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OPTIMAL APPROXIMATION ALGORITHMS FOR DIGITAL FILTER DESIGN

by

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ABSTRACT

Several new algorithms are presented for the optimal approximation and design of various classes of digital filters.

An iterative algorithm is developed for the efficient design of unconstrained and constrained infinite impulse response (IIR) digital filters. Both in the unconstrained and constrained cases, the numerator and denominator of the filter transfer function are designed iteratively by recourse to the Remez algorithm and to appropriate design parameters and criteria, at each iteration. This makes it possible for the algorithm to be implemented by means of a short main program which uses (at each iteration) the linear phase FIR filter design algorithm of McClellan et al. as a subroutine. The approach taken also permits the filter to be designed with a desired ripple ratio. Also, the algorithm determines automatically the minimum passband ripple corresponding to the prescribed orders and band edges of the filter. The filter is designed directly without guessing the passband ripple or stopband ripple.

Another algorithm, based on similar principles, is developed for the design of a nonlinear phase finite impulse response (FIR) filter, whose transfer function optimally approximates a desired magnitude response, there being no constraints imposed on the phase response.
A similar algorithm is presented for the design of two new classes of FIR digital filters, one linear phase and the other nonlinear phase. A filter of either class has significantly reduced number of multiplications compared to the one obtained by its conventional counterpart, with respect to a given frequency response. In the case of linear phase, by introducing the new class of digital filters into the design of multistage decimators and interpolators for narrow-band filter implementation, it is found that an efficient narrow-band filter requiring considerably lower multiplication rate than the conventional linear phase FIR design can be obtained. The amount of data storage required by the new class of nonlinear phase FIR filters is significantly less than its linear phase counterpart.

Finally, the design of a (finite-impulse-response) FIR digital filter with some of the coefficients constrained to zero is formulated as a linear programming (LP) problem and the LP technique is then used to design this class of constrained FIR digital filters. This class includes pulse shaping filters, N-th band filters and nonuniform tap spacing filters, where some of the filter coefficients are constrained to zero. The advantages of this approach, as compared to other methods, with regard to design speed and filter optimality and performance, are described, and illustrated by means of examples.
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CHAPTER I

INTRODUCTION

With the increasing number of applications in the area of digital signal processing, efficient design methods as well as efficient structures for digital filter implementation have assumed great importance. In this dissertation, new optimal approximation algorithms have been developed for the design of various classes of digital filters. The present chapter provides the background for, and a summary of this research work.

1.1 Background

McClellan et al. [1] have proposed an efficient algorithm for the design of linear phase finite impulse response (FIR) digital filters based on the Remez multiple exchange algorithm. Their algorithm is important not only because it is well-developed, but also because it is highly efficient for the design of linear phase FIR digital filters due to the inherent high speed of the Remez algorithm. A linear phase FIR digital filter, although being easy to be realized and designed, usually has lower filtering efficiency than the infinite impulse response (IIR) digital filter with respect to a given desired frequency response [2]. Unfortunately, only limited success has been obtained when the Remez
algorithm is applied to the design of IIR digital filters[3-4].

When the symmetry of the filter coefficients cannot be exploited, such as in some of the applications in adaptive differential pulse-code modulation[5-6], the nonlinear phase FIR digital filter will lead to a more efficient design than the linear phase FIR filter. Goldberg et al.[7-8] proposed an algorithm for the design of a nonlinear phase FIR filter based on the Remez algorithm.

Bellanger et al.[9] found that efficient implementation of narrowband low-pass FIR filters can be obtained by reducing the sampling rate, filtering, and then increasing the sampling rate to the original frequency. Rabiner et al.[10-12] considered optimal design of multistage decimators and interpolators which minimized the multiplication rate or coefficient storage.

The filters, with frequency domain or time domain constraints, are important in the area of communication signal processing. Rabiner[13] considered the design of a linear phase FIR digital filter with a constraint on the step response and Steiglitz[14] considered the design of a linear phase FIR digital filter with monotone passband response, both using the linear programming technique by taking the advantage of its high flexibility. Shenoi and Agrawal[15] proposed an algorithm for the design of an elliptic filter with some of the poles and/or zeros being constrained.
1.2 Outline of the Research Work

In Chapter II, the approximation problem in digital filter design is first reviewed, and then the literature pertinent to the subject of this dissertation is reviewed in some detail.

An efficient iterative algorithm for designing optimal recursive digital filters is presented in Chapter III. Our iterative algorithm designs the filter directly without guessing the passband or stopband ripple. It also has the ability of finding the minimum ripple ratio for the given orders and band edges of the filter so that a solution can always be found. Another advantage of our algorithm is that it can design the filter with a desired ripple ratio.

In Chapter IV, an iterative algorithm is presented for the design of low-pass nonlinear phase FIR filters by decomposing its transfer function into a nonlinear phase part and a linear phase part. Compared to the algorithm of Goldberg et al. [8], our algorithm has the following advantages: (i) Because the set of passband ripple and stopband ripple are not included among the input parameters when calling the Remez algorithm, no "trial and error" method is necessary to design a filter; (ii) The filter designed has stopband zeros on the unit circle and thus the implementation of the filter is simplified; (iii) The algorithm can be applied to design a nonlinear phase FIR filter with high stopband attenuation.
Two new classes of FIR digital filters are presented in Chapter V, one linear phase and the other nonlinear phase, and the iterative algorithm developed in Chapter IV is proposed to design these new classes of FIR filters. By suitably choosing the design parameters, we obtain a filter with significantly reduced number of multiplications compared to its conventional counterpart, with respect to the given frequency response. By introducing these classes of digital filters into the design of multistage decimators and interpolators for narrow-band filter implementation, it is found that a new class of efficient linear phase narrow-band filters requiring considerably fewer multiplications than the conventional linear phase FIR filter design is obtained, and the new class nonlinear phase narrow-band FIR filters need much less storage than the conventional linear phase FIR filters design.

In Chapter VI, the design of an FIR digital filter with some of the coefficients constrained to zero is formulated as a linear programming (LP) problem and the linear programming technique is then used to design this class of constrained FIR digital filters. This class includes pulse shaping filters, N-th band filters and nonuniform tap spacing filters, where some of the filter coefficients are constrained to zero. The advantage of the present approach, as compared to other methods, with regard to design speed and filter optimality and performance, are described, and illustrated by means of examples.
CHAPTER II

A CRITICAL REVIEW OF THE PERTINENT LITERATURE

Approximation techniques play a central role in digital filter design. In this chapter, the approximation problem in digital filter design is reviewed and the available design methods which are closely related to the ones developed in the present dissertation are briefly discussed.

2.1 Preliminary Remarks - The Approximation Problem

2.1.1 Digital Filter Approximation

Let the sequence \( \{y(n)\} \) be the filter response to the input sequence \( \{x(n)\} \). We will consider digital filters which are represented by a linear, shift-invariant operator by a difference equation of the form:

\[
y(n) = \sum_{k=0}^{N} a(k)x(n-k) - \sum_{k=1}^{M} b(k)y(n-k), \tag{2.1}
\]

where \( \{a(k)\} \) and \( \{b(k)\} \) characterize the filter. By taking the \( z \)-transform of (2.1), we get

\[
Y(z) = X(z)[\sum_{k=0}^{N} a(k)z^{-k}] - Y(z)[\sum_{k=1}^{M} b(k)z^{-k}],
\]

where \( Y(z) \) and \( X(z) \) are the \( z \)-transforms of the sequences \( \{y(n)\} \) and \( \{x(n)\} \) respectively. The transfer function of the digital filter is

\[5\]
defined as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} a(k)z^{-k}}{1 + \sum_{k=1}^{M} b(k)z^{-k}},$$  \hspace{1cm} (2.2)$$

where $N$ and $M$ are all positive integers. The format of (2.2) is often used in general filter design problems.

If the \{(b(k))\} in (2.2) are not all zero and all the roots of the denominator are not cancelled exactly by the roots of the numerator, the filter is said to be a recursive filter because the calculation of $y(n)$ in (2.1) requires the values $y(n-1), y(n-2), \ldots, y(n-M)$ in addition to the values $x(n), x(n-1), \ldots, x(n-N)$.

If the \{(b(k))\} in (2.2) are all zero, the calculation of $y(n)$ in (2.1) depends only on the present and past values of the input sequence \{x(n)\}, and then the filter is said to be nonrecursive. From (2.2), the transfer function of a nonrecursive filter can be written as

$$H(z) = \sum_{k=0}^{N} h(k)z^{-k},$$  \hspace{1cm} (2.3)$$

where the impulse response \{h(n)\} of the filter is given by

$$h(n) = \begin{cases} a(n) & 0 \leq n \leq N \\ 0 & \text{elsewhere} \end{cases}$$  \hspace{1cm} (2.4)$$

From (2.4), it is found that the duration of the impulse response of this filter is finite, and thus nonrecursive filters are also called finite-duration impulse response (FIR) filters.
From (2.2), the transfer function of a recursive filter can be written as

\[ H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} = \sum_{k=0}^{N} a(n)z^{-k} \sum_{k=0}^{N} b(k)z^{-k} \]

where \( b(0) = 1 \). The impulse response of a recursive filter has infinite duration, therefore, recursive filters are also called infinite-duration impulse response (IIR) filters.

The frequency response of a digital filter is the transfer function \( H(z) \) evaluated on the unit circle \( z = \exp(j2\pi F) \),

\[ H(F) = H(z)\bigg|_{z=\exp(j2\pi F)} \]

where, and henceforth, we use the abbreviated notation \( P(F) \) for \( P(\exp(j2\pi F)) \) obtained from a given function \( P(z) \). Note that \( H(F) \) is periodic in \( F \) with period equal to 1. \( F \) is called "normalized frequency". Obviously, the function \( H(F) \) depends on the coefficients \( C = [a(0), a(1), \ldots, a(N), b(1), \ldots, b(M)] \). For FIR digital filters, \( C \in \mathbb{R}^{N+1} \), but for IIR digital filters, \( C \) belongs to a subset of \( \mathbb{R}^{M+N+2} \) such that the poles of (2.5) are located inside the unit circle. In this section, this subset will be denoted by \( \Omega \).

In the design of a digital filter, we are interested in approximating a desired real function \( D(F) \geq 0 \) by the magnitude of frequency response \( H(F) \). Usually, the approximation is performed on the compact subset \( \Lambda \) of the union of disjoint closed-intervals, where each closed-
interval is a subset of \([0, 0.5]\). At a given \(F\), the approximation error is

\[
E(F) = D(F) - H(F) .
\]  

(2.6)

Then the filter design problem reduces to choosing coefficients \(C\) such that the norm of the approximation error function \(E(\cdot)\) in (2.6) is minimized. It is customary to choose the \(L_\infty\) (Chebyshev) norm

\[
\|E(F)\|_\infty = \max_{F \in \mathbb{A}} |E(F)| .
\]  

(2.7)

Since the Chebyshev norm is used in most of the filter approximation problems, we will use it and simplify the notation \(\|\cdot\|_\infty = \|\cdot\|\). Then, the filter approximation problem becomes

\[
\min_{C \in \mathbb{Q}} \|E(F)\| = \min_{C \in \mathbb{Q}} [\max_{F \in \mathbb{A}} |E(F)|] .
\]  

(2.8)

2.1.2 **Linear Phase FIR Filter Design**

Let \(\{h(n)\}\) be a causal finite-duration sequence defined over the interval \(0 \leq n \leq N-1\). The transfer function of an FIR digital can be represented in the form of (2.3). The discrete Fourier transform of \(\{h(n)\}\) is given by

\[
H(F) = \sum_{n=0}^{N-1} h(n) \exp(-j2\pi n F) ,
\]  

(2.9)

which is periodic in frequency with period 1, i.e.,

\[
H(F) = H[\exp(j(2\pi F + 2m\pi))], \quad m = 0, \pm 1, \pm 2, \ldots
\]
If \{h(n)\} is restricted to be real and the phase is exactly linear, then the magnitude response \(H^*(F)\) of an FIR filter has symmetric coefficients and can be written in the unified form[16]:

\[ H(F) = P(F) \cdot Q(F) , \quad (2.10) \]

where

\[ P(F) = \sum_{n=0}^{(N-1)/2} a(n) \cos(2\pi n F) , \text{ and } Q(F) = 1 , \text{ (N odd)} , \quad (2.11a) \]

or

\[ P(F) = \sum_{n=0}^{(N/2)-1} b(n) \cos(2\pi F) , \text{ and } Q(F) = \cos(\pi F) , \text{ (N even)} , \quad (2.11b) \]

where \{a(n)\} and \{b(n)\} can be expressed in terms of \{h(n)\} in (2.9).

2.1.3 Characterization of FIR Filter Design

Based on (2.10), the optimal design of a linear phase FIR (LPFIR) filter to match a desired magnitude response can be formulated as a Chebyshev approximation problem. Let \(D(F)\) and \(W(F)\) be respectively the desired frequency response of the filter and the weighting function for the approximation error. Then, the weighted approximation error function in (2.6) is given by

\[ E(F) = W(F) [D(F) - P(F)Q(F)] \]

\[ = \tilde{W}(F) [\tilde{D}(F) - \tilde{P}(F)] , \quad (2.12) \]

where \(\tilde{W}(F) = W(F)Q(F)\), \(\tilde{D}(F) = D(F)/Q(F)\) and \(\tilde{P}(F) = P(F)\).
Now, the Chebyshev approximation problem reduces to choosing the coefficients \{a(n)\} and \{b(n)\} in (2.11) such that the Chebyshev norm of \(E(F)\) is minimized over the frequency intervals over which the approximation is desired.

The well-known alternation theorem, which can be used to obtain a solution to this class of Chebyshev approximation problems, may be stated as follows[16]:

**Theorem 1 (Alternation Theorem)**

Let \(\tilde{\Pi}(F)\) be a linear combination of \(r\) cosine functions, i.e.,
\[
\tilde{\Pi}(F) = \sum_{n=0}^{r-1} a(n) \cos(2\pi n F)
\]
and \(A\) be any compact subset of \([0,0.5]\). Then \(\tilde{\Pi}(F)\) will be the unique, best-weighted Chebyshev approximation to a desired continuous function \(\Pi(F)\) on \(A\) if and only if the weighted error function exhibits at least \((r+1)\) extremal frequencies on \(A\); that is, there must be \((r+1)\) points on \(A\), \(F_1 < F_2 < \cdots < F_{r+1}\), such that \(E(F_i) = -E(F_{i+1})\), \(i = 1,2,\cdots,r\), and \(\max_{F \in A} |E(F)|.\)

2.1.4 The Remez Exchange Algorithm

Based on the Chebyshev formulation of the approximation problem and the alternation theorem described above, the Remez exchange algorithm can be applied to design the optimal filter[1].

From the alternation theorem, the optimal solution must have at least \((r+1)\) extrema of the approximation error. Let a set
\((F_i, i = 0, 1, \cdots, r)\) of such extrema be in the approximation region. Then we can write the following \((r+1)\) equations:

\[
\tilde{W}(F_j) [\tilde{D}(F_j) - \tilde{P}(F_j)] = (-1)^j \delta, \quad j = 0, 1, \cdots, r. \tag{2.13}
\]

These \((r+1)\) linear equations involve \((r+1)\) variables. These are the \(r\) coefficients \(\{a(n)\}\) of the approximation function plus the unknown error \(\delta\). The linear system of these \((r+1)\) equations has a definite solution since the coefficient matrix is invertible based on the fact that the basis of the approximation function satisfies the Harr condition.

In (2.13), the approximation error \(\delta\) is efficiently computed by the formula:

\[
\delta = \frac{a_0 \tilde{D}(F_0) + a_1 \tilde{D}(F_1) + \cdots + a_r \tilde{D}(F_r)}{\tilde{W}(F_0) + \tilde{W}(F_1) + \cdots + (-1)^r \tilde{W}(F_r)}, \tag{2.14}
\]

where

\[
a_k = \prod_{i=0, i \neq k}^{r-1} \frac{1}{x_k - x_i}, \tag{2.15}
\]

and

\[x_i = \cos(2\pi F_i).\]

After calculating \(\delta\), the Lagrange interpolation formula in the barycentric form is used to interpolate \(\tilde{P}(F)\) on the \(r\) points \(F_0, F_1, \cdots, F_{r-1}\).
Now, the procedure for designing the optimal LPFIR filter by using the Remez algorithm can be described as follows. In the first step, the design algorithm chooses \((r+1)\) equally-spaced extremal frequencies \(\{F_i, i = 0, 1, \ldots, r\}\) and solves (2.13) by calculating the approximation error \(\delta\) in (2.14). Then the algorithm interpolates \(\tilde{P}(F)\) on the \(r\) points \(F_0, F_1, \ldots, F_{r-1}\).

The next step is to evaluate \(E(F)\) of (2.12) on a dense grid of frequencies. If \(|E(F)| \leq \delta\) for all frequencies in the dense grid set, then the optimal approximation has been found. If \(|E(F)| > \delta\) for some frequencies in the dense grid set, then a new set of \((r+1)\) frequencies, where the peaks of the resulting error curve occur, are chosen as candidates for the extremal frequencies of the next iteration. Based on this new set of extremal frequencies, (2.13) is solved again. This procedure is continued until \(\delta\) converges to its upper bound and thus the optimal solution to the problem is achieved.

2.1.5 Linear Phase FIR Filter Design by Linear Programming Techniques

The optimal LPFIR filter design described above can also be formulated as a linear programming problem. We can write the problem of LPFIR filter design as

\[
-\delta \leq \tilde{W}(F)(\tilde{D}(F) - \sum_{n=0}^{r-1} a(n) \cos(2\pi n F)) \leq \delta ,
\]

(2.16)

and from this relation, two sets of linear constraints can be obtained by sampling on a dense grid of points in the approximation region.
\[ \tilde{W}_j(F_j) \tilde{D}(F_j) - \sum_{n=0}^{r-1} a(n) \cos(2\pi n F_j) \leq \delta, \quad j = 0, 1, \ldots, M, \]

where there are (M+1) grid points from \( F = 0 \) to \( F = 0.5 \) and a grid density of 16 is generally used in filter design. These equations are equivalent to

\[ -\tilde{W}_j(F_j) \sum_{n=0}^{r-1} a(n) \cos(2\pi n F_j) \leq -\tilde{W}_j(F_j) \tilde{D}(F_j), \quad j = 0, 1, \ldots, M, \quad (2.17a) \]

\[ \tilde{W}_j(F_j) \sum_{n=0}^{r-1} a(n) \cos(2\pi n F_j) \leq \tilde{W}_j(F_j) \tilde{D}(F_j), \quad j = 0, 1, \ldots, M, \quad (2.17b) \]

where the coefficients \( \{a(n)\} \) must be chosen to minimize the \( (-\delta) \) value.

It must be observed that the solution obtained through linear programming is exactly the same as the one obtained from the Remez algorithm. Although the linear programming formulation leads to an algorithm less efficient than the Remez algorithm, it has the advantage of being more flexible. Its flexibility lies essentially on the fact that other constraints can be considered in addition to (2.17). This allows one to easily add other constraints on the characterization of the filter without large increase in calculations. For example, Rabiner[13] considered the design of the LPFIR filter with the constraints on the step response of the filter, and Steiglitz[14] designed the optimal LPFIR filter with monotone passband response, both using linear programming techniques.
2.2 Nonlinear Phase FIR Filter Design

2.2.1 Introduction

Most of the work on FIR digital filter design, thus far, has been limited to LPFIR filters, mainly because it is possible to use the symmetry property of filter coefficients. However, there are applications where the linear phase characteristic may not be necessary. For example, when an FIR filter is used with delta-modulated signals or with adaptive differential PCM signals, there is no term-by-term multiplications as in the convolution, but, rather, sum of coefficients, corresponding to the same incremental term (±1), are added. Also, in implementing a charge-coupled device filter, one is concerned with minimizing the filter length so that the effect of charge transfer inefficiency may be reduced. In such a case, the optimal nonlinear phase FIR (NLPFIR) filter will be more efficient than the optimal LPFIR filter for a given magnitude response since the former allows a reduction of filter length with respect to the latter[8].

2.2.2 Design of Nonlinear Phase FIR Digital Filters

Many authors have shown how to design the NLPFIR filter to meet a desired magnitude response[17-23]. Recently, Goldberg et al.[8] have also proposed a method for designing the optimal NLPFIR filter using an approach similar to that proposed by Herrmann et al[7,17]. Here, we briefly review the algorithm which was first proposed by Herrmann and then extended by Goldberg et al[8].
Let the transfer function of a nonlinear phase FIR filter of order \( N \) be of the form

\[
H(z) = \sum_{n=0}^{N} h(n)z^{-n},
\]

(2.18)

where \( \{h(n)\}, \ n = 0,1,\ldots,N \), constitutes the impulse response of the filter. Define an FIR filter whose transfer function is

\[
\tilde{H}(z) = H(z)H(z^{-1})^{-N} = \sum_{n=0}^{2N} h(n)z^{-n},
\]

(2.19)

where

\[
\tilde{h}(n) = \tilde{h}(2N-n) = \sum_{j=0}^{n} h(j)h(j+N-n), \ n = 0,1,\ldots,N.
\]

(2.20)

Equation (2.20) implies that \( \tilde{H}(z) \) is a linear phase filter since its coefficients are symmetrical. Thus, the zeros of \( \tilde{H}(z) \) in the passband must be reciprocal and conjugate, i.e., if \( z_1 \) is a zero of \( \tilde{H}(z) \), then \( z_1^{-1}, z_1^* \) and \( (z_1^*)^{-1} \) are all zeros of \( \tilde{H}(z) \) where the superscript * denotes the conjugate quantity. The frequency response of \( \tilde{H}(z) \) can be described as

\[
\tilde{H}(F) = H(F)H(-F)e^{-j2\pi NF} = |H(F)|^2e^{-j2\pi NF}.
\]

(2.21)

Since \( |H(F)|^2 \) cannot be negative, and therefore cannot intersect the frequency-axis but be tangent to it, all of the zeros of \( \tilde{H}(z) \) on the unit circle must appear in a multiplicity of two.

Based on (2.21), the following algorithm was used by Goldberg et al.[8] for designing an NLPFIR filter \( H(z) \) of order \( N \) and whose given
magnitude response approximates the desired magnitude response $S(F)$:

(i) Design an LPFIR digital filter $\tilde{H}(z)$ of order $2N$ whose gain response approximates $S^2(F)$; (ii) Find the zeros of $\tilde{H}(z)$; and (iii) Pick one from each pair of reciprocal zeros of $\tilde{H}(z)$ and replace each pair of double zeros on the unit circle by a single zero, and thereby define $H(z)$. (The zeros inside and on the unit circle should be selected if a minimum phase filter is needed).

Let $\delta_p$ and $\delta_s$ be respectively the passband and stopband ripples of $\tilde{H}(z)$ and $\tilde{H}'$, and $\delta_p$ and $\delta_s$ be their corresponding quantities for $\tilde{H}(z)$. Then the relation between the set of $\delta_p$, $\delta_s$ and the set of $\delta_p$, $\delta_s$ can be derived as [8]

$$\frac{\delta_p}{\delta_s} = \frac{\delta_p}{\delta_s}, \quad (2.22a)$$

and

$$\delta_s = (2\delta_s)^{1/2}. \quad (2.22b)$$

If the Remez exchange algorithm in [1] is used to design the prototype LPFIR filter $\tilde{H}(z)$ directly, the set of ripples $\delta_p$ and $\delta_s$ is the output of the design procedure, while the input design parameter is a weighted function $\tilde{W}(F)$. The desired function $\tilde{D}(F)$ and weighting function $\tilde{W}(F)$ can be found as

$$\tilde{D}(F) = 1, (\text{in the passband}), \quad (2.23a)$$
\[ D(F) = \frac{\delta}{2}, \text{(in the stopband)}, \] (2.23b)

and

\[ \tilde{W}(F) = \frac{\delta_s^p}{\delta_s^s} = \frac{25_s^p}{\delta_s^s} = \frac{1}{\frac{25_s^p}{\tilde{W}_s} (25_s^s)^{1/2}} = \text{function of } \delta_s^s. \] (2.24)

Usually, \( \delta_s^s \) is unknown before the design of \( \tilde{P}(F) \) is carried out. Thus \( \tilde{W}(F) \) cannot be obtained and \( \tilde{P}(F) \) cannot be designed. In other words, the ripple \( \delta_p^p \) or \( \delta_s^s \) must be included among the input design parameters. Thus it is necessary to use a "trial-and-error" method when designing the filter.

2.2.3 Optimality Property of the Solution

The optimality of the solution obtained by the algorithm of Goldberg et al. [8] can be examined by investigating the error function in the passband and stopband. Suppose the order of the NLPFIR filter with transfer function \( H(z) \) is \( N_s \). Then the algorithm designs an order \( 2N \) prototype LPFIR filter \( \tilde{H}(z) \) and then constructs \( H(z) \) from \( \tilde{H}(z) \). From the alternation theorem, there exist at least \( (N+2) \) extremal frequencies on the frequency subset \( A_t = [0, F_p] \cup [F_s, 0.5] \) (where \( F_p \) is the passband edge and \( F_s \) the stopband edge of the filter), and the solution thus obtained is the optimal solution to the approximation problem with the desired value 1 and ripple \( (\delta_p^p/2) \) in the passband, and the desired value \( (2\delta_s^s)^{1/2} \) and ripple \( (2\delta_s^s)^{1/2} \) in the stopband.

2.2.4 Construction of NLPFIR Filter from a Prototype LPFIR Filter
After finding the transfer function \( \tilde{H}(z) \) of the prototype LPFIR filter, the transfer function \( H(z) \) of the corresponding NLPFIR filter can be constructed from \( \tilde{H}(z) \). There are several approaches toward this goal. Two of them are discussed below.

(a) **Root finding method**: Schmidt and Rabiner\(^{[24]}\) proposed a technique for finding the zeros of standard LPFIR digital filters. Three algorithms were used for finding the roots of a polynomial: bisection method, modified false position method, and the Newton-Raphson method. As pointed out in \(^{[24]}\), it was found that the Newton-Raphson method is the most accurate in determining the location of the roots, as well as being computationally the most efficient of the three methods.

(b) **Cepstral method**: From (2.19), \( \tilde{H}(z) \) can be written in the form

\[
\tilde{H}(z) = Az^{-N_2} \prod_{j=1}^{N_1} (1 - z_j z^{-1})^2 \prod_{k=1}^{N_2} (1 - z_k z^{-1})(1 - z_k),
\]

(2.25)

with \( 2(N_1 + N_2) = 2N, A > 0, |z_j| < 1 \). Once the term \( z^{-N_2} \) has been removed from \( \tilde{H}(z) \), it is possible to choose as region of convergence of \( \log[z^{-N_2} \tilde{H}(z)] \) the annular region of the \( z \)-plane defined by:

\[1 < |z| < \min_k |z_k^{-1}|, \text{ where } |z_k z^{-1}| < 1, \forall k \text{ and } |z_j z^{-1}| < 1, \forall j.\]

By choosing the region of convergence for \( \log[z^{-N_2} \tilde{H}(z)] \), the minimum phase zeros of \( \tilde{H}(z) \) and the zeros on the unit circle will contribute to the causal components of cepstral sequence \( \tilde{c}(n) \) associated with \( \tilde{H}(z) \); while the maximum phase zeros contribute to the anti-causal components of \( \tilde{c}(n)[25] \). Based on this principle, the cepstral sequence
c(n) associated with H(z) can be found by

\[ c(n) = \begin{cases} 
0 & n < 0 \\
\frac{c(0)}{2} & n = 0 \\
\frac{[c(n) + c(-n)]}{2} & n > 0 
\end{cases} \quad (2.26) \]

At this point, the impulse response \( h(n) \) associated with the minimum phase \( \text{FIR} \) filter can be derived through the recursive formula:

\[ h(n) = \begin{cases} 
0 & n < 0 \\
c(0) & n = 0 \\
\sum_{k=0}^{n-1} \frac{k}{n} c(k) h(n-k) & 0 < n \leq N 
\end{cases} \quad (2.27) \]

Miao and Naimer[20] suggested a procedure for obtaining \( h(n) \) from \( \tilde{h}(n) \) by using FFT techniques. Boite and Leich[19] used a similar approach to construct minimum phase \( \text{FIR} \) filters from the prototype LPFIR filter.

