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STUDIES OF DYNAMIC RESPONSE OF A GUYED TOWER

Rice University

University
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Ph.D.  1982
RICE UNIVERSITY

STUDIES OF DYNAMIC RESPONSE
OF A GUYED TOWER

by

GEORGES ROBERT DARBRE

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE

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HOUSTON, TEXAS

APRIL 1982
ABSTRACT

The objective of this investigation was to contribute to improved understanding of the dynamic response of offshore guyed towers. This was accomplished through studies of the following interrelated problems:

1. Effects of cable and foundation constraints on the natural frequencies and modes of vibration of representative structural systems;
2. Low-amplitude harmonic stiffness and free vibrational characteristics of inclined parabolic cables;
3. Effects of cable-tower interaction on the dynamic response of the tower; and
4. Response of representative guyed tower models to simulated wave loadings.

Both the force method and the displacement method are applied to the analysis of the first problem, and the interrelationship of the two methods is elucidated. A study is made of the factors that affect the natural frequencies and modes of vibration of guyed towers, and comprehensive numerical data are presented for a proposed structure which illustrate the sensitivity of the results.

Closed-form expressions are presented for the low-amplitude harmonic stiffness of a viscously damped, inclined parabolic cable, and extensive numerical data are included to illustrate the effects of the various parameters involved. Both a complete and an approximate solution are considered, and their interrelationship is examined.
In addition, a single-degree-of-freedom model is proposed which reproduces with good accuracy the salient features of the response of the system over a wide range of conditions.

The free-vibrational characteristics of simply supported cables are re-examined, and a simple procedure is presented for evaluating the complete spectrum of natural frequencies. Useful closed-form expressions are included for certain infinite series involving integrals of the natural modes of vibration of cables. Also described is a finite method of analysis for the problem.

The effects of cable-structure interaction are investigated for a harmonic condition of loading using mainly a single-degree-of-freedom representation for the tower. The implications of the results for more complex representations of the tower and for transient excitations are also considered.

The nonlinear response of the guyed tower is evaluated by application of the pseudo-force method in combination with the modal method of analysis. Flow charts are included for a computer program developed to implement the analysis. The response of representative structures is evaluated for simulated wave loadings obtained from a Pierson-Moskowitz wave spectrum and Morison's equation for wave forces, and the results are analyzed and discussed.
ACKNOWLEDGEMENTS

The reported research has been performed under a research project directed by Dr. A.S. Veletzos and sponsored by Brown & Root, Inc., Houston, Texas. The project was under the administrative direction of Dr. D. Nair. The support of Brown & Root, Inc. is gratefully acknowledged, as are the encouragements and useful comments of Dr. Nair. Parts of chapters II, VII and VIII of the present work had been submitted to Brown & Root, Inc., at an earlier date (1,2). Content of chapters IV and V are published elsewhere (3 to 5).

Financial support has been provided by Rice University and the 'Bata Schuh Stiftung', Zürich, Switzerland. The International Communication Agency, Washington, provided travel grant and financial support; this support was placed under the administrative direction of the Institute of International Education. The generosity of these organizations is gratefully acknowledged.

Sincere appreciation is expressed to Dr. A.S. Veletzos, Thesis Advisor, without whom this work would not have been possible, and to the faculty and staff of the Department of Civil Engineering at Rice University for their cooperation.

Finally, the wise comments and help of Dr. A. Kumar are sincerely appreciated.


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CHAPTER ONE

INTRODUCTION

Scope of Study
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SCOPE OF STUDY

Background

The extraction of oil has become of increasing importance in the past few years. The discovery of fields in deep water regions has set new challenges to engineers when it became apparent that traditional offshore structures would be, either in technical or financial terms, inadequate for the exploitation of these fields.

The behavior of platforms in deep water being of a dynamic type, alternatives thought in terms of dynamic response where sought. In shallow waters, traditional fixed platforms have small fundamental period of vibration $T_{st}$, in comparison to the range of periods $T_{wa}$, of the exciting waves. Their response is therefore close to the static response. In deep waters, the same platforms will have a fundamental period of vibration approaching that of the exciting waves, bringing the structures into resonance or near resonance. Their response will therefore be large in comparison to the static response.

Modifying the structures in order to have a sufficient reduction in the period ratio $T_{st}/T_{wa}$, and thereby reducing the response is a way of overcoming the unacceptable resonance phenomenon. Another way is to increase the period ratio $T_{st}/T_{wa}$; the response will then be of comparable magnitude to the static one or lower.

The 'Guyed Tower' is based on this latter concept. The structure is designed to be flexible enough to have a fundamental period of vibration larger than the period of the waves.

It should be noted that the guyed tower is not the only type of
structure satisfying the deep water dynamic requirements, another one being the 'Tension Leg Platform' (1).

A guyed tower is composed of elements having specific characteristics in terms of function and behavior. The principle ones are:

a. The deck;
b. The space frame or the tower;
c. The foundation system; and
d. The guying system.

The deck supports all the drilling and production equipment. It is supported by the tower. Structurally, the tower is a space frame constrained at the base by the foundation system and near the top by the guying system, and supports an almost rigid platform of large weight. Its behavior is mostly in the linear elastic range of deformations while subjected to exciting forces (wave, current, wind) and to a base excitation (earthquake). It is also subjected to its own gravity and buoyancy forces.

The foundation system provides vertical, lateral and torsional support for the tower, but none against overturning. This is consistent with the concept of compliancy mentioned earlier. The foundation provides its constraint in a nonlinear inelastic motion-dependent manner.

The guying system exerts a lateral nonlinear and motion dependent constraint on the tower. It is intended to provide the necessary overturning resistance in order for the structure to remain stable. The stiffness of the guying system however must remain within reasonable limits to avoid the reduction in the value of the fundamental period to
a value near the range of periods of the waves.

Several authors have recently presented concepts or feasibility studies pertinent to the deep water guyed tower (2 to 4). Although this type of structure is intended to be operative in waters of 1000 to 2000 feet deep, only a 370 feet prototype has been installed in 1975. The first full scale guyed tower will be installed by 1984 in 1000 feet of water.

Outline of Thesis.

The purpose of the reported research is to provide a better understanding of the dynamic behavior of the guyed tower and to develop concepts which will be useful in the design of such structures. This is accomplished by studying a selection of vibrational characteristics related primarily to the dynamics of the tower and the guying system. These characteristics are later juxtaposed to account for the harmonic and the transient behavior of the complete guyed tower system.

The contents of the thesis are divided into eight broad sections, presented as self contained chapters, each one of which is followed by its figures, tables and appendices. The scope of these chapters is briefly outlined below:

Consistent Approach to the Study of Free Vibration Characteristics of Systems with Localized Constraints (Chapter 2). Fundamental to the understanding of the dynamic behavior of any structure is the knowledge of its natural frequencies and modes of vibration. For systems constrained at localized points, the free vibration characteristics can advantageously be expressed in terms of the corresponding characteristics of the
same systems in unconstrained or fully constrained conditions. This con-
cept is presented in two complementary manners. The usefulness of such an
approach in problems involving foundation-structure interaction is thor-
oughly investigated. Further, the interrelationship between this approach
and a Rayleigh Ritz procedure is also recognized.

The novelty of this study lies in complete and consistent treatment
of the above concept in investigating the free vibration characteristics
of constrained systems, and in systematic application to the problems of
interest here.

Studies on the Free Vibration Characteristics of a Pile Supported
Guyed Tower (Chapter 3). The sensitivity of the free vibration character-
istics of a pile founded guyed tower to variations in the stiffness of
the tower, the cables and the foundation is assessed in this chapter.
The constraints provided by the cables and the foundation are such that
the concepts of Chapter 2 can be judiciously applied for systematic
investigation of the system in both lateral and torsional vibration.

The model of a pile founded guyed tower to be used in all applica-
tions of the thesis is also introduced in this chapter.

Low Amplitude Dynamic Stiffness of Parabolic Cables (Chapter 4). The
function of guying cables is to provide restraint to the tower-deck sys-
tem. A study establishing how this restraint is altered by a dynamic ex-
citation, is therefore of utmost interest. The subject matter of change
in dynamic stiffness of a parabolic cable under low amplitude harmonic
excitation is investigated here. This study includes a careful assessment
of the parameters involved, the systematic presentation of the funda-
mental results and interpretation of these results leading to a better appreciation of the guying restraint behavior.

**Free Vibration Characteristics of a Supported-Supported Parabolic Cable (Chapter 5).** This chapter deals with the free vibration of a parabolic cable fixed at both ends. This study is an extension of the subject matter reported in the previous chapter.

Similar study has also been conducted by Irvine (5). The emphasis in this chapter is on the interpretation of the results, leading to a simple expression approximating the frequency spectrum.

**Selected Studies on the Low-Amplitude Behavior of Cables (Chapter 6).** The free vibration behavior of a parabolic cable free to move horizontally at one of its supports is investigated in this chapter. The variation of the cable dynamic stiffness is established in a modal manner.

An alternative approach using concepts of finite element procedure is also presented to assess the dynamic stiffness.

**Studies on Cable-Structure Interaction of Guyed Systems with Application to a Pile Supported Guyed Tower (Chapter 7).** The contents of this chapter put together the tower and the cables studied separately thus far. The effect of the inertia of the guying cables on the tower, termed cable structure interaction, is investigated. The parameters pertinent to the whole system are identified and consequences of their variation studied. The analysis procedure established is applied to the pile founded guyed tower under investigation.

**Method for Transient Analysis of a Guyed Tower and Corresponding Computer Program (Chapter 8).** In this chapter, a computationally effici-
ent method for establishing the transient response is formulated. This method is developed for a two dimensional frame and specialized for a stick-like structure resting on a flexible base and constrained by a non-linear guying system. Second order effects are also taken into account. A discussion of the various methods of analysis and their efficiency in terms of applicability and convergence rate, and a flow-chart introducing the computer program which carries out such an analysis are presented.

Response of Guyed Tower to Lateral Excitations (Chapter 9). The transient response of a guyed tower is presented in this last chapter. Various wave excitations are considered, and selected response quantities discussed in terms of nonlinear behavior and modal response.
REFERENCES


CHAPTER TWO

CONSISTENT APPROACH TO THE STUDY OF FREE VIBRATION CHARACTERISTICS OF SYSTEMS WITH LOCALIZED CONSTRAINTS

Introduction
Force and Displacement Methods
Soil-Structure Interaction
Rayleigh-Ritz Procedure
Conclusions
References
Appendices
Figures
INTRODUCTION

Background

This Chapter is concerned with the development of a computationally efficient method for evaluating the natural frequencies and modes of vibration of structures locally constrained by linear elastic springs. For such systems, the influence of the constraints can most efficiently be investigated by disconnecting those ones from the structure. Such a concept is known as 'substructuring' and extensive studies have been reported in this area (1 to 7). In these studies, each subsystem is analyzed separately and the response of the whole system obtained by synthetizing each subsystem by imposing force and/or displacement compatibilities.

In the particular case of the determination of the frequencies and modes of vibration of the system, the Rayleigh-Ritz procedure is most commonly used. The natural modes of vibration of the whole system are assumed to be composed of a limited number of natural modes of vibration of each subsystem and of displacement patterns satisfying specified compatibility conditions at the junctions of each subsystem. These displacement patterns primarily depend upon the substructures having been permitted to move freely at their junctions while computing their modes of vibration, or having been completely restrained.

Scope

In this Chapter, two different methods of analysis based on the concept of substructuring will be formulated. In the first, called 'force method', the natural frequencies and modes of the constrained structure
will be evaluated in terms of the corresponding quantities of the structure in its unconstrained condition. In the second, called 'displacement method', they will be determined in terms of the corresponding quantities of the structure in its fully constrained condition. Both methods will be described and their accuracy and range of applicability discussed. Special emphasis will be given to the utilisation of the displacement method in the analysis of structures interacting with the foundation medium.
FORCE AND DISPLACEMENT METHODS

Force Method

Preliminaries. Thomson (8) and Davies et al. (9), among others, presented a method of determining the natural frequencies and modes of vibration of linear elastic systems constrained at discrete points by linear elastic springs. In this procedure, the constraints exerted by the springs are replaced by harmonic forces and the equations for evaluating the natural frequencies and modes of vibration of the system are determined by requiring that the deformation of the springs be compatible with the deflections of the structure at its points of attachment to the springs. This will be referred to as the 'force method' by analogy to the well-known procedure used in the static analysis of indeterminate structures. This concept will be extended to provide bases for systematic investigations of the free vibration characteristics of any linear elastic constrained structure. This will first be done for a single constraint and then for an arbitrary number of constraints.

