} \tag{5.11j}
\]

\[
\tau_{\theta\theta} = \left[ \frac{a^2}{b^2} - \tau_e \right] \frac{(1 + \frac{a^2}{b^2})}{\frac{r^2}{b^2}} + \tau_e \frac{a^2}{r^2} \tag{5.11k}
\]

\[
\tau_{zz} = - \tau_0 \tag{5.11l}
\]

\[
u = \frac{\nu}{E} \tau_0 r + \frac{\tau_w}{E} \frac{a^2}{r}
\]

\[
\frac{1}{E} \left[ \frac{a^2}{b^2} - \tau_e \right] \frac{1}{(1 - \nu) r + (1 + \nu) \frac{a^2}{r}} \tag{5.11m}
\]

Superimposing the stresses of equations (5.10d), (5.10e), and (5.10j) through (5.10l) gives the rock stresses around an open borehole under uniform overburden loading as

\[
\tau_{rr} = - (1 - \nu) \left[ \frac{a^2}{r^2} \right] \int_a^r \frac{P r d r}{a} + \frac{a^2}{r^2} \tau_w \]

\[
+ \left[ (1 - \nu) \frac{a^2}{b^2} \right] \int_a^b \frac{P r d r}{a} + \frac{a^2}{b^2} \tau_w - \tau_e \frac{1 - \frac{a^2}{b^2}}{(1 - \frac{a^2}{b^2})} \tag{5.12a}
\]
\[
\tau_{ee} = (1 - \nu) \frac{a}{r^2} \int_a^r \frac{r}{a} Prdr + \frac{a^2}{r^2} \tau_w - (1 - \nu) \alpha P \\
+ \left[(1 - \nu) \frac{a}{b^2} \int_a^b \frac{b}{a} Prdr + \frac{a^2}{b^2} \tau_w - \tau_e \right] \frac{(1 + \frac{a^2}{b^2})}{(1 - \frac{a^2}{b^2})} \tag{5.12b}
\]

\[
\tau_{zz} = -\tau_0.
\]

Substituting equation (5.8a) for the pore pressure into the above relations yields the final expressions for the total stresses

\[
\tau_{rr} = -(1 - \nu) \frac{a}{2} \left[(P_w - \frac{C_0}{r^2})(1 - \frac{a^2}{r^2}) + C_0 \ln \frac{r}{a} \right] - \frac{a^2}{r^2} \tau_w \\
+ \left\{ (1 - \nu) \frac{a}{2} \left[P_e - \frac{C_0}{2} - \frac{a^2}{b^2}(P_w - \frac{C_0}{2}) \right] + \frac{a^2}{b^2} \tau_w - \tau_e \right\} \frac{(1 - \frac{a^2}{b^2})}{(1 - \frac{a^2}{b^2})} \tag{5.12d}
\]

\[
\tau_{ee} = (1 - \nu) \frac{a}{2} \left[(P_w - \frac{C_0}{r^2})(1 - \frac{a^2}{r^2}) + C_0 \ln \frac{r}{a} \right] + \frac{a^2}{r^2} \tau_w \\
- \left[(1 - \nu) \alpha (P_w + C_0 \ln \frac{r}{a}) \right] \\
+ \left\{ (1 - \nu) \frac{a}{2} \left[P_e - \frac{C_0}{2} - \frac{a^2}{b^2}(P_w - \frac{C_0}{2}) \right] + \frac{a^2}{b^2} \tau_w - \tau_e \right\} \frac{(1 + \frac{a^2}{b^2})}{(1 - \frac{a^2}{b^2})} \tag{5.12e}
\]
\[ \tau_{zz} = -\tau_0 \]  \hfill (5.12f)

To illustrate the state of stress in the region near the wellbore, equations (5.12d) through (5.12f), (5.8a), and (5.8b) were used to calculate the effective stress distribution for the following numerical example.

\[ \frac{b}{a} = 500 \]  \hfill (5.12g)
\[ \tau_w = p_w \]  \hfill (5.12h)
\[ \frac{\tau_e}{\tau_0} = 1 \]  \hfill (5.12i)
\[ \frac{p_e}{\tau_0} = 0.8 \]  \hfill (5.12j)
\[ \nu = 0.25 \]  \hfill (5.12k)
\[ \alpha = 1. \]  \hfill (5.12l)

The effective stresses \( \frac{\sigma_{rr}}{\tau_0} \), \( \frac{\sigma_{\theta\theta}}{\tau_0} \), and \( \frac{\sigma_{zz}}{\tau_0} \) are shown as a function of the radial position for \( \frac{p_w}{\tau_0} = 0.8, 0.5, \) and 0.1 in Figures 31, 32, and 33. As shown by equation (5.9e), when \( \alpha = 1 \) the effective stress is the sum of the pore pressure and the total stress. Since the total vertical overburden stress is constant, the effective vertical stress varies with the pore pressure. The effective stress is of principal interest since it determines the deformation of the rock. As the well pressure decreases, the compressive effective stresses increase in magnitude and the maximum principal stress difference also increases. Note that the order of the principal stresses
\[ \sigma_{rr} > \sigma_{zz} > \sigma_{\theta \theta} \]

remains the same as the well pressure drops.

c. Coulomb yielding of the rock around an open borehole.

As the well pressure is reduced and the maximum principal stress difference increases the solid material around the borehole may begin to experience plastic deformation. The onset of Coulomb yielding occurs when the maximum and minimum principal stresses satisfy Terzaghi's equation

\[ \sigma_{\theta \theta} = \sigma_{rr} N^2 - 2CN \quad (5.13a) \]

\[ N = \tan(\frac{\pi}{4} + \frac{\phi}{2}) \quad (5.13b) \]

where \( C \) is the cohesive strength of the rock and \( \phi \) is its angle of internal friction. As shown in Figures 31 and 33, the greatest principal stress difference occurs at \( r = a \) where the effective stresses are from equations (5.9e), (5.12d), and (5.12e)

\[ \sigma_{rr} = -\tau_w + p_w \]

\[ \sigma_{\theta \theta} = \tau_w + \nu \alpha p_w \]

\[ + \left\{ (1-\nu) \frac{a}{2} \left[ \frac{C_0}{2} - \frac{a^2}{b^2} \left( p_w - \frac{C_0}{2} \right) \right] + \frac{\tau_w a^2}{b^2} - \tau_e \right\} \frac{2}{\left( 1- \frac{a^2}{b^2} \right)}. \]

\[ (5.13d) \]
Substituting equations (5.13c), (5.13d), and (5.8b) into the Coulomb relation (5.13a) gives the critical well pressure at initial yield as

\[
P_w = \frac{\tau_w [N^2+1+ \frac{2}{(b^2/a^2-1)}] - \frac{2\tau_e}{(1-a^2/b^2)} + 2CN+(1-v)\alpha P_e \left[ \frac{1}{(1-a^2/b^2)} - \frac{1}{2 \ln b/a} \right]}{N^2 - 1 + (1-v)\alpha \left[ 1 + \frac{1}{(b^2/a^2-1)} - \frac{1}{2 \ln b/a} \right]}.
\]

(5.13e)

To illustrate the effect of the rock properties on the critical well pressure, the example described by equations (5.12g) through (5.12l) was again considered. Figures 34 through 37 show the pressure \( \frac{P_w}{\tau_0} \) calculated from equation (5.13e) as a function of the rock cohesive strength \( \frac{C}{\tau_0} \) and the angle of internal friction \( \phi \). It can be seen that the well pressure at yield increases as the cohesive strength and angle of internal friction decrease. That is the weaker the rock, the lower the allowable production rate if plastic deformation around the wellbore is to be avoided.

d. The rock stresses around a cased borehole.

Completed wells are frequently lined with a steel casing that is perforated to allow for the production of pore fluid. The casing provides considerable support to the surrounding rock. This support has been modeled by applying a nonzero effective stress boundary condition at the
wellbore$^{54}$. For example, if the quantity $\tau_w$ used in sections 5.3b and 5.3c is chosen to be greater than the well pressure, then the difference $\tau_w - P_w$ can represent a supporting load provided by a casing. An alternate method for simulating the effect of casing support on the formation is to take the boundary condition at the wellbore to be that of zero displacement. This approach, which represents a rigid, permeable casing, or alternately a slotted or perforated liner, will be used below.

