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RESPONSE OF YIELDING MDF STRUCTURES
TO STOCHASTIC EXCITATION

by

TYSH SHANG JAN

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

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HOUSTON, TEXAS

MAY, 1982
ABSTRACT

Response of Yielding MDF Structures to Stochastic Excitation

by

Tysh Shang Jan

The random response of bilinear hysteretic yielding systems subjected to a stationary Gaussian white noise excitation is studied. The responses considered are the root mean square (rms) interfloor displacements of yielding multi-degree-of-freedom (MDF) shear beam type building structures. Simulation results are presented as well as analytical methods to approximate the response levels. Both stationary response levels and the transient build-up of response for systems which are initially at rest are investigated. Nearly elastoplastic 2DF, 4DF and 10DF systems are studied with different stiffness distribution over the height (uniform or tapered) and with different damping ratios.

In the simulation studies, the stationary rms interfloor displacements of the yielding MDF bilinear hysteretic system are taken as the time averages of responses to a particular time history (assuming ergodicity). Determining the transient rms interfloor displacements, though, involves computing averages across an ensemble of response time histories. Each response time history is obtained by numerical integration of the equations of motion for a particular time history of excitation.

The analytical method presented for predicting the response levels of the yielding MDF system is based on a substitute structure concept. The parameters of the substitute structure are determined from an equivalent third-order system model for single-degree-of-freedom (SDF) structures. That is, each yielding story in the MDF system is replaced by springs and dashpots corresponding to a third-order model for a SDF system. The values of
these linear parameters depend on the expected ductility ratio and are
determined empirically from simulation data for SDF systems. The Liapunov
covariance matrix equation of this higher order linear system is then
established to compute the stationary rms interfloor displacements. The
stationary results obtained by this method compare very well with those
from simulation.

The transient rms interfloor displacements of this higher order linear
system are also investigated. The Liapunov covariance differential equation
is integrated to obtain the transient mean squared responses numerically.
The computed results are in good agreement with those from simulation for
stories with very high or very low stationary expected ductility factors.
For a story with moderate stationary expected ductility factor, the higher
order linear system response builds up too rapidly. An improved approxi-
mation of the initial build-up of response is obtained by using a variable
coefficient linear model with dampings and stiffnesses dependent on the
response levels.
ACKNOWLEDGEMENTS

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I INTRODUCTION

1.1 Background

In the study of structural safety under earthquake-like loading, knowledge of the structural response in the nonlinear range is required since most structures experience inelastic behavior before failure.

Satisfactory deterministic studies\(^{(26-32)(43-44)}\) of response have been performed for both deteriorating and non-deteriorating nonlinear structures subjected to ground motions representative of past earthquakes. These studies certainly reveal much about the structural behavior, but the fact remains that one cannot expect the excitation from some future earthquake to have the same time history as any of those studied. Hence probabilistic studies of the structural response statistics in the yielding range seem to be appropriate.

There have been a number of studies in recent years of the response statistics of structures subjected to random load. For linear structures the theory of the response to random excitation is quite well developed in reference books\(^{(13)(37)}\). Various response statistics can be calculated if the appropriate statistics of the excitation are known.

During the last twenty years, considerable attention has been given to the single-degree-of-freedom (SDF) yielding system subjected to stationary random excitation. The most commonly studied model of yielding is bilinear hysteretic, wherein the stiffness of the structure switches back and forth between the initial stiffness and the yielded stiffness (or second slope), when the structure is in the nonlinear range. No exact theoretical solution has been obtained. However, some approximate analytical techniques have been found both for the bilinear
hysteretic system (1-5) (7) (9) (17) (22-23) (33-36) and for the elasto-plastic system (39-40) (46-47).

Analytical study of random vibration of nonlinear multi-degree-of-freedom (MDF) systems is generally difficult. So far only very few approximate solutions to such systems have been found. Karmopp and Brown (11) have predicted response statistics of a two-degree-of-freedom bilinear hysteretic system by using the power balance method and modal analysis. Unfortunately the only systems considered were limited to a special class of vibration-isolation problems and this method cannot be extended to a general MDF system. An alternative method which has been used in the analysis of a MDF offshore structure with bilinear restoring forces is based on a direct linearization of the Fokker-Planck equation (10). The response statistics were obtained by solving the Liapunov covariance matrix differential equation. The linearized system coefficients were obtained by a heuristic procedure. Examples showed that this method could predict the response of an elastic system very well but over-estimated the response of a yielding system.

One approximate technique developed by Wen (8) has the capability of predicting mean square values of responses of the hysteretic system described by

\[ Q(x, \dot{x}) = g(x, \dot{x}) + z(x) \]  

with

\[ \dot{z} = -r|x|z|z|^{n-1} - \beta \dot{x} |z|^n + A \dot{x} \]  

where \( g \) is the non-hysteretic restoring force component, and \( z \) is the
hysteretic restoring force component. The parameters \( n, A, r, \beta \) are given constants. A nearly elasto-plastic SDF system can be approximated if \( n, A, r, \beta \) are properly selected. An equivalent linearization method (6) was used to linearize equation (1-2) for SDF and MDF systems under random vibration. Comparison of the predicted mean square displacements with Monte-Carlo simulations of equations (1-1) and (1-2) indicated that the method gave satisfactory result for a 2DF system. However the system of equations (1-1) and (1-2) differs substantially from the bilinear hysteretic system in that it has hysteretic energy dissipation for all levels of response, while a real bilinear hysteretic system has none when the displacement of the system is less than its yield level. Hence equations (1-1) and (1-2) cannot be used to truly model a bilinear hysteretic system.

A method based on a substitute structure concept was also used by Wen (41) to compute the statistics of maximum response for multi-story buildings with deteriorating restoring forces. Damping and stiffness of the substitute elements were obtained from empirical results for the SDF system. Results showed that the statistics of the response predicted by this method were qualitatively satisfactory but had substantial quantitative errors.

More recently Brinkmann (14) attempted to use an equivalent linearization approach to predict the responses of bilinear hysteretic 2DF systems. By modifying the stiffness and damping characteristics of the yielding story of the structure, a new linear system was created whose response was compared to that of the original system. The stiffness and damping values were determined theoretically from SDF results. Figure 1.1 shows an example of the responses of the modified linear 2DF
system normalized to the elastic linear system values, along with simulation data for the bilinear hysteretic system subjected to Gaussian normal white noise acceleration of the base. From Figure 1.1 one notes that the modified linear model is able to predict the response level of the yielding first floor for this case of a small second slope in the bilinear stiffness relationship. One also notes, however, that the model is very inaccurate for predicting the response in the elastic story of the structure. The figure also shows a curve (obtained in this study) which removes an error introduced by an approximate computation procedure used by Brinkmann. This does give an improvement, but one is still left with the conclusion that the model is inadequate. A purely interpretative (as constrained to predictive) approach taken by Brinkmann was to attempt to determine empirically the parameters of a linear system whose response levels would match the simulation values for the bilinear hysteretic MDF system. Thus the properties of the linear system, such as natural frequencies and mode shapes, were to be calculated from the empirical (simulation data) response levels, rather than vice versa. He concluded that no meaningful equivalent linear system could be found by this procedure.

Knowledge about the random vibration of elasto-plastic MDF structures is much more scarce than for bilinear hysteretic systems. One approximate method was given by Gazetas\textsuperscript{(45)} to find the ductility factor for a 4DF structural system by using analytical results for the SDF elasto-plastic system\textsuperscript{(47)}.

1.2 **Scope and Objectives**

There are two basic objectives in this study. The first one
is to obtain simulation data on the stationary and nonstationary response of nonlinear MDF systems subjected to random excitation by using a numerical integration procedure. The particular nonlinearity considered is the bilinear hysteretic model of yielding (see Figure 1.2). The statistical responses studied are the root mean squared interfloor displacements of this MDF nonlinear structure. The second objective is to develop an approximate analytical technique to predict the stationary and transient values of root mean squared interfloor displacements for the structure. The excitation considered is Gaussian normal white noise ground acceleration that is taken as a series of Dirac delta functions at uniformly space time intervals. The magnitude of each pulse is a normally distributed random variable.

1.3 Outline

Chapter II describes the digital computer simulation of the stationary root mean squared interfloor displacements of MDF bilinear hysteretic systems. This simulation consists of two steps. First it is necessary to create a sample of normal white noise acceleration at the base, by using a sequence of numbers given by a "random number generator". The second step consists of obtaining stationary rms values of the structural responses to the generated time history of excitation. The response time histories are found by numerical integration of the equations of motion, and the rms values are obtained from long time averages. Results are presented for 2DF, 4DF and 10DF structural systems with uniform and tapered stiffness distribution. Modal damping ratios considered are 5% and 1%. The second slope of the stiffness is chosen to be 1/21 for this study.
Chapter III presents an approximate analytical approach to predict the stationary root mean squared interfloor displacement values of the structures considered. An equivalent higher order linearization method is used. The damping and stiffness constants of the higher-order linearization of the yielding stories are determined from matching the responses of a third order linear system and the bilinear hysteretic SDF system. The method is semi-empirical in the sense that empirical data for the responses of the bilinear hysteretic SDF system are used in this step. A Liapunov covariance matrix differential equation is established and its stationary solution is found for this equivalent higher-order linear MDF system. Results obtained from this higher-order equivalent linear system are compared with data from Monte-Carlo simulation.

Chapter IV presents the digital computer simulation of the nonstationary build-up of the root mean squared interfloor displacements. These nonstationary values are basically evaluated from ensemble averages of zero-start time histories. Actually some time averaging is also used to smooth the curves somewhat. Each time history is obtained by numerical integration of the equations of motion for a particular time history of excitation. The excitation is a series of pulse-like forces as described in Chapter II.

The idea of predicting the nonstationary build-up of the root mean squared interfloor displacements using the higher-order equivalent linear system is discussed in Chapter V. The equivalent linear systems used in Chapter III for finding the stationary rms response levels is now used to find the rms transient responses. Direct numerical integration of the covariance matrix differential equation is used. Finally
a higher-order linear system with varying coefficients is also tried. In this model both the equivalent damping and stiffness are functions of the response ductility factor. Results are compared with those obtained in Chapter IV.

Chapter VI gives conclusions drawn from the investigation reported herein.
II DIGITAL COMPUTER SIMULATION FOR STATIONARY RESPONSE

2.1 Introduction

A structural system with a bilinear hysteretic stiffness property can be illustrated by the portal frame shown in Figure 2.1. If the cross section of the beam is small in comparison with those of the columns then yield hinges at the two ends of the beam may be formed when the displacement $x$ is large. In other words, the frame displaces with its initial stiffness when $x$ is small and then displaces with a reduced stiffness after the end moments of the beam reach the maximum $M_0$. If the loading is reversed before the other two additional hinges form in the columns then the system will again displace with its initial stiffness. Hysteretic energy is dissipated at the end hinges of the beam when the frame displaces with the reduced stiffness. Neglecting the gravitational force due to mass, the restoring force of the frame can be modeled as a system shown in Figures 1.2 and 1.3. The equation of motion of this SDF bilinear hysteretic system can be written as:

$$m \ddot{x} + c \dot{x} + Q(x) = -m \ddot{f}(t)$$

or

$$\ddot{x} + 2 \beta_0 \omega_0 \dot{x} + \frac{1}{m} Q(x) = -\ddot{f}(t) \quad (2-1)$$

where

- $x$ = displacement relative to the base
- $\omega_0$ = small amplitude undamped natural frequency
- $\beta_0$ = small amplitude fraction of critical damping
- $Q(x)$ = restoring force of the system
\[ \ddot{f}(t) = \text{stationary Gaussian white noise acceleration} \]

Dots in equation (2.1) denote mean square derivatives with respect to time of the random process \( X(t) \). No exact analytical solution for the statistics of the response of such a bilinear hysteretic system to random excitation has yet been obtained. The most important statistics of interest are the mean squared level of displacement, \( E(x^2) \), and the mean squared level of velocity \( E(x_2^2) \). Assuming ergodicity of the response, the empirical mean squared values of stationary \( E(x^2) \) and \( E(x_2^2) \) can be taken as the time averages of the squared values of displacement and velocity respectively, after response has reached the stationary level. Such empirical solutions of the stationary \( E(x^2) \) and \( E(x_2^2) \) have been obtained by Iwan and Lutes\(^{49}\) by using an analog computer simulation.

Empirical statistical data for the response of MDF bilinear hysteretic yielding systems is still very scanty. The following sections of this chapter will present some simulation data for this system.

2.2 Description of the MDF System

The systems considered are models representing shear beam type of building structures (see Figure 2.2), the mass of which is assumed to be concentrated at the floor levels, and the floors are rigid as compared with columns. It is assumed that the columns have bilinear hysteretic force deflection relationships (see Figure 1.1). Two-story, four-story and ten-story structures are chosen for study. Two types of pre-yielding stiffness distribution are studied.
Type A: stiffness is uniform over the height

Type B: tapered structure or step-tapered structure. In the tapered structure the stiffness is \( k \) at the top story and has an increase of \( v k \) in each subsequent story from the top ( \( v \) is a constant). The step-tapered structure has the largest stiffness at the bottom story and the stiffness changes at every subsequent two or three stories.

The masses in any particular structure are all the same, except the two-story system of Type A in which the mass at the top story is assumed to be half that of the bottom story. Modal damping values of 1% and 5% of the critical damping are included in this study. Modal natural frequencies and mode shapes of each structure studied herein are given in Tables 1 to 6.

2.3 **Excitation**

An earthquake like excitation to the structure can be modelled as a random process. In this study the excitation is taken to be a stationary "white noise" - a signal whose power spectral density is constant for all frequencies. Three possible methods for simulating a white noise signal are ramp, step, and pulse type approximation, referring to the shape of the acceleration trace in each case. Brinkmann (14) gave comparisons of the simulated mean squared velocity and displacement with theoretical values for an elastic linear single-degree-of-freedom system: (13)
\[ E(x^2) = \frac{\pi S_0}{2\beta_0 \omega_0^2} \quad (2-2) \]

\[ E(\ddot{x}^2) = \frac{\pi S_0}{2\beta_0 \omega_0} \]

The results indicated that all three types of approximation agree with theory very well. Comparative studies were also given by Brinkmann for a SDF nonlinear system. He concluded that the pulse type approximation is much more accurate in computing the mean square levels of response, comparing with the empirical results obtained by Iwan and Lutes\(^{(49)}\).

Because of the better approximation of the pulse type acceleration in the single degree of freedom yielding case, it will be the only approximation used in all subsequent calculation for the MDF structures.

Now the white noise acceleration process can be generated by first using a standard random number generator program to obtain a sequence of random numbers \( R_i \) which are statistically independent and uniformly distributed over the range 0 to 1. This set of numbers determines a new sequence of normal random numbers \( \ddot{R}_i \), which are related to \( R_i \) by the relations

\[ \ddot{R}_i = ( -2 \ln R_i )^{1/2} \cos ( 2 \pi R_{i+1} ) \]

\[ \ddot{R}_{i+1} = ( -2 \ln R_i )^{1/2} \sin ( 2 \pi R_{i+1} ) \quad (2-3) \]

where \( \ln \) is the natural logarithm and the \( i \) are odd numbers. Pertinent expectations of \( \ddot{R}_i \) and \( \ddot{R}_{i+1} \) are
\[ E(\bar{R}_i) = E(\bar{R}_{i+1}) = 0. \]
\[ E(\bar{R}_i^2) = E(\bar{R}_{i+1}^2) = 1. \] (2-4)
and \[ E(\bar{R}_i \cdot \bar{R}_{i+1}) = 0. \]

The acceleration at the base of the structure is taken to be a sequence of uniformly spaced Dirac delta function pulses. The time interval between pulses is \( b \) and the random number \( \bar{R}_i \) is scaled by a constant \( A \) to be the magnitude of the \( i \)th pulse. A wave form of the random excitation process is thus generated by repeating the above procedure. The time interval is taken as some multiple of the step size used in the integration process. The relationship between the pulse magnitudes and the power spectral density of the process is

\[ S_0 = \frac{A^2}{2\pi b} \] (2-5)

This impulse acceleration \( A\bar{R}_i \delta(t-t_i) \) at the base will give an \( A\bar{R}_i \) change in absolute velocity of the base while the absolute velocities of the masses are unchanged. In other words, this impulse acceleration \( A\bar{R}_i \) will give a change, \(-A\bar{R}_i\), in velocity of any mass relative to the base of the structure.