Analysis, Single Constraint. Let consider the discrete structure of figure 1a defined by

1. The mass matrix [M];
2. The natural modes of vibration \( \{ \phi_j \} \);
3. The natural circular frequencies \( \omega_j \); and
4. The number of dynamic degrees of freedom \( n \).

For the same structure constrained by a spring of stiffness \( k_0 \) applied at, and in the direction of the degree of freedom \( x_o \) (Fig. 1b) and vibrating at one of its natural circular frequency \( \omega_i \), the effect of the spring is the same as the one of a harmonic force of suitable amplitude
$F_o$ and frequency of excitation $p_i^*$ applied at the same node. The steady-state deflection of the unconstrained structure under this excitation is

$$
\left\{\begin{array}{l}
\nu_1^* = \sum_{i=1}^{n} \frac{1}{F_o} \frac{F_o \cdot \phi_1^*(x_o)}{\xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi} \frac{1}{1 - \left(\frac{p_i^*}{F_o}\right)^2} e^{j \frac{p_i^* \xi}{F_o}} \\
\end{array}\right.
$$

At the point of location of the spring, it is

$$
\nu_c(x_o) = \sum_{i=1}^{n} \frac{1}{F_o} \frac{F_o \cdot \phi^2_1(x_o)}{\xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi} \frac{1}{1 - \left(\frac{p_i^*}{F_o}\right)^2} e^{j \frac{p_i^* \xi}{F_o}} 
$$

The same force $F_o e^{j p_i^* t}$ opposite in direction acts on the spring (Fig. 1c) and induces a steady-state deflection of

$$
\nu_c(x_o) = -\frac{F_o}{k_o} e^{j \frac{p_i^* \xi}{F_o}}
$$

Displacement compatibility at the point of attachment of the spring to the structure requires to equate Eqs. 2 and 3, and leads to

$$
\frac{1}{k_o} = -\sum_{i=1}^{n} \frac{1}{F_o} \frac{\phi_1^2(x_o)}{\xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi} \frac{1}{1 - \left(\frac{p_i^*}{F_o}\right)^2}
$$

Since, in the absence of any external forces, a system can undergo a steady-state motion only at one of its natural mode of vibration, this equation relates the natural frequencies $p_i^*$ of the constrained structure to the stiffness $k_o$ of the spring thru the natural modes and frequencies of the unconstrained structure.

Once the natural frequencies $p_i^*$ of the constrained system are known, the corresponding modes $\{\psi_i^*\}$ are easily found when it is recognized that they are proportional to the deflection $\{\nu\}$ given by Eq. 1, i.e.
\[ \Psi_{\infty} = \sum_{i=1}^{M} \frac{1}{p_i} \frac{\phi_i^T \mathbf{M} \phi_i}{1 - \left( \frac{p_i}{F_0} \right)^2} \sum \phi_i^2 \]  

(5)

During the derivation of Eqs. 4 and 5, it was implicitly assumed that the modes of vibration of the unconstrained structure were orthogonal with respect to the mass matrix, i.e. \( \phi_i^T \phi_j = 0 \) for \( i \neq j \). In the event that several modes \( \phi_i \) have the same frequency, it would therefore be necessary to choose them so as to satisfy this relationship. It should be emphasized that the expressions for \( p_i^* \) and \( \Psi_{\infty}^* \) are exact as long as the summations are performed over all the modes of the unconstrained structure, including the rigid-body ones if any. Also, the procedure is in no way restricted to structures having only translational degrees of freedom. The force \( F_0 \) should be understood as being either a translational force or a rotational moment, the spring of stiffness \( k_0 \) as being either a translational or a rotational spring, and the degrees of freedom as being translational and/or rotations.

Application. Let us analyze the effect of the addition of a spring of stiffness \( k_0 \) on the natural modes and frequencies of the 3-degrees of freedom system of figure 2a. It is a 3-story shear-building constrained at the upper level. Each story is of stiffness \( k \), with the lower and intermediate floors being of mass \( m \) and the upper of mass \( m/2 \). In its unconstrained condition, the structure has the natural frequencies and modes of vibration of figure 2b.

Application of equation 4 gives:
\[
\frac{1}{k_o} = -\frac{m/g}{.518^2} \frac{1}{1.5m} \frac{1}{1 - \left(\frac{p_1^*}{.518}\right)^2} - \frac{m/g}{1.414^2} \frac{(-1)^2}{1.5m} \frac{1}{1 - \left(\frac{p_1^*}{1.414}\right)^2} - \frac{m/g}{1.932^2} \frac{1}{1.5m} \frac{1}{1 - \left(\frac{p_2^*}{1.932}\right)^2} \tag{6}
\]

This equation is depicted in figure 4. When \(k_o\) is equal to zero, the natural frequencies \(p_1^*\) are the ones of the unconstrained system. Also, when \(k_o\) tends to infinity the natural frequencies \(p_1^*\) are the ones of the 2-DOF system of figure 3. For intermediate values of \(k_o\), the variation of \(p_1^*\) with \(k_o\) is given by Eq. 6. For example, a value of \(k_o=0.5k\) leads to the natural circular frequencies of \(p_1^*=0.720\sqrt{k/m}, p_2^*=1.520\sqrt{k/m}\), and \(p_3^*=2.042\sqrt{k/m}\), respectively.

Once the constrained frequencies \(p_1^*\) have been determined, the analysis can be completed by applying equation 5 so as to find the natural modes of vibration \(\{\psi_i^*\}\). This leads to

\[
\{\psi_i^*\} = \frac{m/k}{.518^2} \frac{1}{1.5m} \frac{1}{1 - \left(\frac{p_1^*}{.518}\right)^2} \left\{ \begin{array}{c} 0.5 \\ 0.866 \end{array} \right\} + \\
+ \frac{m/k}{1.414^2} \frac{1}{1.5m} \frac{1}{1 - \left(\frac{p_1^*}{1.414}\right)^2} \left\{ \begin{array}{c} 1 \\ -0.1 \end{array} \right\} + \\
+ \frac{m/k}{1.932^2} \frac{1}{1.5m} \frac{1}{1 - \left(\frac{p_1^*}{1.932}\right)^2} \left\{ \begin{array}{c} 0.5 \\ -0.366 \end{array} \right\}
\]

These modes are reported in figure 5 for different values of spring stiffness \(k_o\). The way the natural modes of vibration of the unconstrained structure are altered by the presence of the additional spring is to be noticed, more particularly the way they tend to the modes of the 2-DOF system of figure 3 as \(k_o\) tends to infinity. For what the third mode is
concerned, only the node at which the spring is attached will move as the stiffness of the additional spring increases. The third natural circular frequency will therefore tends to \( p_3^* = \frac{\sqrt{k_o}}{m_i Z} \).

\( p_i^* = p_r \): While deriving Eq. 4, the case of \( p_i^* = p_r \) has not been discussed. This represents the case in which the \( i^{th} \) natural frequency of the constrained system is identical to the \( r^{th} \) natural frequency of the unconstrained system. Since equation 4 can be rewritten as

\[
\frac{1}{\kappa_o} \left[ 1 - \left( \frac{p_i^*}{p_r} \right)^2 \right] = \frac{1}{p_r^2} \left( \frac{\phi_{r,i}(\omega_o)}{\phi_{r,i}^*(\omega_o)} \right) = \frac{1}{p_r^2} \left( \frac{\phi_{r,i}(\omega_o)}{\phi_{r,i}^*(\omega_o)} \right) \frac{1}{1 - \left( \frac{p_i^*}{p_r} \right)^2} \]

(9)

\( p_i^* = p_r \) leads to \( \frac{\phi_{r,i}(\omega_o)}{\phi_{r,i}^*(\omega_o)} = \frac{1}{p_r^2} \left( \frac{\phi_{r,i}(\omega_o)}{\phi_{r,i}^*(\omega_o)} \right) \frac{1}{1 - \left( \frac{p_i^*}{p_r} \right)^2} \). This expression is satisfied for \( k_o = 0 \). That is, for the case of the constrained structure being the same as the unconstrained one. If furthermore \( \phi_{r,i}(\omega_o) = 0 \), then any spring stiffness \( k_o \) will satisfy the above expression. That is, if the spring is applied at a node experiencing zero deflection in the \( r^{th} \) mode of the unconstrained structure, then the corresponding natural frequency \( p_r \) is also a natural frequency of the constrained system. Also, there exists a unique value of the spring stiffness \( k_o \) (positive or negative) for which \( p_r \) represents the natural circular frequency of two different modes of the constrained system. By substituting \( \phi_{r,i}(\omega_o) = 0 \) in equation 4 and then setting \( p_i^* = p_r \), this value of spring stiffness is given as

\[
\frac{1}{\kappa_o} = \frac{1}{p_r^2} \left( \frac{\phi_{r,i}(\omega_o)}{\phi_{r,i}^*(\omega_o)} \right) \frac{1}{1 - \left( \frac{p_i^*}{p_r} \right)^2} \]

(10)

Additional Mass. The effect of a concentrated mass can be analyzed in a similar way as the effect of a discrete linear elastic constraint.
As derived from figure 6, it needs only to replace the term \( k_0 \) in equation 4 by \(-m_0 \omega_0^2\), where \( m_0 \) is the value of the inertia of the additional concentrated mass. The equation relating the circular natural frequencies \( \omega_i^2 \) and the value \( m_0 \) of the inertia of the concentrated mass is then

\[
\frac{1}{m_0} = \sum_{i=1}^{\infty} \left( \frac{P_i^*}{P_c} \right)^2 \frac{d_i^2 \omega_i}{d_i^3 \omega_i^2} \frac{1}{1 - (\frac{P_i^*}{P_c})^2}
\]

while the equation for the natural modes of vibration is unchanged (Eq. 5).

**Continuous System.** In the event that it is necessary to investigate a continuous system, Eq. 4, 5 and 11 respectively become

\[
\frac{1}{K_0} = -\sum_{i=1}^{\infty} \frac{1}{P_i^2} \frac{\phi_i^2(x_0)}{\int p(x) \phi_i^2(x) \, dx} \frac{1}{1 - (\frac{P_i^*}{P_c})^2}
\]

\[
\psi_i^0(x) \approx \sum_{i=1}^{\infty} \frac{1}{P_i^2} \frac{\phi_i^2(x_0)}{\int p(x) \phi_i^2(x) \, dx} \frac{1}{1 - (\frac{P_i^*}{P_c})^2} \phi_i(x) \text{ and}
\]

\[
\frac{1}{m_0} = \sum_{i=1}^{\infty} \left( \frac{P_i^*}{P_c} \right)^2 \frac{\phi_i^2(x_0)}{\int p(x) \phi_i^2(x) \, dx} \frac{1}{1 - (\frac{P_i^*}{P_c})^2}
\]

where \( x \) is the coordinate and \( p(x) \) the mass (or inertia) per elementary unit of coordinate \( dx \).

Analysis, Multiple Constraints. Let consider now the same discrete
structure constrained by \( m \) linear elastic springs of stiffness \( k_1 \) to \( k_m \) applied at, and in the direction of the degrees of freedom \( x_1 \) to \( x_m \). This structure will have the circular natural frequencies \( p_1^* \). As seen previously, the effect of the springs on the system vibrating at one of its natural circular frequencies \( p_1^* \) is the same as the one of harmonic forces of suitable amplitude \( F_1 \) to \( F_m \) and frequency of excitation \( p_1^* \) applied at the same constrained nodes. The steady-state deflection of the unconstrained structure under these excitations is

\[
\xi_i = \frac{\sum_{j=1}^{m} \frac{F_j}{k_j} \Phi_i(x_j) + \frac{F_2}{k_3} \Phi_i(x_2) + \ldots + \frac{F_m}{k_m} \Phi_i(x_m)}{\frac{\xi_i \Phi_i^2 \sum_{j=1}^{m} \xi_j \Phi_j^2}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2}} \frac{e^{\frac{i p_1^*}{2} t}}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2} \tag{15}
\]