### 2.2.5 Comments on the Algorithm of Goldberg et al.[8]

The algorithm of Goldberg et al.[8], even though a significant contribution to the design of the NLPFIR filter, is subject to the following limitations:

(i) If we apply the algorithm to design an NLPFIR filter by calling the Remez algorithm directly, the set of passband ripple and stopband ripple must be included among the input design parameters as shown in (2.24).

(ii) As mentioned by Goldberg et al.[8], since even for small values of stopband attenuation, the magnitude response of the prototype LPFIR filter designed by this algorithm is no longer equiripple, it intersects the frequency axis instead of being tangent to it. In order to achieve
the desired stopband attenuation, the design procedure must be slightly modified. The resulting NLPFIR filter, even though it meets the desired stopband attenuation, has not all stopband zeros on the unit circle, so that the filter cannot be implemented by taking advantage of having all stopband zeros on the unit circle and thus it needs more multipliers in the filter implementation.

(iii) The algorithm cannot be applied to design an NLPFIR filter with high stopband attenuation.

2.3 Recursive Digital Filter Design

There are a number of computer-aided techniques[26-33] available for the design of IIR digital filter to match a desired magnitude response. In this section, we very briefly review two methods for the design of a recursive filter with all zeros on the unit circle and all poles in the passband.

2.3.1 Formulation of the Recursive Filter Design Problem

The transfer function of a recursive low-pass filter is of the form

$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{N} a(k)z^{-k}}{\sum_{k=0}^{M} b(k)z^{-k}}, \quad (2.28)$$

where $b(0) = 1$ and the poles of $H(z)$ must be inside the unit circle for the filter to be stable. Unlike in the case of nonrecursive filters, the design of a recursive filter must consider the coefficients in both the
numerator and denominator. This makes it difficult to solve the approximation problem for recursive filter design.

The magnitude-squared function of the filter is obtained by evaluating

\[
\hat{H}(z) = \frac{\hat{N}(z)}{\hat{D}(z)} = \frac{N(z)N(z^{-1})}{D(z)D(z^{-1})} = \frac{c(0) + \sum_{i=1}^{N} c(i)(z^i + z^{-i})}{d(0) + \sum_{i=1}^{M} d(i)(z^i + z^{-i})}, \quad (2.29)
\]

where

\[
c(i) = \sum_{k=0}^{N-1} a(k)a(k+i), \quad 0 \leq i \leq N,
\]

and

\[
d(i) = \sum_{k=0}^{M-1} b(k)b(k+i), \quad 0 \leq i \leq M.
\]

along the unit circle, i.e.,

\[
\hat{H}(F) = \frac{\hat{N}(F)}{\hat{D}(F)} = \frac{|H(F)|^2}{|\hat{D}(F)|^2} = \frac{c(0) + 2\sum_{i=1}^{N} c(i)\cos(2\pi iF)}{d(0) + 2\sum_{i=1}^{M} d(i)\cos(2\pi iF)}. \quad (2.30)
\]

It should be noted that the denominator of \(\hat{H}(F)\) is always positive.

**Theorem 2 (Characterization of a Recursive Filter)**

The function defined in (2.30) is the optimal solution in the minimax sense to the approximation problem where the desired function is 1 with ripple \(\delta_p\) on \([0,F_p]\), \(\delta_s/2\) with ripple \(\delta_s/2\) on \([F_s,0.5]\) and has an
appropriate weighting function to take into account the differences between the ripples if and only if the error function has at least \((N+M+2)\) alternations on \([0, F_p] \cup [F_s, 0.5]\) (where \(F_p\) is the passband edge and \(F_s\) the stopband edge of the filter).

2.3.2 The Martinez-Parks Algorithm for Recursive Filter Design\([34, 35]\)

(a) Algorithm

Based on the squared-magnitude function (2.30) of the recursive filter, Martinez and Parks\([34]\) considered the following approximation problem:

Given filter orders \(N, M, F_p\) (passband edge), \(F_s\) (stopband edge) and \(\hat{\delta}_p\) (passband ripple), find the coefficients \(\{c(i)\}\) and \(\{d(i)\}\) in (2.30) in such a way that \(\hat{\delta}_s\) (stopband ripple) is minimized.

Martinez and Parks proposed an algorithm to design a recursive filter by working iteratively with the numerator and denominator. The design procedure can be outlined as follows\([34-35]\):

a) Set \(\hat{N}(F) = 1, k = 1\) if \(M\) is even or \(0\) if \(M\) is odd, \(\hat{K} = 1\) if \(N\) is even or \(0\) if \(N\) is odd;

b) A denominator \(\hat{D}(F)\) is found by solving the interpolation problem

\[
\hat{d}(0) + \sum_{i=0}^{M} d(i) \cos^{\hat{r}}(2\pi F_j) = \frac{\hat{N}(F_j)}{[1+(-1)^{k+j} \cdot \hat{\delta}_p]}, \text{for } j = 0, 1, \ldots, M,
\]

such that \(\hat{N}(F)/\hat{D}(F)\) is equiripple on \([0, F_p]\) oscillating around 1 with \(M+1\) extremal frequencies;
c) Find extremal values of \( E(F) = 1 - \hat{N}(F)/\hat{D}(F) \) on \([0,F_p]\) and calculate an initial value for \( \hat{s}_S \) by taking the value of \( \hat{N}(F)/\hat{D}(F) \) at the last extremal frequencies on \([F_S,0.5]\) where \( \hat{N}(F)/\hat{D}(F) \neq 0 \). (Actually, this \( \hat{s}_S \) is only a scaling factor);

d) After a value of \( \hat{s}_S \) is found, the numerator \( \hat{N}(F) \) is found by solving the interpolation problem

\[
c(0) + \sum_{i=1}^{N} c(i) \cos^{i}(2\pi F_j) = 0, \text{ for } j = 1,3,5,\ldots,N-k,\]

and

\[
c(0) + \sum_{i=1}^{N} c(i) \cos^{i}\cos(2\pi F_j) = \hat{s}_S \hat{D}(F_j), \text{ for } j = 0,2,4,\ldots,N+k-1,\]

such that \( \hat{N}(F)/\hat{D}(F) \) oscillates on the stopband around \( \hat{s}_S/2 \) with ripple \( \hat{s}_S/2 \) and \( N+1 \) extremal frequencies;

e) Find the extremal values of \( \tilde{E}(F) = \hat{s}_S/2 - \hat{N}(F)/\hat{D}(F) \) on \([F_S,0.5]\);

f) Go to step b) until \( \hat{N}(F)/\hat{D}(F) \) is found to have equiripple behavior in both passband and stopband.

Once the filter coefficients \( \{c(i)\} \) and \( \{d(i)\} \) are known, a stable minimum phase filter transfer function can be constructed.

(h) **Remarks on the Martinez-Parks Algorithm**

While constituting an important forward step in the field of IIR filter design, the Martinez-Parks algorithm presents certain limitations.
In the Martinez-Parks algorithm, either the passband ripple or stopband ripple, which is usually unknown before the design procedure is carried out, has to be prescribed. If the prescribed ripple is too tight, no solution exists and hence their algorithm will not converge. For designing a filter, their algorithm requires the development of an entire new computer program for implementing the algorithm.

Furthermore, unlike the design of the LPFIR filter in [1], the Martinez-Parks algorithm is not geared toward designing a filter with a prescribed ripple ratio (i.e., passband ripple to stopband ripple ratio). Given the filter orders N, M and band edges $F_p, F_s$, there exists a minimum passband ripple which can be designed, but the Martinez-Parks algorithm does not find this minimum passband ripple directly.

Gockler[36] presented an algorithm for overcoming part of the above difficulties, but his algorithm still suffers from the inability to design the recursive filter with a prescribed ripple ratio.

2.3.2 The Shenoi-Agrawal Algorithm for Recursive Filter Design

Shenoi and Agrawal[37] proposed an algorithm to design the recursive filter by a modified Darlington method[38]. Their method is briefly reviewed here.

(a) Problem Formulation

An N-th order recursive filter is described by a rational transfer function
\[ H(z) = \sum_{k=0}^{N} \frac{a(k)z^{-k}}{b(k)z^{-k}} \]  \hspace{1cm} (2.31)

where \( b(0) = 1 \). The squared-magnitude response of \( H(z) \) in (2.31) can be written as a ratio of two polynomials, each of which is expressed as a linear combination of Chebyshev polynomials

\[ \hat{H}(F) = \frac{\Phi(x)}{\Psi(x)} = \frac{\alpha(0) + \alpha(1)T_1(x) + \cdots + \alpha(N)T_N(x)}{\beta(0) + \beta(1)T_1(x) + \cdots + \beta(N)T_N(x)} \]  \hspace{1cm} (2.32)

where \( x = \cos(2nF) \) and \( T_i(x) \) is the \( i \)-th Chebyshev polynomial of the first kind. The transformation \( x = \cos(2nF) \) will map the passband \([0,F_p]\) into \([x_p,1]\), and stopband \([F_s,0.5]\) into \([-1,x_s]\). The nonnegativity of \( \hat{H}(F) \) and infinite attenuation in stopband dictates that

\[ \hat{H}(F) = \frac{\Phi(x)}{\Psi(x)} = \begin{cases} \frac{1}{1 + \frac{P(x)}{Q^2(x)}} & \text{N even} \\ \frac{1}{1 + \frac{P(x)}{Q^2(x)(1+x)}} & \text{N odd} \end{cases} \]  \hspace{1cm} (2.33)

where

\[ \Phi(x) = \begin{cases} Q^2(x) & \text{N even} \\ Q^2(x)(1+x) & \text{N odd} \end{cases} \]

and

\[ \Psi(x) = P(x) + \phi(x) \]

For even \( N \) (odd \( N \)), the design of an elliptic filter reduces to the determination of \( P(x) \) and \( Q(x) \), such that
\[ \frac{P(x)}{Q^2(x)} \frac{P(x)}{Q^2(x)(1+x)} , \]

and

\[ \frac{Q^2(x)}{P(x)} \frac{(1+x)Q^2(x)}{P(x)} , \]

approximate, in a minimax sense, the zero function in the passband and stopband respectively.

The Darlington method[38] is used to obtain \( P(x) \) and \( Q(x) [15,37] \). Once \( \phi(x) \) and \( \psi(x) \) are determined, the poles(zeros) of \( H(z) \) are obtained from the roots of \( \psi(x) (\phi(x)) \) as follows: If \( x_0 \) is a root of \( \psi(x) \) then \( z_0 \) is a pole of \( H(z) \) where

\[ z_0 = x_0 \pm (x_0^2 - 1)^{1/2} , \]

where the \( \pm \) sign is chosen to select a pole inside the unit circle. The details of the design procedure can be found in [15,37].

(b) Remarks on the Shenoi-Agrawal Algorithm

In the Shenoi-Agrawal algorithm described above, the squared-magnitude of the filter is used when working the approximation problem in order to meet the nonnegativity requirement. This may cause numerical difficulties when used in the design of a filter with high stopband attenuation. The algorithm also requires a specially tailored computer program to implement the design algorithm.

Furthermore, given the order \( N \) and band edges of the filter, the algorithm is not capable of determining the minimum passband ripple. Unlike in the design of the LPFIR filter by the program of McClellan et
al.[1], the Shenoι-Agrawal algorithm cannot design a recursive filter with a prescribed ripple ratio.

2.4 Design of Multirate Filters

The implementation of a narrow-band filter constitutes a difficult problem in digital filtering because of the inherently sharp transition band requirements. For this reason, a high order design is required to meet the desired frequency response. The implementation of a high order filter requires a large amount of arithmetic computations and large filter coefficient storage, and high multiplication roundoff error.

An efficient way of implementing a narrow-band filter is to use a multirate filter which consisting of cascaded decimators (with sampling rate reduction) and interpolators (with sampling rate increase)[39-41]. Bellanger et al.[9] used half-band decimators and interpolators to approximate a narrow-band low-pass filter, and showed that a much reduced computation rate can be obtained in this implementation over the direct implementation. Shively[42], Crochiere and Rabiner[10-11], and Rabiner and Crochiere[12] discussed the design of optimal multirate filters which minimize the multiplication rate or coefficient storage. A general approach for the design of a linear phase FIR multirate filter is to distribute the specifications of a narrow-band filter into the requirements of each stage of decimators and interpolators, and then the linear phase FIR decimators and interpolators are designed to meet these requirements. They also suggested procedures for optimally choosing the
decimation ratios at each stage so that the amount of overall computations in the multirate filter is minimized. Chu[43] developed a design procedure of a multirate filter by designing each stage to meet the stopband attenuation and then, at the last stage, including an IIR or FIR compensator to compensate the passband response. This concept also leads to the use of comb filters for the design of multirate filters.

The multirate filters were considered as a special class of periodically time varying filter by Meyer and Burrus[44]. In [45], Meyer and Burrus used an extension of the impulse-invariant method to get a multirate filter with a transfer function of equal numerator and denominator order but having only power of $z^D$ (D the decimation ratio) in the denominator. Bellanger et al.[46] considered the same type of transfer functions and realized the filter with filter banks.

Martinez and Parks[34] proposed a class of IIR multirate filters whose transfer functions contain only powers of $z^D$ in the denominator. Hence, when used for decimation or interpolation by a factor D, the arithmetic operations can be performed at a lower sampling frequency, and thus the computation rate in implementing the filter is reduced.

2.5 Constrained Digital Filter Design

2.5.1 Digital Filters with Constraints in Frequency Domain

In the field of digital filter design, most of the algorithms are limited to finding the optimal digital filter with flat desired passband
magnitude response and flat desired stopband magnitude response. The optimal solution obtained from this kind of algorithms will have equiripple error function in both the passband and stopband. In FIR digital filter design, the zeros in the stopband and in the passband are distributed so that the response of the resulting filter will have equiripple error in both the passband and stopband and the solution is optimal in the minimax sense[47]. In IIR digital filter design, the zeros in the stopband and the poles in the passband are also distributed so that the response of the resulting filter will have equiripple in both the passband and stopband and the solution is also optimal in the minimax sense[30].

In many applications of digital signal processing, especially in communication systems, the desired digital filter is of a prescribed passband shape or prescribed stopband shape. In some types of system designs, there are prescribed zeros or prescribed poles in the system. Then the design objective is to construct a digital filter to compensate (or shape) the response of these prescribed zeros or prescribed poles so that the resulting response of the system will meet the desired magnitude response. For example, in the class E amplifier design[48] used in a transmitter, in order to obtain high harmonic suppression, it is usually necessary to place prescribed stopband zeros of the filter to suppress some harmonics to the desired weak level. Sometimes, the stopband of this type of harmonic-suppression filter is also shaped to match the desired magnitude response of the output stage amplifier. Similarly, in phase-locked loop applications, the loop filter is designed to
suppress the harmonics of the reference signal of the phase detector. Thus, it is preferred that the loop filter have prescribed zeros in the stopband to suppress these harmonics so that the sidebands in the output spectrum of the voltage-controlled oscillator be suppressed to a desired level.

As mentioned by Shenoi and Agrawal[15], usually the transmission requirements of telephone systems demand that the magnitude response, be flat within 0.5 dB over the voiceband (passband). However, the specification cannot be met even if the simplest analog filter with a single pole, is used to limit aliasing and the rest of filtering is done digitally. The difficulty arises because, with the present LSI technology, the actual time constant may vary from the desired value by as much as 100 percent. Thus, it is desired to design a digital filter with shaped passband response to compensate the response of the analog RC filter, so that the total frequency response from input to output will meet the requirements of the system.

In spectrum shaping filter design[49], it is desired to design a filter with a weighted error function in the passband or stopband. In these applications, the filtering operation of the signals for one frequency region is more important than that for other frequency regions, so a filter with higher stopband rejection in one frequency region than other frequency regions is desired. All of these situations lead to a constrained digital filter design problem.
A digital filter which meets the desired passband specifications exactly and exceeds the stopband specifications considerably (or vice versa), could be redesigned by prescribing part of its zeros (or poles) so that the resulting filter will have more attenuation for the signals with some particular frequencies or the filter will be more suitable for hardware implementation.

Shenoi and Agrawal [15] have proposed an algorithm to design constrained IIR filters. Their algorithm designs an elliptic filter with passband-shaped or stopband-shaped magnitude response. In order to meet the nonnegativity requirement of the design algorithm, they worked the approximation problem with squared magnitude of the filter instead of magnitude alone. However, their algorithm, even though constitutes a significant practical approach to the design of constrained digital filters, poses the same limitations of unconstrained filters described in Section 2.3.2.

2.5.2 Linear Phase FIR Filters with Constraints in Time Domain

In the field of LPFIR digital filter design, most of the methods are limited to finding the optimal filter without any restriction on the filter coefficients (for example, see [47]). The coefficients of an FIR filter correspond directly to its impulse response. In many applications, it is required to impose restrictions on the time response (i.e., filter coefficients) as well as on the frequency response when designing the filter. Rabiner [13] considered the design of a low-pass filter with a constraint on the step response of the filter, while maintaining
reasonable control over its frequency response. There are applications where some of the values of the impulse response have to be limited to zero. In this case, the filter design involves constructing the filter by considering the time and frequency responses at the same time. Here, we consider three applications of this type filters and briefly review the methods available for their design.

(a) **FIR Nyquist Filters with Zero Intersymbol Interference**

The design of a Nyquist filter for generating a band-limited pulse for data transmission with the minimum intersymbol interference has always been an important subject[50-53]. Due to the recent developments in advanced large-scale integrated technology, digital Nyquist filters are playing an important role in digital modem systems[54-57]. By digital design techniques, a pulse shaping transversal filter with an exact zero crossing impulse response, which corresponds to zero intersymbol interference, can be obtained. Fig.2.1(a) shows the block diagram of a digital Nyquist filter, where $X(z)$ is the transform of input data signals (i.e., input pulse train) whose speed is the Nyquist rate $F_n$ and the sampling rate for $H(z)$ and $Y(z)$ is $F_r$, where $F_r = M F_n$. $M$ an integer and $z = \exp(j2\pi F_r)$. The frequency response of the digital Nyquist filter will have normalized band edges[58]

$$F_p = \frac{1-\rho}{2M}, \quad (2.34a)$$

$$F_s = \frac{1+\rho}{2M}, \quad (2.34b)$$

where $\rho$ indicates the rolloff rate. The desired FIR Nyquist filter needs
to have an impulse response \( h(n) \) exactly zero crossing at the Nyquist rate except for one point \( n = K \). Fig. 2.1(b) shows an impulse response of a typical digital Nyquist filter.

**The Iterative Chebyshev Approximation Method**

Nakayama and Mizukami[57] proposed a two-step method to design the Nyquist filter described above. The first step is to design the optimal filter by approximating the desired frequency response directly using the Remez algorithm[1] without time constraint on the filter coefficients. The second step is to apply the iterative Chebyshev approximation method in [59] to design the desired Nyquist filter by using the coefficients of the optimal filter obtained in the first step as the initial values of the iterative nonlinear optimization algorithm for finding the Chebyshev solution, where the coefficients \( h(n) \) are modified to zero for \( (n-K) \) \( M = 0 \) and \( n \neq K \).

(b) **Half-Band and \( N \)-th Band FIR Filters**

A half-band filter is a filter whose impulse response is symmetrical about the time origin and crosses the zero-axis at times which are nonzero integral multiples of twice the sampling period and the cutoff frequency is in the middle of the available bandwidth[9, 60]. The half-band filter is an efficient tool for increasing or decreasing the sampling rate by a factor of 2 and is used in multirate digital filtering with nearly two-fold reduction in the computation rate as compared to the symmetric FIR filter of the same length.
Fig. 2.1(a) Block diagram of Nyquist filter.

Fig. 2.1(b) Impulse response of a Nyquist filter.

Fig. 2.1(c) Impulse response of a Class I partial response filter.
As described above, a half-band filter with odd length has a transfer function of the form

$$H(z) = \sum_{-(N-1)/2}^{(N-1)/2} h(n)z^{-n}, \quad (2.35)$$

where the impulse response \( h(n) \) has the property that

$$h(2i) = 0, \text{ for } i \neq 0. \quad (2.36)$$

Since the cutoff frequency is in the middle of the available bandwidth, the low-pass half-band filter satisfies the conditions

$$F_p + F_s = 0.5, \quad (2.37a)$$

and

$$\delta_p = \delta_s, \quad (2.37b)$$

where \( F_p \) is the normalized passband edge, \( F_s \) is the normalized stopband edge, \( \delta_p \) is the passband ripple and \( \delta_s \) is the stopband ripple. The constraint of (2.37a) closely matches the requirements of the filter required for decimation or interpolation by a factor of 2[9,60]. In these cases, a reduction in the computation rate of nearly 50 percent can indeed be achieved.

Mintzer[61] has extended the concept of half-band filters to \( N \)-th band filters for saving computations in decimation and interpolation by a factor other than 2. An \( N \)-th band filter has a transfer function in the form of (2.35), but the filter coefficients are subject to the following constraint:

\[ h(n_i) = 0 \quad \text{for } i \neq 0 \quad . \quad (2.38) \]

Similarly, by taking advantage of the zero filter coefficients, an \( N \)-th band filter can achieve a computational saving of nearly \( 1/N \) when compared to a general symmetrical FIR filter of the same length. It has been shown\[61\] that an \( N \)-th band filter with normalized passband, i.e., \( Nh(0) = 1 \), satisfies the condition:

\[ \sum_{k=0}^{N-1} H(F + \frac{k}{N}) = 1 \quad , \quad (2.39) \]

and has the stopbands

\[ [-F + \frac{k}{N}, F + \frac{k}{N}] \quad , \quad \text{for } k = 1, 2, \ldots, N-1 \quad , \quad (2.40) \]

and satisfies the condition

\[ \delta_p \leq (N-1)\delta_s \quad . \quad (2.41) \]

The band placement specifications of (2.40) closely match the requirements of the filter used for decimation or interpolation by \( N \) [60].

**The \( N \)-th Band Filter Design by the Remez Algorithm**

Mintzer\[61\] designed the suboptimal \( N \)-th band filters by using the computer program of McClellan, Parks and Rabiner\[1\]. The first step is to derive the ripple bound of \( N \)-th band filter from the ripples of the optimal filter and then design the prototype optimal filter, with a suitable choice of error weighting values which most closely reflect the filter requirements, by using the Remez algorithm in [1]. The second step of the design procedure is to obtain the \( N \)-th band filter from the
resulting prototype optimal filter by choosing the appropriate filter coefficients. Then check if the requirement (2.41) is satisfied. If not, return to the first step with a new choice of weighting values. This procedure is repeated until the requirement (2.41) is satisfied and the filter is the desired N-th band filter.

(g) Nonuniform Tap Spacings FIR Filters

Most of FIR filters are designed to have uniform tap spacings regardless of the filter's impulse response. But, in many cases the nonuniform tap spacing (NUTS) filters may constitute a better design than the optimal uniform tap spacing (UTS) filter with respect to the desired frequency response since the former could have smaller passband ripple and/or stopband ripple as compared to the latter for the same number of sample points. In an UTS FIR filter, the coefficients with small magnitude compared to other coefficients will make small contribution to the spectrum shaping of the filter. In particular, if the impulse response of an UTS filter crosses the zero-axis at sampling points, no spectrum shaping contribution can be obtained from these zero impulse response terms. Obviously, if these terms with essentially zero impulse response are forced to zero, the resulting NUTS filter with the same number of sample points will have smaller passband and/or stopband ripples than the corresponding UTS filter. Smith and Farden[62], by using the minimum mean-squared error technique, pointed out that if a filter is designed by thinning an UTS filter, the resulting NUTS filter could have a better frequency response than the corresponding UTS filter with the same
number of sample points.

2.6 Concluding Remarks

After a brief review of the basic digital filter design problem, various specific digital filter design methods described by other authors for solving the design problems considered in this dissertation have been presented in the present chapter. Their limitations of those methods have been pointed out. The algorithms developed in the present dissertation attempt to overcome these difficulties.
CHAPTER III

AN EFFICIENT ITERATIVE ALGORITHM FOR DESIGNING

OPTIMAL RECURSIVE DIGITAL FILTERS

3.1 Introduction

A new iterative algorithm is presented for the design of an optimal
IIR digital filter. This algorithm uses a short main program which
invokes the conventional Remez algorithm of McClellan et al.[1] for the
FIR filter design as a subroutine. As a consequence, the entire design
procedure is efficient and converges to the desired solution within a
few iterations.

The algorithm can design the filter in the form of an unconstrained
or constrained transfer function, where by a "constrained" transfer
function we mean one for which some of the zeros and/or poles are
prescribed as described in Section 2.4.1 of the preceding chapter.

The algorithm presented here designs iteratively the magnitude of
the numerator and the squared magnitude of the denominator of the filter
transfer function by directly calling the Remez algorithm to efficiently
solve the approximation problem in the filter design procedure. If the
filter is constrained, the constraint is incorporated in the weighting
and desired functions when calling the Remez algorithm. The filters
designed by the algorithm then automatically satisfy the constraint.
When designing the denominator, a parameter $d_{0.5}$, which is the desired squared magnitude of the denominator at the frequency $F = 0.5$, is introduced to control the ripple ratio of the filter, where by "ripple ratio" we mean the ratio of passband ripple to stopband ripple. It is found that the ripple ratio is proportional to $3/2$ power of $d_{0.5}$ for a filter possessing a wide transition band and for which the passband and stopband are not too narrow, and the passband and stopband ripples are not too small. Even when this $3/2$ power relation does not hold exactly, an iterative design procedure, which is based on the estimate of the required value of the parameter $d_{0.5}$ obtained from the $3/2$ power relation, can be used for the design of the IIR filter with a desired ripple ratio. This approach also allows one to find the minimum ripple ratio for the given orders of the filter and under the prescribed constraint. Any such filter with ripple ratio larger than this minimum value can be designed by our algorithm. Thus neither the passband nor stopband ripples are required to be prescribed when designing the filter.

Several examples illustrate the design procedure and the main computer program is listed in APPENDIX.

3.2 Problem Formulation

Let the transfer function $H(z)$ of a possibly constrained low-pass recursive digital filter be of the form
$$H(z) = \frac{T_s(z)N(z)}{T_p(z)D(z)} = \frac{T_s(z) \left( \sum_{k=0}^{N} b(k)z^{-k} \right)}{T_p(z) \left( \sum_{k=0}^{M} a(k)z^{-k} \right)},$$

(3.1)

where the functions $T_s(z)$ and $T_p(z)$ are the prescribed functions (or general constraint functions), and $N(z)$ and $D(z)$ are the parts in the transfer function to be designed. We assume that both functions $T_s(z)$ and $N(z)$ have all their zeros on the unit circle in the stopband, while the functions $T_p(z)$ and $D(z)$ have all their zeros inside the unit circle in the passband. The design of a filter is then reduced to the determination of $N(z)$ and $D(z)$, under the given constraint embodied by $T_s(z)$ and $T_p(z)$, such that the filter $H(z)$ will have a desired low-pass frequency response when evaluated along the unit circle.

