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CAUCHY PROBLEM FOR NONLINEAR HYPERBOLIC SYSTEMS OF  
PARTIAL DIFFERENTIAL EQUATIONS

*Rice University*

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CAUCHY PROBLEM FOR NONLINEAR HYPERBOLIC SYSTEMS  
OF PARTIAL DIFFERENTIAL EQUATIONS

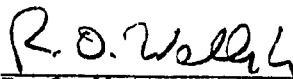
by

VICTORIA YASINOVSKAYA

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HOUSTON, TEXAS

APRIL 1983

МОИМ РОДИТЕЛЯМ

С ЛЮБОВЬЮ

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CAUCHY PROBLEM FOR NONLINEAR HYPERBOLIC SYSTEMS  
OF PARTIAL DIFFERENTIAL EQUATIONS

by

Victoria Yasinovskaya

ABSTRACT

Consider the Cauchy problem for the system of nonlinear partial differential equations

$$\vec{e}(\vec{x}, D^{m\vec{u}}(\vec{x})) = 0$$

(\*)

$$D_{x_0}^j \vec{u}(x_0=0) = \vec{w}_j, \quad j = 0, \dots, m-1$$

where  $\vec{x} \in \mathbb{R} \times \mathbb{R}^N$ ,  $D^{m\vec{u}} = \left\{ D^{\alpha\vec{u}} \mid |\alpha| = |\alpha_0 + \dots + \alpha_N| \leq m \right\}$ ,

$$\vec{e} = (e_1, \dots, e_n), \quad \vec{u} = (u_1, \dots, u_n),$$

and suppose:

$$a) \quad e_k(\vec{x}, \vec{y}) \in C^\infty(\mathbb{R} \times \mathbb{R}^N \times \underbrace{\mathbb{R}^r \times \dots \times \mathbb{R}^r}_{n \text{ times}})$$

n times

where  $\vec{y} = D^{m\vec{u}}$ ,

$$r = \begin{pmatrix} m+N+1 \\ N+1 \end{pmatrix},$$

b) the matrix

$$E = \left( \frac{\partial e_i}{\partial y_{j\alpha}} \right), \quad 1 \leq i, j \leq n$$

is invertible for  $\alpha_0 = m$ , where  $y_{j\alpha} = D^\alpha u_j$ ,

c) the hyperbolicity condition is satisfied, i.e.

$$\det \left[ \sum_{k=0}^m (i\tau)^{m-k} \tilde{A}_k \right]$$

has  $m$  purely imaginary, distinct roots  $i\tau_j(\vec{x}, \vec{\xi})$ ,  $j = 1, \dots, m$ , where

$$\tilde{A}_k = \sum_{\substack{|\alpha|=m \\ \alpha_0=m-k}} \begin{pmatrix} e_{1y_{1\alpha}} & \cdots & e_{1y_{n\alpha}} \\ \vdots & & \vdots \\ e_{ny_{1\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix} \xi^{(\alpha_1, \dots, \alpha_N)}$$

d)  $\vec{w}_j \in H_{loc}^{M-j+1}(\mathbb{R}^N)$  for  $j = 0, \dots, m-1$  and  $M > \frac{N}{2} + 1$ .

Theorem. Provided hypotheses a), b), c), d) are satisfied, the Cauchy problem (\*) for nonlinear partial differential equations has a unique solution  $\vec{u}$ , such that

$$\frac{\partial^j}{\partial x_0^j} \vec{u} \in C([-T, T], H_{loc}^{M-j}(\mathbb{R}^N)) \cap L^\infty([-T, T], H_{loc}^{M-j+1}(\mathbb{R}^N)),$$

$j = 0, \dots, m$ , if  $T$  is small enough.



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I would be happy to have some credit left for myself.

TABLE OF CONTENTS

	page
Abstract . . . . .	i
Acknowledgements . . . . .	iii
INTRODUCTION . . . . .	1
I. HYPOTHESES AND NOTATION . . . . .	4
II. CONSTRUCTION OF A QUASILINEAR SYSTEM . . . . .	7
III. REDUCTION TO A FIRST ORDER SYSTEM . . . . .	12
IV. SOLUTION OF THE FIRST ORDER HYPERBOLIC SYSTEM . . . . .	15
V. CONCLUSION . . . . .	37
References . . . . .	38

CAUCHY PROBLEM FOR NONLINEAR HYPERBOLIC SYSTEMS  
OF PARTIAL DIFFERENTIAL EQUATIONS

0. Introduction

This paper gives an analysis of the local solvability of the Cauchy problem for hyperbolic systems of nonlinear partial differential equations of the general type:

$$\begin{aligned}
 & \vec{e}(\vec{x}, D^{\vec{m}} \vec{u}(\vec{x})) = 0, \\
 (*) \quad & D_{x_0}^j \vec{u}(x_0=0) = \vec{w}_j, \quad j = 0, \dots, m-1,
 \end{aligned}$$

where  $x \in \mathbb{R} \times \mathbb{R}^N$ ,  $D^{\vec{m}} \vec{u} = \{D^{\alpha} \vec{u} \mid |\alpha| \leq m\}$ ,

$$\vec{w}_j \in H_{loc}^{M-j+1}(\mathbb{R}^N), \quad \vec{e} = (e_1, \dots, e_n), \quad \vec{u} = (u_1, \dots, u_n).$$

In the scalar case it has been proven by P. A. Dionne [4] that for any integer  $M > \frac{N}{2} + 1 + m$  there exists a unique solution of (\*) in  $L^\infty([-T, T], H_{loc}^{M + \frac{N}{2} + 1}(\mathbb{R}^N))$  for  $T$  small enough. In Hughes, Kato and Marsden [6] it is shown that local solutions for symmetric hyperbolic first order differential equations exist for  $M > \frac{N}{2} + 1$ .

Bona and Scott have demonstrated in [1] that for  $s \geq 2$  the initial value problem for the Korteweg-de Vries equation

$$\begin{aligned}
 & u_t + uu_x + u_{xxx} = 0, \quad t > 0 \\
 & u(x, 0) = g(x) \in H^s(\mathbb{R})
 \end{aligned}$$

has a unique solution  $u \in C_0([0, \infty), H^S(\mathbb{R}))$ .

All these results imply that for  $M > \frac{N}{2} + 1$  the general Cauchy problem (\*) may have a unique solution. Indeed, our major result is:

Theorem 4. For  $M > \frac{N}{2} + 1$  the Cauchy problem (\*) for hyperbolic nonlinear system has a unique local solution  $\vec{u}$ , such that

$$\frac{\partial^j}{\partial t^j} \vec{u} \in C([-T, T], H_{loc}^{M-j}(\mathbb{R}^N)) \cap L^\infty([-T, T], H_{loc}^{M-j+1}(\mathbb{R}^N))$$

$$j = 0, \dots, m,$$

if  $T$  is small enough, provided

$$a) \quad \vec{e}(\vec{x}, \vec{y}) \in C^\infty(\mathbb{R} \times \mathbb{R}^N, \underbrace{\mathbb{R}^r \times \dots \times \mathbb{R}^r}_{n \text{ times}})$$

where

$$\vec{y} = D^{m+1} \vec{u}, \quad r = \begin{pmatrix} m+N+1 \\ N+1 \end{pmatrix}$$

and

$$b) \quad \text{the matrix } E = \left( \frac{\partial e_i}{\partial y_{j\alpha}} \right), \quad 1 \leq i, j \leq n,$$

is invertible for  $\alpha_0 = m$ , where  $y_{j\alpha} = D^\alpha u_j$ .

This result is sharp, since it was proved by DiPerna [5] in a much more general context, a continuous (with respect to  $t$ ) solution

to the Cauchy problem

$$\begin{aligned}u_t + uu_x &= 0 \\ u(0) &= g\end{aligned}$$

does not exist if  $g \in H^\sigma(\mathbb{R})$ ,  $\frac{1}{2} < \sigma \leq \frac{3}{2}$ .

Establishing Theorem 4 required the following steps. In Section II we construct a quasilinear system equivalent to (\*), applying the method of [4], generalized for systems. Then in Section III we further reduce it to a first order system which is solved in Section IV, where we apply an iterative method and derive energy estimates in order to prove the existence and uniqueness of a solution.

The essential tools that made evaluation of  $M$  possible were the techniques of pseudodifferential operators with  $H^M$  coefficients which were suggested by M. E. Taylor in [7] and paradifferential operators introduced by J. M. Bony in [2].

We consider the Cauchy problem for an  $n \times n$  system:

$$(1) \quad \begin{aligned} (*) \quad \vec{e}(x, D^m \vec{u}(x)) &= 0, & x &= (x_0, \dots, x_N) \in \mathbb{R}^{N+1}, \\ & & \vec{e}, \vec{u} &\text{ take values in } \mathbb{R}^n \\ \\ D_0^i u_j(x_0=0) &= w_{ji}, & i &= 0, \dots, m-1, j = 1, \dots, n \end{aligned}$$

with the following.

