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P-N APPROXIMATION FOR RADIATIVE TRANSFER IN A NONGRAY PLANAR MEDIUM

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P-N APPROXIMATION
FOR RADIATIVE TRANSFER
IN A NONGRAY PLANAR MEDIUM

by

ADNAN YÜCEL

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

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HOUSTON, TEXAS

May 1982
P-N APPROXIMATION
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ABSTRACT

The P-N approximation is extended to treat nongray radiative transfer problems in planar media. The rectangular model is used to characterize the spectral dependence of the absorption coefficient. The P-1 and P-3 approximation formulations are presented. Solutions are obtained for the cases of radiative equilibrium, internal heat generation and combined conduction and radiation. The effects of the temperature dependence of the absorption coefficient are also investigated. The results show good comparison with the existing exact solutions. As would be expected, the P-3 approximation is superior to the P-1 approximation over the whole range of optical thickness.
To my parents

RAİKA and MEHMET
ACKNOWLEDGEMENT

I wish to express my sincere gratitude to Dr. Yıldız Bayazıtöglu for her continued support and guidance throughout my research. I would also like to thank Dr. Alan J. Chapman and Dr. Mary F. Wheeler for serving on the oral examination committees.

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NOMENCLATURE

\( D \)  \( \text{the operator } \frac{d}{dt} \)
\( e_b \)  \( \text{blackbody emissive power} \)
\( i \)  \( \text{intensity of radiation} \)
\( i_b \)  \( \text{blackbody intensity of radiation} \)
\( i_k \)  \( k^{th} \text{ moment of intensity of radiation, } k = 0,1, \ldots, N. \)
\( k \)  \( \text{thermal conductivity} \)
\( l_z \)  \( \text{direction cosine of } z\text{-direction, } \cos \beta \)
\( L \)  \( \text{distance between plates} \)
\( N \)  \( \text{conduction-radiation parameter, } \kappa k/4\sigma T_1^3 \)
\( q_C, Q_C \)  \( \text{conductive heat flux, } Q_C = q_C/\sigma T_1^4 \)
\( q_R, Q_R \)  \( \text{radiative heat flux, } Q_R = q_R/\sigma T_1^4 \)
\( q_T, Q_T \)  \( \text{total heat flux, } Q_T = q_T/\sigma T_1^4 \)
\( \vec{r} \)  \( \text{position vector} \)
\( s, S \)  \( \text{rate of heat generation, } S = (s/\kappa)/\sigma T_1^4 \)
\( T \)  \( \text{temperature} \)
\( T_1 \)  \( \text{wall temperature at } z = 0 \)
\( T_2 \)  \( \text{wall temperature at } z = L \)
\( T_{av} \)  \( \text{average temperature, } (T_1+T_2)/2 \)
\( z \)  \( \text{spatial coordinate} \)
Greek Symbols

\( \beta \)  
\( \Delta \nu, \Delta \omega \)  
\( \varepsilon \)  
\( \theta \)  
\( \theta_1 \)  
\( \theta_2 \)  
\( \kappa \)  
\( \nu \)  
\( \overline{\nu} \)  
\( \overline{\nu}_c \)  
\( \sigma \)  
\( \tau \)  
\( \tau_0 \)  
\( \phi \)  
\( \phi \)  
\( \psi \)  
\( \omega \)  
\( \Omega \)  

Subscripts

\( j \)  
\( k \)  
\( \nu \)
CHAPTER I

INTRODUCTION

The study of radiative heat transfer in a radiating medium has become increasingly important because of the applications in nuclear reactors and industrial furnace designs, laboratory shock tube studies, ablative protection, glass manufacture and cryogenics. In these applications, radiative heat transfer may become the dominant mode of heat exchange. The basis for analyzing a radiation field in an absorbing, emitting, and scattering medium is the equation of radiative transfer, which is an integro-differential equation written in terms of the spectral intensity of radiation. The study of combined conductive and radiative medium requires the solution of the coupled equations for the radiative transfer and for the conservation of energy, to obtain the temperature field and energy transfer. The governing equations are often subject to boundary conditions which are integral expressions.

An exact solution of the equation of radiative transfer may require integrations with respect to time, position, frequency and direction. While such a solution may be accurate, its complexity, for most practical problems, makes it inconvenient. Many investigators have employed assumptions and approximations that led to simple solutions which are restricted in applicability. The most common assumption in the literature is that of a gray medium which
eliminates the frequency dependence. Since absorption in most gases and semi-transparent solids is restricted to specific intervals of the spectrum, the gray gas concept is of limited utility. Earlier attempts to take into account the non-gray behavior led to the definition of various mean absorption coefficients which are restricted to certain optical conditions [1]. The study of radiative transfer in planetary atmospheres and combustion systems have resulted in detailed knowledge of the constituents such as CO, N₂O, H₂O, CH₄, NH₃ and O₃ which emit and absorb significantly in the infrared range. Various line and band models and band absorptance correlations have been developed to be used in theoretical calculations of the transmittance (or absorptance) of these gases [1-29]. Non-gray radiative transfer in semi-transparent solids have also received considerable attention [30].

Even with the gray medium assumption, the mathematical difficulties encountered in working with integro-differential equations have resulted in a number of approximate solutions to the equation of radiative transfer. One of the more elaborate schemes is the spherical harmonics method or the P-N approximation, first suggested by Jeans [31]. The spherical harmonics method was initially employed by astrophysicists in the problem of radiative transfer in stellar atmospheres. It was first applied to thermal radiative studies by Cheng [32, 33], who took the first order intensity approximation. Arpacı and Gözüm [34] and Arpacı and Bayazıtöglu [35] also used the first order approximation in natural convection studies between
parallel plates. The P-1 approximation has been shown to give satisfactory results for one dimensional optically thick medium. But, it will, like the diffusion approximation, tend to overpredict the heat flux in a medium which is optically thin. To overcome this setback, various modifications and improvements have been introduced [36-40]. Despite the increased accuracy, all of their improvements introduce integral expressions which remove the element of simplicity afforded by the P-1 approximation. The analysis becomes more complex if the modifications are extended to multi-dimensional problems. Higher order approximations would retain the differential nature of the approximation and result in more accurate solutions than those obtained from the first order approximation. Increasing the number of terms, however, also complicates the solution process. Higher order approximations were used in solving various one- and two-dimensional radiative transfer problems in planar, spherical and cylindrical media [41-47]. The P-3 approximation yields satisfactory solutions over the whole range of optical thickness in planar geometries [42, 46]. Bayazitoglu and Higenyi [42] have shown that the P-N approximation is less accurate in nonplanar media.

Despite its effectiveness in generating accurate approximate solutions to the gray radiative transfer problem, little work has been done to generalize the P-N approximation for nongray problems. The challenge here lies in the integration of the moment equations over the spectral variable. The differential form of these equations precludes the use of band absorptance correlations which are tailored
for the integral formulation. Modest [48] has combined a box model with the P-1 approximation. The results show good comparison for the case of radiative equilibrium. Kung and Sibulkin [49] have used the differential approximation to obtain approximate solutions in a radiating medium with a two-level absorption coefficient. In another approach, Yuen and Rasky [50] have applied the P-1 approximation to a nongray medium by formulating the problem in terms of the spectral coefficient. Their results obtained by approximating the spectral moments of intensity by specific polynomials compare well with the exact results for the case of radiative equilibrium.

The purpose of this study is to develop and solve the differential relations for nongray radiative transfer in a participating planar medium. The P-1 and P-3 approximations will be used to represent the angular distribution of the radiation intensity. The rectangular model will be employed to characterize the spectral variations of the absorption coefficient. The rectangular model can be readily introduced into the P-N approximation formulation. Though less realistic than other band models, its accuracy can be improved by dividing the spectrum into smaller intervals. Multiband cases are easily handled. It is also applicable to those media for which band absorptance correlations are not available or the use of narrow band approximation is not justifiable. Marshak boundary conditions are used to complete the formulation. Solutions will be obtained for various radiative transfer problems. Results will be compared to the available exact solutions and the accuracy of the approximations will be assessed.
The next five chapters can be separated into two distinct parts: one-dimensional P-1 and P-3 approximation formulation for a nongray planar medium (Chapters 2, 3, and 4) and analysis involving various rectangular models and discussion of results (Chapters 5 and 6). The spectral moment differential equations of radiative transfer and the corresponding boundary conditions for the P-1 and P-3 approximations are developed for a one-dimensional planar medium in Chapter 2. The conservation of energy equation is presented in Chapter 3. The rectangular model is introduced into the moment equations and boundary conditions in Chapter 4. In Chapter 5, various forms of the rectangular model are considered. Radiative transfer and combined conductive and radiative transfer results are presented and compared with the existing exact solutions. Effects of the temperature dependence of the absorption coefficient are presented. Conclusions and suggestions for future work are given in Chapter 6.
CHAPTER II

FORMULATION OF ONE-DIMENSIONAL P-N APPROXIMATION

The physical system under consideration is shown schematically in Figure 2.1. It consists of an absorbing, emitting and nonscattering medium bounded by two infinite parallel plates. The medium is in local thermodynamic equilibrium and has a refractive index of unity. The absorption coefficient of the medium is spectrally dependent and may also vary with temperature. The medium is capable of generating heat at a rate of $s$ per unit of volume. Steady state is assumed.

The plate at $z = 0$ (lower boundary) is maintained at temperature $T_1$ and the other at $z = L$ (upper boundary) at temperature $T_2$. Throughout this work $T_1$ is greater than $T_2$. The opaque boundaries are assumed to be diffusely emitting and reflecting surfaces. In the following section, the one-dimensional problem of radiative heat transfer in this system will be formulated.

2.1 Equation of Radiative Transfer

The equation of radiative transfer governing this one-dimensional planar system is:

$$l_z \frac{di_v}{dz} = -\kappa_v i_v + \kappa_v i_{bv}$$  \hspace{1cm} (2.1)
Figure 2.1 Physical model and coordinate system
where

\[ i_{\nu} = i_{\nu}(z; \beta) \]  \hspace{1cm} (2.2)  
\[ i_{b\nu} = e_{b\nu}/\pi = i_{b\nu}(z) \]  \hspace{1cm} (2.3)  
\[ \kappa_{\nu} = \kappa_{\nu}(T) \]  \hspace{1cm} (2.4)  
\[ l_{z} = \cos \beta \]  \hspace{1cm} (2.5)  
and \( \nu \) denotes the spectral dependence.

2.2 Intensity of Radiation

In the spherical harmonics method, the intensity of radiation is expanded in an orthogonal series of the spherical harmonics

\[ i_{\nu}(\mathbf{r}; \beta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2n+1}{4\pi} P_{n}^{m}(\cos \beta) [A_{n}^{m}(\mathbf{r}, \nu) \cos m\phi \]
\[ + D_{n}^{m}(\mathbf{r}, \nu) \sin m\phi] \]  \hspace{1cm} (2.6)  

where \( P_{n}^{m}(\cos \beta) \) are the associated Legendre polynomials of the first kind. In practice the series is truncated after a certain (N) number of terms (hence the P-N approximation) such that \( A_{n}^{m} = D_{n}^{m} = 0 \) for \( n > N \). The coefficients \( A_{n}^{m} \) and \( D_{n}^{m} \) up to \( N=3 \) are given in Table 2.1.

For one-dimensional geometry under azimuthal symmetry the intensity distributions for the P-1 and P-3 approximations are given below.