2.4 Integration Scheme

The equation of motion for the system considered is

\[ M \ddot{x} + C \dot{x} + Q(x) = -M \{1\} \ddot{f}(t) \] (2-6)

where
\[
M = \begin{pmatrix}
  m_1 & 0 & & \\
  m_2 & 0 & & \\
  & 0 & \ddots & \\
  & & & m_n
\end{pmatrix} \quad X = \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

(2-7)

\[C \text{ is the damping matrix, and } Q \text{ is the vector of restoring forces of the system. The excitation } \ddot{f}(t) \text{ is the stationary white noise acceleration which has been described as a sequence of random impulse base accelerations. When all } x_i \text{ are small } Q(x) \text{ is a linear function of } x_i, \text{ and thus equation (2-6) can be written as}
\]

\[M \ddot{X} + C \dot{X} + K X = -M \ddot{f}(t)
\]

(2-8)

where \(K\) is given as follows for a shear beam type structure

\[
K = \begin{pmatrix}
  k_1 + k_2, & -k_2, \\
  -k_2, & k_2 + k_3, & -k_3, & 0 \\
  & -k_3, & k_3 + k_4, & -k_4 \\
  & & & \ddots \\
  & & & 0, & -k_{n-1}, & k_{n-1} + k_n, & -k_n \\
  & & & & & -k_n, & k_n
\end{pmatrix}
\]

(2-8a)

The modal frequency \(\omega_i\) and the corresponding mode shape \(\phi_i\) can be obtained by solving for the eigen values and eigen vectors of the equation

\[
\left( M \omega^2 - K \right) X = 0
\]

(2-9)

With this calculated modal frequency \(\omega_i\) and mode shape \(\phi_i\), the
assumed modal damping values can be used to determine the damping matrix \( C \), which is (see Appendix A)

\[
C = M \left( \sum_{i=1}^{n} \frac{\beta_i \omega_i}{\bar{M}_i} \phi_i \phi_i^T \right) M\)  \quad (2-10)
\]

where

\[
\bar{M}_i = \phi_i^T M \phi_i
\]  \quad (2-11)

The system in equation (2-6) is assumed to be initially at rest, then suddenly excited by Gaussian, stationary white noise acceleration. Because of the nonlinearity of the restoring force \( Q(x) \), no exact solution to equation (2-6) has been obtained. However, an exact stepwise integration is possible due to the piecewise linear characteristic of the resistance deformation relationship and the assumption that the tangent stiffness is constant within this time interval. Let \( U_i \) denote the interfloor displacement of the \( i \)th story

\[
U_i = x_i - x_{i-1}
\]  \quad (2-12)

\( x_0 = 0 \)

Overall, this stepwise integration procedure is basically illustrated as follows for the time increment \( t_j \) to \( t_{j+1} \)

1. Assume all \( U_i(t) \) are known for \( t = t_0, t_1, t_2, \ldots, t_j \) and so are the hysteretic loops of the whole history up to \( t_j \).
2. Find the restoring force \( F_i(t_j) \) and stiffness slope \( k_i(t_j) \) for each story from the hysteretic loop of the time history up to time \( t_j \).
3. Compute \( U_i(t_{j+1}) \) terms by assuming that the system behaves linearly during the time increment.
4. Compute new $F_i(t_{j+1})$ and $k_i(t_{j+1})$ from new $U_i(t_{j+1})$ and find
the extension of the hysteretic loop.
Repeating step 3 and step 4 the time history of response is obtained.
Since the structure is a shear beam type system, it can be
modelled as masses connected by springs and damping dashpots. Figure
2.2c shows that the spring forces on mass $i$ are from spring $i$ and spring
$i+1$. Referring to Figure 1.2, at any instantaneous beginning of an
integration time interval, the restoring force at spring $i$ can be cal-
culated as follows

$$F_i = k_i (x_i - x_{i-1}) + B_i \quad (2-13)$$

where

$$B_i = \begin{cases} 
\left( \frac{k_i^{(2)} - k_i^{(1)}}{k_i^{(1)}} \right) (U_{i}^Y - Y_i), & \text{if } k_i = k_i^{(1)} \\
\left( k_i^{(1)} - k_i^{(2)} \right) Y_i, & \text{if } k_i = k_i^{(2)} \text{ and } U_i(t_j) > U_i(t_{j-1}) \\
- \left( k_i^{(1)} - k_i^{(2)} \right) Y_i, & \text{if } k_i = k_i^{(2)} \text{ and } U_i(t_j) < U_i(t_{j-1})
\end{cases} \quad (2-14)$$

in which

$U_i$ = distortion of spring $i$

$U_i^Y$ = instantaneous yielding distortion at a particular inte-
gration time interval $t_j$ to $t_{j+1}$ ( $U_i^Y = Y_i$, initially )

$Y_i$ = yielding level of spring $i$

$k_i$ = stiffness of spring $i$, equal to $k_i^{(1)}$ or $k_i^{(2)}$

$k_i^{(1)}$ = initial stiffness of spring $i$

$k_i^{(2)}$ = second slope stiffness of spring $i$

$F_i$ = restoring force of spring $i$
In equation (2-13) the restoring force consists of two terms. The first terms is similar to that for the elastic system with either \( k_i(t_j) = k_i^{(1)} \) or \( k_i(t_j) = k_i^{(2)} \) (depending on \( U_i^y(t_{j-1}) \), \( U_i(t_j) \) and velocity \( \dot{U}_i(t_j) \)), and the second term is a constant depending on the instantaneous yielding distortion \( U_i^y \), and the sign of the distortion at a particular time \( t = t_j \). Similarly the restoring force in spring \( i+1 \) is
\[
F_{i+1} = k_{i+1} (x_{i+1} - x_i) + B_{i+1}
\]
(2-15)

Combining (2-13) and (2-15), the spring force acting on mass \( i \) is obtained as
\[
F_{i+1} - F_i = k_i x_i - (k_i + k_{i+1}) x_i + k_{i+1} x_{i+1} + G_i
\]
(2-16)

where
\[
G_i = B_{i+1} - B_i
\]
(2-17)

Now the equilibrium equation in the particular integration time interval is
\[
M \ddot{X} + C \dot{X} + K X = -M \ddot{f}(t) + \tilde{G}
\]
(2-18)

where \( \tilde{G} \) is a column vector matrix, varied for different time intervals:
\[
\tilde{G}^T = \begin{bmatrix} G_1, G_2, G_3, \ldots, G_h \end{bmatrix}
\]
(2-19)
Let

$$\mathbf{Y}^T = \begin{bmatrix} x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n \end{bmatrix}$$  \hspace{1cm} (2-20)$$

Equation (2-18) is then simplified to a first order linear matrix differential equation which can be solved numerically by the Runge Kutta method or by direct integration with a small integration time interval.

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1} \mathbf{K} & -\mathbf{M}^{-1} \mathbf{C} \end{bmatrix} \mathbf{Y} + \mathbf{G} - \mathbf{M} \ddot{f}(t)$$  \hspace{1cm} (2-21)$$

where

$$\mathbf{I} = \text{identity matrix with dimension } nxn$$

$$\mathbf{K} = \text{defined as (2-8a) in which } k_i = k_i^{(1)} \text{ or } k_i^{(2)}$$

Note that in the numerical solution the $\mathbf{G}$ matrix and $\mathbf{K}$ matrix, which are dependent on the time history, should be determined or calculated at the beginning of every integration step. Then equation (2-21) can be integrated step by step with initial conditions $\mathbf{Y} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$ and a time history of the response at each floor level can be determined. The root mean squared value of response at each floor level is then taken from a time average of the time history of the response after it has had time to build up.
2.5 Results and Discussion

A simulation program using a PDP-11 digital computer was established for simulating the response of a bilinear hysteretic MDF system subjected to Gaussian white noise excitation (see Appendix B). This program was verified by simulating a 4DF uniform linear structure subjecting to a white noise ground acceleration excitation. The damping ratio of each mode was taken to be 5%. The stationary mean square values of the interfloor displacements from this simulation are listed in Table 7, for comparison with those obtained by theoretical analysis. Note that the simulation data are taken as the time averages of response over a time interval containing 44 periods of the fundamental mode, after the system had reached the stationary response condition. The largest error is 3.6% in the bottom story. Some inaccuracy is inevitable since the simulation time is finite. The accuracy achieved was considered to be acceptable.

The simulation results of stationary rms values of interfloor displacement obtained for yielding structures are plotted in Figures 2.3 to 2.27, for different values of damping and for different types of structures considered. All simulation results are plotted in dashed lines or in dots and each rms interfloor displacement is nondimensionalized by normalizing by a factor \( N \) which is given by

\[
N = \sqrt{\frac{2 \, S_0}{\omega_1^3}} \quad (2-22)
\]
where \( \omega_1 \) is the natural frequency of the fundamental mode of the unyielded structure, and \( S_0 \) is the PSD value of the excitation.

The solid curves in Figures 2.3 to 2.27 represent analytical results from a linear model which will be discussed in detail in Chapter III. They are in this chapter only to avoid replotting all the simulation results again in Chapter III for purpose of evaluating the validity of the analytical results. The purpose of this section is only to present and discuss the simulation results, so the solid curves can be ignored for now.

The results are presented in the order of increasing number of stories of the structures: two-story, four-story, then ten-story. Some of the results are for structures in which some stories have an infinite yield level while only one or two stories have finite yield levels. In a few situations these results are reasonable for practical structures, since similar yield levels in all stories may lead to actual yielding only in those stories with larger response levels. In many other situations, though, these results are certainly not representative of realistic structures, but they do allow one to study the effects of yielding in a simplified situations. It is felt that the insight gained from such simplified situations are helpful in understanding the phenomena of yielding and thereby developing an approximate analysis of practical structures.

2DF Structures:

Figure 2.3 and 2.4 show the simulation results for two-degree-of-freedom (2DF) structures of Type A (\( k_1 = k_2 \), \( m_1 = m_2 \)) with the top story kept in the elastic range (i.e. infinite yield level) no matter what the yielding level of the bottom story is. Note that for
### Table 1 Normal Modes for Type A, 2DF Structure

<table>
<thead>
<tr>
<th>Story (i)</th>
<th>Mass ( m_i/m_1 )</th>
<th>Stiffness ( k_i/k_1 )</th>
<th>Mode (j)</th>
<th>Frequency ( \omega_j \sqrt{m_1/k_1} )</th>
</tr>
</thead>
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### Table 2 Normal Modes for Type B, 2DF Structure

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<th>Stiffness ( k_i/k_1 )</th>
<th>Modes (j)</th>
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### Table 3 Normal Modes for Type A, 4DF Structure

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<th>Mass ( m_i/m_1 )</th>
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<th>Mode (j)</th>
<th>Frequency ( \omega_j \sqrt{m_1/k_1} )</th>
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Table 4 Normal Modes for Type B, 4DF Structure

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<th>Stiffness (k_i/k_1)</th>
<th>Mode (j)</th>
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Table 5 Normal Modes for Type A, 10DF Structure

\( m_1=m_2=\cdots=m_{10}, \quad k_1=k_2=\cdots=k_{10} \)

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<th>7</th>
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Table 6  Normal Modes for Type B, 10DF Structure

\( (m_1=m_2=\ldots=m_{10}) \)
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<td>0.576</td>
<td>0.799</td>
<td>0.981</td>
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<td>-0.05</td>
<td>0.01</td>
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Table 7  Normalized RMS Interfloor Displacements for a 4DF Elastic Structure

\( (m_1=m_2=m_3=m_4, k_1=k_2=k_3=k_4, \beta = 5\%) \)

<table>
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<th>Story</th>
<th>Simulation</th>
<th>Theoretical</th>
<th>Error %</th>
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<td>21.49</td>
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<td>2</td>
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<td>9.266</td>
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<tr>
<td>4</td>
<td>2.97</td>
<td>2.993</td>
<td>1.07</td>
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Modified Linear Model
Response Parameters
$\alpha = 0.1$, $\beta = 0.05$

I. $E[X_1^2]^*$
II. $E[(X_2-X)^2]^*$
III. $E[X_1(X_2-X_1)]^*$

*Normalized to linear system values

Bottom Story Yielding
Top story elastic

Fig. 1.1 Responses of a Two-Story Shear Type Building
Fig. 1.2 Bilinear Hysteretic Restoring Force

Fig. 1.3 Bilinear Hysteretic Mechanical Oscillator
Fig. 2.1 Frame With Bilinear Hysteretic Stiffness

Fig. 2.2 MDF Shear Beam Type Building
Fig. 2.3 Normalized RMS Interfloor Displacements
Type A, 2DF, $\beta = 5\%$, Bottom Story Yielding

Fig. 2.4 Normalized RMS Interfloor Displacements
Type A, 2DF, $\beta = 1\%$, Bottom Story Yielding
Fig. 2.5 Normalized RMS Interfloor Displacements
Type A, 2DF, $\beta=5\%$, Top Story Yielding

Fig. 2.6 Normalized RMS Interfloor Displacements
Type A, 2DF, $\beta=1\%$, Top Story Yielding
Fig. 2.7 Normalized RMS Interfloor Displacements
Type A, 2DF, $\beta = 5\%$, Both Story Yielding

Fig. 2.8 Normalized RMS Interfloor Displacements
Type A, 2DF, $\beta = 1\%$, Both Story Yielding
Fig. 2.9 Normalized RMS Interfloor Displacements
Type B, 2DF, $\beta =5\%$, Both Story Yielding

Fig. 2.10 Normalized RMS Interfloor Displacements
Type B, 2DF, $\beta =1\%$. Both Story Yielding
Fig. 2.11 Normalized RMS Interfloor Displacements
Type A, 4DF, $\beta = 5\%$
Fig. 2.12 Normalized RMS Interfloor Displacements
Type A, 4DF, $\beta = 1\%$

--- Simulation

--- Higher Order System
Fig. 2.13  Normalized RMS Interfloor Displacements
Type B, 4DF, $\beta = 5\%$
Fig. 2.14 Normalized RMS Interfloor Displacements
Type B, qDF, \( \beta = 1/5 \)

(a) \( \frac{Y_N}{N} \)

(b) \( \frac{Y_N}{N} \)

(c) \( \frac{Y_N}{N} \)
Fig. 2.15  Normalized RMS Interfloor Displacements
Type A, 10DF, \( \beta = 5\% \), First Story Yielding
Fig. 2.15 Normalized RMS Interfloor Displacements
Type A, 10DF, $\beta=5\%$, First Story Yielding
Fig. 2.15 Normalized RMS Interfloor Displacements
Type A, 10DF, $\beta = 5\%$, First Story Yielding

Fig. 2.16 Normalized RMS Interfloor Displacements
Type A, 10DF, $\beta = 1\%$, First Story Yielding
Fig. 2.16 Normalized RMS Interfloor Displacements
Type A, 10DF, $\beta = 1\%$, First Story Yielding
Fig. 2.16 Normalized RMS Interfloor Displacements
Type A, 1ODF, $\beta = 1\%$, First Story Yielding
Fig. 2.16 Normalized RMS Interfloor Displacements
Type A, 10DF, $\beta = 1\%$, First Story Yielding
a low yielding level, the rms interfloor displacement in the bottom story is far greater than that of the very high yielding level case, while the rms interfloor displacement in the top story is decreased in comparison with the elastic case. In Fig. 2.4 a heavy mark near the left-hand margin is used to represent the limiting value of $\sigma_{u2}$ for $Y = 0$. This, then, corresponds to a data point which falls outside the range of the figure. When $\beta_0 = 0.01$ (Fig. 2.4) both curves have a distinct low point for an intermediate yield level, while for $\beta_0 = 0.05$ (Fig. 2.3) the curve for the yielding story is monotonic, and the curve for the elastic story is almost monotonic. The shape of all the curves in Figs. 2.3 and 2.4 can be explained qualitatively by considering separately two effects of yielding; namely, reduction in stiffness and increase in energy dissipation. One can show that only reducing the stiffness in the bottom story of the elastic structure results in a monotonic increase in the rms response in that story, and a monotone decrease in the response of the second story. The hysteretic energy dissipation, on the other hand, results in a decrease in the response of both stories. This increase in energy dissipation, though, is not monotonic. It is zero for both $Y = 0$ and $Y = \infty$, and reaches a maximum for some intermediate yield level. Thus the effect of yielding on the bottom story response is a superposition of a monotone increase (due to stiffness reduction) and a decrease at intermediate yield levels (due to hysteretic energy dissipation). Similarly, the second story response is a superposition of a monotonic decrease (due to stiffness reduction) and an additional decrease at intermediate yield levels (due to hysteretic energy dissipation). Obviously, the non-monotonic effect of hysteretic energy dissipation is much less pronounced in Fig.
than in Fig. 2.4 ( $\beta_0 = 0.01$ ) since hysteretic energy dissipation is a much smaller fraction of total energy dissipation when viscous damping is larger.

Figures 2.5 and 2.6 show the normalized stationary rms interfloor displacements for the same Type A 2DF system, but with yielding now limited to the top story (infinite yield level in the lower story). Again the results can be interpreted in terms of the stiffness reduction causing increased response in the yielding story and decreased response in the elastic story, along with hysteretic energy dissipation causing reductions in both response quantities for intermediate yield levels. Actually the stiffness reduction effect in this situation is slightly different than in the previous figures. Reducing the second story stiffness in an elastic model does not quite give a monotonic reduction in the first story response, but rather gives a relative minimum when the stiffness is about 16% of its initial unyielded value. This minimum, though, is only about 10% less than the value when the stiffness is reduced to its limiting value of $\alpha$ times the initial stiffness. Thus this effect is not very significant in interpreting the effects of yielding. The increase in response in the yielding story due to stiffness reduction is monotonic as before.

Figures 2.7 and 2.8 present the simulation results for the same structure under the more realistic assumption that the yield level is the same in both stories. Thus the structure may yield simultaneously in both stories when the yield level is small. In Fig. 2.7, for $Y/N > 1.0$, the system has the same responses as those shown in Figure
2.3. Thus only the bottom story is yielding in this region. For $Y/N < 1.0$, the rms response of the bottom story remains monotonically increasing, while that of the top story increases sharply from $Y/N = 1.0$. This increase in $\sigma u_2$ is due to the top story yielding. Similarly, in Figure 2.8, for $Y/N > 1.5$, the system has the same rms responses as those shown in Figure 2.4, but for $Y/N < 1.5$, the responses of both stories are monotonically increasing when $Y/N$ is decreasing. Therefore the conclusion can be made that for $Y/N > 1.5$, yielding is only in the bottom story, and for $Y/N < 1.5$ the system is yielding in both stories.

Figures 2.9 and 2.10 show the simulation results for the 2DF Type B structure ($m_1 = m_2$, $k_1 = 2 k_2$) with the yield level the same for both stories. It is noticeable in Figure 2.9 or 2.10, that for $Y = \infty$, $\sigma u_2$ is significantly greater than $\sigma u_1$ in this system and therefore the second story yielding is more important for high yield levels. For $Y/N > 4.0$ in Fig. 2.9 (5.0 in Figure 2.10) both curves look like those for only the top story yielding and for $Y/N < 4.0$ (5.0 in Figure 2.10), $\sigma u_1$ increases due to the bottom story yielding.

**4DF Structures:**

The results for the 4DF structures are presented in Figures 2.11 to 2.14. The form used here is simpler than the form used for 2DF systems, and the information presented is less detailed. Each of the 4DF structures has been simulated for $Y = \infty$ (elastic) and for two finite values of $Y$, where $Y$ is the deformation yield level in each story.
Figures 2.11 and 2.12 show the distribution of the normalized stationary rms interfloor displacement over the height of a Type A, 4DF structure. The average normalized rms elastic interfloor displacement over the height of the structure in Figure 2.11(a) is 1.3. Thus the finite yield levels $Y/N = 1.3$ and $Y/N = 0.65$, chosen in this study are nearly the average and half the average of the stationary rms elastic interfloor displacements. From Figures 2.11(b) and 2.11(c) one can see that in this bilinear hysteretic system, yielding has a greater effect on the lower stories than on the upper stories. The stationary rms deformation in the bottom story increases while the stationary rms interfloor displacement in the top story decreases when the yield level is moderately low. This is quite different from the results obtained for the elastoplastic system. It is shown by Gazetas\(^{45}\) that in the elastoplastic system yielding may give a bigger interfloor displacement for the top story than for the bottom story even for a uniform MDF system. In Figure 2.12, one can see the effect of reducing the viscous damping to 1%. It is noted that all the stationary rms interfloor displacements for the yielding situations shown are smaller than those for the elastic case. This is due to the hysteretic energy dissipation rate being much larger than the rate at which the energy is dissipated by the viscous dampers. However the stationary rms interfloor displacements would be greater than those for the elastic case if the yield level were very low. For a very low yield level, the hysteretic energy dissipation rate is no longer significant in comparison with the energy dissipation by the viscous dampers.
Figures 2.13 and 2.14 present the stationary rms interfloor displacements of a Type B, 4DF structure. Note that in Figures 2.13(a) and 2.14(a), the interfloor responses are more uniform over the height of the elastic structure than for the Type A structure. Similar to the responses for the uniform stiffness structure, in Figure 2.13 for a moderately or slightly yielding case, the interfloor displacement response of the bottom story increases greatly while that of the top story decreases slightly. In Figure 2.14(b), (c), one finds again that for $\beta_0 = 0.01$ the interfloor responses do not increase at all. Hysteretic energy dissipation brings the stationary rms level of the interfloor displacement down, even though the stiffness of the structure is reduced.