### I. Hypotheses and Notation

We denote:

$$D_i = \frac{\partial}{\partial x_i}, \quad D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \dots D_N^{\alpha_N},$$

$$D_i^{\alpha_i} = \left( \frac{\partial}{\partial x_i} \right)^{\alpha_i}, \quad i = 0, 1, \dots, N$$

$$D^m u_j = \left\{ D^\alpha u_j \mid 0 \leq |\alpha| = |\alpha_0 + \alpha_1 + \dots + \alpha_N| \leq m \right\} . \\ j = 1, \dots, n$$

Note that for each  $j$ ,  $D^m u_j$  can be considered as a function taking values in  $\mathbb{R}^r$ , where

$$r = \binom{m+N+1}{N+1} = \frac{(m+N+1)!}{(N+1)!m!}$$

Let  $y_i = D^m u_i$ , for  $i = 1, \dots, n$ , then  $y_i = (y_{i_1}, \dots, y_{i_r})$ ,  
 $y_{i\alpha} = D^\alpha u_i$ . Then we can rewrite the system (\*) as:

$$(2) \quad \vec{e}(x, y(x)) = 0, \quad \text{where } y = (y_1, \dots, y_n) \text{ takes} \\ \text{values in } \underbrace{\mathbb{R}^r \times \dots \times \mathbb{R}^r}_{n \text{ times}}$$

Let  $e_{ky_{\ell\alpha}}$  be the partial derivative of  $e_k$  by  $y_{\ell\alpha}$ ,  $k = 1, \dots, n$ ;  
 $\ell = 1, \dots, n$ ;  $|\alpha| \leq m$ . Define  $e_{kx_i}(x, y)$  by the equation:

$$(3) \quad \frac{\partial}{\partial x_i} e_k(x, y(x)) \\ = e_{kx_i}(x, y(x)) + \sum_{\ell=1}^n \sum_{|\alpha|=0}^m e_{ky_{\ell\alpha}} \frac{\partial y_{\ell\alpha}}{\partial x_i} .$$

Since we are interested in local properties of the solution of (1), let

$$(x, y) \in \mathbb{R} \times \mathbb{T}^N \times \underbrace{\mathbb{R}^r \times \dots \times \mathbb{R}^r}_{n \text{ times}}$$

We suppose:

$$(a) \quad e_k(x, y) \in C^\infty(X, Y), \quad k = 1, \dots, n \\ \text{where } X = \mathbb{R} \times \mathbb{T}^N \quad \text{and } Y = \underbrace{\mathbb{R}^r \times \dots \times \mathbb{R}^r}_{n \text{ times}}$$

(b) The matrix

$$E = \begin{pmatrix} e_{1y_{1\alpha}} & e_{1y_{2\alpha}} & \cdots & e_{1y_{n\alpha}} \\ e_{2y_{1\alpha}} & e_{2y_{2\alpha}} & \cdots & e_{2y_{n\alpha}} \\ \vdots & \vdots & \ddots & \vdots \\ e_{ny_{1\alpha}} & e_{ny_{2\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix}$$

where  $\alpha = (m, 0, \dots, 0)$ , i.e. for  $\alpha_0 = m$ , is invertible.

(c) The hyperbolicity condition is satisfied, i.e.

$$\det \left[ \sum_{k=0}^m (i\tau)^{m-k} \tilde{A}_k \right]$$

has  $m$  purely imaginary, distinct roots  $i\tau_j(\vec{x}, \vec{\xi})$ ,  $j = 1, \dots, m$ , where

$$\tilde{A}_k = \sum_{\substack{|\alpha|=m \\ \alpha_0=m-k}} \begin{pmatrix} e_{1y_{1\alpha}} & e_{1y_{2\alpha}} & \cdots & e_{1y_{n\alpha}} \\ \vdots & \vdots & \ddots & \vdots \\ e_{ny_{1\alpha}} & e_{ny_{2\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix} \xi^{(\alpha_1, \dots, \alpha_N)}.$$



- (d)  $w_{ij} \in H^{M-j+1}(\mathbb{T}^N)$ , for  $i = 1, \dots, n$ ,  $j = 0, \dots, m-1$ , with  $M \in \mathbb{R}$  to be determined.

## II. Construction of a Quasilinear System.

We want to construct a Cauchy problem for a quasilinear hyperbolic system equivalent to (1). We use the method of [4], generalized for systems. Let

$$(4) \quad A(x,y,D) = \sum_{|\alpha|=0}^m A_\alpha(x,y,D) = \sum_{|\alpha|=0}^m \begin{pmatrix} e_{1y_{1\alpha}} & e_{1y_{2\alpha}} & \cdots & e_{1y_{n\alpha}} \\ e_{2y_{1\alpha}} & e_{2y_{2\alpha}} & \cdots & e_{2y_{n\alpha}} \\ \vdots & \vdots & \ddots & \vdots \\ e_{ny_{1\alpha}} & e_{ny_{2\alpha}} & \cdots & e_{ny_{n\alpha}} \end{pmatrix} D^\alpha,$$

$$(5) \quad B(x,y) = (\vec{b}_{-1}(x,y), \dots, \vec{b}_N(x,y))$$

$$= \begin{pmatrix} e_{10}, & -e_{1x_0}, & -e_{1x_1}, & \cdots, & -e_{1x_N} \\ e_{20}, & -e_{2x_0}, & -e_{2x_1}, & \cdots, & -e_{2x_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{n0}, & -e_{nx_0}, & -e_{nx_1}, & \cdots, & -e_{nx_N} \end{pmatrix}$$

where

$$(6) \quad e_{k0}(x,y) = \sum_{|\alpha|=0}^m \sum_{i=1}^n e_{ky_{i\alpha}} y_{i\alpha} - e_k, \quad k = 1, \dots, n.$$

Let  $\alpha = \beta(\alpha) + \gamma(\alpha)$

where  $\beta(\alpha) = (\beta_0, \beta_1, \dots, \beta_N)$

$\gamma(\alpha) = (\gamma_0, \gamma_1, \dots, \gamma_N)$

such that:  $|\beta(\alpha)| \leq m-1$

$|\gamma(\alpha)| = 0$  if  $|\alpha| \leq m-1$

$|\gamma(\alpha)| = 1$  if  $|\alpha| = m$ ,

and for each  $\gamma(\alpha)$  define  $k(\alpha)$  as follows:

if  $|\alpha| = m$ , then  $\gamma(\alpha) = (0, \dots, 0, \gamma_{k(\alpha)}, 0, \dots, 0)$ ,

if  $|\alpha| \leq m-1$ , then  $k(\alpha) = -1$ .

Let

$$(7) \quad U(x) = (U_{ik}(x)) = (\vec{U}_{-1}(x), \dots, \vec{U}_N(x))$$

$$= \begin{pmatrix} u_1 & u_{1x_0} & u_{1x_1} & \dots & u_{1x_N} \\ u_2 & u_{2x_0} & u_{2x_1} & \dots & u_{2x_N} \\ & & \vdots & & \\ u_n & u_{nx_0} & u_{nx_1} & & u_{nx_N} \end{pmatrix},$$

$i = 1, \dots, n; \quad k = -1, 0, 1, \dots, N;$

and

$$(8) \quad W_j(x_1, \dots, x_N) = (\vec{W}_{-1j}, \dots, \vec{W}_{Nj})$$

$$= \begin{pmatrix} w_{1j} & w_{1j+1} & D_{x_1} w_{1j} & \dots & D_{x_N} w_{1j} \\ w_{2j} & w_{2j+1} & D_{x_1} w_{2j} & \dots & D_{x_N} w_{2j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{nj} & w_{nj+1} & D_{x_1} w_{nj} & \dots & D_{x_N} w_{nj} \end{pmatrix},$$

$$j = 0, \dots, m-1,$$

be vector valued functions in  $\mathbb{R} \times \mathbb{T}^N$  and  $\mathbb{T}^N$  respectively. (Here functions  $w_{1m} = D_0^m u_1(x_0=0)$ ,  $\dots$ ,  $w_{nm} = D_0^m u_n(x_0=0)$ , which appear in (8) when  $j = m-1$ , are determined by (1).)

Now consider the Cauchy problem for the following quasilinear system of order  $m$ :

$$(9) \quad \begin{cases} A(x, D^{\beta(\alpha)} \vec{U}_{k(\alpha)}, D)U = B(x, D^{\beta(\alpha)} \vec{U}_{k(\alpha)}) \\ D_0^j U(x_0=0) = W_j, \text{ where } j = 0, \dots, m-1; W_j \in H^{M-j}(\mathbb{T}^N) \end{cases}.$$

Here the components,  $U_{ik(\alpha)}$ , of  $\vec{U}_{k(\alpha)}$  are entries of the column  $k(\alpha)$  in the matrix (7),  $k(\alpha) = -1, 0, 1, \dots, N$ , and  $D^{\beta(\alpha)} U_{ik(\alpha)}$  are substituted for  $y_{i\alpha}$  in (4) and (5),  $i = 1, \dots, n$ .

Lemma 1. Cauchy problems (1) and (9) are equivalent.

Proof. The  $n(N+2)$  equations in (9) can be separated into two different types.

First type:

$$(10) \quad \sum_{|\alpha|=0}^m \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^{\beta(\alpha)} U_{k(\alpha)}) D^\alpha u_i = e_{\ell 0} (x, D^{\beta(\alpha)} U_{k(\alpha)})$$

for  $\ell = 1, \dots, n$

which is:

$$\begin{aligned} & \sum_{|\alpha|=0}^{m-1} \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^\alpha \vec{u}) y_{i\alpha} \\ & + \sum_{|\alpha|=m} \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^{\beta(\alpha)} U_{k(\alpha)}) D^\alpha u_i \\ & = \sum_{|\alpha|=0}^{m-1} \sum_{i=1}^n e_{\ell y_{i\alpha}} y_{i\alpha} \\ & + \sum_{|\alpha|=m} \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^{\beta(\alpha)} U_{k(\alpha)}) D^{\beta(\alpha)} U_{k(\alpha)} - e_\ell . \end{aligned}$$

Here the first term on the left is equal to the first term on the right side of the equation. By definition of  $\beta(\alpha)$  and  $k(\alpha)$  the second term on the left and right sides are also equal.