**P-1 Approximation**

\[ 4\pi i_{\nu}(z) = i_{0\nu}(z) + 3i_{1\nu}(z)P_{1}^{0}(\cos \beta) \]  \hspace{1cm} (2.7)  

Table 2.1  Coefficients $A^m_n$ and $D^m_n$ in terms of the moments of intensity

<table>
<thead>
<tr>
<th>m</th>
<th>n = 0</th>
<th>n = 1</th>
<th>n = 2</th>
<th>n = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$i_0$</td>
<td>$i_z$</td>
<td>$(3i_{zz} - i_0)/2$</td>
<td>$(5i_{zzz} - 3i_z)/2$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$i_x$</td>
<td>$i_{xz}$</td>
<td>$(5i_{xxx} - i_x)/4$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>$(i_{xx} - i_{yy})/4$</td>
<td>$(i_{zxx} - i_{zyy})/4$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>$(i_{xxx} - i_{xyy})/24$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>m</th>
<th>n = 0</th>
<th>n = 1</th>
<th>n = 2</th>
<th>n = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$i_y$</td>
<td>$i_{zy}$</td>
<td>$(5i_{yzz} - i_y)/4$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>$i_{xy}/2$</td>
<td>$i_{xyz}/2$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>$(3i_{yxx} - i_{yyy})/24$</td>
</tr>
</tbody>
</table>

Note: Coefficients $A^m_n$ and $D^m_n$ can be adapted to cylindrical and spherical geometries by rearranging the subscripts according to the following:

<table>
<thead>
<tr>
<th>Direction cosine</th>
<th>Cartesian</th>
<th>Direction Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin\theta\cos\phi$</td>
<td>$x$</td>
<td>$r$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$\sin\theta\cos\phi$</td>
<td>$y$</td>
<td>$\theta$</td>
<td>$\varphi$</td>
</tr>
<tr>
<td>$\cos\theta$</td>
<td>$z$</td>
<td>$z$</td>
<td>$r$</td>
</tr>
</tbody>
</table>

The following relations also hold:

$$i_0 = i_{xx} + i_{yy} + i_{zz}, \quad i_s = i_{sxx} + i_{syy} + i_{szz}, \quad s = x, y \text{ or } z$$
P-3 Approximation

\[ 4\pi i_\nu(z) = i_0\nu(z) + 3i_1\nu(z)P_1^0(\cos \beta) \]
\[ + \frac{5}{2}[3i_2\nu(z) - i_0\nu(z)]P_2^0(\cos \beta) \]
\[ + \frac{7}{2}[5i_3\nu(z) - 3i_1\nu(z)]P_3^0(\cos \beta). \] \hspace{1cm} (2.8)

In the above equations, the relevant coefficients \( A_n^m \) and \( B_n^m \) have been replaced by their equivalents in terms of the moments of intensity. The \( k^{th} \) moment of intensity is defined as

\[ i_{k\nu}(z) = \int_{\Omega} i_{\nu 1}^k d\Omega, \quad k = 0, 1, \ldots, N. \] \hspace{1cm} (2.9)

Of particular interest is the first moment of intensity which physically signifies the spectral radiative flux in the \( z \)-direction.

\[ i_{1\nu} \equiv q_{\nu 0} = \int_{\Omega} i_{\nu 1} d\Omega \] \hspace{1cm} (2.10)

2.3 Moment Differential Equations

The governing set of differential equations in the P-N approximation are obtained by multiplying the equation of transfer by the appropriate powers of the direction cosine and integrating over all solid angles. Equations 2.11 are obtained for the P-3 approximation.

\[ \frac{di_{1\nu}}{dz} = \kappa_{\nu}(4e_{b\nu} - i_{0\nu}) \] \hspace{1cm} (2.11a)
\[ \frac{di_{2\nu}}{dz} = -\kappa_{\nu}i_{1\nu} \] \hspace{1cm} (2.11b)
\[ \frac{di_{3\nu}}{dz} = \kappa_{\nu}(4e_{b\nu}/3 - i_{2\nu}) \] \hspace{1cm} (2.11c)
\[
\frac{d i_{4v}}{dz} = -\kappa_{i_{3v}}
\]  
(2.11d)

The closure condition for the P-3 approximation is

\[
i_{4v} = 6i_{2v}/7 - 3i_{0v}/35.
\]  
(2.12a)

Equations 2.11a and 2.11b constitute the governing equations for the P-1 approximation, with the closure condition given by

\[
i_{2v} = i_{0v}/3.
\]  
(2.12b)

2.4 **Boundary Conditions**

The boundary conditions obtained by using the Marshalk approach [42, 51] are given below.

**P-1 Approximation**

\[
\varepsilon_{y} i_{0v} \pm 2(2-\varepsilon_{y})i_{1v} = 4\varepsilon_{y} e_{bv}
\]  
(2.13)

**P-3 Approximation**

\[
5\varepsilon_{y} i_{0v} \pm 16(2-\varepsilon_{y})i_{1v} + 15\varepsilon_{y} i_{2v} = 32\varepsilon_{y} e_{bv}
\]  
(2.14a)

\[
(5\varepsilon_{y} - 5)i_{0v} \pm 16(1-\varepsilon_{y})i_{1v} + 15(1+\varepsilon_{y})i_{2v} \pm 32i_{3v} = 32\varepsilon_{y} e_{bv}
\]  
(2.14b)

where the + sign corresponds to the lower boundary and the - sign to the upper boundary.
CHAPTER III

CONSERVATION OF ENERGY

Equations 2.11 can be solved if the temperature distribution is known a priori. However, this is not the case in the majority of the problems where the temperature and the intensity distributions need to be solved simultaneously. The conservation of energy equation is used to relate the emissive power (i.e. the temperature) with the moments of intensity. In the absence of convection, the conservation of energy for a conducting and heat generating medium dictates that

\[
d(q_C + q_R)/dz = s
\]

(3.1)

where the conductive heat flux is given by

\[
q_C = -k \frac{dT}{dz}.
\]

(3.2)

The total radiative flux is

\[
q_R = \int_0^\infty q_{R\nu} d\nu = \int_0^\infty i_{1\nu} d\nu.
\]

(3.3)

From equation 2.11a, we have

\[
\frac{dq_R}{dz} = \int_0^\infty \kappa_{\nu}(4e_{\nu} - i_{0\nu}) d\nu.
\]

(3.4)
CHAPTER IV

P-N APPROXIMATION WITH THE RECTANGULAR MODEL

4.1 Rectangular Model for the Absorption Coefficient

In order to integrate equations 2.11 over the spectral variable $\nu$, the spectral dependence of the absorption coefficient of the medium must be known. In many gases and semi-transparent solids, absorption is restricted to specific intervals of the spectrum. Among the many models, the rectangular model is the simplest one that can be used in such physical situations. In this model, the spectral absorption coefficient is assumed to be constant over certain intervals and zero outside these "windows"

$$\kappa_{\nu} = \kappa_j \text{ for } \nu_{cj} - \Delta \nu_j/2 < \nu < \nu_{cj} + \Delta \nu_j/2,$$

$$j = 1, 2, \ldots, n \quad (4.1)$$

where $\Delta \nu_j$ is the bandwidth and $\nu_{cj}$ is the bandcenter for the $j^{th}$ band. A three band model is shown in Figure 4.1.

In principle any spectral dependence of the coefficient can be accurately approximated by the rectangular model. Its accuracy can be increased by dividing the spectrum into smaller intervals.

4.2 Spectrally Integrated Moment Equations and Boundary Conditions

Given the rectangular model for the spectral variation of the absorption coefficient, the moment equations and the boundary conditions
Figure 4.1 Rectangular model for the spectral absorption coefficient
for the P-1 and P-3 approximations can be integrated over the spectral variable for each band. Introducing the closure conditions given by equations 2.12, the integrated form of the moment equations are given below.

**P-1 Approximation**

\[
\begin{align*}
    d\text{i}_{0j}/dz &= -3\kappa_j\text{i}_{1j} \\
    d\text{i}_{1j}/dz &= \kappa_j(4\text{e}_{bj} - \text{i}_{0j})
\end{align*}
\]  

(4.2a)  
(4.2b)

**P-3 Approximation**

\[
\begin{align*}
    d\text{i}_{0j}/dz &= \kappa_j(-10\text{i}_{1j} + 35\text{i}_{3j}/3) \\
    d\text{i}_{1j}/dz &= \kappa_j(4\text{e}_{bj} - \text{i}_{0j}) \\
    d\text{i}_{2j}/dz &= -\kappa_j\text{i}_{1j} \\
    d\text{i}_{3j}/dz &= \kappa_j(4\text{e}_{bj}/3 - \text{i}_{2j})
\end{align*}
\]  

(4.3a)  
(4.3b)  
(4.3c)  
(4.3d)

where

\[
\begin{align*}
    \text{i}_{kj} &= \int_{\Delta\nu_j} \text{i}_{kv} \, d\nu, \quad k = 0,1, \ldots, N \\
    \text{e}_{bj} &= \int_{\Delta\nu_j} \text{e}_{bv} \, d\nu \quad \text{and} \quad j = 1,2, \ldots, n.
\end{align*}
\]  

(4.4)  
(4.5)

The total radiative flux can now be expressed in the form:

\[
q_R = \sigma T_1^4 - \sigma T_2^4 + \sum_{j=1}^{N} \text{i}_{1j} - \sum_{j=1}^{N} [\text{e}_{bj}(T_1) - \text{e}_{bj}(T_2)]
\]

(4.6)

where

\[
\sigma T^4 = \int_{0}^{\infty} \text{e}_{bv}(T) \, d\nu
\]

(4.7)

is the total emissive power.
Also, combining equations 4.3b and 3.4 yields

\[ \frac{dq_R}{dz} = \sum_{j=1}^{n} di_{1j}/dz = \sum_{j=1}^{n} \kappa_j (4e_{b_j} - i_{0j}). \quad (4.8) \]

To simplify the analysis the boundaries will be assumed to be gray. The respective integrated boundary conditions for the P-1 and P-3 approximations are

\[ \varepsilon i_{0j} \pm 2(2-\varepsilon)i_{1j} = 4\varepsilon e_{b_j} \quad (4.9) \]

and

\[ 3\varepsilon i_{0j} \pm 16(2-\varepsilon)i_{1j} + 15\varepsilon i_{2j} = 32\varepsilon e_{b_j} \quad (4.10a) \]

\[ (3\varepsilon - 5)i_{0j} \pm 16(1-\varepsilon)i_{1j} + 15(1+\varepsilon)i_{2j} = 32\varepsilon e_{b_j}. \quad (4.10b) \]
CHAPTER V

ANALYSIS FOR VARIOUS RECTANGULAR MODELS AND DISCUSSION OF RESULTS

In this chapter, various rectangular model configurations will be considered and the heat transfer results will be presented to show the effects of nongrayness. Wherever possible, the P-1 and P-3 solutions will be compared with the exact solutions and the accuracy of these approximations will be discussed. In the first part of this chapter, pure radiative transfer problems will be considered. The discussion of a temperature dependent absorption coefficient is included. In the second part, the interaction of conduction and radiation will be presented. In order to reduce the number of independent parameters in these problems, the boundaries will be assumed to be black (i.e. \( \varepsilon_1 = \varepsilon_2 = 1 \)) although the expressions for the boundary conditions may retain their general form.