10DF Structures:

Figures 2.15 to 2.27 present results for 10DF structures. Some of these results (Figs. 2.15 to 2.22) present detailed studies of $\sigma u_i/N$ versus $Y/N$, as given for the 2DF structures. These detailed studies were performed, though, only for simulations where only one or two stories were allowed to yield and the other yield levels were given infinite values. The stories given finite yield levels were always those with the largest rms response levels in the elastic situation; i.e. those most prone to yielding. For structures with equal yield levels in all stories, only a few yield levels were considered and the plots (Figures 2.23 to 2.26) show the distribution of deformation over the height of the structure, as was done for 4DF structures. Figure 2.27 compares the distribution of deformation over
the height of the structure for one story yielding, two stories yielding, and all stories yielding for one structure and one yield level.

All the plots in Figures 2.15 to 2.22 are qualitatively the same as those in Figures 2.3 to 2.6 for 2DF structures with only one story yielding. The response level in the one or two stories with finite yield level is increased by yielding, while the response levels in the elastic stories are generally decreased by yielding. For $\beta_0 = 0.01$, in Figures 2.16 and 2.18, there is a much more pronounced low point in the response curves for the elastic stories than is true for $\beta_0 = 0.05$, in Figures 2.15 and 2.17 (where hysteretic energy dissipation is relatively less important).

Comparing Figs. 2.17, 2.18, 2.21 and 2.22 with 2.7 to 2.10 shows one noticeable difference between the behavior of these particular 10DF and 2DF systems. Note that all these figures refer to structures with two yielding stories. For simplicity let the story with the largest rms response in the elastic situation be called the story of primary yielding, and let the other yielding story be called the story of secondary yielding. Thus for the 2DF Type A and Type B structures and the 10DF structures of Type A and Type B, the stories of primary yielding are $1,2,1$, and $5$, respectively, while the stories of secondary yielding are $2,1,2$, and $8$. One might expect, then, that the rms response curves for stories 2 and 8 for the 10DF structures would be shaped like those for the story of secondary yielding in the 2DF structure. In fact, though, they are shaped like those for the story of primary yielding (which are similar to those for a single story yielding). The most probable explanation of this
difference seems to be the similarity in the response levels in the two yielding stories of the 10DF structure. For the elastic situation the rms response in the story of secondary yielding is within about 10% of that in the story of primary yielding for each of the 10DF structures, whereas for the 2DF structure there is much more difference between these rms response levels. Thus it appears that the distinction between stories of primary and secondary yielding may be irrelevant when elastic rms levels are so nearly the same.

The response trends for the 10DF systems with all yield levels equal (Figures 2.23 to 2.26) are basically what one would intuitively expect. Note that Fig. 2.25 gives more different values of Y/N than do the other 3 figures, so trends are most obvious there. As Y/N is decreased from infinity the first noticeable effect is for the response to be increased in the stories which had the largest response values for Y = \infty; i.e., in stories 1 and 2 for the Type A system and stories 5 and 8 for the Type B system. This effect is most clear in Fig. 5.25(c). It is accompanied by slight decreases in the response levels of other stories which are apparently not yielding significantly. As Y/N is further reduced almost all story responses are increased as yielding spreads over the entire structure. A less obvious trend which appears here is a tendency for yielding to increase responses in the lower portion of the structure compared to those in the upper portion. This is obvious in Figs. 5.25(d, e, and f) where the largest response levels are now near the base of the structure, whereas story 5 had the largest elastic response in Figure 5.25(a).
A different type of comparison of the distribution of the stationary rms interfloor displacement over the height of the Type B, IODF structure is shown in Fig. 2.27. The modal damping ratio is 5% each, and the yield level considered always gives $Y/N = 0.43$. The different parts of the figure involve different restrictions as to which stories are allowed to yield. In part (a) all stories are elastic, and in part (b) only the fifth story can be yielding. Hence the rms level at the fifth story is increased. In Figure 2.27(c), the structure is idealized such that only the fifth and the eighth stories can be yielding, and relatively large rms levels occur in these two yielding stories while the rms levels in the elastic stories are comparatively smaller than in Figure 2.27(b). Figure 2.27(d) shows the responses of the structure when all stories have the same yield level. As noted above, the maximum response now occurs in the first story, even though the fifth story had the largest response in the elastic case ($Y = \infty$).

**Criterion for Yielding:**

Interpretation of various results is easier if one can sometimes say that yielding is negligible in a certain story, even though that story has a finite yield level. An obvious sort of criterion for such a statement is to say that yielding is negligible in story $i$ if $\sigma u_i/Y$ is less than some specified level. Commonly people choose to treat as negligible the event that a zero-mean random variable exceeds three times its standard deviation (rms). One basis for this is the fact that the probability of the absolute value of a normal random variable exceeding $3\sigma$ is only 0.0026. This normal distribution is
precisely correct for the responses of the systems studied here when no stories are yielding, but it is certainly only an approximation when any stories are yielding. The following paragraphs investigate the empirical evidence for adopting the rather arbitrary criterion that yielding in story \( i \) has a negligible effect on system response if \( \sigma u_i < \gamma / 3 \).

The plots of \( \sigma u_i / \gamma \) versus \( \gamma / N \) (Figs. 2.3 to 2.10, and 2.15 to 2.22) all show that the system response is the same as for an elastic system if \( \sigma u_i < \gamma / 3 \) in all stories of the structure. This, then, is a justification that yielding is negligible in that situation. It is desirable also to see whether it is valid to neglect yielding in a particular story, based on the \( \sigma u_i < \gamma / 3 \) criterion, for a system where yielding in other stories is definitely not negligible. The easiest test of this idea for the data here is to compare Figs 2.7 and 2.8 with Figs. 2.3 and 2.4. One does indeed find in Figs 2.7 and 2.8 that when \( \sigma u_2 < \gamma / 3 \) for these 2DF systems then the response curves are indistinguishable from those for \( \gamma = \infty \) in story 2, as given in Figs 2.3 and 2.4, even when yielding in story 1 is significant. The results in Figs. 2.15 to 2.18 for Type A 10DF systems are not inconsistent with this conclusion, but they are much less conclusive. In this case the elastic response levels in stories 1 and 2 are nearly the same so that there is almost no range of yield levels for which story 1 yields significantly while story 2 does not. The results are similarly inconclusive for yielding in stories 5 and 8 of the Type B 10DF system (Figs. 2.19 to 2.22). For 1% damping in this system, Figs. 2.22(e) and (h) show that there is a very small range of yield levels where story 5 yields
sufficiently that $\sigma_{u_5}$ is increased while $\sigma_{u_8}$ is decreased, indicating that yielding in story 8 is probably insignificant. All of these results are consistent with the idea of neglecting yielding when $\sigma_{u} < Y/3$.

The curves in Figs. 2.23 to 2.26 can also be investigated for compatibility with the yielding criterion suggested above. Fig. 2.23(a), for example, shows that all stories should remain elastic for $Y/N > 2.2$, since the maximum value of $\sigma_{u_1}/N(\text{story } 1)$ for the elastic structure is one-third of this value. From curve (a) one would predict that reducing $Y/N$ to a level of 2.1 would result in yielding in the first story, but not in any other story. Curve (b) confirms this prediction for $Y/N = 2.1$, in that $\sigma_{u_1}$ is increased while $\sigma_{u_2}$ is slightly decreased. Continuing with the same logic, one can predict from curve (b) that reducing $Y/N$ to 1.8 should result in yielding in stories 2 and 3, as well as story 1. The increased responses for these stories in curve (c) confirms the prediction. Overall, it appears that for this particular system, there is no yielding if $Y/N > 2.2$; only story 1 yields if $2.1 < Y/N < 2.2$; only stories 1 and 2 yield if $2.0 < Y/N < 2.1$; and stories 1, 2 and 3 yield for $1.8 < Y/N < 2.0$. The results do seem to agree with the criterion that yielding is negligible if $\sigma_{u_1}/Y < 1/3$.

The results in Figs. 2.24 to 2.26 also seem to be consistent with the hypothesized yield criterion when studied in detail, as in the preceding paragraph for Fig. 2.23. For the structure of Fig. 2.24 (Type A 10DF with 1% damping) there is no yielding if $Y/N > 5.0$, curve (b) has only story 1 yielding for $Y/N = 4.4$, and curve (c) has only
stories 1 and 2 yielding for $Y/N = 4.2$. For the Type B 10DF structure, initial yield is in story 5, followed by story 8, then the lower stories. Based on Fig. 2.25 one can conclude for $\beta = 5\%$ that there is no yielding if $Y/N > 2.1$, curve (b) has only story 5 yielding for $Y/N = 1.96$, curve (c) has only stories 5 and 8 yielding for $Y/N = 1.7$, and curves (d, e, and f) have almost all stories yielding for $Y/N \leq 0.6$.

Overall, it does appear to be a valid conclusion that if $\sigma u_i < Y/3$, then yielding in story $i$ has a negligible effect on any rms structural response level.
III ANALYTICAL DETERMINATION OF STATIONARY RESPONSE

3.1 Introduction

In the study of structural safety two of the most important response statistics of structures are rms values of displacement and velocity. For example rms values are necessary factors in approximating the first passage probability or the fatigue damage accumulation.

For a linear elastic single-degree-of-freedom system, with mass $m$, stiffness $k$ and damping constant $c$, subjected to a white noise base acceleration $\ddot{f}(t)$, the equation of motion can be written as

$$m \ddot{x} + c \dot{x} + k x = -m \ddot{f}(t)$$

or

$$\ddot{x} + 2\beta_0 \omega_0 \dot{x} + \omega_0^2 x = -\ddot{f}(t)$$

(3-1)

where $\beta_0 = c/(2m\omega_0)$, $\omega_0^2 = k/m$, and $x(t)$ is the motion relative to the base. If the power spectral density function of the excitation is $S_0$, the autocorrelation function of the excitation is given by

$$E(\ddot{f}(t_1) \ddot{f}(t_2)) = 2\pi S_0 \delta(t_1-t_2)$$

(3-2)

and the stationary rms of displacement and velocity are found (13) to be

$$\sigma_x = \left[\frac{\pi S_0}{2\beta_0 \omega_0^3}\right]^{1/2}$$

(3.3a)

and
\[ \sigma_X = \left( \frac{\pi}{\omega_0} \frac{S_0}{\omega_0} \right)^{1/2} \] (3.3b)

For a linear n degree-of-freedom system, the equation of motion can be written as

\[ [M] \ddot{X} + [C] \dot{X} + [K] X = -[M] \{1\} \ddot{f}(t) \] (3.4)

where \([M]\) = mass matrix, \([C]\) = damping matrix, \([K]\) = stiffness matrix, \([1]\) = unit column matrix, \(X\) = displacement vector relative to the base and \(\ddot{f}(t)\) = white noise excitation as defined before.

Modal analysis and direct solution of the equation for the Liapunov covariance matrix provide the two basic ways of finding the rms values of the displacements of a system governed by the equation (3-4). Straightforward usage of the modal method requires that the damping matrix \(C\) satisfy the condition for uncoupling the governing equation (3-4). To introduce modal analysis let

\[ \{X\} = [\phi] \{q(t)\} \] (3.5)

where \([\phi]\) is the normal mode matrix, columns of \([\phi]\) are eigen vectors of \([M]^{-1}[K]\) (i.e. mode shapes) and \(q_i(t)\) are corresponding modal amplitudes. Substituting equation (3.5) into (3.4) gives

\[ [M] [\phi] \ddot{q} + [C] [\phi] \dot{q} + [K] [\phi] q = -[M] \{1\} \ddot{f}(t) \] (3-6)
Pre-multiplying equation (3-6) by $[\phi]^T$ always results in diagonal matrices on the $\ddot{q}$ and $q$ terms. If $[\phi]^T[C][\phi]$ is also diagonal then the simultaneous differential equations can be uncoupled. The uncoupled equation for the $i^{th}$ mode can be written as

$$\ddot{q}_i + 2\beta_i\omega_i \dot{q}_i + \omega_i^2 q_i = -s_i \ddot{f}(t) \quad (3-7)$$

where $\beta_i$ and $\omega_i$ are the damping ratio and natural frequency of mode $i$, and $s_i$ is the modal participation factor of the $i^{th}$ mode. The $s_i$ modal participation factor is given by

$$s_i = -[\phi^T] M [l] / ([\phi^T] M [\phi]) \quad (3-8)$$

where

$[\phi_i]$ is column $i$ of $[\phi]$.

The solution of equation (3-7) for zero initial conditions can be obtained by Duhamel's integral as

$$q_i(t) = -s_i \int_0^t \ddot{f}(\tau) h_i(t-\tau) \, d\tau \quad (3-9)$$

where $h_i(t)$ is the impulse response function

$$h_i(t) = \{ e^{-\beta_i \omega_i t} \sin p_i t / p_i \} u(t) \quad (3-9a)$$

in which $\omega_i$ is the undamped circular frequency, $u(t)$ is the unit step function and...
\[ p_i = \omega_i (1 - \beta_i^2)^{1/2} \]  

(3-9b)

Note that the quantities most closely related to the structural damage are the story distortions rather than the displacements of each story. Hence the rms values of the story distortions are of primary interest in this study. Let \( U_i \) denote the distortion of the \( i \)th story

\[ U_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n \]  

(3-10)

where \( x_0 = 0 \)

Combining equation (3-10), (3-9) and (3-5), the story distortions can be written as

\[ U_1 = \sum_1^n s_j (\phi_{i,j} - \phi_{i-1,j}) \int_0^t \ddot{y}(\tau) h_j(t-\tau) \, d\tau. \]  

(3-11)

The rms story distortion can be obtained by taking the expectation of the square of equation (3-11)

\[ \sigma_{U_i}^2 = \mathbb{E}[U_i^2(t)] = \sum_{j=1}^n \sum_{k=1}^n s_j s_k (\phi_{i,j} - \phi_{i-1,j}) (\phi_{i,k} - \phi_{i-1,k}) \]

\[ \cdot \int_0^t \int_0^t \mathbb{E}[\ddot{y}(\tau_1) \ddot{y}(\tau_2)] h_j(t-\tau_1) h_j(t-\tau_2) \, d\tau_1 \, d\tau_2 \]  

(3-12)

With equation (3-2) and the assumption that \( \beta_1 = \beta_2 = \cdots = \beta_n = \beta \), the above equation can be simplified as
\[
\sigma_{u_i}^2 = 2 \pi S_0 \sum_{j=1}^{n} \sum_{k=1}^{n} s_j s_k (\phi_{i,j} - \phi_{i-1,j}) (\phi_{i,k} - \phi_{i-1,k}) \cdot T_{j,k}
\]

(3-12a)

where

\[
T_{j,k}(t) = \int_0^t h_j(t-\tau) h_k(t-\tau) \, d\tau
\]

\[
= 2 \beta \left( \omega_j + \omega_k \right) \left[ \left( \omega_j - \omega_k \right)^2 + 4 \beta^2 \omega_j \omega_k \right]^{-1} + \frac{e^{-\beta(\omega_j + \omega_k)t}}{2 p_j p_k} \left[ \frac{\beta \cos(p_j+p_k)t - (p_k/p_j) \sin(p_j+p_k)t}{(\omega_j + \omega_k)} \right. + \\
\left. \frac{-\beta(\omega_j + \omega_k) \cos(p_k-p_j)t + (p_k-p_j) \sin(p_k-p_j)t}{(\omega_k - \omega_j)^2 + 4 \beta^2 \omega_j \omega_k} \right]
\]

(3-12b)

The rms of the story distortion approaches a stationary value as \( t \) approaches \( \infty \), i.e.

\[
\sigma_{u_i}^2 = 2 \pi S_0 \sum_{j=1}^{n} \sum_{k=1}^{n} s_j s_k (\phi_{i,j} - \phi_{i-1,j}) (\phi_{i,k} - \phi_{i-1,k}) \cdot T_{j,k}
\]

(3-13)

and

\[
T_{j,k} = 2 \beta \left( \omega_j + \omega_k \right) \left[ \left( \omega_j - \omega_k \right)^2 + 4 \beta^2 \omega_j \omega_k \right]^{-1}
\]

(3-14)
As an alternative to using the time domain integration presented above one could use frequency domain integration. The response power spectral density could be written as the product of $S_0$ and the squared amplitude of the harmonic response function. This total harmonic response function would be the sum of modal harmonic response functions. Once the response power spectral density is found, its integral gives the mean squared response. The results, of course, would be the same as equations (3-13) and (3-14).

The alternative of obtaining the rms of the displacement in each story by integrating directly the equation for the Liapunov covariance matrix can be used whether or not equation (3-4) can be uncoupled by modal analysis. To do this, rewrite equation (3-4) as

$$\begin{bmatrix} \{\dot{x}\} \\ \{\ddot{x}\} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \{x\} \\ \{\dot{x}\} \end{bmatrix} - \begin{bmatrix} \{0\} \\ \{1\} \end{bmatrix} \ddot{f}(t) \tag{3-15}$$

Letting

$$\{Y\} = \begin{bmatrix} \{x\} \\ \{\dot{x}\} \end{bmatrix}, \quad [G] = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

and

$$\{F\} = \begin{bmatrix} \{0\} \\ \{1\} \end{bmatrix} \ddot{f}(t) \tag{3-16a}$$

Equation (3-15) may be written more concisely as follows:
\[
\frac{d\{Y\}}{dt} = [g] \{Y\} + \{F\} \quad (3-16b)
\]

From (3-16b) the Liapunov covariance matrix equation can be obtained as

\[
\frac{d[\hat{S}]}{dt} = [g] [\hat{S}] + [\hat{S}] [g]^T + [B] \quad (3-17)
\]

where

\[
[\hat{S}] = E \{\{Y\} \{Y\}^T \}
\]

and

\[
[B] = 2 E [\{F\} \{Y\}^T ]
\]

For a white noise excitation of the base \( \ddot{f}(t) \), \([B]\) is obtained as a constant matrix, i.e.

\[
[B] = 2\pi S_0 \begin{bmatrix} 0 & 0 \\ \hline 0 & [1] \end{bmatrix} \quad (3-18)
\]

where \([1] = n \times n \) unit square matrix. It can be seen that the covariance matrix \( \hat{S} \) builds up from the initial condition \( \hat{S}(t=0) = 0 \).