At the points of application of the springs, the steady-state deflections of the structure are

\[
U_1(x_1) = \frac{\sum_{j=1}^{m} \frac{F_j}{k_j} \Phi_i(x_j) + \frac{F_2}{k_3} \Phi_i(x_2) + \ldots + \frac{F_m}{k_m} \Phi_i(x_m)}{\frac{\xi_i \Phi_i^2 \sum_{j=1}^{m} \xi_j \Phi_j^2}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2}} \frac{e^{\frac{i p_1^*}{2} t}}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2} \Phi_i(x_1) \tag{16}
\]

\[
U_2(x_2) = \frac{\sum_{j=1}^{m} \frac{F_j}{k_j} \Phi_i(x_j) + \frac{F_2}{k_3} \Phi_i(x_2) + \ldots + \frac{F_m}{k_m} \Phi_i(x_m)}{\frac{\xi_i \Phi_i^2 \sum_{j=1}^{m} \xi_j \Phi_j^2}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2}} \frac{e^{\frac{i p_1^*}{2} t}}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2} \Phi_i(x_2) \tag{16}
\]

\[
\vdots
\]

\[
U_m(x_m) = \frac{\sum_{j=1}^{m} \frac{F_j}{k_j} \Phi_i(x_j) + \frac{F_2}{k_3} \Phi_i(x_2) + \ldots + \frac{F_m}{k_m} \Phi_i(x_m)}{\frac{\xi_i \Phi_i^2 \sum_{j=1}^{m} \xi_j \Phi_j^2}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2}} \frac{e^{\frac{i p_1^*}{2} t}}{1 - \left(\frac{p_1^*}{p_j^*}\right)^2} \Phi_i(x_m) \tag{16}
\]

or

\[
\begin{align*}
U_1(x_1) &= \left[ A_{11} F_1 + A_{12} F_2 + \ldots + A_{1m} F_m \right] e^{\frac{i p_1^*}{2} t} \\
U_2(x_2) &= \left[ A_{21} F_1 + A_{22} F_2 + \ldots + A_{2m} F_m \right] e^{\frac{i p_1^*}{2} t} \tag{16} \\
&\vdots \tag{16} \\
U_m(x_m) &= \left[ A_{m1} F_1 + A_{m2} F_2 + \ldots + A_{mm} F_m \right] e^{\frac{i p_1^*}{2} t}
\end{align*}
\]
with

$$A_{HF} = \sum_{j=1}^{M} \frac{1}{F_j^2} \frac{\phi_j(x_j) \cdot \phi_j(x_j)}{\sum_{i=1}^{M} \sum_{k=1}^{M} \phi_i \phi_k} \frac{1}{1 - (p_j^*)^2}$$  \hspace{1cm} (17)

The same forces $F_1 e^{ip_1^* t}$ to $F_m e^{ip_m^* t}$ opposite in direction act on the springs and induce the steady-state harmonic displacements

$$u(x_1) = -\frac{F_1}{K_1} e^{ip_1^* t}$$

$$u(x_2) = -\frac{F_2}{K_2} e^{ip_2^* t}$$

$$...$$

$$u(x_m) = -\frac{F_m}{K_m} e^{ip_m^* t}$$  \hspace{1cm} (18)

Requiring displacement compatibility and therefore equating equations 16 and 18 leads to

$$(A_{11} + \frac{1}{K_1}) F_1 + A_{12} F_2 + ... + A_{1m} F_m = 0$$

$$A_{21} F_1 + (A_{22} + \frac{1}{K_2}) F_2 + ... + A_{2m} F_m = 0$$

$$...$$

$$A_{m1} F_1 + A_{m2} F_2 + ... + (A_{mm} + \frac{1}{K_m}) F_m = 0$$  \hspace{1cm} (19)

This system of equations represents an eigenvalue problem of size $m$. Once a specific circular natural frequency $p_j^*$ of the constrained structure
is assumed, the coefficients A are determined by Eq. 17 and the solution of Eqs. 19 leads to the eigenvalues k and the corresponding eigenvectors \( \{ F_i \} \). The different values of k represent the spring stiffness for which \( p_i^* \) is a circular natural frequency of the constrained system and the eigenvectors \( \{ F_i \} \) introduced into Eq. 15 give the corresponding natural modes of vibration

\[
\{ \phi_i \} \approx \sum_{n=1}^{\infty} \frac{1}{r_{1n}^2} \frac{\phi_i \cdot \phi_n(x_1) + \phi_i \cdot \phi_n(x_2) + \cdots + \phi_i \cdot \phi_n(x_n)}{\phi_i \cdot \phi_n(x_n)} \phi_n \cdot \phi_n \frac{1}{1 - \left( \frac{p_n}{p_1} \right)^2} \tag{20}
\]

**Application.** Application of the above equations to analyze the 3-DOF system of figure 2b now constrained by a spring of stiffness \( k_l \) at the lower level and one of stiffness \( k_2 \) at the upper level (figure 7) leads to the characteristic equation

\[
\left( \frac{1}{k_{1l}} + \frac{1}{k_{2l}} \right) \left( \frac{1}{k_{1s}} + \frac{1}{k_{2s}} \right) = 0
\]

If it is furthermore specified that \( k_1 = 2 \cdot k_2 \), this becomes a quadratic equation in \( k_2 \) (or \( k_1 \)) which solution is plotted in Fig. 8. It should be noticed that, in this example, two different values of spring stiffnesses \( k_2 \) can induce the same natural frequency; for example, \( k_2/k = 0.25 \) gives \( p_1^* = 0.707 \sqrt{\text{k/m}} \), \( p_2^* = 1.581 \sqrt{\text{k/m}} \), and \( p_3^* = 2.0 \sqrt{\text{k/m}} \), while \( k_2/k = 1 \) gives \( p_1^* = 1.7 \sqrt{\text{k/m}} \), \( p_2^* = 2.0 \sqrt{\text{k/m}} \), and \( p_3^* = 2.236 \sqrt{\text{k/m}} \). This was to be expected since the characteristic equation is quadratic.

**Particularities.** In comparison with the particularities discussed for a single spring constraint, only the case of a constrained frequency \( p_i^* \) being equal to an unconstrained frequency \( p_i \) needs to be revised. In a manner similar to what Meek (10) did in the problem of soil-structure
interaction, such a case can be investigated by rewriting Eq. 15 as

\[
\{\omega\} = \frac{1}{P^2} \sum_{j \neq r} \frac{F_{ij} \phi_j(x_i) + F_{ij} \phi_j(x_j) + \ldots + F_{nj} \phi_j(x_m)}{\varepsilon d_1^{1/3} n_2^{1/3} d_2^{1/3}} \{d_{ji}\} \frac{e^{i \gamma \varepsilon r}}{1 - (\frac{\varepsilon}{\gamma})^2} + C_r \{\phi_r\} \frac{e^{i \gamma \varepsilon r}}{1 - (\frac{\varepsilon}{\gamma})^2} \tag{21}
\]

where

\[
C_r = \frac{1}{P^2} \frac{F_{ir} \phi_r(x_i) + F_{ir} \phi_r(x_m) + \ldots + F_{nr} \phi_r(x_m)}{\varepsilon d_1^{1/3} n_2^{1/3} d_2^{1/3}} \frac{1}{t - (\frac{\varepsilon}{P})^2} \tag{22}
\]

is the participation of the \(r\)\(^{th}\) unconstrained mode when \(p_i^*\) is equal to \(p_r\). In order for expression 21 to remain finite, it is necessary that

\[
F_{ir} \phi_r(x_i) + F_{ir} \phi_r(x_m) + \ldots + F_{nr} \phi_r(x_m) = 0 \tag{23}
\]

This represents a new condition on \(F_{ir}\) to \(F_{nr}\). Also, Eq. 19 can then be written as

\[
(A_{i1,r} + \frac{1}{\nu_k}) F_i + A_{i2,r} F_2 + \ldots + A_{im,r} F_m + C_r \phi_r(x_i) = 0
\]

\[
A_{i2,r} F_2 + (A_{i2,r} + \frac{1}{\nu_k}) F_2 + \ldots + A_{im,r} F_m + C_r \phi_r(x_i) = 0
\]

\[
\ldots \ldots
\]

\[
A_{im,r} F_m + A_{im,r} F_m + \ldots + (A_{im,r} + \frac{1}{\nu_k}) F_m + C_r \phi_r(x_m) = 0
\]

where

\[
\nu_k \phi_r = \frac{1}{P^2} \sum_{i \neq r} \frac{F_{ij} \phi_j(x_i) \cdot \phi_j(x_r)}{\varepsilon d_1^{1/3} n_2^{1/3} d_2^{1/3}} \frac{1}{1 - (\frac{\varepsilon}{\gamma})^2} \tag{25}
\]

Applied to the example of figure 8 for \(p_r = \sqrt{1.932} \text{ k/m}\), equations 24 and 25 give a value of \(k_2 = 0.866 \text{ k}\) in addition to the trivial solution \(k_2 = 0\).
If the investigation of a continuous system or of the effects of multiple additional concentrated masses is necessary, substitutions identical to the ones done for single constraints have to be made.

**Displacement Method**

**Preliminaries.** In the present section, it will be shown how the natural frequencies and modes of vibration of a fully-constrained structure can be used to evaluate the corresponding quantities of a partially constrained structure. The spring constraints will be treated as rigid supports undergoing harmonic motion, and the frequency equation will be determined by requiring that equilibrium holds at the points of contact of the springs with the structure. By analogy to the procedure used in the static analysis of indeterminate structure, this method will be referred to as the 'Displacement Method'. The procedure for a single constraint will first be presented and then expanded to include multiple constraints. This latter case will be limited to structures having a diagonal mass matrix.

**Analysis, Single Constraint.** Let first consider the discrete structure of figure 9a defined by the mass matrix \([M']\), the stiffness matrix \([K']\), and the number of degrees of freedom \(n'\). Let also consider the same structure with an additional support at the node \(x_0\) (figure 9b) defined by

- The mass matrix \([M]\);
- The stiffness matrix \([K]\);
- The natural modes of vibrations \(\{\psi_i\}\);
- The natural circular frequencies \(p_j\); and
- The number of dynamic degrees of freedom \(n\).
The mass and stiffness matrices of both systems are related by

\[
[K'] = \begin{bmatrix} k_n & k_{n,2} \\ k_{n,1} & k_{n,2} \end{bmatrix} \quad [K] = \begin{bmatrix} k_n \end{bmatrix}
\]
\[\begin{bmatrix} \mathbf{m}' \end{bmatrix} = \begin{bmatrix} m_n & m_{n,2} \\ m_{n,1} & m_{n,2} \end{bmatrix}, \quad [\mathbf{m}] = \begin{bmatrix} m_n \end{bmatrix}
\]

and the number of degrees of freedom by

\[n' = n + 1\]

(26b)

The subscript '2' refers to the node at which the additional support is applied (i.e., node 'x_o'), and the subscript '1' to the remaining nodes of the system. When the latter structure, called 'fully constrained', is submitted to a harmonic support excitation of frequency \(p_1^*\) and amplitude \(y(x_o)\), it is known from the theories pertinent to support excited systems (11) that the steady-state deflection of the structure can be decomposed into two parts (Fig. 9c). The first one is the deflection

\[\{q_2^2\} \cdot y_o \cdot e^{ip_1^*t}\]

(27)

with \(\{q_2^2\}\) being the static deflection of the system induced by a unit displacement of the support (Fig. 9d). The second one corresponds to the steady-state deflection

\[\{u_2^2\} \cdot y_o \cdot e^{ip_1^*t}\]

(28)

of the fully constrained structure (Fig. 9c) due to the force excitation

\[p_2^* [M] \{q_2^2\} y_o e^{ip_1^*t}\]

(29)

The latter deflection is therefore equal to

\[\{u_2^2\} y_o e^{ip_1^*t} = \frac{\sum \{q_2^2\} y_o e^{ip_1^*t}}{1 - (\frac{p_1^*}{p_2^*}) y_o e^{ip_1^*t}}\]

(30)
where the notation \( \{ \phi_j \} \) has been introduced for sake of consistency in the size of the matrices. The mode \( \{ \phi_j \} \) is equal to the mode \( \{ \phi_j \} \) to which a dummy node with zero deflection has been added at the support, i.e.

\[
\{ \phi_j \} = \{ \phi_j \} \quad \text{(31)}
\]

If the support reaction associated with the \( j \)th mode of the fully constrained structure is defined as \( F_j \), then the support reaction \( F_{o \cdot \varepsilon_{p_{o} \varepsilon_{p}}} \) due to the deflection \( \{ u \}_{o \cdot e_{p_{o} e_{p}}} \) is equal to

\[
F_{o \cdot \varepsilon_{p_{o} \varepsilon_{p}}} = \sum_{i=1}^{n} \frac{r_i^{* 2}}{d_i^{* 2}} \{ \varepsilon_{p_{o} \varepsilon_{p}} \}^T \{ M_{o} \} \{ \phi_j \} \frac{F_j}{1 - \left( \frac{r_i^{* 2}}{d_i^{* 2}} \right)^2 \{ \varepsilon_{p_{o} \varepsilon_{p}} \}^T \{ \gamma_{o} \}} \quad \text{(32)}
\]

Also, as shown in Fig. 9c and 9b, the forces required to produce respectively a harmonic support displacement \( \{ u \}_{o \cdot e_{p_{o} e_{p}}} \) and a static support displacement \( \{ \gamma_{o} \} \) are \( F_{o \cdot \varepsilon_{p_{o} \varepsilon_{p}}} \) and \( F_{o \cdot \gamma_{o}} \), respectively, related by the expression

\[
F_{o}^* = F_{o} - F_{o \cdot \varepsilon_{p_{o} \varepsilon_{p}}} \quad \text{(33)}
\]

Let assume now that the structure of Fig. 16a is constrained by a linear elastic spring of stiffness \( k_o \) at the node \( x_o \) (Fig. 10a). All the derivations done for a fixed support remain valid in that case if the force compatibility at the junction of the spring and the structure is duly considered. This latter reads (Fig. 10b)

\[
F_{o}^* + k_o = 0 \\
\text{or by substituting Eq. 33}
\]

\[
k_o = F_{o} + \frac{r_i^{* 2}}{d_i^{* 2}} \{ \eta_{21} \} \{ \eta_{21} \} \quad \text{(34)}
\]
Since a structure on which no external forces are applied can undergo a steady-state harmonic motion only at one of its natural modes of vibration, Eq. 34 is indeed the frequency equation of the partially constrained system of Fig. 10a. This expression relates the spring stiffness $k_0$ and the natural circular frequencies $p^2_1$ of any linear elastic partially constrained structure through the natural frequencies and modes of vibration of the fully constrained structure.