5.1 Purely Radiating Medium

When there is uniform heat generation and conduction is negligible, equation 5.1 reduces to

\[
\frac{d q_R}{dz} = s = \sum_{j=1}^{n} \kappa_j (4 \varepsilon_{b_j} - i_{b_j})
\]

(5.1)

with the total radiative heat flux given by equation 4.6.
5.1.1 \textbf{n Bands with Identical Absorption Coefficients}

A special case merits attention when conduction is negligible. If the absorption coefficients for each band are identical, i.e.

\[ \kappa_j = \kappa = \text{constant} \]

the moment equations and the boundary conditions for each band can be summed to yield a single set of differential equations and boundary conditions for the total quantities

\[ i_{kt} = \sum_{j=1}^{n} i_{kj}, \quad k = 0, 1, \ldots, N \quad (5.2a) \]

and

\[ e_{bt} = \sum_{j=1}^{n} e_{bj} \quad (5.2b) \]

In general this system of equations is nonlinear. If the absorption coefficient is independent of the temperature, the system becomes linear and superposition can be used to divide the general problem into two subproblems [24].

\begin{itemize}
  \item \textbf{Subproblem I} \quad Lower Boundary at \( T_1 \)
  \item \quad Upper Boundary at \( T_2 \)
  \item \quad No Heat Generation
  \item \textbf{Subproblem II} \quad Lower Boundary at \( T_2 \)
  \item \quad Upper Boundary at \( T_2 \)
  \item \quad Uniform Heat Generation
\end{itemize}
P-1 Formulation

We define

\[
\begin{align*}
\phi_b &= (e_1 - e_2) \phi_b + (s/\kappa) \psi_b + e_2 \\
i_{0t} &= (e_1 - e_2) \phi_0 + (s/\kappa) \psi_0 + 4e_2 \\
i_{1t} &= (e_1 - e_2) \phi_1 + (s/\kappa) \psi_1
\end{align*}
\]  

(5.3a), (5.3b), (5.3c)

The dimensionless quantities \( \phi \) and \( \psi \) that correspond to subproblems I and II respectively are defined as follows:

\[
\begin{align*}
\phi_b &= [(e_{bt})_I - e_2]/(e_1 - e_2) \\
\phi_0 &= [(i_{0t})_I - 4e_2]/(e_1 - e_2) \\
\phi_1 &= (i_{1t})_I/(e_1 - e_2)
\end{align*}
\]  

(5.4a), (5.4b), (5.4c)

\[
\begin{align*}
\psi_b &= [(e_{bt})_II - e_2]/(s/\kappa) \\
\psi_0 &= [(i_{0t})_II - 4e_2]/(s/\kappa) \\
\psi_1 &= (i_{1t})_II/(s/\kappa)
\end{align*}
\]  

(5.5a), (5.5b), (5.5c)

and \( e_1 = e_{bt}(T_1), \ e_2 = e_{bt}(T_2). \)  

(5.6)

Substituting equations 5.3 in the moment equations and the boundary conditions, we obtain

\[
\begin{align*}
DI_0 &= -3I_1 \quad \text{(5.7a)} \\
DI_1 &= F \quad \text{(5.7b)}
\end{align*}
\]

\[
\begin{align*}
I_0 + 2(2/\varepsilon_1 - 1)I_1 &= 4B \quad \text{at } \tau = 0 \quad \text{(5.8a)} \\
I_0 - 2(2/\varepsilon_2 - 1)I_1 &= 0 \quad \text{at } \tau = \tau_0 \quad \text{(5.8b)}
\end{align*}
\]

where \( \tau = \kappa z, \ D \equiv (d/d\tau), \ \tau_0 = \kappa L. \)

Conservation of energy yields
\[ 4I_b - I_0 = F. \]  

(5.9)

In the above expressions, \( I, B \) and \( F \) correspond to:

<table>
<thead>
<tr>
<th>Subproblem I</th>
<th>Subproblem II</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( \phi )</td>
</tr>
<tr>
<td>( B )</td>
<td>1</td>
</tr>
<tr>
<td>( F )</td>
<td>0</td>
</tr>
<tr>
<td>( \psi )</td>
<td>1</td>
</tr>
</tbody>
</table>

Equations 5.7 can be integrated to yield:

\[ I_0(\tau) = -3F\tau^2/2 - 3c_1\tau + c_2 \]
\[ I_1(\tau) = F\tau + c_1 \]

The integration coefficients \( c_1 \) and \( c_2 \) are evaluated through the boundary conditions 5.8.

\[ c_1 = (4B - 3F\tau_0^2/2 - 2E_2) / (3\tau_0 + 2E_1 + 2E_2) \]
\[ c_2 = 4B - 2E_1 c_1 \]

where \( E_1 = 2/\varepsilon_1 - 1 \), \( E_2 = 2/\varepsilon_2 - 1 \).

**P-3 Formulation**

In addition to definitions of \( e_{bt}, i_{0t} \) and \( i_{1t} \) given by equations 5.3, we define

\[ i_{2t} = (e_1 - e_2)\phi_2 + (s/\kappa)\psi_2 + 4e_2/3 \]  
\[ i_{3t} = (e_1 - e_2)\phi_3 + (s/\kappa)\psi_3 \]  

(5.3d)

(5.3e)

where

\[ \phi_2 = \left[ (i_{2t})_1 - 4e_2/3 \right] / (e_1 - e_2) \]  
\[ \phi_3 = (i_{3t})_1 / (e_1 - e_2) \]  

(5.4d)

(5.4e)
\[
\psi_2 = \frac{(i_{2t})_{II} - 4e_2/3}{(s/\kappa)} \quad (5.5d)
\]
\[
\psi_3 = \frac{(i_{3t})_{II}}{(s/\kappa)} \quad (5.5e)
\]

Substituting equations 5.3 in the moment differential equations and the boundary conditions for the P-3 approximation, we obtain

\[
D I_0 = -10 I_1 + (35/3) I_3 \quad (5.10a)
\]
\[
D I_1 = F \quad (5.10b)
\]
\[
D I_2 = -I_1 \quad (5.10c)
\]
\[
D I_3 = 4 I_b/3 - I_2 \quad (5.10d)
\]

and

\[
3\varepsilon_1 I_0 + 16(2-\varepsilon_1)I_1 + 15\varepsilon_1 I_2 = 32\varepsilon_1 B \quad (5.11a)
\]
\[
(3\varepsilon_1-5) I_0 + 16(1-\varepsilon_1)I_1 + 15(1+\varepsilon_1) I_2 + 32 I_3 = 32\varepsilon_1 B
\]

at \( \tau = 0 \) \quad (5.11b)

\[
3\varepsilon_2 I_0 - 16(2-\varepsilon_2) I_1 + 15\varepsilon_2 I_2 = 0 \quad (5.11c)
\]
\[
(3\varepsilon_2-5) I_0 - 16(1-\varepsilon_2) I_1 + 15(1+\varepsilon_2) I_2 - 32 I_3 = 0
\]

at \( \tau = \tau_0 \) \quad (5.11d)

The conservation of energy equation is identical to equation 5.9.

Equations 5.10 can be combined to yield a single fourth order differential equation for \( I_0 \)

\[
D^4 I_0 - 35 D^2 I_0/9 = 35 F/3. \quad (5.12)
\]

The general solution of this equation is

\[
I_0(\tau) = -3F\tau^2/2 + D_1 + D_2 \tau + D_3 \cosh \left(\sqrt{35}\tau/3\right)
\]
\[
+ D_4 \sinh \left(\sqrt{35}\tau/3\right) \quad (5.13)
\]
where \( D_1 \) through \( D_4 \) are integration constants to be determined via the boundary conditions given in equations 5.11. Details can be found in Appendix A.

**Results.** From equations 5.7 and 5.10 we observe that the function vectors \( \phi \) and \( \psi \) are identical to those that arise in the gray case \([42, 46]\). Therefore \( \phi \) and \( \psi \) are general functions representing solutions of a class of problems for which the gray case is a limiting solution. The same also holds for the exact formulation of the problem in which the energy equation is reduced to the same linear Fredholm-type integral equation of the second kind that governs the temperature field in a gray medium \([24]\).

Hence when the absorption coefficient has only one nonzero value, the dimensionless heat flux \((\phi_1, \psi_1)\) and emissive power \((\phi_b, \psi_b)\) distributions given by the P-N approximation will be the same whether the medium is gray or "fractionally" gray. The influence of the nongrayness occurs in the functions \( e_1 \) and \( e_2 \).

Since the emissive power and radiative flux distributions for the gray case have been thoroughly discussed in the literature \([42, 46]\), they will be briefly mentioned here. As would be expected, P-3 approximation is superior to the P-1 approximation over the whole range of the optical thickness in predicting the emissive power and the radiative flux. P-1 approximation approaches the exact solution as the medium becomes optically thick. The largest errors occur at the boundaries where both approximations tend to underestimate the emissive power near the hot boundary and
Table 5.1 Radiative flux $\phi_R$ of subproblem I

<table>
<thead>
<tr>
<th></th>
<th>$\tau_o = .1$</th>
<th>$\tau_o = 1.$</th>
<th>$\tau_o = 10.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact[26]</td>
<td>0.91570</td>
<td>0.55340</td>
<td>0.11675</td>
</tr>
<tr>
<td>P-3</td>
<td>0.92166</td>
<td>0.55576</td>
<td>0.11691</td>
</tr>
<tr>
<td>P-1</td>
<td>0.93023</td>
<td>0.57143</td>
<td>0.11765</td>
</tr>
</tbody>
</table>
overestimate it in the vicinity of the cold boundary. In Table 5.1 the radiative flux results for the case of radiative equilibrium (subproblem I) are compared with the exact results.

The largest deviations in the results are for the optically thin medium. It should be noted, however, that the P-N approximations yield the correct results as the optical thickness approaches zero.

To show the effect of nongrayness, two simple models for the absorption coefficient will be considered. These are shown in Figure 5.1. Notice when $\nu_{c0}$, the cut-off frequency goes to infinity in model A or approaches zero in model B, the medium becomes gray.

In Table 5.2, the radiative flux results for the case of radiative equilibrium are compared with the exact results [26, 52]. Once the functions $\phi_b$ and $\psi_b$ are known, the medium temperature can be determined by solving equation 5.3a.