Equations of the second type are:

$$(11) \quad \sum_{|\alpha|=0}^m \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^{\beta(\alpha)} U_{k(\alpha)}) D^\alpha u_{ix_j} = -e_{\ell x_j} (x, D^\alpha \vec{u})$$

where  $\ell = 1, \dots, n$ ;  $j = 0, \dots, N$ .

We rewrite the last equation as:

$$\begin{aligned} & \sum_{|\alpha|=0}^{m-1} \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^\alpha \vec{u}) D^\alpha u_{ix_j} \\ & + \sum_{|\alpha|=m} \sum_{i=1}^n e_{\ell y_{i\alpha}} (x, D^{\beta(\alpha)} \vec{U}_{k(\alpha)}) D^\alpha u_{ix_j} \\ & = -e_{\ell x_j} (x, D^\alpha \vec{u}) . \end{aligned}$$

Since  $D^{\beta(\alpha)} \vec{U}_{k(\alpha)} = D^\alpha \vec{u}$  for  $|\alpha| = m$ , where  $\vec{u}$  is a column  $(u_1, \dots, u_n)^t$ , we use (3) to see that equation (11) means

$$\frac{\partial}{\partial x_i} e_{\ell} (x, D^\alpha \vec{u}) = 0.$$

We conclude that (10) and (11) are equivalent to (1). The second condition in (9) is equivalent to hypothesis I(d).

Q.E.D.

### III. Reduction to a First Order System.

In order to solve (9) we reduce it to a first order system as follows. Let us change notations:  $(x_0, x_1, \dots, x_N) = (t, x)$ . Multiply both sides of (9) by  $E^{-1}$  on the left (see hypothesis I(b)) and rewrite it in the form:

$$(12) \quad \left[ \frac{\partial^m}{\partial t^m} I_n - \sum_{i=0}^{m-1} A_{m-i}(t, x, D^{\beta(\alpha)} U_{k(\alpha)}, D_x) \frac{\partial^i}{\partial t^i} \right] U \\ = f(t, x, D^{\beta(\alpha)} U_{k(\alpha)}),$$

where  $A_{m-i}(t, x, y, D_x)$  is a differential operator of order  $m-i$  with top order symbol  $\tilde{A}_{m-i}$  and

$$f(t, x, D^{\beta(\alpha)} U_{k(\alpha)}) = E^{-1} B(t, x, D^{\beta(\alpha)} U_{k(\alpha)}).$$

Since the next step in the transformation of equation (12) will involve a pseudodifferential operator of a certain class (see [7]), we will give the following

Definition A. Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ ,  $m \in \mathbb{R}$ . We define the symbol class  $S_{1,0}^m(\Omega)$  to consist of the set of  $p(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^N)$  with the property that, for any compact  $K \subset \Omega$ , any multi-indices  $\alpha, \beta$ , there exists a constant  $C_{K, \alpha, \beta}$  such that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{K, \alpha, \beta} (1 + |\xi|)^{m - |\alpha|}$$

for all  $x \in K$ ,  $\xi \in \mathbb{R}^N$ . In this case the operator  $p(x,D)$  is said to belong to  $\text{OPS}_{1,0}^m(\Omega)$ .

If, moreover, there are smooth functions  $p_{m-j}(x,\xi)$ , homogeneous of degree  $m-j$  in  $\xi$  for  $|\xi| \geq 1$  such that

$$p(x,\xi) \sim \sum_{j \geq 0} p_{m-j}(x,\xi)$$

where the asymptotic condition means that

$$p(x,\xi) - \sum_{j=0}^M p_{m-j}(x,\xi) \in S_{1,0}^{m-M-1}(\Omega),$$

then we say that symbol  $p(x,\xi) \in S^m(\Omega)$  and the operator  $p(x,D) \in \text{OPS}^m(\Omega)$ .

We can now define the concept of a pseudodifferential operator on a manifold  $M$ .

Definition B.  $p(x,D): C_0^\infty(M) \rightarrow C^\infty(M)$  belongs to  $\text{OPS}_{1,0}^m(M)$  ( $\text{OPS}^m(M)$ ) if  $p(x,\xi)$  is smooth on  $M$  and if for any coordinate neighborhood  $U$  in  $M$  with  $\chi: U \rightarrow \mathcal{O}$  a diffeomorphism onto an open subset  $\mathcal{O}$  of  $\mathbb{R}^N$ , the map of  $C_0^\infty(\mathcal{O})$  into  $C^\infty(\mathcal{O})$  given by  $u \mapsto p(x,D)(u \circ \chi) \circ \chi^{-1}$  belongs to  $\text{OPS}_{1,0}^m(\mathcal{O})$  ( $\text{OPS}^m(\mathcal{O})$ ).

Now let

$$U^j = \left( \frac{\partial}{\partial t} \right)^{j-1} \Lambda^{m-j} U, \quad j = 1, \dots, m,$$

and

$$\Lambda = (1-\Delta)^{\frac{1}{2}} \in \text{OPS}^1.$$

Then (12) is equivalent to

$$(13) \quad \frac{\partial}{\partial t} \begin{pmatrix} U^1 \\ \cdot \\ \cdot \\ \cdot \\ U^m \end{pmatrix} = \begin{pmatrix} O_n & \Lambda_n & O_n & \cdots & O_n \\ & O_n & \Lambda_n & \cdot & \vdots \\ & & & \cdot & \Lambda_n \\ (b_1) & (b_2) & (b_3) & \cdots & (b_m) \end{pmatrix} \begin{pmatrix} U^1 \\ \cdot \\ \cdot \\ \cdot \\ U^m \end{pmatrix} + \begin{pmatrix} O_n \\ \vdots \\ O_n \\ f \end{pmatrix}$$

where  $b_j = A_{m-j+1}(t, x, P_{j1}(U^1 \cdots U^m)^t, \dots, P_{jv}(U^1, \dots, U^m)^t, D_x) \Lambda^{j-m}$

with  $P_{j\mu} \in OPS^s$ ,  $s \leq 0$ .

Each  $U^j$  in (13) is an  $n$ -vector, hence

$$\begin{pmatrix} U^1 \\ \cdot \\ \cdot \\ \cdot \\ U^m \end{pmatrix} \text{ is an } mn\text{-vector.}$$

Each entry in the matrix of equation (13) is an  $n \times n$  matrix, for example,

$$\Lambda_n = \Lambda I_n, \quad O_n = 0 I_n$$

so, (13) is an  $(m \cdot n) \times (m \cdot n)$  system. Changing notation, calling the column vector  $(U^1, \dots, U^m)^t$   $U$  and the matrix in (13)  $K$ , write (13) with the initial data as

$$(14) \quad \begin{aligned} \frac{\partial}{\partial t} U &= K(t, x, P_1 U, \dots, P_\ell U, D_x) U + f(t, x, P_1 U, \dots, P_\ell U) \\ U(0) &= g \end{aligned}$$



where

$$g = \begin{pmatrix} g_1 \\ \cdot \\ \cdot \\ \cdot \\ g_m \end{pmatrix} = \begin{pmatrix} \Lambda^{m-1} W_0 \\ \cdot \\ \cdot \\ \cdot \\ W_{m-1} \end{pmatrix}, \text{ i.e. } g_j = \Lambda^{m-j} W_{j-1},$$

and  $W_j$ 's are given by (8), and where  $P_j \in OPS^0$  and  $K(t,x,U_1, \dots, U_\ell, \xi) \in S^1$ .

#### IV. Solution of the First Order Hyperbolic System.

By hypothesis I(c) system (14) is hyperbolic, and in order to solve it we will use an iterative method as follows. Given  $U$  on  $\mathbb{R} \times \mathbb{T}^N$  with  $U(0) = g$ , we define  $FU = V$  to be the solution to the system

$$(15) \quad \frac{\partial}{\partial t} V = K(t,x,P_1 U, \dots, P_\ell U, D_x) V + f(t,x,P_1 U, \dots, P_\ell U)$$

$$V(0) = g$$

and find a fixed point of  $F$ , i.e. a function  $U$  such that  $FU = U$  on  $(-T,T) \times \mathbb{T}^N$ . In order to treat this problem, it will be necessary to introduce pseudodifferential operators with less than  $C^\infty$  symbols (see [7]) and paradifferential operators (see [2]).

Definition 1. We say  $p(x, \xi) \in H^M S_{1,0}^m(\mathbb{T}^N)$  and  $p(x, D) \in OPH^M S_{1,0}^m(\mathbb{T}^N)$  provided

$$\|D_\xi^\alpha p(x, \xi)\|_{H^M(\mathbb{T}^N)} \leq C(1+|\xi|)^{m-|\alpha|} \quad \text{for } |\alpha| \leq M.$$

If, moreover, there are functions  $p_{m-j}(x, \xi) \in H^M S_{1,0}^{m-j}(\mathbb{T}^N)$ , homogeneous of degree  $m-j$  in  $\xi$  for  $|\xi| \geq 1$  such that

$$p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$$

then we say that  $p(x, \xi) \in H^M S^m(\mathbb{T}^N)$  and  $p(x, D) \in OPH^M S^m(\mathbb{T}^N)$ .