As time becomes larger the matrix \( \hat{S} \) becomes stationary and hence (3-17) reduces to simultaneous algebraic equations:

\[
[g] [\hat{S}] + [\hat{S}] [g]^T + [B] = 0 \quad (3-19)
\]

Thus (3-19) gives a direct simple method which is free from the problem caused by the damping matrix \( C \), in uncoupling the vibration differential
equations. It should be noted, though, that it is a fairly time consuming method of obtaining the solution. For example, the $\hat{S}$ matrix is $20 \times 20$ in size for a 10DF system and because of symmetry of $\hat{S}$, (3-19) gives 210 simultaneous algebraic equations of which only ten are trivial. It takes about ten seconds to obtain the solution by using ITEL-IBM-370 computer.

Some solid lines in Figures 2.11 to 2.14 show the rms interfloor displacements of the 4DF elastic systems, computed by the above theoretical method. Some other solid lines in Figures 2.23 to 2.27, representing the rms interfloor displacements of the 10DF elastic systems were also obtained by this method.

Various approximate techniques have been developed in recent years to predict the statistics of response of nonlinear SDF systems subjected to random excitations (7)(12)(17)(25)(46). Among these, various equivalent linearization techniques have been widely used by many investigators. Of course, the basic idea is to seek an equivalence between linear and nonlinear systems. Equivalent linearization techniques may be valuable in two ways. First the physical significance of experimental results for a nonlinear system may be more obvious if a linear system can be found which has approximately the same response to the same excitation. This is because the dynamic properties of linear systems are more widely known than are those of nonlinear systems. Second, if an analytical method is available for choosing the parameters of the equivalent linear system without first knowing the dynamic response of the nonlinear system, then the method can be
used to predict the nonlinear system response. The relative mathematical simplicity of linear analysis makes this method very useful. The results of some simple equivalent linearization schemes for the stationary response of SDF systems with the bilinear hysteretic nonlinearity considered in this study have been summarized by Lutes(17). A higher order equivalent linear system, used by Hsieh(25), to investigate the stationary and transient responses statistics for SDF system with bilinear hysteretic nonlinearity, was shown to be a quite successful equivalent linearization technique also.

It is quite difficult to directly apply to MDF yielding systems as approximate technique developed specifically for SDF systems. One of the primary reasons for this difficulty is lack of knowledge concerning the effect of yielding in one story on the response in the other stories.

A limited number of papers dealing with MDF yielding systems have appeared recently. Kaul & Penzien(10) and Asano (51) used different mathematical models for statistical linearization and obtained obtained different analytical results. No comparison has been made between their results.

The substitute structure method proposed by Gulkan and Sozen (30) and Shibata and Sozen(31) for seismic design of reinforce concrete structures is similar to the idea of equivalent linearization used for SDF yielding systems. Later, Wen(41) obtained some practical results by an analytical-empirical method for MDF yielding system using a substitute structure concept. The substitute structure parameters were
determined from empirical results for SDF nonlinear systems. Expected ductility values were found for R.C. structures and hysteretic steel frames subjected to earthquake loading. Comparison with empirical results was qualitatively satisfactory.

The analytical advantage of using linear models for dynamic analysis, and the success obtained by the substitute structure method motivated the use of such models for the investigation presented in this study.

The remainder of this chapter presents an approximate analytical method for predicting the stationary rms interfloor displacements of a bilinear hysteretic MDF system subjected to a white noise acceleration at the base. Linear substitute structure concepts are used. Two categories of substitute elements are investigated; namely simple spring-dashpot substitute elements and higher order substitute elements. Liapunov covariance matrix equations are used to solve for the stationary rms interfloor displacements.

The analysis used here consists of (1) determination of equivalent stiffness and damping for each element as a function of the rms ductility, (2) formulation and solution of the Liapunov covariance matrix equation, and (3) iterating the above procedure until a convergent set of rms interfloor displacements are obtained. Results are compared with the simulation data given in chapter II.
3.2 Simple Equivalent Linearization

Brinkmann\(^{14}\) used a simple equivalent linearization method to approximate the response statistics of a 2DF structure in which only one story exhibited nonlinear characteristics. The method was to find an equivalent simple linear stiffness and an equivalent damping for the yielding story of the 2DF structure, under the assumptions that the effects of the yielding can be completely determined by changing the stiffness and damping of the yielding story only, and that the relationships between the original and modified stiffness and damping for the yielding story are the same as for a SDF system.

Based on results for SDF systems, the ratio of the modified linear stiffness to the original stiffness was taken as

\[
\frac{k_e}{k_0} = 1 - (1 - \alpha) \exp \left( - \frac{\gamma^2}{2 \sigma^2} \right) \quad (3.20)
\]

where \(k_0\) is the initial stiffness of the yielding story, \(\alpha\) is the second slope of the bilinear stiffness, \(\gamma\) is the yielding level, and \(\sigma\) is the standard deviation of the distortion of the yielding story. From the power balance equation of the SDF yielding system given by Takemiya and Lutes\(^4\), the equivalent viscous damping for the yielding story was written as

\[
C_e = C_0 \left[ 1 + \frac{1}{\sqrt{2\pi}} \frac{1}{\beta_0} \frac{k_0}{k_e} (1-\alpha) \sqrt{\alpha} \frac{\gamma}{\sigma} \text{erfc} \left( \frac{\gamma}{\sqrt{2\sigma}} \right) \right] \quad (3.21)
\]
where \( C_e \) = equivalent viscous damping

\( C_0 \) = the original viscous damping of the yielding story

\( \text{erfc} \) = complementary error function

\( \beta_0 \) = critical damping ratio of SDF system

For the two-story structure, Brinkmann took \( \beta_0 \) to be the modal damping ratio, which was considered to be the same for both modes.

Having determined the equivalent stiffness and viscous damping for this yielding 2DF system, the governing differential equation of this vibrating equivalent linear system can be written in the form of equation (3-4). It is noted that the \( C \) matrix, in (3-4), for this equivalent linear system cannot be diagonalized and thus the governing equations (3-4) for this equivalent linear system cannot be uncoupled. Brinkmann, though, replaced the coupled damping matrix by an uncoupled matrix, with values based on equating the energy dissipated by the system with a coupled damping matrix to that of the uncoupled system. This approximate technique had previously been used by Penzien to uncouple vibration equations so that modal superposition could be used. In order to determine whether a significant error resulted from the diagonalization approximation, the writer determined the solution for the coupled system by using the Liapunov covariance equation (equations 3-15 to 3-19).

3.3 Results by Simple Equivalent Linearization Method

A typical structure with masses and stiffness shown in Table 1, and with \( \beta_0 = 0.05, \alpha = 0.1 \) was studied. Only the bottom story of the system was allowed to yield (the yield level of the top
story was taken to be $\infty$). The stationary mean squared values of distortion obtained by the approximate modal superposition method and from the Liapunov covariance matrix equation are shown in Fig. 1.1. It is noticeable that for this particular structure, diagonalization of the C matrix gave very low values for the mean squared distortion in the second story at intermediate yield levels, in comparison with the result obtained from the Liapunov covariance equations. Therefore, the diagonalization of the C matrix is inadequate for this system.

Results were also obtained by the writer for the same structure with second stiffness of $\alpha = 1/21$, by solving the Liapunov covariance matrix equation. Results are shown in Figures 2.3 to 2.6. In Figures 1.1, 2.3 and 2.4 (bottom story yielding), one can see that the predicted stationary mean squared interfloor displacement of the bottom story agrees well with the simulation, while the prediction of the response in the top story is quite poor.

Figures 2.5 and 2.6 show that when the top story is yielding, this simple equivalent linearization method gives good predictions of the mean squared interfloor displacements only when the yield level is quite high or low, and has significant errors for the intermediate yielding level systems.

Overall, Brinkmann's model is insufficient to predict the rms values of the yielding MDF system. A higher order equivalent linear system will be discussed in the following sections.
3.4 Third Order Equivalent Linear System For SDF Yielding Oscillator

The purpose of this section is to find an equivalence between a linear third order system (Figure 3.1) and a bilinear hysteretic yielding oscillator (Figure 1.3) when both are subjected to a stationary Gaussian random white noise excitation. It is noted that the Coulomb slider in Figure 1.3 is replaced by a dashpot connected by a spring as shown in Figure 3.1. The obvious difference between the Coulomb slider-spring element and the dashpot-spring element is that the force across the Coulomb slider is nonlinear, while that across dashpot is proportional to \( \dot{z} \). Hence the system shown in Figure 3.1 is linear.

The procedure used to find the equivalence is to choose the ductility dependent constants \( k_2 \) and \( c_2 \) in the linear system (Figure 3.1) such that the stationary rms values of displacement and velocity of this linear third order system will match those obtained by simulation for the bilinear hysteretic yielding oscillator subjected to the same Gaussian random white noise excitation. Figures 3.2 and 3.3 show normalized simulation values of rms displacement and velocity for a nearly elasto-plastic yielding system subjected to stationary Gaussian random white noise excitation. These particular figures are for \( \beta = 0 \), and the normalization factor is

\[
N = \sqrt{2} \frac{S_0}{\omega_1^3}
\]

Now consider the system shown in Figure 3.1, the governing equation of motion for this third order linear system can be written as

\[
\begin{align*}
m \ddot{x} + c \dot{x} + k_1 x + k_2 (x - z) &= -m \ddot{y}(t) \\
c_2 \dot{z} &= k_2 (x - z)
\end{align*}
\]

(3-22)
Intuitively, one can recognize that this system with $c_2 = \infty$ and $k_1 + k_2 = k$, corresponds to the original system with $Y = \infty$; both describing a second order linear system with a stiffness $k$. Similarly, the third order linear system with $c_2 = 0$, and $k_1 = \alpha k$ corresponds to the original system with $Y = 0$; both describing a linear system with the reduced stiffness $\alpha k$. Let

$$\frac{k_1}{m} = \omega_1^2$$

(3-23)

$$\frac{k_2}{m} = \omega_2^2$$

(3-24)

$$\frac{c}{m} = 2 \beta_0 \omega_0$$

(3-25)

and

$$r = \frac{\omega_0 c_2}{k_2}$$

(3-26)

where $\omega_1$ is the undamped natural frequency when $c_2 = 0$, $\omega_2$ is a measure of the increase in frequency due to $k_2$, $\beta_0$ is the damping ratio when $c_2 = \infty$, and $r$ is a dimensionless measure of the $c_2$ damping.

Then equation (3-22) becomes

$$\ddot{x} + 2 \beta_0 \omega_0 \dot{x} + \omega_1^2 x + \omega_2^2 (x - z) = -\ddot{f}(t)$$

$$x = \frac{r}{\omega_0} \dot{z} + z$$

(3-27)

Alternatively, (3-27) can be written as a differential equation of 1st
order as

\[
\frac{d}{dt} \begin{pmatrix} x \\ z \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{\omega_0}{r} & \frac{\omega_0}{r} & 0 \\ -(\omega_1^2 + \omega_2^2) & \omega_2^2 & -2\beta_0\omega_0 \end{pmatrix} \begin{pmatrix} x \\ z \\ \dot{x} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ddot{f}(t) \tag{3-28}
\]

Since (3-28) is an elastic system and \( \ddot{f}(t) \) is a Gaussian normal random "white noise" process, the mean values of \( x, z \) and \( \dot{x} \) can be shown to be zero. Let

\[
[g] = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{\omega_0}{r} & \frac{\omega_0}{r} & 0 \\ -(\omega_1^2 + \omega_2^2) & \omega_2^2 & -2\beta_0\omega_0 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ddot{f}(t) \quad (3-29)
\]

Then the covariance matrix \([\hat{S}]\) of the system satisfies the following equation, when the response is stationary:

\[
[g] \begin{bmatrix} \hat{S} \end{bmatrix} + \begin{bmatrix} \hat{S} \end{bmatrix} [g]^T + [B] = 0 \tag{3-30}
\]

where

\[
[\hat{S}] = \begin{pmatrix} E[x^2] & E[xz] & E[x\dot{x}] \\ E[xz] & E[z^2] & E[\dot{x}z] \\ E[\dot{x}x] & E[\dot{x}z] & E[\dot{x}^2] \end{pmatrix} \tag{3-31a}
\]
\[ [B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\pi S_0 \end{bmatrix} \quad (3-31b) \]

and \( S_0 \) is the power spectral density function of \( \dot{f}(t) \). Expanding (3-30), the following equation can be obtained,

\[
\begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 0 \\
\frac{\omega_0}{r} & -\frac{\omega_0}{r} & 0 & 0 & 1 & 0 \\
0 & 2\frac{\omega_0}{r} & -2\frac{\omega_0}{r} & 0 & 0 & 0 \\
-(\omega_1^2 + \omega_2^2) & \omega_2^2 & -2\beta_1 \omega_1 & 0 & 1, & 0 \\
0 & -(\omega_1^2 + \omega_2^2) & \omega_2^2 + \frac{\omega_0}{r} & -(\frac{\omega_0}{r} + 2\beta_1 \omega_1) & 0 & 0 \\
0 & 0 & 0 & -2(\omega_1^2 + \omega_2^2) & 2\omega_2^2 & -4\beta_0 \omega_0 \\
\end{bmatrix}
\begin{bmatrix}
\text{E}[x^2] \\
\text{E}[xz] \\
\text{E}[z^2] \\
\text{E}[x\dot{x}] \\
\text{E}[xz] \\
2\pi S_0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} = 0
\]

(3-32)

From (3-32) the covariance terms are obtained as follows

\[
\text{E}[x^2] = \frac{\pi S_0 \left[ 1 + 2\beta_0 r + \omega_1^2 \left( \frac{r}{\omega_0} \right)^2 \right]}{\omega_1^2 \left\{ \omega_0^2 \omega_2^2 + 2 \beta_0 \omega_0 \left[ 1 + 2 \beta_0 r + \omega_1^2 \left( \frac{r}{\omega_0} \right)^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right] \right\}}
\]

(3-33)

\[
\text{E}[\dot{x}^2] = \frac{\pi S_0 \left[ 1 + 2\beta_0 r + 4\omega_1^2 \left( \frac{r}{\omega_0} \right)^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right]}{\omega_2 \omega_0 \left\{ \omega_0^2 \omega_2^2 + 2 \beta_0 \omega_0 \left[ 1 + 2 \beta_0 r + \omega_1^2 \left( \frac{r}{\omega_0} \right)^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right] \right\}}
\]

(3-34)

\[
\text{E}[xz] = \frac{\pi S_0 \left[ 1 + 2\beta_0 r \right]}{\omega_1^2 \omega_2 \omega_0 \left\{ \omega_0^2 \omega_2^2 + 2 \beta_0 \omega_0 \left[ 1 + 2 \beta_0 r + \omega_1^2 \left( \frac{r}{\omega_0} \right)^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right] \right\}}
\]

(3-35)
\[ E[z^2] = E[xz] \quad (3-36) \]

\[
E[xz] = -\pi S_0 \cdot \frac{r}{\omega_0} \left\{ \omega_2^2 \frac{r}{\omega_0} + 2 \beta_0 \omega_0 \left[ 1 + 2 \beta_0 r + \omega_1^2 \left( \frac{r}{\omega_0} \right)^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right] \right\} \quad (3-37)
\]

\[ E[xx] = 0 \quad (3-38) \]

Since the second slope of the stiffness in the bilinear hysteretic system is \( \alpha k \), let

\[ k_1 = \alpha k \quad (3-39) \]

\[ \omega_1 = \sqrt{\alpha} k \quad (3-40) \]

then (3-33) and (3-34)

\[
E[x^2] = \frac{\pi S_0 (1 + 2 \beta_0 r + \alpha r^2)}{\omega_0^2 \alpha \left\{ \omega_2^2 \frac{r}{\omega_0} + 2 \beta_0 \omega_0 \left[ 1 + 2 \beta_0 r + \omega_1^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right] \right\} } \quad (3-41)
\]

\[
E[x^2] = \frac{\pi S_0 [1 + 2 \beta_0 r + \alpha r^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2]}{\left\{ \omega_2^2 \frac{r}{\omega_0} + 2 \beta_0 \omega_0 \left[ 1 + 2 \beta_0 r + \alpha r^2 + \omega_2^2 \left( \frac{r}{\omega_0} \right)^2 \right] \right\} } \quad (3-42)
\]

Hsieh(35) obtained the same results by integration of the power spectral density functions of displacement and velocity. Finding the third order linear system which is equivalent to the bilinear hysteretic
yielding system for a particular yield level \( Y \), then consists of solving equations (3-41) and (3-42) for the equivalent constant \( \omega_2^2 \) and \( r \) as functions of the \( E[x^2] \) and \( E[\dot{x}^2] \) values read from Figures 3.2 and 3.3. Since both \( E[x^2] \) and \( E[\dot{x}^2] \) vary as functions of the yield level \( Y \), it is clear that \( \omega_2^2 \) and \( r \) will also depend on \( Y \). Let

\[
\omega_a^2 = \frac{E[\dot{x}^2]}{E[x^2]} \tag{3-43}
\]

Solving (3-41) and (3-42) gives

\[
r = \frac{2 \left( \omega_a^2 - \alpha \omega_0^2 \right) E[\dot{x}^2]}{\omega_a \left\{ \pi \omega_0^2 N^2 - 4 \beta_0 E[\dot{x}^2] \right\}} \tag{3-44}
\]

or

\[
r = \frac{2 \left( \omega_a^2 - \alpha \omega_0^2 \right) E[\dot{x}^2]}{\pi N^2 \omega_0^2 - 4 \beta_0 \omega_a^2 E[\dot{x}^2]} \tag{3-45}
\]

and

\[
\omega_2^2 = \frac{(1 + 2 \beta_0 r + \alpha r^2) (\omega_a^2 - \alpha \omega_0^2)}{\alpha r^2} \tag{3-46}
\]
Curves A and C in Fig. 3.4 show the values of the dimensionless parameters $r$ and $\omega_2^2 / \omega_0^2$ determined from equations (3-45) and (3-46) using the response values in Figures 3.2 and 3.3 for the undamped yielding system. Values for 5% damping showed that $r$ and $\omega_2^2 / \omega_0^2$ are almost unaffected by this change in viscous damping. The only significant change is shown in curve B of Fig. 3.4, where in the region $1.0 < Y/\sigma < 3.0$, the $\omega_2^2 / \omega_0^2$ value for 5% damping slightly deviates from that for the undamped system. It is obvious that $r$ and $\omega_2^2 / \omega_0^2$ are simply functions of the rms ductility factor $\sigma/Y$, if one neglects the slight shifting of curve A in the region $1.0 < Y/\sigma < 3.0$.

