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RICE UNIVERSITY

SELF-CONSISTENT MAGNETOSPHERE FOR A STATIC ALIGNED ROTATOR

by

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A THESIS SUBMITTED
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Abstract

Self-Consistent Magnetosphere for a Static Aligned Rotator

by Juergen Krause-Polstorff

Since the Goldreich and Julian model for a pulsar magnetosphere was introduced attempts to form a self-consistent theory based on their model have met with failure. This thesis contends that the Goldreich and Julian magnetosphere is not unique, that more plausible magnetospheres exist and that they are characterized by vacuum gaps separating the charge-separated plasmas in the pulsar magnetosphere. This work models a number of such magnetospheres using discrete charges that are generated from the pulsar surface charge. The models are physically differentiated by their net charge, with one model having the net charge of the Goldreich and Julian model and the others less than this in regularly spaced intervals. It is demonstrated that static solutions exist which possess corotational regions above the polar regions and both a corotational and superrotational region about the equator. The result is in accord with Pilipp's theorem concerning the impossibility of having only a corotational magnetosphere with vacuum gaps. The finite size of the space-charge configuration places the magnetosphere well within the light cylinder of all but one of the known radio pulsars (assuming
that a pulsar has the standard radius). The confinement of the space-charge within the light cylinder raises questions about the present assumptions made for pulsars. The configurations are totally static as the surface has been made emission-free in the aligned case, and the results suggest that one must at least look at the non-aligned case to explain the pulsar emission process. Possibly some of the assumptions about radio pulsars and their environment must be revised.
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I. Introduction

Pulsars are astronomical objects believed to be rapidly rotating neutron stars. Neutron stars are stars that have collapsed to the point that only nuclear degeneracy pressure supports them against self-gravitation and prevents them from becoming black holes. Standard radii for neutron stars are believed to be about 10 km and the mass is about 1.4 solar masses. Magnetic fields in a pulsar are believed to be of the order of $10^{12}$ gauss, an estimate which is based on the conservation of flux. Pulsars emit regular radio-frequency pulses in the UHF frequency area and the pulsational periodicity (order of a second) is extremely regular with a 'Q' of about $10^{11}$.

Since the introduction of the aligned rotator model of the pulsar (a rotating neutron star aligned with its dipolar magnetic field) surrounded by a charge-separated plasma magnetosphere (Goldreich and Julian, 1969), attempts to use this model as a basis for a self-consistent physical theory for pulsars have met with considerable difficulties. Briefly, the Goldreich and Julian model supposes a rotating magnetized neutron star which is surrounded everywhere by a charge-separated plasma. The neutron star itself is assumed to be an excellent conductor (Baym et al., 1968). The plasma
loaded field lines are assumed to have essentially infinite conductivity and are therefore equipotentials. The equation for the dipole magnetic field lines where $\Theta$ is the colatitude is given by:

$$ f = \sin^2\Theta/r $$

(1)

from which it follows that the potential $V$ induced by the rotation is:

$$ V = \omega \sin^2\Theta/r $$

(2)

where $\omega$ is the angular velocity, and the charge density $\rho$ is:

$$ \rho = \omega(1 - 3\cos^2\Theta)/2\pi r^3 $$

(3)

Since the field lines are presumed to be equipotentials, the charge separated magnetosphere everywhere rotates rigidly with the angular velocity of the star, and this corotation leads to a light cylinder at some distance from the axis of rotation. The light cylinder is that cylinder upon which $\omega r \sin \theta = c$. The physics in the vicinity of such a light
cylinder poses considerable difficulty and is not well understood. Particles on field lines that close within the light cylinder are assumed corotating—those on open field lines beyond the light cylinder are assumed to stop corotation at some distance short of the light cylinder. The open field lines have plasma streaming along them out to infinity. This scenario leads to some of the problems associated with the Goldreich and Julian model in that positron or proton currents are required to pass through regions of negative space charge to satisfy the condition that the net current flowing out from the system is zero. The complications inherent in having a light cylinder have so far prevented presentation of a global model of a pulsar that is free of physical contradictions and that can provide the basis for further refinements. The problem of the light cylinder can be avoided if one can model an aligned rotator with static charge distributions. This implies a finite extent magnetosphere which lies well within the light cylinder, yet fulfills the condition of an emission-free surface for the star. For a further discussion of the problems inherent in the Goldreich and Julian model see Michel(1982).

The Goldreich and Julian solution is only unique (for an \( E \times B = 0 \) surface for the star) if one allows a completely charge-filled magnetosphere. It is the purpose of this work
to explain and show that explicit static solutions to the aligned rotator exist and that they differ topologically from the Goldreich and Julian in that they possess partially filled magnetospheres characterized by finite size vacuum gaps. The existence of such configurations was first discussed by Rylov (1976) who then went on to postulate an unknown mechanism to allow particles to escape in the equatorial region. The contention here is that such configurations are naturally static.
Goldreich-Julian Magnetosphere

Figure 1.
II. Physics of the Aligned Rotator

The existence of static solutions to the aligned rotator hinges on the possibility of having regions surrounding the star where the charge density is zero \( \rho_-=\rho_+=0 \). This possibility arises with the introduction of the vacuum gap concept in connection with charge-separated pulsar magnetospheres. The origin of the vacuum gap concept can be found in a paper by Holloway (1973) which concerned itself with the depletion of charge in the Goldreich and Julian model in a region where it could not be resupplied, thus causing a vacuum region to develop. The first detailed exposition of the vacuum gap concept is given in a paper by Michel (1979) where the properties of the plasma/vacuum discontinuities are discussed. The existence of a vacuum gap requires the concepts of:

1) Density discontinuities (as defined below)
2) Force-free surfaces \( \mathbf{E} \cdot \mathbf{B}=0 \)

A density discontinuity occurs when the density of a magnetically threaded one-signed plasma drops abruptly to zero from some finite value. The locus where this occurs then defines a surface. The side of the discontinuity where the space charge resides has \( \mathbf{E} \cdot \mathbf{B}=0 \) but the vacuum side (absence of plasma) does not in general have \( \mathbf{E} \cdot \mathbf{B}=0 \). This
means that on the vacuum side there is a component of the electric field parallel to the magnetic field which acts to move space charge (of the appropriate sign) that has been perturbed across the discontinuity surface to the vacuum side back to the plasma side of the discontinuity. The space charge is trapped on one side of the discontinuity surface. This happens because the plasma in this case is charge-separated whereas the usual case in plasma physics is that of a quasi-neutral plasma that cannot be confined in the fashion described above. The idea of force-free surfaces arises from the observation that the vacuum field models (no space charge) possess surfaces on which $E \cdot B = 0$. These surfaces offer a potential well for particles of the appropriate sign and therefore provide an accumulation locus for space charge. The introduction of the space charge at these loci gives rise to density discontinuities there. The aligned magnetic dipole rotator in vacuum possesses two distinct force-free surfaces:

1) The geometric $E \cdot B = 0$ surface that lies in the equatorial plane and is apparent from symmetry.
2) The other type of $E \cdot B = 0$ surface is a pair of spheres lying above and below the equatorial plane and tangent to it at $r = 0$.