As in sections 5.3b and 5.3c consider a circular cylinder of porous rock with a concentric hole. The wellbore lies along $r = a$ and the reservoir edge along $r = b$. The pore pressure distribution is again given by equations (5.8a) and (5.8b). The boundary conditions on the cylinder are now

\begin{align}
  u &= 0 \quad \text{at } r = a \quad (5.14a) \\
  \tau_{rr} &= -\tau_e \quad \text{at } r = b \quad (5.14b) \\
  \tau_{zz} &= -\tau_0 \quad \text{for all } r. \quad (5.14c)
\end{align}

Two poroelasticity problems will be analyzed and the results superimposed. For the first problem the boundary conditions and fluid pressure are

\begin{align}
  u &= 0 \quad \text{at } r = a \quad (5.14d) \\
  \tau_{rr} &= 0 \quad \text{at } r = b \quad (5.14e) \\
  \tau_{zz} &= 0 \quad \text{for all } r \quad (5.14f)
\end{align}
\[ P = P_w + C_0 \ln \frac{r}{a}. \]  
\text{(5.14g)}

The general solution to this plane stress problem is given by equations (5.10a) through (5.10c). Applying the conditions (5.14d) and (5.14e) to the general solution leads to the total stresses and displacement

\[ \tau_{rr} = -(1-v) \frac{\alpha}{r^2} \int_a^r r \Prdr + \left[ (1-v) \frac{a^2}{b^2} \int_a^b r \Prdr \right] \frac{[(1+v) + (1-v)\alpha^2]}{[(1+v) + (1-v)\alpha^2]}. \]  
\text{(5.14h)}

\[ \tau_{\theta\theta} = (1-v) \frac{\alpha}{r^2} \int_a^r r \Prdr - (1-v)\alpha P + \left[ (1-v) \frac{a^2}{b^2} \int_a^b r \Prdr \right] \frac{[(1+v) - (1-v)\alpha^2]}{[(1+v) + (1-v)\alpha^2]}. \]  
\text{(5.14i)}

\[ u = \frac{(1-v^2)}{E} \frac{\alpha}{r} \int_a^r r \Prdr + \frac{(1-v^2)}{E} \int_a^b r \Prdr \left( r - \frac{a^2}{r} \right) \frac{[(1+v) - (1-v)\alpha^2]}{[(1+v) + (1-v)\alpha^2]}. \]  
\text{(5.14j)}

For the second poroelasticity problem the boundary conditions are

\[ u = 0 \quad \text{at} \quad r = a \]  
\text{(5.15a)}

\[ \tau_{rr} = -\tau_e \quad \text{at} \quad r = b \]  
\text{(5.15b)}

\[ \tau_{zz} = -\tau_0 \quad \text{for all} \quad r \]  
\text{(5.15c)}
and the pore pressure is zero. The general solution to this generalized plane strain problem is given by equations (5.11f) through 5.11i). Applying the conditions (5.15a) through (5.15c) to the general solution leads to the total stresses and displacement

\[
\tau_{rr} = \frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{r^2} - \left[ \tau_e + \frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{b^2} \right] \frac{[1 + \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{r^2}]}{[1 + \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{b^2}]} \quad (5.15d)
\]

\[
\tau_{\theta\theta} = -\frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{r^2} - \left[ \tau_e + \frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{b^2} \right] \frac{[1 - \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{r^2}]}{[1 + \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{b^2}]} \quad (5.15e)
\]

\[
\tau_{zz} = -\tau_0 \quad (5.15f)
\]

\[
u \tau_0 \frac{a^2}{E'} \left[ \frac{\tau_e + \frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{b^2}}{[1 + \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{b^2}]} \right] (r - \frac{a^2}{r}). \quad (5.15g)
\]

Superimposing the stresses of equations (5.14h), (5.14i), and (5.15d) through (5.15f) gives the rock stresses around a cased borehole under uniform overburden loading as
\[
\tau_{rr} = - (1-v) \frac{a}{r^2} \int_{a}^{b} Prdr + \frac{\nu \tau_0}{(1+v)} \frac{a^2}{r^2} 
\]
\[
+ \frac{[(1-v) \frac{a}{b^2} \int_{a}^{b} Prdr - \eta - \frac{\tau_0}{(1+v)} \frac{a^2}{b^2}]}{[1 + \frac{(1-v) a^2}{(1+v) b^2}]}
\]
\[
(5.15h)
\]

\[
\tau_{\theta\theta} = (1-v) \frac{a}{r^2} \int_{a}^{b} Prdr - \frac{\nu \tau_0}{(1+v)} \frac{a^2}{r^2} - (1-v) \alpha P 
\]
\[
+ \frac{[(1-v) \frac{a}{b^2} \int_{a}^{b} Prdr - \eta - \frac{\tau_0}{(1+v)} \frac{a^2}{b^2}]}{[1 + \frac{(1-v) a^2}{(1+v) b^2}]}
\]
\[
(5.15i)
\]

\[
\tau_{zz} = - \tau_0. 
\]
\[
(5.15j)
\]

Substituting equation (5.8a) for the pore pressure into the above relations yields the final expressions for the total stresses

\[
\tau_{rr} = - (1-v) \frac{a}{2} \left[ (P_e - \frac{C_0}{2})(1-\frac{a^2}{r^2}) + C_0 \ln \frac{r}{a} \right] + \frac{\nu \tau_0}{(1+v)} \frac{a^2}{r^2} 
\]
\[
+ \frac{[(1-v) \frac{a}{b^2} \left[ P_e - \frac{C_0}{2} - (P_e - \frac{C_0}{2}) \frac{a^2}{b^2} \right] - \eta - \frac{\tau_0}{(1+v)} \frac{a^2}{b^2}]}{[1 + \frac{(1-v) a^2}{(1+v) b^2}]}
\]
\[
(5.16a)
\]
\[ \tau_{\theta\theta} = (1-\nu) \frac{a}{2} \left[ (P_w - \frac{C_0}{2})(1 - \frac{a^2}{r^2}) + C_0 \ln \frac{r}{a} \right] - \frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{r^2} \]

\[- (1-\nu) a (P_w + C_0 \ln \frac{r}{a}) \]

\[ + \frac{(1-\nu)}{2} \left[ P_e - \frac{C_0}{2} - \frac{C_0 a^2}{b^2} \right] \tau_e - \frac{\nu \tau_0}{(1+\nu)} \frac{a^2}{b^2} \frac{1 - \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{r^2}}{1 + \frac{(1-\nu)}{(1+\nu)} \frac{a^2}{b^2}} \]

\[ \tau_{zz} = - \tau_0^* \quad (5.16c) \]

To illustrate the state of stress in the region near the wellbore, equations (5.16a) through (5.16c), (5.8a), and (5.8b) were used to calculate the effective stress distribution for the following numerical example:

\[ b/a = 500 \quad (5.16d) \]
\[ \tau_e/\tau_0 = 1 \quad (5.16e) \]
\[ P_e/\tau_0 = 0.8 \quad (5.16f) \]
\[ \nu = 0.25 \quad (5.16g) \]
\[ \alpha = 1. \quad (5.16h) \]

The effective stresses \( \frac{\sigma_{rr}}{\tau_0}, \frac{\sigma_{\theta\theta}}{\tau_0}, \) and \( \frac{\sigma_{zz}}{\tau_0} \) are shown as a function of the nondimensional radius \( \frac{r}{a} \) for well pressures \( \frac{P_w}{\tau_0} = 0.8, 0.5, \) and 0.1 in Figures 38, 39, and 40. As the well pressure decreases, the compressive effective stresses increase in magnitude. The largest
principal stress difference occurs at the lowest value of the well pressure, and the order of the principal stresses changes as the well pressure drops. At low well pressures the order of the principal effective stresses is

\[ \langle \sigma_{\theta \theta} \rangle > \langle \sigma_{rr} \rangle > \langle \sigma_{zz} \rangle. \]

e. Coulomb yielding of the rock around a cased borehole.