Specifically, in Section 3.3, an iterative algorithm is proposed to design the filter with the general constraint as in (3.1). The particular types of constraint considered in this paper and their relation to the general constraint functions in (3.1) are listed as follows:

(A) **Unconstrained filter**: $T_s(z) = T_p(z) = 1$;

(B) **Constrained filter**:

(a) All the stopband zeros are prescribed: $T_p(z) = N(z) = 1$ and $T_s(z) =$ the function obtained from the prescribed stopband zeros;

(b) All the passband poles are prescribed: $T_s(z) = D(z) = 1$ and $T_p(z) =$ the function obtained from the prescribed passband poles;
(c) Only some of the stopband zeros are prescribed: \( T_p(z) = 1 \) and 
\( T_s(z) = \) the function obtained from the prescribed stopband zeros;

(d) Only some of the passband poles are prescribed: \( T_p(z) = 1 \) and 
\( T_p(z) = \) the function obtained from the prescribed passband poles;

(e) Only some of the stopband zeros and some of the passband poles are 
preserved: \( T_p(z) = \) the function obtained from the prescribed 
passband poles and \( T_s(z) = \) the function obtained from the 
prescribed stopband zeros;

(f) Stopband shaping: \( T_s(F) = T_p(F) = 1 \) in the passband, but 
\( T_s(F) = 1 \) and \( T_p(F) = Q(F) \) in the stopband, where \( Q(F) \) is the stop-
band shaping function defined only in the stopband;

(g) Passband shaping: If \( R(F) \) is a function of normalized frequency \( F \) 
in \([0,0.5]\), let us define the operators as \( \Delta_s \) and \( \Delta_p \) by

\[
\Delta_s R(F) = \begin{cases} 
1 & 0 \leq F \leq F_p \\
R(F) & F_s \leq F \leq 0.5
\end{cases}
\] (3.2a)

and

\[
\Delta_p R(F) = \begin{cases} 
R(F) & 0 \leq F \leq F_p \\
1 & F_s \leq F \leq 0.5
\end{cases}
\] (3.2b)

Then, the constraint of passband shaping is described by:
\( T_p(F) = 1 \) and \( T_s(F) = \Delta_p R(F) \), where \( R(F) \) is the prescribed function.
For cases (f) and (g), \( T_s(z) \) and \( T_p(z) \) are for shaping and thus they do 
not necessarily have zeros respectively in the stopband and passband.

In (3.1), \( N(z) \) corresponds to a linear phase FIR (LPFIR) filter
but \( D(z) \) does not. Let us define an FIR filter whose transfer function
is

\[ \tilde{D}(z) = D(z)D(z^{-1})z^{-M} = \sum_{m=0}^{2M} \tilde{b}(m)z^{-m}, \quad (3.3) \]

where

\[ \tilde{b}(m) = \tilde{b}(2M-m) = \sum_{j=0}^{M} b(j)b(j+M-m), \quad m = 0, 1, \cdots, M. \quad (3.4) \]

Equation (3.4) implies that \( \tilde{D}(z) \) is the transfer function of an LIPFIR filter since its coefficients are symmetrical. The amplitude response of \( \tilde{D}(z) \) is related to that of \( D(z) \) by

\[ |\tilde{D}(F)| = |D(F)D(-F)e^{-2j\pi MF}| = |D(F)|^2, \quad (3.5) \]

where, and henceforth, we use the abbreviated notation \( P(F) \) for \( P(e^{2j\pi F}) \) obtained from a given function \( P(z) \). Let \( \tilde{T}_p(z) \) be defined in terms of \( T_p(z) \) in a way analogous to that of \( \tilde{D}(z) \). Introduce a transfer function

\[ \tilde{H}(z) = \frac{T_p(z)N(z)}{\tilde{T}_p(z)\tilde{D}(z)}. \quad (3.6) \]

For convenience of notation, let \( T_s(F), T_p(F), N(F), D(F) \) and \( \tilde{D}(F) \) denote respectively only the magnitude parts of the transfer functions \( T_s(z), T_p(z), N(z), D(z) \) and \( \tilde{D}(z) \) evaluated along the unit circle. In this paper, it is assumed that both \( T_s(F) \) and \( T_p(F) \) are normalized to 1 at \( F = 0 \).

In the algorithm described in next section, we design the constrained filter \( H(z) \) in (3.1) by designing \( N(z) \) and \( \tilde{D}(z) \) instead of \( N(z) \) and \( D(z) \). After we have found \( \tilde{D}(z) \), then \( D(z) \) can be constructed.
directly from \( \tilde{\delta}(z) \) [19-20, 24]. The relation between the specifications of 
\( \Pi(z) \) and those of \( \tilde{\Pi}(z) \) is found to be [8]

\[
\delta_p = \frac{\tilde{\delta}_p}{2}, \quad (\text{if } \tilde{\delta}_p \ll 1),
\]

\[
\delta_s = \tilde{\delta}_s,
\]

where \( \delta_p \) and \( \delta_s \) are the passband ripple and stopband ripple respectively 
corresponding to the optimal solution obtained from the design procedure, and \( \delta_p \) and \( \delta_s \) are respectively the passband ripple and stopband 
ripple of \( |\Pi(F)| \), and the ripple ratio \( K \) is

\[
K = \frac{\delta_p}{\delta_s} = \frac{\tilde{\delta}_p}{\tilde{\delta}_s} = \frac{\tilde{\delta}_p}{2\tilde{\delta}_s}.
\]

3.3 Iterative Design Algorithm

Suppose \( N(z) \) and \( D(z) \) in (3.1) have orders \( N \) and \( M \) respectively.
Let the subscript \( k \) denote the \( k \)-th iteration (\( k = 1, 2, \ldots \)) in the fol-
lowing description of the algorithm. The first step in our algorithm is 
to design \( N(F) \) on the band \( A_s = [F_s, 0.5] \). In order to obtain a non-
trivial solution, it is necessary to have a starting point on the 
passband. Let us assign \( N_k(0) = d_0, 0 \) (which is assumed to be equal to 1) 
and equate the weighting value \( W_{0,0} \) for this point to a constant. In 
other words, we design the filter on \( \tilde{A}_s = [0]U A_s \). Since \( N(z) \) is an 
LPFIR filter, the Remez algorithm program of McClellan et al. [1] can be 
applied to design \( N(F) \) so that the stopband ripple is minimized with
respect to the given specifications. In order to design the filter on \( \tilde{A}_s \), it is necessary to take care of the function \( |\tilde{D}_{k-1}(F)|^{1/2} \) on \( A_s \). The approximation error function on \( A_s \) at the \( k \)-th iteration becomes (for the sake of brevity only \( N \) even is considered, a similar analysis being applicable to the case in which \( N \) is odd):

\[
E_{s,k}(F) = \frac{|T_{s,k}(F)|}{|T_{p}(F)||\tilde{D}(F)|^{1/2}_{k-1}[0 - N_k(F)]}
\]

\[
= \tilde{W}_{s,k}(F)[\tilde{D}_{s,k}(F) - \tilde{P}_{s,k}(F)] , \quad F \in A_s , \quad (3.10)
\]

where \( \tilde{P}_{s,k}(F), \tilde{D}_{s,k}(F) \) and \( \tilde{W}_{s,k}(F) \) are respectively the designed, desired, and weighting functions on \( A_s \). From (3.10), we have

\[
\tilde{W}_{s,k}(F) = \frac{|T_{s,k}(F)|}{|T_{p}(F)||\tilde{D}_{k-1}(F)|^{1/2}} , \quad F \in A_s , \quad (3.11a)
\]

\[
\tilde{P}_{s,k}(F) = N_k(F) , \quad F \in A_s , \quad (3.11b)
\]

\[
\tilde{D}_{s,k}(F) = 0 , \quad F \in A_s . \quad (3.11c)
\]

In designing \( N_k(F) \), the passband is considered as a transition band except for the very narrow band near the origin. In (3.11a), in order to remove the ill-conditioning, a small positive value is assigned to the weighting function when \( F \) is in the neighborhood of the zeros of \( T_s(F) \) so that the weighting function is a positive definite function. This assumption also applies to the same cases discussed later.
After designing $N_k(F)$ on $\tilde{A}_p$, we return to design $\tilde{D}_k(F)$ on the passband $A_p = [0, F^p]$. Recall that $\tilde{D}(z)$ is an LPFIR filter with all its zeros off the unit circle in the passband, so the Remez algorithm can be applied to design $\tilde{D}(z)$. But we include the point $F = 0.5$ when designing $\tilde{D}_k(F)$ as the point $F = 0$ when designing $N_k(F)$. For convenience, we denote the desired value at the point $F = 0.5$ by $d_{0.5}$ (Note that since $\tilde{D}(z)$ is of even order, it is always possible to have this assignment); and the weighting value $\tilde{W}_{0.5}$ (both are assumed constant). In other words, we design $\tilde{D}_k(F)$ on $\tilde{A}_p = A_p \cup [0, 0.5]$. However, it is necessary to take care of the function $|N_k(F)|^2$ on $A_p$. The approximation error function on $A_p$ at the $k$-th iteration becomes

$$E_{p,k}(F) = \frac{|T_{D}(F)|^2}{|T_{S}(F)|^2 |N_k(F)|^2} \left( \frac{|T_{D}(F)|^2 |N_k(F)|^2}{|T_{P}(F)|^2} - \tilde{D}_k(F) \right)$$

$$= \tilde{W}_{p,k}(F) [\text{Des}_{p,k}(F) - \tilde{P}_{p,k}(F)] , \quad F \in A_p \quad \text{(3.12)}$$

where $\tilde{P}_{p,k}(F)$, $\text{Des}_{p,k}(F)$ and $\tilde{W}_{p,k}(F)$ are respectively the designed, desired, and weighting functions on $A_p$. According to (3.12), we have

$$\tilde{W}_{p,k}(F) = \frac{|T_{D}(F)|^2}{|T_{S}(F)|^2 |N_k(F)|^2} , \quad F \in A_p \quad \text{(3.13a)}$$

$$\tilde{P}_{p,k}(F) = \tilde{D}_k(F) , \quad F \in A_p \quad \text{(3.13b)}$$

$$\text{Des}_{p,k}(F) = \frac{|T_{D}(F)|^2 |N_k(F)|^2}{|T_{P}(F)|^2} , \quad F \in A_p \quad \text{(3.13c)}$$
It is easy to see that, in the above iterative design procedure, the weighting function depends on the result of the previous iteration when designing \( N_k(F) \), and on the result of the current iteration when designing \( \tilde{D}_k(F) \), and the desired function also depends on the result of the current iteration when designing \( \tilde{D}_k(F) \). Usually, \( \tilde{D}_0(F) \), the initial value of \( \tilde{D}_k(F) \), is set equal to 1.

The Remez algorithm is applied to design (3.10) and (3.12) iteratively. A dense grid of frequency points is used to find the set of \( r+1 \) extremal frequencies where the approximation error function is forced to have magnitude \( \delta \) with alternating signs (for \( N(F) \) design, \( r = (N/2)+1 \) for even \( N \); and for \( \tilde{D}(F) \) design, \( r = M+1 \)). From the initial specifications of the problem, the requirement says that for the given set of extremal frequencies \([F_j, j = 0, 1, 2, \cdots, r]\), the following set of equations must be solved

\[
\tilde{w}_k(F_j) [\tilde{D}_k s_k(F_j) - \tilde{p}_k(F_j)] = (-1)^j \delta, \quad j = 0, 1, \cdots, r. \quad (3.14)
\]

When designing \( N_k(F) \), the stopband ripple \( \delta_{s,k} \) in (3.14), after the \( k \)-th iteration is completed, is calculated analytically as[16] (also see (3.8))

\[
\delta_{s,k} = \frac{a(0) d_0 + a(1) \tilde{D}_k s_k(F_1, k) + \cdots + a(r) \tilde{D}_k s_k(F_r, k)}{\frac{\tilde{w}_k}{a(0)} - \frac{\tilde{w}_k}{a(1)} + \cdots + (-1)^r \frac{\tilde{w}_k}{a(r)}}. \quad (3.15)
\]

By substituting the corresponding quantities in (3.11) into (3.15), we have
\[\delta_{s,k} = \frac{a(0) \cdot d_{0,0}}{\sum_{n=0}^{\infty} \frac{a(n) \cdot |T_{p}(F_{1,k})| \cdot |\tilde{D}_{k-1}(F_{1,k})|^{1/2}}{|T_{s}(F_{1,k})|}} + \ldots\]

\[\delta_{p,k} = \frac{a(n+1) \cdot |T_{p}(F_{N+1,k})| \cdot |\tilde{D}_{k-1}(F_{N+1,k})|^{1/2}}{|T_{s}(F_{N+1,k})|} + \ldots + (-1)^{N+1} \frac{a(N+1) \cdot |T_{p}(F_{N+1,k})| \cdot |\tilde{D}_{k-1}(F_{N+1,k})|^{1/2}}{|T_{s}(F_{N+1,k})|}\]  

(3.16)

where the subscript \(k\) denotes the corresponding values at the \(k\)-th iteration and \(\{F_{j,k}\} j = 1, 2, \ldots, r\), are the extremal frequencies on \(A_{s}\) after the \(k\)-th iteration is completed.

When designing \(\tilde{D}(F)\), the passband ripple \(\delta_{p,k}\) in (3.14) after the \(k\)-th iteration is completed, is calculated analytically as (also see (3.7))

\[\delta_{p,k} = \frac{a(r) \cdot d_{0,5} + a(0) \cdot \tilde{D}_{s, p, k}(F_{0,k}) + \ldots + a(r-1) \cdot \tilde{D}_{s, p, k}(F_{r-1,k})}{(-1)^{r} \frac{a(r) \cdot d_{0,5} + a(0) \cdot \tilde{D}_{s, p, k}(F_{0,k}) + \ldots + a(r-1) \cdot \tilde{D}_{s, p, k}(F_{r-1,k})}{\sum_{n=0}^{\infty} a(n) \cdot |T_{p}(F_{0,k})| \cdot |\tilde{D}_{s, p, k}(F_{0,k})|^{1/2}} + \ldots}} \]

(3.17)

By substituting the corresponding quantities in (3.13) into (3.17), we have

\[\delta_{p,k} = \frac{a(M+1) \cdot d_{0,5} + a(0) \cdot |T_{p}(F_{0,k})|^{2} \cdot |N_{s}(F_{0,k})|^{2}}{|T_{p}(F_{0,k})|^{2}} + \ldots\]

\[\frac{(-1)^{M+1} \frac{a(M+1) \cdot d_{0,5} + a(0) \cdot |T_{p}(F_{0,k})|^{2} \cdot |N_{s}(F_{0,k})|^{2}}{|T_{p}(F_{0,k})|^{2}} + \ldots}} \]
\[
\begin{align*}
\cdots + \frac{a(M) \, |T_s(F_{M,k})|^2 \, |N_p(F_{M,k})|^2}{|T_p(F_{M,k})|^2} \\
\cdots + (-1)^N \frac{a(M) \, |T_s(F_{M,k})|^2 \, |N_p(F_{M,k})|^2}{|T_p(F_{M,k})|^2}
\end{align*}
\]

where the subscript \(k\) denotes the corresponding values at the \(k\)-th iteration, and \(\hat{F}_{j,k}, j = 0,1,\ldots,r-1\) are the extremal frequencies on \(A_p\) after the \(k\)-th iteration is completed, and

\[
a(i)_k = \prod_{l=0}^{r} \frac{1}{\cos(2\pi F_{i,k}) - \cos(2\pi F_{1,k})}, \quad i = 0,1,\ldots,r, \quad (3.19a)
\]

and

\[
\tilde{a}(i)_k = \prod_{l=0}^{r} \frac{1}{\cos(2\pi \hat{F}_{i,k}) - \cos(2\pi \hat{F}_{1,k})}, \quad i = 0,1,\ldots,r. \quad (3.19b)
\]

After the 1st iteration, the function \(|N_1(F)|\) will change monotonically from the point \(F = 0\) (where \(|N_1(F)|\) has a desired value 1) to the stopband. But in the stopband, \(|N_1(F)|\) will have the approximation error weighted by \(|T_s(F)|/|T_p(F)|\). \(|D_1(F)|\) will follow \(|T_s(F)|/|N_1(F)|/|T_p(F)|\) in the passband and then change monotonically from the passband edge \(F = F_p\) to the point \(F = 0.5\). After the 2nd iteration, the function \(|N_2(F)|\) will have approximately the same shape as \(|N_1(F)|\) in the passband. In the stopband, \(|N_2(F)|\) will have the approximation error weighted by \(|T_s(F)|/|T_p(F)|\). Same as \(|D_1(F)|\), the function \(|D_2(F)|\) will follow \(|T_s(F)|/|N_2(F)|/|T_p(F)|\) in the passband and then change monotonically from passband edge \(F = F_p\) to the point \(F = 0.5\). As mentioned above,
\[ |N_2(F)| \] does not change appreciably from \[ |N_1(F)| \] in the passband so that \[ |D_2(F)| \] will not have an appreciable change from \[ |D_1(F)| \]. As the algorithm advances, \[ |N_{k+1}(F)| \] will not differ appreciably from \[ |N_k(F)| \], and \[ |D_{k+1}(F)| \] will not differ appreciably from \[ |D_k(F)| \] either for \( k > 2 \). As the iterative design algorithm proceeds further, it will converge and the optimal solution will occur. An algorithm, similar to that in [36], has been designed to detect the occurrence of the optimal solution and then stop the design procedure if the optimal solution has occurred. Good convergence property of our general algorithm has been found from our experience with it. The high convergence speed of the algorithm can also be seen from the independence of passband and stopband in the design procedure. When the design algorithm converges, the ripples \( \delta_{s,k} \) and \( \delta_{p,k} \) must also converge to their final values \( \delta_s \) and \( \delta_p \) respectively, and (3.16) and (3.18) become

\[
\delta_s = \frac{a(0) \delta_0}{a(0) - \frac{a(1)}{W_0} \frac{T_s(F_s)}{|D(F_s)|}^{1/2}} + \cdots
\]

\[
\left( \frac{N_{+1}}{2} \right) \frac{a(N_{+1})}{T_s(F_{+1})} \frac{D(F_{+1})^{1/2}}{T_s(F_{-1})} + \cdots + (-1)^{N+1} \frac{a(N_{-1})}{T_s(F_{-1})} \frac{D(F_{-1})^{1/2}}{T_s(F_{+1})}
\]

(3.20)

and
\[
\delta_p = \frac{1}{2} a(M+1)d_{0.5} + \frac{\tilde{a}(0)|T_p(\tilde{F}_0)|^2|N(\tilde{F}_0)|^2}{|T_p(\tilde{F}_0)|^2} + \ldots
\]

\[
\delta_p = \frac{1}{2} a(M+1)d_{0.5} + \frac{\tilde{a}(0)|T_p(\tilde{F}_0)|^2|N(\tilde{F}_0)|^2}{|T_p(\tilde{F}_0)|^2} + \ldots
\]

\[
\frac{\tilde{a}(M)|T_s(\tilde{F}_M)|^2|N(\tilde{F}_M)|^2}{|T_s(\tilde{F}_M)|^2} + \frac{\tilde{a}(M)|T_s(\tilde{F}_M)|^2|N(\tilde{F}_M)|^2}{|T_s(\tilde{F}_M)|^2} + \ldots
\]

\[
\cdot (-1)^M \frac{\tilde{a}(M)|T_s(\tilde{F}_M)|^2|N(\tilde{F}_M)|^2}{|T_s(\tilde{F}_M)|^2}
\]

\[
\delta_s = \frac{a(0)d_{0.0}}{W_{0.0}} + f(d_{0.5})
\]

\[
\delta_p = \frac{1}{2} a(M+1)d_{0.5} + a
\]

\[
f(d_{0.5}) = -\frac{a(1)|T_p(F)|\left|\tilde{D}(F)\right|^{1/2}}{|T_s(F_1)|} + \ldots
\]

\[
\cdot (-1)^{N+1} \frac{a(N+1)|T_p(F_N)|\left|\tilde{D}(F_N)\right|^{1/2}}{|T_s(F_N)} + \ldots
\]

\[
a = \frac{\tilde{a}(0)|T_p(\tilde{F}_0)|^2|N(\tilde{F}_0)|^2}{|T_p(\tilde{F}_0)|^2} + \frac{\tilde{a}(M)|T_p(\tilde{F}_M)|^2|N(\tilde{F}_M)|^2}{|T_p(\tilde{F}_M)|^2} + \ldots
\]
\[
\beta = (-1)^{(M+1)} \frac{w_{0.5}^{(M+1)}}{w_{0.5}} + \frac{a(0)|T_s(F_0)|^2|N(F_0)|^2}{|T_p(F_0)|^2} + \ldots \\
\ldots + (-1)^M \frac{\bar{a}(M)|T_s(F_M)|^2|N(F_M)|^2}{|T_p(F_M)|^2}.
\] (3.26)

From (3.9), (3.22) and (3.23), the ripple ratio \( K \) can be expressed as

\[
K = \frac{\delta_d}{\delta_s} = \frac{1}{2} \left[ \frac{\bar{a}(M+1)d_{0.5} + \bar{a}(0) + f(d_{0.5})}{a(0)d_{0.5}} \right].
\] (3.27)

Given a \( d_{0.5} \), we obtain a corresponding \( K \) value from (3.27). Suppose \( d_{0.5} \) is changed to \( d'_{0.5} \). Then the ripple ratio is changed from \( K \) to \( K' \)

\[
K' = \frac{\delta_d}{\delta_s} = \frac{1}{2} \left[ \frac{\bar{a}'(M+1)d_{0.5} + \bar{a}'(0) + f(d'_{0.5})}{a'(0)d_{0.5}} \right].
\] (3.28)

The \( a(0) \) and \( \bar{a}(M+1) \) in (3.27) are functions of extremal frequencies in the stopband and passband respectively, and \( f(d_{0.5}) \) is a function of extremal frequencies in the stopband and of the \( |T_s(F)|, |T_p(F)| \) and \( |\bar{B}(F)|^{1/2} \) values evaluated at these extremal frequencies. The \( \alpha \) and \( \beta \) values are functions of extremal frequencies in the passband and the \( |T_s(F)|^2, |T_p(F)|^2 \), and \( |N(\bar{e})|^2 \) evaluated at these extremal frequencies. Suppose \( d_{0.5} \) is changed to \( d'_{0.5} \). If we assume that the change of \( d_{0.5} \) is small compared to \( d_{0.5} \), the changes of positions of extremal frequencies of the optimal solution in the passband and stopband due to the change in the \( d_{0.5} \) value have negligible effects on \( \alpha, \beta, f(d_{0.5}), a(i)'s \) and \( \bar{a}(i)'s \). Then the \( |\bar{B}(F)| \) values evaluated in the stopband will be
changed accordingly, but the change of $|N(F)|$ in the passband will be
negligible if the filter is not a very narrow stopband filter. Recall
that both $T_s(F)$ and $T_p(F)$ are prescribed functions and do not change due
to the change in $d_{0.5}$. Thus the ripple ratio $K$ in (3.27) only depends
on $d_{0.5}$ through the terms $\tilde{a}(M+1)d_{0.5}$ and $f(d_{0.5})$, i.e., we have the fol-
lowing approximate relation

$$K = \text{function of } d_{0.5} \text{ only}$$

$$= \frac{1}{2} \left[ \frac{a(M+1)d_{0.5}}{\beta} + \frac{a(0)}{a(0)d_{0.5}} + f(d_{0.5}) \right]. \quad (3.29)$$

Suppose a filter with a specific ripple ratio $K$ is desired. The
required $d_{0.5}$ to obtain the desired $K$ can be estimated by (3.29). Up to
this point, we do not have a closed form expression for $f(d_{0.5})$ in terms
of $d_{0.5}$ only. Hence, we recalculate the corresponding $d_{0.5}$ according to
(3.29) and design the filter again.

If the stopband is not too narrow, we have the following approxima-
tions:

$$|\tilde{B}(F_1)|^{1/2} \approx \left( \frac{d_{0.5}}{d_{0.5}} \right)^{1/2} |\tilde{B}(F_1)|_{d_{0.5}}^{1/2}, \quad |\tilde{B}(F_2')|^{1/2} \approx \left( \frac{d_{0.5}}{d_{0.5}} \right)^{1/2} |\tilde{B}(F_2')|_{d_{0.5}}^{1/2},$$

$$\ldots, |\tilde{B}(F_2(N+1))|^{1/2} \approx \left( \frac{d_{0.5}}{d_{0.5}} \right)^{1/2} |\tilde{B}(F_2(N+1))|_{d_{0.5}}^{1/2}, \quad (3.30)$$

where $|\tilde{B}(F_i)|_{d_{0.5}}$ denotes the optimal $|\tilde{B}(F)|$ evaluated at the extremal
$d_{0.5}$.
frequency \( F = F_i \) \((i = 1, 2, \dots, (N/2)+1)\) when the desired value of \( |\tilde{D}(F)| \) at the point \( F = 0.5 \) is \( d_{0.5} \). Combining (3.24) and (3.30) and assuming reasonable shapings of \(|T_s(F)|\) and \(|T_p(F)|\), we have

\[
f(d_{0.5}) = c_0 (d_{0.5})^{1/2},
\]

(3.31)

where \( c_0 \) is a constant. By substituting (3.31) into (3.29), we have

\[
K = \frac{1}{2} \left[ \frac{a(M+1)d_{0.5} + a_{0} + c_0 (d_{0.5})^{1/2}}{\beta} \right]
\]

(3.32)

Now, given the desired \( K \), the required \( d_{0.5} \) can be obtained by solving (3.32).

In particular, if the desired filter has a rather wide transition band, not too narrow passband and stopband, and not too small passband and stopband ripples, so that

\[
|a(M+1)|d_{0.5} \gg a,
\]

(3.33)

then the sum of the terms \( a(i)|T_p(F_i)||\tilde{D}(F_i)|^{1/2}/|T_s(F_i)| \), \( i = 1, 2, \dots, (N/2+1) \) (assuming \( N \) even) in the denominator of (3.20) is large compared to the value \( a(0)/W_{0.0} \), and assuming reasonable shapings of \(|T_s(F)|\) and \(|T_p(F)|\) (also note that all the terms included in \( f(d_{0.5}) \) have the same sign) we have

\[
f(d_{0.5}) \gg \frac{a(0)}{W_{0.0}}.
\]

(3.34)

Under conditions (3.33) and (3.34), the ripple ratio \( K \) becomes
The ripple ratio $K$ is defined according to the second member of (3.35),

$$K = \frac{1}{2} \frac{a(M+1)}{\beta} \frac{c_0(d_{0.5})^{1/2}}{a(0)d_{0.0}} = c_1(d_{0.5})^{3/2}$$

where $c_1$ is a constant defined according to the second member of (3.35). Equation (3.35) tells us that ripple ratio is a function of $d_{0.5}$ only, i.e. the ripple ratio $K$ is proportional to the $3/2$ power of $d_{0.5}$. With (3.35), the desired $d_{0.5}$ can be easily found if the desired $K$ is given, and then the filter with the desired ripple ratio can be obtained in one step.

It is important to note that even if, in the general filter design, the ripple ratio $K$ does not follow (3.35), this relation still provides us with a good estimate of $d_{0.5}$ for the desired $K$, which can be used in the design procedure. Given the desired $K$, an estimate of $d_{0.5}'$ is obtained from (3.35), and then a filter with this estimated $d_{0.5}'$ is designed. The resultant $K'$ from the first design cycle, although not exactly the desired $K$, will be close to the desired $K$. Then the same procedure is repeated to obtain the estimated $d_{0.5}''$ value based on the filter with ripple ratio $K'$, and the filter design procedure is repeated again to design the filter with $d_{0.5}'''$. The resultant ripple ratio $K''$ from the second design cycle will be even closer to the desired $K$. By this iterative procedure, a filter with the desired ripple ratio $K$ can be obtained within a few design cycles.
3.4 Minimum Ripple Ratio and Optimality Properties

Being different from the elliptic filters, the filter designed by our algorithm can have more zeros than poles. This flexible adjustment property of the filter's numerator and denominator allows this class of filters to have more stopband attenuation than the elliptic filter with the same sum of numerator and denominator as discussed in [34].

In this section, the minimum passband ripple property and the optimality property of the unconstrained filter designed by the proposed algorithm are discussed.