The temperature distributions for the case with $S = 0$, $\theta_2 = 0.5$, $\tau_0 = 1$ and $\nu_{c0} = 3$ are shown in Figure 5.2, along with the exact results [24]. The curves are almost identical in the central part of the medium. It should be noted that both the exact and the P-1 and P-3 approximation solutions yield the identical result $\phi_b(\tau_0/2) = 0.5$ for all $\tau_0$. We observe that the discrepancy in the emissive powers at the walls is reflected in the temperature distributions. The relative errors defined by
Figure 5.1 Models A and B for the spectral absorption coefficient:  a) model A  
  b) model B
Table 5.2 Effect of cut-off frequency $\bar{\nu}_c$ on the dimensionless radiative flux $Q_R$: model A and B, $\theta_2 = .5$ (radiation equilibrium)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\bar{\nu}_c$</th>
<th>$\tau = 1$</th>
<th>$\tau = 1.$</th>
<th>$\tau = 10.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex$^+$</td>
<td>0.9254</td>
<td>0.8733</td>
<td>0.8104</td>
<td></td>
</tr>
<tr>
<td>2. P-3</td>
<td>0.9262</td>
<td>0.8736</td>
<td>0.8105</td>
<td></td>
</tr>
<tr>
<td>P-1</td>
<td>0.9275</td>
<td>0.8759</td>
<td>0.8106</td>
<td></td>
</tr>
<tr>
<td>Ex</td>
<td>0.9089</td>
<td>0.7860</td>
<td>0.6378</td>
<td></td>
</tr>
<tr>
<td>3. P-3</td>
<td>0.9109</td>
<td>0.7868</td>
<td>0.6379</td>
<td></td>
</tr>
<tr>
<td>P-1</td>
<td>0.9138</td>
<td>0.7921</td>
<td>0.6381</td>
<td></td>
</tr>
<tr>
<td>Ex</td>
<td>0.8791</td>
<td>0.6282</td>
<td>0.3257</td>
<td></td>
</tr>
<tr>
<td>5. P-3</td>
<td>0.8832</td>
<td>0.6298</td>
<td>0.3258</td>
<td></td>
</tr>
<tr>
<td>P-1</td>
<td>0.8892</td>
<td>0.6407</td>
<td>0.3264</td>
<td></td>
</tr>
<tr>
<td>Ex</td>
<td>0.8585</td>
<td>0.5188</td>
<td>0.1095</td>
<td></td>
</tr>
<tr>
<td>Gray</td>
<td>0.8641</td>
<td>0.5210</td>
<td>0.1096</td>
<td></td>
</tr>
<tr>
<td>P-1</td>
<td>0.8721</td>
<td>0.5357</td>
<td>0.1103</td>
<td></td>
</tr>
<tr>
<td>Ex</td>
<td>0.8706</td>
<td>0.5831</td>
<td>0.2365</td>
<td></td>
</tr>
<tr>
<td>2. P-3</td>
<td>0.8753</td>
<td>0.5849</td>
<td>0.2366</td>
<td></td>
</tr>
<tr>
<td>P-1</td>
<td>0.8821</td>
<td>0.5974</td>
<td>0.2372</td>
<td></td>
</tr>
<tr>
<td>Ex</td>
<td>0.8871</td>
<td>0.6703</td>
<td>0.4091</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>0.8906</td>
<td>0.6717</td>
<td>0.4092</td>
<td></td>
</tr>
<tr>
<td>3. P-3</td>
<td>0.8958</td>
<td>0.6811</td>
<td>0.4096</td>
<td></td>
</tr>
<tr>
<td>P-1</td>
<td>0.9169</td>
<td>0.8261</td>
<td>0.7212</td>
<td></td>
</tr>
<tr>
<td>Ex</td>
<td>0.9183</td>
<td>0.8287</td>
<td>0.7213</td>
<td></td>
</tr>
<tr>
<td>5. P-3</td>
<td>0.9204</td>
<td>0.8326</td>
<td>0.7214</td>
<td></td>
</tr>
</tbody>
</table>

*Exact results obtained from [26, 52]*
Figure 5.2 Dimensionless temperature distributions for models A and B: $\nabla c_0 = 3, \tau_0 = 1, \theta_2 = 0.5$ (radiative equilibrium)
\[
|\% \text{ error}| = \left| \frac{P - N - \text{EXACT}}{\text{EXACT}} \right| \cdot 100\% \tag{45}
\]

and evaluated at the walls are presented in Table 5.3.

Both approximations can be considered to be highly accurate. Figure 5.3 displays the effect of optical thickness on the temperature. The temperature distribution for model A is always located under the gray solution. Conversely, the temperature for model B lies above the gray solution. As in the gray case, the temperature jump at the boundaries disappears with increasing optical depths.

5.1.2 Two Bands with Different Absorption Coefficients

In order to demonstrate the effects of a more general function of \( \kappa_v \), two simple models will be employed. These models are shown in Figure 5.4.

A two-level model for the absorption coefficient similar to model E was employed in the study of radiating shock layers \([53, 54]\) and also in stellar atmosphere problems \([49]\).

Note that models E and F reduce to models A and B respectively when \( \alpha = 0 \), while they both correspond to the gray model when \( \alpha = 1 \).

Solution Method. Unlike the previous case, the problem becomes nonlinear under the multi-level absorption coefficient model. Analytical solutions cannot be obtained. Equations 4.2 or 4.3 together with their respective boundary conditions constitute a nonlinear two-point boundary value problem with a nondifferential
Table 5.3 Percentage errors in the P-1 and P-3 approximation results for dimensionless temperature: $\tau_o = 1$, $\theta_2 = 0.5$ (radiative equilibrium)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\bar{v}_{\infty}$</th>
<th>$\theta_0$ (P-3)</th>
<th>$\theta_0$ (P-1)</th>
<th>$\theta(\tau_o)$ (P-3)</th>
<th>$\theta(\tau_o)$ (P-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>P-3</td>
<td>P-1</td>
<td>P-3</td>
<td>P-1</td>
</tr>
<tr>
<td>.1</td>
<td>0.75</td>
<td>2.50</td>
<td>1.08</td>
<td>3.54</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>0.65</td>
<td>2.19</td>
<td>1.11</td>
<td>3.64</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>0.58</td>
<td>1.95</td>
<td>1.11</td>
<td>3.62</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>0.48</td>
<td>1.61</td>
<td>1.08</td>
<td>3.49</td>
<td></td>
</tr>
<tr>
<td>Gray</td>
<td></td>
<td>0.39</td>
<td>1.35</td>
<td>1.05</td>
<td>3.37</td>
</tr>
<tr>
<td></td>
<td>2.</td>
<td>0.37</td>
<td>1.25</td>
<td>0.97</td>
<td>3.12</td>
</tr>
<tr>
<td>3.</td>
<td>0.33</td>
<td>1.12</td>
<td>0.88</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>0.25</td>
<td>0.86</td>
<td>0.69</td>
<td>2.20</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>0.15</td>
<td>0.51</td>
<td>0.42</td>
<td>1.34</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.3 Effect of optical thickness $\tau_0$ on the temperature distributions for models A and B: $\bar{v_{co}} = 3$, $\theta_z = 0.5$ (radiative equilibrium; P-1 approximation results)
Figure 5.4 Models E and F for the spectral absorption coefficient:  a) model E  b) model F
constraint given by equation 5.1.

The nonlinear problem was solved by the modified quasilinearization algorithm [55]. In this algorithm, the resulting linear two point boundary value problem is solved by the method of particular solutions [56]. Hamming's method was used to integrate the first order differential equation system. Details of the solution method are given in Appendix C. In determining the convergence, a performance index \( P \) was used. The algorithm was stopped when \( P \leq 10^{-16} \). Rapid convergence was achieved in all cases.

Radiative Equilibrium. In Tables 5.4 and 5.5, the P-1 and P-3 approximation results for the temperature and radiative heat flux are compared with the exact results [25] for various values of \( \alpha \). It is observed that both approximations are quite accurate for the whole range of \( \alpha \). Figure 5.5 shows the variation of the dimensionless heat flux with \( \alpha \). Maximum error in the P-1 approximation is about 3.35%. It is less than 1% for the P-3 approximation. The P-3 approximation matches the exact results within the accuracies associated with plotting Figure 5.5.

Uniform Heat Generation. When the medium generates heat at a constant rate, the local radiative flux is given by

\[
Q_R(\tau) = S(\tau-\tau_0) + C
\]  \hspace{1cm} (5.14)

where \( C \) is the integration constant.

Dimensionless radiative flux evaluated at the cold wall (i.e. \( Q_R(\tau_0) = C \)) are tabulated in Table 5.6 and plotted in Figure 5.6.
Table 5.4 Dimensionless temperatures and radiative flux $Q_R$: model E
\[ \vec{v}_{co} = 3, \tau_0 = 1, \theta_2 = 0.5 \text{ (radiative equilibrium)} \]

<table>
<thead>
<tr>
<th>( \tau / \tau_0 )</th>
<th>( a = 0.0 )</th>
<th>( a = 0.05 )</th>
<th>( a = 0.1 )</th>
<th>( a = 0.3 )</th>
<th>( a = 0.5 )</th>
<th>( a = 0.7 )</th>
<th>( a = 0.9 )</th>
<th>( a = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_x^+ )</td>
<td>0.9072</td>
<td>0.9086</td>
<td>0.9098</td>
<td>0.9164</td>
<td>0.9235</td>
<td>0.9353</td>
<td>0.9377</td>
<td></td>
</tr>
<tr>
<td>( P-3 )</td>
<td>0.9019</td>
<td>0.9039</td>
<td>0.9051</td>
<td>0.9110</td>
<td>0.9184</td>
<td>0.9313</td>
<td>0.9340</td>
<td></td>
</tr>
<tr>
<td>( P-1 )</td>
<td>0.8895</td>
<td>0.8930</td>
<td>0.8951</td>
<td>0.9018</td>
<td>0.9090</td>
<td>0.9222</td>
<td>0.9250</td>
<td></td>
</tr>
<tr>
<td>( E_x )</td>
<td>0.8600</td>
<td>0.8682</td>
<td>0.8734</td>
<td>0.8852</td>
<td>0.8922</td>
<td>0.9014</td>
<td>0.9031</td>
<td></td>
</tr>
<tr>
<td>( P-3 )</td>
<td>0.8609</td>
<td>0.8690</td>
<td>0.8738</td>
<td>0.8849</td>
<td>0.8921</td>
<td>0.9020</td>
<td>0.9036</td>
<td></td>
</tr>
<tr>
<td>( P-1 )</td>
<td>0.8540</td>
<td>0.8624</td>
<td>0.8677</td>
<td>0.8792</td>
<td>0.8863</td>
<td>0.8965</td>
<td>0.8985</td>
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</tr>
<tr>
<td>( E_x )</td>
<td>0.7977</td>
<td>0.8146</td>
<td>0.8247</td>
<td>0.8421</td>
<td>0.8484</td>
<td>0.8532</td>
<td>0.8537</td>
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</tr>
<tr>
<td>( P-3 )</td>
<td>0.7977</td>
<td>0.8146</td>
<td>0.8250</td>
<td>0.8428</td>
<td>0.8489</td>
<td>0.8533</td>
<td>0.8537</td>
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</tr>
<tr>
<td>( P-1 )</td>
<td>0.7977</td>
<td>0.8132</td>
<td>0.8232</td>
<td>0.8417</td>
<td>0.8483</td>
<td>0.8532</td>
<td>0.8537</td>
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</tr>
<tr>
<td>( E_x )</td>
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<td>0.7532</td>
<td>0.7674</td>
<td>0.7900</td>
<td>0.7954</td>
<td>0.7948</td>
<td>0.7939</td>
<td></td>
</tr>
<tr>
<td>( P-3 )</td>
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<td>0.7524</td>
<td>0.7674</td>
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</tr>
<tr>
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<td>0.7721</td>
<td>0.7971</td>
<td>0.8032</td>
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</tr>
<tr>
<td>( E_x )</td>
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<td>0.7408</td>
<td>0.7357</td>
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</tr>
<tr>
<td>( P-3 )</td>
<td>0.6757</td>
<td>0.7020</td>
<td>0.7190</td>
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<td>0.7511</td>
<td>0.7438</td>
<td>0.7411</td>
<td></td>
</tr>
<tr>
<td>( P-1 )</td>
<td>0.6925</td>
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<td>0.7324</td>
<td>0.7612</td>
<td>0.7670</td>
<td>0.7609</td>
<td>0.7581</td>
<td></td>
</tr>
<tr>
<td>( Q_R )</td>
<td>0.7860</td>
<td>0.7604</td>
<td>0.7376</td>
<td>0.6644</td>
<td>0.6101</td>
<td>0.5336</td>
<td>0.5188</td>
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<tr>
<td>( P-3 )</td>
<td>0.7868</td>
<td>0.7640</td>
<td>0.7426</td>
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<td>0.6141</td>
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<td>0.5210</td>
<td></td>
</tr>
<tr>
<td>( P-1 )</td>
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<td>0.7526</td>
<td>0.6848</td>
<td>0.6305</td>
<td>0.5512</td>
<td>0.5357</td>
<td></td>
</tr>
</tbody>
</table>