Also, the corresponding natural modes of vibration of the partially constrained structure are given by Eqs. 27 and 28, namely
\[
\{q_c^1\} = \{q\} + \{u_c^0\} \quad (35)
\]

**Application.** Let apply these expressions to analyze the free vibration characteristics of the 3 DOF shear structure, partially constrained at the upper level, of Fig. 2a. The natural frequencies and modes of the structure in its fully constrained condition and the modal support reactions $F_1 = K$ and $F_2 = -K$ are shown in Fig. 3. The force $F_0$ required to induce a unit static deflection at the upper level is equal to $F_0 = 1/3 K$, and the expression $\sum n_i n_2 \{q_i\}$ to $\sum n_i n_2 \{q_i\} = \frac{1}{2} m$. The expression $\sum n_i n_2 \{q_i\}$ is equal respectively to $\{q_1\} n_2 \{y_1\} = m$ for the first mode and $\{q_2\} n_2 \{y_2\} = -\frac{m}{2}$ for the second one. Applying Eq. 34 and substituting Eq. 32 therefore leads to
\[
k_0 = \frac{p_c^2}{l^2 \cdot K/m} \cdot \frac{m}{2} \cdot \frac{1}{1 - \left(\frac{p_c}{l_c^2}\right)^2 \frac{m}{M}} + \frac{p_c^2}{l^2 \cdot K/m} \cdot \frac{1}{2} \cdot \frac{-\frac{1}{2} m}{1 - \left(\frac{p_c}{l_c^2}\right)^2 \frac{m}{M}}
\]
\[
+ p_c^2 \cdot \frac{1}{2} \cdot m - \frac{1}{2} \cdot \kappa \quad (36)
\]

The equation leads to the same results as Eq. 6, depicted in Fig. 4.
Special Cases. As done for the force method, particular cases can be investigated and the expressions extended to continuous systems. The case of the mass matrix being diagonal is worth noting for its physical explanation of Eq. 34. If $M_{22}$ is replaced by $m_0$, the value of the inertia of the mass at the node $x_0$, the equation becomes

$$k_0 = F_n + p_x^2 m_0 - F_0$$

(35)

As seen in figure 11, this equation expresses the equilibrium of the spring when the concentrated mass $m_0$ is assumed to be part of the constraining system.

Also, if the effect of a concentrated mass of total inertia $m_0$ at the node $x_0$ was to be investigated, setting $k_0 = 0$ in Eq. 35 would directly lead to the frequency equation.

Analysis, Multiple Constraints. Finally, let us consider the discrete structure of Fig. 12a defined by the diagonal mass matrix $[M']$, the stiffness matrix $[K']$ and its number of dynamic degrees of freedom $n'$. When constrained by $m$ linear elastic springs of stiffness $k_1$ to $k_m$ applied at the degrees of freedom $x_1$ to $x_m$ (Fig. 12b), the same structure will have the circular natural frequencies $p^*_i$. In order to find their relationship with the values of the spring stiffness by application of the displacement method, let us also consider the fully constrained structure of Fig. 12c defined by

- The diagonal mass matrix $[M]$;
- The stiffness matrix $[K]$;
- The natural modes of vibration \( \xi \); 
- The natural circular frequencies \( p_j^* \); and 
- The number of dynamic degrees of freedom \( n \).

Eq. 26a relates both mass and stiffness matrices, while

\[
\mathbf{n} = n + m 
\]  
(36)

Again, the total deflection can be decomposed into a 'static' part
\[
\mathbf{\xi} \mathbf{v} = \sum_{k=1}^{m} \mathbf{p}_k \mathbf{v}_k \]  
and a 'dynamic' part \( \mathbf{\xi} \mathbf{u} \mathbf{e}^* \). By calling \( \mathbf{\xi} \mathbf{q}_k \) the static deflection of the fully constrained structure due to an unit displacement of the support \( x_k \), \( F_k(x_k) \) the necessary static force to produce this unit deflection and \( F_r(x_k) \) the corresponding reaction at the support \( x_r \) \((r \neq k)\), then the equilibrium of each constraining element can be written as

\[
\begin{align*}
\mathbf{p}_1 \mathbf{m}_1 \mathbf{y}(x_1) + \mathbf{F}_1(x_1) \mathbf{y}(x_1) & = \mathbf{F}_1(x_1) \mathbf{y}(x_1) + \mathbf{F}_2(x_1) \mathbf{y}(x_2) + \ldots + \mathbf{F}_m(x_1) \mathbf{y}(x_m) = \mathbf{k}_1 \mathbf{y}(x_1) \\
\mathbf{p}_2 \mathbf{m}_2 \mathbf{y}(x_2) + \mathbf{F}_1(x_2) \mathbf{y}(x_2) & = \mathbf{F}_2(x_2) \mathbf{y}(x_2) + \ldots + \mathbf{F}_m(x_2) \mathbf{y}(x_m) = \mathbf{k}_2 \mathbf{y}(x_2) \\
& \quad \vdots \\
\mathbf{p}_m \mathbf{m}_m \mathbf{y}(x_m) + \mathbf{F}_1(x_m) \mathbf{y}(x_m) & = \mathbf{F}_2(x_m) \mathbf{y}(x_m) + \ldots + \mathbf{F}_m(x_m) \mathbf{y}(x_m) = \mathbf{k}_m \mathbf{y}(x_m)
\end{align*}
\]  
(37)

where \( m_r \) is the inertia of the concentrated mass at the node \( x_r \), and \( F_{uk}(x_r) \) is the support reaction at the same node induced on the fully constrained structure by the inertia forces associated with the deflection \( \mathbf{\xi} \xi \); i.e.

\[
F_{uk}(x_r) = \sum_{j=1}^{m} \frac{m}{n} \frac{\mathbf{p}_j^* \mathbf{p}_j}{\mathbf{p}_j^* \mathbf{p}_j - \mathbf{1}} \left[ \mathbf{q}_k \mathbf{m}_j \mathbf{y}(x_j) \right] \frac{1}{1 - (\mathbf{p}_j^* \mathbf{p}_j)^2} \mathbf{F}_j(x_r)
\]  
(38)

with \( F_j(x_r) \) being the \( j^{th} \) modal reaction at the node \( x_r \) of the structure in its fully constrained condition. Substituting Eq. 38 into Eq. 37.
leads to the system of Eq. 39 representing an eigenvalue problem of size m.

\[ (B_{11} - \kappa_1) \ddot{y}(x_1) + B_{12} \ddot{y}(x_2) + \ldots + B_{1m} \ddot{y}(x_m) = 0 \]

\[ B_{21} \ddot{y}(x_1) + (B_{22} - \kappa_2) \ddot{y}(x_2) + \ldots + B_{2m} \ddot{y}(x_m) = 0 \]

\[ \ldots \]

\[ B_{m1} \ddot{y}(x_1) + B_{m2} \ddot{y}(x_2) + \ldots + (B_{mm} - \kappa_m) \ddot{y}(x_m) = 0 \]  

(39)

where

\[ B_{r+i} = F_{r+i}(x_r) + \sum_{j=1}^{n} \frac{F_{r+i}}{F_{r+j}} \frac{\eta_j \phi_j^T \Gamma \eta_j \phi_j^T}{\xi \phi_j^T \Gamma \eta_j \phi_j^T} \frac{1}{1 - (\frac{F_{r+i}}{F_{r+j}})^2} F_{r+i}(x_r) \]

(40)

\[ B_{r+r} = F_{r+r}(x_r) + \sum_{j=1}^{n} \frac{F_{r+r}}{F_{r+j}} \frac{\eta_j \phi_j^T \Gamma \eta_j \phi_j^T}{\xi \phi_j^T \Gamma \eta_j \phi_j^T} \frac{1}{1 - (\frac{F_{r+r}}{F_{r+j}})^2} F_{r+r}(x_r) \]

The solution of this eigenvalue problem is indeed the characteristic equation relating the circular natural frequencies \( \omega \) of the partially constrained structure with the stiffness \( k_1 \) to \( k_m \) of its constraining spring through the natural frequencies and modes of the same structure in its fully constrained condition.

**Force Method versus Displacement Method**

As stated previously, both the forces and the displacement methods lead to the exact frequency equations when the summations are performed over all the modes of the unconstrained and fully constrained structure, respectively. When only a limited number of modes are considered, the
expressions will be approximations of the exact characteristic equations. This is particularly true for continuous systems where in no case the summations can be performed over an infinite number of modes. Two questions arise then:

1. How many modes need to be considered so as to get sufficiently accurate expressions; and

2. Which of the two methods will give the most accurate expressions, considering equivalent number of modes.

Let investigate these two questions in light of the convergence in the solutions of the characteristic equations of respectively an uniform shear beam and an uniform flexural beam, with a fixed support at the end and a lateral spring constraint at the other end. The solutions for the fundamental frequency of both systems are shown in Fig. 13 and 14 respectively for both the force and displacement methods. It should be remembered that a zero value of the spring stiffness $k_0$ means the absence of any spring, and that an infinite value means the presence of a rigid support. Studies of the fundamental frequency curves show that in order to have sufficient convergence over a wide range of spring stiffness, it is necessary to consider at least 2 modes of the unconstrained and fully constrained system, respectively. This is consistent with the rule generally adopted in the Rayleigh-Ritz procedure. Also, the force method will be more accurate for low values of spring stiffness while the displacement method will be more accurate for large values. This was to be expected since the force method uses the unconstrained modes as basis of analysis, while the displacement method uses the fully constrained ones.
It should be remembered that, in the displacement method, not only the modes of the fully constrained structure are taken, but also additional 'static' deflections. The comparison should take this into account and the consideration of t modes in the displacement method should be compared with the consideration of t+1 modes in the force method (when a single constraint is involved).

It appears that, in general, it is necessary to consider at least twice as many modes as desired in the solution for both methods, and that the force method offers better convergence for low values of spring stiffness while the displacement method offers better convergence for large values of spring stiffness. Thomson (8) presented a way to increase the convergence of the force method; this however requires to work with a set of quartic linear equations.
SOIL-STRUCTURE INTERACTION

Simplified Form of Displacement Method

Whenever the fully constrained system is statically determinate, the displacement method can be simplified. Instead of computing the modal support reactions through the deformations of the structure, the inertia forces associated with the individual modes are introduced and the overall equations of equilibrium written. Also, the forces required to produce unit static displacements at the supports are zero and so are the corresponding other support reactions. By contrast, the force method applied to a similar case would not be simplified; it would require to first get the natural modes and frequencies of an unstable system, what might present a slight difficulty. An example of such a system is a structure resting on a elastic foundation medium. Bielak (12) presented limited results on the natural modes and frequencies of a multistory shear building vibrating laterally, while Papadopoulos and Trujillo (13) investigated the natural frequencies of a Timoshenko beam on flexible base. In this section, the displacement method will be applied to systematically investigate the effect of the base flexibility on the lateral natural modes and frequencies of multistory shear buildings. The possibility of modelling the response of the higher modes by one of equivalent single story buildings will also be briefly discussed. The procedure will also be extended to include the torsional vibration of a structure resting on a flexible medium.

Lateral Vibration. Let consider the fixed-base multistory shear building of Fig. 15 defined by
- The diagonal mass matrix \([M]\);
- The natural lateral modes of vibration \(\xi_{i}^{2}\);
- The corresponding natural circular frequencies \(\omega_{j}\);
- The elevation from the base of each story \(\xi_{j}\); and
- The number of lateral dynamic degrees of freedom \(n\), which is equal to the number of stories.