(a) Minimum Passband Ripple Property

Since the constrained filters designed by our algorithm described above will have all its zeros on the unit circle, this fact places a lower limit on the passband ripple of the filter. In other words, given orders and band edges of the filter and the prescribed constraint, there exists a minimum ripple ratio for which the filter can be designed by the present algorithm. Recall that each time we design the denominator, the algorithm will yield a passband magnitude response of \( H(z) \) with the minimum attainable passband ripple and each time we design the numerator, the algorithm will result in a stopband magnitude response of \( H(z) \) with maximum attainable stopband attenuation for a given \( d_{0.5} \) under the given constraint. For filters of even \( M \), it is always possible, without imposing a restriction at \( F = 0.5 \), to design the filter with the minimum
\( d_{0.5} \) (or the minimum ripple ratio) for given filter orders, band edges, and constraint. For filters of odd \( N \), the minimum \( d_{0.5} \) can be estimated by means of a filter of \( N+1 \). If the minimum ripple ratio is smaller than the minimum attainable ripple ratio, then the order \( N \) should be increased.

(h) **Optimality Property**

For the unconstrained filter, our algorithm designs \( N(F) \) with the Remez algorithm on \( A_s \). If the order \( N \) of \( N(z) \) is odd, then according to the alternation theorem, there are at least \( (N+1)/2+1 \) extremal frequencies on \( A_s \). Therefore, there are at least \( (N+1)/2 \) extremal frequencies on \( A_s \), and \( H(F) = |N(F)/|D(F)| | \) will oscillate around zero between \(-\delta_s \) and \(+\delta_s \) at least \((N+1)/2\) times, or \(|H(F)|^2 \) will oscillate around \( \delta_s^2/2 \) between zero and \( \delta_s^2 \) at least \( N+1 \) times with ripple \( \delta_s^2/2 \). If the order \( N \) is even, then there are at least \((N/2)+2\) extremal frequencies on \( A_s \). Therefore, there are at least \((N/2)+1\) extremal frequencies on \( A_s \), and \( H(F) \) will oscillate around zero between \(-\delta_s \) and \(+\delta_s \) at least \((N/2)+1\) times. Hence \(|H(F)|^2 \) will oscillate around \( \delta_s^2/2 \) between zero and \( \delta_s^2 \) at least \( N+1 \) times with ripple \( \delta_s^2/2 \). Furthermore, when designing \( D(F) \), there are at least \( M+2 \) extremal frequencies on \( A_p \). Therefore there are at least \( M+1 \) extremal frequencies on \( A_p \), and then \(|H(F)|^2 \) will oscillate between \( 1 - \delta_p \) and \( 1 + \delta_p \) at least \( M+1 \) times with ripple \( \delta_p \). We thus conclude that \(|H(F)|^2 \) will have at least \( N+M+2 \) extremal frequencies on \( A_t = [0,F_p] U [F_s,0.5] \). In other words, our design algorithm will yield the optimal solution under the specification of the filter in the
minimax sense[34]. We thus have the following result.

**Proposition**

The algorithm described in Section 3.3 will yield the optimal solution $|H(F)|^2$ in the minimax sense to the approximation problem where the desired values are one on $A_p$ with ripple $\tilde{\delta}_p$ (if $\tilde{\delta}_p \ll 1$) and $\tilde{\delta}_s^2/2$ on $A_s$ with ripple $\tilde{\delta}_s^2/2$ and has an appropriate ripple ratio $K$ to take into account the difference between the ripples. The solution is optimal because the error function will have at least $N+M+2$ alternations on $A_t$.

### 3.5 Design Examples

The iterative design algorithm described in Section 3.3 has been applied to design the filters with constraints described in Section 3.2. Several design examples are given below to illustrate the design procedure described in preceding sections. The first example concerns the unconstrained filter design, Example 2 to Example 5 consider the design of filters with constraints representing various passband and/or stopband shapings, while the last example designs a constrained filter with high stopband attenuation. Coefficients of some of the filters designed in this section are listed in Table 3.1 for reference. A DEC PDP-11/55 computer with double precision was used in the calculations.

**Example 1:**

An unconstrained filter with $N = 8$, $M = 7$, $F_p = 0.2$ and $F_s = 0.25$ was designed.
### TABLE 3.1
Filter coefficients $a(k)$'s and $b(k)$'s as defined in equation (3.1)
for example 3.2 through 3.6.

<table>
<thead>
<tr>
<th>filters coefficients</th>
<th>Example 3.2</th>
<th>Example 3.3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>filter 1</td>
<td>filter 2</td>
</tr>
<tr>
<td>$a(0)$</td>
<td>0.669328e+01</td>
<td>0.689124e+01</td>
</tr>
<tr>
<td>$a(1)$</td>
<td>-0.172626e+02</td>
<td>-0.176059e+02</td>
</tr>
<tr>
<td>$D(z)$</td>
<td>0.205433e+02</td>
<td>0.205586e+02</td>
</tr>
<tr>
<td>$a(3)$</td>
<td>-0.120621e+02</td>
<td>-0.117307e+02</td>
</tr>
<tr>
<td>$a(4)$</td>
<td>0.308703e+01</td>
<td>0.286540e+01</td>
</tr>
<tr>
<td>$b(0)$</td>
<td>0.439366e+00</td>
<td>0.427425e+00</td>
</tr>
<tr>
<td>$b(1)$</td>
<td>-0.333459e+00</td>
<td>-0.297446e+00</td>
</tr>
<tr>
<td>$N(z)$</td>
<td>0.749705e+00</td>
<td>0.703936e+00</td>
</tr>
<tr>
<td>$b(3)$</td>
<td>-0.333459e+00</td>
<td>-0.297446e+00</td>
</tr>
<tr>
<td>$b(4)$</td>
<td>0.439366e+00</td>
<td>0.427425e+00</td>
</tr>
<tr>
<td></td>
<td>Example 3.4</td>
<td>Example 3.5</td>
</tr>
<tr>
<td>$a(0)$</td>
<td>0.797337e+01</td>
<td>0.461613e+01</td>
</tr>
<tr>
<td>$a(1)$</td>
<td>-0.210778e+02</td>
<td>-0.113012e+02</td>
</tr>
<tr>
<td>$D(z)$</td>
<td>0.253584e+02</td>
<td>0.133550e+02</td>
</tr>
<tr>
<td>$a(3)$</td>
<td>-0.151104e+02</td>
<td>-0.772920e+01</td>
</tr>
<tr>
<td>$a(4)$</td>
<td>0.387212e+01</td>
<td>0.208729e+01</td>
</tr>
<tr>
<td>$b(0)$</td>
<td>0.749274e+00</td>
<td>0.895299e+00</td>
</tr>
<tr>
<td>$b(1)$</td>
<td>-0.126082e+01</td>
<td>-0.167307e+01</td>
</tr>
<tr>
<td>$b(2)$</td>
<td>0.198793e+01</td>
<td>0.235235e+01</td>
</tr>
<tr>
<td>$b(3)$</td>
<td>-0.126082e+01</td>
<td>-0.167307e+01</td>
</tr>
<tr>
<td>$N(z)$</td>
<td>0.749274e+00</td>
<td>0.895299e+00</td>
</tr>
<tr>
<td>$b(4)$</td>
<td>0.749274e+00</td>
<td>0.895299e+00</td>
</tr>
<tr>
<td>$b(5)$</td>
<td>0.215464e+00</td>
<td></td>
</tr>
<tr>
<td>$b(6)$</td>
<td>0.112454e+00</td>
<td></td>
</tr>
<tr>
<td>$b(7)$</td>
<td>0.345356e+01</td>
<td></td>
</tr>
<tr>
<td>$b(8)$</td>
<td>0.478389e-02</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.2 (Example 3.1)

The relation between \( d_05 \), corresponding ripples and ripple ratios of filters with \( N = 8, M = 7, F_p = 0.2 \) and \( F_s = 0.25 \).

<table>
<thead>
<tr>
<th>( d_05 )</th>
<th>( \delta_p )</th>
<th>( \delta_s )</th>
<th>( \delta_p / \delta_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50.0</td>
<td>0.000164</td>
<td>0.001642</td>
<td>0.10</td>
</tr>
<tr>
<td>100.0</td>
<td>0.000317</td>
<td>0.001169</td>
<td>0.27</td>
</tr>
<tr>
<td>500.0</td>
<td>0.001541</td>
<td>0.000526</td>
<td>2.93</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.003071</td>
<td>0.000372</td>
<td>8.25</td>
</tr>
<tr>
<td>5000.0</td>
<td>0.015310</td>
<td>0.000156</td>
<td>91.95</td>
</tr>
<tr>
<td>8000.0</td>
<td>0.024486</td>
<td>0.000132</td>
<td>185.92</td>
</tr>
<tr>
<td>10000.0</td>
<td>0.030655</td>
<td>0.000118</td>
<td>260.23</td>
</tr>
</tbody>
</table>
The algorithm was applied to design the filter with different \( d_{0.5} \) values. The passband ripples and stopband ripples of the resultant filters and the corresponding \( d_{0.5} \) and ripple ratio \( K \) are listed in Table 3.2. It is easy to see that the ripple ratio \( K \) is approximately proportional to \( 3/2 \) power of \( d_{0.5} \).

Suppose we have already designed a filter with \( d_{0.5} = 500.0 \) (corresponding to \( K = 2.93 \)) and it is desired to design a filter with \( K = 10.0 \). The required \( d_{0.5}^{' \prime} \) value to obtain a filter with this desired \( K \) value was found to be 1133.9. With this \( d_{0.5} \) value, the resultant filter has a ripple ratio \( K = 9.98 \) which is close to the desired value.

**Example 2:**

In this example we illustrate, in some detail, the various points involved in the design procedure. A passband-shaped filter with parameters \( M = 4 \), \( N = 4 \), \( F_p = 0.125 \), \( F_s = 0.15625 \) was designed to compensate the passband droop of an FIR filter with a simple zero at \( F_0 = 0.25 \).

The FIR filter with a single zero at \( F_0 = 0.25 \) has a transfer function \( R(z) = 0.5 + 0.5 z^{-2} \) and magnitude function \( R(F) = \cos(2\pi F) \). From the discussions in Section 3.2, it is required to design a filter with constraints \( T_s(F) = \Delta R(F) \) and \( T_p(F) = 1 \). By applying the method of finding the minimum passband ripple with respect to given orders and constraint of the filter, the minimum \( \delta_p \), \( \delta_s \) and ripple ratio \( K \) were found to be 0.0003123, 0.22919 and 0.00136 respectively. So the filter with ripple ratio larger than 0.00136 can be designed with the present
Fig. 3.1(a) Amplitude response of $N(z)/D(z)$ of filter 1 designed in Example 2.

Fig. 3.1(b) Amplitude response of filter 1 designed in Example 2.
algorithm. Before proceeding to design a filter with a desired ripple ratio, we design a filter with \( d_{0.5} = 1000.0 \). The \( \delta_p \), \( \delta_s \) and \( K \) are found to be 0.01072, 0.07189 and 0.149 respectively. Suppose a filter with ripple ratio \( K = 1.0 \) is desired. By applying our design procedure, a filter (called filter 1 in Table 3.1) with \( \delta_p = 0.03879 \) (0.33 dB), \( \delta_s = 0.03848 \) and \( K = 1.008 \) is obtained after the second design cycle. This ripple ratio is very close to the desired value. The reason why the iterative design algorithm converges so fast is that its ripple ratio approximately follows the \( 3/2 \) power relation of (3.35). The amplitude responses of \( N(z)/D(z) \) and \( H(z) \) evaluated along the unit circle are shown in Figs.3.1(a) and (b) respectively.

As mentioned in Section 3.3, the filter with weighted error in the passband can also be designed by the proposed algorithm. The amplitude response of the filter whose passband ripple is weighted by \( 1/N(F) \) (called filter 2 in Table 3.1) is shown in Fig.3.1(c) with \( \delta_p = 0.93 \) dB.

If the constraint of this example is replaced by \( T_s(F) = R(F), T_p(F) = 1, \) and our algorithm is applied to design the filter again, the amplitude response of the resultant filter (called filter 3 in Table 3.1) with \( \delta_p = 0.33 \) dB is shown in Fig.3.1(d). By comparing the responses of Figs.3.1(b) and (d), it is found that the latter has a lower maximum stopband ripple but higher minimum stopband ripple than the former (Note that both filters have the same \( \delta_p \). The reason is that, in the latter filter design, we take care of the weighting of \( R(F) \) in the stopband so that the design algorithm will try to
Fig. 3.1(c) Amplitude response of the filter 2 designed in Example 2.

Fig. 3.1(d) Amplitude response of the filter 3 designed in Example 2.
minimize the maximum value of the stopband ripple by sacrificing the minimum value of the stopband ripple.

Example 3:

A filter with $M = 4$, $N = 0$, $F_p = 0.125$, $F_s = 0.15625$, was designed with three stopband zeros prescribed at $F_0 = 0.1778$, $F_1 = 0.25$ and $F_2 = 0.3125$.

The transfer function $R(z)$ of the filter can be found from the given prescribed zeros. This is equivalent to designing a filter with all stopband zeros prescribed, i.e., $T_p(z) = N(z) = 1$, and $T_s(z) = R(z)$. Our algorithm was applied to design the filter. After having the minimum $d_{0.5}$ value, a filter with $d_{0.5} = 1000.0$ was designed and $\delta_p$ was found to be 0.068 dB. The amplitude response of the filter and its equiripple behavior in the passband are shown in Fig.3.2.

Example 4:

A filter with parameters $M = 4$, $N = 4$, $F_p = 0.125$, $F_s = 0.15625$ was designed for a stopband shaping of $1/[(\tan(\pi F))^2]$.

The constraint of this problem implies that it is required to design a filter with $T_s(F) = A_s Q(F)$, where $Q(F) = 1/[(\tan(\pi F))^2]$ (Note that in this case $T_s(z)$ is used only for shaping the stopband response and is not included in the transfer function of the filter). A filter designed with $d_{0.5} = 5000.0$ has $\delta_p = 0.46$ dB. The amplitude response of the resultant filter is shown in Fig.3.3.
Fig. 3.2 Amplitude response of the filter designed in Example 3.

Fig. 3.3 Amplitude response of the filter designed in Example 4.
Example 5:

A filter with parameters $M = 4$, $N = 4$, $F_p = 0.125$, $F_s = 0.15625$ was designed with two prescribed passband poles at $F_0 = 0.05625$ (with radius 0.52) and $F_1 = 0.1125$ (with radius 0.7) and one prescribed stopband zero at $F_2 = 0.3125$.

This filter has prescribed poles in the passband and zeros in the stopband. The requirement is to design a filter with constraint functions $T_s(z)$ and $T_p(z)$, where $T_s(z) = 0.36162 + 0.27677z^{-1} + 0.36162z^{-2}$ and $T_p(z)$ is derived from the prescribed poles as $T_p(z) = (3.39360 - 3.31123z^{-1} + 0.91763z^{-2})(2.35055 - 2.50232z^{-1} + 1.15177z^{-2})$. The filter designed with $d_{0.5} = 1528.0$ has ripple ratio $K = 10.0$ and $\delta_p = 0.27$ dB. The amplitude response of the resultant filter is shown in Fig.3.4.

Example 6:

In order to show that the proposed algorithm can be applied to design a constrained filter with high stopband attenuation, a filter with $M = 8$, $N = 4$, $F_p = 0.25$, $F_s = 0.4$, was designed to compensate the droop of an FIR filter with a simple zero at $F_0 = 0.4375$.

The transfer function of the filter with zero at $F = 0.4375$ is $R(z) = 0.25989 + 0.48022z^{-1} + 0.25989z^{-2}$. It is required to design the filter with $T_s(F) = \Delta R(F)$ and $T_p(F) = 1$. The filter designed with $d_{0.5} = 2.903$ has ripple ratio $K = 1.0 \times 10^5$ and $\delta_s = -125.1$ dB. The amplitude response of the resultant filter is shown in Fig.3.5.
Fig. 3.4 Amplitude response of the filter designed in Example 5.

Fig. 3.5 Amplitude response of the filter designed in Example 6.
3.6 Conclusions

For unconstrained and constrained IIR filters, our iterative algorithm designs the filter directly without guessing the passband ripple or stopband ripple. As compared to the algorithms in [15,34-35], the proposed algorithm has a second advantage, namely the capability of finding the minimum ripple ratio for the given order and band edges of the filter so that a solution can always be found. The third advantage is that it can design the filter with a desired ripple ratio as in the design algorithm of LPFIR filters by McClellan et al.[1]. For the benefit of potential users, we have listed the filter design computer program in the appendix. An extension of the approach used here to the design of nonlinear phase FIR digital filters is described in next chapter.
CHAPTER IV

A DESIGN ALGORITHM FOR OPTIMAL LOW-PASS NONLINEAR PHASE
FIR DIGITAL FILTERS

4.1 Introduction

In this chapter, an algorithm is proposed for the design of nonlinear phase FIR (NLPFIR) filters. The design rationale is similar to that in our iterative algorithm for optimal recursive filter design described in the preceding chapter.

In Section 4.2, the approximation problem is formulated from filter specifications. This leads to the development of the design algorithm. In Section 4.3, the iterative design algorithm is described. In Section 4.4, the minimum ripple ratio property is discussed and the optimality of the solution is established. The algorithm is extended to the design of a constrained filter in Section 4.5. Section 4.6 contains an analysis of the roundoff noise of the cascade filter implementation. Several examples are given in Section 4.7. Concluding remarks are presented in Section 4.8.
4.2 Formulation of the Approximation Problem

Suppose it is desired to design a low-pass NLPFIR filter with transfer function \( H(z) \) of order \( N \), and phase response unspecified. Let \( H_p(z) \) and \( H_s(z) \) be the transfer functions which include all zeros of \( H(z) \) respectively in the passband and stopband and \( h_p(n) \)'s and \( h_s(n) \)'s their corresponding impulse responses. Then \( H(z) \) can be decomposed as

\[
H(z) = \sum_{n=0}^{N} h(n)z^{-n} = H_p(z)H_s(z) = (\sum_{n=0}^{N_p} h_p(n)z^{-n})(\sum_{n=0}^{N_s} h_s(n)z^{-n}),
\]

where \( N_p \) and \( N_s \) are the number of zeros of \( H(z) \) respectively in the passband and stopband. Define a filter whose transfer function is

\[
\tilde{H}_p(z) = H_p(z)H_p(z^{-1})z^{-N_p} = \sum_{n=0}^{2N_p} \tilde{h}_p(n)z^{-n},
\]

where

\[
\tilde{h}_p(n) = h_p(2N_p-n) = \sum_{j=0}^{N_p} h_p(j)h_p(j+N_p-n), n = 0, 1, \ldots, N_p.
\]

The amplitude response of \( \tilde{H}_p(z) \) evaluated along the unit circle is described by

\[
|\tilde{H}_p(F)| = |H_p(F)||H_p(-F)e^{-j2\pi N_F}| = |H_p(F)|^2,
\]

where, and henceforth, we use the abbreviated notation \( P(F) \) for \( P(e^{j2\pi F}) \) obtained from a given function \( P(z) \) evaluated along the unit circle. Introduce a transfer function \( \tilde{H}(z) \) by

\[
\tilde{H}(z) = H_s(z)\tilde{H}_p(z).
\]
For designing the NLPFIR filter $H(z)$ in (4.1) with passband ripple $\delta_p$ and stopband ripple $\delta_s$, it is only necessary to design a filter $\tilde{H}(z)$ in (4.5) with $\tilde{\delta}_p$ (≈ 28°), and $\tilde{\delta}_s$ (= $\delta_s$) and with desired values 1 in passband and 0 in stopband. Then $H_p(z)$ can be constructed from $\tilde{H}_p(z)$ and thus $H(z)$ is obtained from (4.1).

### 4.3 Iterative Design Algorithm

Let $F_p$ and $F_s$ be respectively the passband and stopband edges of the filter. The first step in our algorithm consists of designing $H_s(F)$ on the band $A_s = [0][F_s, 0.5]$ by taking care of $\tilde{H}_p(F)$ on this band. The approximation error function on $A_s$ at the $k$-th iteration ($k=1,2,\cdots$) becomes (for brevity, only even $N_s$ is considered in this section and Section 4.5, the results extending in an obvious way to the case of odd $N_s$):

$$E_{s,k}(F) = [\tilde{H}_{p,k}(F)]^{1/2}[0 - H_{s,k}(F)]$$

$$= \tilde{W}_{s,k}(F)[D\tilde{s}_{s,k}(F) - \tilde{P}_{s,k}(F)], F \in A_s,$$  \hspace{2cm} (4.6)

where $\tilde{P}_{s,k}(F)$, $D\tilde{s}_{s,k}(F)$ and $\tilde{W}_{s,k}(F)$ are called respectively the designed, desired, and weighting functions on $A_s$, and are defined accordingly. The desired and weighting values at $F = 0$, denoted respectively by $d_{0,0}$ and $W_{0,0}$, are usually set equal to 1 and $\tilde{H}_{p,0}(F)$ in (4.6), the initial value of $\tilde{H}_p(F)$, is also set equal to 1.

After designing $H_s(F)$ in the stopband, we return to design $\tilde{H}_p(F)$ in the passband, with the point $F = 0.5$ included in the design procedure as
the point \( F = 0 \) when designing \( H_s(F) \). For convenience, we denote the desired value of \( \tilde{H}_p(0.5) \) by \( d_{0.5} \); and the weighting value \( W_{0.5} \) at \( F = 0.5 \) is set equal to 1. In other words, we design \( \tilde{H}_p(F) \) on the frequency subset \( \tilde{A}_p = [0, F_p] \cup [0.5] \) by taking care of \( |H_s(F)|^2 \) on \( A_p \). The approximation error function on \( A_p \) at the \( k \)-th iteration becomes

\[
E_{p,k}(F) = \frac{|H_{s,k}(F)|^2}{|H_{s,k}(F)|^2} - \tilde{H}_{p,k}(F) = \tilde{W}_{p,k}(F) [\tilde{D}_{p,k}(F) - \tilde{P}_{p,k}(F)], \quad F \in \tilde{A}_p,
\]

where \( \tilde{P}_{p,k}(F), \tilde{D}_{p,k}(F) \) and \( \tilde{W}_{p,k}(F) \) are called respectively the designed, desired and weighting functions on \( A_p \) and are defined accordingly.

The Remez algorithm[1] is applied to design \( H_s(F) \) and \( \tilde{H}_p(F) \) in (4.6) and (4.7) iteratively. After the 1st iteration, \( H_{s,1}(F) \) will change monotonically from the point \( F = 0 \) to the stopband, and \( |H_{p,1}(F)| \) will approximate \( 1/H_{s,1}(F) \) in the passband, and then change monotonically from the passband edge to the point \( F = 0.5 \). \( |H_1(F)| \) is equiripple in the passband and not in the stopband, so the 2nd iteration is required to repeat the design procedure.

After the 2nd iteration, \( H_{s,2}(F) \) will have approximately the same shape as \( H_{s,1}(F) \) in the passband. But \( H_{s,2}(F) \) will have the error weighted by \( |H_{p,1}(F)| \) in the stopband, so that \( |H_{2}(F)| \) will be even closer to equiripple in this band. Same as \( |H_{p,1}(F)| \), the function \( |H_{p,2}(F)| \) will approximate \( 1/H_{s,2}(F) \) in the passband and then change
monotonically from the passband edge to the point \( F = 0.5 \). As mentioned above, \( |H_{s,2}(F)| \) does not change appreciably from \( |H_{s,1}(F)| \) in the passband so that \( |H_{p,2}(F)| \) will not change appreciably from \( |H_{p,1}(F)| \). In the passband, \( |H_2(F)| \) will be equiripple again, but in the stopband it will not. \( |H_2(F)| \) is even closer to equiripple in the stopband. The design procedure is repeated, if necessary, for more iterations.

As the algorithm advances, \( |H_{s,k+1}(F)| \) will not differ appreciably from \( |H_{s,k}(F)| \) and \( |H_{p,k+1}(F)| \) will not differ appreciably from \( |H_{p,k}(F)| \) for \( k > 2 \). As the design algorithm proceeds further, \( |H_k(F)| \) is found to have equiripple behavior in both the passband and stopband, and thus the algorithm converges to the optimal solution. A procedure has been formulated, similar to the one in [36], for the purpose of detecting the occurrence of the optimal solution and in case of such an occurrence, stopping the design procedure.

From our experience, we have found good convergence of our algorithm for arbitrary orders of the filter provided the dynamic range of the decomposed spectra is within the range of the program in [1]. Compared to the algorithm in [8], the dynamic range of the decomposed spectra is greatly reduced in the present algorithm for the same desired filter, particularly for wide transition band and high stopband attenuation filters. It has also been found that the algorithm usually converges within two to four iterations.
To analyze the role played by the ripple ratio in our algorithm, we present the following developments. These developments, even though similar to those in the IIR filter design case described in Chapter III, are included here for the sake of clarity and because they lead to a formula different from the one in Chapter III. When the design algorithm converges, the ripples $\delta_s$ and $\delta_p$ can be expressed as [16]

$$\delta_s = \frac{\sum_{n=0}^{\frac{N}{2}+1} a(n)d_{0.5} + \sum_{n=0}^{\frac{N}{2}+1} \frac{(-1)^n}{\sqrt{H_p(F_{1/2})}} + \sum_{n=0}^{\frac{N}{2}+1} \frac{(-1)^n}{\sqrt{H_p(F_{N/2})}}}{a(0)d_{0.5} + \sum_{n=0}^{\frac{N}{2}+1} a(n)d_{0.5} + \sum_{n=0}^{\frac{N}{2}+1} \frac{(-1)^n}{\sqrt{H_p(F_{1/2})}} + \sum_{n=0}^{\frac{N}{2}+1} \frac{(-1)^n}{\sqrt{H_p(F_{N/2})}}}$$

and

$$\delta_p = \frac{1}{2} \frac{\sum_{n=0}^{N/2+1} \frac{(-1)^n}{\sqrt{H_s(F_0)}} + \sum_{n=0}^{N/2+1} \frac{(-1)^n}{\sqrt{H_s(F_{N/2})}}}{a(N+1)d_{0.5} + \sum_{n=0}^{\frac{N}{2}+1} \frac{(-1)^n}{\sqrt{H_s(F_0)}} + \sum_{n=0}^{\frac{N}{2}+1} \frac{(-1)^n}{\sqrt{H_s(F_{N/2})}}}$$

where $\{F_j, j=1,2,\ldots,(N_s/2+1)\}$ and $\{F_p, j=0,1,\ldots,N_p\}$ are the extremal frequencies respectively in the stopband and passband and

$$a(i) = \prod_{i=0}^{N_s/2+1} \frac{\cos(2\pi F_i) - \cos(2\pi F_{1/2})}{\cos(2\pi F_{i}) - \cos(2\pi F_{1/2})}, \quad i = 0,1,\ldots,(N_s/2)+1.$$  

Let

$$\tilde{a}(i) = \prod_{i=0}^{N_p+1} \frac{\cos(2\pi F_i) - \cos(2\pi F_{1/2})}{\cos(2\pi F_{i}) - \cos(2\pi F_{1/2})}, \quad i = 0,1,\ldots,N+1.$$
\[
\delta_s = \frac{a(0)d_{0,0}}{W_{0,0}} + f(d_{0,5})
\]

(4.11)

and

\[
\delta_p = \frac{1}{2} \frac{a(N+1)d_{0,5}}{\beta} + \alpha
\]

(4.12)

where

\[
f(d_{0,5}) = -\frac{\tilde{a}(1)}{|H_p(F_1)|^{1/2}} + \cdots + (-1)^{N-\frac{3}{2}+1} \frac{\tilde{a}(N)}{|H_p(F_N)|^{1/2}}
\]

(4.13)

\[
\alpha = \frac{\tilde{a}(0)}{|H_s(F_0)|^2} + \cdots + \frac{\tilde{a}(N)}{|H_s(F_N)|^2}
\]

(4.14)

\[
\beta = (-1)^N \frac{\tilde{a}(N+1)}{W_{0,5}} + \frac{\tilde{a}(0)}{|H_s(F_0)|^2} + \cdots + (-1)^N \frac{\tilde{a}(N)}{|H_s(F_N)|^2}
\]

(4.15)

In terms of (4.11) and (4.12), the ripple ratio \( K \) is of the form

\[
K = \frac{\delta_p}{\delta_s} = \frac{1}{2} \left[ \frac{a(N+1)d_{0,5}}{\beta} + \frac{a(0)}{a(0)d_{0,0}} \right]
\]

(4.16)

Here, \( a(0) \) and \( a(N+1) \) are functions of extremal frequencies in the stopband and passband respectively and \( f(d_{0,5}) \) is a function of extremal frequencies in the stopband and of the \( |H_p(F)|^{1/2} \) values evaluated at these extremal frequencies. The \( \alpha \) and \( \beta \) values are functions of extremal frequencies in the passband and \( |H_s(F)|^2 \) evaluated at these extremal frequencies. Suppose \( d_{0,5} \) is changed to \( d'_{0,5} \). If we assume that the
change of $d_{0.5}$ is small compared to $d_{0.5}$, the changes of positions of extremal frequencies of the optimal solution in the passband and stopband due to the change in $d_{0.5}$ have negligible effects on $\alpha, \beta, f(d_{0.5}), a(i)'s$ and $\tilde{a}(i)'s$. The $|\tilde{H}_p(F)|$ values evaluated in the stopband will change accordingly but the change of $|H_s(F)|$ in the passband will be negligible if the filter is not a very narrow stopband filter. Thus the ripple ratio $K$ in (4.16) only depends on the $d_{0.5}$ value through the terms $a(N+1)d_{0.5}$ and $f(d_{0.5})$.