As was done for the open hole completion, it is possible to determine a critical well pressure for given values of cohesive strength and angle of internal friction. The Coulomb yield equation of Terzaghi is

\[ \langle \sigma_3 \rangle = \langle \sigma_1 \rangle N^2 - 2CN \]

where \( \langle \sigma_1 \rangle \) and \( \langle \sigma_3 \rangle \) are the maximum and minimum principal effective stresses. In this problem, at low well pressures

\[ \langle \sigma_1 \rangle = \langle \sigma_{\theta \theta} \rangle \]
\[ \langle \sigma_3 \rangle = \langle \sigma_{zz} \rangle \]

and Terzaghi's equation is

\[ \langle \sigma_{zz} \rangle = \langle \sigma_{\theta \theta} \rangle N^2 - 2CN. \] (5.17a)

As shown in Figure 40, the largest principal stress difference occurs at \( r = a \) where the effective stresses are
given by equations (5.9e), (5.16b), and (5.16c) as

\[ \langle \sigma_{zz} \rangle = -\tau_0 + \alpha P_w \]  \hspace{1cm} (5.17b)

\[ \langle \sigma_{\theta\theta} \rangle = \frac{-\nu \tau_0}{(1+\nu)} + \nu \alpha P_w \]

\[ + \left\{ (1-\nu) \frac{\alpha}{2} \left[ \frac{C_0}{2} - \frac{C_0 a^2}{b^2} \right] - \tau_0 - \nu \frac{\tau_0}{(1+\nu)} \right\} \frac{a^2}{b^2} \frac{[1 - (1-\nu)]}{[1 + \frac{(1-\nu) a^2}{b^2}]} \]

(5.17c)

Substituting equations (5.17b), (5.17c), and (5.17b) into the Coulomb relation (5.17a) gives the critical well pressure at initial yield as

\[ \tau_0 \left[ 1 - \frac{\nu N^2}{(1+\nu)} \right] - 2CN + N^2 K_0 \left\{ (1-\nu) \frac{\alpha}{2} \left[ \frac{1 - (1-a^2/b^2)}{2 \ln b/a} \right] - \nu \frac{\tau_0}{(1+\nu)} \frac{a^2}{b^2} \right\} \]

\[ P_w = \frac{\alpha \left\{ 1 - \nu N^2 - K_0 N^2 \frac{(1-\nu)}{2} \frac{(1-a^2/b^2)}{2 \ln b/a} \right\} \]

(5.17d)

where

\[ K_0 = \frac{\left[ 1 - \frac{(1-\nu)}{(1+\nu)} \right]}{\left[ 1 + \frac{(1-\nu) a^2}{(1+\nu) b^2} \right]} \]  \hspace{1cm} (5.17e)

To illustrate the effect of rock properties on the critical well pressure, the example described by equations
(5.16d) through (5.16h) was again considered. Figures 41 through 44 show the pressure \( \frac{P_w}{\tau_0} \) calculated from equation (5.17d) as a function of the angle of internal friction \( \phi \) and rock cohesive strength \( \frac{C}{\tau_0} \). It can be seen that the well pressure at yield increases as the cohesive strength and angle of internal friction decrease. When these results are compared to the open hole analysis of Section 5.3c, it is apparent that a much larger drawdown is required to yield the rock around a cased hole. For example, with \( \frac{C}{\tau_0} = 0.2 \) and \( \phi = 0^\circ \), the well pressures \( \frac{P_w}{\tau_0} \) at yield are 0.80 and 0.15 for the open and cased holes respectively.

e. Conclusions

In the preceding analysis the elastic stresses in rock around open and cased boreholes were derived under the condition of uniform overburden loading. It was shown that well pressure drawdown can produce Coulomb yielding in both cased and open holes. As expected, higher production rates are required to yield the rock around a borehole if the formation is supported by a casing. Conditions around a perforated casing are expected to lie somewhere between the extremes of the open hole and the rigid, permeable casing considered here. Thus the value of the critical well pressure for a perforated casing may be bounded by the pressures calculated from equations (5.13c) and (5.17d). However it should be noted that this analysis does not include the
effects of the flow restriction present when the pore fluid is channeled through only a small percentage of the surface area of a perforated casing. Attention must be directed to this effect in order to better relate the physical problem of interest to the wellbore models considered here.
CHAPTER 6.
THE EFFECTS OF FLUID FLOW
ON EXTRUSION OF A POROUS SOLID

6.1 Introduction.

Collapse of casings in weak chalk reservoirs has been preceded by fluidization or extrusion of rock through casing perforations\textsuperscript{16}. The extrusion of materials obeying a Tresca yield condition has been studied by Ford\textsuperscript{55} and Prager\textsuperscript{56}. It is of principal interest in the present problem to determine the effect of a flowing pore fluid on the extrusion process. Reviewing the existing extrusion solutions shows that the effect of a steady state Darcy's law flow field may be included in the problem. Thus it is possible to determine the reduction in required extrusion pressure resulting from the simultaneous effects of pore fluid flow and boundary loading.

6.2 Plastic deformation of a porous material.

Consider a porous solid experiencing plastic deformation and subjected to a nonuniform pore pressure distribution. If the solid is rigid-perfectly plastic and incompressible, obeys a Tresca yield condition, and is in a state of plane strain, the governing equations for the deformation are as follows.
The yield condition may be written in terms of the effective stress \( \langle \sigma_{ij} \rangle \)

\[
\langle \sigma_{ij} \rangle = \tau_{ij} - \delta_{ij}p
\]

(6.1)

where \( \tau_{ij} \) is the total stress, \( P \) is the pore pressure, \( \delta_{ij} \) is the Kronecker delta, and compression is taken as positive. The yield condition is

\[
\frac{1}{4}(\langle \sigma_{xx} \rangle - \langle \sigma_{yy} \rangle)^2 + \langle \sigma_{xy} \rangle^2 = s^2
\]

(6.2a)

where \( s \) is the yield strength in pure shear and the deformation is in the \( xy \) plane. Using equation (6.1), the yield condition is alternately

\[
\frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2 = s^2.
\]

(6.2b)

The equilibrium equations are, in terms of the total stress

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0
\]

(6.3a)

\[
\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0.
\]

(6.3b)

The last three equations alone have been used to determine the stress field for some problems. However, in general, boundary conditions involve displacement rates and therefore a stress-strain relation must be introduced.
The Levy-Mises flow rule written in terms of the deviatoric effective stress \((\sigma'_{ij})\) is \(^{55}\)

\[
\dot{e}_{ij} = \lambda \sigma'_{ij}
\]  
(6.4a)

where the \(\dot{e}_{ij}\) are the strain rates for the porous solid, \(\lambda\) is a positive proportionality function, and

\[
\sigma'_{ij} = \sigma_{ij} - \left(\frac{1}{3}\right) \delta_{ij} \sigma_{\alpha\alpha}.
\]  
(6.4b)

The repeated index \(\alpha\) indicates a sum over \(\alpha = x, y, z\).

Equation (6.4b) may be compared with the definition of the deviatoric total stress \((\tau'_{ij})\)

\[
\tau'_{ij} = \tau_{ij} - \left(\frac{1}{3}\right) \delta_{ij} \tau_{\alpha\alpha}.
\]  
(6.4c)

Equation (6.1) shows that

\[
\sigma_{\alpha\alpha} = \tau_{\alpha\alpha} - 3 \mathbf{p}.
\]  
(6.4d)

Substituting (6.1) and (6.4d) into (6.4b) yields

\[
\sigma'_{ij} = \tau_{ij} - \frac{1}{3} \delta_{ij} \tau_{\alpha\alpha}
\]  
(6.4e)

so that the deviatoric effective and deviatoric total stresses are equal. Hence the flow rule (6.4a) can be written

\[
\dot{e}_{ij} = \lambda \tau'_{ij}.
\]  
(6.4f)
The preceding analysis demonstrates that for the problem under study the yield condition and flow rule written in terms of the effective stress reduce to the conventional forms (6.2b) and (6.4f) despite the introduction of a non-uniform pore pressure. Thus the total stress distribution in the mixture may be found from a conventional method of characteristics solution\textsuperscript{55}.