The type 2 force-free surface owes its existence to the central charge acting together with the surface charge and
the magnetic field along with the electric quadrupole moment induced by the rotating magnetic field (assumed dipolar). Changing the central charge, for example, changes the force-free surface of type 2 but leaves type 1 unchanged. The introduction of space charge to the system causes formation of the discontinuity surfaces as charges of the appropriate sign are accumulated on a force-free surface. The surface splits open and one now has an \( E \cdot B = 0 \) volume now filled with space charge and bounded by the charge discontinuity surfaces. The introduction of space charge creates discontinuity surfaces which are deformations of the original force-free surfaces. With the introduction of a charge-separated plasma in a magnetosphere the surface of the star can be made to satisfy \( E \cdot B = 0 \). The stellar surface is no longer electrically distinguishable from its surroundings and is everywhere enclosed by the magnetosphere, and once this requirement is met one obtains a static aligned rotator.

The interior of the star is assumed to be an excellent conductor with \( E \cdot B = 0 \) everywhere in the star. This assumption forces one to have a central charge for the star if one assumes the magnetic field to be dipolar (Michel and Pellat, 1980). For a magnetic dipole, \( E \cdot B = 0 \) has the form:

\[
E_r \cdot 2 \cdot \cos \theta + E_\theta \cdot \sin \theta = 0 \tag{4}
\]
which gives:

\[ E_r = -\frac{E_\theta}{2} \times \tan \theta \]  \hspace{1cm} (5)

where \( \theta \) is the colatitude. Note that both \( \tan \theta \) and \( E_\theta \) change sign for \( \theta = \pi/2 \), \( E_\theta \) for reasons of symmetry since \( E \times B = 0 \) in the equatorial plane and \( B \) is a magnetic dipole. Hence, \( E_r \) is one-signed over a centered sphere of arbitrary radius (if one has a dipole magnetic field) and by Gauss’s Law that implies the existence of an enclosed central charge. This central charge is only located at the center of the star if one takes a point magnetic dipole as the source of the magnetic field. If on the other hand one assumes a uniform interior magnetization, the inside of the vacuum rotator actually acquires a negative charge while the surface becomes positively charged by factors of two and three times the net charge, respectively. The important point is that the net charge of the pulsar is not a free parameter but is determined by the dipole magnetic field and the assumption \( E \times B = 0 \) in the interior of the star. For the whole system, consisting of star and magnetosphere, to be neutral the magnetosphere must contain a net amount of charge equal and opposite to the stellar charge. Given the Goldreich and Julian charge density in the interior of the star and the \( E \times B = 0 \) condition at \( r = 1 \), the potential for \( r < 1 \) is given by:

\[ V = 15 \sin^2 \theta/r \]  \hspace{1cm} (6)
for \( r=1 \) it is:

\[
V = 10/r - (6/r^3 + 4/r^2)P_2
\]  \( (7) \)

(Michel, 1979)

and for the case \( r>1 \) it is:

\[
V = 10/r - 10P_2/r^3
\]  \( (8) \)

The factor of 15 is chosen for \( \omega \) to avoid fractional coefficients and the \( P_2 \) are the Legendre polynomials.

Examining the potential for \( r=1 \) it is apparent that the first term is the contribution from the central charge, the second term is the induced quadrupole due to the rotating dipole and the last term arises from the surface charge if the star is in vacuum. If the surface charge cannot be maintained this charge must reside in the magnetosphere which, if the stellar interior is to maintain \( E \times B = 0 \), must contribute an external quadrupole term to the potential. The condition of perfect conductivity imposed by the interior of the neutron star (see Baym et al. 1969) is satisfied by any magnetosphere which provides the \( 4P_2 \) term in the potential at the surface. The potential for \( r<1 \) is just that of Goldreich and Julian since high conductivity along the field lines is assumed to hold throughout the interior. The
potential for \( r > 1 \) is a consequence of the field point now lying outside of the surface charge layer as is apparent from the changing of \( r^2 \) to \( r^{-3} \) in the expressions for the potential. It is instructive to look at the force-free surfaces created by the potential of equation (8):

\[
(1 - 3\cos^2 \theta / r^2)4\cos \theta / 3r^5 = 0
\]  

(9)

This expression is obtained by taking the gradient of equation 8 and dotting it into the dipole \( \mathbf{B} \). There are two solutions to this equation: \( \theta = \pi / 2 \) (equatorial plane) and \( r = (3)^{1/2} \cos \theta \) which is the surface of a sphere penetrating the star at about 55° and having a radius of \( (3/2)^{1/2} \) and is the type 2 surface described earlier. The two surfaces provide the natural accumulation points for the surface charge as it is emitted from the surface. Moreover, as a point magnetic dipole field is assumed, the \( \mathbf{E} \cdot \mathbf{B} = 0 \) boundary condition at the stellar surface precludes the existence of images or surface charge there if the boundary condition is satisfied. The surface charge in terms of \( \mathbf{E} \cdot \mathbf{B} \) on the surface is given by (see Appendix B):

\[
\sigma = \mathbf{E} \cdot \mathbf{B} / (8\pi \cos \theta) \quad (\theta \neq \pi / 2)
\]  