Definition 2. Let  $\chi(\theta, \eta)$  be a  $C^\infty$  function defined on  $\mathbb{R}^N \times \mathbb{R}^N \setminus 0$ , homogeneous of degree 0, such that for some small  $\epsilon_1, \epsilon_2$ ,  $0 < \epsilon_1 < \epsilon_2$   $\chi(\theta, \eta) = 1$  for  $|\theta| \leq \epsilon_1 |\eta|$ ,  $\chi(\theta, \eta) = 0$  for  $|\theta| \geq \epsilon_2 |\eta|$  and let  $s(\eta)$  be a  $C^\infty$  function from  $\mathbb{R}^N$  to  $\mathbb{R}$  which is 0 in some neighborhood of 0 and 1 outside of some compact subset of  $\mathbb{R}^N$ . Let  $\ell(x, \xi)$  be a function homogeneous of degree  $m$  in  $\xi$ ,  $C^\infty$  in  $\xi$  for  $\xi \neq 0$ , with compact support with respect to  $x$ , and  $C^\rho$  in  $x$  for  $\rho$  non-integer. Define the operator  $T_\ell$  on the space of distributions,  $\mathcal{D}'$ , as

$$(T_\ell u)^\wedge(\xi) = \int \chi(\xi - \eta, \eta) \hat{\ell}(\xi - \eta, \eta) s(\eta) \hat{u}(\eta) d\eta$$

where  $\hat{\ell}(\theta, \xi)$  is the Fourier transform of  $\ell(x, \xi)$  with respect to the first variable.

Definition 3. a) For any  $\Omega \subset \mathbb{R}^N$ ,  $m \in \mathbb{R}$ ,  $\rho > 0$  and non-integer, we define  $\Sigma_\rho^m(\Omega)$  as the set of all functions  $l(x, \xi)$  in  $\Omega \times (\mathbb{R}^N \setminus \{0\})$  such that

$$l(x, \xi) = l_m(x, \xi) + l_{m-1}(x, \xi) + \dots + l_{m-[\rho]}(x, \xi)$$

where  $l_{m-k}(x, \xi)$  is homogeneous of degree  $m-k$  in  $\xi$ , is  $C^\infty$  in  $\xi$ , and  $C_{loc}^{\rho-k}$  in  $x$ .

b) If  $l^{(i)} \in \Sigma_\rho^{m_i}$ ,  $i = 1, 2$ , we define  $l^1 \# l^2 \in \Sigma_\rho^{m_1+m_2}$  as

$$l^1 \# l^2 = \sum_{|\alpha|+k_1+k_2 \leq [\rho]} \sum_{|\alpha|+k_1 \leq [\rho]} \sum_{|\alpha|+k_2 \leq [\rho]} \frac{1}{\alpha!} D_\xi^\alpha l_{m_1-k_1}^1 D_x^\alpha l_{m_2-k_2}^2 .$$

c) If  $l \in \Sigma_\rho^m$ , we define  $l^* \in \Sigma_\rho^m$  as

$$l^* = \sum_{|\alpha|+k \leq [\rho]} \sum_{|\alpha|+k \leq [\rho]} \frac{1}{\alpha!} D_\xi^\alpha D_x^\alpha \bar{l}_{m-k} .$$

Definition 4. Let  $\Omega$  be an open subset of  $\mathbb{R}^N$  and let  $L$  be a linear transformation in  $\mathcal{D}'(\Omega)$ , which is properly supported, i.e. for any compact  $K$  in  $\Omega$  there is compact  $\hat{K}$  in  $\Omega$  such that

$$\text{supp } u \subset K \implies \text{supp } L u \subset \hat{K}$$

and

$$(\text{supp } u) \cap \hat{K} = \emptyset \implies \text{supp } L u \cap K = \emptyset .$$

Then we call  $L$  a paradifferential operator of order  $m$  and of class

$C^\rho$  in  $\Omega$  and write  $L \in OP(\Sigma_\rho^m)(\Omega)$  if there exists  $\ell \in \Sigma_\rho^m(\Omega)$  such that for any compact  $K \subset \Omega$  and any  $\chi \in C_0^\infty(\Omega)$  which is equal to 1 in a neighborhood of  $K$ , the operator  $L - \chi T_{\chi\ell}$  is a continuous map of elements of  $H^S$  with support in  $K$  to  $H^{S-m+\rho}$ . In this case  $\ell$  is called a symbol of  $L$ .

The proofs of the following results are given in [2].

Theorem 1.

a) If  $L \in OP(\Sigma_\rho^m)(\Omega)$ , then  $L: H_{loc}^S(\Omega) \rightarrow H_{loc}^{S-m}(\Omega)$ .

b) If  $L \in OP(\Sigma_\rho^m)(\Omega)$ , there exists a unique symbol  $\ell$  of  $L$ , i.e.  $\sigma(L) = \ell$ .

c) If  $L^j \in OP(\Sigma_\rho^{m_j})(\Omega)$ ,  $j = 1, 2$ , then

$$L^1 L^2 \in OP(\Sigma_\rho^{m_1+m_2})(\Omega)$$

and  $\sigma(L^1 \cdot L^2) = \sigma(L^1) \# \sigma(L^2)$ .

d) If  $L \in OP(\Sigma_\rho^m)(\Omega)$ , then

$$\sigma(L^*) = (\sigma(L))^*$$

where  $L^*$  is adjoint of  $L$ .

e) If  $L$  is a classical pseudodifferential operator of order  $m$ , properly supported in  $\Omega$ , with symbol

$$\ell(x, \xi) \sim \sum_{j \geq 0} \ell_{m-j}(x, \xi),$$

then for any  $\rho > 0$ ,  $L \in \text{OP}(\Sigma_\rho^m)$  with symbol  $\sigma(L) = \sum_{0 \leq j \leq [\rho]} \ell_{m-j}$ .

NOTE. If  $\ell(x, \xi)$  is such that  $\ell_{m-j}(x, \xi)$  is in  $C^{\gamma-j}$  as a function of  $x$ , then  $L \in \text{OP}(\Sigma_\gamma^m)$  and  $\sigma(L) = \sum_{0 \leq j \leq [\gamma]} \ell_{m-j}$  where  $\gamma$  is not an integer.

To return to our iterative method (15), we assume that  $U \in C([-T, T], H^M(\mathbb{T}^N))$ ,  $\frac{\partial}{\partial t} U \in C([-T, T], H^M(\mathbb{T}^N))$  for  $M$  to be estimated. Then  $K(t, x, P_1 U, \dots, P_\ell U, D_x) \in \text{OPH}_{1,0}^{M-1}$  and in order to solve the quasilinear hyperbolic system

$$(16) \quad \begin{aligned} \frac{\partial}{\partial t} V &= KV + f \\ V(0) &= g \end{aligned}$$

$$\begin{aligned} \text{with } f &\in C([-T, T], H^M(\mathbb{T}^N)), \\ g &\in H^M(\mathbb{T}^N) \end{aligned}$$

we need to construct what is called a symmetrizer for  $K$ .

Definition 5. Let  $K \in \text{OPS}_{1,0}^1$ . Then a symmetrizer for  $(\frac{\partial}{\partial t}) - K$  is a smooth one parameter family of operators  $R = R(t) \in \text{OPS}^0$ , such that

$$(17) \quad R_0(t, x, \xi) \text{ is a positive definite matrix for } |\xi| \geq 1,$$

$$(18) \quad RK + (RK)^* \in \text{OPS}_{1,0}^0.$$

If such a symmetrizer exists, one says  $\frac{\partial}{\partial t} - K$  is symmetrizable.

The proof of the following proposition can be found in [2].

Proposition 1. Any strictly hyperbolic first order system  $\frac{\partial}{\partial t} - K$ , where  $K \in OPS_{1,0}^1$ , has a symmetrizer  $R$  and we have:

$$(19) \quad R(t, x, D_x) = \sum_{j=1}^k P_j(t, x, D) {}^*P_j(t, x, D)$$

$$(20) \quad \sigma_{RK}(t, x, \xi) = i \sum_{j=1}^k \lambda_j(t, x, \xi) P_j(t, x, \xi) {}^*P_j(t, x, \xi) \pmod{S^0}$$

where  $i\lambda_j(t, x, \xi)$  are eigenvalues of  $K_1(t, x, \xi)$ , the principal symbol of  $K$ ,  $\lambda_1(t, x, \xi) < \lambda_2(t, x, \xi) < \dots < \lambda_k(t, x, \xi)$ , and  $P_j(t, x, \xi) \in S^0$  are the projections onto the associated eigenspaces of  $i\lambda_j(t, x, \xi)$ :

$$P_j = \frac{1}{2\pi i} \int_{\gamma_j} (\zeta - K_1(t, x, \xi))^{-1} d\zeta$$

where  $\gamma_j$  is the circle around  $\lambda_j$  only.

Now we are ready to prove the following result.

Proposition 2. If  $M > \frac{N}{2} + 1$ , then, given  $K \in OPH^M S^1(\mathbb{T}^N)$  the hyperbolic system

$$(21) \quad \frac{\partial}{\partial t} V = KV + f$$

$$V(0) = g$$

has a unique solution  $V \in C([-T, T], H^M(\mathbb{T}^N))$  for  $g \in H^M(\mathbb{T}^N)$  and  $f \in C([-T, T], H^M(\mathbb{T}^N))$ . Such a solution satisfies the estimate

$$\|V(t)\|_{H^M}^2 \leq (1+C|t|) [\|g\|_{H^M}^2 + \int_0^t \|f(\tau)\|_{H^M}^2 d\tau], \quad |t| \leq T$$

where  $C$  depends on finitely many seminorms of  $K \in \text{OPH}^M S^1(\mathbb{T}^N)$ ,  $R \in \text{OPH}^M S^0(\mathbb{T}^N)$ ,  $\frac{\partial}{\partial t} R \in \text{OPH}^M S^0(\mathbb{T}^N)$ , and  $RK + (RK)^* \in \text{OPH}^L S^0(\mathbb{T}^N)$  for some  $L < M$ , and on  $N$  but not on the order of the system.