Given the solution for a problem involving rigid-perfectly plastic, plane strain deformation of an incompressible Tresca material, the effects of pore fluid flow may be obtained by taking the pore pressure from an appropriate potential flow solution to be related to the total stress distribution by equation (6.1). The flow field may be obtained from Darcy's law and conformal mapping methods. In the next section the approach outlined above is applied to an extrusion problem.

6.3 The effects of pore fluid flow on extrusion.

Consider the extrusion of a wide billet of solid material through a smooth die, with the extruded sheet having half the thickness of the original billet (see Figure 45)\textsuperscript{56}. The half-thickness of the billet is $h$, while the piston and extruded sheet velocities are $u$ and $U$ respectively. The horizontal stress applied to the die along line $AD$ is given by Ford\textsuperscript{55} as
\[ \tau_{xx} = 2s(1 + \frac{\pi}{2}) \]  

(6.5a)

Since the die is taken to be smooth, the average force on the upper half of the piston (per unit length in the \( z \) direction) is then

\[ F = \tau_{xx} \frac{h}{2} = hs(1 + \frac{\pi}{2}). \]  

(6.5b)

Now take the extruded material to be porous, and subjected to steady flow of an incompressible fluid from the direction of the permeable piston towards the die opening \( AO \). The walls of the die are taken to be impermeable. The pore pressure distribution for this flow field is (see Appendix B)

\[ P = \frac{Q \mu}{\pi KL} \text{Im} \left\{ \sin^{-1} \left[ \sqrt{2} \sin \left( \frac{\pi}{2h} (y - ix) \right) \right] \right\} \]  

(6.6)

where \( Q \) is the fluid volumetric flowrate, \( L \) is the billet length in the \( z \) direction, \( \mu \) is the fluid viscosity, \( K \) is the billet permeability, \( \text{Im} \) denotes the imaginary part of a complex number, and \( i = \sqrt{-1} \). The total stress on line \( AD \) is again given by equation (6.5a) but it is composed of an effective stress and a pore pressure

\[ \tau_{xx} = \langle \sigma_{xx} \rangle + P \]  

(6.7a)
which with equation (6.5a) becomes

\[ \langle \sigma_{xx} \rangle = 2s(1 + \frac{\pi}{2}) - P. \] \hspace{1cm} (6.7b)

The average force per unit length on the upper half of the permeable piston required to extrude the porous solid is estimated by integrating the effective stress (6.7b) over the line AD

\[ F = \int_{h/2}^{h} \langle \sigma_{xx} \rangle \, dy \] \hspace{1cm} (6.7c)

\[ F = hs(1 + \frac{\pi}{2}) - \int_{h/2}^{h} Pdy. \] \hspace{1cm} (6.7d)

where the pore pressure is given by equation (6.6) with \( x = 0 \). Integrating this equation (see Appendix B) gives

\[ F = hs(1 + \frac{\pi}{2}) - 0.3466 \frac{Quh}{\pi KL}. \] \hspace{1cm} (6.7e)

Equation (6.7e) gives one half the reduced average extrusion force per unit length of the piston as a function of the flowrate through the porous solid. The nondimensional average pressure pressure on the piston face is

\[ \frac{F}{hs} = 1 + \frac{\pi}{2} - 0.3466 \frac{Qu}{\pi KLS}. \] \hspace{1cm} (6.7f)
which shows that large flowrates and viscosities and small permeabilities tend to maximize the influence of fluid flow in reducing the required extrusion pressure.

As a numerical example, consider the typical reservoir conditions

\[
P_e = 6500 \text{ psi} \quad (6.8a)
\]

\[
P_w = 3500 \text{ psi} \quad (6.8b)
\]

\[
\frac{b}{a} = 500 \quad (6.8c)
\]

where \( P_w \) and \( P_e \) are the pore pressures at the well (\( r = a \)) and reservoir edge (\( r = b \)). For steady flow\(^{24} \)

\[
\frac{Q_{ui}}{2\pi Kh} = \frac{P_e - P_w}{\ln \left( \frac{b}{a} \right)} \quad (6.8d)
\]

and with

\[
h = L \quad (6.8e)
\]

equations (6.8) give

\[
\frac{Q_{ui}}{\pi KL} = 965 \text{ psi.} \quad (6.9a)
\]

For a weak rock with a shear strength of

\[
s = 500 \text{ psi} \quad (6.9b)
\]

equations (6.7f) and (6.9) give the nondimensional extrusion pressure as
\[ \frac{P}{h_s} = 1.90. \] (6.9c)

This represents a twenty-six percent reduction in the required extrusion pressure as compared to the case of no flow (equation (6.7f) with \( Q = 0 \)).

6.4 Conclusion.

The extrusion of chalk through casing perforations is facilitated by pore fluid flow. High production rates may significantly reduce the boundary loads required to produce extrusion, particularly with viscous fluids and low permeability rock in the region of the perforation. A quantitative estimate of the effects of pore fluid flow has been obtained for a Tresca material. It is of interest to investigate the extrusion of a porous solid obeying the Coulomb yield condition, which better characterizes rock behavior.
CHAPTER 7.

APPROXIMATE SOLUTIONS FOR THE POTENTIAL FLOW
OF A PORE FLUID INTO A SLOTTED CYLINDER

Darcy's law and conformal mapping may be used to determine the pore pressure distribution produced by two-dimensional, steady, incompressible, laminar flow into a slotted casing. The general problem of describing the three-dimensional and two-dimensional flow fields around slotted liners has been solved by Dodson and Cardwell\textsuperscript{19}. In this section a simplified method is used to approximate the flow field for the two-dimensional, single slot case, for large reservoir radius to casing radius ratios. The conformal transformations used are from Churchill, Brown, and Verhey\textsuperscript{15}.

In the $z$ plane the problem is to find the pore pressure $P$ such that $\nabla^2 P = 0$ and

\begin{align}
P = P_e & \quad \text{for } r = r_e, \quad 0 \leq \theta \leq \pi \tag{7.1a} \\
P = P_w & \quad \text{for } r = r_c, \quad 0 \leq \theta \leq \alpha \tag{7.1b}
\end{align}

where $\nabla^2$ is the Laplace operator, $r_c$ and $r_e$ are the casing and reservoir radii, $P_w$ and $P_e$ are the pressure at the well and the reservoir edge, $\alpha$ is half the slot angle, and $z = x + iy$ with $i = \sqrt{-1}$. Only the upper half plane is considered since the problem is symmetric. The remaining boundaries are no flow boundaries, since the casing surface lies along
\[ r = r_c, \quad \alpha \leq \theta \leq \pi \]  

and the lines

\[ r_c \leq r \leq r_e, \quad \theta = 0 \]  
\[ r_c \leq r \leq r_e, \quad \theta = \pi \]

are streamlines due to symmetry.

This region is mapped into the \( w \) plane using the transformation (see Appendix C)

\[ w = \sin^{-1} \left[ \frac{1}{(1 - \cos \alpha)} \left( \frac{z}{r_c} + \frac{r_c}{z} - 1 - \cos \alpha \right) \right] \]  

where

\[ w = u + iv. \]

The boundary given by equation (7.1b) becomes

\[ w = \sin^{-1} \left\{ \frac{e^{i\theta} + e^{-i\theta} - 1 - \cos \alpha}{1 - \cos \alpha} \right\}, \quad 0 \leq \theta \leq \alpha \]

or

\[ w = \sin^{-1} \left\{ \frac{2 \cos \theta - 1 - \cos \alpha}{1 - \cos \alpha} \right\}, \quad 0 \leq \theta \leq \alpha \]  

which is the line

\[ v = 0, \quad \frac{-\pi}{2} \leq u \leq \frac{\pi}{2} \]

in the \( w \) plane. The boundary given by equation (7.1c) becomes
\[ w = \sin^{-1} \left[ \frac{2 \cos \theta - 1 - \cos \alpha}{1 - \cos \alpha} \right], \quad \alpha \leq \theta \leq \pi. \quad (7.4c) \]

Since
\[ \sin w = \sin u \cosh v + i \cos u \sinh v \quad (7.4d) \]
equation (7.4c) is the line
\[ u = \frac{-\pi}{2}, \quad 0 \leq v \leq v_1 \quad (7.4e) \]
in the w plane, where
\[ v_1 = \cosh^{-1} \left[ \frac{3 + \cos \alpha}{1 - \cos \alpha} \right]. \quad (7.4f) \]