Tajimi (14), among others, related the natural frequencies and modes of the structure vibrating laterally on a flexible base in terms of the corresponding quantities of the same structure in its fixed-base condition. This is written as

\[
\begin{bmatrix}
\Delta K_{s,\text{str}} - \Delta K_{s,\text{eff}} \\
\end{bmatrix}
\begin{bmatrix}
\dot{X} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
\end{bmatrix} \tag{41}
\]

where

\[
\Delta K_{s,\text{str}} = \begin{bmatrix}
\kappa_{\text{xx}} & \kappa_{\text{xy}} \\
\kappa_{\text{yx}} & \kappa_{\text{yy}} \\
\end{bmatrix} \tag{42}
\]

and

\[
\Delta K_{s,\text{eff}} = \begin{bmatrix}
\kappa_{x} & \kappa_{\phi} \\
\kappa_{\phi x} & \kappa_{\phi \phi} \\
\end{bmatrix} \tag{43}
\]

The elements of the stiffness matrix \([K_{s,\text{str}}]\) are defined as

\[
\kappa_{\text{xx}} = \rho_{c}^2 \left[ \frac{n_1}{\sum_{i=1}^{n} \left( \frac{p_{c}^{2}}{P_{i}^{2}} \right) \frac{m_{i}^{2}}{\xi_{i}^{2}} \frac{1}{\xi_{i}^{2} - \left( \frac{P_{c}^{2}}{P_{i}^{2}} \right)^{2}} \frac{1}{1 - \left( \frac{P_{c}^{2}}{P_{i}^{2}} \right)^{2}} \right] + m_{o} \tag{44}
\]

\[
\kappa_{\text{xy}} = \kappa_{\text{yx}} = \rho_{c}^2 \left[ \frac{n_1}{\sum_{i=1}^{n} \left( \frac{p_{c}^{2}}{P_{i}^{2}} \right) \frac{m_{i}^{2}}{\xi_{i}^{2}} \frac{1}{\xi_{i}^{2} - \left( \frac{P_{c}^{2}}{P_{i}^{2}} \right)^{2}} \frac{1}{1 - \left( \frac{P_{c}^{2}}{P_{i}^{2}} \right)^{2}} \right] \cdot s_{o} \tag{44}
\]

\[
\kappa_{\text{xx}} = \rho_{c}^2 \left[ \frac{n_1}{\sum_{i=1}^{n} \left( \frac{p_{c}^{2}}{P_{i}^{2}} \right) \frac{m_{i}^{2}}{\xi_{i}^{2}} \frac{1}{\xi_{i}^{2} - \left( \frac{P_{c}^{2}}{P_{i}^{2}} \right)^{2}} \frac{1}{1 - \left( \frac{P_{c}^{2}}{P_{i}^{2}} \right)^{2}} \right] \cdot I_{o} \tag{44}
\]

\(k_{X}, k_{\phi}\) and \(k_{X\phi}\) are respectively the lateral, rocking and coupled lateral-
rocking stiffness of the foundation system. \( \bar{m}_j \) is the \( j^{th} \) modal mass defined as

\[
\bar{m}_j = \xi_j \bar{d}_j^T \bar{E} \bar{M} \bar{J} \xi_j \xi_j
\]  

and \( \bar{E}_j \) is the \( j^{th} \) first modal mass moment of inertia with respect to the base defined as

\[
\bar{E}_j = \xi_j \bar{d}_j^T \bar{E} \bar{M} \bar{J} \bar{D}_3
\]

\( m_o \) is the total mass of the building, including the base; \( S_o \) is the first mass moment of inertia of the building with respect to the base; and \( I_o \) is the second mass moment of inertia of the structure with respect to the base, including the discrete values lumped at the floors and base (Fig. 15). Eq. 41 is an eigenvalue equation of size 2, with the base lateral displacement \( X \) and the rocking angle \( \Theta \) - identical at each floor - being the elements of the eigenvector. Once this equation has been solved for the eigenvalues \( p_i^* \) and the eigenvectors \( \{\xi_i\} \), the corresponding flexible base modes of vibration can be written as

\[
\{\xi_i\} \simeq \left[ \begin{array}{cc}
N a_i \left( \frac{p_i^*}{p_i} \right)^2 & \frac{\bar{m}_j}{\xi_j \bar{d}_j^T \bar{E} \bar{M} \bar{J} \xi_j} \\
N a_i \left( \frac{p_i^*}{p_i} \right)^2 & \frac{\bar{E}_j}{\xi_j \bar{d}_j^T \bar{E} \bar{M} \bar{J} \bar{D}_3}
\end{array} \right] \cdot \left[ \begin{array}{c}
\xi_j \xi_j + \xi_j \xi_j \\
\xi_j \xi_j + \xi_j \xi_j + \frac{1}{1 - (\frac{p_i^*}{p_i})^2}
\end{array} \right] \cdot X + \left[ \begin{array}{cc}
N a_i \left( \frac{p_i^*}{p_i} \right)^2 & \frac{\bar{E}_j}{\xi_j \bar{d}_j^T \bar{E} \bar{M} \bar{J} \bar{D}_3}
\end{array} \right] \cdot \Theta
\]  

\( \{\xi_i\} \) is the total displacement vector for the upper stories, whose \( j^{th} \) element is defined as \( X + \Theta \cdot u_j - u_j \) in Fig. 16. The procedure is valid for both frequency-dependent and frequency-independent foundation stiffnesses. In further applications, frequency-independence will be assumed. Also, the summation can be performed over a limited number of fixed-base
modes, and the above equations become approximations to the true expressions.

The case $p_i = p_r$ has been discussed by Meek (10) who showed that a mode other than the $r^{th}$ fixed-base mode occurs when the following relationship is satisfied between $\chi$ and $\theta$:

$$\chi + \frac{\bar{g}_r}{m_r} \theta = 0$$  \hspace{1cm} (48)

**Application.** Let apply the above equations to systematically analyze the 3-story shear building of Fig. 17 whose fixed-base modes and frequencies have previously been presented (Fig. 2b). The structure rests on an homogeneous elastic halfspace through a rigid circular foundation of radius $r$. The constraint exerted by the soil is assumed to be frequency-independent, and equivalent to the static values $k_x = \frac{G\sigma r}{2(1-\nu)}$, $k_\theta = \frac{G\sigma r^3}{3(1-\nu)}$ and $k_{x\theta} = 0$. $G$ is the shear modulus of the soil and $\nu$ the poisson's ratio. The frequency curves for the values of $I = \frac{h r^4}{2}$, $h = 3$, and $\nu = 0.45$ are depicted in Fig. 18 and selected modes of vibration in Fig. 19. A value of $k/Gr$ equal to zero corresponds to an infinitely stiff soil. For that case, three first natural frequencies are the ones of the structure in its fixed-base condition while the two higher ones are infinite. From Fig. 19 it can be seen that, for very stiff soils, the two higher modes are (1) a lateral motion of the base without rocking nor lateral motion of the upper floors, and (2) a rocking of the base without any lateral motion. Under the assumption of a shear building, each floor will experience the same rocking motion as the base. Therefore, the frequency curves for the two higher modes in a region of stiff soil can be approximated by
\[ \rho^* = \sqrt{\frac{k_G}{m_G}} \quad \text{and} \quad \rho^* = \sqrt{\frac{k_G}{I_G + \frac{m_G}{g} I_I}} \] (49a)  
(49b)

For very soft soils, i.e. \( k/Gr \to \infty \), the two lower modes are rigid body modes with zero frequency while the three higher ones are deformation modes with non-zero frequency. The way the frequencies and modes are modified for intermediate values of soil flexibility is depicted in Figs. 18 and 19. Of particular interest are the 3\textsuperscript{rd} and 4\textsuperscript{th} frequencies in the vicinity of \( k/Gr = 2 \) (Fig. 18), where both frequencies "cross" each other. This phenomenon can best be explained with reference to equivalent single story buildings presented next. It will be shown that this represents the region where the base mass and the second mass moments of inertia start having a significant importance for what the 3\textsuperscript{rd} mode is concerned.

**SDOF Representation**

As a simplification, the deformation of the flexible base structure is sometimes assumed to be proportional to the one of the fixed-base structure in the configuration of the mode investigated, i.e. equivalent single story buildings are considered (see, for example, Ref. 15). The extent to which such an assumption is valid is considered here through the analysis of an 8-story shear building. Each floor is of mass \( m \) and of second mass moment of inertia \( I \), except for the top one whose same quantities have the values \( m/2 \) and \( I/2 \). All stories are of height \( h \) and stiffness \( k \). The rigid and circular foundation is of radius \( r \), mass \( m_b \) and second mass moment of inertia \( I_b = I \). The stiffness of the foundation
system is given by \( k^x = \frac{3Gr}{2-\nu} \), \( k^\varphi = \frac{Ghr^3}{3(1-\nu)} \) and \( k^x \approx 0 \).

**Exact.** Fig. 20 shows the exact frequency curves for \( h/r = 1 \) and \( \nu = 0.45 \) and (1) \( m_b = 0, I = 0 \); (2) \( m_b = m, I = 0 \); (3) \( m_b = m, I = \frac{mr^2}{2} \). In the first case \( (m_b = 0, I = 0) \), the natural frequencies are affected more significantly in the lower modes than in the higher ones. The higher modes experiencing more deformation than the lower ones, the ratio of the strain energy accumulated in the superstructure versus the strain energy stored in the foundation medium will be larger for the higher modes than for the lower ones. Therefore, the higher modes will be less affected by the foundation medium than the lower ones. In the second case \( (m_b = m, I = 0) \), an additional dynamic degree of freedom is introduced by the presence of the base mass \( m_b \). As seen previously this additional mode is, for stiff soils, a lateral base displacement without any other motion and its circular natural frequency is approximately given by equation 49a. In the stiff soil region, this mode is therefore affected only by the stiffness of the soil and the mass of the base, and not by the properties of the superstructure; i.e. it depends directly on the soil flexibility. When the frequency as approximated by Eq. 49a is of the same order of magnitude as the deformation frequencies, then these latter frequencies start being strongly affected by the soil flexibility. This is not apparent in figure 20 since that happens for \( k/Gr \) larger than 2, but can be seen in the same diagram for the third case \( (m_b = m, I = \frac{mr^2}{2} \) where a further dynamic degree of freedom has been introduced by the presence of second moments of inertia. The natural frequency associated with the base lateral displacement is the 10th one, while the one
associated with rocking (Eq. 49b) is the 9\textsuperscript{th} frequency. As seen in the figure, the effect of rocking starts affecting the higher deformation modes near $k/Gr = 0.6$.

**SDOF Representation.** Let us now investigate this 8-story shear building by assuming that the deformation modes can be approximated by equivalent single story modes and that the two higher frequencies are given by expression 49. The quantities characterizing the fixed-base single story building modelling the $j$\textsuperscript{th} fixed-base mode of the $n$ story building are defined as (see, for example, Ref. 16)

$$
\omega_j^* = \left( \frac{1 + \frac{k_j}{k_0} + \frac{I_j^*}{I_{j_0}}}{} \right)^{1/2} \hspace{1cm} (50a)
$$

$$
\xi_j^* = \frac{h_j^* \xi_n^*}{\xi \kappa^*} \hspace{1cm} (50b)
$$

$$
\kappa_j^* = \kappa^* \hspace{1cm} \text{and by extension} \hspace{1cm} (50c)
$$

$$
I_{j_0}^* = \sum_{i=1}^{n} I_i \hspace{1cm} (50d)
$$

The natural circular frequencies of the equivalent single story shear building on a flexible base are then (23)

$$
P_{c_i}^* = P_{i_0} \sqrt{1 + \frac{k_i^*}{k_0} + \frac{I_{j_0}^*}{k_0}} \hspace{1cm} \text{for } M_b = 0 \text{ and } I_i^* = I_{j_0}^* = 0 \hspace{1cm} (51a)
$$

$$
P_{c_i}^* = P_{i_0} \sqrt{1 + \frac{k_i^*}{(k_0 - M_b P_{i_0}^2) + I_{j_0}^*}} / k_0 \hspace{1cm} \text{for } I_i^* = I_{j_0}^* = 0 \text{ and } M_b \neq 0 \hspace{1cm} (51b)
$$

$$
P_{c_i}^* = P_{i_0} \sqrt{1 + \frac{k_i^*}{(k_0 - M_b P_{i_0}^2) + I_{j_0}^*}} / (k_0 - (I_{j_0}^* + I_{j_0}^*) P_{i_0}^2) \hspace{1cm} \text{for } M_b \neq 0 \text{ and } I_i^* + I_{j_0}^* \neq 0 \hspace{1cm} (51c)
$$

When neither the base mass nor the mass moments of inertia are equal to
zero, Eq. 51a is still often used as an approximation even though only Eq. 51c represents the exact solution for the equivalent single story building and would give the three natural frequencies of the structure on a flexible base.