(10)

which implies that if \( \mathbf{E} \cdot \mathbf{B} = 0 \) the surface charge is zero and there are no image charges. The \( \mathbf{E} \) field is therefore
continuous throughout the space occupied by the static aligned rotator. The surface of the star is no longer a discontinuity and loses its identity from an electrical point of view. The static aligned rotator problem is that of finding the magnetospheres, finite in extent, that can provide a smooth joining of the electric fields at the star surface. The static rotator can exist owing to the stability of the discontinuity surfaces which act to contain the one-signed plasma regions of the magnetosphere which are separated by the regions of zero plasma density—the vacuum gaps—which it follows are also stable regions. One is then left with the task of finding solutions to the problem posed: finite extent magnetospheres that are in equilibrium and satisfy $E \cdot B = 0$ at $r=1$ in order to eliminate the difficulty of the light cylinder and any emission from the surface so as to obtain a static rotator.
Figure 2

$E \cdot B = 0$ surfaces for vacuum model
Figure 3
III. Modeling a Static Aligned Rotator

Since the physics of the problem of the static aligned rotator suggests that solutions should exist, the question arises as to how one may construct such solutions. In a conventional electrostatics problem the aim is to solve the Poisson equation in the regions of non-zero charge density and the Laplace equation in the regions where the charge density is zero. As regards the static aligned rotator these equations are underdetermined. The charge density everywhere is not known — by Pilipp's theorem (1974) the solution having vacuum gaps cannot have Goldreich and Julian charge density in the whole of the space charge regions. Furthermore, the boundaries between the space charge region and the vacuum are not specified beforehand. What is known is that the boundary between the space charge region and the vacuum region must satisfy the boundary condition \( \mathbf{E} \cdot \mathbf{B} = 0 \) (stability condition which means no surface charge) and inside the space charge region the magnetic field lines are equipotentials. Further the structure of the magnetic field is assumed to be dipolar and one therefore obtains the potential contributed by the star itself. The most one can say at this point is that there is a space charge region above the poles of one sign and another of the opposite sign surrounding the equatorial region, both possibly enclosing vacuum regions (so-called vacuum bubbles) and a vacuum gap.
separating the two space charge regions. This much can be surmised by looking at the stellar potential and magnetic field. To actually solve the problem one must discard the differential equation approach used in solving the usual electrostatics problem.

In order to find the solutions to the static aligned rotator a method was chosen that was guided by how a vacuum rotator would establish a magnetosphere as the surface charge is emitted. The idea behind the model is to create a magnetosphere of discrete charges that simulate a continuous distribution of charge and as closely as possible impose the \( E \cdot B = 0 \) condition at \( r=1 \), the stellar surface. The number of discrete charges has to be rather small due to the computational effort required by the approach (despite the high degree of symmetry). The starting point for the model is the potential given by equation (8) from which the magnetosphere is to be constructed by the emission of charge elements from the surface. Charge elements are constrained to move along field lines on which they are launched (analogous to beads on wires). The charge elements are situated at \( E \cdot B = 0 \) positions with respect to the stellar field and the sum total field of the other charge elements. The assumption that these charges should physically be in equilibrium is based on the result — see for instance Rylov (1976) — that because of the curvature of the magnetic field
lines and the magnitude of that field the kinetic energy the particles receive from the longitudinal electric field will be the major contribution to the perpendicular energy which is radiated away as magnetobrehmstrahlung. The characteristic time for a charge like the electron to damp its oscillations along the field lines can be shown to be the order of the radius of curvature of the magnetic field line divided by the speed of light (Rylov, 1976). Once the charge element is situated at an equilibrium position iteration is performed over the positions of all the charge elements until they are in equilibrium with each other—each charge element residing at an $E \cdot B = 0$ position.

The formation of charge elements is determined by the surface charge which resides on the stellar surface prior to $E \cdot B = 0$ there. Initially, the surface charge arises from the discontinuity contributed by the $P_2$ term in the expression for the normal derivative of the electric potential at $r < 1$ and $r > 1$ (see eqs. (6) and (8)) and this is emitted from the star in terms of quantized charge elements. After these have been emitted and equilibrated they induce a potential at the surface and the new surface charge distribution is chosen so that this induced potential is cancelled and a $-4P_2$ potential is again present at the surface. What has been accomplished is to again make the surface satisfy $E \cdot B = 0$ by creation of a surface charge which has been translated into
surface charge elements. These surface charges are then
emitted and the process is repeated until the potential
induced by the charge elements in the magnetosphere is
sufficiently close to $-4P_2$ ($\mathbf{E} \cdot \mathbf{B} = 0$). 'Sufficiently close' here means that under the scheme employed the total surface
charge that would be created is less than the charge
quantization. This indicates a natural optimum for the
chosen charge size quantization and the system is static for
the chosen quantization—to guarantee that the system is
static is after all the reason behind the $\mathbf{E} \cdot \mathbf{B} = 0$ condition at
the surface. Once this condition is met a static aligned
rotator has been constructed. The secondary magnetic field
created by the charges rotating about the star azimuthally
is ignored since its magnitude is of the order of $\beta^2 B$ where
$\beta = R \sin \theta / c$.

Now that the concept of the model has been discussed one
can examine the model in more detail. The identity of the
charge elements is almost forced in view of the azimuthal
symmetry that is required. The mathematical construction for
the charge element is a ring of charge $q$ which has the sym-
metry required but has the drawback that its potential and
field cannot be represented in closed form. The ring charge
element concept has two problems associated with it: the
first, already mentioned, regarding representation of the
potential and the field and the second, that the self-energy
and self force of a ring are infinite. The self-force is the force that acts on the ring to blow it apart and the self-energy is the energy required to assemble the ring.

Two choices for the representation of the ring potential and field are in the form of elliptic integrals (complete) or an infinite series in Legendre polynomials. The representation of the potential I by elliptic integrals is simple:

\[ I(r, \theta) = qk \frac{K(k)}{A} \quad (11) \]

where \( A = \pi (rR \sin \theta \sin \alpha)^{1/2} \)

where the ring coordinates are \( R, \alpha \) and the field point coordinates are \( r, \theta \) and \( k^2 \) is given by:

\[
K(k) = \int_0^{\pi/2} \frac{dx}{(1-k^2 \sin^2 x)^{1/2}}
\]

\[ k^2 = \frac{4rR \sin \theta \sin \alpha}{(r^2 + R^2 - 2Rr \cos(\theta + \alpha))} \quad (12) \]

The representation of the field by elliptic integrals is a good deal more complicated as can be seen below:
\[ E_r = \frac{qk/2A}{4R \sin \theta \sin a} \left( K/r + (1-R^2/r^2)k^2E \right) \quad (13) \]

\[ E_\theta = \frac{qk/2A}{4R \sin \theta \sin a} \left( K \cot \theta/r + 2R \cos \alpha/\rho \sin \theta - (1+R^2/r^2)\cot \theta k^2E \right) \]

The expressions above consist of both elliptic integrals \( E(k) \) and \( K(k) \) which increases the computational effort required for evaluation.