The following inequality from the theory of ordinary differential equations will be useful in the proof of Proposition 2.

Lemma 2. (Gronwall's inequality). If  $y \in C^1$  and  $y'(t) + f(t)y \leq g(t)$ , then

$$y(t) \leq e^{-\int_0^t f(\tau) d\tau} \left[ y_0 + \int_0^t g(\tau) e^{\int_0^\tau f(\sigma) d\sigma} d\tau \right].$$

Proof of Proposition 2. Since  $K \in \text{OPH}^M S^1(\mathbb{T}^N)$ , it is clear from (19) that the symmetrizer  $R \in \text{OPH}^M S^0(\mathbb{T}^N)$ . Then we have the following asymptotic expansions of the symbols  $K(t, x, \xi) \in H^M S^1$  and  $R(t, x, \xi) \in H^M S^0$ :

$$K(t, x, \xi) \sim \sum_{j \geq 0} K_{1-j}(t, x, \xi)$$

$$R(t, x, \xi) \sim \sum_{j \geq 0} R_{-j}(t, x, \xi)$$

where  $K_{1-j}(t, x, \xi)$  and  $R_{-j}(t, x, \xi)$  are homogeneous functions in  $\xi$  of degree  $1-j$  and  $-j$  respectively and are in  $H^M(\mathbb{T}^N)$  as functions of  $x$ . Then, by the Sobolev imbedding theorem  $K_{1-j}$  and  $R_{-j}$  belong to  $C^{M-\frac{N}{2}}(\mathbb{T}^N)$  and we can apply Theorem 1 to conclude that:

$$K(t, x, D_x) \in \text{OP}(\Sigma_{M-\frac{N}{2}}^1)$$

with 
$$\sigma(K) = \sum_{0 \leq j \leq [M-\frac{N}{2}]} K_{1-j}(t, x, \xi)$$

and 
$$R(t, x, D_x) \in \text{OP}(\Sigma_{M-\frac{N}{2}}^0)$$

with 
$$\sigma(R) = \sum_{0 \leq j \leq [M-\frac{N}{2}]} R_{-j}(t, x, \xi)$$

for  $M - \frac{N}{2}$  non-integer.

NOTE. In the case when  $M - \frac{N}{2}$  is an integer we can find some small  $\epsilon > 0$ , such that  $M - \frac{N}{2} - \epsilon > 1$  (since by hypothesis  $M - \frac{N}{2} > 1$ ) and replace  $M - \frac{N}{2}$  by non-integer  $M - \frac{N}{2} - \epsilon$  for all symbols above.

It follows that  $RK \in \text{OP}(\Sigma_{M-\frac{N}{2}}^1)$  with



$$\sigma(\text{RK}) = R_0 K_1 + \sum_{1 \leq |\alpha| + j_1 + j_2 \leq [M - \frac{N}{2}]} \frac{1}{\alpha!} D_\xi^\alpha R_{-j_1} D_x^\alpha K_{1-j_2}$$

where for each  $|\alpha|, j_1, j_2$  the corresponding term is homogeneous of degree  $1 - (|\alpha| + j_1 + j_2)$  in  $\xi$  and belongs to

$$C^{[M - \frac{N}{2}] - (|\alpha| + j_1 + j_2)}(\mathbb{T}^N) \text{ in } x. \text{ Hence}$$

$$\begin{aligned} \sigma((\text{RK})^*) &= \sum_{|\beta| + |\alpha| + j_1 + j_2 \leq [M - \frac{N}{2}]} \frac{1}{\alpha! \beta!} D_\xi^\beta D_x^\beta \overline{D_\xi^\alpha R_{-j_1} D_x^\alpha K_{1-j_2}} \\ &= R_0 \bar{K}_1 \\ &+ \sum_{1 \leq |\beta| + |\alpha| + j_1 + j_2 \leq [M - \frac{N}{2}]} \frac{1}{\alpha! \beta!} D_\xi^\beta D_x^\beta D_\xi^\alpha R_{-j_1} D_x^\alpha \bar{K}_{1-j_2} \end{aligned}$$

and since  $K_1(t, x, \xi)$  has purely imaginary distinct eigenvalues for each  $(t, x, \xi)$ , we have:

$$(22) \quad \sigma(\text{RK} + (\text{RK})^*) = \sigma(\text{RK}) + \sigma((\text{RK})^*)$$

$$\begin{aligned} &= \sum_{1 \leq |\alpha| + j_1 + j_2 \leq [M - \frac{N}{2}]} \frac{1}{\alpha!} D_\xi^\alpha R_{-j_1} D_x^\alpha K_{1-j_2} \\ &+ \sum_{1 \leq |\beta| + |\alpha| + j_1 + j_2 \leq [M - \frac{N}{2}]} \frac{1}{\alpha! \beta!} D_\xi^\beta D_x^\beta D_\xi^\alpha R_{-j_1} D_x^\alpha \bar{K}_{1-j_2} . \end{aligned}$$

Thus  $\text{RK} + (\text{RK})^* \in \text{OP}(\Sigma^0_{M - \frac{N}{2} - 1})$ , which justifies the hypothesis that

$M > \frac{N}{2} + 1$ . Applying Theorem 1 again, we conclude that

$$\begin{aligned}
& RK + (RK)^* : H^s(\mathbb{T}^N) \rightarrow H^s(\mathbb{T}^N) \\
\text{as well as} & \quad R : H^s(\mathbb{T}^N) \rightarrow H^s(\mathbb{T}^N), \\
(23) \quad & \quad \frac{\partial}{\partial t} R : H^s(\mathbb{T}^N) \rightarrow H^s(\mathbb{T}^N), \\
\text{and} & \quad K : H^s(\mathbb{T}^N) \rightarrow H^{s-1}(\mathbb{T}^N) \\
\text{for any} & \quad s \in \mathbb{R}^1.
\end{aligned}$$

Now we can obtain the energy estimate for  $V$  solving (21). We write

$$\begin{aligned}
(24) \quad \frac{d}{dt} (RV, V) &= (RV_t, V) + (RV, V_t) + (R_t V, V) \\
&= (RKV + Rf, V) + (RV, KV+f) + (R_t V, V) \\
&= ((RK+K^*R)V, V) + (Rf, V) + (RV, f) + (R_t V, V) \\
&\leq C_1 \|V\|_{L^2}^2 + C_2 \|f\|_{L^2}^2 \leq C(RV, V) + C \|f\|_{L^2}^2
\end{aligned}$$

Applying Gronwall's inequality (Lemma 2) to this yields

$$(25) \quad \|V(t)\|_{L^2}^2 \leq (1+C|t|) [\|g\|_{L^2}^2 + \int_0^t \|f(\tau)\|_{L^2}^2 d\tau]$$

for  $t \in [-T, T]$ .

More generally, differentiating

$$\|V(t)\|_{H^s(\mathbb{T}^N)}^2 = \|\Lambda^s V(t)\|_{L^2(\mathbb{T}^N)}^2$$

yields

$$(26) \quad \|V(t)\|_{H^s}^2 \leq (1+C|t|) [\|g\|_{H^s}^2 + \int_0^t \|f(\tau)\|_{H^s}^2 d\tau]$$

for  $t \in [-T, T]$ , which is bounded in a bounded interval  $-T \leq t \leq T$ .  $C$  is a function of finitely many seminorms of  $K \in \text{OPH}^M S^1$ ,  $R \in \text{OPH}^M S^0$ ,  $\frac{\partial}{\partial t} R \in \text{OPH}^M S^0$ , and  $RK + (RK)^* \in \text{OPH}^L S^0$  for some  $L < M$  which we need not specify, and of  $N$ . Inequality (26) is valid whenever  $u \in H^1([-T, T], H^{S+1}(\mathbb{T}^N))$ ,  $f \in L^2([-T, T], H^S(\mathbb{T}^N))$ ,  $g \in H^{S+1}(\mathbb{T}^N)$  and (21) is satisfied. We shall obtain the solution of the initial value problem (21) as a limit of solutions to the problem

$$(27) \quad \begin{aligned} \frac{\partial}{\partial t} V &= KJ_\epsilon V + f \\ V(0) &= g \end{aligned}$$

where  $J_\epsilon$  is a Friedrich's mollifier on  $\mathbb{T}^N$  defined as the following:

Definition 6. A Friedrich's mollifier on  $\mathbb{T}^N$  is a family,  $J_\epsilon$ , of scalar pseudodifferential operators,  $0 \leq \epsilon \leq 1$ , such that:

$$J_\epsilon \in \text{OPS}^{-\infty}(\mathbb{T}^N) \quad \text{for each } \epsilon \in (0, 1],$$

$$\{J_\epsilon : 0 < \epsilon \leq 1\} \text{ is a bounded subset of } \text{OPS}_{1,0}^0(\mathbb{T}^N),$$

and  $J_\epsilon u \rightarrow u$  in  $L^2(\mathbb{T}^N)$  as  $\epsilon \rightarrow 0$  for each  $u \in L^2(\mathbb{T}^N)$ .