Suppose a filter with a specific ripple ratio $K$ is desired. Then the required $d_{0.5}$ to obtain the desired $K$ can be estimated by (4.16). Since, thus far in the above developments, we do not have a closed form expression for $f(d_{0.5})$ in terms of $d_{0.5}$ only, we need to recalculate the corresponding $d_{0.5}$ value according to (4.16) and design the filter again.

If the filter stopband is not too narrow, then we have the following approximations:

$$|\tilde{H}_p(F_1)|^{1/2} \approx \left( \frac{d_{0.5}}{d_{0.5}} \right)^{1/2} |\tilde{H}_p(F_1)|^{1/2}, |\tilde{H}_p(F_2)|^{1/2} \approx \left( \frac{d_{0.5}}{d_{0.5}} \right)^{1/2} |\tilde{H}_p(F_2)|^{1/2},$$

$$\cdots, |\tilde{H}_p(F_{N_{s}+1})|^{1/2} \approx \left( \frac{d_{0.5}}{d_{0.5}} \right)^{1/2} |\tilde{H}_p(F_{N_{s}+1})|^{1/2},$$

(4.17)

where $|\tilde{H}_p(F_i)|$, denotes the optimal $|\tilde{H}_p(F)|$ evaluated at the extremal $d_{0.5}$ frequency $F = F_i$ $(i=1,2,\cdots,(N_{s}/2)+1)$ when the desired value of $|\tilde{H}_p(F)|$
at the point $F = 0.5$ is $d'_{0.5}$. Combining (4.13) and (4.17), we get

$$f(d_{0.5}) = c_0 (d_{0.5})^{-1/2},$$  \hspace{1cm} (4.18)

where $c_0$ is a constant. By substituting (4.18) into (4.16), we obtain

$$K = \frac{1}{2} \frac{a(N+1)d_{0.5}}{\beta} + \alpha \frac{z(0)}{w(0)} + c_0 (d_{0.5})^{-1/2}$$  \hspace{1cm} (4.19)

and, given the desired $K$, the required $d_{0.5}$ can be found by solving (4.19).

In particular, if the desired filter has the following property:

$$c_0 (d_{0.5})^{-1/2} \gg \frac{z(0)}{w(0)},$$  \hspace{1cm} (4.20)

the ripple ratio becomes

$$K = \frac{1}{2} \frac{a(N+1)d_{0.5}}{\beta} + c_1 (d_{0.5})^{-1/2} + c_2 (d_{0.5})^{-1/2},$$  \hspace{1cm} (4.21)

where $c_1$ and $c_2$ are constants. Furthermore, if the filter also has the following property:

$$a \gg \frac{z(N+1)d_{0.5}}{p},$$  \hspace{1cm} (4.22)

then the ripple ratio reduces to

$$K = \frac{1}{2} \frac{a(N+1)d_{0.5}}{\beta} + c_3 (d_{0.5})^{-1/2},$$  \hspace{1cm} (4.23)

where $c_3$ is an appropriate constant. Equation (4.23) tells that the ripple ratio is related simply to the parameter $d_{0.5}$. With (4.23), the suitable value of $d_{0.5}$ can be easily found if the desired $K$ is given;
and thus the filter with the desired ripple ratio may be obtained in one step.

It is important to note that even if, in the general filter design, the ripple ratio $K$ does not follow (4.23), this relation still provides us with a good estimate of $d_{0.5}$ for the desired $K$, which can be used in the design procedure. Given the desired $K$, an estimate of $d_{0.5}'$ is obtained from (4.23), and then a filter with this estimated $d_{0.5}'$ is designed. The resultant $K'$ from the first design cycle, although not exactly the desired $K$, will be close to it. This procedure may be iterated to give a new value $d_{0.5}''$ of $d_{0.5}'$ to which there corresponds a ripple ratio $K''$ which is even closer to $K$ than $K'$. In fact, by this iterative procedure, a filter with the desired ripple ratio $K$ can be obtained within a few design cycles. Usually, only two design cycles are required to design an NLPFIR filter with the ripple ratio near the desired value.

4.4 Minimum Ripple Ratio and Optimality Properties

(a) Minimum Ripple Ratio Property

Filters designed by our algorithm will have all their stopband zeros on the unit circle. This fact implies that given the filter orders $N_p$ and $N_s$ and band edges of the filter, there exists a minimum ripple ratio of the filter which can be realized by the present algorithm. For filters of even $N_p$, it is always possible, without imposing a restriction at $F = 0.5$, to design the filter with the minimum $d_{0.5}$ (or the
minimum ripple ratio) for given filter orders, band edges, and constraint. For filters with odd \( n_p \), the minimum \( d_{0.5} \) can be estimated by means of a filter of \( n_p + 1 \). If the desired ripple ratio is smaller than this minimum value, then the order \( n_p \) should be increased.

If the filter order \( n \) is given, the initial estimates of \( n_p \) and \( n_s \) can be obtained by evaluating the filter length by the procedure in [63,64] (specifically by setting, in eq. (4.7) of [63], respectively \( F_s = 0.5 \) and \( F_p = 0 \) and then normalizing it by \( N \)).

Compared to the algorithm in [8], although it is still necessary to find the zeros of a polynomial in our algorithm, the order of the polynomial is reduced from \( 2N \) to \( 2n_p \) and thus the ill-conditioning in the numerical processes of finding the zeros can be lessened in the proposed algorithm.

(b) Optimality Property

Note that the optimality of the solution obtained by the algorithm of Goldberg et al. [8] can be examined by investigating the behavior of the error function. Let the transfer function of NLPFIR filter be \( H(z) \) as in (4.1). Then from alternation theorem in [16], the optimal solution \( |H(F)|^2 \) obtained by Goldberg's algorithm will have at least \( N+2 \) extremal frequencies on \( A_t \) (=\( [0,F_p] \cup [F_s,0.5] \)).

In our algorithm, \( \tilde{H}(z) \) is designed by iteratively designing \( H_s(z) \) and \( \tilde{H}_p(z) \). If \( n_s \) is odd, there are at least \( (n_s+1)/2 \) extremal frequen-
cies on $A_s$, and $H(F) = \frac{H_s(F)}{|H_p(F)|^2}$ will oscillate around zero between $-\tilde{\delta}_s$ and $+\tilde{\delta}_s$ at least $(N_s+1)/2$ times, or $|H(F)|^2$ will oscillate around $(\tilde{\delta}_s)^2/2$ at least $N_s+1$ times with ripple $(\tilde{\delta}_s)^2/2$. If $N_s$ is even, then there are at least $(N_s/2)+1$ extremal frequencies on $A_s$, and $H(F)$ will oscillate around zero between $-\tilde{\delta}_s$ and $+\tilde{\delta}_s$ at least $(N_s/2)+1$ times. Hence, $|H(F)|^2$ will oscillate around $(\tilde{\delta}_s)^2/2$ between zero and $(\tilde{\delta}_s)^2$ at least $N_s+1$ times with ripple $(\tilde{\delta}_s)^2/2$. Furthermore, when designing $\tilde{H}_p(F)$, there are at least $N_p+1$ extremal frequencies on $A_p$, and $|H(F)|^2$ will oscillate between $1-\tilde{\delta}_p$ and $1+\tilde{\delta}_p$ at least $N_p+1$ times with ripple $\tilde{\delta}_p$. We thus conclude that $|H(F)|^2$ will have at least $N+2$ extremal frequencies on $A_t$ and thus the filter design algorithm proposed in this paper will yield the optimal solution under the specifications of the filter.

### 4.5 Design of Constrained NLPFIR Filters

In Section 4.3, the iterative algorithm is applied to the design of NLPFIR filter with flat desired passband and stopband magnitude response. In many applications, such as spectrum shaping filtering[40–41] and hardware implementation of digital communication systems[15], the desired filter is required to have a prescribed passband or stopband shaping. In such cases, our algorithm may be modified to be applicable to the design of appropriately constrained NLPFIR filters.

Let $H_c(z)$ be the transfer function of a constrained low-pass NLPFIR filter
$$H_c(z) = T_s(z)T_p(z)H_s(z)H_p(z), \tag{4.24}$$

where the functions $T_s(z)$ and $T_p(z)$ are the prescribed (or constraint) functions, and $H_s(z)$ and $H_p(z)$ are the functions to be determined. Both $T_s(z)$ and $H_s(z)$ are assumed to have all their zeros on the unit circle in the stopband, while both $T_p(z)$ and $H_p(z)$ have all their zeros in the passband. The design of a constrained filter is then reduced to the determination of $H_s(z)$ and $H_p(z)$ in (4.24), under the given constraint embodied by $T_s(z)$ and $T_p(z)$, so that the filter $H_c(z)$ will have a desired low-pass magnitude response when evaluated along the unit circle. In this section, it is assumed that both $T_s(F)$ and $T_p(F)$ in (4.24) have been normalized to 1 at $F = 0$.

By the same procedure as in Section 4.2, let us introduce a transfer function $\tilde{H}_c(z)$ by

$$\tilde{H}_c(z) = T_s(z)H_s(z)\tilde{T}_p(z)\tilde{H}_p(z). \tag{4.25}$$

When designing the filter on $\tilde{A}_s$, the approximation error function in (4.6) becomes

$$E_{s,k}(F) = |T_s(F)T_p(F)|[\tilde{H}, k-1(F)]^{1/2}[H_{s,k}(F) - 0], \quad F \in \tilde{A}_s. \tag{4.26}$$

If there exists ill-conditioning in the weighting function of (4.26), a small positive value is assigned to this weighting value so that it will be a positive definite function.

After designing $H_s(F)$, we return to the design of $\tilde{H}_p(F)$ on $\tilde{A}_p$. The approximation error function in (4.7) becomes
\[ E_{p,k}(F) = \left| T_{s}(F) T_{p}(F) H_{s,k}(F) \right|^2 \left[ \frac{1}{T_{s}(F) T_{p}(F) H_{s,k}(F)} - \frac{1}{|T_{s}(F) T_{p}(F) H_{s,k}(F)|^2} \right], \quad F \in \mathbb{A}_p \]  

(4.27)

The Remez algorithm[1] is applied to design (4.26) and (4.27) iteratively. Good convergence has been found from the experience with this algorithm, for arbitrary constraint function shapings. A similar analysis of the relation between ripple ratio and \( d_{0.5} \) can be made and it is found that (4.23) in Section 4.3 still holds.

### 4.6 Roundoff Noise in Filter Cascade Implementation

An NLPFIR filter cannot be efficiently implemented in direct form due to the lack of symmetry of the filter coefficients. But, as discussed in [8], if it is implemented as a cascade of its subfilters \( H_p(z) \) and \( H_s(z) \), with both subfilters realized in direct form, it is possible to take advantage of the symmetry of the coefficients of \( H_s(z) \). For the NLPFIR filter with large \( N_s \), a large number of multipliers can be saved when implemented in cascade.

Let the NLPFIR filter be implemented in direct form and assume that fixed point arithmetic and roundoff quantization are used. If all multiplication products are rounded after they are summed, then the means of output roundoff noise is clearly zero and its variance \( \sigma_0^2 \) is given by

\[ \sigma_0^2 = \frac{\sigma^2}{12}. \]  

(4.28)
where $Q$ is the quantization step size.

If the NLPFIR filter is implemented as cascade of $H_s(z)$ and $H_p(z)$ and scaling multipliers are included in each stage to prevent overflow, the transfer function $H_i(z)$ of each stage is defined as

$$H_i(z) = S_i H_{i,n}(z), \ i = s, p,$$  \hspace{1cm} (4.29)

where $S_i$ is the scaling factor for the $i$-th stage and $H_{i,n}(z)$ is the normalized transfer function with unit passband gain or $H_{i,n}(0) = 1$.

The overall transfer function $H(z)$ is then

$$H(z) = \beta H_n(z),$$  \hspace{1cm} (4.30)

where $H_n(z)$ is the normalized overall transfer function and $\beta = S_1 S_2$ is chosen to prevent overflow at each stage.

Assume that the input $\{x(n)\}$ is a deterministic sequence and define the $L_p$ norm ($p \geq 1$) of a Fourier transform $A(F)$ as

$$|A|_p = \left[ \int_{-0.5}^{0.5} |A(F)|^p \, dF \right]^{1/p}. \hspace{1cm} (4.31)$$

Then, following the treatments in [43] and [65], the output noise variances $\sigma_{sp}^2$, with $H_s(z)-H_p(z)$ cascaded, and $\sigma_{ps}^2$, with $H_p(z)-H_s(z)$ cascaded, can be determined as

$$\sigma_{sp}^2 = \frac{\left(\sigma_s^2 |H_s(F)|^2 \right)^2}{12} + \frac{\left(\sigma_p^2 |H_p(F)|^2 \right)^2}{12}, \hspace{1cm} (4.32)$$

or
\[ \sigma^2_{ps} = \left( \frac{Q}{12} \right) \frac{2}{|H_n(F)|^2_p |H_p,F_p|^2} + \frac{Q^2}{12}. \] (4.33)

If \( p = 2 \), we have, according to the Schwarz inequality,

\[ |H_n(F)|^2_p \leq |H_p,F_p|^2 |H_s,F_s|^2. \] (4.34)

Actually, because the rapidly rising characteristic of \( H_{p,n}(F) \) on the band \([F_p,0.5]\), the inequality in (4.34) always holds. The roundoff noise of cascade filter implementation is always larger than that of direct form implementation. One method to reduce this noise is to reduce the quantization step size \( Q \) by increasing the wordlengths.

### 4.7 Design Examples

The algorithm described in Section 4.3 and Section 4.5 has been used to design various type of NLPFIR filters. Two of these design examples are included here to illustrate the design procedure. A DEC PDP 11/55 computer with double precision was used in the calculations.

**Example 1:**

An NLPFIR filter with \( N_s = 13, N_p = 4, F_p = 0.12, F_s = 0.34 \) was designed with the desired ripple ratio = 3600.0.

The minimum \( \delta_p \) and \( \delta_s \) were determined to be 0.0065 dB and -109.4 dB respectively. Based on the \(-1/2\) power relation, the optimal filter with the desired ripple ratio was found after 4 design cycles. The passband ripple and the stopband ripple of the resulting filter are 0.048 dB and
-116.3 dB respectively and its amplitude response is shown in Fig.4.1(a).

On the other hand, when the algorithm in [8] was used to solve the design problem, the amplitude response of the resulting filter was found to be as shown in Fig.4.1(b), with stopband attenuation up to only about 70 dB.

Note also that, in order to construct \( H_p(z) \) from \( \tilde{H}_p(z) \) by our algorithm, we need to determine the zeros of a polynomial of order \( 2N_p = 8 \). This number is dramatically less than the number of zeros of a polynomial of order \( 2(N_p + N_s) = 34 \) which need to be determined by the algorithm in [8] in order to construct \( H(z) \) from \( \tilde{H}(z) \).

**Example 2:**

A filter with \( N_s = 23, N_p = 4, F_p = 0.1, F_s = 0.2 \) was designed with a desired ripple ratio \( K = 2000.0 \).

The minimum \( \delta_p \) and \( \delta_s \) were found to be 0.095 dB and -79.6 dB respectively. Using our algorithm to design the filter with the ripple ratio \( K = 2000.0 \), we obtain for the \( \delta_p \) and \( \delta_s \) of the the resultant filter respectively 0.46 dB and -91.5 dB. The amplitude response of this filter is shown in Fig.4.2(a).

The algorithm in [8] was also applied to this case. Fig.4.2(b) shows the amplitude response of the resulting filter, from which we again deduce that the performance of our algorithm is superior to that
Fig. 4.1(a) Amplitude response of the filter designed by the proposed algorithm.

Fig. 4.1(b) Amplitude response of the filter designed by the algorithm of Goldberg et al.
of the algorithm in [8].

Suppose one of the stopband zeros of the filter in this example is prescribed on $F = 0.25$ so that the filter will have higher stopband attenuation to the signals with this frequency. This is a constrained filter design problem and it is equivalent to designing a filter with the constraint function $T_s(z) = 0.5 + 0.5z^{-2}$. We applied our algorithm to design a filter with $\delta_p = 0.45$ dB. Fig. 4.2(c) shows the amplitude response of the resulting filter. By comparing Figs. 4.2(a) and (c), it is found that in this example the constrained filter exhibits the same performance as the unconstrained filter in the stopband (both have the same $\delta_p = 0.46$ dB), but the former has an exact zero at $F = 0.25$.

If the desired constrained filter is not required to be equiripple in the stopband, the filter can be designed with $T_s(z) = 0.5 + 0.5z^{-2}$ in the passband and $T_s(z) = 1$ in the stopband. The amplitude response of the resulting filter with $\delta_p = 0.46$ dB is shown in Fig. 4.2(d). This filter has more attenuation to the signals near the frequency $F = 0.25$ than the former.

4.8 Concluding Remarks

An iterative algorithm has been proposed for designing low-pass NLPFIR filter by decomposing its transfer function into a nonlinear phase part and a linear phase part. The algorithm iteratively designs the linear phase part and the nonlinear phase part by directly calling
Fig. 4.2(a) Amplitude response of the filter designed in Example 2 by the proposed algorithm.

Fig. 4.2(b) Amplitude response of the filter designed in Example 2 by the algorithm of Goldberg et al.
Fig. 4.2(c) Amplitude response of the constrained filter designed in Example 2 with $T_s(z) = 0.5 + 0.5 z^{-2}$ in the stopband.

Fig. 4.2(d) Amplitude response of the constrained filter designed in Example 2 with $T_s(z) = 1$ in the stopband.
the Remez algorithm. When designing the nonlinear phase part, a scheme is incorporated into the algorithm so that it can design the NLPFIR filter with a desired ripple ratio.

Given the orders, the filter with the minimum ripple ratio can be obtained by designing the filter without imposing a restriction at \( F = 0.5 \). If the desired ripple ratio is smaller than the minimum value, then the order of the nonlinear phase part should be increased.

Since the filter designed by the proposed algorithm will have all its stopband zeros on the unit circle, the cascade implementation will be more efficient than its direct form realization. Analysis of the roundoff noise reveals however that the cascade filter implementation usually needs higher wordlengths than its direct form counterpart for the same noise performance.

Our method is based on the decomposition of the component spectra of the filter. The performance of the approximations depends on the spectral dynamic range. The dynamic range of the component spectra depends on the width of the transition band. The component spectral dynamic range is larger for smaller widths. The introduction of the squared-magnitude of the nonlinear phase part doubles the dynamic range of the component spectra and thus deteriorates the performance of the approximation. This limits the types of NLPFIR filters that can be designed using our method.
CHAPTER V

NEW CLASSES OF FIR DIGITAL FILTERS AND THEIR APPLICATION TO
THE EFFICIENT DESIGN OF MULTIRATE DIGITAL FILTERS

5.1 Introduction

In this chapter, two new classes of FIR filters are proposed, one linear phase, the other nonlinear phase. An iterative algorithm is presented to the design of these classes of FIR filters to meet the given frequency response. In the case of linear phase filters, by introducing the new class of FIR filters as the elements of decimators and interpolators, it is found that the multiplication rate of the resulting multirate filter is greatly reduced compared to its conventional linear phase FIR counterpart. For the nonlinear phase FIR filters, the amount of data storage is significantly reduced.

In Section 5.2, a new class of linear phase FIR filters is proposed and an iterative design algorithm for the design of this class of linear phase FIR filters is described. The order estimation of the new filter is discussed. In Section 5.3, the new class of linear phase FIR filters is then introduced as the elements of a multirate filter and a greatly reduced computation rate is obtained with this new class of multirate filters. In Section 5.4, a new class of nonlinear phase FIR filters is proposed. In Section 5.5, the advantages of introducing this class of nonlinear phase FIR filters to the design of multirate filters are
described. In both linear and nonlinear phase cases, several design examples are given to illustrate the efficiency of the new classes of multirate filters. Conclusions are presented in Section 5.6.

5.2 A New Class of Linear Phase FIR Digital Filters

5.2.1 A New Class of Linear Phase FIR Filters

Consider a class of linear phase FIR digital filters whose transfer function can be decomposed into the form

\[
H(z) = H_s(z) H_p(z^D) = \left[ \sum_{n=0}^{N_s \text{LP}} h_s(n) z^{-n} \right] \left[ \sum_{n=0}^{N_p \text{LP}} h_p(n) z^{-nD} \right], \tag{5.1}
\]

where \(D\) is a positive integer and \(H_s(z)\) is a transfer function whose zeros consist of all zeros of \(H(z)\) on the unit circle, and \(H_p(z^D)\) is a transfer function which includes the remaining zeros of \(H(z)\). Note that if \(D = 1\), then \(H(z)\) is reduced to the conventional linear phase FIR digital filters and both \(H_p(z^D)\) and \(H_s(z)\) represent linear phase FIR filters.

The filter frequency response of \(H(z)\) is obtained by evaluating (5.1) on the unit circle, i.e.,

\[
H(F) = H(z) \bigg|_{z = e^{j2\pi F}} = H_p(DF) H_s(F), \tag{5.2}
\]

where, and henceforth, we use the abbreviated notation \(P(F)\) for \(P(e^{j2\pi F})\). The transfer function in (5.1) will be suitable for describing the class of linear phase FIR filters whose passband edge is smaller
than 0.5/D because $H_p(DF)$ is a function of DF. If $H(F)$ exhibits equiripple behavior both in the passband and stopband, then all the zeros of $H_s(z)$ must be distributed on the unit circle in the $z$-plane and all the zeros of $H_p(z^D)$ will be distributed in the passband in the $z^D$-plane, so that the product of $H_p(DF)$ and $H_s(F)$ will yield the equiripple behavior of $H(F)$.

By suitably choosing the integer $D$, this new class of linear phase FIR filters will yield a more efficient design, with respect to a given frequency response, than the conventional linear phase FIR filters. Here, by efficient, we mean "less multipliers".

5.2.2 Iterative Design Algorithm

Let $F_p$ and $F_s$ be respectively the passband and stopband edges of the desired filter. Here, an iterative algorithm is proposed for the design of the filter. The algorithm iteratively designs $H^*_s(z)$ and $H^*_p(z^D)$ according to (5.1) by directly calling the Remez algorithm[1] to efficiently solve the approximation problem.

The first step in our algorithm consists of designing $H_s(F)$ on the frequency band $\tilde{A}_s = [0] \cup A_s = [0] \cup [F_s, 0.5]$, where, in order to obtain a nontrivial solution, the starting point $F = 0$ is included in the design procedure. Let $d_{0,0}$ and $w_{0,0}$ be respectively the desired and weighting functions when designing $H_s(F)$. The approximation error function $E_{s,k}(F)$ on $A_s$ at the $k$-th iteration ($k = 1, 2, \ldots$), by taking care of $H_p(DF)$ on
this band, becomes (for brevity, only even $N_s$ are considered here, the results extending in an obvious way to the case of odd $N_s$)

$$E_{s,k}(F) = [H_{p,k}(DF)][0 - H_{s,k}(F)]$$

$$= \tilde{W}_{s,k}(F)[D_{s,k}(F) - \tilde{P}_{s,k}(F)] , F \in A_s , \quad (5.3)$$

where $\tilde{P}_{s,k}(F)$, $D_{s,k}(F)$ and $\tilde{W}_{s,k}(F)$ are called respectively the designed, desired and weighting functions on $A_s$, and are defined accordingly. The $H_{p,0}(DF)$ in (5.3), the initial value of $H_p(DF)$, is usually set equal to 1.

After designing $H_s(F)$ in the stopband, we return to design $H_p(DF)$ in the passband. Since $H_p(z^D)$ contains only powers of $z^D$, the frequency response of $H_p(DF)$ repeats $D$-fold in the frequency interval $[0,F_s]$, where $F_s$ is the sampling frequency. In order to meet the requirements of $H_s(F)$, the function $H_s(F)$ must be modified in order to compensate its response when designing $H_p(DF)$. In addition to the passband frequency interval, the point $F = 0.5/D$ is included in this design procedure. The desired and weighting values of $H_p(0.5/D)$ are denoted by $d_{0.5}$ and $\tilde{W}_{0.5}$ respectively. In other words, we design $H_p(DF)$ on the frequency subset

$$A_p = A_p[0.5/D] = [0,F_p/D]U[0.5/D] ,$$

The approximation error function on $A_p$ at the $k$-th iteration becomes

$$E_{p,k}(F) = [H_{s,k}(F)][\frac{1}{H_{s,k}(F)} - H_{p,k}(DF)]$$

$$= \tilde{W}_{p,k}(F)[D_{p,k}(F) - \tilde{P}_{p,k}(F)] , F \in A_p , \quad (5.4)$$
where $\tilde{P}_{p,k}(F)$, $\tilde{D}_{p,k}(F)$ and $\tilde{W}_{p,k}(F)$ are called respectively the designed, desired and weighting functions on $A_p$ and are defined accordingly.

The Remez exchange[1] algorithm is applied to design $H_s(F)$ and $H_{p}(DF)$ in (5.3) and (5.4) respectively. After the first iteration, $H_s(F)$ will change monotonically from the point $F = 0$ to the stopband, and $H_{p}(DF)$ will approximate $1/H_s(F)$ in the passband, and then change monotonically from the passband edge to the point $F = 0.5/D$. In the frequency interval $[0.5/D, 1/D]$, $H_{p}(DF)$ is symmetric with respect to it values in the interval $[0, 0.5/D]$. Then $H_{p}(DF)$ will repeat $D$-fold in the remaining interval $[1/D, 1]$. Since $H(F)$ after the first iteration is equiripple in the passband and not in the stopband, the second iteration is required to repeat the design. This procedure is continued until $H(F)$ has equiripple behavior both in the passband and stopband and thus the algorithm converges to the optimal solution. A procedure has been formulated for the purpose of detecting the occurrence of the optimal solution and in case of such occurrence, stopping the design procedure. From our experience, we have found good convergence of this algorithm.

Based on this approach, a filter with the desired ripple ratio can be designed by controlling the $d_{0.5}$ used in the design of $H_{p}(DF)$. The design procedure is similar to that described in Chapter III.

The filter designed by our algorithm has minimum passband ripple property as described in chapter III. In other words, given the orders
$N_p$, $N_s$ and band edges of the filter, there exists a minimum passband ripple filter which can be designed by our algorithm.