It is the purpose of this study to use the values of $r$ and $\omega_2^2 / \omega_0^2$ obtained from a SDF yielding bilinear hysteretic system to determine a linear substitute structure for a MDF bilinear hysteretic system subjected to the same Gaussian normal random white noise acceleration of the base. A similar third order element is used to replace the spring of each yielding story. In other words, the original bilinear hysteretic yielding spring of the yielding story is replaced by a linear spring (stiffness $k_1 = \alpha k$) and an equivalent linear spring (stiffness $k_2 = (\omega_2 / \omega_0)^2 k$) connected by a dashpot (viscous damping $c_2$) without changing the original viscous damper. Note that $c_2$ can be determined by (3-25) in which $\omega_0$ will be approximated by the apparent frequency of the yielding story. More details about this apparent frequency will be discussed in the next section.
3.5 Apparent Frequency of Vibration

The frequency of the interstory displacement of the MDF system is more complicated than that of the SDF vibrating system. More or less the higher modes contribute to the frequency of the interstory displacement, when the excitation is white noise acceleration of the base. Wen (41) considered the frequency of the interfloors displacement to be the apparent frequency $\frac{\sigma \dot{u}_i}{\sigma u_i}$, where $\sigma \dot{u}_i$ is the rms interfloor velocity of the $i$th story and $\sigma u_i$ is the rms interfloor displacement of the $i$th story. This apparent frequency can be shown to be exactly the same as the frequency obtained from the rate of zero crossing for any normal process.

It is also the natural frequency of a SDF linear system excited by a normal white noise excitation. This apparent frequency $\frac{\sigma \dot{u}_i}{\sigma u_i}$ will be used in this study to substitute for $\omega_0$ in (3-26) for finding the equivalent viscous damper, $(C_e)_i$, of the $i$th story. It is noted that $\sigma \dot{u}_i$ and $\sigma u_i$ can be obtained from equation (3-19) for the elastic situation. Thus

$$\left( C_e \right)_i = r_i \left( k_e \right)_i \left( \frac{\sigma u_i}{\sigma \dot{u}_i} \right)_{\text{elastic}} \quad (3-47)$$

where

$C_e$: equivalent to $C_2$ in (3-26)

$k_e$: equivalent to $k_2$ in (3-26)

$r$: as defined in (3-26)
3.6 **Linear Substitute Structure for Yielding MDF Structure**

Fig. 3.5 shows a higher order linear system which is assumed to be equivalent to a 2DF structure with yielding in the bottom story. The original stiffnesses of the structure are, respectively, $k_1$ and $k_2$ for the bottom story and the second story, while $c_1$, $c_2$, and $c_3$ are the viscous dampers required to give $\beta_0$ modal damping in both modes of the unyielded structure (see equation 2.10). It is postulated that if the bottom story is yielding then the nonlinear element of the bottom story can be replaced by a spring with a reduced stiffness $\alpha k_1$ and a dashpot-spring system whose constant $k_{e1}$ is dependant on the expected ductility factor $\frac{\sigma}{\sigma_{u1}} / \gamma_1$ and the apparent frequency $\omega_1$ of the bottom story of the unyielded structure. Thus $k_{e1}$ and $C_{e1}$ can be computed as follows

\[
K_{e1} = k_{i} \cdot \left( \frac{\omega_{2}}{\omega_{0}} \right)_{i}, \quad i=1 \quad (3-48)
\]

\[
C_{e1} = k_{e1} \cdot \left( \frac{\sigma_{u1}}{\sigma_{u1}} \right) . r_{i}, \quad i=1 \quad (3-47)
\]

where $r_i$ and $\left( \frac{\omega_{2}}{\omega_{0}} \right)^2$ are read from the curves shown in Figure 3.4. Similarly, Figure 3.6 shows the equivalent higher order linear system for the same 2DF structure with yielding in the top story. Figure 3.7 describes a higher order linear system corresponding to a 2DF structure which may be yielding in both stories, and Figure 3.8 is the higher order linear system substituted for a 4DF structure yielding in every story. The viscous dampers of the elastic system are obtained from equation (2-10) for given modal damping ratios.
3.7 Liapunov Covariance Matrix Differential Equation for Linear Substitute Structure

The governing equation of motion of the linear substitute system (see figure 3.8) for an n story bilinear hysteretic yielding structure can be written as

\[
\begin{bmatrix}
M & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{v}_1 \\
\dot{v}_n
\end{bmatrix}
+ \begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix}
\begin{bmatrix}
\ddot{v}_1 \\
\ddot{v}_n
\end{bmatrix}
+ \begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix}
\begin{bmatrix}
\ddot{v}_1 \\
\ddot{v}_n
\end{bmatrix}
= \begin{bmatrix}
M & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{v}_1 \\
\dddot{v}_n
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\dddot{f}(t)
\]  
(3-49)

where

\[\begin{align*}
v_1 &= x_1 \\
v_2 &= x_2 \\
\vdots &= \vdots \\
v_n &= x_n \\
v_{n+1} &= z_1 \\
\vdots &= \vdots \\
v_{2n} &= z_n
\end{align*}\]  
(3-50)

\(x_j\) and \(z_j\) are displacements relative to the base.

\[
M = \begin{bmatrix}
m_1 & 0 \\
m_2 & \ddots \\
0 & \ddots & \ddots \\
& \ddots & \ddots & m_n
\end{bmatrix}
\]
\( C_{11} \) : viscous damping matrix obtained from (2-10) for given modal damping ratios of the elastic system

\( C_{12} \) : null matrix of order \( n \times n \)

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
-c_{e2} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -c_{en} & 0
\end{bmatrix}
\]

(3-51)

\[
\begin{bmatrix}
c_{e1} & 0 & \cdots & 0 & 0 \\
0 & c_{e2} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & c_{en}
\end{bmatrix}
\]

(3-52)

\[
\begin{bmatrix}
k_{e1}+\alpha(k_1+k_2), & -\alpha k_2-k_{e2}, & \cdots, & 0, & 0 \\
-\alpha k_2, & k_{e2}+\alpha(k_2+k_3), & \cdots, & 0, & 0 \\
0, & -\alpha k_3, & \cdots, & 0, & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & \cdots, & -\alpha k_n, & \alpha k_n+k_{en}
\end{bmatrix}
\]

(3-53)

\[
\begin{bmatrix}
-k_{e1} & k_{e2}, & \cdots, & 0, & 0 \\
0, & -k_{e2}, & \cdots, & 0, & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & \cdots, & 0, & -k_{en}
\end{bmatrix}
\]

(3-54)
\[ K_{21} = \begin{bmatrix} -k_{e1} & 0 & \cdots & 0 \\ 0 & -k_{e2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -k_{en} \end{bmatrix} \] (3-55)

and \[ K_{22} = -K_{21} \] (3-56)

Introducing new variables \( y_i \)

\[
y_1 = v_1, \quad y_2 = v_2, \quad \ldots, \quad y_n = v_n
\]

\[
y_{n+1} = \dot{v}_{n+1}, \quad y_{n+2} = \dot{v}_{n+2}, \quad \ldots, \quad y_{2n} = \dot{v}_{2n}
\] (3-57)

\[
y_{2n+1} = \ddot{v}_1, \quad y_{2n+2} = \ddot{v}_2, \quad \ldots, \quad y_{3n} = \ddot{v}_n
\]

then equation (3-49) can be reduced to a set of \( 3n \) equations of the first order. They can be written concisely in the following form.

\[
\frac{d}{dt} Y = g \dot{Y} + F
\] (3-58)

where

\[
Y^T = \{ y_1, y_2, \ldots, y_{2n}, \ldots, y_{3n} \}
\] (3-59)

\[
F^T = -\{ 0, 0, \ldots, 0, 1, 1, \ldots, 1 \} \tilde{f}(t)
\] (3-60)

\[
g = \begin{bmatrix} 0 & 0 & I_{nxn} \\ g_{21} & g_{22} & g_{23} \\ -M^{-1}K_{11} & -M^{-1}K_{12} & -M^{-1}C_{11} \end{bmatrix}
\] (3-61)
\[
g_{21} = \begin{bmatrix}
\frac{k_{e1}}{c_{e1}} & & & \\
& \frac{k_{e2}}{c_{e2}} & & \\
& & 0 & \\
& & & \frac{k_{en}}{c_{en}}
\end{bmatrix}
\quad \quad \quad \quad \quad \quad
\]
\[
g_{23} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]
\[
g_{22} = -g_{21}
\]

where \(I_{n \times n}\) is an identity matrix of order \(n\) and the \(g\) matrix is of order \(3n \times 3n\). The excitation \(\ddot{f}(t)\) is a stationary Gaussian white noise random acceleration and the system is linear, therefore the mean displacements are zero

\[
E[Y_i] = 0, \quad i = 1, 2, \cdots, 3n
\]

(3-62)

The system (3-58) will start to vibrate from rest and the mean squared displacements and velocities build up gradually. When time is large the stationary mean squared values of the responses will satisfy the following Liapunov covariance matrix equation.

\[
g \hat{S} + \hat{S} \hat{G}^T + B = 0
\]

(3-63)

in which
\[
\hat{S} = \begin{bmatrix}
E[y_1^2], E[y_1y_2], \ldots, E[y_1y_{3n}] \\
E[y_1y_2], E[y_2^2], \ldots, E[y_2y_{3n}] \\
\vdots \\
E[y_{3n}y_1], E[y_{3n}y_2], \ldots, E[y_{3n}^2]
\end{bmatrix}
\]

(3-64)

and

\[
B = 2 \pi S_0 \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{n \times n}
\end{bmatrix}
\]

(3-65)

where \(S_0\) is the power spectral density function of \(\dot{y}(t)\). The covariance of the displacements can be obtained by solving (3-63), and the mean squared values of the story distortions are obtainable with the aid of the relation

\[
E[u_i^2] = E[x_i^2] + E[x_{i-1}^2] - 2 E[x_i x_{i-1}]
\]

(3-66)

Note in (3-66), for \(i=1\), \(E[x_0^2] = 0\) and \(E[x_1 x_2] = 0\). A computer program has been developed (see Appendix C) to compute the mean squared inter-story displacements of the system described by (3-49), given the masses, original stiffnesses, modal damping ratios and the parameters \(K_{ei}\) and \(C_{ei}\).

It is worthwhile to note that though (3-63) is very compact, a large computer is required to solve the equation. For a 10DF
structural system, (3-63) will give 465 simultaneous equations. Approximately two minutes is required to solve these 465 equations with ITEL, IBM 370 computer.

To reduce the computation time, equation (3-57) and (3-61) are rewritten for some special systems considered in this study. For example, for a 1ODF structure with yielding only in the fifth story, the matrices $g$ and $Y$ are rewritten as

$$
g = \begin{bmatrix}
0 & 0 & \cdots & I_{nxn} \\
-g_{21} & g_{22} & g_{23} \\
M^{-1}K_{11} & M^{-1}K_{12} & -M^{-1}C
\end{bmatrix} \quad (3-67)
$$

where

$$
g_{21} = \begin{bmatrix} 0, 0, 0, 0, \frac{ke_5}{ce_5}, 0, 0, 0, 0 \end{bmatrix}
$$

$$
g_{22} = -g_{21}
$$

$$
g_{23} = \begin{bmatrix} 0, 0, 0, 1, 0, 0, 0, 0, 0 \end{bmatrix}
$$

and

$$
Y^T = \begin{bmatrix} x_1, x_2, \ldots, x_{10}, z_5, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{10} \end{bmatrix} \quad (3-68)
$$

where $k_{11}$ and $k_{12}$ become
\[
\begin{bmatrix}
    k_1 + k_2, & -k_2, \\
    -k_2, & k_2 + k_3, & -k_3, \\
    & -k_3, & k_3 + k_4, & -k_4, \\
    & & -k_4, & \alpha k_5 + k_4, & -k_e + \alpha k_5, \\
    & & & -\alpha k_5, & \alpha k_5 + k_e + k_6, & -k_6, \\
    & & & & -k_6, & k_6 + k_7, & -k_7, \\
    & & & & -k_7, & k_7 + k_8, & -k_8, \\
    0 & & & & -k_8, & k_8 + k_9, & -k_9 \\
    & & & & -k_9, & k_9 + k_{10}, & -k_{10} \\
    & & & & -k_{10}, & k_{10} \\
\end{bmatrix}
\]

(3-69)

and

\[
k_{12}^T = - \begin{bmatrix} 0, 0, 0, k_e, k_e, 0, 0, 0, 0 \end{bmatrix}
\]

(3-70)

The number of variables in (3-68) is decreased to 21. Hence only 231 simultaneous equations are obtained from (3-63) and it took only 10 seconds to solve these equations.

Combining equation (3-61) to (3-66), the rms values of the interfloor displacements can be computed. However, recall that \(c_{e_i}, k_{e_i}\) in (3-61) are dependent on the rms value of the \(i^{th}\) interfloor displacement (see 3-47 and 3-48), therefore iteration is required to predict the rms values of the interfloor displacements.

3.8 Comparison of Numerical Results with Simulation Data

All the systems simulated in Chapter II have also been
investigated by use of the higher-order linear substitute structure concept. The results are shown, along with the simulation data, in Figs. 2.3 to 2.27.

As noted in section 3.3, the Type A 2DF system was also analyzed by the simple equivalent linearization concept. From Figs. 2.3 to 2.6 one can note that the results from the higher-order linearization are generally in somewhat better agreement with the simulation data, than are the simple linearization results. In some situations the improvement is quite significant. For example, when $\beta = 5\%$ and yielding is restricted to the top story (Fig. 2.5) the simple linear system predicts a very abrupt onset of yielding at $Y/N = 3$, whereas the simulation data and the higher-order linear system results show a much more gradual transition.

From Figs. 2.3 to 2.10 one can conclude the results of the higher-order system exhibit all the significant trends of the simulation data for 2DF systems. Monotonic simulation curves are matched by monotonic curves for the higher-order system; as are curves with distinct low points. As mentioned above, the fairly gradual onset of yielding in Fig. 2.5 is well matched, but so also is the rather abrupt onset of yielding in story 2 in Figs. 2.7 and 2.8. Overall, the ability of the higher-order model to predict the shape of the various curves seems to be quite remarkably good.

The higher-order model does not, of course, predict results which agree exactly with the simulation data. In fact the differences are quite significant in some instances, but the largest percentage errors generally are associated with the smaller (and therefore less significant) story distortions. For example, in Figs. 2.3 and 2.4 the large response in story 1 is predicted quite accurately, while the
much smaller response of story 2 sometimes has an error of 35%. In fact, this error in the upper story of a structure with yielding only in the bottom story was the largest error found for 2DF systems.

The general conclusions drawn for 2DF systems are also seen to apply to the 4DF systems in Figs. 2.11 to 2.14. The general shapes of the curves from the higher-order system agree with the simulation data. The percentage errors are small for the larger responses of stories 1 and 2 and sometimes are large (as much as 60%) for the smaller responses of stories 3 and 4. These same conclusions also apply to the 10DF results in Figs. 2.15 to 2.27. Significant percentage errors (as for 4DF systems) sometimes occur in the upper stories of structures with all stories yielding (for example Figs. 2.25 d, e and f). The poorest prediction of the root mean squared interfloor displacement is in the top story as shown in Figure 2.25(f). The predicted value is 39% off the simulated response level for this story, but that level, in turn, is only one tenth of that for the bottom story. The results are generally better when only one or two stories are yielding.

Overall it appears that the higher-order linear substitute system works quite well. It predicts the proper trends both in terms of $\sigma_{u_i}/N$ versus $Y/N$, and also in terms of the distribution of $\sigma_{u_i}$ over the height of the building. The larger percentage errors are generally for small responses, while the more significant responses are usually relatively accurate.
IV DIGITAL COMPUTER SIMULATION FOR NONSTATIONARY RESPONSE

4.1 Introduction

The response of a structure to an earthquake-like excitation generally starts from rest, so there is a definite interval of time during which the response is clearly transient as it builds up. This chapter presents simulation results for the build-up of rms interfloor displacements for a class of bilinear hysteretic MDF yielding systems.

The structures investigated herein include the 2DF, 4DF, and 10DF systems for which the stationary rms interfloor displacements have been studied in chapter II. The excitation considered is stationary Gaussian white noise acceleration of the base. Characteristics of the structures are shown in Tables 1 to 6.

Of course, a real earthquake acceleration is not white noise. However, a fairly reasonable model of an earthquake acceleration can be produced by passing white noise through high frequency and low frequency filters\(^{(34)}\). A study of the transient response to this simple white noise excitation should, therefore, at least in a qualitative sense, give a considerable insight into the transient response to an earthquake loading.

4.2 Statistical Accuracy

It is important to investigate how the accuracy of the simulation values of rms interfloor displacement will be affected by the size of the ensemble of samples used in the simulation. Note that this problem of ensemble size did not arise for simulation of stationary response levels, since time averages from a single sample were used
to approximate expectations in that situation.

Let \( U \) be one of the distortions of the bilinear hysteretic MDF yielding system. An approximation of the statistical accuracy can be obtained by considering \( U \) to be normal. Of course one should not anticipate that the response of the nonlinear system would be strictly normal, even though the response of a linear system with normal excitation is normal. Results obtained by Shah\(^{(33)} \) for the SDF bilinear hysteretic system with normal white noise excitation did show that the response was approximately normal. Thus the normal approximation will also be used here for the MDF yielding system.

At a certain time \( t \), let the ensemble of sample values for a given story distortion be \( U_1, U_2, \ldots, U_n \). Since the mean of the white noise excitation is taken as zero, one also has a zero mean value for \( U \). Thus \( E[U^2] \) and \( \sigma_u^2 \) are the same and the usual estimator for this unknown quantity is

\[
S^2 = \frac{1}{n} \sum_{i=1}^{n} U_i^2
\]

(4-1)

Taking expectation on both sides, one obtains

\[
E[S^2] = \sigma_u^2
\]

(4-2)

where \( \sigma_u^2 \) is the variance of \( U \). Equation (4-2) shows that \( S^2 \) is an unbiased estimator of \( \sigma_u^2 \). Dividing (4-2) by \( \sigma_u^2 / n \), one obtains
\[
\frac{n}{\sigma_u^2} s^2 = \sum_{i=1}^{n} \frac{U_i^2}{\sigma_u^2}
\]  

(4-3)

In (4-3) the \( U_i \) are independent and \( U_i / \sigma_u \) has a standard normal distribution. Therefore \( n s^2 / \sigma_u^2 \) has a \( \chi^2 \) distribution with \( n \) degrees of freedom. The mean and variance of the \( n \) degree of freedom \( \chi^2 \) distribution are known to be \( n \) and \( 2n \) respectively. One can further show, by the central limit theorem, that \( n s^2/\sigma_u^2 \) tends to a normal distribution with mean = \( n \) and variance = \( 2n \), as \( n \) becomes large (say \( n \geq 30 \)). In other words, \( s^2 \) approaches a normal distribution with mean and variance as

\[ E[S^2] = \sigma_u^2 \]  

(4-4)

and

\[ \text{Var}[S^2] = \frac{2}{n} \sigma_u^4 \]  

(4-5)

when \( n \) tends to infinity. In this study 50 samples or 150 samples are used in finding the ensemble average \( S^2 \). Therefore \( S^2 \) considered herein essentially has a normal distribution

To find the effect of ensemble size on the accuracy of the rms interfloor displacement, one can use the relationship between the ensemble size \( n \) and the confidence level \( C \). One can say with confidence \( C \) that the true mean squared value of \( \sigma_u^2 \) is in a given interval \( S^2 - d \leq \sigma_u^2 \leq S^2 + d \); where \( d \) is half the confidence interval, which is dependent on the confidence \( C \) and the standard deviation of \( \sigma_u \) (or say with
confidence C that the true rms interfloor displacement is in a given interval \( \sqrt{S^2-d} \leq \sigma_u \leq \sqrt{S^2+d} \).