To construct a Friedrich's mollifier, we multiply  $\epsilon^{-|\epsilon\xi|^2}$  by a partition of unity for some coordinate patches on  $\mathbb{T}^N$ .

Now for each  $\epsilon > 0$ ,  $K_\epsilon = KJ_\epsilon$  is a continuous linear operator on  $H^S$ . Hence (27) can be considered as a Banach-space valued ordinary

differential equation and to solve it we apply the Picard iteration method (see Dieudonne [3]). Thus, given  $g \in H^{S+1}(\mathbb{T}^N)$  and  $f \in C([-T, T], H^{S+1}(\mathbb{T}^N))$ , we can solve (27), producing a solution  $V_\epsilon \in C^1([-T, T], H^{S+1}(\mathbb{T}^N))$ . Now since  $\{K_\epsilon: 0 < \epsilon \leq 1\}$  is a bounded subset of  $OPH^M S_{1,0}^1(\mathbb{T}^N)$  and  $\{RK_\epsilon + (RK_\epsilon)^*: 0 < \epsilon \leq 1\}$  is a bounded subset of  $OPH^L S_{1,0}^0(\mathbb{T}^N)$  for some  $L < M$ , we get the estimate (26) for  $V_\epsilon$ :

$$(28) \quad \|V_\epsilon\|_{H^S}^2 \leq (1 + C|t|) [\|g\|_{H^S}^2 + \int_0^t \|f(\tau)\|_{H^S}^2 d\tau]$$

with  $C$  independent of  $\epsilon$ ,  $0 < \epsilon \leq 1$ . Now by this estimate,  $\{V_\epsilon: 0 < \epsilon \leq 1\}$  is a bounded subset of  $C([-T, T], H^S(\mathbb{T}^N))$  given

$$g \in H^{S+1}(\mathbb{T}^N) \quad \text{and} \quad f \in C([-T, T], H^{S+1}(\mathbb{T}^N)).$$

Since  $V_\epsilon' = K_\epsilon V_\epsilon + f$  and  $\{K_\epsilon: 0 < \epsilon \leq 1\}$  is a bounded subset of  $OPH^M S_{1,0}^1(\mathbb{T}^N)$ , it follows that  $\{V_\epsilon': 0 < \epsilon \leq 1\}$  is a bounded subset of  $C([-T, T], H^{S-1}(\mathbb{T}^N))$ . Hence  $\{V_\epsilon: 0 < \epsilon \leq 1\}$  is a bounded subset of  $C^1([-T, T], H^{S-1}(\mathbb{T}^N))$ . Furthermore, for each  $t_0 \in [-T, T]$ ,  $\{V_\epsilon(t_0): 0 < \epsilon \leq 1\}$ , being a bounded subset of  $H^S(\mathbb{T}^N)$ , is a relatively compact subset of  $H^{S-1}(\mathbb{T}^N)$ . Hence, by Ascoli's theorem ([3]), there is a sequence  $\epsilon_n \rightarrow 0$  such that  $V_{\epsilon_n}$  converges, in  $C([-T, T], H^{S-1}(\mathbb{T}^N))$ , to a limit we call  $V$ , which satisfies (21) in the sense of distributions.

Now let  $g_j \in H^{S+4}(\mathbb{T}^N)$  with  $g_j \rightarrow g$  in  $H^S(\mathbb{T}^N)$  and let  $f_j \in C([-T, T], H^{S+4}(\mathbb{T}^N))$ , with  $f_j \rightarrow f$  in  $C([-T, T], H^S(\mathbb{T}^N))$ . The

argument above also produces solutions  $V_j$  to (21) with  $g, f$  replaced by  $g_j, f_j$  with  $V_j \in C([-T, T], H^{s+2}(\mathbb{T}^N))$ . Since  $V_j' = KV_j + f_j$ , it follows that  $V_j \in C^1([-T, T], H^{s+1}(\mathbb{T}^N))$ . Hence we can apply the energy inequality (26) and conclude that  $\{V_j\}$  is a Cauchy sequence in  $C([-T, T], H^s(\mathbb{T}^N))$ . The limit  $V$  solves our system. As for uniqueness, since any  $V \in C([-T, T], H^s(\mathbb{T}^N))$ , solving (21) must belong to  $C^1([-T, T], H^{s-1}(\mathbb{T}^N)) \subset H^1([-T, T], H^{s-1}(\mathbb{T}^N))$ , the energy inequality (26) with  $s$  replaced by  $s-2$  applies for a difference of two solutions and we see that the solution is unique.

Q.E.D. (for Prop. 2).

To return to our iterative method (15), we suppose  $g \in H^{M+1}(\mathbb{T}^N)$ ,  $U \in C([-T, T], H^M(\mathbb{T}^N))$ , and  $\frac{\partial}{\partial t} U \in C([-T, T], H^{M-1}(\mathbb{T}^N))$ , where  $M > \frac{N}{2} + 1$ . Then  $K(t, x, P_1 U, \dots, P_\ell U, D_x) \in OPH^M S^1$ ,  $f(t, x, P_1 U, \dots, P_\ell U) \in C([-T, T], H^M(\mathbb{T}^N))$ , and by Proposition 2 system (15) has a unique solution  $V(t, x) \in C([-T, T], H^M(\mathbb{T}^N))$ , such that  $\frac{\partial}{\partial t} V(t, x) \in C([-T, T], H^{M-1}(\mathbb{T}^N))$ . In order to prove convergence of the iterative method (15), we will construct equations for various derivatives of  $V$ . Set

$$\begin{aligned} V_{0\alpha} &= D_x^\alpha V, & V_{1\alpha} &= \frac{\partial}{\partial t} D_x^\alpha V \\ U_{0\alpha} &= D_x^\alpha U, & U_{1\alpha} &= \frac{\partial}{\partial t} D_x^\alpha U. \end{aligned}$$

Applying the chain rule to (15) yields

$$\begin{aligned}
(29) \quad \frac{\partial}{\partial t} V_{0\alpha} &= K(t, x, P_1 U_{00}, \dots, P_\ell U_{00}, D_x) V_{0\alpha} \\
&+ \sum_{\substack{\gamma+\sigma+\delta=\alpha, \sigma<\alpha \\ \delta=\delta_1^1+\dots+\delta_{\mu_1}^1+\dots+\delta_1^\ell+\dots+\delta_{\mu_\ell}^\ell}} C_{\sigma\gamma\mu_1 \dots \mu_\ell} \delta_1^1 \dots \delta_{\mu_1}^1 \dots \delta_1^\ell \dots \delta_{\mu_\ell}^\ell \cdot \\
&\quad \cdot P_1 U_{0\delta_1^1} \dots P_1 U_{0\delta_{\mu_1}^1} \dots P_\ell U_{0\delta_1^\ell} \dots P_\ell U_{0\delta_{\mu_\ell}^\ell} K_{\gamma\mu_1 \dots \mu_\ell} V_{0\sigma} \\
&+ \sum_{\substack{\gamma+\delta=\alpha \\ \delta=\delta_1^1+\dots+\delta_{\mu_1}^1+\dots+\delta_1^\ell+\dots+\delta_{\mu_\ell}^\ell}} C'_{\sigma\gamma\mu_1 \dots \mu_\ell} \delta_1^1 \dots \delta_{\mu_1}^1 \dots \delta_1^\ell \dots \delta_{\mu_\ell}^\ell \cdot \\
&\quad \cdot P_1 U_{0\delta_1^1} \dots P_1 U_{0\delta_{\mu_1}^1} \dots P_\ell U_{0\delta_1^\ell} \dots P_\ell U_{0\delta_{\mu_\ell}^\ell} f_{\gamma\mu_1 \dots \mu_\ell}
\end{aligned}$$

and,

$$\begin{aligned}
(30) \quad \frac{\partial}{\partial t} V_{1\beta} &= K(t, x, P_1 U_{00}, \dots, P_\ell U_{00}, D_x) V_{1\beta} \\
&+ \sum_{i=1}^{\ell} P_i U_{10} K_i V_{0\beta} + K_t V_{0\beta} \\
&+ \sum_{\gamma+\delta+\sigma=\beta} C_{\sigma\gamma\mu_1 \dots \mu_\ell} \delta_1^1 \dots \delta_{\mu_1}^1 \dots \delta_1^\ell \dots \delta_{\mu_\ell}^\ell \\
&\quad \left[ \left( P_1 U_{1\delta_1^1} P_1 U_{0\delta_2^1} \dots P_1 U_{0\delta_{\mu_1}^1} \dots P_\ell U_{0\delta_1^\ell} \dots P_\ell U_{0\delta_{\mu_\ell}^\ell} \right. \right. \\
&\quad \left. \left. + \dots + P_1 U_{0\delta_1^1} \dots P_\ell U_{0\delta_{\mu_\ell-1}^\ell} P_\ell U_{1\delta_{\mu_\ell}^\ell} \right) K_{\gamma\mu_1 \dots \mu_\ell} V_{0\sigma} \right]
\end{aligned}$$