*Exact [25]*
Table 5.5 Dimensionless temperatures and radiative flux $Q_R$: model F

\[ \bar{\nu}_{co} = 3, \tau_0 = 1, \theta_2 = 0.5 \text{ (radiative equilibrium)} \]

<table>
<thead>
<tr>
<th>$\tau/\tau_0$</th>
<th>$\alpha=0.$</th>
<th>$\alpha=.1$</th>
<th>$\alpha=.4$</th>
<th>$\alpha=.5$</th>
<th>$\alpha=.6$</th>
<th>$\alpha=.8$</th>
<th>$\alpha=1.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex</td>
<td>0.9483</td>
<td>0.9426</td>
<td>0.9364</td>
<td>0.9359</td>
<td>0.9359</td>
<td>0.9365</td>
<td>0.9377</td>
</tr>
<tr>
<td>P-3</td>
<td>0.9452</td>
<td>0.9389</td>
<td>0.9319</td>
<td>0.9315</td>
<td>0.9316</td>
<td>0.9325</td>
<td>0.9340</td>
</tr>
<tr>
<td>P-1</td>
<td>0.9377</td>
<td>0.9315</td>
<td>0.9235</td>
<td>0.9228</td>
<td>0.9227</td>
<td>0.9234</td>
<td>0.9250</td>
</tr>
<tr>
<td>.2</td>
<td>Ex</td>
<td>0.9195</td>
<td>0.9125</td>
<td>0.9043</td>
<td>0.9034</td>
<td>0.9029</td>
<td>0.9027</td>
</tr>
<tr>
<td>P-3</td>
<td>0.9200</td>
<td>0.9129</td>
<td>0.9043</td>
<td>0.9033</td>
<td>0.9030</td>
<td>0.9031</td>
<td>0.9036</td>
</tr>
<tr>
<td>P-1</td>
<td>0.9156</td>
<td>0.9090</td>
<td>0.8996</td>
<td>0.8985</td>
<td>0.8979</td>
<td>0.8978</td>
<td>0.8985</td>
</tr>
<tr>
<td>.5</td>
<td>Ex</td>
<td>0.8781</td>
<td>0.8710</td>
<td>0.8604</td>
<td>0.8585</td>
<td>0.8571</td>
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</tr>
<tr>
<td>P-3</td>
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<td>0.8581</td>
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<td>0.8537</td>
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<td>0.8715</td>
<td>0.8606</td>
<td>0.8586</td>
<td>0.8570</td>
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<tr>
<td>.8</td>
<td>Ex</td>
<td>0.8272</td>
<td>0.8114</td>
<td>0.8019</td>
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<tr>
<td>P-3</td>
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<td>0.8207</td>
<td>0.8086</td>
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<td>0.8024</td>
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<tr>
<td>P-1</td>
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<td>0.8275</td>
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<td>0.8101</td>
<td>0.8050</td>
<td>0.8005</td>
</tr>
<tr>
<td>1.</td>
<td>Ex</td>
<td>0.7744</td>
<td>0.7712</td>
<td>0.7558</td>
<td>0.7554</td>
<td>0.7500</td>
<td>0.7414</td>
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<tr>
<td>P-3</td>
<td>0.7812</td>
<td>0.7779</td>
<td>0.7663</td>
<td>0.7621</td>
<td>0.7577</td>
<td>0.7492</td>
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</tr>
<tr>
<td>P-1</td>
<td>0.7962</td>
<td>0.7925</td>
<td>0.7815</td>
<td>0.7776</td>
<td>0.7737</td>
<td>0.7658</td>
<td>0.7581</td>
</tr>
<tr>
<td>$Q_R$</td>
<td>Ex</td>
<td>0.6703</td>
<td>0.6427</td>
<td>0.5846</td>
<td>0.5702</td>
<td>0.5576</td>
<td>0.5362</td>
</tr>
<tr>
<td>P-3</td>
<td>0.6717</td>
<td>0.6464</td>
<td>0.5882</td>
<td>0.5734</td>
<td>0.5604</td>
<td>0.5386</td>
<td>0.5210</td>
</tr>
<tr>
<td>P-1</td>
<td>0.6811</td>
<td>0.6585</td>
<td>0.6037</td>
<td>0.5891</td>
<td>0.5761</td>
<td>0.5538</td>
<td>0.5357</td>
</tr>
</tbody>
</table>

*Exact [25]*
Figure 5.5 Dimensionless radiative flux $Q_R$ vs. $\alpha$ for Models E and F: $\vec{V}_c = 3$, $\tau_0 = 1$, $\theta_2 = 0.5$ (radiative equilibrium)
and Figure 5.7 for various values of $S$, the dimensionless rate of heat generation.

The variations in the radiative flux for model E are shown in Figure 5.6. The curves for $S = 0$ represent the radiative equilibrium results, given in Figure 5.5. For $S \neq 0$, it is observed that the curves shift almost parallel to each other with increasing values of $S$. For the limiting values of $\alpha = 1$ (gray case) and $\alpha = 0$ (model A), the problem reduces to the linear case discussed in the previous section 5.1.1. In this case, the value of the dimensionless radiative flux is directly proportional to the rate of heat generation as given by equation 5.3c. For a gray medium with uniform heat generation, Ratzel [46] has shown that the P-1 and P-3 approximations have the same order of accuracy as in the case of radiative equilibrium. As discussed in section 5.1.1, this is also true for the case $\alpha = 0$ (model A). Considering the established accuracy of the approximations for the limiting cases, a) $\alpha = 0$ and $\alpha = 1$, $S \geq 0$, and b) $S = 0$, $0 \leq \alpha \leq 1$ and judging from the behavior of the curves in Figure 5.6, the P-1 and P-3 approximations can be expected to be accurate in the range $0 \leq \alpha \leq 0$ for $S > 0$. Similar trends in the results for model F presented in Figure 5.7 lead to the same conclusion.

Significant changes occur in the variation of temperature with $\alpha$ when there is uniform heat generation in the medium. Variations in the medium temperatures at the walls are given in Table 5.7 and Figure 5.8 for model E and in Table 5.8 and Figure 5.9 for model F.
Table 5.6 Dimensionless radiative flux $Q_R$ at $\tau = \tau_0$:
models E and F, $\overline{v}_{co} = 3, \tau_0 = 1, \theta_2 = 0.5$
(uniform heat generation)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$S=0.$</th>
<th>$S=0.1$</th>
<th>$S=1.$</th>
<th>$S=10.$</th>
</tr>
</thead>
<tbody>
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<td><strong>E</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.7868</td>
<td>0.8568</td>
<td>1.2868</td>
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</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.7921</td>
<td>0.8421</td>
<td>1.2921</td>
<td>5.7921</td>
</tr>
<tr>
<td></td>
<td>.1</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P-3</td>
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<td>1.2464</td>
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<tr>
<td></td>
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<tr>
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<td></td>
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<tr>
<td></td>
<td>P-3</td>
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<td>0.6025</td>
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<tr>
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<td>0.7311</td>
<td>1.1811</td>
<td>5.6811</td>
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Figure 5.6 Dimensionless radiative flux $Q_R$ at $\tau = \tau_0$ vs. $\alpha$ for model E: $\overline{\nu_c} = 3, \tau_0 = 1, \theta_2 = 0.5$ (uniform heat generation)
Figure 5.7 Dimensionless radiative flux at $\tau = \tau_0$ vs. $\alpha$ for model F: $\bar{V}_{co} = 3$, $\tau_0 = 1$, $\theta_2 = 0.5$ (uniform heat generation)
For model E, when \( S = 0 \), \( \theta(0) \) decreases almost linearly with \( \alpha \) from the gray case (\( \alpha = 1 \)) to the result for model A (\( \alpha = 0 \)). On the other hand, \( \theta(\tau_0) \) has a value greater than the gray case for \( \alpha > 0.26 \) [25]. When \( S \neq 0 \), \( \theta(0) \) shows a rapid increase at small values of \( \alpha \) with increasing values of \( S \). Observe that the curve for \( S = 0.1 \) shows a minimum with respect to \( \alpha \) at about \( \alpha = 0.2 \) which disappears for \( S \geq 1.0 \). The maximum in \( \theta(\tau_0) \) with respect to \( \alpha \) is also displaced. Both \( \theta(0) \) and \( \theta(\tau_0) \) become bounded from below by the gray case with increasing heat generation in the medium. The variations in the temperatures for model F show similar behavior, although they are not as strong as those for model E.

From Figures 5.8 and 5.9 we observe that the temperatures for model A (\( \alpha = 0 \) in model E) surpass those for model B (\( \alpha = 0 \) in model F) with increasing rate of heat generation, while they are both higher than those for the gray case. The reason for this is that in a medium with a single band, only a certain spectral interval (rather than the whole spectrum in a gray medium) has to account for the heat generation in the medium. This results in higher temperatures in a nongray medium with significant heat generation.

5.1.3 Effect of Temperature Dependent Absorption Coefficient

In reality, nongray radiative transfer problems in high temperature systems are complicated by the fact that the radiation properties also vary with the temperature. Realistic solutions are very difficult to obtain in this case as the problem becomes highly
Table 5.7 Dimensionless temperatures $\theta(0)$ and $\theta(\tau)$: model E, $\bar{V}_{co}=3$, $\tau_0=1$, $\theta_2=0.5$ (uniform heat generation; P-3 approximation results)