The potential and the field in the Legendre representation are straightforward but suffer from slow convergence in some cases. The potential for the ring in Legendre polynomials is:

\[ I = q \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos \theta) P_n(\cos \alpha), R > r \quad (14) \]

\[ I = q \sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} P_n(\cos \theta) P_n(\cos \alpha), r > R \]

with variables as before. The second problem faced by the ring as the charge element is the property that it has an infinite self-energy as well as an infinite self-force. This means that the effect of the ring on itself must be compensated for in an ad hoc fashion. This is a drawback in the use of the ring as a charge element but as one goes to a system of more and more rings the effect on the system of
the self-interaction of each ring should become less and less. An obvious way to introduce the self-interaction term is to consider the rings as consisting of a finite number of point charges. The field due to k-1 point charges of the ring on the kth point charge is given by:

\[
E_r = \frac{q}{(4kR^2 \sin \theta)} \sum_{n=1}^{k-1} \csc \left( \frac{n \pi}{k} \right) \\
E_\theta = \frac{q \cos \theta}{(4kR^2 \sin^2 \theta)} \sum_{n=1}^{k-1} \csc \left( \frac{n \pi}{k} \right) \\
= \left( \frac{q \cos \theta}{(4kR^2 \sin^2 \theta)} \right) S_k
\]

(15)

where \( S_k \) is an asymptotic expansion for the sum of a series of cosecants (Watson, 1916). The first few terms in the series for \( S_k \) are given below:

\[
\pi S_k = 2k \ln(2k) + 2k(\gamma - \ln \pi) - 0.0873\pi/k + 0.0104/k^3
\]

(16)

where \( \gamma \) = Euler's constant.

This provides an easy way to compute the fields in the case of a large number of point charges making up a ring. As it turned out a ring was configured to consist of four particles for the purpose of the self-interaction calculation on the basis of a near neighbor rule. That is, a ring was found to have four near neighbor rings on average
so extending this to the substructure of the ring gives one rings consisting of four particles (although the term is not that sensitive to the number of particles since it goes as the log of the number). A more direct idea was to have the number of particles making up the ring equal the total number of rings but this tended to greatly exaggerate the importance of the self-force for outlying rings. The symmetry about the equatorial plane in the problem means that one can save computing time if the rings are added in pairs, one above the equatorial plane, the other below the plane at the image position. For the Legendre representation the potential of the two rings together has no odd terms. This occurs because $P_{2k-1}(-\cos\theta) = -P_{2k-1}(\cos\theta)$. Savings in computational effort are important since the computational effort increases as $n^2$ with the scheme employed. The method of adding rings in pairs has the drawback that since the ring and its twin never 'see' each other an interaction term must be added in the expressions. This term is most important in the equatorial region where the ring and its twin are close together.

The next aspect of the model to be examined is how the charge to be added to the system is determined. Initially the star is surrounded by a vacuum but possesses an unstable surface charge which acts to maintain $E\cdot B = 0$ at the surface and in the interior. As indicated earlier, the potential just inside the star is given by:
\[ V = 15 \sin^2 \theta / r \]  

(6)

and outside the star it is given by:

\[ V = 10/r - 10P_2/r^3 \]  

(8)

The surface charge that arises from the discontinuity in \( E_r \) is given by:

\[ \sigma = -5/\pi P_2 \]  

(18)

and it is this charge that the star originally surrenders to the magnetosphere in the model. Additionally, in order to have a model where one can alter the net charge, one takes the star to be surrounded by a uniform shell of charge in the initial step - charge is conserved in subsequent stages. This is employed to give an unbiased method for giving the magnetosphere a net charge as this charge is combined with the surface charge. The shell is viewed as being slightly above the surface so that \( E \cdot R = 0 \) is obtained at \( r=1 \). The net surface charge of each sign is integrated along the surface and a ring is created when the surface charge has been integrated to the chosen quantized charge size. The question
of which magnetic field line to associate with the ring deserves some discussion. The straightforward scheme would be to simply take the midpoint of the integration interval as the foot of the field line to which the ring (assumed infinitely stretchable) is constrained. However, the problem with this idea is that if, for instance, the integration interval is long and the first fourth of the interval contributes 75 percent of the total charge making up the ring, the ring will be created and launched at a position that does not reflect the distribution of the surface charge correctly. A much better scheme is to associate that magnetic field line with the ring at that point where the value of the integral reaches half of the chosen charge quantization. This elaboration is only necessary if the charge quantization is not sufficiently small. Since the integrated charge is related to the surface charge times the surface area the scheme creates a region (conical in shape) that is devoid of charges. This happens since the integration starts at $\Theta=0$ and one must integrate in $\Theta$ for some measure before a charge is formed. To possibly minimize this artificial feature a bias method was introduced into the model. That is, if a ring is created between $0$–$10$ degrees or some other upper limit depending on the chosen charge size, it is alternately associated with the field line where it was created or the theta of that field line divided by a constant larger than one (i.e. two). Once the rings
created in the first step have been equilibrated in the magnetosphere the new surface charge is calculated so as to cancel the potential on the surface due to the rings and give a $-4P_2$ term on the surface. In other words the surface charge is again chosen to make the surface and interior satisfy $E \cdot B = 0$. The surface charge is again not in equilibrium and is shed into the magnetosphere where the rings are equilibrated with respect to each other. This procedure is repeated until the condition of no surface charge is satisfied to the extent that no more rings of the chosen quantization can be formed. The general expression for sigma is given as:

$$\sigma = \frac{1}{4\pi} \left( 5C_2P_2 + 9C_4P_4 + \ldots + (4k+1)C_{2k}P_{2k} \right)$$  \hspace{1cm} (19)$$

where $k$ is the number of rings (See Appendix A). The term $C_{2n}$ is given by:

$$C_{2n} = -2 \sum_{n=1}^{k} \frac{q_iP_{2n}(x_i)}{R_i^{2n+1}} + 2(\text{if } n=1)$$  \hspace{1cm} (20)$$