For example, when \( n = 150 \), \( S^2 \) has a normal distribution with standard deviation 0.1155 \( S^2 \) ( \( \sigma^2 \) is substituted by \( S^2 \)). With confidence \( C = 90\% \), \( \sigma_u^2 \) is in the interval 0.81 \( S^2 \leq \sigma_u^2 \leq 1.19 \) \( S^2 \).

Similarly, for \( n = 50 \), \( S^2 \) has a normal distribution with standard deviation of 0.2 \( S^2 \). One can have 90\% confidence that \( \sigma_u^2 \) is in the interval 0.67 \( S^2 \leq \sigma_u^2 \leq 1.329 \) \( S^2 \) and 95\% confidence that \( \sigma_u^2 \) is in the interval 0.608 \( S^2 \leq \sigma_u^2 \leq 1.392 \) \( S^2 \).

The width of the confidence interval indicates that a large ensemble size would be desirable if a simple ensemble average is used to estimate \( \sigma_u^2 \). Note that doubling the ensemble size (and consequently doubling the computer costs) would only reduce the standard deviation of \( S^2 \) by about 30\%.

In this study, the standard deviation of the average response \( \sigma_u^2 \) was reduced by using a combination of time and ensemble averages, rather than a simple ensemble average. A time average of \( U(t) \) was taken over the time interval \( t - \pi/(2\omega_1) \leq t \leq t + \pi/(2\omega_1) \) before the ensemble average (see equation (4-2)) was taken. Note that \( \omega_1 \) is the fundamental natural frequency of the MDF elastic system, so that the time interval used for averaging corresponds to one half cycle of the fundamental mode. The most obvious advantage of this combination average is that if the response is dominated by the fundamental mode then the averaging eliminates the effect of the random phase of the various members of the ensemble. The width of the confidence interval is not
known for these combination averages, but empirical evidence shows that they are much less variable than simple ensemble averages.

4.3 Integration Scheme

The equation of motion for the bilinear hysteretic MDF yielding system is

\[
M \ddot{X} + C \dot{X} + K X = -M \ddot{f}(t) + \ddot{g}
\]  

(2-18)

where matrices M, C, K, and \( \ddot{g} \) are defined in Chapter II. The system is assumed to be initially at rest then suddenly excited by a Gaussian, stationary random white noise acceleration at the base. Equation (2-18) is modified (see equation 2-22) such that the direct numerical integration procedure can be used. Two per cent of the fundamental mode period of the elastic structure has been used as the integration time interval. It is assumed that within this time interval the stiffness of each story is constant, being either the initial stiffness or the second slope stiffness. The excitation is a pulse like force applied at the beginning of a certain number of the integration steps. The amplitude of the impulse force is a random number generated by the normal random number generator. Fifty or one hundred and fifty time histories of the interfloor displacements of this yielding structure, with different random impulse forces, are generated. The estimator of the mean squared value of a certain interfloor displacement, is then taken as the ensemble average of the time average of each time history at time \( t \). Note that the time average is taken over a given time
interval $t - \pi/(2\omega_f) \leq t \leq t + \pi/(2\omega_f)$, in which $\omega_f$ is the fundamental period of the system with small response.

4.4 Results and Discussion

Figures 4.1 to 4.41 present the normalized nonstationary rms interfloor displacements. Fewer situations are considered than for stationary response because of increased cost of computer simulation, as well as the number of figures required to show the results. Note that the curves are presented in some detail for 2DF systems, in much less detail for 4DF systems, and only one simple 10DF system is studied. Simulation results are shown by curve A in each figure. Analytical curves in Figures 4.1 to 4.41 will be discussed in chapter V. They are in this chapter only to avoid reploting all the simulation results again in chapter V, for the purpose of comparing with the analytical results. The purpose of this section is only to present and discuss the simulations, so the analytical curves can be ignored for now.

Figures 4.1 to 4.4 show the simulation results for the Type A, 2DF system with modal damping ratios of 5% each. It is assumed that the top story is kept elastic (yield level $Y = \infty$). Fig. 4.1 shows that it takes about three times the fundamental period of the elastic system for the response to build up. For a moderately yielding system it is shown in Figure 4.2 that the response of the yielding story builds up a little more even after a long time interval of nearly stationary vibration. Figures 4.3 and 4.4 show the nonstationary responses for the low yield level system. Curves obtained from ensemble average of 50
samples (shown by A) are still fluctuating after the response has built up. This may be attributed to the increase of the fundamental period which reduces the advantages of the combination average. Fortunately, curves from combination averages of 150 samples (shown by \( \bar{A} \)) show more stationarity. Figures 4.5 to 4.8 show similar results for the same structure with modal damping ratios of 1% each.

Figures 4.9 to 4.12 show the simulation results for Type A, 2DF system with modal damping ratios of 5% each. It is idealized that only the top story of the system can be yielding. Figures 4.13 to 4.17 show the simulation results for the system (uniform stiffness, top story yielding) with modal damping ratios of 1% each. Looking at Figures 4.9 and 4.13, one can see that the response of the system with less viscous damping builds up more slowly than that of the system with higher viscous damping, when there is relatively little yielding taking place.

Figures 4.18 to 4.26 present the simulation results for the responses of Type B, 2DF structures. Similar to Figure 4.3, Curve A (combination average of fifty samples) in Figure 4.25 fluctuates after the response has built up, while curve \( \bar{A} \) (combination average of 150 samples) is more stable.

Curves \( A_1 \) in Figures 4.27 to 4.29 are the simulation results for Type A, 4DF structures with modal damping ratios of 5% each. In Figure 4.27, all stories of the structure are elastic. In Figures 4.28 and 4.29 only the fourth story remains elastic and all other stories are more or less yielding. Note that the response of the fourth story builds up very fast, while those of the other three stories build up
rather slowly, especially those for the moderately yielding stories (stories 2 and 3). Figures 4.30 to 4.32 give the similar results for the Type A, 4DF structure with damping ratios of 1% each. In Figure 4.31 the structure has a moderate yield level which is the approximate average over the height of the standard deviation of the stationary responses of the elastic system to the same excitation. Figure 4.32 shows the response for a rather low yielding level structure. Again, the responses of the moderately yielding stories (stories 2 and 3) build up very slowly.

The simulation results for the Type B, 4DF structure are presented by curves $A_1$, $A_2$, $A_3$, $A_4$ in Figures 4.33 to 4.38. Modal damping ratios of 5% and 1% are considered. Figures 4.39 to 4.41 present the simulation results for the Type A 10DF structure with 5% modal damping ratios. Figure 4.39 shows the transient responses for the elastic system. Figures 4.40 and 4.41 present the normalized transient rms interfloor displacements of the structure under the assumption that only the bottom story can be yielding. Looking at these two figures, one can see that the response of the yielding first story builds up more slowly than do the other elastic stories.

It is concluded that simulation curves are all fairly smooth when the system is elastic (see Figures 4.1, 4.5, 4.18, 4.22, 4.27, 4.30, 4.33, 4.36, 4.39); and when yielding is restricted to only one story, the curve for that story is generally much more erratic than those for the elastic stories (see Figures 4.3, 4.6 to 4.8, 4.9 to 4.12, and 4.14 to 4.17). In fact, curves for the elastic stories are usually quite smooth (except Fig 4.9). When the yielding level
is quite low, the curves from an ensemble of size 150 show less fluctuation than those from an ensemble of size 50 (see Figures 4.3, 4.4, 4.7 and 4.8). Note that for a certain yield level, the responses of the elastic stories generally build up faster than those of the yielding stories.

The time for responses to reach some fixed fraction of their stationary levels is affected by the viscous damping $\beta_0$ for the elastic structure, and by the yielding. The responses of systems with greater $\beta_0$ build up more rapidly than those with smaller $\beta_0$. (see Figures 4.1 and 4.5). The yielding increases the hysteretic damping and thus may tend to reduce the time for the response to reach the stationary level. On the other hand, yielding also increases the period of the system and hence may tend to increase the time for the response to build up. Figures 4.5 to 4.8, for example, show that the effect due to the increase of hysteretic damping is more pronounced than the effect due to the increase of the period.
5.1 Introduction

It will be desirable to begin by summarizing some important results concerning the nonstationary build-up of response of elastic systems, and some approximate methods for predicting the nonstationary response of a SDF yielding system.

Consider the system described by (3-1)

\[ \ddot{X} + 2\beta_0\omega_0 \dot{X} + \omega_0^2 X = - \ddot{f}(t) \]  

(3-1)

Caughey and Stumpf \(^{(24)}\) first obtained the mean squared level of transient response of this system initially at rest and then subjected to a stationary white noise excitation having a power spectral density of \(S_0\) as

\[ E[X^2(t)] = \frac{\pi S_0}{2\beta_0\omega_0^3} \left(1 - e^{-2\beta_0\omega_0 t}\left[ \frac{\omega_0^2}{p^2} - \frac{\beta_0^2\omega_0^2}{p^2} \cos 2p_0 t + \frac{\beta_0\omega_0}{p_0} \sin 2p_0 t \right] \right) \]

(5-1)

For a linear \(n\) degree-of-freedom system the equation of motion can be written as follow:

\[ [M] \{\ddot{X}\} + [C] \{\dot{X}\} + [K] \{X\} = - [M] \{1\} \ddot{f}(t) \]

(3-4)

The transient mean squared value of the interfloor displacement of this system to a white noise excitation having power spectral density of \(S_0\)
is given by (3-12), or can be obtained by direct numerical integration of (3-17). It can be expected that the transient build-up of the distortion of each story will contain oscillatory terms, as well as tend exponentially to the stationary value.

For a yielding SDF system, since the response builds up from rest, the stiffness of the structure is initially elastic and then switches back and forth between the initial stiffness and the yielding stiffness after the response exceeds the yielding level. It is difficult to find an exact theoretical transient response for this yielding system. Some equivalent linearization schemes based on matching the stationary response of the bilinear hysteric systems were used by Lutes and Shah\(^{(3)}\) to predict transient responses. Results from these equivalent linear methods agreed quite well with the simulation data for the moderately nonlinear systems (\(\alpha = 0.5\)), however significant discrepancies sometimes occurred for nearly elasto-plastic systems (\(\alpha = 1/21\)). An equivalent linear system with only an equivalent damping (\(\omega = \omega_0\) and \(\beta_e\) determined by matching the stationary response of displacement) underestimated the rate of build-up of the transient response. Another equivalent linear system with reduced stiffness as well as increased damping (\(\beta_e\) and \(\omega_e\) determined by matching the stationary response of displacement and velocity simultaneously) gave transient displacements which built up too rapidly, while the velocity built up too slowly.

A variable coefficient (\(\beta_e\) and \(\omega_e\)) equivalent linear scheme was also used by Lutes and Shah\(^{(3)}\). The parameters \(\omega_e\) and \(\beta_e\) were considered constants only within each integration time interval. Transient response levels obtained by this scheme agreed very well with
the simulation data for both the moderately bilinear hysteretic system and the nearly elasto-plastic system.

Other equivalent linear systems have included a third order linear system and a two mode linear system investigated by Hsieh \((35)\) for the SDF yielding system. He concluded that none of these constant coefficient linear systems is as good as the variable coefficient linear system proposed by Lutes and Shah\((3)\), for predicting transient response. The third order linear system and the two mode linear system, however both can predict the transient response fairly well.

The higher order equivalent linear substitute structure studied in chapter III will be used to predict the transient response of the interstory displacements of the MDF yielding system. Initially the equivalent stiffness and the equivalent viscous damping will be kept constant throughout the time required for the response to build up. The integration procedure and results will be discussed in section 5.2.

An attempt to predict the transient response of the MDF bilinear hysteretic yielding system by using a higher order equivalent linear substitute structure with variable equivalent viscous damping and variable equivalent stiffness will also be given. Details will be given in section 5.3.

5.2 Equivalent Constant Coefficient Higher Order MDF Linear System

The equivalent higher order linear MDF systems used to predict the stationary rms interstory displacements in chapter III are now used to calculate the rms values of the transient interfloor displacements. Since this equivalent system is linearly elastic, the mean squared
levels of displacement at each story can be obtained by integrating the Liapunov covariance matrix differential equation.

\[
\frac{d \hat{S}}{dt} = g \hat{S} + \hat{S} g^T + B
\]  
(3-17)

where \( g \), \( \hat{S} \), and \( B \) matrices are defined in (3-61), (3-64) and (3-65) respectively. The equivalent coefficients \( (Ce)_1 \) and \( (Ke)_1 \) in (3-61) are assumed to be constant throughout the time history while the system response builds up from rest. Note that there is an intuitive difference between the real structure and the substitute system under this assumption. The real structure is definitely stiffer than the higher order linear MDF system up until the time when the response of the yielding story exceeds the yield level of that story. However both the real structure and the substitute higher order linear system have responses which build up to similar stationary values, sooner or later. Of course, any difference between the stationary values obtained from simulation (chapter 2) and from the substitute linear system (chapter 3) will show up as different asymptotes for the two corresponding transient rms curves. In some situations this difference is sufficient to make comparisons somewhat difficult for the transient curves. For example, Figure 2.6 showed that the higher order linearization scheme did give a good qualitative prediction of the effects of top-story yielding on the stationary rms responses of the Type A 2DF structure with 1% damping. Nonetheless, the actual values of \( \sigma_{u_2} \) predicted by the linear model were significantly smaller than the simulation values for \( 0.3 \leq Y/N \leq 1.5 \), being about 25% low for \( Y/N = 0.7 \). Such a 25% error in the asymptote will look quite large in a comparison of transient response curves,
even though it is not directly related to the ability of the linear model
to predict the rate of build-up of the response.

Curves B in Figures 4.1 to 4.8 show the predictions of the
rms interfloor transient responses of the Type A (uniform stiffness),
two-story structures assuming that only the bottom story can be yielding.
The transient responses predicted by this constant coefficient higher
order linear system agree with the simulation data (curve A) very well
except for some moderately yielding systems for which the response of
the yielding story builds up too fast while the response of the elastic
story (top story) builds up too slowly (Figures 4.2, 4.3, 4.6 and 4.7).

Curves B in Figures 4.9 to 4.17 present the prediction of the
rms interfloor transient responses of the same type of two-story struc-
ture, but with only the top story allowed to yield. The transient
responses obtained from this constant coefficient higher order linear
system agree with the simulation data very well for the system with
yield level either very high or very low, while the predicted transient
responses for the moderate yield level system build up faster than the
simulation data. The curves for $\sigma_{y2}$ in Figures 4.11, 4.12, 4.15, 4.16
and 4.17 illustrate the point made earlier, that discrepancies in
stationary response levels show up much more clearly in these transient
response curves than they did in the figures of chapter 2.

Curves B in Figures 4.18 to 4.26 show the predictions of the
transient rms interfloor responses of type B (nonuniform stiffness) two-
story structures from this constant coefficient equivalent linear system.
Both the bottom story and the top story have the same yield level. Again
the transient responses predicted by this method build up faster than the
simulation data for the moderate yield level systems.

Curve $B_i$ ($B_1, B_2, B_3, B_4$) in Figures 4.27 to 4.32 gives the predicted transient rms interfloor displacement of the $i^{th}$ story of the Type A four-story structures having the same yield level in all stories. Curves $B_i$ (transient response of the first story) predicted by this method agree with the simulation data quite well. The major discrepancies for $B_2$, $B_3$, and $B_4$ (transient responses of the second, the third and the fourth stories) in some situations (for example Figure 4.28) may be attributed to poor prediction of stationary interfloor displacement of these stories by the equivalent linear system.

Similar results are also obtained for the Type B (tapered stiffness) four-story structures. They are given in curves $B_i$ in Figures 4.33 to 4.38. For a low viscous damping and only slightly yielding system (see Figure 4.37), the responses from simulation build up very slowly, while the responses from the constant coefficient higher order linear system build up within eight times the fundamental period. The predicted responses for other systems agree with the simulation data fairly well.

Dotted lines in Figures 4.39 to 4.41 show the predicted results for a specific ten-story structure in which only the first story can be yielding. The rms transient responses of the interfloor displacements obtained by this method are in good agreement with those from simulation.

5.3 Variable Coefficient Higher Order MDF Linear System

The transient response of the constant coefficient higher order linear systems, shown in the preceding section, built up too rapidly for
some moderate yield level bilinear hysterrtic systems. A simplified approach to the varying coefficient equivalent higher order linear system will be attempted in this section. In this approach \((Ke)_i\) and \((Ce)_i\) in the matrix \(g\) of (3-61) are kept constant within a given fixed time interval and instantaneously changed at the beginning of the next time interval. The constant \((Ke)_i\) and \((Ce)_i\) values for a particular time interval are based on the level of \(\sigma_{u_i}/Y_i\) at the beginning of that time interval. The curves shown in Figure 3.4 are used to determine the values of \((Ke)_i\) and \((Ce)_i\) for the \(i^{th}\) story following the procedure discussed in section 3.4. The difference from the preceding section is simply that of using instantaneous rather than stationary values of \(\sigma_{u_i}/Y_i\) to determine the parameters of the linear system.

Since \((Ke)_i\) and \((Ce)_i\) are constant within each time interval, the covariance matrix \(\hat{S}\) at the end of the time interval can be obtained by integrating (3-17), given \(\hat{S}\) at the beginning of that particular interval and given the constant \(S_0\) of excitation. The time interval used was one half percent of the fundamental period of the elastic system. In order to put the curves in Figures 3.4 into the computer, each curve was separated into several segments, each of which was approximated by a second or third order polynomial (see appendix D). Curves C presented in Figures 4.1 to 4.26 show the normalized rms transient story distortions obtained by these variable coefficient higher order linear systems for 2DF yielding systems. From comparison with the simulation data, and with the results obtained from the constant coefficient higher order linear systems, it is concluded that this method can predict the transient response better than the constant coefficient linear system.
can during the first few response periods of the fundamental mode. Figures 4.2, 4.6 and 4.7, for example, show fairly clearly that the variable coefficient system can give improved approximations of the initial build-up of response for both yielding and elastic stories.