Fig. 21 shows the curves obtained by applying equation 51c to compute the deformation frequencies of the 8-story shear building with finite base mass and second mass moments of inertia, and equation 49 to compute the two higher frequencies. The single story approximation gives accurate results for all frequencies as long as the single story model is not affected by the rocking of the structure.

When applying Eq. 51b instead (Fig. 22), then larger discrepancies appear when the base lateral motion strongly affects the deformation mode of the equivalent single story building; the effect of the second mass moments of inertia lumped at the floors is also totally disregarded, what modifies the true picture of the higher deformation modes.

When approximation 51a is used ($m_b = 0$ and $I^* = I_b = 0$; Fig. 23), then the single story model is valid for soil stiffness for which the frequencies are only slightly affected by rocking and base lateral motion.

In any case, the accuracy of the single story model does not depend much on the formula used but more on the region of soil stiffnesses. For soil stiffness so as the rocking or base lateral motion affecting greatly the response of the structure, the single story building approximation is unacceptable. A limit for this region can easily be found by reporting Eqs. 49 ($p^* = \sqrt{\kappa_x / m_b}$ and $p^* = \sqrt{\omega / (L_b + \frac{E}{J})}$) in Figs. 21 to 23. For
ratios of \( k/Gr \) leading to deformation frequencies of the single story
models of magnitude so as to remain on the left hand side of both curves,
the single story approximation gives accurate results. All three equations
51a to 51c give good results in the region of soil stiffness of concern,
with the one putting the less emphasis on the base lateral motion and
rocking (Eq. 51a, \( m_b = I_b = \sum_j N_j, I_j = 0 \)) being the most accurate. The single
story structure is therefore a perfectly acceptable model of interacting
multistory buildings when the analysis has to be performed for soil
stiffnesses of magnitude so as to remain in the region just defined.

**Torsion**

In a manner similar to what has been done for lateral motion, torsional
motion can also be investigated. Let consider a fixed-base discrete
structure with uncoupled torsional modes defined by

- The diagonal mass matrix \( [M] \), where 'mass' is to be understood as
  'polar mass moment of inertia';
- The natural torsional modes of vibration \( \xi_t \);
- The natural circular frequencies \( \omega_j \); and
- The number of torsional dynamic degrees of freedom \( n \), which is equal
to the number of stories.

The characteristic frequency equation of the flexible base structure in
torsion is then, by applying the "simplified" deformation method,

\[
\kappa = \frac{\beta^2}{\zeta} \sum_{i=1}^{n} \left( \frac{\rho_i \bar{\phi}_i}{\mu_i} \right)^2 \frac{\bar{t}_i^2}{\bar{M}_{i}^2 \bar{d}_i^4} \frac{1}{1 - \left( \frac{\rho_i \bar{\phi}_i}{\mu_i} \right)^2 + \omega_i^2} \quad (52)
\]

where \( \rho_i \) is the \( i \)th circular natural frequency of the flexible base
structure, \( \mu_j \) the \( j \)th modal polar mass moment of inertia of the fixed-
-base structure as defined by equation 45, and \( m_0 \) the total polar mass moment of inertia of the structure, including the base. \( k_\phi \) is the rotational stiffness provided to the structure by the foundation medium, as described by Veletso and Nair for example (17). The flexible base torsional modes of vibration are then

\[
\{\psi_1, \psi_2, \psi_3, \psi_4\} = \begin{bmatrix} M_2 \left( \frac{R^2}{r^2} \right)^2 & \frac{\bar{m}_2}{\bar{d}_1} & \frac{1}{\bar{d}_2} & \frac{\bar{d}_3}{\bar{d}_4} \end{bmatrix} \begin{bmatrix} \bar{d}_1 \bar{d}_2 \bar{d}_3 \bar{d}_4 \end{bmatrix}^{1/2} \begin{bmatrix} \phi_1 \phi_2 \phi_3 \phi_4 \end{bmatrix}
\]

(53)

\( \{\psi_3\} \) is the total rotation of each node of the superstructure, while \( \phi \)
is the rotation of the base.
RAYLEIGH-RITZ PROCEDURE

The Rayleigh-Ritz procedure is the method most commonly used in substructuring. It is shown here that when suitable displacement functions are chosen, it is equivalent to the force method, and the simplified displacement method when applied to soil structure interacting systems. The Rayleigh-Ritz procedure will then be used to derive the characteristic frequency equation of a flexural building resting on a flexible base.

Equivalence to Force Method

After the procedure, the following fundamental relationships hold (18):

\[
\{\psi^*_i\} = \sum_{j=1}^{n} a_j \{\phi^*_j\} \quad (54)
\]

\[
P^* = \frac{V}{T} \quad \text{and} \quad (55)
\]

\[
P^* = \frac{\partial V / \partial \alpha}{\partial T / \partial \alpha} \quad \text{for} \quad r = 1 \text{ to } n \quad (56)
\]

\{\phi^*_j\} are the displacement vectors introduced to approximate the mode \{\psi^*_i\}, \[j\] are unknown coefficients to be determined from equation 56, \(V\) is the maximum strain energy of the system in the configuration of the mode \{\phi^*_j\}, \(T\) is the corresponding maximum pseudo kinetic energy defined as the true maximum kinetic energy divided by \(p^*_i\), and \(n\) is the number of independent displacement vectors \{\phi^*_j\}.

Applying above equations to a structure constrained by a single spring and using the unconstrained modes as displacement functions leads to, with the notation used in the section on the force method,

\[

\text{eqn} (57) \quad \text{is no longer valid}
\]
\[ T = \frac{1}{2} \left\{ \psi_r^* \right\}_c \left\{ M \right\}_c \left\{ \psi_r^* \right\}_c \quad \text{and} \quad (57b) \]

\[ P_r^f = \frac{\partial V}{\partial \alpha_r} = \frac{\alpha_r \left\{ \psi_r^* \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c}{\left\{ \psi_r \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c} \quad \text{for } r = 1 \ldots n \quad (57c) \]

or

\[ \alpha_r = \frac{\kappa_0 \psi_r(x_0) \phi_r(x_0)}{P_r^f \left\{ \psi_r \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c} \quad (57d) \]

Recognizing that, for the unconstrained modes, the following relationship holds exactly:

\[ P_r^f = \frac{\left\{ \psi_r \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c}{\left\{ \psi_r \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c} \quad \text{for } r = 1 \ldots n \quad (58) \]

finally leads to

\[ \alpha_r = -\kappa_0 \psi_r(x_0) \frac{1}{P_r^f} \frac{\phi_r(x_0)}{\left\{ \psi_r \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c} \frac{1}{1 - \left( \frac{P_r^f}{P_r^*} \right)^2} \quad \text{and} \quad (59) \]

\[ \left\{ \psi_r \right\}_c \frac{1}{P_r^f} \frac{1}{\left\{ \psi_r \right\}_c \left\{ M \right\}_c \left\{ \psi_r \right\}_c} \frac{1}{1 - \left( \frac{P_r^f}{P_r^*} \right)^2} \left\{ \psi_r \right\}_c \quad (5) \]

This equation is identical to equation 5. The force method is therefore equivalent to the Rayleigh-Ritz procedure when the unconstrained modes are used as displacement functions. This can be shown to be also true for the force method applied to multiconstrained structures.

**Equivalence to Displacement Method**

The Rayleigh-Ritz procedure is also equivalent to the displacement method applied to flexible base shear buildings. Using the fixed-base
lateral modes of vibrations as deformation functions leads to, with the
notation used in the section on soil-structure interaction,

\[
\{q^*_e\} = \sum_{j=1}^{n} a_j \{\phi_j\} + x_1 \{1\} + \Theta \{h\} \quad \text{for the superstructure} \quad (60a)
\]

\[x = k \quad \text{and} \quad \Theta = \Theta \quad \text{for the base} \quad (60b)
\]

\[
T = \frac{1}{2} \sum_{j=1}^{n} \left[ a_j \{\phi_j\}^T \{k\} \{\phi_j\} + \frac{1}{2} k \Theta^2 + k \Theta \right] + \sum_{j=1}^{n} \left[ \frac{1}{2} k \Theta^2 \right]
\]

\[\text{and} \quad (61a)
\]

\[
T = \frac{1}{2} \sum_{j=1}^{n} \left[ \phi_j^T \{h\} \phi_j + \frac{1}{2} \omega_\phi \phi_j^T \phi_j + \frac{1}{2} \left( I_3 + \frac{2}{3} I_4 \right) \Theta^2 \right]
\]

\[\text{and} \quad (61b)
\]

\[a_1 \text{ to } a_n, \ x \text{ and } \Theta \text{ are to be found from Eq. 56.} \text{ This leads to the same}
\]

characteristic equation as expression 41. \text{ The Rayleigh-Ritz procedure}

and the displacement method are therefore equivalent when applied to soil

structure interacting systems.

**Flexural Building**

As presented earlier, the analysis of soil structure interacting

systems was limited to shear buildings with a single degree of freedom

per node. \text{ By using the Rayleigh-Ritz procedure, buildings with two degrees}

of freedom per node, \text{i.e.} translational and rocking, can be analyzed

readily. Let us consider the fixed-base multistory building defined by

- The diagonal mass matrix \([M]\);
- The natural modes of vibration \(\{\phi_j\}\);
- The natural circular frequencies \(\nu_j\);
- The elevation from the base of each story \(\{h\}\); and
-The number of dynamic degrees of freedom \( n \), \( n/2 \) of them being associated with lateral motion and the remaining ones with rocking motion. \( n/2 \) is also the number of stories.

Identifying the lateral and rocking degrees of freedom respectively by the subscripts '1' and 'r', the displacement \( \mathbf{w} \) of the superstructure can be written as

\[
\begin{bmatrix}
\mathbf{w}^l \\
\mathbf{w}^r
\end{bmatrix} = \mathbf{X} \begin{bmatrix}
\mathbf{1} \\
\mathbf{0}
\end{bmatrix} + \mathbf{\Theta} \begin{bmatrix}
\mathbf{l} \\
\mathbf{r}
\end{bmatrix} + \begin{bmatrix}
\mathbf{u}^l \\
\mathbf{u}^r
\end{bmatrix}
\]  

(62)

the deformation \( \{u\} = \begin{bmatrix}
\mathbf{u}^l \\
\mathbf{u}^r
\end{bmatrix} \) being equal to

\[
\{u\} = \frac{n}{L} \{\phi\}; \quad \{\phi\} = \begin{bmatrix}
\mathbf{1} \\
\mathbf{0}
\end{bmatrix} \{\phi^l\}
\]

(63)

Applying Eqs. 54 and 56 and assuming that the mass and the second mass moment of inertia of the base are uncoupled then leads to the same characteristic equation as in the case of shear buildings, i.e.

\[
\begin{bmatrix}
\mathbf{K}_{s,l}
\end{bmatrix} - \begin{bmatrix}
\mathbf{K}_{s,11}
\end{bmatrix} \{\mathbf{X}\} = \{\mathbf{0}\}
\]

(41)

The only differences reside in the definitions of \( \overline{m}_j \) and \( \overline{S}_j \). \( \overline{m}_j \) is the \( j \)th translational modal mass and is given by

\[
\overline{m}_j = \{\phi^l\}^T \mathbf{M} \{\phi^l\}
\]

(64)

while \( \overline{S}_j \) is the \( j \)th first modal mass moment of inertia defined as

\[
\overline{S}_j = \begin{bmatrix}
\{\phi^l\}^T \\
\{\phi^r\}^T
\end{bmatrix} \left[ \begin{bmatrix}
\mathbf{M}^p \\
\mathbf{M}^r
\end{bmatrix} + \mathbf{r}^T \right] \begin{bmatrix}
\{\phi^l\} \\
\{\phi^r\}
\end{bmatrix}
\]

(65)
Notice also that $[\eta']$ is in fact the diagonal mass matrix used in the case of shear buildings, while $[\Lambda']$ is a diagonal inertia matrix whose elements are the second mass moments of inertia of each story. The derivation of above expressions is presented in appendix A.
COMPUTER PROGRAMS

Programs 'UNCON'

A set of computer programs has been developed to evaluate the natural frequencies and modes of any linear elastic system constrained by a single linear elastic spring. On the basis of informations on the unconstrained structure, the programs will compute:

. The stiffnesses of the constraining spring corresponding to natural frequencies of the constrained structure; and
. The natural frequencies and modes of vibration of the constrained structure corresponding to a specific spring stiffness.

While the first computation is a direct application of Eqs. 4 and 10, the second requires the use of an iterative procedure to evaluate the natural frequencies.