### 5.2.3 Filter Order Estimation

For designing a filter with transfer function defined in (5,1) to satisfy the given specifications, the filter order can be estimated by a procedure similar to the one in [64]. If the $H_s(F)$ is considered to have one passband ripple and to be equiripple in the stopband, with stopband ripple $\delta_s$ and taking the value 1 at $F=0$, then a value $M_1$ can be obtained from [64]

$$
M_1 = \cosh^{-1}(1/\delta_s),
$$

where

$$
X_0 = \frac{3 - \cos(2\pi F_s)}{1 + \cos(2\pi F_s)}.
$$

Similarly, if $H_s(F)$ is considered to be equiripple as described above, but taking a value 1 at $F = F_p$, then a value $M_2$ can be obtained from [64]

$$
M_2 = \cosh^{-1}(1/\delta_p),
$$

where

$$
X_p = \frac{X_0 + 1}{2} \cos(2\pi F_p) + \frac{X_0 - 1}{2}.
$$

For the new class filters considered in this chapter, $H_{s,LP}(F)$ is
usually not equiripple in the stopband and thus the filter order \( N_{s,LP} \) does not follow \( M_1 \) or \( M_2 \). From all the filters designed, we have observed that good initial values of the filter orders are \( N_{s,LP} = 3(M_1 + M_2)/2 \) and \( N_{p,LP} = 2N_z \), where \( N_z \) is the number of passband extremal frequencies of conventional linear phase FIR filter which satisfies the same specifications. A filter can be designed starting from these values. If the desired passband ripple is not satisfied with these \( N_{s,LP} \) and \( N_{p,LP} \), then it is necessary to increase \( N_{p,LP} \). Once the desired passband ripple is satisfied, the stopband ripple is then checked to determine whether it satisfies its desired value. If not, then \( N_{s,LP} \) is increased. It is important to note that, by increasing \( N_{s,LP} \), although \( \delta_s \) will be reduced, the \( \delta_p \) could be increased. For this case, \( N_{p,LP} \) may increase again. In this way, the parameters \( N_{s,LP} \) and \( N_{p,LP} \) can be adjusted until the specifications are satisfied. Before designing the filter, we have to calculate the filter orders. This is not a serious inconvenience since the filter design algorithm converges relatively fast.

5.2.4 Zeros Locations

The new class filter has a transfer function of the form (5.1), the zeros of \( H_s(z) \) being located on the unit circle. The transfer function \( H_p(z^D) \) has the form

\[
H_p(z^D) = \sum_{n=0}^{N_{p,LP}} h(n)z^{-nD},
\]
Multiplying $H_p(z^D)$ by $z^{-N_{p,LP}D}$, we have

$$\hat{H}_p(z^D) = \sum_{n=0}^{N_{p,LP}} h(n)z^{-D(n_{p,LP}-n)}.$$ 

By changing the variable $\alpha = z^D$, we have

$$\frac{\hat{H}_p(\alpha)}{\hat{H}_p(0)} = \sum_{n=0}^{N_{p,LP}} h(n)\frac{\hat{H}_p(1)}{\hat{H}_p(0)} + \frac{\hat{H}_p(1)}{\hat{H}_p(0)} + \cdot \cdot \cdot + h(N_{p,LP}).$$

If $\alpha(k) = \alpha_0\exp(j\theta_k)$ is a root of $\hat{H}_p(\alpha)$, then $z_i = \alpha_0^{1/D}\exp[j(2\pi i/D+\theta_k)]$, $i = 1, 2, \cdots, D-1$, are roots of $\hat{H}_p(Z^D)$, where $\alpha_0$ is positive. Thus, the zeros of $H_p(z^D)$ are located on a circle of radius $\alpha_0^{1/D}$ with angle of $2\pi/D$ radians between them.

### 4.2.5 Design Examples

**Example 1:**

A low-pass filter was designed with the specifications: $F_p = 0.15$, $F_s = 0.22$, $\delta_p = 0.001$ and $\delta_s \leq 0.00316$.

The linear phase FIR filter ($D=1$) that satisfies the specifications has length 44 with multiplication rate 22 multiplies/sample. Its frequency response is shown in Fig. 5.1. A new class linear phase FIR filter with $D=2$ was designed to meet the specifications. The estimated filter order is $N_{s,LP} = 15$. We can take $N_{s,LP} = 15$ and $N_{p,LP} = 16$ as initial values. For these values, the required stopband ripples $\delta_s$ can not be attained. Hence, it is necessary to increase $N_{s,LP}$. For $N_{s,LP} = 16$ and $N_{p,LP} = 16$, the required specifications are satisfied. The multiplication rate of
the resulting filter, implemented as cascade of $H_s(z)$ and $H_p(z^D)$, is found to be 18 mults/sample. Compared to the value of the conventional class linear phase filter (D=1), a saving of 18.2% is obtained. The number of filter coefficient storages is also reduced from 22 to 18. The plots of the magnitude response of $H_s(z)$ and $H_p(z^D)$ are shown in Fig.5.2(a) and the frequency response of the resulting filter designed in this example is shown in Fig.5.2(b). From Fig.5.2(b), it is found that the number of zeros of $H_s(z)$ near $F = 0.5$ is reduced compared to its conventional linear phase FIR counterpart by introducing the new class linear phase FIR filter. In other words, the spectrum shaping of $H_p(z^D)$ in the stopband is one of the reasons for the reduced multiplication rate of the new class linear phase FIR filter. Fig.5.2(c) shows the zero locations of the resulting filter.

**Example 2:**

The filter designed in this example has the specifications: $F_p = 0.2$, $F_s = 0.275$, $\delta_p = 0.001$ and $\delta_s \leq 0.005$.

A linear phase FIR filter (D=1) with length 39 (multiplication rate = 20 mults/sample) satisfies the specifications. Its frequency response is shown in Fig.5.3. The order estimate of the new class linear phase FIR filter (D=2) is $N_{s,LP} = 9$. We designed the filter with the initial values $N_{s,LP} = 9$ and $N_{p,LP} = 22$. The resulting filter satisfies the required specifications. The multiplication rate of the filter, implemented as the cascade of $H_s(z)$ and $H_p(z^D)$, is thus equal to 16 mults/sample. The number of filter coefficient storages has been reduced.
from 20 to 17. The plots of magnitude responses of $H_s(z)$ and $H_p(z^2)$ are shown in Fig.5.4(a) the magnitude response of the resulting filter is shown in Fig.5.4(b).

The coefficient sensitivity of the type of filter designed in this example is also examined. Fig.5.5(a) and (b) show the frequency response of the filter in the passband and stopband respectively by assuming that $H_s(z)$ and $H_p(z^2)$ are cascaded and both are implemented in direct form. From these results, we can see that a suitable wordlength is 16 bits.

5.3 Design of Multistage Decimators with New Class of Linear Phase FIR Filters

5.3.1 Conventional Decimator Design

As stated previously, a narrow-band filter can be realized in an efficient way if implemented as a cascade of multistage decimators and multistage interpolators. Fig.5.6(a) shows a block diagram of a $K$-stage decimators. A frequency domain interpretation of this multistage operation is shown in Fig.5.6(b), (c) and (d). The input sampling ratio $F_{r0}$, the final output sampling rate is $F_{rK}$; the intermediate sampling frequencies are $F_{r1}$, $F_{r2}$, \ldots, $F_{r,K-1}$. The decimation ratio at each stage is denoted as $D_i$, $i = 1,2,\ldots,K$, and the relations among the intermediate sampling frequencies are

$$F_{ri} = \frac{F_{r}(i-1)}{D_i}, \quad i = 1,2,\ldots,K. \quad (5.7)$$
Fig. 5.1 Frequency response of the conventional FIR filter designed in Example 1.

Fig. 5.2(a) Frequency responses of $H_S(z)$ and $H_P(z^D)$ designed in Example 1.
Fig. 5.2(c) Zero location for the new class linear phase FIR filter designed in Example 1.

Fig. 5.2(b) Frequency response of the new class filter designed in Example 1.
Fig. 5.3 Frequency response of the conventional FIR filter designed in Example 2.

Fig. 5.4(a) Frequency responses of $H_S(z)$ and $H_p(z^D)$ designed in Example 2.
Fig. 5.4(b) Frequency response of the new class filter designed in Example 2.

Fig. 5.7 Filter frequency response requirements at the i-th of a K-stage decimator ($D_i=5$).
Fig. 5.5 Frequency response of the new class linear phase FIR filter designed in Example 2: (a) passband, (b) stopband.
where $D_i$ is the decimation ratio at the $i$-th stage. Then the overall decimation ratio is given by

$$D = \prod_{i=1}^{K} D_i.$$  \hfill (5.8)

Fig. 5.6(b), (c) and (d) show not only the frequency response at each stage but also the aliased component after decimation.

Conventional design of multirate filters uses multiple-stopband low-pass filter as the filter in each stage of Fig. 5.6(a). Fig. 5.7 shows the filter requirements at the $i$-th stage. If the overall filter is to have a passband ripple less than $\delta_p$, this requirement is distributed into each stage, usually uniformly, i.e., $\delta_{p,i} = \delta_p / K$, $i = 1, 2, \ldots, K$. The overall filter must have the stopband ripple less than $\delta_s$ for enough aliasing attenuation, then the $i$-th stage filter should have the same stopband attenuation, i.e., $\delta_{s,i} = \delta_s$. The stopband is the region which will alias into the baseband $[-F_s, F_s]$ after decimation. The don't care regions will alias to the current transition region $[F_s, F_{r1} - F_s]$ and is subject to further filtering. It is found that the optimal ratios are integers and can vary over quite a wide range with little change in efficiency. Actually, by examples, experience, and investigating multistage filter efficiency, it is found[43] that the optimal number of stages is two or three, and for most cases depends on how large the total decimation ratio $D$ is. The optimal ratios can be found by factoring $D$ into $K$ integers with decreasing magnitude, $D_1$ is the largest integer, $D_2$ the second, \ldots and $D_K$ the least. The method generally gives the optimal or near optimal ratios regardless of the type of filter.
Commutation Rule[43]: A filter with transfer function \( H(z^D) \) followed by a sampler with decimation ratio \( D \) as shown in Fig.5.8(a) is equivalent to a sampler with decimation ratio \( D \) followed by a filter with transfer function \( H(z) \) as shown in Fig.5.8(b).

From the multistage decimator shown in Fig.5.6(a), the transfer function of a multistage decimator can be written as

\[
H(z) = H_1(z)H_2(z^{D_1})H_3(z^{1/D_2})\cdots H_K(z^{D_1D_2\cdots D_{K-1}}), \tag{5.9}
\]

The frequency response of a multistage decimator is thus given by

\[
H(F) = H_1(F)H_2(D_1F)H_3(D_1D_2F)\cdots H_K(D_1D_2\cdots D_{K-1}F), \tag{5.10}
\]

The frequency response (5.10) is in terms of input sampling frequency. All of the frequency components in the stopband will be aliased to the baseband on the decimated output.

One of the reasons for the higher efficiency of the multistage multirate filter than a single stage is that the filter in the multistage case has a much wider transition band than the single stage one. The other reason is that in the multistage case, the filter with a narrow transition band corresponds to the lowest sampling rate.

5.3.2 Decimator Design with New Class of Linear Phase FIR Filters

By introducing the new class of linear phase FIR filters into each stage of the decimator in Fig.5.6(a), the resulting decimator filter is shown in Fig.5.9(a). By using the commutation rule, this multistage
Fig. 5.6 Illustration of a K stage decimator.
decimator can be transformed into the decimator structure shown in Fig. 5.9(b). From this decimator structure, the transfer function of the equivalent decimator can be written as

\[
H(z) = [H_{s,1}(z)]^{D_1} [H_{s,2}(z)]^{D_2} [H_{p,1}(z)]^{D_1} [H_{p,2}(z)]^{D_2} \ldots \]

\[
\ldots [H_{s,K}(z)]^{D_1} [H_{p,K}(z)]^{D_2} \] , \quad (5.11a)
\]

or

\[
H(z) = [H_{s,1}(z)][H_{s,2}(z)]^{D_1} [H_{p,1}(z)][H_{s,3}(z)]^{D_2} [H_{p,1}(z)]^{D_1} [H_{p,2}(z)]^{D_2} \ldots \]

\[
\ldots [H_{p,K-1}(z)]^{D_1} [H_{s,K}(z)]^{D_2} [H_{p,K}(z)]^{D_1} [H_{p,K}(z)]^{D_2} \ldots \] , \quad (5.11b)
\]

where the brackets are used here to indicate the combination of the filter structure. Equation (5.11a) corresponds to the decimator structure in Fig. 5.9(a), while (5.11b) to that in Fig. 5.9(b). The frequency response of the new decimator is thus given by

\[
H(F) = [H_{s1}(F)][H_{s2}(D_1Z)][H_{p1}(D_1F)][H_{p2}(D_1D_2F)] \ldots \]

\[
\ldots [H_{s,K}(D_1D_2 \ldots D_{K-1} F)][H_{p,K}(DF)] , \quad (5.12a)
\]

or

\[
H(F) = [H_{s1}(F)][H_{s2}(D_1F)][H_{p1}(D_1F)][H_{s3}(D_1D_2F)] \ldots \]

\[
\ldots [H_{p,K-1}(D_1D_2 \ldots D_{K-1} F)][H_{s,K}(D_1 \ldots D_{K-1} F)][H_{p,K}(DF)] . \quad (5.12b)
\]

The frequency response of (5.12) is in terms of the input frequency. All of the frequency components in the stopband will be aliased to the
baseband on the decimated output.

5.3.2 The Design Problem

The optimal multistage decimator, given the order at each stage, will be the filter that optimizes the response of (5.12). In other words, the most efficient multistage multirate filter, given the type of filter in each stage, will be the filter that has the minimum total arithmetic rate and the filter response of (5.10) satisfies the specifications. Usually this is the filter with minimum filter order at each stage. Since the frequency response of a latter stage will reflect to the earlier stages, an underestimate of the filter order of an earlier stage resulting in not enough gain difference between the passband and stopband(s) cannot be made up by using the filter order of a latter stage.

Since the earlier a stage, the more important is this stage (due to higher sampling rate) to the overall filter efficiency, the optimization scheme (if exists) for the filter response of (5.10) will be first finding a set of a filter orders that the optimal response of (5.10) satisfies the specifications, then reducing the order of the second stage until for the first time it cannot be further reduced and the specifications still hold, and then the order of the second stage being reduced, etc.
In the new structure of the decimator in Fig.5.9, the transfer function of the filter at the i-th stage is defined as

$$H_i(z) = H_{s,i} H_{p,i}^{D_i}(z^{i})$$  \hspace{1cm} (5.13)

The difference between the new decimator structure and the conventional decimator structure is that in the former, we optimize the filter frequency response with the new class of linear phase FIR filters with transfer function defined in (5.13), while in the latter structure, the conventional linear phase FIR filters are used. At the i-th stage of the new decimator structure, the \( H_{p,i}^{D_i}(z^{i}) \) can be moved through the sampler with decimation ratio \( D_i \) to the (i+1)-st stage. From the viewpoint of the i-th stage, both structures in Fig.5.9(a) and (b) have the same arithmetic rate, however, the structure of Fig.5.9(b) uses less number of storages and is readily in the form to save computations.

Compare the new decimator structure in Fig.5.9(b) with the conventional decimator structure in Fig.5.6(a), it is found that the former could be designed with a lower arithmetic rate than the the latter due to the fact that the \( H_{p,i}^{D_i}(z^{i}) \) at the i-th stage in the former structure can be moved through the sampler with decimation \( D_i \), in other words, it has been moved to a lower sampling rate.

For the new decimator structure in Fig.5.9(b), the design problem is reduced to finding the optimal multistage filter with (5.10) as the transfer function at its i-th stage, so that the frequency response (5.12b) will meet the specifications in (5.10). If the specifications of
the overall filter are distributed uniformly into each stage, then the passband ripple $\delta_{p,i}$ and stopband ripple $\delta_{s,i}$ at the $i$-th stage of the new decimator are given by

$$\delta_{p,i} = \frac{\delta_p}{K}, \text{ and } \delta_{s,i} = \delta_s, \ i = 1, 2, \ldots, K,$$

(5.14)

where $\delta_p$ and $\delta_s$ are respectively the passband and stopband ripples of decimator. If filter with transfer function in (5.13) is used in each stage, then the design algorithm in Section 5.2 is used to find the filters to meet the requirements at each stage.

5.3.4 Design Examples

Example 3:

The filter designed in this example has the specifications: $F_p = 0.0225$, $F_s = 0.025$, $\delta_p = 0.05$ and $\delta_s \leq 0.005$.

This example is from [9]. The length of a linear phase FIR filter that satisfies above specifications is estimated to be 653, which gives a multiplication rate of 16.3 mults/sample. If two-stage decimation ($D_1=10$ and $D_2=2$) is used to implement the filter with linear phase FIR filter ($D=1$), the required multiplication rate was found to be 3.65 mults/sample. The frequency response of the two-stage conventional linear phase FIR multirate filter is shown in Fig.5.10.

Now, the filter is designed with two-stage new class linear phase FIR filters. The specifications for each stage are described as follows
First Stage: \( D_1 = 10 \), input sampling frequency \( F_{r0} = 1 \), output sampling frequency \( F_{r1} = 1/10 \), \( F_{p,1} = 0.0225 \), multiple stopband \([0.075, 0.125], [0.175, 0.225], [0.275, 0.325], [0.375, 0.425]\) and \([0.475, 0.5]\), \( \delta_{p,1} = 0.025 \) and \( \delta_{s,1} \leq 0.005 \). A filter with transfer function of the form \( H_s(z)H_p(z^{10}) \) and \( N_{s1,LP} = 26, N_{p1,LP} = 2 \) satisfies these specifications. The frequency response of the filter \( H_1(z) \) is shown in Fig. 5.11(a).

Second Stage: \( D_2 = 2 \), input sampling frequency \( F_{r1} = 1/10 \), output sampling frequency \( F_{r2} = 1/20 \), \( F_{p,2} = 0.0225 \), \( F_{s,2} = 0.025 \), \( \delta_{p,2} = 0.025 \) and \( \delta_{s,2} \leq 0.005 \). A new class of linear phase FIR filter with transfer function of the form \( H_s(z)H_p(z^2) \) was designed to satisfy this specifications. The resulting filter has \( N_{s2,LP} = 13 \) and \( N_{p2,LP} = 40 \), which gives the multiplication rate 2.90 multiplies/sample. The frequency response of \( H_2(z) \) is shown in Fig. 5.11(b). The magnitude response of the filter with two stage implementation is shown in Fig. 5.11(c).

If the filter in this example is implemented as three-stage with \( D_1 = 5 \), \( D_2 = 2 \) and \( D_3 = 2 \). The three-stage design with conventional linear phase FIR filters needs a multiplication rate of 4.05 multiplies/sample. Its frequency response is shown in Fig. 5.11(d). Now the filter is implemented with the new class of linear phase FIR filters. The specifications for each stage are described as follows:
Fig. 5.9 New decimator structure: (a) new structure
(b) equivalent structure.
Fig. 5.10 Frequency response of the two stage conventional FIR filter designed in Example 3.

Fig. 5.11(a) Frequency response of $H_1(z)$ of two stage design in Example 3.
Fig. 5.11(b) Frequency response of $H_2(z)$ of two stage design in Example 3.

Fig. 5.11(c) Frequency response of the narrow-band filter implemented with two stage in Example 3.
Fig. 5.8 Illustration of the commutative rule.

Fig. 5.11(d) Frequency response of the three-stage conventional linear phase FIR filter designed in Example 3.
Fig. 5.12(a) Frequency response of $H_1(z)$ of three stage design in Example 3.

Fig. 5.12(b) Frequency response of $H_2(z)$ of three stage design in Example 3.
**First Stage**: \( D_1 = 5 \), input sampling frequency \( F_{r0} = 1 \), output sampling frequency \( F_{r1} = 1/5 \), \( F_{p1} = 0.0225 \), multiple stopbands \([0.175, 0.225] \) and \([0.375, 0.425] \), \( \delta_{p,1} = 0.0167 \) and \( \delta_{s,1} \leq 0.005 \). The new class filter with transfer function \( H_{s}(z)H_{p}^{5}(z) \) that satisfies these specifications has \( N_{s1,LP} = 10 \) and \( N_{p1,LP} = 2 \). The frequency response of \( H_1(z) \) is shown in Fig.5.12(a).

**Second Stage**: \( D_2 = 2 \), input sampling frequency \( F_{r1} = 1/5 \), output sampling frequency \( F_{r2} = 1/10 \), \( F_{p,1} = 0.0225 \), \( F_{s,2} = 0.075 \), \( \delta_{p,2} = 0.0167 \) and \( \delta_{s,2} \leq 0.005 \). The new class filter with transfer function of the form \( H_{s}(z)H_{p}^{2}(z) \) that satisfies this specifications has \( N_{s2,LP} = 4 \) and \( N_{p2,LP} = 2 \). The frequency response of \( H_2(z) \) is shown in Fig.5.12(b).

**Third Stage**: \( D_3 = 2 \), input sampling frequency \( F_{r2} = 1/10 \), output sampling frequency \( F_{r3} = 1/20 \), \( F_{p,2} = 0.0225 \), \( F_{s,3} = 0.025 \), \( \delta_{p,3} = 0.0167 \) and \( \delta_{s,3} \leq 0.005 \). The new class filter with the transfer function of the form \( H_{s}(z)H_{p}^{2}(z) \) that satisfies this specifications has orders \( N_{s3,LP} = 13 \) and \( N_{p3,LP} = 42 \). Fig.5.12(d) shows the frequency response of \( H_3(z) \).

The multiplication rate of the filter with three stage implementation is thus found to be 32.25 multiplies/sample. The results obtained in this example are summarized in TABLE 5.1. Also, the results obtained by Chu's approach[43] are included in this table. The magnitude response of this filter is shown in Fig.5.13.
From TABLE 5.1, it is found that, the multiplication rate of the present approach is much lower than that obtained by using the conventional linear phase FIR filters as the procedure described by Rabiner and Crochiere's approach[10], but is a little bit higher than that obtained by Chu's approach[43]. The reason is that, in the present approach, the compensators \( H_{p,i}(z) \)'s are distributed at each stage instead of at the final stage as employed in Chu's approach. But, in the present approach, it is easier to design the compensators because the spectrum needed to be compensated is not as sharp as in Chu's approach. Finally, in the present case, no "trial and error" method is required to design a decimator.

**Example 4:**

The specifications of the filter designed in this example are: \( F_p = 0.1, F_s = 0.2, \delta_p = 0.001, \delta_s \leq 0.0025. \)

A two-stage implementation with \( D_1 = 2 \) and \( D_2 = 2 \) is used. If this two-stage decimation is implemented with conventional linear phase FIR filter (\( D=1 \)), the required multiplication rate is found to be 10,000 multi/sample. Fig.5.14 shows its frequency response.

Now, the filter is implemented with two-stage (\( D_1 \) and \( D_2 \)) of new class linear phase filter. The specifications at each stage are described as follows:
First Stage: \( D_1 = 2 \), input sampling frequency \( F_{r0} = 1 \), output sampling frequency \( F_{r1} = 1/5 \), \( F_{p,1} = 0.1 \), \( F_{s,1} = 0.38 \), \( \delta_{p,1} = 0.0005 \) and \( \delta_{s,1} \leq 0.0025 \). The new class filter with transfer function \( H_{s,1}(z)H_{p,1}(z^2) \) that satisfies these specifications has length \( N_{s1,LP} = 5 \), \( N_{p1,LP} = 6 \). The frequency response of \( H_1(z) \) is shown in Fig. 5.15(a).

Second Stage: \( D_2 = 2 \), input sampling frequency \( F_{r2} = 1/2 \), output sampling frequency \( F_{r2} = 1/4 \), \( F_{p,2} = 0.0225 \), \( F_{s,2} = 0.13 \), \( \delta_{p,2} = 0.0005 \) and \( \delta_{s,2} \leq 0.0025 \). The new class filter with transfer function \( H_{s}(z)H_{p}(z^2) \) that satisfies these specifications has length \( N_{s2,LP} = 11 \) and \( N_{p2,LP} = 30 \). The frequency response of \( H_2(z) \) is shown in Fig. 5.15(b).

The multiplication rate of the filter with two-stage implementation is thus found to be 8.0 mults/samples. A 20.0% saving of multiplication rate is obtained in this example by implementing the filter with new class linear phase FIR filters compared to that of general linear phase FIR filters. The magnitude response of the resulting filter is shown in Fig. 5.15(c).

5.4 New Class of Nonlinear Phase FIR Digital Filters

5.4.1 New Class of NLPEIR Digital Filters

In Chapter IV, the transfer function of a nonlinear phase FIR filters was decomposed into the product of a passband transfer function \( H_p(z) \) corresponding to all the zeros in the passband and a stopband
Fig. 5.12(c) Frequency response of $H_3(z)$ of three stage design in Example 3.

Fig. 5.13 Frequency response of the narrow-band filter implemented with three-stage new class of FIR filters.
<table>
<thead>
<tr>
<th>Type of filter</th>
<th>Decimation ratio</th>
<th>Filter order</th>
<th>Number of coeff's</th>
<th>Mult. rate</th>
<th>Data storage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-stage FIR</td>
<td>D₁ = 10</td>
<td>N₁ = 36</td>
<td>55</td>
<td>3.65</td>
<td>109</td>
</tr>
<tr>
<td></td>
<td>D₂ = 2</td>
<td>N₂ = 73</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Two-stage linear phase new filter</td>
<td>D₁ = 10</td>
<td>Nₛ₁ = 26</td>
<td>Nₚ₁ = 2</td>
<td>44</td>
<td>2.90</td>
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<td></td>
<td>D₂ = 2</td>
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<td>Nₚ₂ = 40</td>
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<td></td>
</tr>
<tr>
<td>Two-stage FIR (by Chu)</td>
<td>D₁ = 10</td>
<td>N₁ = 25</td>
<td>N₂ = 10</td>
<td>39</td>
<td>2.60</td>
</tr>
<tr>
<td></td>
<td>D₂ = 2</td>
<td>N₃ = 41</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two-stage nonlinear phase new filter</td>
<td>D₁ = 10</td>
<td>Nₛ₁ = 26</td>
<td>Nₚ₁ = 2</td>
<td>49</td>
<td>3.15</td>
</tr>
<tr>
<td></td>
<td>D₂ = 2</td>
<td>Nₛ₂ = 12</td>
<td>Nₚ₂ = 24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three-stage FIR</td>
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<td>N₁ = 14</td>
<td>53</td>
<td>4.05</td>
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</tr>
<tr>
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<td>N₂ = 7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D₃ = 2</td>
<td>N₃ = 80</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Three-stage linear phase new filter</td>
<td>D₁ = 5</td>
<td>Nₛ₁ = 10</td>
<td>Nₚ₁ = 2</td>
<td>42</td>
<td>3.25</td>
</tr>
<tr>
<td></td>
<td>D₂ = 2</td>
<td>Nₛ₂ = 4</td>
<td>Nₚ₂ = 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D₃ = 2</td>
<td>Nₛ₃ = 13</td>
<td>Nₚ₃ = 42</td>
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<td></td>
</tr>
</tbody>
</table>
Fig. 5.14 Frequency response of two stage conventional FIR filter designed in Example 4.

Fig. 5.15(a) Frequency response of $H_1(z)$ of two stage design in Example 4.
Fig. 5.15(b) Frequency response of $H_2(z)$ of two stage design in Example 4.

Fig. 5.15(c) Frequency response of the narrow-band filter implemented with two stage new class filters in Example 4.
transfer function $H_s(z)$ corresponding to all the zeros in the stopband.

Now, consider a new class of nonlinear phase FIR filters with the transfer function of the form

$$H(z) = H_s(z)H_p(z^D) = \left[ \sum_{n=0}^{N_{s,\text{NLP}}} h_s(n)z^{-n} \right] \left[ \sum_{n=0}^{N_{p,\text{NLP}}} h_p(z^{-nD}) \right], \quad (5.15)$$

where $D$ is a positive integer and $H_p(z^D)$ is a transfer function whose zeros consist of all the zeros of $H(z)$ in the passband, and $H_s(z)$ is a transfer function which includes all the zeros of $H(z)$ on the unit circle. Note that if $D = 1$, then $H(z)$ in (5.12) is reduced to the conventional class of nonlinear phase FIR filters discussed in Chapter IV.