A major difficulty was encountered with the variable coefficient model, though. In some situations the transient response levels obtained appear to diverge in an oscillatory manner, rather than to converge to the stationary levels (see Figures 4.2, 4.9 and especially 4.23). It can be reasoned that in the first few periods of the response, the yielding system is still basically linear and hence this variable coefficient linear system is more representative of the original system than is the constant coefficient linear system, however after the response significantly exceeds the yield level, the system may become unstable. In essence the model of equation (3-17) becomes a feedback system with the structural parameters ($g$) depending on the response level ($S$). A general investigation of the stability of the system is beyond the scope of this study, but it is obvious from the results noted above that the problem is not trivial.

Because of the instability in some cases, the rms transient responses from this variable coefficient higher order linear system are reliable only for relatively small values of $\omega_1 t$. For the data obtained, it appears that the approximation is valid for $\omega_1 t \leq 5 \pi$. The study of the transient response for 4DF and 10DF structures by this variable coefficient linear system is omitted due to this instability problem.
VI SUMMAR Y AND CONCLUSIONS

A digital computer was used to empirically investigate the stochastic response of bilinear hysteretic multiple-degree-of-freedom (MDF) shear beam type structures subjected to an excitation which was approximately Gaussian and white. The second slope ratio $\alpha$ of the bilinear hysteretic restoring force curve was chosen to be $1/21$ in order to study nearly elasto-plastic systems which had previously been studied for SDF systems. Two-story, four-story and ten-story structures with viscous damping ratios of $\beta = 5\%$ and $\beta = 1\%$ were chosen for study. The characteristics of the response which were determined in the investigation were root mean squared interfloor displacements.

Empirical curves showing the stationary root mean squared interfloor displacements for the bilinear MDF systems were presented, and compared with corresponding analytical results. A practical analytical method for finding the stationary responses of a MDF bilinear hysteretic system was obtained. The method was based on an equivalent linear structure concept. The yielding story was replaced by an equivalent third order linear counterpart with ductility dependent stiffnesses and dampings. The Liapunov covariance matrix equations of this equivalent higher order linear system were established to compute the root mean squared interfloor displacements. The following paragraphs summarize the results of the above stationary response studies.

1. For a two-degree-of-freedom system with only one story allowed to yield, the shift of energy dissipation from the elastic story to the yielding story caused the root mean squared interfloor
displacement of the elastic story to decrease while that for the yielding story usually increased, as compared to the response of the same system with infinite yield level to the same excitation. However, for low viscous damping structures, decreasing the yield level of the yielding story may reduce the responses of both the elastic story and the yielding story. This is due to the increase in hysteretic energy dissipation due to yielding having more effect on the responses of the system than does the reduction in stiffness.

2. The root mean squared interfloor displacements of Type A (stiffness uniform over the height) elastic structures varied from the largest at the bottom story to the smallest at the top story, while those for the Type B (tapered stiffness) elastic structures were more uniform over the height. Yielding of a Type A structure usually started at the bottom story while yielding of a Type B structure might start at any story, depending on the distribution of the stiffness. It was found that the bottom story of the Type A structure always had the largest root mean squared interfloor displacement for any value of the yield level, while the story of the maximum interfloor displacement of a Type B structure might be changed if the yield level was varied.

3. Using the third order linear system as the equivalent substitute counterpart of the yielding story, one can predict very well the root mean squared interfloor displacements of a MDF bilinear yielding structure. For the structures considered in this study, the largest percentage error of the predicted root mean square interfloor displacement usually occurred in an elastic story in which the interfloor displacement was insignificant in comparison with that of the yielding story. For example, in the two-degree-of-freedom structure with only
one story yielding, the maximum error of the predicted root mean squared interfloor displacement was 35% (see Fig. 2.4), while the predicted root mean squared displacement of the yielding story was within about 10% of the simulated value. However, the root mean squared interfloor displacement of the elastic story was only one sixth of that in the yielding story. Overall it is concluded that this equivalent higher order linear system is a good equivalent linear system to predict the root mean squared interfloor displacements of a MDF bilinear hysteretic yielding structure. Comprehensiveness of the physical model and simplicity of the mathematical formulation are the greatest advantages of this approach.

The digital computer simulation program was also extended to investigate the transient response of the bilinear hysteretic MDF structures with zero initial conditions, subjected to the same approximately (digitally simulated) white, stationary and Gaussian excitation. Mean squared interfloor displacements were evaluated from ensemble averages, although some time averaging was also used to smooth the curves somewhat. In each case either fifty or one hundred and fifty samples were generated and ensemble averages were taken at a time interval which was one-half the fundamental period of the unyielded structure. These empirical transient root mean squared interfloor displacement curves were compared with corresponding analytical results. Two analytical approximations were attempted to evaluate the transient response. One was to integrate the Liapunov covariance matrix differential equation for the equivalent constant coefficient higher order
linear system which was used to find the stationary interfloor displacements of the MDF structure. The other was to integrate the corresponding covariance matrix differential equation but with the coefficients for the equivalent higher order linear system varying with time, as functions of the response levels. Some of the significant findings can be summarized as follows:

1. Empirical results showed that if some stories were yielding and some stories were elastic in a structure then the transient interfloor displacements of the elastic stories built up faster than those for the yielding stories, and the transient response of stories with moderate yielding built up slower than those of stories with large yielding.

2. In general the transient interfloor displacements of a system with moderate yield levels built up more slowly than those of very high yield level or very low yield level systems.

3. The transient response of a system with large viscous damping built up faster than did that of the same system with smaller viscous damping.

4. An equivalent higher order linear system can predict the transient interfloor response fairly well for a system with very high or very low yield level, especially in those situation where the equivalent higher order system gives good prediction of the stationary root mean squared interfloor displacement. For a moderate yield level system, the interfloor displacements of this equivalent higher order linear system usually built up somewhat too fast in a yielding story and too slow in an elastic story.
5. The higher order linear system modified to have coefficients varying with time can predict the transient interfloor displacements more accurately than the constant coefficient model, provided the time is kept fairly small \( \omega t \leq 5 \pi \). Unfortunately, in some cases, the interfloor displacements predicted by this method diverged, rather than converging to the stationary values. Therefore results obtained by this method are only valuable for the initial portion of the response.
REFERENCES


27. Veletsos, A. S. and Newmark, N. M., "Effect of Inelastic Behavior on the Response of Simple Systems to Earthquake Motions", Proceeding of


Fig. 2.17  Normalized RMS Interfloor Displacements $\frac{\sigma_u}{N}$
Type A, 10DF, $\beta = 5\%$, 1st and 2nd Stories Yielding
Fig. 2.17 Normalized RMS Interfloor Displacements \( \frac{\sigma_{u}}{N} \)
Type A, 10DF, \( \beta = 5\% \), 1st and 2nd Stories Yielding
Fig. 2.17 Normalized RMS Interfloor Displacements
Type A, 10DF, $\gamma = 5\%$, 1st and 2nd Stories Yielding

Fig. 2.18 Normalized Interfloor Displacements
Type A, 10DF, $\beta = 1\%$, 1st and 2nd Stories Yielding
Fig. 2.18 Normalized RMS Interfloor Displacements
Type A, 10DF, β = 1%, 1st and 2nd Stories Yielding
Fig. 2.18 Normalized RMS Interfloor Displacements

Type A, 1ODF, $\beta = 1\%$, 1st and 2nd Stories Yielding
Fig. 2.19  Normalized RMS Interfloor Displacements \( \frac{\sigma_{u1}}{N} \), Type B, 1ODF, \( \beta = 5\% \), Fifth Story Yielding
Fig. 2.19  Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 5\%$, Fifth Story Yielding
Fig. 2.19 Normalized RMS Interfloor Displacements \( \frac{\sigma_{}\text{u19}}{N} \)
Type B, 10DF, \( \beta =5\% \), Fifth Story Yielding

Fig. 2.20 Normalized RMS Interfloor Displacements
Type B, 10DF, \( \beta =1\% \), Fifth Story Yielding
Fig. 2.20 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 1\%$, Fifth Story Yielding
Fig. 2.20 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 1\%$, Fifth Story Yielding
Fig. 2.21  Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta$ = 5%, 5th and 8th Stories
Yielding
Fig. 2.21 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta =5\%$, 5th and 8th Stories Yielding
Fig. 2.21 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 5\%$, 5th and 8th Stories Yielding

Fig. 2.22 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 1\%$, 5th and 8th Stories Yielding

\[ \frac{\sigma_{u\gamma}}{N} \]

\[ \frac{\sigma_{u^{10}}}{N} \]

\[ \frac{\sigma_{u_1}}{N} \]

\[ \frac{\sigma_{u_2}}{N} \]
Fig. 2.22 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 1\%$, 5th and 8th Stories Yielding
Fig. 2.22 Normalized RMS Interfloor Displacements
Type B, 10DF, $\beta = 1\%$, 5th and 8th Stories
Yielding
Fig. 2.23 Normalized RMS Interfloor Displacements, Type A, 10DF, $\beta = 5\%$
Fig. 2.24  Normalized RMS Interfloor Displacements, Type A, 10DF, $\beta = 1\%$
Fig. 2.25 Normalized RMS Interfloor Displacements, Type B, 10DF, $\beta=5\%$
Fig. 2.25 Normalized RMS Interfloor Displacements, Type B, 10DF, $\beta = 5\%$
Higher Order Linear System

--- Simulation

\[ \frac{Y}{N} = \infty \]

(a)

\[ \frac{Y}{N} = 3.2 \]

(b)

Fig. 2.26 Normalized RMS Interfloor Displacements, Type B, 10DF, \( \beta = 1\% \)
Fig. 3.1 Third Order Linear Mechanical Oscillator
Fig. 3.2 RMS Displacement of SDF system.

Fig. 3.3 RMS Velocity Response of SDF System.
Fig. 3.4 Parameters of \( \left( \frac{\omega_2^2}{\omega_0^2} \right) \) and \( r \)

A: \( \left( \frac{\omega_2}{\omega_0} \right)^2, \beta = 0\% \)
B: \( \left( \frac{\omega_2}{\omega_0} \right)^2, \beta = 5\% \)
C: \( r \)
Fig. 3.5 Equivalent Linear System for Bottom Story Yielding, 2DF Structure

Fig. 3.6 Equivalent Linear System for Top Story Yielding, 2DF Structure

Fig. 3.7 Equivalent Linear System for Both Stories Yielding, 2DF Structure

Fig. 3.8 Equivalent Linear System for 4DF Yielding Structure
Type A, 2DF, $\beta = 5\%$

Bottom story yielding

$\frac{\gamma}{N} = 10.11$

$\frac{\sigma_{U1}}{N}$

$\frac{\sigma_{U2}}{N}$

Fig. 4.1 Normalized Transient RMS Interfloor Displacements
Fig. 4.4 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 5\%$
Bottom Story Yielding
$\frac{\gamma}{N} = 0.469$
Type A, 2DF, $\beta = 1\%$

Bottom Story Yielding

$Y = 11.15$

Fig. 4.5 Normalized Transient RMS Interfloor Displacements
Fig. 4.6 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 1\%$

Bottom Story Yielding

$\frac{Y}{N} = 2.654$

$\frac{\sigma_u_1}{N}$

$\frac{\sigma_u_2}{N}$
Fig. 4.7 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 1\%$

Bottom Story Yielding

$\frac{\sigma_{U1}}{N} = 0.922$

$\frac{\sigma_{U2}}{N}$
Fig. 4.8 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 1\%$
Bottom Story Yielding
$\frac{\gamma}{N} = 0.615$
Type A, ZDF, $\beta = 5\%$

Top Story Yielding

$N = 2.698$

Fig. 4.9 Normalized Transient RMS Interfloor Displacements
Type A, 2DF, $\beta = 5\%$
Top Story Yielding
$\frac{Y}{N} = 1.319$

Fig. 4.10 Normalized Transient RMS Interfloor Displacements
Fig. 4.11 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 5\%$
Top Story Yielding
$\frac{y}{N} = 0.415$

$\frac{\sigma_{u2}}{N}$
$\frac{\sigma_{u1}}{N}$

$\omega_1 t / \pi$
Fig. 4.12 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 5\%$
Top Story Yielding
$\frac{\gamma}{N} = 0.255$
Fig. 4.13 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 1\%$
Top Story Yielding

$N = 5.66$

$\sigma_{u1}/N$

$\sigma_{u2}/N$
Type A, 2DF, $\beta = 1\%$
Top Story Yielding
$\frac{Y}{N} = 1.62$

Fig. 4.14 Normalized Transient RMS Interfloor Displacements
Fig. 4.15 Normalized Transient RMS Interfloor Displacements
Fig. 4.16 Normalized Transient RMS Interfloor Displacements

Type A, 2DF, $\beta = 1\%$

Top Story Yielding

$\frac{\sigma_{u2}}{N} = 0.485$

$\frac{\sigma_{u1}}{N}$
Type A, 2DF, $\beta = 1\%$

Top Story Yielding

$\frac{\gamma}{N} = 0.313$

Fig. 4.17 Normalized Transient RMS Interfloor Displacements
Fig. 4.18 Normalized Transient RMS Interfloor Displacements

Type B, 2DF, $\beta = 5\%$
Both Stories Yielding
$Y = \frac{20.75}{N}$
Fig. 4.19  Normalized Transient RMS Interfloor Displacements

Type B, 2DF, $\beta = 5\%$
Both Stories Yielding
$\frac{\gamma}{N} = 6.916$
Fig. 4.20 Normalized Transient RMS Interfloor Displacements
Fig. 4.21 Normalized Transient RMS Interfloor Displacements

Type B, 2DF, $\beta = 5\%$
Both Stories Yielding
$
\frac{Y}{N} = 0.692$

$\frac{\sigma_{u_2}}{N}$
$\frac{\sigma_{u_1}}{N}$
Fig. 4.22 Normalized Transient RMS Interfloor Displacements

\[ \frac{\sigma_{U2}}{N} \]
\[ \frac{\sigma_{U1}}{N} \]

Type B, 2DF, \( \beta = 1\% \)
Both Stories Yielding
\( \frac{Y}{N} = 20.75 \)
Type B, 2DOF, $\beta = 1\%$
Both Stories Yielding
$\frac{Y}{N} = 7.0$

Fig. 4.23 Normalized Transient RMS Interfloor Displacements
Type B, 2DF, $\beta = 1\%$
Both Stories Yielding
$\gamma = 0.692$

Fig. 4.25 Normalized Transient RMS Interfloor Displacements
Fig. 4.26 Normalized Transient RMS Interfloor Displacements

Type B, 2DF, \( \beta = 1\%

Both Stories Yielding

\[
\frac{\gamma}{N} = 0.346
\]
Type A, 4DF, $\beta = 5\%$

$Y = \infty$

Fig. 4.27 Normalized Transient RMS Interfloor Displacements
Fig. 4.28 Normalized Transient RMS Interfloor Displacements
Fig. 4.30 Normalized Transient RMS Interfloor Displacements
Fig. 4.31 Normalized Transient RMS Interfloor Displacements
Fig. 4.33 Normalized Transient RMS Inter-Floor Displacements
Fig. 4.34 Normalized Transient RMS Interfloor Displacements
Fig. 4.35 Normalized Transient RMS Interfloor Displacements

Type B, 4DF, β = 5%, Y/N = 0.634
Type B, 4DF, $\beta = 1\%$, $Y/N = 3.17$

Fig. 4.37 Normalized Transient RMS Interfloor Displacements
Type A, 10DF, $\beta = 5\%$, $\gamma/N = \infty$

Fig. 4.39 Normalized Transient RMS Interfloor Displacements
Type A, 10DF, $\beta = 5\%$, $Y/N = 0.69$

Fig. 4.40 Normalized Transient RMS Interfloor Displacements
Type A, 10DF, \( \beta = 5\% \), \( Y/N = 0.23 \)

Fig. 4.41 Normalized Transient RMS Interfloor Displacements
APPENDIX A

The method of evaluating the damping matrix associated with any given set of modal damping ratios can be obtained by considering the complete diagonal matrix of generalized damping coefficients, which may be obtained by pre- and post-multiplying the damping matrix by the mode-shape matrix $\phi$.

$$\Gamma = \phi^T c \phi = 2 \begin{bmatrix} \beta_1 \omega_1 \ddot{M}_1 & 0 & \cdots & 0 \\ 0 & \beta_2 \omega_2 \ddot{M}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n \omega_n \ddot{M}_n \end{bmatrix}$$  \hspace{1cm} (A-1)

Where $\phi$ is the mode shape matrix and $\ddot{M}_i$ is an element of the diagonal matrix, $\ddot{M} = \phi^T M \phi$. The damping matrix $c$ can then be obtained by pre-multiplying and postmultiplying (A-1) by the inverse of the mode-shape matrix or its transpose

$$(\phi^T)^{-1} \Gamma (\phi)^{-1} = (\phi^T)^{-1} \phi^T c \phi \phi^T = c$$  \hspace{1cm} (A-2)

Premultiplying the generalized mass matrix by its inverse

$$\ddot{M}^{-1} \ddot{M} = (\ddot{M}^{-1} \phi^T M ) \phi = I = \phi^{-1} \phi$$  \hspace{1cm} (A-3)

gives the inverse of the mode shape matrix as

$$\phi^{-1} = \ddot{M}^{-1} \phi^T M$$  \hspace{1cm} (A-4)
Substituting (A-4) into (A-2) gives

\[ c = (M \phi \bar{M}^{-1} ) \Gamma ( \bar{M}^{-1} \phi^T M ) \]  

(A-5)

Since both \( \bar{M} \) and \( \Gamma \) are diagonal matrices, the multiplying result of \( \bar{M}^{-1} \Gamma \bar{M}^{-1} \) is also a diagonal matrix. Let

\[ \tilde{\alpha} = \bar{M}^{-1} \Gamma \bar{M}^{-1} = 2 \begin{bmatrix} \frac{\beta_1 \omega_1}{\bar{M}_1}, 0, & \ldots, & 0 \\ 0, \frac{\beta_2 \omega_2}{\bar{M}_2}, & \ldots, & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0, 0, & \ldots, & \frac{\beta_n \omega_n}{\bar{M}_n} \end{bmatrix} \]  

(A-6)

and Equation (A-5) can be written as

\[ c = M \phi \tilde{\alpha} \phi^T M \]  

(A-7)