This procedure creates an external charge distribution that, in the limit of an infinite number of rings of infinitesimal charge contributes a $-4P_2$ term to the potential on the surface and one obtains $E \cdot B = 0$ exactly. The
equilibrium position of a ring along a field line is determined by finding the closest (to the star) \( E \cdot B = 0 \) position which is stable for the sign of the ring charge. \( E \) represents the field due to the star and all other rings including those on the surface which have not been placed in the magnetosphere, plus the self-interaction of the ring. The iteration steps performed after a 'layer' of surface charge has been ejected in the form of rings do not include the effects of the new surface charge since it is a minor perturbation for the charges in the magnetosphere already. The effect of the surface charge is only taken into account in each launching step. The \( B \) represents the field due to a magnetic dipole centered in the star. The equilibrium position for a particle is found as a root of the \( E \cdot B = 0 \) equation in the interval where it has an appropriate change of sign along the field line. A part of the program written for the model devotes itself to finding the interval without having to poll a large number of points each time a root is being looked for. This is done via a tree search which checks for the sign of the charge and the sign of \( E \cdot B = 0 \) at the previous position which gives an indication in which direction the root has 'moved'. The root finder employed is a high order method of regula falsi and hyperbolic interpolation (Anderson and Bjoerck, 1973).
Once the equilibrium position has been found for one ring it is found for the next ring until all the surface rings have been launched. The positions of all rings are then iterated until they settle down. In practice this involves two iterations for each ring. Finally, when no more rings can be formed on the surface, iteration is performed until all are simultaneously stable— all the while checking to see that no new rings can be formed on the surface. The configuration is presumed stable when the positions change less than .005 from the previous value. In order to make this scheme work one needs a relatively well behaved $F(R)=0$ function and there is a deficiency as regards the Legendre representation. Looking at the expression for $E_r$:

$$E_r = \begin{cases} 
-q \sum_{n=1}^{\infty} \frac{n(r^{n-1}/R^{n+1})}{r} P_n(\cos\theta)P_n(\cos\alpha) R > r \\
-q \sum_{n=1}^{\infty} \frac{(n+1)(R^n/r^{n+2})}{r} P_n(\cos\theta)P_n(\cos\alpha) r > R 
\end{cases} \quad (21)$$

it becomes clear that the rapidity of the convergence of the series is solely dependent on the ratio of the $r$ field coordinate to the $R$ source coordinate. When the ratio is close to one (1) the behavior of the partial series sum is highly oscillatory until $k$ is large enough to give the ratio
a damping effect on the contributions of $k$ times the $k$th Legendre polynomial. Since the root finder searches through an interval of $r$ values it is quite possible for an $r$ value to be close to $R$ of a source or indeed the equilibrium $r$ may be close to the $R$ of a source leading to erratic behavior of the function ($E^*B=0$) as a ring is to be added or iterated — a root is found where it really should not be — a false $E^*B=0$ position because the truncated series is far from the value the infinite series has. At the least this produces a radical shift of position, as new rings are added (along with new terms in the series), in the locations of the pre-existing rings. The procedure needs to be smoother. In an effort to control this behavior various weighting schemes were tried but were unsatisfactory. The unsatisfactory feature of the Legendre representation of the $E$ field led to the implementation of the elliptic integral representation of the $E$ field. The complete elliptic integrals of the first and second kind can be evaluated very accurately by the use of low order Chebyshev polynomials (see Hastings, 1955).
IV. Results and Discussion

The results presented here consist of a set of models that differ from each other by having a different amount of net charge as an initial condition. Since the formation of the model magnetosphere preserves charge this results in a series of models differing in net charge. The net charge of the models was chosen to vary by steps of 2 from 2 to 10. The model 10 has the same net charge as the Goldreich and Julian model. In contrast, though, the Goldreich and Julian model has space-charge regions extending out to infinity, whereas the 10 model has charge confined to finite regions well within the light cylinder (see figure 1a). The amount of negative or positive space charge in the Goldreich and Julian model is infinite (the positive and negative charge amounts go with $\ln r$) although the positive and negative space charge cancel exactly except at the origin where there is a net charge +10. The 10 model has the vacuum gaps that characterize all the models presented here.

A complete set of five models was computed using the same charge size for the rings in each model. For some additional models smaller charge sizes were used—the 2 model was also constructed using the bias feature described in the last chapter with the bias constant equal to two. Another
parameter that was changed in the different models was the 'scrub parameter'. This is a value set in the program to decide when a charge element is close enough to the surface so that it is no longer in the magnetosphere but part of the surface charge. It determines how close a ring may be in equilibrium to the star before it should be reabsorbed. The reason for this feature is to avoid having (relatively) large charges too close to the surface where they produce anomalies in the surface charge distribution. If the charge size were sufficiently small the precision of the computer would take over this function. The value of the scrub parameter in the various models varied from $8 \times 10^{-3}$ to $5 \times 10^{-4}$. It was determined in range by calculation and trial and error. Values larger than $8 \times 10^{-3}$ produced instabilities in the method and values smaller than $5 \times 10^{-4}$ defeated the purpose of having the parameter in the first place. The instability arises when a particle is created but its equilibrium position lies closer to the surface than the scrub parameter so that it would be absorbed and promptly recreated to have the same thing happen once again. For comparison purposes the potentials of all the models were adjusted to zero additive constant to agree with the standard Goldreich and Julian model. This is accomplished by subtracting off the constant term in the potential contributed by each ring. This is the same as the constant term in the Legendre series expansion for the potential of the ring.
The results from each of the models are summarized in three plots stressing various aspects of the model. The first plot for each model shows the location of each ring relative to the star. The second plot is of the surface potential at the positions from which a ring was launched during the construction of the magnetosphere. These potentials are compared with those of the Goldreich and Julian model at the surface (or stated differently with the surface potential of a spherical force-free rotator) which is given by \( V = 15 \sin^2 \theta \). The degree of agreement indicates how well the \( E \cdot H = 0 \) boundary condition is being met by the model. In general it is met rather well except for the conical vacuum region that lies above the surface from about \( \theta = 0^\circ \) to \( 15^\circ \) where the agreement trails off. This vacuum region above the polar cap is due to the finite charge quantization and can be reduced by reducing the charge size as will be discussed later. The third plot is a comparison of the equipotentials of the constructed model to those of the Goldreich and Julian model for selected values of the potential. For example the equipotential labeled by 10 is the field line which leaves the star where the charge density of the star goes from - to + and indicates the boundary of the corotating positive region. Equipotentials less than 10 lead to negative regions on the surface and if they were part of the corotating positive region these equipotentials would have to be occupied with positive
charges right to the negative charge surface. The positive charges on the equipotential could not be stable in such a configuration and thus the equipotentials less than 10 must be open circuited by vacuum before they reach the equatorial plane. When the equipotentials of the model coincide with the Goldreich and Julian model the region that is threaded by them is corotating. If the two equipotentials coincide all the way back to the surface of the star the region threaded by them corotates with the star and has the Goldreich and Julian charge density. If the two equipotentials coincide and then separate it indicates that the model has a vacuum gap bounded there. However, further along the equipotential coincidence may again exist with a different equipotential (dipole magnetic field line) of the Goldreich and Julian model and that section of the equipotential would have a constant E x B drift which is not the same as the rotational velocity of the star (see figure 2c for the 4 equipotential near the equatorial plane). It is known on theoretical grounds that the stellar surface cannot be adjacent to the vacuum region if \( E \times B \) is to be zero on the surface. If the surface were adjacent to the vacuum in a neighborhood of points then since the potential at the surface is given along with its derivative, the potential can be uniquely continued into the vacuum (Kellogg, 1929). The potential is given below:
\[ V = 10/r - (6/r^3 + 4r^2)P_2 \]  \hspace{1cm} (22)