The boundary given by equation (7.2a) becomes
\[ w = \sin^{-1} \left[ \frac{1}{(1 - \cos \alpha) \left( \frac{r}{r_c} + \frac{r_c}{r} - 1 - \cos \alpha \right)} \right], \quad r_c \leq r \leq r_e \quad (7.5a) \]
which is the line
\[ u = \frac{\pi}{2}, \quad 0 \leq v \leq v_2 \quad (7.5b) \]
in the w plane, where
\[ v_2 = \cosh^{-1} \left[ \frac{1}{(1 - \cos \alpha) \left( \frac{r_e}{r_c} + \frac{r_c}{r_e} - 1 - \cos \alpha \right)} \right]. \quad (7.5c) \]

The boundary given by equation (7.2b) becomes
\[ w = \sin^{-1} \left[ \frac{1}{(1 - \cos \alpha)} \left( \frac{r}{r_c} - \frac{r_e}{r_c} + 1 + \cos \alpha \right) \right], \quad r_c \leq r \leq r_e \]  
\[(7.6a)\]

which is the line

\[ u = \frac{-\pi}{2}, \quad v_1 \leq v \leq v_3 \]  
\[(7.6b)\]

in the \( w \) plane, where

\[ v_3 = \cosh^{-1} \left( \frac{r_e}{r_c} + \frac{r_c}{r_e} + 1 + \cos \alpha \right) \].  
\[(7.6c)\]

For large \( \frac{r_e}{r_c} \) ratios, which are of interest here,

\[ v_2 = v_3. \]  
\[(7.7a)\]

For example, for \( \frac{r_e}{r_c} = 500 \) and \( \alpha = 45^\circ \),

\[ v_2 = 8.132 \]
\[ v_3 = 8.139. \]

Therefore for large \( \frac{r_e}{r_c} \) ratios, the boundary given by equation (7.1a) becomes approximately the line

\[ v = v_2, \quad \frac{-\pi}{2} \leq u \leq \frac{\pi}{2} \]  
\[(7.7b)\]

in the \( w \) plane. Hereafter equation (7.7b) will be assumed to describe the reservoir edge boundary.
In the \( w \) plane the approximate pore pressure is given by Darcy's law for linear flow as
\[
P = P_w + \frac{Qu}{2\pi kL} \nu
\] (7.8)

where \( Q \) is the volumetric flowrate, \( \nu \) is the viscosity, \( k \) is the permeability, and \( h \) is the formation thickness. To obtain the pore pressure in the \( z \) plane, it is necessary to express \( \nu \) in terms of \( x \) and \( y \) using the transformation equation (7.3). To do this let \( \Omega = \sin w \), or
\[
\Omega = \frac{1}{(1 - \cos \alpha)} \left( \frac{z}{r_c} + \frac{r_c}{z} - 1 - \cos \alpha \right)
\] (7.9a)

where \( \Omega = \xi + i\eta \).

Using equations (7.3) and (7.4a) and equating the real and imaginary parts of \( \Omega \) and \( \sin w \) yields
\[
\xi = \sin u \cosh \nu \quad (7.9b)
\]
\[
\eta = \cos u \sinh \nu. \quad (7.9c)
\]

Combining these equations to eliminate \( u \) results in
\[
\frac{\xi^2}{\cosh^2 \nu} + \frac{\eta^2}{\sinh^2 \nu} = 1. \quad (7.9d)
\]

For any fixed \( \nu \), this is an ellipse in the \( \Omega \) plane with foci at the points \((\xi, \eta) = (\pm1, 0)\). (See reference 15, p. 97). The sum of the distances from the foci to any point
on the ellipse in the first quadrant is given by (See reference 20, p. 383)

\[ 2 \cosh v = \left[ (\xi + 1)^2 + \eta^2 \right]^{1/2} + \left[ (\xi - 1)^2 + \eta^2 \right]^{1/2}. \]

Therefore

\[ v = \cosh^{-1} \left\{ \frac{\left[ (\xi + 1)^2 + \eta^2 \right]^{1/2} + \left[ (\xi - 1)^2 + \eta^2 \right]^{1/2}}{2} \right\}. \quad (7.9e) \]

Writing \( z = re^{i\theta} \) and equating the real and imaginary parts of equation (7.9a) gives

\[ \xi = \frac{1}{(1 - \cos \alpha)} \left[ \left( \frac{r}{r_C} \right) \cos \theta - 1 - \cos \alpha \right] \quad (7.10a) \]

\[ \eta = \frac{1}{(1 - \cos \alpha)} \left( \frac{r}{r_C} - \frac{r_C}{r} \right) \sin \theta. \quad (7.10b) \]

Substituting equations (7.10a) and (7.10b) into equation (7.9e) allows equation (7.8) to be written
\[ p = p_w + \frac{Q_u}{2\pi kh} \cosh^{-1} \left( \frac{1}{2} \left( \frac{\left( \frac{r}{r_c} - \frac{r_c}{r} \right) \sin \theta - \left( \frac{r}{r_c} + \frac{r_c}{r} \right) \cos \theta - 1 - \cos \alpha}{1 - \cos \alpha} \right)^2 + 1 \right)^{1/2} \]

This is the approximate pore pressure distribution in the \( z \) plane, for large \( r_e/r_c \) ratios.

The streamlines of the flow field are represented by the curves

\[ u = \text{constant} . \quad (7.11a) \]

The constant pressure lines of the flow field are represented by the curves

\[ v = \text{constant} . \quad (7.11b) \]

The variable \( v \) can be considered a nondimensional pore pressure, since from equation (7.8)

\[ v = \frac{P - p_w}{\frac{Q_u}{2\pi kh}} \quad (7.11c) \]
Streamlines and constant pressure lines in the \( z \) plane are shown in Figure 46 for a slot half-angle (\( \alpha \)) equal to 45°.

Equations (7.4c) through (7.4f) show that on the casing surface

\[
v = \cosh^{-1} \left( \frac{1 + \cos \alpha - 2 \cos \theta}{1 - \cos \alpha} \right). \tag{7.12a}
\]

If equation (7.12a) is substituted into equation (7.8), the pore pressure variation along the casing surface is described by

\[
P = P_w + \frac{Q_w}{2 \pi k h} \cosh^{-1} \left( \frac{1 + \cos \alpha - 2 \cos \theta}{1 - \cos \alpha} \right). \tag{7.12b}
\]

Figure 47 shows the variation of the nondimensional pore pressure \( v \) with \( \theta \), along the surface of a casing with \( \alpha = 45° \), as given by equation (7.12a). The maximum pore pressure occurs at \( \theta = \pi \), where equation (7.12a) gives \( v_{\text{max}} = 3.23 \). The slot is located along \( 0 \leq \theta \leq 45° \). On this part of the casing surface, \( P = P_w \) and therefore \( v = 0 \).

Figure 48 displays the pore pressure distribution of equation (7.12a) in a polar plot. A nonsymmetric pore pressure distribution is of interest since the effective stress in the rock is a function of pore pressure, and since
the pore pressure determines one component of the total stress exerted on the casing.

The preceding analysis may be compared with the results of Dodson and Cardwell\(^\text{19}\). They give the volumetric flowrate into a slotted liner as

\[
Q = \frac{2\pi kh \left( P_e - P_w \right)}{\mu \left( \ln \frac{r_e}{r_c} + \frac{2}{N} \ln \frac{2}{\pi f} \right)} \tag{7.12c}
\]

where \(N\) is the number of slots and \(f\) is the open fraction of the pipe. For the example consider previously, below equation (7.7a),

\[
N = 1 \tag{7.12d}
\]

and

\[
f = \frac{1}{4}. \tag{7.12e}
\]

With these values Dodson and Cardwell's expression yields

\[
\frac{Q_l}{2\pi kh} = 8.08. \tag{7.12f}
\]

The value of 8.13 calculated below (7.7a) from the approximate method of this chapter represents an error of less than one percent.