Since \( \tilde{\alpha} \) is a diagonal matrix with \( \tilde{\alpha}_i = 2 \beta_i \omega_i / \bar{M}_i \), \( c \) can be considered as being the sum of the modal contributions:

\[ c = M \left[ \sum_{i=1}^{n} \frac{2 \beta_i \omega_i}{\bar{M}_i} \phi_i \phi_i^T \right] M \]  

(A-8)
APPENDIX B

C PURPOSE OF THIS PROGRAM IS TO COMPUTE THE RMS OF
C RESPONSES OF THE MDF YIELDING SYSTEM SUBJECTED TO WHITE NOISE
C EXCITATION BY RUNGE-KUTTA METHOD.
C INPUT DATA INCLUDES MASSES M(I), STIFFNESS K1(I) AND
C DAMPING RATIOS OF EACH MODE BTA(I)
INTEGER SI,STAR,STEP
REAL K1(4),M(4),U(8),K2(4),IF
DIMENSION Y(8),DY(8),DY1(8),DY2(8),DY3(8),G(8,8),G1(8,8)
DIMENSION FKO(8),FK1(8),FK2(8),FK3(8),C(4,4),BTA(4)
DIMENSION SX(8),SX2(8),UU(8),TUU(8),TX(8),TX(8),P(4)
DIMENSION PU(4),UMAX(4),YL(4),FB(4),BB(8),YLD(4,10)
PRINT 5
5 FORMAT(‘ GIVE ME N, ST,STAR,NDOT,MN,IX,IIY,STEP,FST,AO,TIME’) READ(6,6)N, ST,STAR,NDOT,MN,IX,IIY,STEP,FST,AO,TIME
6 FORMAT(8I4,4F10.3)
N2=2*N
DO 2 I=1,N2
B1(I)=0.
DO 2 J=1,N2
2 G(I,J)=0.
C INPUT DATA MASS, STIFFNESS, AND DAMPING RATIO
PRINT 942
942 FORMAT(‘ GIVE ME M(I), I=1,N’) READ(6,901)M(I),I=1,N
901 FORMAT(10F8.3)
PRINT 943
943 FORMAT(‘ GIVE ME K1(I), I=1,N’) READ(6,901)K1(I),I=1,N
PRINT 130
130 FORMAT(‘ GIVE ME K2(I), I=1,N’) READ(6,901)K2(I),I=1,N
PRINT 100
100 FORMAT(‘ GIVE ME YLD(I,J), I=1,N, J=1,MN’) DO 11 J=1,MN
11 READ(6,901)YLD(I,J),I=1,N,J=1,MN
PRINT 944
944 FORMAT(‘ GIVE ME BTA(I), I=1,N’) READ(6,901)BTA(I),I=1,N
C FIND MODE SHAPES AND EIGEN VALUES
CALL CIJ4(N,M,K1,BTA,C,P)
C FILL THE G MATRIX
DO 41 I=1,N
G(I,N+I)=1.
DO 41 J=1,N
41 G(I+N,J+N)=-C(I,J)/M(I)
DO 42 I=1,N
J=N-I+1
J1=J-1
TK=K1(I)
G(N+J,J)=-TK/M(J)+G(N+J,J)
IF(J.EQ.1) GO TO 42
G(N+J-1,J1)=-TK/M(J1)
G(N+J-1,J)=TK/M(J1)
G(N+J,J1)=TK/M(J)

42 CONTINUE

C INTEGRATION PARAMETERS
J=1
DO 43 I=1,N
43 IF(P(J).LE.P(I)) J=I
TT=6.2831852/P(J)
DT=TT/STEP
PRINT 920,TT,DT

920 FORMAT(//"PERIOD OF THE HIGHEST MODE='F9.4,'SEC'//
$ TIME INCREMENT DT='F9.4'/)
DO 115 I=1,N2
DO 115 J=1,N2
115 G1(I,J)=G(I,J)
DO 81 KK=1,MN
DO 116 I=1,N2
DO 116 J=1,N2
116 G(I,J)=G1(I,J)
DO 120 I=1,N
YL(I)=YLD(I,KK)
PU(I)=0.
FB(I)=0.
120 UMAX(I)=YL(I)
NP=0
FNP=0.
T=0.
IX=IX
IY=IY
DO 45 I=1,N2
Y(I)=0.
SX(I)=0.
SX2(I)=0.
UU(I)=0.
TX(I)=0.
TUU(I)=0.
TX2(I)=0.
BB(I)=0.
45 CONTINUE

47 CALL RANDU(IX,IY,R1)
CALL RANDU(IX,IY,R2)
TR1=(-2*ALOG(R1))*0.5
R=TR1*SIN(6.2831852*R2)
DO 46 I=1,N
46 Y(I+N)=AO*R
PRINT 926, (YL(I),I=1,N)
PRINT 927
PRINT 928,(I,I=1,N)
926 FORMAT(1H1,' RUNGGE KUTTA METHOD'/1X,'YL(I)='10F8.3/)
50 DO 51 I=1,N2
   DY1(I)=0.
   DO 52 J=1,N2
51   DY1(I)=DY1(I)+G(I,J)*Y(J)
      DY1(I)=(DY1(I)+BB(I))*DT
51   Y(I)=Y(I)+DY1(I)
    NP=NP+1
    FNP=FNP+1.
    T=T+DT
   IF(NDOT,EQ.1) PRINT 924,T,(Y(I),I=1,N)
924   FORMAT(1H ,F7.2,4X,20(F10.3))
   U(1)=Y(1)
   U(N+1)=Y(N+1)
   DO 61 I=2,N
   U(I)=Y(I)-Y(I-1)
61   U(I+N)=Y(I+N)-Y(I+N-1)
   G(N2,N)=0.
   DO 201 I=1,N
    J=N-I+1
    J1=J-1
    IF(U(J),GT,PU(J)) GO TO 202
    IF(U(J),GT,UMAX(J)-2*YL(J)) GO TO 203
    TK=K2(J)
    UMAX(J)=2*YL(J)+U(J)
    FB(J)=(K2(J)-K1(J))*YL(J)
    GO TO 204
202   IF(U(J),LT,UMAX(J)) GO TO 203
    TK=K2(J)
    UMAX(J)=U(J)
    FB(J)=(K1(J)-K2(J))*YL(J)
    GO TO 204
203   TK=K1(J)
    FB(J)=(K2(J)-K1(J))*(UMAX(J)-YL(J))
204   PU(J)=U(J)
    G(N+J,J)=G(N+J,J)-TK/M(J)
    IF(J,EQ.1) GO TO 201
    G(N+J,J)=TK/M(J1)
    G(N+J,J1)=-TK/M(J1)
    G(N+J,J1)=TK/M(J)
201   CONTINUE
   DO 210 J=1,N
    IF(J,EQ.N) GO TO 211
    BB(N+J)=(FB(J+1)-FB(J))/M(J)
    GO TO 210
211   BB(N2)=-FB(J)/M(J)
210   CONTINUE
   IF(FNP.LE.FST) GO TO 65
   DO 62 I=1,N2
   UU(I)=UU(I)+U(I)**2
   SX(I)=SX(I)+Y(I)
62   SX2(I)=SX2(I)+Y(I)**2
   IF(NP.LE,NP/TAR*TAR) GO TO 65
   IF=FNP-FST
DO 63 I=1,N2
TUU(I)=UU(I)/IF
TX2(I)=SX2(I)/IF
63 TX(I)=SX(I)/IF
PRINT 931,T(I),TX2(I),TX2(I+N),TUU(I),TUU(I+N),TX(I),TX(I+N),
$I=1,N)
931 FORMAT(1H0,F7.2,3X,I2,3X,E12.4,2X,E12.4,2X,E12.4,2X,E12.4,
$2X,E12.4,2X,E12.4/))
65 IF(NP.EQ.32001) NP=NP-32000
IF(T.GT.TIME) GO TO 81
IF(NP.NE.NP/ST*ST) GO TO 50
CALL RANDU (IX,IY,R1)
CALL RANDU (IX,IY,R2)
TR1=(-2*ALOG(R1))***0.5
R=TR1*SIN(6.2831852*R2)
DO 70 I=1,N
70 Y(I+N)=Y(I+N)+AO*R
GO TO 50
81 CONTINUE
STOP
END

THE PURPOSE OF THIS SUBROUTINE IS TO FIND THE DAMPING
MATRIX GIVEN DAMPING RATIO OF EACH MODE, MASS, AND STIFFNESS
SUBROUTINE CIJ4(N,M,K1,BTA,C,P)
REAL M(4),K1(4),A(4,4),B(4,4)
DIMENSION BTA(4),C(4,4),PHI(4,4),P(4)
DO 7 I=1,N
DO 7 J=1,N
B(I,J)=0.
7 A(I,J)=0.
DO 10 I=1,N
B(I,I)=M(I)
J=N-I+1
J1=J-1
A(J,J)=A(J,J)+K1(J)
IF(J.EQ.1) GO TO 10
A(J1,J1)=K1(J)
A(J1,J)=-A(J1,J1)
A(J,J1)=-K1(J)
10 CONTINUE
PRINT 911
911 FORMAT(1H1, ' M INVERSE K MATRIX')
   DO 12 I=1,N
12 PRINT 907,(A(I,J), J=1,N)
907 FORMAT(1H1,10FB.3)
   CALL NROOT(N,A,B,P PHI)
   DO 8 I=1,N
8 P(I)=SQRT(P(I))
   C NORMALIZE EIGEN VECTOR
   DO 18 J=1,N
   PHIN=PHI(1,J)
   DO 18 I=1,N
18 PHI(I,J)=PHI(I,J)/PHIN
   PRINT 916,(P(I),I=1,N)
   DO 17 I=1,N
17 PRINT 917,(PHI(I,J),J=1,N)
916 FORMAT(1HO/ ' CIRCULAR NATURAL FREQUENCIES AND MODES SHAPES'
   $/20FB.3)
917 FORMAT(1HO/20FB.3)
   C COMPUTE DAMPING MATRIX
   DO 35 I=1,N
35   DO 30 J=1,N
30      AA=0.
   DO 31 J=1,N
31      AA=AA+M(J)*PHI(J,I)**2
      AA=2*BTA(I)*P(I)/AA
   DO 32 J=1,N
32      L=1,N
   DO 32 L=1,N
32      C(J,L)=C(J,L)+M(J)*AA*PHI(J,I)*PHI(L,I)*M(L)
   CONTINUE
   PRINT 918
918 FORMAT(' DAMPING MATRIX')
   DO 22 I=1,N
22 PRINT 919,(C(I,J),J=1,N)
919 FORMAT(1H1,20FB.4)
   RETURN
END
APPENDIX C

THIRD ORDER SYSTEM FOR N DEGREE OF FREEDOM
REAL K(I), N(I), WK(I), K(I)
COMPLEX Z(30, 30), WK(10)
DIMENSION E(30, 30), B(465), NNC(30, 30), BTA(10)
DIMENSION (10, 10), P(10), PHIC(10, 10), EDZ(10)
DIMENSION DOHC(465, 465), WKK(465), EXC(30, 30)
DIMENSION UC(10), DU(10)
READ(C, I, 500, A, I)

1 FORMAT(10, 2F10.3)
READ(C, I, 500, A, I)

2 FORMAT(10, 2F10.3)
READ(C, I, 500, A, I)

3 FORMAT(10, 2F10.3)
READ(C, I, 500, A, I)

4 FORMAT(10, 2F10.3)
READ(C, I, 500, A, I)

5 FORMAT(10, 2F10.3)
READ(C, I, 500, A, I)

6 FORMAT(10, 2F10.3)
READ(C, I, 500, A, I)

7 AC, I = 0
DO 10 I = 1, N
AC, J = AC, J + Z(I, J)
10 CONTINUE

11 CONTINUE
N2 = 2*N
N3 = N*3

12 CONTINUE

13 CONTINUE

14 CONTINUE

15 CONTINUE

16 CONTINUE

17 CONTINUE

18 CONTINUE

19 CONTINUE
WRITE(6,915)(P(I),I=1,N)
DO 17 I=1,N
17 WRITE(6,917)(PHI(I,J),J=1,N)
916 FORMAT(6,918),* CIRCULAR NATURAL FREQUENCY AND MODE SHAPES*/20F8.3)
917 FORMAT(1H10,20F6.3)
C
COMPUTE DAMPING MATRIX
DO 35 I=1,N
DO 35 J=1,N
35 CC(I,J)=0.
DO 30 I=1,N
AA=0.
DO 30 J=1,N
30 AA=AA+M(J)*PHIC(J,I)**2
AA=2*BT*(I)*P(I)/AA
DO 32 J=1,N
DO 32 I=1,N
32 CC(I,J)=CC(I,J)+M(J)*AA*PHIC(J,I)*PHIC(L,I)*M(I)
30 CONTINUE
C
WRITE(6,918)
918 FORMAT(1H10,20F8.5)
DO 22 I=1,N
22 WRITE(6,919)CC(I,J),J=1,N
DO 23 I=1,N
DO 23 J=1,N
23 NI=N+J-2*N=CC(I,J)/M(I)
DO 24 I=1,N
G(I+N+1,J+2*N)=G(I+N+1,J+N)*G(J+N+1,J+2*N)+1.
24 CONTINUE
DO 26 I=1,N
26 WRITE(6,920)(P(I),I=1,N)
READ(5,2)(P(I),I=1,N)
READ(5,2)(CE(I),I=1,N)
IF(C(I+1,E)=0.0) GO TO 160
WRITE(6,22)(P(I),I=1,N)
WRITE(6,22)(CE(I),I=1,N)
G(I+N+1,I)=G(I+N+1)+P(I)*CE(I)
G(I+N+1,I+N)=G(I+N+1)+P(I)*CE(I)
G(I+N+2,I+1)=G(I+N+2)+P(I)*CE(I)
IF(C(I+1,E)=0.N) GO TO 29
G(I+N+2,I+1)=G(I+N+2)+P(I)*CE(I)
IF(C(I+1,E)=0.0) GO TO 34
G(I+N+2,I+1)=G(I+N+2)+P(I)*CE(I)
34 CONTINUE
L=I
DO 200 J=1,N
DO 201 L=1,N
201 IF(L+J) GO TO 203
203 NNJ=NNJ+1
L=L+1
GO TO 201
DO 203 NNJ=NNJ(J,I)
201 CONTINUE
200 CONTINUE
N2=N3*(N3+1)/2.
DO 25 I=1,N2
DO 25 J=1,N2
25 DDH(I,J)=0.
C
FIND DDHC(I,J) MATRIX
NL=1
DO 350 L=1,N3
DO 350 I=1,N3
350 IF(L+L) GO TO 350
DO 350 J=1,N3
NK=NNJ+L
DDH(NL,NK)=G(I,J)+DDH(NL,NK)
320 CONTINUE
NL = NL + 1
CONTINUE

DO 360 L = 1, N3
DO 361 I = 1, N3
IF (L .GT. I) GO TO 361
DO 362 J = 1, N3
NK = NK + 1
DDH(NL, NK) = DDH(NL, NK) + G1(L, J)
CONTINUE
NL = NL + 1

CONTINUE
N20 = N21 - N2*(N + 1)/2
DO 87 I = 1, N21
IF (I .GT. N20) B(I) = 6.28 + 50
IF (I .LE. N20) B(I) = 0.
CONTINUE
CALL LEQT1 = (DDH, 1, 1, N21, N21, B, D, WKK, IER)
DO 89 I = 1, N3
DO 90 J = 1, N3
IF (J .GT. I) GO TO 90
L = I + (J - 1)*N3 - J*(J - 1)/2
EX(I, J) = B(I)
CONTINUE
DO 91 I = 1, N3
DO 92 J = 1, N3
IF (J .LE. I) GO TO 91
EX(I, J) = EX(J, I)
CONTINUE
WRITE (6, 92)
FORMAT ('EX(I,J) MATRIX')
N4 = N3
L = 0
IF (N4 .EQ. 10) 111, 111, 112
112 DO 113 I = 1, N3
113 WRITE (6, 94) (EX(I, J + L), J = 1, 10)
L = L + 10
111 DO 93 I = 1, N3
93 WRITE (6, 94) (EX(I, J + L), J = 1, N4)
94 FORMAT (1X, 10F10.3/
DO 400 I = 1, N
EX(1, 1) = 1.0
DC(I) = EX(I, 1) + EX(I, I) = 2*EX(I, I)
J = I + 2*N
GO TO 410
410 DC(I) = EX(J, J) + EX(J, J + 1) = 2*EX(J, J + 1)
GO TO 400
400 CONTINUE
WRITE (6, 420) (DC(I), I = 1, N)
FORMAT ('DUM(I) = 1.0/N')
WRITE (6, 430) (DC(I), I = 1, N)
GO TO 27
STOP
END
APPENDIX D

Curve A is approximated as:

\[ z = 0.9524 \quad x \geq 3.5 \]
\[ z = -0.0051 \ x^2 + 0.04295 \ x + 0.8645 \quad 2.0 \leq x \leq 3.5 \]
\[ z = -0.3573 \ x^2 + 1.6504 \ x - 0.94175 \quad 1.3 \leq x \leq 2.0 \]
\[ z = 0.1287 \ x^3 - 0.03297 \ x^2 - 0.004602 \ x + 0.3789 \quad 0.4 \leq x \leq 1.3 \]
\[ z = 6.66 \ x^3 - 6.995 \ x^2 + 2.5321 \ x + 0.0501 \quad 0.1 \leq x \leq 0.4 \]
\[ z = -10.0 \ x^2 + 4.3 \ x - 0.08 \quad 0.05 \leq x \leq 0.1 \]
\[ z = 17.5 \ x^2 + 1.35 \ x - 0.00125 \quad x \leq 0.05 \]

where \( z = \left( \frac{\omega_2 \tau^2}{\omega_0^2} \right) \) and \( x = \frac{Y}{\sigma} \)

Curve C is approximated as:

\[ r = 1000. \ ( x - 3.5 )^{3.45} \quad x \geq 3.5 \]
\[ r = 186.7 \ x^2 - 550.15 \ x + 473.5 \quad 2.0 \leq x \leq 3.5 \]
\[ r = 117.857 \ x^2 - 242.5 \ x + 133.57 \quad 1.3 \leq x \leq 2.0 \]
\[ r = 24.3 \ x^3 - 43.665 \ x^2 + 32.81 \ x - 4.74 \quad 0.4 \leq x \leq 1.3 \]
\[ r = -25.01 \ x^3 + 25.007 \ x^2 - 1.251 \ x + 1.0501 \quad 0.1 \leq x \leq 0.4 \]
\[ r = 50. \ x^2 - 16.5 \ x + 2.3 \quad 0.05 \leq x \leq 0.1 \]
\[ r = 1125. \ x^2 - 110. \ x + 4.2875 \quad x \leq 0.05 \]

where \( x = \frac{Y}{\sigma} \)