and it is seen to fail on physical grounds at infinity. The Laplacian is zero as expected. The \( E \cdot B = 0 \) surfaces of this potential are given by:

\[ 5r^2(1-r^3) - 9(1-r^5)\cos^2\theta = 0 \]  \hspace{1cm} (23)

(Michel, 1979)

where the surface \( r=1 \) is readily apparent but there is also a surface attaching to the star at approximately 55° and fanning out to an angle of 42° as \( r \) goes to infinity. The preceding potential therefore excludes vacuum gaps that lie between the stellar surface and the magnetosphere since the vacuum potential does not have an \( E \cdot B = 0 \) surface closing a finite distance above the surface.

The argument presented does not exclude the existence of vacuum regions nested entirely in the charge regions of the magnetosphere—a vacuum bubble. The existence of these vacuum bubbles is in doubt on other theoretical grounds (see for example, Asseo et al., 1982). Further, the results presented show no evidence for vacuum bubbles in the regions where the results may be considered valid.
The first two models to be discussed in detail are the 10 model with charge size of .25 and the 10 model with charge size of .175. A look at figures 1a and 2a shows clearly the conical region above the poles that does not contain any charges. It is apparent however that the size of the cone is somewhat narrower in figure 2a. That this is indeed so is supported by the plots of the surface potential in figures 1b and 2b. The agreement of 2b with the theoretical curve is better in the 10°–20° range than that of 2a, at 10° by about 33 percent. If this cone were made still smaller by going to a smaller charge size the agreement would become even better.

As figures 1c and 2c show, the field lines and the equipotentials coincide near the surface and separate when they are further away from the star. What this means is that in the 10 model case the space charge density is the same as the Goldreich and Julian density for those regions where the magnetic field line and equipotential coincide back to the surface. Comparison of 1a to 1c and 2a to 2c shows that the entire negative region is Goldreich and Julian but that the positive region is only Goldreich and Julian out to r=1.5 in the equatorial plane and is bounded by the 10 equipotential as mentioned earlier (the last totally corotating field line). There is space charge beyond this r value in the equatorial plane but it is not
electrically connected to the star, being open-circuited by the vacuum gap.

As figures 1c and 2c show, the equipotentials crossing the equatorial plane beyond $r=1.5$ are closer together than they would be if they were in corotation with the star. This means that the space charge lying beyond $r=1.5$ is superrotating (generally by about 20 percent) since the $E$ field (gradient of the potential) is stronger in the equatorial region than if corotation existed—thus the $E \times B$ drift there is faster. There are then two regions in the 10 model that rotate with the star, the negative region and the part of the positive region bounded by the 10 equipotential. The positive region lying beyond $r=1.5$ (which is where the 10 equipotential crosses the equatorial plane) is in superrotation. This result avoids conflict with Pilipp's theorem (Pilipp, 1974) which states that it is impossible to have a solution with vacuum gaps that has all the space charge corotating. The 10 model neatly avoids the problem by providing a superrotating region. A non-corotating region (and hence a vacuum gap model) could exist otherwise only if vacuum bubbles were present in the space charge region to open circuit field lines. It will be noticed that the superrotating region in figures 1a and 2a is contiguous to the corotating positive region. If one then looks at figures 3a–9a one sees only 5a, the 4 model with a .25 charge size,
also having this feature. In the rest the superrotating region has a part that is non-contiguous with the rest. Sometimes there are even two isolated superrotating regions. This feature appears to be a result of the charge quantization and the method by which the net charge of the system is changed - the shell of uniform charge that is superimposed upon the surface charge.

The 8 and 6 models are presented in figures 3a–c and 4a–c primarily for the sake of completeness. They give an idea of how the negative region expands as the net charge of the system is reduced and show that the vacuum gap model is not limited to certain net charges.

The 4 model is examined in figures 5a–c, 6a–c and 7a–c. Figures 5a–c the 4 model with the standard .25 charge size. Examining figure 5a one notes show that the negative region spreads out at an almost constant angle as one moves out from the star. That this effect is a consequence of the charge size is indicated by figures 6a or 7a with a .15 charge size. Here the negative region stops spreading out after a certain distance from the star. Figure 5c indicates that there are problems with the 4 model insofar as the 2 equipotential is concerned. It bends to the left, away from the corotational equipotential (field line). That the model is not good in this region is already apparent by looking at 5b which shows the required condition at the surface begins
to be violated at greater than 20°. A dramatic improvement in the equipotential may be noted by examining figure 6c or 7c where the charge size has been reduced to .15 for \( a = 4 \) model. The 2 equipotential near the star agrees much more closely with the corotating equipotential. It will be noted that the 2 equipotential does not cross the corotating field line within the plot but this does not imply that the negative space charge region extends beyond the right of the plotted region. Rather, the equipotential is biased to the left because of the conical vacuum that exists about the axis. To get an idea of where the plasma region on a given field line terminates one needs to plot an equipotential whose value is not 2 but which coincides more closely with the 2 corotating field line/equipotential.