Further work on this problem indicated that a description of the flow field around a slotted casing could
be found that was mathematically simpler than that just described. As a result the conformal mapping described below was developed by combining some basic transformations given by Churchill, Brown, and Verhey.\footnote{15}

Let the physical plane, or $z$ plane, be an infinite medium with a circular hole of radius $r_c$ at its center, where $r_c$ is the casing radius. The polar coordinates $r$, $\theta$ and the rectangular coordinates $x$, $y$ are related to the complex variable $z$ by

$$ z = r e^{i\theta} = x + iy $$  \hspace{1cm} (7.13)

where $i = \sqrt{-1}$. If the casing contains a slot along an arc centered at $z = -r_c$ and subtending an angle $2\beta$, then there exists a finite line sink for the pore fluid along

$$ r = r_c, \quad \pi - \beta \leq \theta \leq \pi + \beta. $$  \hspace{1cm} (7.14)

The pore pressure is taken to be a constant along the slot.

The region $R_z$ outside the casing in the physical plane is mapped into the region $R_{z_1}$ inside the circle $|z_1| = 1$ in the $z_1$ plane using

$$ z_1 = \frac{r_c}{z}. $$  \hspace{1cm} (7.15)

Next the transformation

$$ z_2 = \sqrt{z_1} $$  \hspace{1cm} (7.16)
maps \( R_{z_1} \) into the region \( R_{z_2} \) bounded by the upper half of the circle \(|z_2| = 1\) and the real axis in the \( z_2 \) plane. Finally \( R_{z_2} \) is mapped into the lower half of the \( w \) plane with

\[
w = z_2 + \frac{1}{z_2}.
\]  

(7.17)

Combining equations (7.15) through (7.17) shows that

\[
w = \sqrt{\frac{z}{r_c}} + \sqrt{\frac{r_c}{z}}
\]  

(7.18)

transforms the flow region in the physical plane into the lower half of the \( w \) plane. The slot described by equations (7.14) is located in the \( w \) plane along

\[
v = 0, \ -2a \leq u \leq 2a
\]  

(7.19)

where

\[
w = u + iv
\]  

(7.20)

and

\[a = \sin \frac{\beta}{2}.
\]  

(7.21)

Now that the flow field has been transformed into the \( w \) plane, it is possible to make use of Muskat's solution for the complex potential of the flow into a finite line sink. The complex potential in the \( w \) plane is proportional to
\[ F(w) = \cosh^{-1} \left( \frac{w}{2a} \right). \]  
(7.22)

With equation (7.18), the complex potential for the flow field in the physical plane is proportional to

\[ F(z) = \cosh^{-1} \left( \frac{2}{r} + \sqrt{\frac{L}{z}} \right). \]  
(7.23)

The real part of \( F(z) \) is proportional to the fluid pressure and is given by

\[ 2 \text{Re}(F(z)) = F(z) + \overline{F(z)} \]  
(7.24)

where the bar denotes the complex conjugate. Using equation (7.24) it is possible to describe the variation of the pore pressure in the physical plane in a mathematically simpler form.
CHAPTER 8.
THE EFFECT OF A PORE FLUID ON
THE ULTIMATE STRENGTH OF AUSTIN CHALK

8.1 Introduction.

Field experience has indicated that Dania chalk is significantly weakened by exposure to water and oil. Should this be the case, extrusion or fluidization of Dania chalk might be expected in the presence of high pore fluid pressure gradients.

In order to investigate the effects of pore fluids on the strength of a weak rock, samples of Austin chalk soaked in oil (viscosity 150/42 Saybolt Univ., Sec. at 100/210°F), water, and acetone were subjected to conventional compression tests. The results of these tests were compared to those obtained using dry Austin chalk samples.

8.2 Experimental procedure.

The cylindrical rock samples used were nominally one inch long and one-half inch in diameter. Because Austin chalk exhibits anisotropy due to the presence of a bedding plane, each sample was labeled with the direction in which it was cored. For example, a sample labeled X was cored with its cylindrical axis in the X direction, where X, Y, and Z are the directions of a rectangular coordinate system. A sample labeled XY was cored with its cylindrical axis in
the XY plane at 45° from the X or Y axis. Samples of types X, Y, Z, XY, XZ and YZ were utilized. The treated samples were immersed in fluid for approximately four days before testing.

The compression experiments were performed on a 60,000 lb. Riehle testing machine. Axial displacement was measured using a Hewlett Packard linear variable-differential-transformer and a dial indicator. Four dry samples and three each of the oil, water, and acetone treated samples were loaded until fracture. Figures 52 through 64 show axial compressive stress versus axial strain for the Austin chalk samples. Each figure gives the direction in which the sample was cored and the fluid with which it was treated, if any. The last data point on each graph represents the ultimate strength of the tested material, i.e. the largest load the sample was able to support. The first sample only was unloaded at 1000 psi and then reloaded to observe the hysteresis effect.

8.3 Results and conclusions.

Figure 65 shows the ultimate strengths obtained from the experiments for various core orientations and fluids. The results indicate that the XY plane is the plane of bedding and that the chalk used is symmetric with respect to the Z axis. Hence the X, Y, and XY samples are equivalent and can be considered as a single sample group.
Because of the aforementioned symmetry the $XZ$ and $YZ$ samples are likewise equivalent. The experiments showed that oil had a limited and water a substantial degrading effect on the ultimate strength of Austin chalk. Only two of the three acetone treated samples showed a reduced ultimate strength, making it difficult to reach a conclusion on the effect of this fluid.

Figure 66 shows in a nondimensional form the relative strengths of the dry and fluid treated samples. Equivalent sample types are combined into a single row on the chart. The strength of each treated sample type is given as a percentage of the dry strength of that sample type. Chart entries that represent more than one sample are obtained by using average values.

This experiment indicates that the presence of a pore fluid can reduce the ultimate strength of Austin chalk by as much as one third. A more detailed description of the effects of pore fluids on the material properties of Austin or Dania chalk could be obtained from triaxial tests. These results are consistent with those given by Farmer\textsuperscript{60} for the ultimate strength in unconfined compression tests of partially saturated porous rocks.
SUMMARY AND CONCLUSIONS

The elastic and plastic stresses in a porous solid containing a flowing pore fluid have been studied and the influence of fluid flow on the stresses around spherical and cylindrical cavities has been determined. The following conclusions are indicated:

(1) The porous solid and fluid are best described by a mixture model which accounts for the coupling of the stress and strain in the two constituents. This concept provides a basis with which to compare and evaluate the work which has been done on porous media.

(2) Although various methods are used to separate the stress tensor into distinct parts, the decomposition which provides the most physical insight is that of Biot, which identifies the stresses carried by the fluid and the solid constituents averaged over a unit area of the mixture.

(3) The objective of defining an effective stress is to isolate that part of the total stress which determines the deformation of the porous solid. Different decompositions of the stress tensor may lead to the same effective stress.

(4) The effective stress laws of Cheatham and Paslay, Nur and Byerlee, and Lubinski for an elastic solid constituent are equivalent. If the solid grains are incompressible,
their formulation agrees with Terzaghi's empirical relation. Carroll has correlated these and other effective stress relations for an isotropic solid.

(5) Biot's model of an elastic fluid and an elastic anisotropic solid provides a general definition for the effective stress in the elastic region.

(6) The application of Drucker's postulate to Biot's mixture model under homogeneous deformations leads to a formulation identical to that for nonporous materials, except that the appropriate stress variable is the total stress.

(7) For plastic homogeneous deformation of a porous medium, the stress in the plastic flow law is the total stress in the mixture.

(8) The loading and unloading cycle employed by Johnson and Green appears to be a special case of the cycle described by applying Drucker's postulate to an elastic-plastic mixture.

(9) The plastic stress state around a spherical cavity is significantly affected by the pore fluid flow field, with the critical flowrate that tends to expand the existing cavity a function of the solid material properties and the fluid reservoir conditions.
(10) Conditions for the initial yielding of the material around a cylindrical cavity may be specified as a function of the solid and fluid material properties, the pore fluid flow field, the in situ stress state of the porous medium, and the degree of external support provided to the cavity walls.

(11) The plane strain model of fluid reservoirs may be improved by accounting for uniform overburden loading, allowing the prediction of initial subsidence as a function of reservoir depletion.