<table>
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<tr>
<th>$\alpha$</th>
<th>S=0.</th>
<th>S=.1</th>
<th>S=1.</th>
<th>S=10.</th>
</tr>
</thead>
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<td>0.9628</td>
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<td>$\theta(1)$</td>
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<tr>
<td>0.05</td>
<td>$\theta(0)$</td>
<td>0.9039</td>
<td>0.9523</td>
<td>1.2761</td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$</td>
<td>0.7020</td>
<td>0.7682</td>
<td>1.1576</td>
</tr>
<tr>
<td>0.1</td>
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<td>0.9460</td>
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<td>$\theta(1)$</td>
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<td>0.7765</td>
<td>1.1101</td>
</tr>
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<td>0.2</td>
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<td>0.9340</td>
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</tr>
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<td>0.7855</td>
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<tr>
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<td>0.9384</td>
<td>1.1238</td>
</tr>
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<td>$\theta(1)$</td>
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<td>0.7885</td>
<td>1.0339</td>
</tr>
<tr>
<td>0.5</td>
<td>$\theta(0)$</td>
<td>0.9183</td>
<td>0.9403</td>
<td>1.0929</td>
</tr>
<tr>
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<td>$\theta(1)$</td>
<td>0.7511</td>
<td>0.7869</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.8</td>
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<td>0.9285</td>
<td>0.9460</td>
<td>1.0727</td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$</td>
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<td>0.7782</td>
<td>0.9692</td>
</tr>
<tr>
<td>1.0</td>
<td>$\theta(0)$</td>
<td>0.9340</td>
<td>0.9498</td>
<td>1.0657</td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$</td>
<td>0.7411</td>
<td>0.7716</td>
<td>0.9546</td>
</tr>
</tbody>
</table>
Table 5.8 Dimensionless temperatures $\theta(0)$ and $\theta(\tau_0)$: model F, $\overline{V}_{co} = 3$, $\tau_0 = 1$, $\theta_2 = 0.5$ (uniform heat generation; P-3 approximation results)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$S=0.1$</th>
<th>$S=0.1$</th>
<th>$S=1.0$</th>
<th>$S=10.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$\theta(0)$ 0.9452 0.9655</td>
<td>1.1016</td>
<td>1.6153</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$ 0.7812 0.8216</td>
<td>1.0257</td>
<td>1.5963</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>$\theta(0)$ 0.9389 0.9582</td>
<td>1.0911</td>
<td>1.6051</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$ 0.7779 0.8153</td>
<td>1.0135</td>
<td>1.5857</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>$\theta(0)$ 0.9316 0.9488</td>
<td>1.0723</td>
<td>1.5824</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$ 0.7621 0.7945</td>
<td>0.9815</td>
<td>1.5586</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>$\theta(0)$ 0.9325 0.9488</td>
<td>1.0675</td>
<td>1.5730</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$ 0.7492 0.7803</td>
<td>0.9644</td>
<td>1.5452</td>
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</tr>
<tr>
<td>1.0</td>
<td>$\theta(0)$ 0.9340 0.9498</td>
<td>1.0657</td>
<td>1.5682</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta(1)$ 0.7411 0.7716</td>
<td>0.9546</td>
<td>1.5375</td>
<td></td>
</tr>
</tbody>
</table>
Figure S.8 Dimensionless temperatures $\theta(0)$ and $\theta(\tau_0)$ vs. $\alpha$ for model E: $\bar{\nabla}_{co} = 3$, $\tau_0 = 1$, $\theta_2 = 0.5$ (uniform heat generation; P-3 approximation results)
Figure 5.9 Dimensionless temperatures \( \theta(0) \) and \( \theta(\tau_2) \) vs. \( \alpha \) for model F: \( \bar{\omega}_{co} = 3, \tau_2 = 1, \theta_2 = 0.5 \) (uniform heat generation; P-3 approximation results)
nonlinear. Most studies on the subject have assumed isothermal behavior of the medium properties. The presence of large temperature differences restricts the applicability of these analyses.

In this section the effects of the temperature dependence of the radiation properties will be discussed. In particular, carbon monoxide gas will be taken to serve as an illustrative example. Only the fundamental band of carbon monoxide will be considered as the overtone bands may be neglected [16].

Band intensity of the CO fundamental band, defined as the area under the absorption coefficient curve, i.e.

\[ \sigma(T) = \int \frac{\langle \kappa \rangle}{P} d\omega \tag{5.15} \]

is evaluated as [8]

\[ \sigma(T) = 237(300/T) \text{ cm}^{-2}\text{atm}^{-1}. \tag{5.16} \]

The 'Box model' for the absorption coefficient, introduced by Penner [8] to calculate gas emissivities, is equivalent to the rectangular model in that it assumes the absorption coefficient within an absorption band is constant over an effective bandwidth \( \Delta\omega_e \).

From Penner [8], the effective bandwidth for the CO fundamental band may be expressed by

\[ \Delta\omega_e = 214(T/300)^{1/2} \text{ cm}^{-1} \text{ (centered at } \omega_c = 2143 \text{ cm}^{-1}) \tag{5.17} \]

Combining equations 5.15-5.17 one obtains

\[ \langle \kappa/P \rangle = \frac{237}{214} \left(\frac{300}{T}\right)^{1/2} \text{ cm}^{-1}\text{atm}^{-1}. \tag{5.18} \]
In the following, the effects of temperature variations in $\kappa$ and $\Delta \omega_e$ are investigated for the two subproblems discussed in section 5.1. Only the $P-1$ approximation is employed to demonstrate the qualitative effects.

**Subproblem I.** When $\kappa$ and $\Delta \omega_e$ are constant and evaluated at the average temperature $T_{av}$

$$T_{av} = (T_1 + T_2)/2 \quad (5.19)$$

the $P-1$ approximation yields

$$\phi_b(\tau) = 1 - (3\tau + 2)/(3\tau_0 + 4) \quad (5.20)$$

where $\tau = \kappa(T_{av}) \cdot z$.

Note that $\phi_b(\tau) = 0.5$ for all $\tau_0$. As noted before, this is the exact result [24]. Therefore, given $T_1$ and $T_2$, the centerline temperature is constant regardless of the optical thickness. This fact is depicted by the straight lines in Figure 5.10. Temperature variations in $\kappa$ and $\Delta \omega_e$ of the form given by equations 5.17 and 5.18 result in an increase in the temperature level. This effect is more pronounced at large optical thicknesses. The temperature tends to a constant value in the optically thin limit. The effect on the temperature profile is shown in Figure 5.11. Temperatures near the hot wall are observed to increase. The most significant changes occur near the cold wall. Because of the particular variation of $\kappa$, the "colder" gas absorbs more energy than in the constant $\kappa$ case. This yields higher temperature jumps at the cold wall.
Figure 5.10 Effect of temperature dependent absorption coefficient on the centerline temperature: CO fundamental band, $\omega_c = 2143$ cm$^{-1}$ (radiative equilibrium)
Figure 5.11 Effect of temperature dependent absorption coefficient on the temperature distribution: CO fundamental band, $\omega_c = 2143 \text{ cm}^{-1}$, $\tau_0 = 2$ (radiative equilibrium)
Subproblem II. For constant $\kappa$ and $\Delta \omega_e$, the P-1 approximation yields

$$
\psi_b(\tau) = \frac{1}{4} + \frac{\tau_0}{4} + \frac{3}{8} \tau_0^2 \left[ \frac{(\tau/\tau_0) - (\tau/\tau_0)^2}{(\tau/\tau_0) - (\tau/\tau_0)^2} \right].
$$

(5.21)

From Cess et al. [16], the exact solution is

$$
\psi_b(\tau) = \frac{1}{3} + \frac{\tau_0}{4} + \frac{3}{8} \tau_0^2 \left[ \frac{(\tau/\tau_0) - (\tau/\tau_0)^2}{(\tau/\tau_0) - (\tau/\tau_0)^2} \right].
$$

(5.22)

In the exact solution, $\psi_b$ is defined as

$$
\psi_b(\tau) = \frac{[e_{b0}(T, \omega_c) - e_{b0}(T_{1, \omega_c})] \cdot \Delta \omega_e}{s/\kappa(T_1)}
$$

(5.23)

i.e. the narrow band approximation has been employed. The two solutions are identical except for a constant. In fact, the P-1 approximation yields the correct optically thin limit, i.e. $\psi_b(\tau) = \frac{\kappa}{4}$ for $\tau_0 \rightarrow 0$. The discrepancy in the exact solution arises from the use of the exponential kernel approximation in its derivation. Both solutions indicate that the emissive power is proportional to the second power of the optical thickness at large path lengths. This leads to very high temperatures with increasing heat generation (Figure 5.12).

It is observed that the temperature dependent $\kappa$ and $\Delta \omega_e$ increase the temperature for small optical thicknesses. However, there is a sharp decrease in the temperature level at large path lengths. These effects are better understood by studying the variations in Figure 5.13. It is seen that the temperature profiles are flattened in the variable $\kappa$ and $\Delta \omega_e$ case. As discussed for subproblem I, this is a direct consequence of the particular variation of $\kappa$. 
Figure 5.12 Effect of temperature dependent absorption coefficient on the centerline temperature: CO fundamental band, $\omega_c = 2143$ cm$^{-1}$, $T_1 = T_2 = 1000$ K (uniform heat generation)
Figure 5.13 Effect of temperature dependent absorption coefficient on the temperature distribution: CO fundamental band, $\omega_c = 2145 \text{ cm}^{-1}$, $T_1 = T_2 = 1000 \text{ K}$, $\tau_0 = 2$ (uniform heat generation)
5.2 Conducting and Radiating Medium

In the presence of conduction, the dimensionless form of the conservation equation is given as

\[ 4N D^2 \theta - DQ_R + S = 0 \]  \hspace{1cm} (5.24)

subject to the boundary conditions

\[ \theta(0) = 1, \hspace{0.5cm} \theta(R_0) = \theta_2. \]  \hspace{1cm} (5.25)

The conduction-radiation parameter \( N \)

\[ N = k\kappa/4\sigma T_1^3 \]  \hspace{1cm} (5.26)

signifies the relative importance of conduction compared to radiation. Obviously \( N = 0 \) represents a purely radiating medium.

Equation 5.24, coupled with the moment equations can be solved to obtain the temperature distribution. Once the solution is found, the dimensionless total heat flux defined by

\[ Q_T = (q_C + q_R)/\sigma T_1^4 = -4N D\theta + Q_R \]  \hspace{1cm} (5.28)

can be evaluated. Note that, from equation 3.1, the total heat flux is constant when there is no heat generation in the medium.

Solution Method. The nonlinear two point boundary value problem consisting of the conservation and moment equations was solved by the modified quasilinearization algorithm of Appendix C.

The solution is more straightforward in this case because the energy equation governing the temperature in the medium is now a second order differential equation that can be cast into a first order system. That is, the temperature is now a state variable.
rather than being a control as in the purely radiating case. The convergence criterion used was $P < 10^{-16}$ as before.

With this method, solutions could not be obtained for small values of $N$ ($N < 0.1$ for P-1 and $N < 1$ for P-3) at large optical depths. In this case, the energy equation was retained as a second order equation. The problem was attacked by transforming the moment equations into a second order system in terms of $i_{0j}$ for the P-1 approximation and $i_{0j}$ and $i_{2j}$ for the P-3 approximation. The resulting system of second order ordinary differential equations was then solved by the collocation method. Cubic Hermite polynomials were employed as the interpolating functions. The modified quasi-linearization algorithm as applied to nonlinear system of equations [56] was used for the nonlinear iteration (see Appendix D).

Results. In order to keep the number of independent parameters to a minimum, the dimensionless cold wall temperature was set equal to 0.5 and the heat generation is neglected. Table 5.9 presents the total heat flux for models A and B. In Table 5.10 the effect of the cutoff frequency on the total heat flux is shown. Both approximations agree well with the exact results [27]. Comparative results for the dimensionless medium temperature are given in Table 5.11. For relatively high values of $N$ ($N \geq 0.1$) the agreement is excellent. Maximum errors in the temperature profiles correspond to those of a purely radiating medium, i.e. $N \to 0$.