In the 4 model the superrotating region is present (as it is in all models) and figures 6a and 7a show the noncontiguous nature of the region. Some of the particle groups that are superrotating are vacuum gapped from each other. The .25 model does not have the noncontiguous feature of the superrotating regions as noted earlier. The difference between the two 4 models with charge quantization of .15 is that the scrub parameter for the model in figures 6a–c is \( 5 \times 10^{-4} \) and in figures 7a–c it is \( 2 \times 10^{-3} \). One notes in 6a that the positive region is extended almost to \( r = 4 \) whereas in 7a it is a good deal closer to the star, at less
than \( r=3 \). This is in part due to the fact that a larger scrub parameter is useful in preventing large anomalies in the surface charge distribution. That is, a single particle is not able to effect a major change in the surface charge, because it has to be a certain minimum distance away from the surface and the further away this is the less its effect (which goes as \( k/r^2 \)). It also gives an idea of how the change in the scrub parameter by a little more than an order of magnitude changes the resulting model. Clearly, the charge quantization is a much more important factor than the scrub parameter in the model.

The last model to be presented is the 2 model and the reason that it is the last is that with the relatively large charge size employed the method used can become unstable. During an iteration step the whole system could become negatively charged (a number of positive particles 'crash' into the surface and since surface charge is not taken into account during the iteration process they are lost for the step) which would cause the system to become unbound. In a real physical case one would expect a neutral system since if the system had a positive charge one would expect electrons to stream in on the polar field lines from infinity to neutralize it. In the limit of infinitely small charge size the model would be a neutral system with the negative region extending to infinity on the rotation axis.
The major feature to notice in figure 8a is how dramatically the size of the negative region has increased in the vertical direction as shielding of the positive charge becomes more dominant.

The 2 model was also used to try the efficacy of the bias method described in the previous chapter. The results are summarized in figures 9a–c. Figure 9c shows that the effect of biasing was less than satisfactory. The 2 equipotential goes more to the left in the biased model than in the unbiased model and the bias method was instituted to do the exact opposite.

Something that deserves comment with respect to all the models presented is that, if one counts the total charge of the particles inside the positive corotation zone, the charge is greater than the amount of charge one would have if the charge density were Goldreich and Julian. The value assuming the Goldreich and Julian charge density is 1.9 whereas the models have about 2.75. This is because we are using discrete charges to model the field due to a continuous charge distribution. With a continuous charge distribution the same field is modeled by a charge of 1.9 inside the corotation region and the rest of the space charge outside. In other words, with a discrete system of rather large charges the fundamental quantity is the field in a region and not the amount of charge used therein to construct the field.
The results of the model presented here indicate a 
corotating negative magnetospheric region separated from the 
positive region by a vacuum gap. The positive region is 
made up of a corotating region bounded by the magnetic field 
line that crosses the equatorial plane at $r=1.5$ and the 
models naturally produce a superrotating region that 
prevents violation of Pilipp's theorem. The exact form of 
this superrotating region does not appear to be critical as 
the negative charge region can vary in shape to account for 
different positive regions. Figures 10 and 11 show the 
magnetospheric envelopes (where the space charge drops 
abruptly to 0) for the 10 and 4 models, respectively. The 
envelope is computed by locating the $E\cdot B=0$ contour that lies 
outside of the regions occupied by the particles. This 
contour defines the boundary of the space charge regions one 
would have if a continuous charge distribution had been 
used.

Figure 12 shows an idealized configuration corresponding 
to a very small charge size and a near neutral system. Note 
that the surface is totally surrounded by the magnetosphere 
so that at most one point of the surface adjoins the vacuum. 
The negative and positive regions connect where the charge 
density of the star goes through zero. The magnetosphere 
lies well within the light cylinder of all radio pulsars 
except possibly the recently discovered superf"
(Backer et al., 1982) whose light cylinder is at about 7.5 stellar radii assuming it has a radius of about 10 km. Its rotation speed is 20 times faster than the second fastest pulsar and its magnetic field is apparently quite weak. If its radius is less than assumed it might also lie within the light cylinder. Too little is yet known to make a more detailed examination as to how these results apply to that curious object. Certainly the aligned models are static for any of the other known pulsar rotation rates insofar as no appreciable conversion of rotational energy to electromagnetic radiation can be expected from them.
V. Conclusions

The existence of vacuum gap models generated by surface emission of particles to a magnetosphere in a self-consistent manner raises serious doubts about the wisdom of using the Goldreich and Julian model as the basis for a pulsar theory. The models presented here satisfy the Goldreich and Julian criterion at the stellar surface and are static once this configuration has been reached. All the charges sit at equilibrium points and rotate about the star at velocities well below the speed of light. This means they produce no appreciable electromagnetic radiation.

The Goldreich and Julian solution is not self-consistent because it extends beyond the light cylinder in all cases and cannot consistently handle the current flow problems that arise there. It does not in any of its forms satisfy the condition that the average net current from the pulsar be zero (Michel, 1982). Even the models developed here might well not be a correct picture of a pulsar but investigations should be carried out of cases where there is a small degree of non-alignment between rotational and magnetic axes. If these also are essentially static it would indicate that there is something missing or incorrect about the basic pulsar assumptions that have been made.
In future work the magnetospheres should be modeled by a continuous charge distribution in the areas that are Goldreich and Julian and some amount of positive charge placed in the superrotating region using the models developed here as a guide. Then the envelope of the negative magnetosphere would be perturbed till \( E \cdot B = 0 \) holds on the envelope and one obtains a stable configuration.
Figure 1a. This plot shows the locations of the rings in the +10 model for a charge size of .25.

Figure 1b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15\sin^2\theta$ which is the Goldreich and Julian value on the surface of the star.

Figure 1c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +10 model (solid lines). The bottom numbers indicate value of equipotentials starting from the left and going clockwise.
Figure 2a. This plot shows the locations of the rings in the +10 model for a charge size of .175.

Figure 2b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15\sin^2\theta$ which is the Goldreich and Julian value on the surface of the star. Charge size is .175.