(12) The required mechanical loads for extrusion may be significantly reduced by the simultaneous effects of fluid flow through a porous extruded material.

(13) Darcy's law and conformal mapping may be used to approximate the two-dimensional pore pressure distribution around a cylinder with a single longitudinal slot. The resulting expression gives the pore pressure as an explicit function of position.

(14) The ultimate strength of Austin chalk in compression may be significantly degraded by the presence of water or oil as a pore fluid.

(15) The application of the previous analysis to the particular problem of a perforated casing allows the
estimation of the well pressure that initiates plastic flow around such a casing, and the well pressure that tends to enlarge the existing perforation cavities. The elastic analysis discussed in (10) above may be applied to well bores which are: (a) unsupported, and (b) supported by a rigid liner. This establishes upper and lower bounds for the well pressure at initial yield for a perforated casing. The plastic analysis in (9) above has shown that lower well pressures may then lead to the loss of formation support over a part of the casing surface. The loss of formation support results in unsymmetric loading and possibly casing collapse. This analysis provides a means of estimating allowable pressures for testing and production of wells under various reservoir and \textit{in situ} stress states, and in various rock formations.

Extension of the above work, particularly in evaluating the influence of strain-hardening of the porous solid and asymmetry of the pore fluid flow field, should provide additional understanding of the basic problems discussed in this thesis.
REFERENCES


28. England, p. 44


31. Ibid., p. 30.

32. Ibid., p. 138.

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34. Ibid., p. 32.

35. Ibid., p. 139.

36. Ibid., p. 38.

37. Ibid., p. 29.

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39. Ibid., p. 139.

40. Ibid., p. 111.

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42. Ibid., p. 112.

43. Ibid., p. 10.
44. Ibid., p. 19.
45. Mushkhelishvili, p. 466.
47. Ibid., p. 113.
48. Ibid., p. 103.
49. Ibid., p. 102.
50. Ibid., p. 114.
51. Ibid., p. 64.
52. Malvern, pp. 293–294.
APPENDIX A

THE STRESS-STRAIN RELATIONS OF POROELASTICITY

The elastic stress-strain relations given by Paslay and Cheatham\textsuperscript{3} are

\begin{align*}
E \sigma_{rr} &= \sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz}) - (1 - 2\nu) \beta P \quad (A1) \\
E \sigma_{\theta\theta} &= \sigma_{\theta\theta} - \nu(\sigma_{zz} + \sigma_{rr}) - (1 - 2\nu) \beta P \quad (A2) \\
0 &= \sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta}) - (1 - 2\nu) \beta P \quad (A3)
\end{align*}

where \( E \) and \( \nu \) are the elastic modulus and Poisson's ratio for the porous solid.

Solving equation (A3) for \( \sigma_{zz} \) and substituting this expression into equations (A1) and (A2) gives

\begin{align*}
\frac{E \sigma_{rr}}{\nu(1 + \nu)} &= \frac{(1 - \nu)}{\nu} \sigma_{rr} - \sigma_{\theta\theta} - \frac{(1 - 2\nu)}{\nu} \beta P \quad (A4) \\
\frac{E \sigma_{\theta\theta}}{(1 - \nu^2)} &= -\frac{\nu}{(1 - \nu)} \sigma_{rr} + \sigma_{\theta\theta} - \frac{(1 - 2\nu)}{(1 - \nu)} \beta P. \quad (A5)
\end{align*}

Adding these two equations and rearranging results in

\begin{equation}
\sigma_{rr} - \beta P = \frac{E(1 - \nu)}{(1 - 2\nu)(1 + \nu)} \left( \frac{\sigma_{rr}}{\nu} + \frac{\sigma_{\theta\theta}}{(1 - \nu)} \right). \quad (A6)
\end{equation}

The Lame coefficients \( \lambda \) and \( \mu \) for the porous solid are related to \( E \) and \( \nu \) by
\[ \lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \quad \text{(A7)} \]

\[ \mu = \frac{E}{2(1 + \nu)} = \frac{\lambda(1 - 2\nu)}{2\nu}. \quad \text{(A8)} \]

The relations (A7) and (A8) allow equation (A6) to be written as

\[ \bar{\sigma}_{rr} - \beta P = \lambda (e_{rr} + e_{\theta\theta}) + 2\mu e_{rr}. \quad \text{(A9)} \]

In a similar manner it can be shown that

\[ \bar{\sigma}_{\theta\theta} - \beta P = \lambda (e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta}. \quad \text{(A10)} \]
APPENDIX B
THE PORE PRESSURE DISTRIBUTION
FOR PLANE STRAIN EXTRUSION

The flow field in the physical plane (z plane) is that corresponding to steady flow through a rectangular billet into a constant pressure sink along the die opening (line AO in Figure 45). The conformal mapping approach used to obtain the flow field is explained in detail in reference 15. The required mappings for this problem are as follows.

First the physical plane is rotated 90° clockwise to the z' plane with the mapping\(^{15}\)

\[ z' = -iz \]  
\[ \text{(B1)} \]

where

\[ z = x + iy \]  
\[ \text{(B2)} \]

with \(x\) and \(y\) rectangular coordinates in the physical plane and \(i = \sqrt{-1}\).

Next the flow region in the \(z'\) plane is mapped onto the upper half of the \(z''\) plane with\(^{15}\)

\[ z'' = \sin\left(\frac{\pi}{2h}z'\right). \]  
\[ \text{(B3)} \]

Lastly the upper half of the \(z''\) plane is mapped onto the strip

\[ u \geq 0, \quad -\frac{\pi}{2} \leq v \leq \frac{\pi}{2} \]  
\[ \text{(B4)} \]
in the w plane using

\[ w = \sin^{-1} (\sqrt{2} z^\nu) \tag{B5} \]

where

\[ w = u + iv . \tag{B6} \]

Substituting (B1), (B2), and (B3) into (B5) gives

\[ w = \sin^{-1} [\sqrt{2} \sin\left(\frac{\pi}{2h} (y - ix)\right)] . \tag{B7} \]

Equation (A7) shows that the constant pressure line AO in the physical plane is mapped onto

\[ v = 0, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \tag{B8} \]

in the w plane. Thus the pore pressure in the w plane is given by Darcy's law for linear flow as

\[ U_o = \frac{K}{\mu} \frac{dP}{dv} \tag{B9} \]

where \( U_o \) is the flux of fluid into the sink, \( K \) is the formation permeability, and \( \mu \) is the fluid viscosity.

Since

\[ U_o = \frac{Q}{\pi L} \tag{B10} \]

where \( Q \) is the volumetric flow rate of fluid into the sink and \( L \) is the thickness of the porous solid perpendicular to the w plane, equation (B9) is now
\[ \frac{Q_u}{\pi kL} = \frac{dP}{dv}. \quad (B11) \]

Integrating this equation for steady flow,

\[ P = \frac{Q_u v}{\pi kL} \quad (B12) \]

where the pore pressure was taken to be zero along the sink.

Using (B6) and (B7) this can be written

\[ P = \frac{Q_u}{\pi kL} \text{Im} \left\{ \sin^{-1} \left[ \sqrt{2} \sin \left( \frac{\pi}{2h} (y - ix) \right) \right] \right\} \quad (B13) \]

where \( \text{Im} \) denotes the imaginary part of a complex function.

Along line AD in Figure 45

\[ x = 0, \quad \frac{h}{2} \leq y \leq h. \quad (B14) \]

The integral of the pore pressure along this line is

\[ I = \frac{Q_u}{\pi kL} \int_{\frac{h}{2}}^{h} \text{Im} \left\{ \sin^{-1} \left[ \sqrt{2} \sin \left( \frac{\pi y}{2h} \right) \right] \right\} \, dy \quad (B15) \]

which can be written

\[ I = \frac{Q_u h}{\pi kL} \int_{\frac{1}{2}}^{1} \text{Im} \left\{ \sin^{-1} \left[ \sqrt{2} \sin \left( \frac{\pi}{2} \hat{y} \right) \right] \right\} \, d\hat{y} \quad (B16) \]
where
\[ \hat{y} = \frac{y}{h} \tag{B17} \]

Numerical evaluation of the integral in (B16) gives
\[ I = .3466 \frac{O_{uh}}{u_{KL}} \tag{B18} \]
APPENDIX C

CONFORMAL TRANSFORMATIONS

The conformal transformation

\[ w = \sin^{-1} \left( \frac{1}{1 - \cos \alpha} \left( \frac{z}{r_c} + \frac{r_c}{z} - 1 - \cos \alpha \right) \right) \]  \hspace{1cm} (C1)

is obtained by applying in sequence two transformations which are modified versions of mappings given by Churchill, Brown, and Verhey\textsuperscript{15}.