Programs 'BASE'

Another set of computer programs has been developed to evaluate the natural frequencies and modes of any linear elastic shear building vibrating laterally on a foundation medium defined by the frequency-independent stiffnesses $k_x$, $k_\theta$ and $k_{x\theta}$. On the basis of informations on the fixed base structure, the programs will compute:

. The stiffnesses of the foundation springs corresponding to the natural frequencies of the flexible base structure; and
. The natural frequencies and modes of vibration of the flexible base structure corresponding to specific base stiffnesses.

While the first computation is a direct application of Eq. 41, the second also requires the use of an iterative procedure to evaluate the natural frequencies.
CONCLUSIONS

Two methods thought in term of substructuring, the force and the displacement methods, were developed to permit the analysis of the effect of localized constraints on the free vibration characteristics of structural systems.

In the force method, the constraints are replaced by harmonic pseudo forces. The natural frequencies and modes are then found by considering the compatibility at the levels of the constraints. In the displacement method, the constraints are replaced by rigid supports and pseudo-statical deflections are introduced. For the case of statically determinate structures, a typical example being buildings resting on a flexible base, overall equilibrium considerations appear to be an efficient way to simplify the latter method. It was shown that the force method and the simplified displacement method as applied to shear buildings are in effect equivalent to the Rayleigh-Ritz procedure when the appropriate displacement functions are used.
REFERENCES


APPENDIX A

The derivation of Eq. 41 for the case of a soil structure interacting system with two degrees of freedom per node, i.e. translational and rocking, is presented.

Let write the expression for the maximum strain energy, by making use of the orthogonality relationships of the modes, as

\[ \mathcal{T} = \frac{1}{2} \{ \mathbf{u} \} \mathbf{D} \{ \mathbf{u} \} + \frac{1}{2} \mathbf{k} \{ \mathbf{x} \} \{ \mathbf{x} \} + \frac{1}{2} \mathbf{k}_0 \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \mathbf{k}_{x \theta} \{ \mathbf{x} \} \{ \mathbf{\theta} \} 
= \frac{1}{2} \sum_{i=1}^{n} a_i \{ \phi_i \} \{ \phi_i \} \mathbf{k} \{ \phi_i \} \{ \phi_i \} + \frac{1}{2} \mathbf{k}_0 \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \mathbf{k}_{x \theta} \{ \mathbf{x} \} \{ \mathbf{\theta} \} \]  \hspace{1cm} (A1)

The maximum pseudo-kinetic energy is similarly obtained as

\[ \mathcal{T} = \frac{1}{2} \{ \mathbf{u} \} \mathbf{D} \{ \mathbf{u} \} + \frac{1}{2} \mathbf{k}_0 \{ \mathbf{x} \} \{ \mathbf{x} \} + \frac{1}{2} \mathbf{k}_0 \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \mathbf{k}_{x \theta} \{ \mathbf{x} \} \{ \mathbf{\theta} \} 
+ \frac{1}{2} \mathbf{\theta} \mathbf{\theta} \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \frac{1}{2} \mathbf{\theta} \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \mathbf{k}_{x \theta} \{ \mathbf{x} \} \{ \mathbf{\theta} \} 
+ \frac{1}{2} \mathbf{\theta} \mathbf{\theta} \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \frac{1}{2} \mathbf{\theta} \{ \mathbf{\theta} \} \{ \mathbf{\theta} \} + \mathbf{k}_{x \theta} \{ \mathbf{x} \} \{ \mathbf{\theta} \} \]  \hspace{1cm} (A2)

where the following notation has been introduced:

\[ \mathbf{m} = \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} + \mathbf{m}_0 = \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} + \mathbf{m}_0 \]

= total mass of the structure, including the base;

\[ \mathbf{I}_0 = \{ \mathbf{\phi} \} \mathbf{I} \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} + \mathbf{I}_b = \{ \mathbf{\phi} \} \mathbf{I} \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} + \mathbf{I}_b \]

= total second mass moment of inertia of the structure with respect to the base;

\[ \mathbf{S}_0 = \{ \mathbf{\phi} \} \mathbf{S} \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} = \{ \mathbf{\phi} \} \mathbf{S} \{ \mathbf{\phi} \} \{ \mathbf{\phi} \} \]

= total first mass moment of inertia of the structure with respect
to the base;

\[ \bar{m}_j = \frac{1}{2} \dot{\phi}_j \times \dot{\phi}_j \]

= \( j \)th modal mass; and

\[ \bar{p}_j = \frac{1}{2} \dot{\phi}_j \times \dot{\phi}_j \]

= \( j \)th first mass moment of inertia with respect to the base.

Differentiating Eqs. A1 and A2, as required by Eq. 56, gives

\[ \frac{\partial v}{\partial x} = a_n \left[ e_{u_i} \right]_{u_j} \bar{m}_n \bar{\phi}_n \]

\[ \frac{\partial v}{\partial x} = \theta \cdot k_x + \theta \cdot k_{x0} \]

\[ \frac{\partial v}{\partial \theta} = \theta \cdot k_x + \chi \cdot k_{x0} \]

\[ \frac{\partial a_n}{\partial x} = a_n \left[ e_{u_i} \right]_{u_j} \bar{m}_n + \chi \cdot \bar{m}_n + \theta \cdot \bar{\phi}_n \]

\[ \frac{\partial a_n}{\partial \theta} = \theta \cdot I_0 + \chi \cdot \bar{S}_n + \frac{\partial \bar{S}_n}{\partial \theta} \]

Application of Eq. 56 finally gives

\[ p_c^2 = \frac{\partial v}{\partial x} = \frac{a_n \left[ e_{u_i} \right]_{u_j} \bar{m}_n + \chi \cdot \bar{m}_n + \theta \cdot \bar{\phi}_n}{\alpha_n \left[ e_{u_i} \right]_{u_j} \bar{m}_n + \chi \cdot \bar{m}_n + \theta \cdot \bar{\phi}_n} \]

or

\[ a_n = \left( \frac{p_c^2}{p_h^2} \right) \frac{X \cdot \bar{m}_n + \theta \cdot \bar{\phi}_n}{\alpha_n \left[ e_{u_i} \right]_{u_j} \bar{m}_n + \chi \cdot \bar{m}_n + \theta \cdot \bar{\phi}_n} \]

(\text{A3})

\[ p_c^2 = \frac{\partial v}{\partial x} = \frac{X \cdot k_x + \theta \cdot k_{x0}}{\chi \cdot k_x + \theta \cdot k_{x0}} \]

(\text{A4})

\[ p_c^2 = \frac{\partial v}{\partial \theta} = \frac{\theta \cdot I_0 + \chi \cdot \bar{S}_n}{\theta \cdot I_0 + \chi \cdot \bar{S}_n} \]

(\text{A5})
Finally substituting Eq. A3 into Eqs. A4 and A5 leads to Eq. 41.
Fig. 1a - Discrete Structure
Fig. 1b - Constrained Structure
Fig. 1c - Structure and Constraint

Fig. 2a - 3 DOF System with Spring
Fig. 2b - Modes of Unconstrained 3 DOF System

Fig. 3a - 2 DOF System
Fig. 3b - Modes of 2 DOF System
Fig. 4 - Natural Frequencies of Constrained
3 DOF System
Fig. 5 - Modes of 3 DOF Constrained System
(cont'd on next page)
Fig. 5 - Continued  
(cont'd on next page)
Fig. 5 - Concluded
Fig. 6a - Discrete Structure
Fig. 6b - Structure with Additional Mass
Fig. 6c - Structure and Additional Mass

Fig. 7 - 3DOF System with Two External Constraints
Fig. 8 - Natural Frequencies of 3 DOF System with Two External Constraints
Fig. 9a - Discrete Structure
Fig. 9b - Structure with Additional Support
Fig. 9c - Decomposition of Deflection

Fig. 9d - "Static" Deflection
Fig. 9e - "Dynamic" Deflection and Excitation

Fig. 10a - Structure with Spring Support
Fig. 10b - Force Compatibility
Fig. 11 - Equilibrium of the Constraining System

Fig. 12a - Discrete Structure
Fig. 12b - Partially Constrained Structure
Fig. 12c - Fully Constrained Structure
Fig. 13a - Uniform Shear Beam by the Force Method

Fig. 13b - Uniform Shear Beam by the Displacement Method
Fig. 14a - Uniform Flexural Beam by the Force Method

Fig. 14b - Uniform Flexural Beam by the Displacement Method
Fig. 15 - Multistory Shear Building

\[
[M] = \begin{bmatrix}
    m_1 & m_2 & \cdots & m_n
\end{bmatrix}
\]

\[
\{h\} = \begin{bmatrix}
    h_1 \\
    h_2 \\
    \vdots \\
    h_n
\end{bmatrix}
\]

\[
m_0 = m_1 + \frac{m_2}{2} \bar{h}^2
\]

\[
\bar{h} = \frac{\sum_{i=1}^{n} h_i m_i}{\sum_{i=1}^{n} m_i}
\]

\[
I_0 = I_1 + \frac{m_1}{\bar{h}^2} (I_2 - m_1 \bar{h}^2)
\]

Fig. 16 - Deformation of Structure

Fig. 17 - 3 Story Shear Building
Fig. 18 - Natural Frequencies of 3 Story Shear Building Resting on Homogenous Elastic Halfspace

Parameters:

\[ \zeta = \frac{m^2}{\pi^2} \]

\[ \frac{1}{\xi} = 1 \]

\[ \nu = 0.48 \]
Fig. 19 - Modes of 3 Story Shear Building
(cont'd on next page)
Fig. 19 - Continued
(cont'd on next page)
Fig. 19 - Concluded
Fig. 20 - Natural Frequencies of 8 Story Building Resting on Homogenous Elastic Halfspace
Fig. 21 - Natural Frequencies of 8 Story Shear Building
Fig. 22 - Natural Frequencies of 8 Story Shear Building
Fig. 23 - Natural Frequencies of 8 Story Shear Building
CHAPTER THREE

STUDIES ON THE FREE VIBRATIONAL CHARACTERISTICS OF A PILE-SUPPORTED GUYED TOWER

Introduction
Models
Presentation and Analysis of Results
Conclusions
Tables
Figures
INTRODUCTION

Knowledge of the free vibrational characteristics of a structure is certainly important to the understanding of its response to a dynamic excitation.

The primary objective of this study is to identify the parameters which influence the free vibrational characteristics of a pile-supported guyed tower and to assess their influence. More specifically, it is desired to evaluate the influence of the following factors:

The stiffness of the guying system;
The stiffness of the tower; and
The stiffness of the foundation system.

The study is made by application of the force and displacement methods described in the preceding Chapter, making use of the computer programs described therein.

The study refers to a particular structure, but the results are interpreted in general terms and may be adapted to other guyed tower systems. It is shown that the guying system affects significantly only the fundamental modes of lateral vibration, whereas the lateral stiffness of the foundation and the stiffness of the tower itself affect primarily the higher order lateral modes and the predominantly torsional modes, the latter being also controlled by the torsional resistance of the foundation. It is further shown that, within realistic limits, changes in the rocking stiffness of the foundation have minimal consequences on the modes of vibration.
MODELS

Structure Considered

The system investigated was developed in the course of a preliminary feasibility study by Brown & Root, Inc., Houston TX. Shown in Fig.1, it is a 3-dimensional steel frame supported on piles and constrained laterally by a guying system.

Tower. The tower is 1,695 ft high and it is to be located in a water depth of 1,600 ft. Excepting the upper part of the structure which is somewhat longer in one direction, its plan dimensions are 120 ft by 160 ft, as shown in the figure.

Disregarding the unsymmetrical arrangement of the transverse framing, the tower may be considered symmetric in the x-direction. In the y-direction, the structure is unsymmetric because of its longer dimensions near the deck and the eccentric arrangement of the main piles and conductors.

All structural members are tubular and circular. The dimensions of the main piles vary from a diameter $D = 54''$ and wall thickness $t = 1.25''$ near the top to $D = 76''$ and $t = 1.75''$ near the base. The corresponding dimensions of the horizontal members vary from $D = 30''$ and $t = 0.75''$ to $D = 36''$ and $t = 1.375''$, the larger members being located at the base and at the level of the guying system. The horizontal members supporting the main piles have an uniform diameter of $D = 48''$ and vary in wall thickness from $t = 1''$ at the top to $t = 1.5''$ at the base. Finally, the diagonal members have a constant diameter of $D = 36''$ and their wall thickness varies from $t = 0.875''$ at the top to $t = 1.375''$ at the bottom.
A group of 35 conductors located on the side of the structure opposite to the piles and of dimensions $D = 20''$ and $t = 0.635''$ also contribute to the lateral resistance.