Figure 2c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +10 model (solid lines) with charge size .175. The bottom numbers indicate value of equipotentials starting from the left and going clockwise.
Fig. 2a

10

8

6

4

2

0 2 4 6 8 10

EQUATORIAL PLANE

POLAR AXIS

Fig. 2b

15

MODEL CHARGE = -10

SURFACE POTENTIAL

0 30

COLATITUDE

Fig. 2c

+10 (.175)

2.5

2.0

1.5

1.0

0.5

0

0.0 0.5 1.0 1.5 2.0 2.5 3.0

2,000 4,000 6,000 8,000 10,000 12,000
Figure 3a. This plot shows the locations of the rings in the $+8$ model for a charge size of .25.

Figure 3b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15 \sin^2 \theta$ which is the Goldreich and Julian value on the surface of the star.

Figure 3c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the $+8$ model (solid lines). The bottom numbers indicate value of equipotentials starting from the left and going clockwise.
Figure 4a. This plot shows the locations of the rings in the +6 model for a charge size of .25.

Figure 4b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15\sin^2\Theta$ which is the Goldreich and Julian value on the surface of the star.

Figure 4c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +6 model (solid lines). The bottom numbers indicate values of equipotentials starting from the left and going clockwise.
Figure 5a. This plot shows the locations of the rings in the +4 model for a charge size of .25.

Figure 5b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15 \sin^2 \theta$ which is the Goldreich and Julian value on the surface of the star.

Figure 5c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +4 model (solid lines). The bottom numbers indicate values of equipotentials starting from the left and going clockwise.
Fig. 5a

Fig. 5b

Fig. 5c
Figure 6a. This plot shows the locations of the rings in the +4 model for a charge size of .15 and scrub set to $5 \times 10^{-4}$.

Figure 6b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15 \sin^2 \theta$ which is the Goldreich and Julian value on the surface of the star. Charge size is .15 and scrub set to $5 \times 10^{-4}$.

Figure 6c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +4 model (solid lines) with charge size of .15 and scrub set to $5 \times 10^{-4}$. The bottom numbers indicate values of equipotentials starting from the left and going clockwise.
Figure 7a. This plot shows the locations of the rings in the +4 model for a charge size of .15 and scrub set to $2 \times 10^{-3}$.

Figure 7b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15 \sin^2 \theta$ which is the Goldreich and Julian value on the surface of the star. Charge size is .15 and scrub set to $2 \times 10^{-3}$.

Figure 7c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +4 model (solid lines) with charge size of .15 and scrub set to $2 \times 10^{-3}$. The bottom numbers indicate values of equipotentials starting from the left and going clockwise.
Figure 8a. This plot shows the locations of the rings in the +2 model for a charge size of 0.25.

Figure 8b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15 \sin^2 \theta$ which is the Goldreich and Julian value on the surface of the star.

Figure 8c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +2 model (solid lines). The bottom numbers indicate values of the equipotentials starting from the left and going clockwise.
Figure 9a. This plot shows the locations of the rings in the +2 model for a charge size of .25 and the model uses the bias feature.

Figure 9b. This plot compares the value of the potential at the ring launch sites on the surface to that of $15\sin^2\theta$ which is the Goldreich and Julian value on the surface of the star.

Figure 9c. This plot compares the equipotentials of the Goldreich and Julian model (dotted lines) to the equipotentials of the +2 model with bias (solid lines). The bottom numbers indicate values of the equipotentials starting from the left and going clockwise.
Fig. 9a

Fig. 9b

Fig. 9c
Figure 10

+10 model with charge size = .175

$r = 4.2$

$\Theta = 55^\circ$

$\text{Star}$

$r = 1$  \quad r = 1.8$
$r = 1.86$

$\theta = 55^\circ$

$4$ model with charge
size $= .15$

Figure 11
Possible configuration for a near neutral system with continuous charge distribution

Figure 12
Appendix A

Derivation of the General Formula for the Surface Charge

The surface charge is defined by:

$$\sigma = -1/4\pi (d\phi/dr) \quad (A1)$$

and the derivation of formula (19) in the body of the thesis is most easily understood by following the example of two rings and their mirror images (which give the factor of two in the potential) in some detail. The potential of ring 1 plus its image in the equatorial plane is:

$$V_{\text{ring}1} = 2q_1(P_2 r^2 P_2(x_1)/R_1^3 + P_4 r^4 P_4(x_1)/R_1^5 + \ldots) \quad (A2)$$

where $x_1 = \cos \theta$

The expression for ring 2 is similar except for subscripts.

We wish that surface charge that at $r=1$ cancels the potential of all rings and satisfies $E \cdot H = 0$ on the surface. The potential of the surface charge has to cancel the potential of the rings and impose a $4P_2$ potential at the surface.
This is accomplished term by term by defining coefficients $C_{2n}$:

$$C_{2n} = -2 \sum_{n=1}^{k} -q_{i}P_{2n}(x_{i})/R_{i}^{2n+1} + 2 \text{ (if } n=1)$$  \hspace{1cm} (A3)

where $k$ is the number of terms in the Legendre series for each ring. The surface charge potential is then given by:

$$\phi = \begin{cases} 
C_{2}P_{2}r^{2} + C_{4}P_{4}r^{4} + C_{6}P_{6}r^{6} + \ldots \text{ for } r<1 \\
C_{2}P_{2}/r^{3} + C_{4}P_{4}/r^{5} + C_{6}P_{6}/r^{7} + \ldots \text{ for } r>1
\end{cases}$$  \hspace{1cm} (A4)

and then employing the definition of the surface charge one arrives at the general formula:

$$\sigma = 1/4\pi (5C_{2}P_{2} + 9C_{4}P_{4} + \ldots + (4k+1)C_{2k}P_{2k})$$  \hspace{1cm} (19)
Appendix B

Relationship of $E \times B$ to Surface Charge

At $r=1$, the surface of the star the charge density is given by:

$$4\pi \sigma = E_r > 1 - E_r < 1$$  \hspace{1cm} (B1)

Now multiplying both sides by $B_r$ for a dipole gives:

$$4\pi \sigma = B_r \times (E_r > 1 - E_r < 1)$$  \hspace{1cm} (B2)

Adding $(E_\theta - E_\theta) \times B_\theta$ to both sides gives:

$$4\pi \sigma B_r = (E_r > 1) - E_r < 1) \times B$$  \hspace{1cm} (B3)

Since $E (r < 1) \times B = 0$ inside the star the expression for the surface charge for a dipole $B$ field is:

$$\sigma = (E_r > 1) \times B / (8\pi \cos \theta)$$  \hspace{1cm} (B4)
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