P-1 approximation results for the total heat flux are presented in Figures 5.14 and 5.15 where the more general models E and F were
Table 5.9 Dimensionless total heat flux $Q_T$ in a conducting and radiating medium: models A and B, $\overline{V}_c = 3$, $\theta_2 = 0.5$

(a) Model A

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\tau_a = 0.1$</th>
<th>$\tau_a = 1.$</th>
<th>$\tau_a = 10.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20.9171 2.0872 0.8401</td>
<td>20.9203 2.8100 0.8401</td>
<td>20.9237 2.8234 0.8407</td>
</tr>
<tr>
<td>.1</td>
<td>2.9171 1.0061 0.6596</td>
<td>2.9203 1.0087 0.6596</td>
<td>2.9237 1.0215 0.6602</td>
</tr>
<tr>
<td>.01</td>
<td>1.1171 0.8200 0.6407</td>
<td>1.1202 0.8217 0.6407</td>
<td>1.1236 0.8315 0.6411</td>
</tr>
</tbody>
</table>

(b) Model B

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\tau_a = 0.1$</th>
<th>$\tau_a = 1.$</th>
<th>$\tau_a = 10.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20.9003 2.7029 0.6125</td>
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<td>20.9117 2.7299 0.6135</td>
</tr>
<tr>
<td>.1</td>
<td>2.9003 0.9041 0.4324</td>
<td>2.9057 0.9085 0.4324</td>
<td>2.9117 0.9305 0.4333</td>
</tr>
<tr>
<td>.01</td>
<td>1.1005 0.7131 0.4126</td>
<td>1.1059 0.7161 0.4127</td>
<td>1.1119 0.7321 0.4134</td>
</tr>
</tbody>
</table>

$^+$Exact [26]
Table 5.10 Effect of cut-off frequency on dimensionless total heat flux $Q_T$ in a conducting and radiating medium: models A and B, $\theta_2 = 0.5$

<table>
<thead>
<tr>
<th>Model</th>
<th>$\tilde{u}_\infty$</th>
<th>$\zeta=1.0$</th>
<th>$\zeta=10.0$</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td>$N=0.005$</td>
<td>$N=0.05$</td>
</tr>
<tr>
<td>Ex$^+$</td>
<td>0.890</td>
<td>0.981</td>
<td>0.822</td>
</tr>
<tr>
<td>2.</td>
<td>P-3</td>
<td>0.890</td>
<td>0.983</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.894</td>
<td>0.988</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Ex</td>
<td>0.805</td>
<td>0.905</td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.809</td>
<td>0.907</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.817</td>
<td>0.920</td>
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<tr>
<td>5.</td>
<td>Ex</td>
<td>0.656</td>
<td>0.763</td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.659</td>
<td>0.768</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.675</td>
<td>0.791</td>
</tr>
<tr>
<td>Gray</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Ex</td>
<td>0.552</td>
<td>0.666</td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.554</td>
<td>0.670</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.575</td>
<td>0.700</td>
</tr>
<tr>
<td>2.</td>
<td>Ex</td>
<td>0.614</td>
<td>0.727</td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.617</td>
<td>0.729</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.635</td>
<td>0.755</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Ex</td>
<td>0.696</td>
<td>0.803</td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.701</td>
<td>0.807</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.715</td>
<td>0.828</td>
</tr>
<tr>
<td>5.</td>
<td>Ex</td>
<td>0.849</td>
<td>0.942</td>
</tr>
<tr>
<td></td>
<td>P-3</td>
<td>0.850</td>
<td>0.942</td>
</tr>
<tr>
<td></td>
<td>P-1</td>
<td>0.857</td>
<td>0.952</td>
</tr>
</tbody>
</table>

$^+$Exact [27]
Table 5.11 Dimensionless temperatures in a conducting and radiating medium: models A and B, $\overline{\nu_{co}} = 3$, $\tau_0 = 1$, $\theta_2 = 0.5$

<table>
<thead>
<tr>
<th>$\tau/\tau_0$</th>
<th>Model A</th>
<th>Model B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N=1.$</td>
<td>$N=.1$</td>
</tr>
<tr>
<td>Ex$^+$</td>
<td>0.9497</td>
<td>--</td>
</tr>
<tr>
<td>P-3</td>
<td>0.9496</td>
<td>0.9463</td>
</tr>
<tr>
<td>P-1</td>
<td>0.9495</td>
<td>0.9451</td>
</tr>
<tr>
<td>Ex</td>
<td>0.8999</td>
<td>--</td>
</tr>
<tr>
<td>P-3</td>
<td>0.8998</td>
<td>0.8976</td>
</tr>
<tr>
<td>P-1</td>
<td>0.8997</td>
<td>0.8963</td>
</tr>
<tr>
<td>Ex</td>
<td>0.8009</td>
<td>--</td>
</tr>
<tr>
<td>P-3</td>
<td>0.8010</td>
<td>0.8071</td>
</tr>
<tr>
<td>P-1</td>
<td>0.8010</td>
<td>0.8072</td>
</tr>
<tr>
<td>Ex</td>
<td>0.7019</td>
<td>--</td>
</tr>
<tr>
<td>P-3</td>
<td>0.7020</td>
<td>0.7164</td>
</tr>
<tr>
<td>P-1</td>
<td>0.7023</td>
<td>0.7186</td>
</tr>
<tr>
<td>Ex</td>
<td>0.6018</td>
<td>--</td>
</tr>
<tr>
<td>P-3</td>
<td>0.6019</td>
<td>0.6168</td>
</tr>
<tr>
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<td>0.6023</td>
<td>0.6196</td>
</tr>
<tr>
<td>Ex</td>
<td>0.5512</td>
<td>--</td>
</tr>
<tr>
<td>P-3</td>
<td>0.5512</td>
<td>0.5611</td>
</tr>
<tr>
<td>P-1</td>
<td>0.5515</td>
<td>0.5630</td>
</tr>
</tbody>
</table>

$^+$ Exact [52]
Figure 5.14 Dimensionless total heat flux vs. conduction-radiation parameter N: model $E, \overline{v}_c = 3, 0.5$ (0.1 approximation results)
employed respectively. The solution for a given value of $\alpha$ lies between the two limiting solutions $\alpha = 1$ (gray) and $\alpha = 0$ (models A and B). Note that, for fixed $\tau_0$ and $N$, the total heat flux is larger for model A than for model B. Clearly, the influence of nongrayness becomes more pronounced at large optical depths.
CHAPTER VI

SUMMARY AND CONCLUSIONS

Radiative energy transfer and its interaction with conduction was considered in this study for a nongray absorbing and emitting medium. The rectangular model was utilized to represent the spectral dependence of the absorption coefficient. The P-1 and P-3 spherical harmonics approximations to the radiation intensity were used for energy transfer in one-dimensional planar geometry. In a purely radiating medium with a single valued absorption coefficient, the problem was reduced to the gray problem. A two-level absorption coefficient model was also employed. Results displayed the strong influence of the absorption bands and the limitations of the gray assumption. The temperature distribution and radiative flux were also strongly affected by the temperature dependence of the radiative properties. The same absorption coefficient models were applied to the combined radiation and conduction problem.

Contrary to the exact formulations, the differential character of the approximations enabled the use of conventional methods and fully developed algorithms in obtaining numerical solutions. Results were compared to the available exact solutions. Deviations from the exact solutions were observed in situations in which the medium is
optically thin or radiation is the dominant transfer mechanism. Although the P-3 approximation yielded more accurate results, the overall performance of the P-1 approximation is commendable. While higher order approximations offer improved accuracy, the increased number of terms demand more computer time and storage. Thus a trade-off with respect to accuracy and computational requirements is in order. In view of the number of independent parameters involved in the problems, the P-1 approximation may be the practical choice in making rapid qualitative and quantitative assessments. It may also be used to obtain initial estimates of the temperature for exact numerical solutions.

A possible continuation of this study would be the extension to two-dimensional enclosures or to nonplanar media. Gray results [42, 43] indicate that the P-1 approximation is not accurate enough in cylindrical and spherical geometries. Therefore, higher order approximations should be considered in these problems. Alternately, an extension can be made to include interactions of radiative transfer with convection, important in the study of boundary layers and duct flows. A more general case may involve combustion effects. For the sake of simplifying the analysis, the present study was limited to gray boundaries. The spectral variations in the surface properties can also be accommodated through the rectangular model. The present solution methods can handle heat transfer boundary conditions without any complications.
REFERENCES


52. Crosbie, A. L., Personal communication of results not given in references [24] and [26].


APPENDIX A

COEFFICIENTS $D_i$ IN THE P-3 APPROXIMATION

FORMULATION FOR A PURELY RADIATING MEDIUM

The general solution to the fourth order equation

$$D^4I_0 - 35D^2I_0/9 = 35F/3$$  \hspace{1cm} (A.1)

is

$$I_0 = -3Ft^2/2 + D_1 + D_2t + D_3 \cosh \left(\sqrt{35}t/3\right)$$
$$+ D_4 \sinh \left(\sqrt{35}t/3\right).$$  \hspace{1cm} (A.2)

$I_1$, $I_2$, and $I_3$ can be evaluated through the moment equations 5.10

$$I_1 = 3D^3I_0/35 - DI_0/3 = Ft - D_2/3$$  \hspace{1cm} (A.3)
$$I_2 = -11F/21 + I_0/3 - 3D^2I_0/35$$
$$= -(4/15 + t^2/2)F + (D_1+D_2t)/3$$  \hspace{1cm} (A.4a)
$$I_3 = 3DI_0/35 + 6I_1/7$$
$$= 3Ft/5 - D_2/5 + [D_3 \sinh(\sqrt{35}t/3)]/\sqrt{35}$$
$$+ D_4 \cosh(\sqrt{35}t/3)/\sqrt{35}$$  \hspace{1cm} (A.5a)

Substituting A.2-A.5 in the boundary conditions 5.11, one obtains the following system of equations of the form

$$\sum_{j=1}^{4} A_{ij}D_j = E_i, \quad i = 1,2,3,4.$$  \hspace{1cm} (A.6)

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Coefficients $A_{ij}$ and $E_j$ are given below.

$A_{11} = A_{21} = 8\varepsilon_1$, $A_{31} = A_{41} = 8\varepsilon_2$

$A_{12} = -16(2-\varepsilon_1)/3$

$A_{22} = -16(1-\varepsilon_1)/3 - 32/5$

$A_{32} = 16(2-\varepsilon_2)/3 + 8\varepsilon_2\tau_0$

$A_{42} = 16(1-\varepsilon_2)/3 + 8\varepsilon_2\tau_0 + 32/5$

$A_{13} = 5\varepsilon_1$, $A_{23} = 5\varepsilon_2 - 5$, $A_{33} = 3\varepsilon_2 C$

$A_{43} = (3\varepsilon_2 - 5)C - 32S/\sqrt{35}$

$A_{14} = 0$, $A_{24} = 32/\sqrt{35}$, $A_{34} = 3\varepsilon_2 S$

$A_{44} = (3\varepsilon_2 - 5)S - 32C/\sqrt{35}$

$E_1 = 32\varepsilon_1 B + 4\varepsilon_1 F$

$E_2 = 32\varepsilon_1 B + 4(1+\varepsilon_1) F$

$E_3 = [4\varepsilon_2 + 16(2-\varepsilon_2)\tau_0 + 12\varepsilon_2\tau_0^2] F$

$E_4 = [4(1+\varepsilon_2) + [16(1-\varepsilon_2) + 96/5]\tau_0 + 12\varepsilon_2\tau_0^2] F$

and \( C = \cosh (\sqrt{35}\tau_0/3) \), \( S = \sinh (\sqrt{35}\tau_0/3) \)

Given $\varepsilon_1$, $\varepsilon_2$ and $\tau_0$, equations A.6 can be solved for $D_1$, $D_2$, $D_3$ and $D_4$. 
APPENDIX B

BLACKBODY FUNCTION

The function $e_{bj}$ has been defined as

$$e_{bj}(T) = \int_{\Delta v_j} e_{bv}(T) dv$$

where

$$e_{bv}(T) = \frac{2\pi h}{c^2} \frac{v^3}{\exp(hv/kt) - 1}$$

is the Planck function. Equation B.1 can be written in the form

$$e_{bj} = \int_0^{v_{j2}} e_{vj} dv - \int_0^{v_{j1}} e_{bv} dv$$

where $v_{j1}$ and $v_{j2}$ are the lower and upper limits of the jth absorption band.