Define a filter whose transfer function is

$$H_p(z^D) = H_p(z^D)H_p(z^{-D})z^{-2N_{p,\text{NLP}}D} = \sum_{n=0}^{2N_{p,\text{NLP}}} h_p(n)z^{-nD}, \quad (5.16)$$

where

$$h_p(n) = \sum_{j=0}^{N_{p,\text{NLP}}} h_p(j)h_p(j-N_{p,\text{NLP}}), n=0,1,\ldots,N_{p,\text{NLP}}. \quad (5.17)$$

The amplitude response of $H_p(z^D)$ evaluated along the unit circle is described by

$$|\tilde{H}_p(DF)| = |H_p(DF)| |H_p(-DF)e^{-j2\pi N_{p,\text{NLP}}DF}| = |H_p(DF)|^2, \quad (5.18)$$

Introduce a transfer function $\tilde{H}(z)$ by

$$\tilde{H}(z) = H_s(z)\tilde{H}_p(z^D). \quad (5.19)$$

It has been shown that, for designing the NLPFIR filter $H(z)$ in (5.15)
with passband ripple $\delta_p$ and stopband ripple $\delta_s$, it is only necessary to design a filter $\tilde{H}(z)$ in (5.19) with $\tilde{\delta}_p$ ($\approx 2\delta_p$), and $\tilde{\delta}_s$ ($= \delta_s$).

The transfer function in (5.16) will be suitable for describing the class of nonlinear phase FIR filters whose passband edge are smaller than 0.5/D due to the fact that $H_p(DF)$ is a function of DF. By suitably choosing the integer D, the new class of nonlinear phase FIR filters will yield a more efficient design than the conventional nonlinear phase FIR filter design with respect to a given magnitude response.

### 5.4.2 Design Algorithm

The design algorithm described in Section 5.2 is easily modified to the design of nonlinear phase FIR filters of this class. But in this case, we design $\tilde{H}(z)$ in (5.19) instead of (5.15). After finding $\tilde{H}_p(z^D)$, the $H_p(z^D)$ is constructed from $\tilde{H}_p(z^D)$. The approximation error function at the k-th iteration in the band $A_s$ becomes

$$E_{s,k}(F) = [\tilde{H}_{p,k}(DF)]^{1/2}[0 - H_{s,k}(F)]$$

Since $H_p(z^D)$ contains only power of $z^D$, the frequency response of $H_p(DF)$ repeats D-fold in the frequency interval $[0, F_s]$, where $F_s$ is the sampling frequency. As mentioned in Section 5.2, the transfer function $H_s(F)$ must be modified in order to compensate its frequency response when designing $\tilde{H}_p(z^D)$. In other words, we design $\tilde{H}_p(DF)$ in the frequency subset $A_p$. The approximation error function on $A_p$ at the k-th iteration becomes
\[ E_{p,k}(F) = |H_{s,k}(F)|^2 \left( \frac{1}{|H_{s,k}(F)|^2 - \bar{H}_{p,k}(DF)} \right) , \quad F \in A_p \]  

(5.21)

The Remez algorithm is applied to design \( H_{s,k}(F) \) and \( \bar{H}_{p,k}(DF) \) in (5.20) and (5.21) respectively. Based on this approach, the filter with the desired ripple ratio can be designed by controlling the \( d_{0.5} \) value.

### 5.4.3 Design Examples

**Example 5:**

A filter with the same specifications described in Example 1 is designed as a nonlinear phase FIR filter belonging to the new class.

A filter with orders \( N_{s,\text{NLP}} = 15 \) and \( N_{p,\text{NLP}} = 11 \) satisfies the specifications and gives the multiplication rate 20.0 mults/sample. The magnitude response of the resulting filter is shown in Fig.5.16. As mentioned in [5-6], when the symmetry of the filter coefficients cannot be exploited, such as in some of the applications in adaptive differential pulse-code modulation, the nonlinear phase FIR filter designed in this example gives 28 mults/sample, while its linear phase counterpart needs 34 mults/sample.

**Example 6:**

A filter with the same specifications as described in Example 2 is designed with the new class of nonlinear phase FIR filters.
Fig. 5.16 Frequency response of the nonlinear phase new class FIR filter designed in Example 5.

Fig. 5.17 Frequency response of the nonlinear phase new class FIR designed in Example 6.
A new class nonlinear phase FIR filter with lengths $N_p, N_l = 9$ and $N_p, N_l = 14$ satisfies the specifications. The multiplication rate is thus found to be 20 mults/sample. The magnitude response of the filter is shown in Fig.5.17. In this example, the multiplication rate, when the symmetry of the filter coefficients, cannot be exploited, is reduced from 33 mults/sample for the linear phase FIR filter to 25 mults/sample for the nonlinear phase FIR filter.

5.5 Design of Multirate Filters with New Class of NLPFIR Filters

5.5.1 Conventional NLPFIR Decimator Design

The decimator structure in Fig.5.6(a) can be used as the structure of nonlinear phase FIR decimator where the filter $H_i(z)$ $i = 1, 2, \ldots, K$, at the $i$-th stage has the transfer function in the form of (5.15). The design procedure of this class of NLPFIR decimator is equivalent to designing an NLPFIR filter at each stage so that the filter at each stage will meet its requirements. If the overall specifications of the filter is distributed into each stage, then the passband and stopband ripples at the $i$-th stage will be $\delta_{p,i} = \delta_p / K$ and $\delta_{s,i} = \delta_s / i = 1, 2, \ldots, K$, where $\delta_p$ and $\delta_s$ are the passband and stopband ripples requirements of the overall decimator filter respectively. The design procedure of the NLPFIR decimator is the same as that of linear phase FIR decimator except that in the former, the design algorithm developed in Chapter IV can be employed to design the filter at each stage.
5.5.2 Design of Decimators with New Class of NLFIR Filters

By introducing the new class of nonlinear phase FIR filters into each stage of a nonlinear phase FIR decimator, the resulting decimator filter is the same as that shown in Fig. 5.9(a) except that the \( H_i(z) \) at the \( i \)-th stage \( (i = 1, 2, \ldots, K) \) is realized with a nonlinear phase FIR filter. The multistage nonlinear phase FIR filter in Fig. 5.9(a) is easily transformed into a decimator structure same as that in Fig. 5.9(b) except that \( H_p(z) \) at the \((i+1)\)-st stage is a nonlinear phase FIR filter.

The frequency response of the nonlinear phase FIR decimator has the same form as that of linear phase FIR filter in (5.12). Comparing the new decimator structure in Fig. 5.9(b) with the conventional decimator structure in Fig. 5.6(a), it is found that the former could be designed with lower arithmetic rate than the latter due to the fact that the \( H_{p,i}^{D_i}(z^i) \) in the \( i \)-th stage of the former structure can be moved through the sampler \( D_i \). In other words, it has been moved to a lower sampling rate.

For the new decimator structure in Fig. 5.9(b), the design problem is reduced to designing the nonlinear phase FIR filter with transfer function in the form (5.15) at each stage so that the overall frequency response of the decimator will meet the specifications.

5.5.3 Filter Order Estimation
For all the filters designed, it is found that the filter order $N_{s,NLP}$ is larger than $M_1 + M_2$ but less than $3(M_1 + M_2)/2$, where $M_1$ and $M_2$ are defined in (5.5) and (5.6) respectively. A good initial value for filter order is taking $N_{p,NLP} = 2N_z$ where $N_z$ is the number of passband extremal frequencies of conventional linear phase FIR filter that satisfies the same specifications. The order $N_{s,NLP}$ and $N_{p,NLP}$ can be adjusted until the specifications are satisfied. The procedure is the same as the one described in Section 5.2.

The zero locations of the filter will have the same distribution as the that of linear phase FIR filter discussed in Section 5.2 except that in this case, the zero distribution of $H_p(z^D)$ will depend on the choice of the zeros when constructing $H_p(z^D)$ from $\tilde{H}_p(z^D)$.

### 5.5.4 Design Example

**Example 2:**

The same filter specifications in Example 3 is used for the design of the new class nonlinear phase FIR filters.

A two-stage ($D_1 = 10$, $D_2 = 2$) implementation is considered here. The filter at the $i$-th stage has the transfer function of the form $H_{p,i}(z)\tilde{H}_{s,i}(z^D)$, $i = 1,2$. The orders of the filter that satisfies the specifications of the first stage and the second stage are $N_{s1,NLP} = 26$, $N_{p1,NLP} = 2$ and $N_{s2,NLP} = 12$, $N_{p2,NLP} = 24$ respectively. The multiplication rate is thus found to be 3.15 muls/sample. The magnitude response
Fig. 5.18 Frequency response of the narrow-band filter implemented with two stage new class nonlinear phase FIR filters in Example 7.
of the resulting filter is shown in Fig. 5.18.

The results of this example are also included in TABLE 5.1. It is found from TABLE 5.1 that the amount of data storage of the new class nonlinear phase FIR decimator is less than that obtained by Chu's approach [43] and its linear phase counterpart.

As mentioned in [5-6], when the symmetry of the filter coefficients cannot be exploited, the multiplication rate is reduced from 5.75 mults/sample for the linear phase FIR filter, to 4.90 mults/sample for its nonlinear phase counterpart.

5.6 Conclusions

Two new classes of FIR digital filters with transfer functions of the form $H(z)H_p(z^D)$ have been presented and an efficient iterative algorithm for the design of these classes of FIR filters has also been given. With respect to a given frequency response, a filter with significantly reduced number of multiplications compared to the conventional design has been obtained by suitably choosing the integer $D$. By introducing these classes of filters into the design of multistage multirate decimators and interpolators, we have obtained an efficient implementation of a narrow-band FIR filter compared to its conventional linear phase counterpart due to the facts that $H_p(z^D)$ have been moved through the stage with a lower sampling rate and the spectrum shaping of $H_p(z^D)$ in the stopband. Several design examples have shown that the performance
of these classes of FIR filter is better than the conventional FIR design with respect to the computational rate and the amounts of storages. The distribution of zeros of the new classes of FIR filters has also been discussed with the aid of the given examples.
CHAPTER VI

DESIGN OF OPTIMAL NYQUIST, PARTIAL RESPONSE, N-th BAND AND NONUNIFORM TAP SPACING FIR DIGITAL FILTERS USING LINEAR PROGRAMMING TECHNIQUES

6.1 Introduction

In this chapter, the design of linear phase FIR filters with some of the coefficients constrained to be zero is considered. Specifically, we will focus on the following objectives in this chapter:

(a) \textit{design of optimal digital Nyquist and Class 1 partial response FIR filters};

(b) \textit{design of optimal half-band and N-th band FIR filters};

(c) \textit{design of optimal nonuniform tap spacing FIR filters}.

In each case, the filter design problem is formulated as a linear program so that the LP approach can be employed to design the filter. A number of examples are presented to illustrate the concept and the efficiency of the design techniques.

In Section 6.2, the LP formulation and solution of the constrained FIR filter design problem is presented, and it is found that LP techniques are particularly suitable for designing the above types of constrained FIR filters due to their high degree of flexibility. In Section 6.3, the design of optimal Nyquist and Class 1 partial response FIR pulse shaping filters, both with zero intersymbol interference is considered, and the design efficacy is compared with that of other design
methods. Section 6.4 describes the design of optimal half-band and N-th band FIR filters. The nonuniform tap spacing FIR filter, obtained by thinning the high order uniform tap spacing FIR filter and designed via the LP approach, is described in Section 6.5. The performance with respect to finite wordlength of constrained filters designed by using the LP techniques discussed here is examined by means of examples in Section 6.6. Concluding remarks are presented in Section 6.7.

6.2 Linear Programming Solution of Constrained Linear Phase FIR Filter Design

Let the transfer function of a linear phase constrained FIR filter with odd-length N be of the form

\[ H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}, \tag{6.1} \]

where \( I_c \) denotes the set of indices of \( h(n) \)'s which are constrained to be zero. The magnitude response \( H^*(F) \) of the constrained filter can be written in the form[16]

\[ H^*(F) = \sum_{n=0}^{(N-1)/2} \tilde{h}(n)\cos(2\pi nF), \tag{6.2} \]

where, and henceforth, we use the abbreviated notation \( P(F) \) for \( P(e^{j2\pi F}) \) obtained from a given function \( P(z) \), and \( \tilde{h}(n) \) can be derived from \( h(n) \) as
\[ \tilde{h}(0) = 2h(\frac{N-1}{2}) \neq 0, \text{ and } \tilde{h}(n) = 2h(\frac{N-1}{2} - n), n=1,2,\ldots,(N-1)/2, m \in I_c. \] (6.3)

The number of filter coefficients becomes

\[ NFILT = N - N_c, \] (6.4)

where \( N_c \) is the number of elements in \( I_c \) and the number of independent \( \tilde{h}(n) \)'s is thus \( m = (NFILT+1)/2 \).

The design of a constrained FIR filter described by (6.1) consists of finding \( \tilde{h}(n) \)'s, \( n = 0,1,2,\ldots,(N-1)/2, m \in I_c \), such that \( H^*(F) \) in (6.2) is the best approximation to the desired frequency response, where by the best approximation we mean the one which minimizes the maximum absolute error between \( H^*(F) \) and \( D(F) \) over the frequency bands to which the approximation applies.

By means of (6.2), the above design objectives may be formulated as a linear programming problem. By taking \( N_g \) grid points from 0 to \( 2\pi \) radians/sample (usually a grid density of 16 is used), the magnitude response of the filter at the grid point \( k \) is

\[ H^*(F_k) = \sum_{n=0}^{(N-1)/2} \tilde{h}(n) \cos(2\pi n F_k), \quad k = 0,1,\ldots,N_g/2. \] (6.5)

Let \( D(F_k) \) and \( W(F_k) \) denote respectively the desired frequency response and the desired weighting value for the approximating error at the grid point \( k \), and let \( \delta \) be the maximum allowable approximation error. Accordingly, it has to satisfy the following set of linear inequalities\[66\]

\[ -\delta \leq W(F_k)[D(F_k) - H^*(F_k)] \leq \delta, \quad k = 0,1,\ldots,N_g/2, \] (6.6)
Taking into account the fact that $h^*(F)$ is a linear combination of $r$ cosine functions, we may use (6.5) to formulate the following linear program:

$$\text{Maximize}(-\delta),$$

subject to:

$$(6.7a)$$

$$-w(F_k) \sum_{n=0}^{(N-1)/2} h(n) \cos(2\pi n F_k) - \delta \leq -w(F_k) D(F_k), \quad k = 0, 1, \ldots, N/2, \quad n \notin I_c$$

$$(6.7b)$$

$$w(F_k) \sum_{n=0}^{(N-1)/2} h(n) \cos(2\pi n F_k) - \delta \leq w(F_k) D(F_k), \quad k = 0, 1, \ldots, N/2, \quad n \notin I_c$$

$$(6.7c)$$

This is the "primal problem" with variables $h(n)'s, n = 0, 1, \ldots, (N-1)/2, n \notin I_c$ and $(-\delta)$.

By the duality principle, (6.7) can be shown to be mathematically equivalent to the "dual problem", which is a linear program in standard form. The standard form is the most natural form for digital filter design, and thus is commonly used for obtaining the desired numerical solution. The primal problem is transformed into the dual problem by replacing each inequality in (6.7) by an equality and a slack (nonnegative) variable. Since the nonnegativity constraints may be taken into account without increasing the volume of computations, the preceding transformation may be interpreted as replacing an inequality by an equality at the cost of adding one variable. The dual problem of (6.7) has one equality constraint for each of the unconstrained variables.
\(H(n)'s\) (\(n = 0, 1, \ldots, (N-1)/2, N/2 I\)), \(-\delta\) in (6.7) and one nonnegative variable for each of the inequality constraints in (6.7).

The solution to the above LP problem with \((r+1)\) variables and \(N/2+1\) inequality constraints occurs when at least \((r+1)\) of the \(N/2+1\) equations are solved with equality (instead of inequality); the remaining inequalities being strict with inequalities. For the optimal filter design problem this implies that there are at least \((r+1)\) frequencies at which the ripple achieves a maximum. The number of variables in the linear program of (6.7) is \((r+1)\), where \(r\) independent coefficients and \(\delta\) are variables.

It is usual in these problems to solve the dual problem by the revised simplex algorithm[63]. The tableau used in the revised simplex algorithm for solving the dual problem can be obtained from (6.7). Based on this tableau, the steps involved in the revised simplex algorithm at each pivot are outlined as follows:

**Step 1:** Determine an initial program;

**Step 2:** Price each column by calculating its relative cost \(\bar{c}_j\). If no \(\bar{c}_j\) is negative, the present program is the optimal solution; if not, change the basis; the minimum \(\bar{c}_j\) determines the column \(k\) to enter the basis so as to maximize the absolute change of the objective function. This is equivalently accomplished by using the maximum entry criterion.
For implementing the design procedure on a digital computer, a suitable small number must be used in this step to test the optimality of the program. If it is too small, invalid pivots will be induced due to the accumulated roundoff errors; if it is too large, valid pivots will be overlooked and suboptimal solution produced.

**Step 3.** The secondary variable \( x_k \) associated with column \( k \) becomes a basic variable and the basic variable \( x_1 \) associated with column 1 becomes a secondary variable. Generate column \( k \) by using an \( m \times m \) inverse-basis matrix which is carried along from pivot to pivot. The quantities associated with the old pivot is transformed to the corresponding quantities associated with the new pivot.

**Step 4.** Use the usual ratio test to choose a row \( l \), which then determines the column which is to leave the basis by the minimum exit criterion. If a tie occurs, break the tie by choosing one corresponding to the largest pivot from among the rows where the tie occurs. Same as pricing operation, a suitable small number must be used in the ratio test.

**Step 5.** Update the \( m \times m \) inverse-basis matrix by pivoting operation, then go to Step 2.

The two-phase method is used in the program, where, in phase 1, one determines whether any feasible solution exists and obtains one if it does exist, and in phase 2, one proceeds from a feasible solution determined by phase 1 to the optimal solution. The program in [14] is modi-
fied to allow the design of the constrained filter formulated in (6.7).

It must be observed that the solution obtained through the LP approach is exactly the same as the one obtained using the Remez algorithm. The well-known simplex algorithm used in the solution of the linear programming problems, can be viewed as a single exchange algorithm. Thus, the linear programming formulation leads to an algorithm less efficient than the Remez multiple exchange algorithm. But the linear programming approach has the advantage of being more flexible. Its flexibility is based essentially on the fact that other constraints can be considered in addition to (6.7). For designing the constrained filter discussed in this paper, the constraint inequalities in (6.7) keep the same form as that in the unconstrained filter design. Details on the revised simplex algorithm for solving the dual program can be found in the references on linear programming (see, e.g., see [67]).

6.3 Design of Pulse Shaping FIR Filters

6.3.1 Design of FIR Nyquist Filters with Zero Intersymbol Interference

Let the transfer function of the FIR Nyquist filter be \( H(z) \) in (6.1) and the desired frequency response be \( D(F) \). In the design of \( H(z) \), \( D(F) \) takes the following values:

\[
D(F) = 1, \ 0 \leq F \leq \frac{1-r}{2M},
\]

\[
= 0, \ \frac{1+r}{2M} \leq F \leq 0.5, \quad (6.8)
\]
with the constraints:

\[ h(n) = 0, \text{for } ((n - \frac{N-1}{2})_M = 0, \text{and } n \neq \frac{N-1}{2}, n = 0, 1, \cdots, N-1, \] (6.9)

where \((x)_M\) denotes modular operation with mod \(M\). The design of this digital filter can be represented as the following minimax approximation problem with (6.9) as constraints on filter coefficients

\[ W(f)[|H(f)| - 1] \leq \delta, \quad 0 \leq F \leq F_p, \] (6.10a)

and

\[ W(f)|H(f)| \leq \delta, \quad F_s \leq F \leq 0.5, \] (6.10b)

where \(W(f)\) is the weighting function for controlling the ripples in the passband and stopband.

The constrained approximation problem of (6.9) and (6.10) can be solved by using the LP techniques described in Section 6.2.

**6.3.2 Optimal Design of Class I Partial Response Filters**

Time domain Class I partial response data transmission systems have received considerable attention recently since they can be used at an increased bit rate under a prescribed available bandwidth for data transmission[68]. It has been shown that the bit insensitivity of the partial response system is such that the sampling rate of the system can be varied by 43 percent for the given filter configuration without changing the three-level eye pattern[58].
Fig. 2.1(c) shows the impulse response of a typical digital Class 1 partial response filter. The impulse response of a digital Class 1 partial response filter should be large and have the same value at main two adjacent sample points and the response at other sample points should be zero for zero intersymbol interference if the process of decision-directed cancellation of intersymbol interference is used [69]. Let the transfer function of the FIR Class 1 partial response filter be denoted by $H(z)$. Then $H(z)$ can be designed by approximating the desired frequency response under the following constraints on the filter coefficients:

$$h(n) = 0, \left((n - \frac{N-1}{2} \pm \frac{3M}{2})\right) \mod M = 0, n \geq \frac{N-1+3M}{2}, \text{ or } n \leq \frac{N-1-3M}{2}, \quad (6.12)$$

where $M$ is defined in (a). The filter design can be formulated as the following minimax approximation problem:

$$W(F)[|H(F)| - 1] \leq \delta, \quad F = 0, \quad (6.13a)$$

and

$$W(F)[|H(F)|] \leq \delta, \quad F_s \leq F \leq 0.5, \quad (6.13b)$$

where $F_s = 1/(2M)$, $W(F)$ is the weighting function for controlling the ripples in the passband and stopband, and $\delta$ is the maximum weighted approximation error. The LP techniques described in Section 6.2 can again be employed to design these Class 1 partial response filters.

6.3.3 Design Examples
The first two examples are from [58]. A DEC PDP11/55 computer with double precision was used for all calculations in designing the filters in this paper.

Example 1:

An FIR Nyquist filter with length $N = 23$, $\rho = 0.3$, $M = 4$ was designed.

The band edges $F_p$ and $F_s$ of the filter are calculated from (2.34) as $F_p = 0.0875$ and $F_s = 0.1625$ and the constraint in $I_c$ are obtained from the given $N$ and $M$ values. The LP technique was modified and thus used to design the filter. Filters with different passband and stopband ripples were designed by controlling the weighting values of passband and stopband ripples in (6.7). The impulse response and frequency response of a resulting filter with stopband ripple $A_s = 38.0$ dB and passband ripple 0.34 dB are shown in Fig.6.1(a) and (b) respectively.

The corresponding optimal filter with length 23 and stopband ripple $A_s = 38.0$ dB has a passband ripple of 0.19 dB which is less than that of the above Nyquist filter. This is because some of the coefficients of the latter filter with small values but not exactly equal to zero are constrained to zero.

Example 2:

A Nyquist filter same as in Example 1 was designed but with the parameters : $N = 39$, $\rho = 0.15$ and $M = 4$. 
Fig. 6.1(a) Impulse response of the filter designed in Example 1.

Fig. 6.1(b) Amplitude response of the filter designed in Example 1.
Nyquist filters corresponding to different passband and stopband ripples were designed. Figs. 6.2(a) and (b) show respectively the impulse response and frequency response of a resultant filter with stopband ripple $A_s = 33.0$ dB and passband ripple 0.45 dB. The corresponding optimal filter with length 39 has a passband ripple of 0.35 dB with the same stopband ripple.

Example 3:

A Class 1 partial response FIR filter with the following parameters was designed: $N = 23$ and $M = 4$.

The filter with $F_s = 0.125$ was designed to have equal passband and stopband ripples. Fig. 6.3(a) and (b) show the impulse response and frequency response with a stopband ripple of 33.2 dB and a passband ripple of 0.19 dB.

6.3.4 Comparison with the Iterative Chebyshev Approximation Method[57,59]

In our algorithm, when designing a filter with appropriate filter coefficients constrained to zero, it is only required to run the LP program one time for obtaining the zero intersymbol interference pulse shaping filter with the desired ripple ratio.
Fig. 6.2(a) Impulse response of the filter designed in Example 2.

Fig. 6.2(b) Amplitude response of the filter designed in Example 2.
Fig. 6.3(a) Impulse response of the filter designed in Example 3.

Fig. 6.3(b) Amplitude response of the filter designed in Example 3.
6.4 Design of Half-Band and N-th Band FIR Filters

6.4.1 Approximation Problem Formulation

The optimal half-band filters described in Section 2.4 can be designed by using the LP techniques described in Section 6.2. The parameters to the program can be calculated from (2.41a) and the weighting values for both the passband and stopband are set equal so that the same ripple values can be obtained at both bands and then the filter is designed with the constraint (2.36) on the filter coefficients.

The optimal N-th band filter can be designed in a similar way as that of half-band filter. The stopband edges of the filter are calculated from (2.40) and the ripple relation (2.41) can be assured by setting

\[ W_s \leq (N-1)W_p, \quad (6.14) \]

where \( W_p \) and \( W_s \) denote the weighting values respectively for passband and stopband ripples in (6.7). Then the N-th band filter is designed with the constraint (2.38) on the filter coefficients.

6.4.2 Design Examples

Example 4:

A third band FIR filter is designed with a passband from 0.0 to \( F_p = 0.1 \) and filter length 23.
The stopband edges of the desired third band filter are calculated from (2.40) to be 0.233333 and 0.433333. The weighting values between passband and stopband used are \( W_s = 2W_p \) so that (2.41) can be satisfied. By using these parameters as the inputs to our LP program in Section II, the third band filter was designed with the constraint (2.38) on the filter coefficients. The impulse response and frequency response of the resulting third band filter are shown in Fig. 6.4(a) and (b) respectively.

**Example 5:**

A low-pass fifth band filter was designed with passband edge \( F_p = 0.05 \) and a maximum passband ripple of 0.001.

The stopbands of the fifth band filter are calculated from (2.40) to be \([0.15, 0.25]\) and \([0.35, 0.45]\). In order to satisfy the requirement (2.41) and the maximum passband ripple, a filter of length 29 is required. The LP program described in Section 6.2 was used to design the filter. The passband and stopband ripples of the resulting fifth band filter are 0.031 dB and -61.03 dB respectively. Fig. 6.5(a) and (b) show its impulse response and frequency response.

6.4.3 Comparison with the Remax Algorithm

Mintzer[61] designed the suboptimal \( N \)-th band filters by using the computer program of McClellan, Parks and Rabiner[1] as described in Section 2.4. By using the LP techniques introduced in Section 6.2, the
Fig. 6.4(a) Impulse response of the filter designed in Example 4.

Fig. 6.4(b) Amplitude response of the filter designed in Example 4.
Fig. 6.5(a) Impulse response of the filter designed in Example 5.

Fig. 6.5(b) Amplitude response of the filter designed in Example 5.
optimal half-band, and N-th band filters can be obtained with only one run of the LP program. Furthermore, the optimality of the resulting N-th filter makes the LP technique more attractive than the one above.

6.5 Design of Nonuniform Tap Spacing FIR Filters

In this section, the LP techniques are employed to design the NUTS filter by thinning the high order UTS filter and using the minimax approximation error criterion instead of the WMSSE criterion so that the resulting filter will be optimal in the minimax sense.

6.5.1 Impulse Response of an FIR Digital Filter

The odd length impulse response of an ideal low-pass filter with bandwidth B is of infinite duration and of the form

\[ h_d(n) = \frac{\sin(2\pi B n)}{\pi n}, \quad -\infty < n < \infty. \]  

(6.15)

A practical finite duration impulse response of odd length N can be considered as the ideal impulse response \( h_d(n) \) times a finite duration window function \( W(n) \), i.e.,

\[ h(n) = h_d(n)W(n). \]  

(6.16)

where \( W(0) \) and \( W(N) = 0 \) for \( |n| > (N-1)/2 \). The impulse response of a practical odd length low-pass filter is thus

\[ h_{LP}(n) = \frac{W(n)}{\pi n} \sin(2\pi B n). \]  

(6.17)

From (6.17), if the desired filter has the bandwidth such that \( h_{LP}(n) \) is
essentially zero for some values of \( n \) as compared to other coefficients, then these corresponding filter coefficients can be thinned and the resulting NUTS filter could have smaller passband ripple and/or stopband ripple than the UTS filter of the same number of sample points.