In the \( z \) plane the flow region is the area in the upper half plane bounded by circles with radii \( r_c \) and \( r_e \) (Figure 49). The angle \( \alpha \) is half the slot angle. This region is mapped into the \( \Omega \) plane using (See reference 15, p. 320)

\[ \Omega = \frac{z}{r_c} + \frac{r_c}{z} - 1 - \cos \alpha \]  \hspace{1cm} (C2)

where \( \Omega = \xi + i\eta \). The transformed region (Figure 50) is bounded by the \( \xi \) axis and an ellipse centered at

\[ (\xi, \eta) = (-1 - \cos \alpha, 0). \]  \hspace{1cm} (C3)

A second transformation is now applied (See reference 15, p. 317):

156
\[ w = \sin^{-1} \left[ \frac{\Omega}{(1 - \cos \alpha)} \right] . \]  

(C4)

For large casing radius to reservoir radius ratios, the flow region is approximately a rectangle in the \( w \) plane (Figure 51).

Substitution of equation (C2) into equation (C4) gives the conformal transformation (C1).

Figures 49 through 51 serve to illustrate the mappings and are not scaled. Corresponding points in the \( z \), \( \Omega \), and \( w \) planes are labeled with the same letter.
FIGURES
Casing diameter = $d_c = 7$ inches
Perforation diameter = $d_p = 0.5$ inches

Figure A.
A perforated casing.
Figure B.

A hemispherical cavity in a porous solid.
$\theta$ = Angular width of the slot.

Figure C.
A slotted casing.
\[ z = x - iy = re^{i\theta}. \]

Figure D.
A cylindrical cavity supported by a slotted liner.
Figure E.
A cylindrical cavity in a porous solid.
STRESS DISTRIBUTION - OPEN WELL BORE - PW=0.855

Nondimensional radius

Radial Stress

Tangential Stress

Vertical Stress

FIGURE 2
STRESS DISTRIBUTION - OPEN WELL BORE - PW=.597

-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
0.1

NODIMENSIONAL RADIUS

1  2  3  4  5

- - - RADIAL STRESS
- - - TANGENTIAL STRESS
... VERTICAL STRESS

FIGURE 3
STRESS DISTRIBUTION - CASED WELL BORE - PW= .735

--- RADIAL STRESS

--- TANGENTIAL STRESS

... VERTICAL STRESS

FIGURE 5
STRESS DISTRIBUTION - CASED WELL BORE - PW=.855

- RADIAL STRESS
- TANGENTIAL STRESS

... VERTICAL STRESS

NONDIMENSIONAL RADIUS
STRESS DISTRIBUTION - CASED WELL BORE - PW=.597

- RADIAL STRESS
- TANGENTIAL STRESS
- VERTICAL STRESS

FIGURE 7
Critical Well Pressure - Perforated Casing

\[ \frac{\sqrt{K}}{P_{\infty}} = 0.344 \]

\[ C = 0.0845 \]

Upper Bound

Lower Bound

Dimensionless Rock Strength (B/P_{\infty})

Figure 13
SLOTTED CASING: PW = 0.597 AND THETA = 0 DEG

\( \Phi = 135 \) DEGREES

- RADIAL STRESS
- TANGENTIAL STRESS
- SHEAR STRESS

NONDIMENSIONAL RADIUS

FIGURE 16
SLOTTED CASING: PW = .597 AND THETA = 90 DEG

PHI = 135 DEGREES

- RADIAL STRESS
- TANGENTIAL STRESS
- SHEAR STRESS

NONDIMENSIONAL RADIUS

FIGURE 18
SLOTTED CASING: PW = .597 AND THETA = 180 Deg

PHI = 135 DEGREES

RADIAL STRESS
TANGENTIAL STRESS
SHEAR STRESS

Nondimensional Radius

Figure 20
SLOTTED CASING: PW = .855 AND R = 2

PHI = 135 DEGREES

- RADIAL STRESS
- TANGENTIAL STRESS
... SHEAR STRESS

FIGURE 22
Figure 26: Nondimensional Effective Stresses for $\frac{P_w}{P_e} = 0.9$

- Radial Stress
- Tangential Stress
- Vertical Stress

Nondimensional Radius $R/A$
Figure 31

Open hole - effective stresses for PL/To = 8, PE/To = .8

- Radial stress
- Tangential stress
- Vertical stress

Non-dimensional radius R/A

Effective stress
OPEN HOLE - EFFECTIVE STRESSES FOR PW/TO= .5, PE/TO=.8

---

RADIAL STRESS
TANGENTIAL STRESS
VERTICAL STRESS

NONDIMENSIONAL RADIUS R/A

FIGURE 32
OPEN HOLE - EFFECTIVE STRESSES FOR $PW/TO=0.1, PE/TO=0.8$

- RADIAL STRESS
- TANGENTIAL STRESS
- VERTICAL STRESS

FIGURE 33
Figure 36

OPEN HOLE - WELL PRESSURE AT YIELD (C/TO = 3)

ANGLE OF INTERNAL FRICTION (DEGREES)
OPEN HOLE - WELL PRESSURE AT YIELD (C/TO= .2)

CRITICAL WELL PRESSURE

0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0

ANGLE OF INTERNAL FRICTION (DEGREES)

FIGURE 37
Cased hole - well pressure at yield (C/To = .01)

Critical well pressure

Angle of internal friction (degrees)

Figure 44
Figure 45.

Plane Strain Extrusion.
Figure 46, Flow field around a slotted casing, for a slot half-angle of $45^\circ$

(Scale: 1 cm = 1 casing radius)
Figure 47. Pore pressure distribution on the surface of a slotted casing, for a slot half-angle of 45°.
Figure 48. Polar plot of the pore pressure distribution on the surface of a slotted casing, for a slot half-angle of 45°
Figure 49. Flow region in the z plane
\[ \Omega = \xi + i\eta \]

\[ \Omega = \frac{z}{r_C} + \frac{r_C}{z} - 1 - \cos\alpha \]

Figure 50. Flow region in the \( \Omega \) plane
\[ w = u + iv \]

\[ w = \sin^{-1}\left(\frac{\Omega}{1 - \cos \alpha}\right) \]

Figure 51. Flow region in the w plane, for \( r_e \gg r_c \)
Figure 56: Stress vs. Strain Graph for Austin Chalk - X Axis - Oil
Figure 59: Stress vs. Strain (Percent) for Austin Chalk - Y Axis - Water
AUSTIN CHALK - Z AXIS - ACETONE

FIGURE 63
Figure 64
AUSTIN CHALK

ULTIMATE STRENGTH (psi) IN COMPRESSION

<table>
<thead>
<tr>
<th>FLUID</th>
<th>DRY</th>
<th>OIL</th>
<th>WATER</th>
<th>ACETONE</th>
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<tr>
<td>X</td>
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<td>2598</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>2706</td>
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<td>2043</td>
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<tr>
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</tr>
<tr>
<td>YZ</td>
<td></td>
<td>3001</td>
<td>2429</td>
<td>2444</td>
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</tbody>
</table>

FIGURE 65
AUSTIN CHALK

RELATIVE STRENGTHS OF SAMPLES

(PERCENT)

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<tr>
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<th>DRY</th>
<th>OIL</th>
<th>WATER</th>
<th>ACETONE</th>
</tr>
</thead>
<tbody>
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<td>75</td>
<td>108</td>
</tr>
<tr>
<td>XZ or YZ</td>
<td>100</td>
<td>81</td>
<td>66</td>
<td>66</td>
</tr>
<tr>
<td>Z</td>
<td>100</td>
<td>—</td>
<td>—</td>
<td>81</td>
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FIGURE 66