Defining $u = hv/kt$, one obtains after some algebra

$$e_{bj}(T) = \sigma T^4 [F(u_{j2}) - F(u_{j1})].$$

The function $F(u)$, defined as

$$F(u) = \frac{15}{\pi^4} \int_0^u \frac{x^3 dx}{e^x - 1}$$

can be approximated as follows [58]
\[ F(u) = \frac{15}{\pi^4} \sum_{n=1,2,\ldots} \frac{e^{-nu}}{n^4} \left\{ (nu+3)nu+6]nu+6 \right\}, \quad u \geq 2 \]  

(B.6)

\[ F(u) = 1 - \frac{15}{\pi^4} u^3 \left\{ \frac{1}{3} - \frac{u}{8} + \frac{u^2}{60} - \frac{u^4}{5040} + \frac{u^6}{272160} \right\}, \quad u < 2. \]  

(B.7)
APPENDIX C

MODIFIED QUASILINEARIZATION METHOD FOR

SOLVING NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS [55]

As an illustrative example, consider the P-1 approximation formulation for a purely radiating medium with two absorption bands.

The moment equations in dimensionless form are

\[ I'_{0j} = -3\alpha_j \tau_0 I_{1j} \]  \hspace{1cm} (C.1)
\[ I'_{1j} = \alpha_j \tau_0 (B_j - I_{0j}) \]  \hspace{1cm} (C.2)

where \[ I_{kj} = i_{kj}/4\sigma T_1^4, \quad B_j = e_{bj}/\sigma T_1^4 \]
\[ \alpha_j = \kappa_j/\kappa_m, \quad \kappa_m = \text{Max}(\kappa_j), \quad j = 1,2 \]
\[ \tau_0 = \kappa_m L. \]

The prime denotes differentiation with respect to \( t \equiv z/L \).

The boundary conditions are

\[ I_{0j}(0) + 2I_{1j}(0) - B_j(1) = 0 \]  \hspace{1cm} (C.3)
\[ I_{0j}(1) - 2I_{1j}(1) - B_j(1) \theta_2 = 0. \]  \hspace{1cm} (C.4)

The moment equations are coupled with the scalar conservation of energy equation

\[ \sum_{j=1}^{2} \alpha_j (B_j - I_{0j}) = 0 \]  \hspace{1cm} (C.5)

where internal heat generation is omitted.

The moment equations can be written in simplified form using
the vector notation:

\[ x' = \phi(x, \theta, t), \quad 0 \leq t \leq 1 \]  \hspace{1cm} (C.6)

subject to the boundary conditions

\[ g[x(0)] = 0, \quad h[x(1)] = 0 \]  \hspace{1cm} (C.7)

and to the constraining energy equation

\[ f(x, \theta, t) = \sum_{j=1}^{2} \alpha_j (B_j - I_{0j}) = 0. \]  \hspace{1cm} (C.8)

Here \( x \equiv (I_{01} I_{02} I_{11} I_{12})^T \), \( \phi \), \( g \), \( h \) are vectors of dimension 4, 4, 2 and 2 respectively.

A performance index \( P \) is defined as

\[ P = \int_{0}^{1} \left[ ((x'-\phi)^T (x'-\phi)) + f^2 \right] dt + g^T g + h^T h \]  \hspace{1cm} (C.9)

The scalar \( P \) is a measure of the cumulative error in the equations and the boundary conditions. \( P = 0 \) for any \( x(t), \theta(t) \) satisfying equations C.6-C.8 and \( P > 0 \) otherwise.

Consider the nominal functions \( x(t), \theta(t) \) and the varied functions \( \bar{x}(t) \) and \( \bar{\theta}(t) \) such that

\[ \bar{x}(t) = x(t) + \Delta x(t), \quad \bar{\theta}(t) = \theta(t) + \Delta \theta(t) \]  \hspace{1cm} (C.10)

where \( \Delta x(t), \Delta \theta(t) \) denote the perturbations in \( x \) and \( \theta \) at \( t \). The variation in the performance index to the first order is
\[ \delta P = 2 \int_0^1 [(x' - \phi)^T \delta (x' - \phi) + f \delta f] dt + g^T \delta g + h^T \delta h. \] (C.11)

Considering the system of variations defined by

\[ \delta (x' - \phi) = -k (x' - \phi), \quad \delta f = -kf \] (C.12)
\[ \delta g = -kg, \quad \delta h = -kh, \quad 0 < k \leq 1 \] (C.13)

the first variation of the performance index becomes

\[ \delta P = -2kP. \] (C.14)

For any \( x(t), \ \delta(t) \) not satisfying equations C.6-C.8, \( \delta P < 0 \), which is the basic descent property of the algorithm: it guarantees that

\[ \tilde{P} < P \] (C.15)

if \( k \) is sufficiently small.

Introducing the auxiliary variables \( A(t) = \Delta x(t)/k \) and \( C(t) = \Delta \theta(t)/k \), one obtains from equations C.12 and C.13

\[ A' = \phi^T_x A + \phi^T_g C - (x' - \phi) \] (C.16)
\[ C = -(f^T_x A + f)/f_g \] (C.17)
\[ g_{x(0)}^T A(0) + g = 0, \quad h_{x(1)}^T A(1) + h = 0. \] (C.18)

This linear first-order ordinary differential equation system, which is independent of \( k \), can be integrated by various superposition methods. Method of particular solutions [55] was employed in
this study. Once $A(t)$ and $C(t)$ are known, one obtains the family of solutions in terms of a single parameter $k$, the stepsize,

$$\tilde{x}(t) = \tilde{x}(t) + kA(t), \quad \tilde{\theta}(t) = \theta(t) + kC(t). \quad (C.19)$$

For this single parameter family, the performance index $P$ becomes a function of $k$, i.e. $\tilde{P} = \tilde{P}(k)$. In order to find an acceptable value of $k$, one first assigns the value $k = 1$ to the stepsize. The acceptable value of stepsize is the one which satisfies

$$\tilde{P}(k) < \tilde{P}(0) = P. \quad (C.20)$$

Otherwise, the previous value of $k$ is replaced by some smaller value (e.g. using a bisection process) until the above inequality is met. This is guaranteed by the descent property. The algorithm is terminated when the stopping condition

$$P \leq \varepsilon \quad (C.21)$$

is satisfied. Here $\varepsilon$ is a small preselected number.
APPENDIX D

SOLUTION METHODS USED IN COMBINED RADIATION-CONDUCTION PROBLEMS

As an illustrative example consider the P-1 approximation formulation for a conducting and radiating medium with one absorption band. The moment equations and the boundary conditions in dimensionless form are

\[ I_0' = -5 \tau_0 I_1 \]
\[ I_1' = \tau_0 (B-I_0) \]
\[ I_0(0)+2I_1(0)-B(1) = 0, \quad I_0(1)-2I_1(1)-B(\theta_2) = 0 \]

The conservation of energy equation in dimensionless form is

\[ N\theta'' = \tau_0^2 (B-I_0) \]

which can be rewritten as a first-order system

\[ \theta' = w \]
\[ w' = \tau_0^2 (B-I_0)/N \]

subject to the boundary conditions

\[ \theta(0) = 1, \quad \theta(1) = \theta_2. \]
**Modified Quasilinearization Algorithm**

Defining \( x = (\theta I_0 \omega I_1)^T \), equations D.1, D.2, D.5, D.6 can be written in vector notation as

\[
x' = \phi(x,t).
\]  

(D.8)

Similarly the boundary conditions are

\[
g[x(0)] = 0, \ h[x(1)] = 0.
\]  

(D.9)

This nonlinear two-point boundary value problem can be solved by the modified quasilinearization algorithm explained in Appendix C. Note that this system is not subjected to a constraining energy equation as in the purely radiating case: the energy equation itself is a differential equation. Therefore, \( \theta \) (and its derivative \( \omega \)) becomes a state variable. The solution is more straightforward. In this case, the performance index is defined as

\[
P = \int_0^1 (x'-\phi)^T(x'-\phi)dt + g^Tg + h^Th.
\]  

(D.10)

**Collocation Method**

For small values of \( N \), the first order system becomes "stiff" because of the large eigenvalues resulting from equation D.6. Solutions could not be obtained with the above method. In this case, the energy equation was retained in the form given by equation D.4. The moment equations were combined to yield
\[ I_0'' + 3\tau_0^2 (B - I_0) = 0 \quad (D.11) \]
\[ I_0(0) - 2I_0'(0)/3\tau_0 = B(1), \quad I_0(1) + 2I_0'(1)/3\tau_0 = B(\theta_2). \quad (D.12) \]

Solutions to equations D.4 and D.11 were obtained by the collocation method following the procedure of [57]. Employing the cubic Hermite basis, we assume the following approximate solutions in the range \( t_i \leq t \leq t_{i+1}, \ i = 0, 1, \ldots, N-1. \)

\[ \theta(t) = \theta(t_i)H_{i0}^0(t) + \theta'(t_i)H_{i0}^1(t) + \theta(t_{i+1})H_{i+1,1}^0(t) + \theta'(t_{i+1})H_{i+1,1}^1(t) \quad (D.13) \]
\[ I_0(t) = I_0(t_i)H_{i0}^0(t) + I_0'(t_i)H_{i0}^1(t) + I_0(t_{i+1})H_{i+1,1}^0(t) + I_0'(t_{i+1})H_{i+1,1}^1(t) \quad (D.14) \]

where the cubic Hermite polynomials are given as

\[ H_{i0}^0 = 1 - 3s^2 + 2s^3 \quad (D.15a) \]
\[ H_{i0}^1 = h_i(s - 2s^2 + s^3) \quad (D.15b) \]
\[ H_{i+1,1}^0 = 3s^2 - 2s^3 \quad (D.15c) \]
\[ H_{i+1,1}^1 = h_i(s^3 - s^2) \quad (D.15d) \]

and \( s = (t - t_i)/h_i, \ h_i = t_{i+1} - t_i. \)

These approximate solutions are introduced into the governing equations and are forced to satisfy the equations at the collocation points.
\[ s_{i1} = t_i + (\frac{1}{2} - \frac{1}{\sqrt{12}})h_i \quad \text{and} \]
\[ s_{i2} = t_i + (\frac{1}{2} + \frac{1}{\sqrt{12}})h_i \]

This results in \( 4N \) equations for the \( 4N+4 \) unknowns. Boundary conditions constitute the remaining of the required equations. The \( 4N+4 \) nonlinear equations were solved by the modified quasilinearization algorithm [56], which is similar to its counterpart used in solving the nonlinear two-point boundary value problems. Denoting by vector \( \mathbf{x} \) the unknowns and by \( f \) the equations

\[ f(x) = 0. \quad \text{(D.16)} \]

The performance index \( P \) is defined as

\[ P = f^T f \quad \text{(D.17)} \]

and its first variation is

\[ \delta P = -2f^T \delta f. \quad \text{(D.18)} \]

Considering

\[ \delta f = -kf, \quad 0 < k \leq 1 \quad \text{(D.19)} \]

one obtains

\[ \delta P = -2kf < 0 \quad \text{(D.20)} \]

which represents the descent property. Let \( A = \Delta x/k \). Then, from equation D.19
\[ f^T_A = -f \]  

which is a linear system in \( A \) and independent of \( k \). Determining \( A \), one forms

\[ \tilde{x} = x + kA \]  

and finds the value of \( k \) such that \( \tilde{P} < P \). The algorithm is repeated until

\[ P \leq \varepsilon \]  

where \( \varepsilon \) is a small, preselected number.