### 6.5.2 Optimal Coefficients Selection

The construction of the NUTS filter involves the procedure for the optimal selection of filter coefficients from a high order UTS filter. In other words, given a high order filter with length \( M \), we would like to optimally select \( N \) coefficients from \( M \) coefficients \((M \geq N)\) so that the resulting NUTS with length \( N \) will be the best approximation to the desired frequency response in the minimax sense.

Let \( R \) represent the set of \( M \) coefficients \( h_1, \ldots, h_M \) of high order filter and \( S_i, S_j, \ldots \) be the subsets of \( R \), and let \( P(R) \) be the set of such subsets: \( S_i, S_j, \ldots \subset P(R) \). Let \( A_s \) be the stopband attenuation which is defined on \( P(R) \). The objective is to design the two filters so that they have the same passband ripple, and thus the performances of the two filters may be evaluated by \( A_s \). Let \( P_N(R) \) where \((1 \leq N \leq M)\) be the set of subsets of \( R \) with exactly \( N \) distinct elements.

The problem is to maximize the function \( A_s \) on \( P_N(R) \). If the elements of \( P_N(R) \) are denoted \( S_i^N, S_j^N, \ldots \), this amounts to finding an element of \( P_N(R) \), say \( S_0^N \), such that \( A_s(S_0^N) \geq A_s(S_i^N) \) for any \( S_i^N \in P_N(R) \). Clearly, the function \( A_s \) satisfies the following condition:
\[ A_s(S_i) \leq A_s(S_j) \text{, if } S_i \supseteq S_j \quad (6.18) \]

Then the selection of coefficients can follow the search algorithm which is based on (6.18) by Boyce[70]. The search algorithm consists of an exhaustive search in \( P_N(R) \), constrained by bounding rules and guided by a search-ordering procedure. The bounding rules allow certain sets to be discarded without computing their stopband attenuations \( A_s \). The search-ordering procedure is used to choose the next set to be examined, among all those that cannot be discarded by the bounding rules. The efficiency of the bounding rules depends to some extent on the search-ordering procedure.

(1) **Bounding Procedure**

Define the unconditional threshold of a coefficient \( h_k \) as

\[ U(h_k) = A_s(R - \{h_k\}) \text{, } k = 1, \cdots, M \quad (6.19) \]

and let \( S_{nc}^N \) be the best known of the stopband attenuation \( A_s \) on \( P_N(R) \).

Define the conditional threshold of the coefficient \( h_k \) given the coefficients \( h_{k_1}, h_{k_2}, \cdots, h_{k_a} \) as

\[ C(h_k | h_{k_1}, \cdots, h_{k_a}) = A_s(R - \{h_k, h_{k_1}, \cdots, h_{k_a}\}) \quad (6.20) \]

Suppose that the sets excluding \( h_{k_1}, \cdots, h_{k_a} \) are now to be examined, other sets including them having been examined previously and rejected. Then if
\[ A_s(S_c^N) \supset C(h_k| h_{k_1}, \ldots, h_{k_a}) \]  \hspace{0.5cm} (6.21)

the value of the stopband attenuation \( A_s \) on sets excluding \( h_{k_1}, \ldots, h_{k_a} \) can be improved only if the coefficient \( h_k \) is included. Hence, the sets excluding \( h_k, h_{k_1}, \ldots, h_{k_a} \) can be discarded.

(2) Search-Ordering Procedure

The search-ordering procedure includes the following steps:

(i) Begin with an arbitrary subset, say \( S_0^N = \{ h_0, \ldots, h_N \} \);

(ii) Delete the coefficient that causes the smallest decrease in the value of the \( A_s \), and that has not yet been deleted;

(iii) Add the coefficients that cause the largest increase in the value of the \( A_s \). If the result is larger than \( A_s(S_0^N) \), retain the new set as the best known set, and proceed as with \( S_0^N \) i.e., return to step (ii). If not, carry on the procedure of deleting one coefficient at a time from \( S_0^N \). If all the coefficients have been tried without success, start dropping two items at a time from \( S_0^N \), etc.

6.5.3 Practical Design Procedure

The optimal selection of the coefficients of the NUTS filter from a high order UTS filter is described in (b). It is the procedure for the optimal selection but is cumbersome for general filter design, especially for very high order filters. For practical design, it is indeed possible in some cases to design the filter by examining the zero-crossing
of the high order response and forcing zeros at the sampling times nearest these points. The design of the corresponding NUTS filter will pass its impulse response smoothly through these zeros to optimize the chosen design criterion. This will save considerable time for selecting the coefficients of the NUTS filter and usually, although not always, will yield the optimal or near-optimal selection.

After determining the tap spacings, the LP technique in Section 6.2 is employed to design the NUTS filter.

6.5.4 Design Examples

Several NUTS filters have been designed by thinning the high order UTS filters. Two examples are included here to illustrate the efficacy of the algorithm proposed here.

Example 6:

A low-pass NUTS filter with \( F_p = 0.15, F_s = 0.35 \) and 13 sample points was designed by thinning a length 19 UTS filter.

Basically, this is a half-band filter discussed in Section 6.5 of this paper. The UTS filter of length 13 is designed with passband ripple 0.049 dB and stopband ripple -44.8 dB. The optimal selection rules in (b) of this section was applied to search for the coefficients of the NUTS filter from the length 19 UTS filter, and then the NUTS filter with 13 points was designed with equal weighting values for the passband and stopband ripples. The resulting filter has a passband ripple of 0.0027
dB and a stopband ripple of -70.3 dB. Fig. 6.6 shows the frequency responses of the resultant UTS filter (light) and NUTS filter (dark). It is found that the UTS filter has a stopband ripple 25.2 dB lower than that of the corresponding UTS filter with the same number of sample points.

Example 7:

A band-pass filter with lower stopband edge $F_{LS} = 0.18$, lower passband edge $F_{LP} = 0.21$, upper passband edge $F_{UP} = 0.39$, upper stopband edge $F_{US} = 0.42$, and the number of sample points $N = 55$.

The NUTS filter was designed by thinning a length 61 UTS filter. Both filters are designed to have the same passband ripple of 0.67 dB. Fig. 6.7 shows the frequency responses of the UTS filter (light) and the NUTS filter (dark). It can be seen the NUTS filter has a stopband ripple 6.9 dB lower than that of the UTS filter corresponding to the same number of sampling points.

6.6 Coefficient Quantization Sensitivity

We wish to show, by means of examples, the coefficient quantization sensitivities of some of the constrained filters designed in this paper. Since the direct form realization is the most commonly used for FIR filters, this will be assumed in the subsequent comparisons. In this case, what is of main interest is the minimum attenuation in the stopband caused by the quantization of FIR filter coefficients.
Fig. 6.6 Amplitude responses of the UTS and NUTS filters designed in Example 6.

Fig. 6.7 Amplitude responses of the UTS and NUTS filters designed in Example 7.
Fig. 6.8 shows the minimum attenuations in the stopband for the unconstrained filter (curves with o and v) and constrained filter (curves with Λ and x) designed in Example 1 and 2. They exhibit almost the same performance.

A similar result is found in Figs. 6.9 and 6.10, where the minimum attenuation in the stopband for the filter designed respectively in Examples 4 and 5 and the suboptimal filter designed by Mintzer [61] using the Remez algorithm are shown.

From Fig. 6.11, where the minimum attenuation in the stopband for the filter of Example 7 and the uniform tap spacing filter are shown, it can be seen that both filters display almost the same performance with respect to the coefficient quantization.

6.7 Concluding Remarks

We have presented LP techniques for the design of an FIR digital filter with some of its coefficients constrained to zero. The first application discussed was that involving the design of digital Nyquist and partial response FIR filters with zero intersymbol interference, which are used in data transmission systems. The advantage of this approach over the two-step iteration Chebyshev method is its high design speed.

Another application discussed by us was to the design of half-band, and N-th band filters used in multirate digital filtering. The
Fig. 6.8 Minimum stopband attenuation caused by coefficient quantization of the filters in Example 1 and 2.

Fig. 6.9 Minimum stopband attenuation caused by coefficient quantization of the filters in Example 4.
Fig. 6.10 Minimum stopband attenuation caused by coefficient quantization of the filter.

Fig. 6.11 Minimum stopband attenuation caused by coefficient quantization of the filter in Example 6.
Fig. 6.12 Minimum stopband attenuation caused by coefficient quantization of the filters in Example 7.
optimality of the resulting filter makes our design algorithm more attractive than the Remez algorithm which yields suboptimal \( N \)-th band filters for designing \( N \)-th band filters.

The last application concerned the design of NUTS FIR filters by thinning the impulse response of high order UTS FIR filters. In some cases, with the same sampled data points for both filters, the former was found to show better performance than the latter with respect to the desired frequency response.

The sensitivity to coefficient quantization of the constrained filters was discussed by means of examples, and it was found that the constrained filters have almost the same coefficient quantization sensitivity as that of the corresponding unconstrained filter based on the direct form realization.

Examples have illustrated that the LP techniques are suitable for the design of FIR filters with some of their coefficients constrained to zero. Although only three applications of the constrained filters are considered in this paper, it is believed that many other applications could be found in the area of digital signal processing for which the LP design approach presented here would be especially suitable.
CHAPTER VII
CONCLUSIONS

Two new algorithms have been developed for the design of IIR and NLPFIR filters with attenuation zeros on the unit circle. In each case, the algorithm directly calls the linear phase FIR filter design algorithm to iteratively design the linear phase part and nonlinear phase part of the filter transfer function. Both algorithms have also been extended to the design of filters with constraints in the frequency domain. Linear programming techniques have also been developed to design FIR filters with constraints making some of the filter coefficients are restricted to zero.

For unconstrained and constrained IIR filters, our iterative algorithm designs the filter directly without guessing the passband ripple or the stopband ripple. As compared to the algorithms in [24,27], the proposed algorithm has a second advantage, namely the capability of finding the minimum ripple ratio for the given order and band edges of the filter so that a solution can always be found. The third advantage is that it can design the filter with the desired ripple ratio as in the design algorithm of LPFIR filter by McClellan at al[1].

For the design of NLPFIR filters, the transfer function of the filter is first decomposed into a nonlinear phase part and a linear phase part and then the linear phase part and the nonlinear phase part
are designed iteratively. The design algorithm has the following advantages over other algorithms used to solve the same design problem such as the one in [8]: (i) In the present algorithm, it is not necessary to guess the passband ripple or the stopband ripple when designing the filter; (ii) the algorithm permits one to design an NLPFIR filter with high stopband attenuation, (iii) the stopband zeros of the resulting NLPFIR filter are exactly on the unit circle even for a filter with high stopband attenuation, this fact simplifies the implementation of the resulting NLPFIR filter due to the symmetry of the coefficients of linear phase part; (iv) the algorithm can design an NLPFIR filter with a prescribed ripple ratio.

Two new classes of FIR digital filters with transfer functions of the form \( H_s(z)H_p(z^D) \) have been proposed, one linear phase and the other nonlinear phase and an iterative algorithm for their design has been presented. By suitably choosing the integer \( D \), a filter can be designed with significantly reduced number of multiplications compared to its conventional linear phase FIR counterpart, with respect to a given magnitude response. By introducing these new classes of FIR filters into the design of multistage decimators and interpolators for implementing the narrow-band filters, it has been found that an efficient narrow-band filter requiring considerably lower multiplication rate than the conventional linear phase FIR design can be obtained because, in this way, the transfer function \( H_p(z^D) \) can be realized at a lower sampling rate and the spectrum shaping of \( H_p(z^D) \) in the stopband.
Linear Programming techniques, when applied to the design of pulse shaping filters, have the advantage of high design speed over the two-step-iteration-based Chebyshev approximation method. The optimality property of the N-th band filter, designed by the linear programming techniques, makes it superior to that obtained by Mintzer's algorithm which is partly based on the Remez procedure as explained in Section 6.4. The nonuniformly tap spacing FIR filter, by thinning the high order impulse response of the uniform tap spacing FIR filter, in some cases, is found to have better performance than the corresponding uniform tap spacing filter of the same sampled data points. In this dissertation, an optimal coefficients selection rule has been presented to choose the coefficients of the filter. Also, the sensitivity to coefficient quantization of this type of constrained FIR filter has been shown to be almost the same as that of the corresponding unconstrained FIR filter.
Reference:


APPENDIX

COMPUTER PROGRAM FOR DESIGNING AN OPTIMAL RECURSIVE DIGITAL FILTER

THE MAIN PROGRAM FOR THE DESIGN OF A CERTAIN CLASS OF IIR
(RECURSIVE) DIGITAL FILTERS BY CALLING THE REMEZ ALGORITHM
OF MCCLELLAN ET AL. (IEEE TRANS. AUDIO ELECTROACOUST., VOL.
AU-21, PP. 506-526, DEC. 1973) TO ITERATIVELY DESIGN THE
NUMERATOR AND DENOMINATOR OF THE FILTER TRANSFER FUNCTION.

THIS MAIN PROGRAM WILL YIELD THE COEFFICIENTS OF N(Z) AND
D(Z). THE USERS OF THIS PROGRAM ARE REQUIRED TO
CONSTRUCT D(Z) FROM D(Z).

THE SUBROUTINE REMEZ CALLED BY THIS PROGRAM CONSISTS OF THE
ABOVE-MENTIONED ALGORITHM OF J.H. MCCLELLAN ET AL., AND IS
NOT INCLUDED HERE.

THE USERS OF THIS PROGRAM ARE REQUIRED TO PROVIDE THE PROGRAM
FOR GENERATING THE AMPLITUDE RESPONSES HS(F) AND HP(F) OF
THE CONSTRAINED FUNCTIONS TS(Z) AND TP(Z) RESPECTIVELY.

COMMON PI2, AD, DEV, X, Y, GRID, DES, WT, ALPHA, IEXT, NFCNS, NGRID
DIMENSION IEXT(66), AD(66), ALPHA(66), X(66), Y(66)
DIMENSION H(66), H1(66), H2(66)
DIMENSION DES(1024), GRID(1024), WT(1024)
DOUBLE PRECISION PI2, PI, HF, F, TSF, HS, EPSDS, EPSIT, DRATIO
DOUBLE PRECISION AD, DEV, X, Y, H, H1, H2, ALPHA, DES, GRID, WT, DIFF
DOUBLE PRECISION D05MIN, RATMIN, RMULT, RN02, DEV1, DEV2
PI2 = 6.283185307179586
PI = 3.141592653589793

INPUT THE REQUIRED PARAMETERS OF THE FILTER:
FP: PASSBAND EDGE OF THE FILTER;
FS: STOPBAND EDGE OF THE FILTER;
NNUM: ORDER OF THE NUMERATOR N(z);
NDEN: ORDER OF THE DENOMINATOR D(z);
DRATIO: DESIRED RIPPLE RATIO;
WO0: WEIGHTING VALUE AT F = 0.0;
DO0: DESIRED VALUE AT F = 0.0;
W05: WEIGHTING VALUE AT F = 0.5;
ITMAX: MAXIMUM NUMBER OF DESIGN CYCLES TO DESIGN A
FILTER WITH THE DESIRED RIPPLE RATIO;
IEPSDS: CRITERION SET TO STOP THE DESIGN CYCLES;
IEPSIT: CRITERION SET TO STOP THE ITERATIONS WITHIN
EACH DESIGN CYCLE;
RMULT: CHOOSE THE INITIAL D05 AS THIS MULTIPLE OF D05MIN;
RN02: PARAMETER TO CONTROL THE INITIAL D05 WHEN NDEN IS ODD;
119 FORMAT(2F10.5)
READ(5,119) FP, FS

111 FORMAT(2I3)
READ(5,111) NNUM, NDEN
N1 = NNUM+1
N2 = NDEN+1
NFF2 = N2/2
NODD2 = N2-2*NFF2
N2 = 2*N2-1

118 FORMAT(F20.10)
READ(5,118) DRATIO

112 FORMAT(3F10.5)
READ(5,112) D00,W00,W05

122 FORMAT(F10.5)
READ(5,122) RMULT
IF(NODD2 .NE. 0) GO TO 123
READ(5,122) RN02

994 FORMAT(3I5)

123 READ(5,994) IEPSDS, IEPSIT, ITDMAX
EPSDS = 10. **(IEPSDS)
EPSIT = 10. **(IEPSIT)
ITDS = 1
JTYPE = 1
LGRID = 16
NEG = 0
ND05 = 1
NF1 = N1/2
NO1 = N1-2*NF1
NF1 = NF1+NO1

C IF DENOMINATOR ORDER NDEN IS ODD, ESTIMATE THE D05MIN FROM C THE FILTER WITH DENOMINATOR ORDER NDEN+1.

C IF(NODD2 ,EQ. 1) GO TO 805
NF2 = N2/2+1
NO2 = N2-2*NF2+2
NF2 = NF2+NO2
H2(NF2) = 1.D0
GO TO 801

805 NF2 = N2/2
NO2 = N2-2*NF2
NF2 = NF2+NO2
DO 806 I=1,N2

806 H2(I) = 0.D0
H2(NF2) = 1.D0
NODD2 = 1
GO TO 801

C THE FILTER WITH THE DESIRED RIPPLE RATIO FOUND ? C

800 RATIO = DEV2/DEV1
DIFF = DABS(DRATIO-RATIO)/DRATIO
IF(DIFF .LT. EPSDS) GO TO 610
ITDS = ITDS+1
850  FORMAT(’ITDS = ’, I5)
      WRITE(6,850) ITDS

C THE MAXIMUM NUMBER OF DESIGN CYCLES EXCEEDED?
C   IF(ITDS .GT. ITDMAX) GO TO 600
C
C CALCULATE THE NEW D05 VALUE BASED ON THE 3/2 POWER RELATION.
C   D05 = ((DRATIO/RATIO)**0.66667)*D05

801  NSTATE = 1
1001  IF(NSTATE .EQ. 2) GO TO 1010
C
C DESIGN OF THE NUMERATOR N(Z) STARTING HERE.
C
C
NFILT = N1
NODD = NO1
NFCNS = NF1

C SPECIFY THE DESIRED VALUE AND WEIGHTING VALUE AT F = 0.0.
C
DEL = 0.5/FLOAT(LGRID*NFCNS)
GRID(1) = 0.0
DES(1) = D00
WT(1) = W00

C CALCULATE THE DESIRED FUNCTION AND WEIGHTING FUNCTION
C ON THE GRID IN THE STOPBAND OF THE FILTER.
C HSF(F) AND HPF(F) ARE RESPECTIVELY THE AMPLITUDE
C VALUES OF THE CONSTRAINED FUNCTIONS TS(Z) AND TP(Z)
C EVALUATED AT FREQUENCY F.
C
1020  DES(J) = 0.0
      F = GRID(J)
      WT(J) = 1./DSQRT(HF(F,H2,NO2,NF2))
      TSF = DABS(HSF(F))
      TPF = HPF(F)
      WT(J) = TSF*WT(J)/TPF
      J = J+1
      GRID(J) = GRID(J-1)+DEL
      IF(GRID(J) .LT. 0.5) GO TO 1020
      GRID(J) = 0.5
      F = GRID(J)
      DES(J) = 0.0
      WT(J) = 1./DSQRT(HF(F,H2,NO2,NF2))
      TSF = DABS(HSF(F))
      TPF = HPF(F)
      WT(J) = TSF*WT(J)/TPF
      NGRID = J
      C
CALL RENDEZ ALGORITHM TO DESIGN NUMERATOR N(Z).

GO TO 1030

DESIGN OF THE DENOMINATOR \( \tilde{D}(z) \) STARTING HERE.

1010 NFILT = N2
    IF(NODD .EQ. 0) NFILT = N2+2
    NODD = NO2
    NFCNS = NF2

CALCULATE THE DESIRED FUNCTION AND WEIGHTING FUNCTION ON THE GRID IN THE PASSBAND OF THE FILTER.
BSF(F) AND HPF(F) ARE RESPECTIVELY THE AMPLITUDE RESPONSES OF THE CONSTRAINED FUNCTIONS TS(Z) AND TP(Z) EVALUATED AT FREQUENCY F.

DELF = 0.5/FLOAT(LGRID*NFCNS)
J = 1
GRID(J) = 0.0D0

1040 F = GRID(J)
    T = HF(F,H1,NO1,NF1)
    T = T*T
    DES(J) = T
    TPF = HPF(F)*HPF(F)
    DES(J) = DES(J)/TPF
    TSF = HSF(F)*HSF(F)
    DES(J) = TSF*DES(J)
    WT(J) = 1./DESC(J)
J = J+1
GRID(J) = GRID(J-1)+DELF
IF(GRID(J) .LT. FP) GO TO 1040
GRID(J) = FP
T = HF(FP,H1,NO1,NF1)
T = T*T
DES(J) = T
TPF = HPF(FP)*HPF(FP)
DES(J) = DES(J)/TPF
TSF = HSF(FP)*HSF(FP)
DES(J) = TSF*DES(J)
WT(J) = 1./DESC(J)
IF(ND05 .EQ. 1) GO TO 1025

INCLUDE THE POINT F = 0.5 IN THE DESIGN PROCEDURE OF D(Z).

J = J+1
GRID(J) = 0.5
DES(J) = D05
WT(J) = W05

1025 NGRID = J
C SET UP A NEW APPROXIMATION PROBLEM WHICH IS EQUIVALENT
C TO THE ORIGINAL PROBLEM.
C
1030 IF(NEG .NE. NODD) GO TO 165
     IF(GRID(NGRID) .GT. (0.5-DELP)) NGRID = NGRID-1
     CONTINUE
C
165 IF(NODD .EQ. 1) GO TO 200
     DO 175 J = 1, NGRID
         CHANGE = DCOS(P1*GRID(J))
         DES(J) = DES(J)/CHANGE
     175 WT(J) = WT(J)*CHANGE
C
C INITIAL GUESS FOR THE EXTREMAL FREQUENCIES --
C EQUALLY SPACED ALONG THE GRID.
C
200 TEMP = FLOAT(NGRID-1)/FLOAT(NFCNS)
     DO 210 J = 1, NFCNS
     210 IEXT(J) = (J-1)*TEMP+1
     IEXT(NFCNS+1) = NGRID
     NMI = NFCNS-1
     NZ = NFCNS+1
C
C CALL THE REMEZ ALGORITHM TO DO THE APPROXIMATION PROBLEM.
C
CALL REMEZ
C
C CALCULATE THE IMPULSE RESPONSE OF THE FILTER.
C
300 IF(NODD .EQ. 0) GO TO 310
     DO 305 J = 1, NMI
     305 H(J) = 0.5*ALPHA(NZ-J)
     H(NFCNS) = ALPHA(1)
     GO TO 350
310 H(I) = 0.25*ALPHA(NFCNS)
     DO 315 J = 2, NMI
     315 H(J) = 0.25*(ALPHA(NZ-J) + ALPHA(NFCNS+2-J))
     H(NFCNS) = 0.5*ALPHA(1)+0.25*ALPHA(2)
     CONTINUE
C
C DOES THE SOLUTION CONVERGE?
C
IF(NSTATE .EQ. 2) GO TO 1100
     DEV1 = DEV
     DO 1050 I = 1, NFCNS
     1050 H1(I) = H(I)
     NSTATE = 2
     GO TO 1001
1100 DO 1060 I = 1, NFCNS
     1060 H2(I) = H(I)
     DEV = DEV/2.DO
     DDEV2 = DABS(DEV-DEV2)/DEV
     IF(DDEV2 .GT. EPSIT) GO TO 1080
     IF(DDEV2 .LT. EPSIT .AND. ND05 .NE. 1) GO TO 800
C FIND THE D05MIN AND RATMIN OF THE FILTER AND SET D05 = RMULT*D05MIN (OR D05 = RMULT*D05MIN/RNO2 WHEN NDEP IS ODD).
C
D05MIN = IF0.5,H2,N02,NF2)
RATMIN = DEV2/DEV1

375 FORMAT(' D05MIN = ', D20.10, ', RATMIN = ', D20.10)
WRITE(6,375) D05MIN, RATMIN
IF(DRATIO .LT. RATMIN) GO TO 1070
D05 = RMULT*D05MIN
IF(NODD2 .EQ. 0) D05 = D05/RNO2
ND05 = 2
IF(NODD2 .EQ. 0) GO TO 805
GO TO 801

888 FORMAT('DRATIO IS LESS THAN RATMIN, NO SOLUTION EXISTS. ')
1070 WRITE(6,888)
GO TO 1000
1080 DEV2 = DEV
NSTATE = 1
GO TO 1001

C OUTPUT SECTION STARTING HERE.
C
C WRITE THE COEFFICIENTS OF NUMERATOR N(Z).
C
110 FORMAT('COEFFICIENTS OF NUMERATOR N(Z) = ')
109 FORMAT(I5,D20.10)
120 FORMAT('COEFFICIENTS OF DENOMINATOR D(Z) = ')
898 FORMAT('****MAXIMUM NO. OF DESIGN CYCLES EXCEEDED****')
600 WRITE(6,898)
GO TO 320
998 FORMAT('****FILTER WITH DESIRED RIPPLE RATIO FOUND****')
610 WRITE(6,998)

C WRITE NO. OF DESIGN CYCLES USED.
C
605 FORMAT('THE NUMBER OF DESIGN CYCLES = ',I5)
WRITE(6,605) ITDS

C WRITE THE D05 VALUE USED IN DESIGNING THE DESIRED OPTIMAL FILTER.
C
606 FORMAT(' D05 = ', D20.10)
WRITE(6,606) D05

C WRITE THE PASSBAND RIPPLE AND STOPBAND RIPPLE OF RESULTANT FILTER.
C
607 FORMAT('PASSBAND RIPPLE = ', D20.10, ', STOPBAND RIPPLE = ',
D20.10, ')
WRITE(6,607) DEV2, DEV1

C WRITE THE COEFFICIENTS OF NUMERATOR N(Z).
C
320 WRITE(6,110)
DO 401 I=1,NF1
401 WRITE(6,109) I,H1(I)
NFFCNS = NF1 - NO1
DO 400 J=1,NFFCNS
   K = NFFCNS+1-J
   KPJ = NF1+J
400 WRITE(6,109) KPJ,H1(K)
C
C WRITE THE COEFFICIENTS OF DENOMINATOR \( \tilde{D}(Z) \).
C
WRITE(6,120)
DO 402 I=1,NF2
402 WRITE(6,109) I,H2(I)
NFFCNS = NF2 - NO2
DO 405 J=1,NFFCNS
   K = NFFCNS+1-J
   KPJ = NF2+J
405 WRITE(6,109) KPJ,H2(K)
1000 STOP
END
C
C FUNCTION HF PROVIDES THE FUNCTION VALUE OF TRANSFER FUNCTION
C \( H(Z) \) EVALUATED ALONG THE UNIT CIRCLE AT THE POINT \( F \).
C
DOUBLE PRECISION FUNCTION HF(F,H,NODD,NFNS)
DOUBLE PRECISION H(66)
DOUBLE PRECISION PPI2
PPI2 = 6.283185307179586
NML = NFNS-NODD
HF = 0.
DO 20 J = 1,NML
   IF(NODD .EQ. 1) GO TO 30
   DO 20 J = 1,NML
20   HF = HF+DCOS(PPI2*F*(J-0.5))*H(NML+1-J)
   GO TO 50
30   DO 40 J = 1,NML
40   HF = HF+DCOS(PPI2*F*J)*H(NML+1-J)
50   HF = HF*2
   IF(NODD .EQ. 1) HF = HF+H(NFNS)
RETURN
END