$$\begin{aligned}
& + P_1 U_{0\delta_1^1} \cdots P_\ell U_{0\delta_{\mu_\ell}^\ell} \left[ \sum_{i=1}^{\ell} P_i U_{10} K_{\gamma\mu_1 \cdots \mu_{i+1} \cdots \mu_\ell} + K_{\gamma\mu_1 \cdots \mu_\ell} t \right] V_{0\sigma} \\
& \quad + P_1 U_{0\delta_1^1} \cdots P_\ell U_{0\delta_{\mu_\ell}^\ell} K_{\gamma\mu_1 \cdots \mu_\ell} V_{1\sigma} \Big] \\
& + \sum_{\gamma+\delta=\beta} C'_{\sigma\gamma\mu_1 \cdots \mu_\ell} \delta_1^1 \cdots \delta_{\mu_1}^1 \cdots \delta_1^\ell \cdots \delta_{\mu_\ell}^\ell \cdot \\
& \quad \cdot \left[ \left( P_1 U_{1\delta_1^1} P_1 U_{0\delta_2^1} \cdots P_1 U_{0\delta_{\mu_1}^1} \cdots P_\ell U_{0\delta_1^\ell} \cdots P_\ell U_{0\delta_{\mu_\ell}^\ell} \right. \right. \\
& \quad + \cdots + \left. \left. P_1 U_{0\delta_1^1} \cdots P_\ell U_{0\delta_{\mu_\ell}^\ell} P_{\ell-1} U_{1\delta_{\mu_\ell}^{\ell-1}} \right) f_{\gamma\mu_1 \cdots \mu_\ell} (t, x, P_1 U, \cdots, P_\ell U) \right. \\
& \quad + P_1 U_{0\delta_1^1} \cdots P_1 U_{0\delta_{\mu_1}^1} \cdots P_\ell U_{0\delta_1^\ell} \cdots P_\ell U_{0\delta_{\mu_\ell}^\ell} \cdot \\
& \quad \cdot \left. \left[ \sum_{i=1}^{\ell} P_i U_{10} f_{\gamma\mu_1 \cdots \mu_{i+1} \cdots \mu_\ell} + f_{\gamma\mu_1 \cdots \mu_\ell} t \right] \right]
\end{aligned}$$

Here  $K_{\gamma\mu_1 \cdots \mu_\ell} = K_{\gamma\mu_1 \cdots \mu_\ell} (t, x, \phi_1, \cdots, \phi_\ell) = D_x^\gamma D_{\phi_1}^{\mu_1} \cdots D_{\phi_\ell}^{\mu_\ell} K(t, x, \phi_1, \cdots, \phi_\ell)$ .

$$K_i = K_i(t, x, \phi_1, \cdots, \phi_\ell) = D_{\phi_i} K(t, x, \phi_1, \cdots, \phi_\ell), \quad i = 1, \cdots, \ell$$

$$\text{and } K_{\gamma\mu_1 \cdots \mu_\ell} t = D_t K_{\gamma\mu_1 \cdots \mu_\ell}.$$

$f_{\gamma\mu_1 \cdots \mu_\ell}$  and  $f_{\gamma\mu_1 \cdots \mu_\ell} t$  are defined similarly.

Now we replace  $U_{j\sigma}$  and  $V_{j\sigma}$  by  $P_{j\sigma}(\tilde{U})$  and  $P_{j\sigma}(\tilde{V})$  respectively where

$$(31) \quad \tilde{U} = \{U_{0\alpha}, U_{1\beta} : 0 \leq |\alpha| \leq M, 0 \leq |\beta| \leq M-1\},$$

$$P_{j\sigma}(\tilde{U}) = \Lambda^{-(M-j-|\sigma|)} \sum_{|\beta|=M-j} C_{\sigma\beta}(x, D_x) U_{j\beta}.$$

$\tilde{V}$  and  $P_{j\sigma}(\tilde{V})$  are defined similarly. After these replacements system (29), (30) for  $\tilde{V}$  becomes

$$(32) \quad \frac{\partial}{\partial t} \tilde{V} = K(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) \tilde{V}$$

$$+ \Phi(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, \tilde{V}).$$

Here  $K(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) \in \text{OPH}^M S^1$  and  $\Phi(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x)$  contains terms of order  $\leq 0$ .

Now since  $U, V \in \mathbb{R}^{mn}$ , it follows that  $\tilde{U}, \tilde{V} \in \mathbb{R}^K$ , where

$$K = mn \left\{ \binom{M+N}{N} + \binom{M+N-1}{N} \right\} \quad \text{and we have the following.}$$

Lemma 3. Assuming  $M > \frac{N}{2} + 1$ , we find that

$$(33) \quad \Phi : \mathbb{R} \times \mathbb{T}^N \times [L^2(\mathbb{T}^N)]^{2mn \left\{ \binom{M+N}{N} + \binom{M+N-1}{N} \right\}}$$

$$\rightarrow [L^2(\mathbb{T}^N)]^{mn \left\{ \binom{M+N}{N} + \binom{M+N-1}{N} \right\}}$$

is a Lipschitz continuous map.



Proof. Consider the components of  $\Phi$  of the form

$$\psi_1 = \prod_{i=1}^{\ell} \prod_{\mu=1}^{\mu_i} P_i P_{0\delta_\mu^i} \tilde{U} K_{\gamma_{\mu_1} \dots \mu_\ell} (t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) P_{0\sigma} \tilde{V}$$

and

$$\psi_2 = \prod_{i=1}^{\ell} \prod_{\mu=1}^{\mu_i} P_i P_{0\delta_\mu^i} \tilde{U} f_{\gamma_{\mu_1} \dots \mu_\ell} (t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U})$$

where  $\left| \gamma + \sigma + \sum_{i=1}^{\ell} \sum_{\mu=1}^{\mu_i} \delta_\mu^i \right| \leq M, \quad |\sigma| < M.$

The map  $\tilde{U} \rightarrow K_{\gamma_{\mu_1} \dots \mu_\ell} (t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x)$  is a Lipschitz map of  $L^2(\mathbb{T}^N)$  into  $OPH^M S^1$  since  $K_{\gamma_{\mu_1} \dots \mu_\ell} (t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x)$  is a  $C^\infty$  function of its arguments. Consequently, according to (23),  $K_{\gamma_{\mu_1} \dots \mu_\ell} (t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) : H^s(\mathbb{T}^N) \rightarrow H^{s-1}(\mathbb{T}^N)$  for any  $s \in \mathbb{R}^1$ . Since  $P_{0\sigma} \tilde{V} \in H^{M-|\sigma|}(\mathbb{T}^N)$  and  $P_{0\delta_\mu^i} \tilde{U} \in H^{M-|\delta_\mu^i|}(\mathbb{T}^N)$ , we conclude that  $\psi_1 \in L^2(\mathbb{T}^N)$  and, being multilinear, is a Lipschitz function of its arguments. Similarly, the map  $\tilde{U} \rightarrow f(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U})$  is a Lipschitz map of  $L^2(\mathbb{T}^N)$  into itself since  $f(t, x, P_1 P_{00} \tilde{U}(x), \dots, P_\ell P_{00} \tilde{U}(x)) \in H^M(\mathbb{T}^N) \subset L^2(\mathbb{T}^N)$ . Hence,  $\psi_2 \in L^2(\mathbb{T}^N)$  and is a Lipschitz function of its arguments. A similar argument controls all the other terms of  $\Phi$ .

Q.E.D.

Next, we construct a positive definite symmetrizer

$R(t, x, w_1, \dots, w_\ell, D_x)$  for  $K(t, x, w_1, \dots, w_\ell, D_x)$  and substitute

$P_i P_{00} \tilde{U}$  for  $w_i, i = 1, \dots, \ell$ , which gives us  $R(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) \in OPH^M S^0$ . Now we can write

$$\begin{aligned}
 (34) \quad \frac{d}{dt} (R\tilde{V}, \tilde{V}) &= (R(K\tilde{V} + \phi), \tilde{V}) + (R\tilde{V}, K\tilde{V} + \phi) \\
 &+ \left( \left[ \frac{\partial}{\partial t} R \right] \tilde{V}, \tilde{V} \right) = ((RK + K^*R)\tilde{V}, \tilde{V}) + (R\phi, \tilde{V}) + (R\tilde{V}, \phi) \\
 &+ \left( R_t + \sum_{i=1}^{\ell} P_i P_{10} \tilde{U} R_{w_i} \right) \tilde{V}, \tilde{V}
 \end{aligned}$$

By the argument (23) the last three terms are bounded by

$$B(\|\tilde{U}\|_{L^2}) \|\tilde{V}\|_{L^2}^2 + B(\|\tilde{U}\|_{L^2})$$

for  $t \in [-T, T]$ , where  $B$  is some function of its arguments. By the same argument and since  $\tilde{U} \rightarrow (RK + (RK)^*)(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x)$  is a Lipschitz map of  $L^2(\mathbb{T}^N)$  into  $OPH^L S^0$  for some  $L < M$ , we obtain the bound

$$((RK + K^*R)\tilde{V}, \tilde{V}) \leq B(\|\tilde{U}\|_{L^2}) \|\tilde{V}\|_{L^2}^2.$$

Consequently, (34) becomes

$$\frac{d}{dt} (R\tilde{V}, \tilde{V}) \leq B'(\|\tilde{U}\|_{L^2}) (R\tilde{V}, \tilde{V}) + B'(\|\tilde{U}\|_{L^2})$$

and Gronwall's inequality yields

$$(35) \quad \|\tilde{V}(t)\|_{L^2}^2 \leq e^{\int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})d\tau} \left[ \|\tilde{V}(0)\|_{L^2}^2 + \int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})d\tau \right]$$

where  $\tilde{V}(0)$  is defined similar to (31) with

$$V_{0\alpha}(0) = D_x^\alpha g, \quad V_{1\beta}(0) = D_x^\beta (K(t,g)g) + D_x^\beta f,$$

$$|\alpha| \leq M, \quad |\beta| \leq M-1.$$

Note that  $V_{0\alpha}(0) \in L^2(\mathbb{T}^N)$  since  $g \in H^M(\mathbb{T}^N)$ ; and  $V_{1\beta}(0) \in L^2(\mathbb{T}^N)$  since  $f \in H^M(\mathbb{T}^N)$  and by the argument used in the proof of Lemma 3.

So, we can rewrite (35) as

$$(36) \quad \|\tilde{V}(t)\|_{L^2} \leq e^{\int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})d\tau} \left[ C_0^2 \|g\|_{H^M}^2 + \int_0^t B'(\|\tilde{U}(\tau)\|_{L^2})d\tau \right].$$

Since by assumption  $U \in C([-T, T], H^M(\mathbb{T}^N)) \cap C^1([-T, T], H^{M-1}(\mathbb{T}^N))$ , then  $\tilde{U} \in C([-T, T], L^2(\mathbb{T}^N))$ . Suppose the norm of  $\tilde{U}$  in this space is  $\leq A_0$  where we pick  $A_0 \geq 2C_0 \|g\|_{H^M} + 1$ . Pick the  $t$  interval fairly small, as follows. Suppose  $T \leq T_0$  where  $T_0$  is so small that, with  $B_1 = \sup \{ |B'(\lambda)| : |\lambda| \leq A_0 \}$

$$(37) \quad e^{T_0 B_1} [C_0^2 \|g\|_{H^M}^2 + T_0 B_1] \leq A_0^2.$$

It follows from (36) that, under this assumption

$$\|\tilde{V}\|_{L^2} \leq A_0, \quad |t| \leq T_0.$$

Consequently, for such a small  $t$  interval, the mapping  $\tilde{V} = \tilde{F}\tilde{U}$ , arising from (15) maps the set

$$\{\tilde{U} \in C([-T, T], L^2(\mathbb{T}^N)) : \|\tilde{U}\|_{L^2} \leq A_0\}$$

into itself.

To check the convergence of  $V_K = F^K U$ , we need to estimate the difference between  $\tilde{V} = \tilde{F}(\tilde{U})$  and  $\tilde{V}_1 = \tilde{F}(\tilde{U}_1)$ . From (32) we get

$$\begin{aligned} (38) \quad \frac{\partial}{\partial t} (\tilde{V} - \tilde{V}_1) &= K(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) \tilde{V} \\ &\quad - K(t, x, P_1 P_{00} \tilde{U}_1, \dots, P_\ell P_{00} \tilde{U}_1, D_x) \tilde{V}_1 \\ &\quad + \Phi(t, x, \tilde{U}, \tilde{V}) - \Phi(t, x, \tilde{U}_1, \tilde{V}_1) \\ &= K(t, x, P_1 P_{00} \tilde{U}, \dots, P_\ell P_{00} \tilde{U}, D_x) (\tilde{V} - \tilde{V}_1) \\ &\quad + [K(t, x, P_i P_{00} \tilde{U}, D_x) - K(t, x, P_i P_{00} \tilde{U}_1, D_x)] \tilde{V}_1 \\ &\quad + \Phi(t, x, \tilde{U}, \tilde{V}) - \Phi(t, x, \tilde{U}_1, \tilde{V}_1) = K(t, x, P_i P_{00} \tilde{U}, D_x) (\tilde{V} - \tilde{V}_1) + \Delta \end{aligned}$$

where, with the aid of Lemma 3, we have

$$(39) \quad \|\Delta\|_{L^2(\mathbb{T}^N)} \leq C \|\tilde{U} - \tilde{U}_1\|_{L^2(\mathbb{T}^N)} [\|\tilde{V}_1\|_{H^1(\mathbb{T}^N)} + C_1] + C_2 [\|\tilde{V}\|_{L^2(\mathbb{T}^N)} + \|\tilde{V}_1\|_{L^2(\mathbb{T}^N)}]$$

Now, if we set  $\hat{U} = \{U_{0\alpha}, U_{1\beta} : |\alpha| \leq M+1, |\beta| \leq M\}$  and define  $\hat{V}$  similarly, (15) gives rise to  $\hat{V} = \hat{F}\hat{U}$  and the proof of (35) extends to

$$\|\hat{V}(t)\|_{L^2} \leq e^{\int_0^t B'(\|\hat{U}(\tau)\|_{L^2})d\tau} \left[ C_1^2 \|g\|_{H^{M+1}}^2 + \int_0^t B'(\|\hat{U}(\tau)\|_{L^2})d\tau \right].$$

Pick  $A_1 \geq 2C_1 \|g\|_{H^{M+1}} + 1$ , supposing  $g \in H^{M+1}(\mathbb{T}^N)$ . Let  $T_1$  be such that if  $\|\hat{U}(t)\|_{L^2} \leq A_1$  for  $|t| \leq T_1$ , then

$$(40) \quad \|\hat{V}(t)\|_{L^2} \leq A_1, \quad |t| \leq T_1.$$

This implies

$$(41) \quad \|\tilde{V}(t)\|_{H^1} \leq CA_1, \quad |t| \leq T_1,$$

and furthermore  $\tilde{F}^\nu \tilde{U}$  satisfies the estimate (41) for  $\nu = 1, 2, 3, \dots$ .

Now (38), (39) and (41) yield

$$(42) \quad \begin{aligned} \|\tilde{V}(t) - \tilde{V}_1(t)\|_{L^2} &\leq A_2 \int_0^t \|\tilde{U}(\tau) - \tilde{U}_1(\tau)\|_{L^2} d\tau, \quad |t| \leq T_1 \\ &\leq A_2 T_1 \sup_{|\tau| \leq T_1} \|\tilde{U}(\tau) - \tilde{U}_1(\tau)\|_{L^2}. \end{aligned}$$

It follows from (42) that  $\tilde{F}^\nu \tilde{U}$  will converge to a limit as  $\nu \rightarrow \infty$ , in  $C([-T_0, T_0], L^2(\mathbb{T}^N))$ , provided  $T_0 \leq \min(T, T_1)$  and  $A_2 T_0 < 1$ . The limit,  $\tilde{w}$ , must be of the form  $\{w_{0\alpha}, w_{1\beta} : |\alpha| \leq M, |\beta| \leq M-1\}$  for some

$w \in C([-T_0, T_0], H^M(\mathbb{T}^N)) \cap C^1([-T_0, T_0], H^{M-1}(\mathbb{T}^N))$  and  $w$  must solve (14). Since the terms  $\hat{F}^\nu \hat{U}$  are bounded in  $L^\infty([-T_0, T_0], L^2(\mathbb{T}^N))$ , we can see that  $\hat{w} \in L^\infty([-T_0, T_0], L^2(\mathbb{T}^N))$ , i.e.  $w \in L^\infty([-T_0, T_0], H^{M+1}(\mathbb{T}^N))$  and  $\frac{\partial}{\partial t} w \in L^\infty([-T_0, T_0], H^M(\mathbb{T}^N))$ . To prove uniqueness, let  $w_1$  be another solution to (14) with similar regularity. Then (42) implies

$$\|\tilde{w}(t) - \tilde{w}_1(t)\|_{L^2} \leq A_2 \int_0^t \|\tilde{w}(\tau) - \tilde{w}_1(\tau)\|_{L^2} d\tau$$

which immediately gives us  $\tilde{w}(t) = \tilde{w}_1(t)$  or  $w(t) = w_1(t)$ .

We summarize the following.

Theorem 2. Given  $M > \frac{N}{2} + 1$ , let  $g \in H^{M+1}(\mathbb{T}^N)$ . Then for  $T$  sufficiently small, the iterative method (15) converges to a unique solution  $U$  of (14) with

$$U \in C([-T, T], H^M(\mathbb{T}^N)) \cap L^\infty([-T, T], H^{M+1}(\mathbb{T}^N))$$

$$\frac{\partial}{\partial t} U \in C([-T, T], H^{M-1}(\mathbb{T}^N)) \cap L^\infty([-T, T], H^M(\mathbb{T}^N))$$

provided (14) is either symmetric hyperbolic or strictly hyperbolic.

### V. Conclusion.

Now we can solve the Cauchy problem (9).

Theorem 3. System (9), being symmetric hyperbolic or strictly hyperbolic, has a unique solution  $U$  on  $[-T, T] \times \mathbb{T}^N$ , provided  $M > \frac{N}{2} + 1$  and  $T$  small enough, and

$$\frac{\partial^j}{\partial t^j} U \in C([-T, T], H^{M-j-1}(\mathbb{T}^N)) \cap L^\infty([-T, T], H^{M-j}(\mathbb{T}^N))$$

$$0 \leq j \leq m-1.$$

Coming back to the original problem (1), we have the following result:

Theorem 4. Provided  $M > \frac{N}{2} + 1$ , the Cauchy problem for a system (1) of nonlinear equations with hypotheses (a), (b), (c), (d) has a unique solution  $\vec{u}$  on  $[-T, T] \times \mathbb{T}^N$ , such that

$$\frac{\partial^j}{\partial t^j} u_i \in C([-T, T], H^{M-j}(\mathbb{T}^N)) \cap L^\infty([-T, T], H^{M-j+1}(\mathbb{T}^N))$$

$$i = 1, \dots, n, \quad j = 0, \dots, m,$$

if  $T$  is small enough.

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