**Foundation.** The foundation system is composed of a group of 9 main piles arranged around, and at center of, a circle on one side of the tower, and a group of 8 shear piles uniformly spaced around the tower. The main piles are designed to provide vertical support to the structure, whereas the shear piles are designed to provide lateral and torsional support.

The main piles are attached to the top of the tower and extend vertically through the tower to a depth of 400 ft into the ground. The shear piles are 150 ft long. The main piles as well as the shear piles have a diameter of $D = 66''$ and a wall thickness of $t = 3''$ to $t = 2''$.

In addition to the vertical resistance, the main piles offer lateral resistance but because of their close spacing offer no significant resistance to rocking. Being fairly short, the shear piles offer practically no vertical or rocking resistance but offer lateral resistance and the principal source of the torsional resistance of the foundation. Also contributing to the lateral and torsional resistances of the foundation are the conductors.

The resistance of the foundation system was obtained as follows. The load-deflection curves for the individual piles and conductors were computed by application of a program in which these elements are treated as beams on an elastic nonlinear foundation. The secant stiffnesses cor-
responding to the expected levels of displacement and rocking at ground level were then evaluated, and were used as the linearized stiffnesses of the individual piles and conductors.

Twelve pairs of identical coated steel cables of the spiral strand type are uniformly arranged around the structure with points of attachment to the tower at an elevation of 1,480 ft from the base. Each individual cable is composed of three segments as illustrated in Fig. 2. They are

1. A cable segment attached to the tower whose characteristics are:
   (a) diameter of 5" with corresponding steel area of 14.5 in²;
   (b) Young's modulus of 21,300 ksi; (c) breaking stress of 200 ksi;
   (d) pretensioning stress of 30% of the breaking stress; and (e)
   unstretched length of 3,480 ft;
2. A clump weight of 940 kips under water distributed over a length of 157 ft; and
3. A cable segment whose back end is anchored to the sea floor. Its characteristics are the same as those of the first segment except that its length is 1,800 ft.

The length of the horizontal projection of the first segment when the system is at rest is 3,170 ft.

The clump weight is designed so as to minimize the increment in tension at large tower deflections and to reduce the vertical component of the reaction at the anchor.

The role of the guying system is to provide lateral resistance to the tower. A torsional resistance is also provided. It comes from the
deformation of the cables out of their vertical plane and from the eccentric arrangement of the cables with respect to the instantaneous center of rotation of the tower.

The force-displacement relationships of the guying systems were obtained as follows. The cable array was assumed to be connected to a rigid frame free to move horizontally and fixed vertically. A lateral displacement of increasing magnitude was then applied to the frame and the lateral and vertical forces required to maintain this displacement were computed by application of a finite element program which takes due account of the nonlinearities due to changes in cable geometry and tension. The resulting force-displacement curves are shown in Fig.3. The initial lateral stiffness of the cable system is 275 kips/ft. Unless otherwise indicated, this value is utilized in the analyses that follow.

Discrete Model

The actual three dimensional structure was idealized for purposes of analysis as a discrete, cantilever, linear, stick model with nine nodes. The nodes are distributed along the height of the structure as indicated in Fig.4. Three degrees of freedom are assigned to each node, one in each of the lateral directions and a rotational one in the horizontal plane.

Flexibility Matrix. The flexibility matrix of the model was determined from an analysis of the actual system. Specifically, the flexibility coefficients in the x-direction were determined by applying unit lateral forces in the x-direction at the elevation of the nodes at points corresponding to the center of the group of main piles and computing the
deflections of those points in the x-direction. The effects of both the foundation constraints and of the cable system were provided for in this evaluation. The resulting flexibility matrix is given in Table 1.

The flexibility matrix in the y-direction was expressed in the form

\[ D_y = \begin{bmatrix} D_{yy} & D_{yt} \\ D_{ty} & D_{tt} \end{bmatrix} \]  

where \( D_{yy} \) represents the displacements in the y-direction of the center of the group of main piles at the elevations of the nodes due to unit horizontal forces in the y-direction at those points, and \( D_{ty} \) represents the corresponding rotations of the cross section of the structure at those elevations. Similarly, \( D_{tt} \) represents the submatrix of the rotations of the cross section of the structure at the elevations of the nodes due to unit torques applied at those levels. Again, due account was taken in these evaluations of the effects of the foundation constraints and the resistance of the cable system. The submatrices \( D_{yy}, D_{tt}, \text{ and } D_{ty} \) are given in Tables 2a, 2b, and 2c, respectively.

It should be noted that the P-\( \Delta \) effects, representing the effects of the vertical forces acting through the lateral deflections, were not considered in the formulation of the flexibility matrix. These effects are studied in Chapter Nine.

**Inertia Matrix.** The inertia matrix of the model was determined by lumping at the individual nodes the actual structural masses, entrained masses of water, and virtual added masses. The inertia matrix associated with the x-direction is then diagonal, the mass coefficients being listed
in Table 3. The inertia matrix corresponding to motion in the y-direction was expressed in the form

\[ m_\Delta = \begin{bmatrix}
m_{yy} & m_{yt}
m_{ty} & m_{ty}
\end{bmatrix} \]  

(2)

where the submatrices \( m_{yy} \), \( m_{tt} \) and \( m_{yt} = m_{ty} \) are diagonal. The elements of \( m_{tt} \) represent the mass moments of inertia of the individual nodes about an axis through the center of the main group of piles, and the elements of \( m_{yt} \) represent the products of \( m_{yy} \) and the horizontal distances along the x-axis of the center of the group of piles and the centers of mass for the individual nodes. The elements of \( m_{yy} \), \( m_{tt} \) and \( m_{yt} \) are also listed in Table 3. Note that the values of \( m_{xx} \) and \( m_{yy} \) are different. This is due to the fact that the virtual masses of water are different in the two directions of motion.

**Natural Periods and Modes.** With the flexibility and inertia matrices of the model established, the natural periods and modes are obtained readily from the solution of the appropriate eigenvalue problem.

For motion in the x-direction, the values of the first three natural periods turn out to be 26.7 sec., 4.82 sec. and 2.30 sec., respectively. These periods and the associated modes are shown in Fig.5a. The fundamental mode is essentially a straight line increasing from almost zero at the base to a maximum at the top. There is practically no structural deformation associated with it, the displacement being due almost exclusively to base rocking. The second and third modes are deformational modes with significant base lateral displacement.
The modes in the y-direction are coupled lateral-torsional modes. The first four periods are 27.0 sec., 8.81 sec., 6.41 sec. and 3.49 sec., respectively. These periods and the associated modes are shown in Fig. 6. The rotational components of the motion are expressed in terms of the displacements of the center of the cross section. These displacements are obtained by multiplying the angles of rotation by 60 ft, which is the distance from the center of the main piles to the center of the cross section. The first mode is a predominantly lateral mode similar to the corresponding mode in the x-direction; the second mode is a torsional mode; and the third one is a lateral deformational mode similar to the second mode in the x-direction.

Modified Models

Flexibility Matrices of Modified Models. Application of the force method of analysis described in the preceding Chapter requires knowledge of the natural frequencies and modes of vibration of the structure without the guying system. The first part of this section describes briefly the derivation of the flexibility of the relaxed system for motions in the symmetric, x-direction.

The element $(D_{xx})_{ij}$ of the flexibility matrix presented in Table 1 represents the displacement at node i produced by an unit horizontal force applied at the $j^{th}$ node of the actual system with the guying system. The corresponding element of the matrix without the guying system, $(D_{xx}^{wo})_{ij}$, may be obtained by superposition as follows. Consider an unit force applied at node j of the system with the cable-spring at node 4. The reaction at the point of attachment of the spring is then $k(D_{xx})_{4j}^,$
where \( k \) is the stiffness of the spring. Imagine that a force \( F \) is applied at node 4 such that \( F \) minus the resulting reaction at that node is equal and opposite to \( k (D_{xx})_{4j} \). The force \( F \) is defined by the equation

\[
F \left[ 1 - k (D_{xx})_{44} \right] = k (D_{xx})_{4j}
\]

whence

\[
F = \frac{k (D_{xx})_{4j}}{1 - k (D_{xx})_{44}}
\]  

(4)

The desired flexibility elements of the system without the spring are then

\[
(D_{xx}')_{ij} = (D_{xx})_{ij} + \frac{k (D_{xx})_{4j}}{1 - k (D_{xx})_{44}} (D_{xx})_{4i}
\]  

(5)

The values of the flexibility coefficients \((D_{xx}')_{ij}\) are reported in Table 4.

In the application of the displacement method presented in the preceding Chapter, one similarly needs the natural frequencies and modes of vibration of the structure without the cables but with the base fixed against both rocking and deflection. Let \((D_{xx}^f)_{ij}\) be the ij element of the flexibility matrix of the structure for this condition of support for motion in the symmetric, x-direction.

The elements \((D_{xx}^f)_{ij}\) may be determined from the corresponding elements of the elastically supported structure, \((D_{xx}')_{ij}\), as follows. Consider an unit horizontal force applied at the \( j^{th} \) node, located at a distance \( h_j \) from the base. The base shear is then unity, and the base moment is \( h_j \). Let \( d_{xx} \) be the horizontal displacement of the foundation due to an unit force at that level, and \( \delta_{xx} \) be the corresponding rotation of the
foundation. Similarly let \( d_{\theta \theta} \) be the rotation of the foundation due to an unit rocking moment and \( d_{x \theta} = d_{\theta x} \) be the corresponding horizontal displacement. It follows that the base displacement

\[
x = \left[ d_{xx} + d_{\theta x} h_i \right]
\]

and the base rotation

\[
\theta = \left[ d_{\theta x} + d_{\theta \theta} h_i \right]
\]

Then \( (D^f)_{xx} \) is related to \( (D^{wo})_{xx} \) by the equation

\[
(D^f)_{ij} = (D^{wo})_{ij} - \left[ d_{xx} + d_{\theta x} h_i \right] h_j - \left[ d_{\theta x} + d_{\theta \theta} h_i \right] h_j
\]  

The values of \( d_{xx} \), \( d_{\theta \theta} \) and \( d_{x \theta} = d_{\theta x} \) for the system under consideration were determined to be

\[
d_{xx} = 1.714 \cdot 10^{-4} \text{ ft/kips} ;
\]
\[
d_{\theta \theta} = d_{\theta x} = 1.072 \cdot 10^{-7} \text{ 1/kips} ; \text{ and}
\]
\[
d_{x \theta} = 3.675 \cdot 10^{-8} \text{ 1/kips ft} ,
\]

and the values of the flexibility coefficients \( (D^f)_{xx} \) are reported in Table 5 .

In general, the base displacement and base rotation, \( x \) and \( \theta \), and the base shear and base moment, \( V \) and \( M \), are interrelated by the equation

\[
\begin{bmatrix}
x \\
\theta
\end{bmatrix} =
\begin{bmatrix}
d_{xx} & d_{x \theta} \\
d_{\theta x} & d_{\theta \theta}
\end{bmatrix}
\begin{bmatrix}
V \\
M
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
V \\
M
\end{bmatrix} =
\begin{bmatrix}
K_{xx} & K_{x \theta} \\
K_{\theta x} & K_{\theta \theta}
\end{bmatrix}
\begin{bmatrix}
x \\
\theta
\end{bmatrix}
\]

The values of the stiffness coefficients in the latter equation are
\[
K_{xx} = \frac{d^2 \phi}{d x^2 \phi_0 - d \phi_0} = 5.845 \times 10^3 \text{ kips/ft} \quad ;
\]

\[
K_{\phi \phi} = - \frac{d^2 \phi_0}{d x^2 \phi_0 - d \phi_0} = -1.705 \times 10^4 \text{ kips} \quad ; \text{ and}
\]

\[
K_{\phi x} = \frac{d \phi}{d x \phi_0 - d \phi_0} = 2.726 \times 10^7 \text{ kips ft}.
\]

**Periods and Modes of Modified Models.** The first three natural periods of vibration in the x-direction of the model without the cables are 130.3 sec., 4.83 sec. and 2.30 sec., respectively. These periods and associated modes are shown in Fig.5b. The modes are essentially the same as those of the model with the cables.

The corresponding periods of the model without the cables and fixed at the base are 21.9 sec., 3.56 sec. and 1.32 sec., respectively. These periods and the associated modes are shown in Fig.5c. All modes are now deformational and are associated with no base deflection or rotation.
PRESENTATION AND ANALYSIS OF RESULTS

Response in Symmetric Direction

The natural frequencies of the system in the symmetric, x-direction were evaluated for a range of cable and foundation stiffnesses and for several different locations of the cable.

The analysis on the effect of guying stiffness was performed in accordance with the force method described in the preceding Chapter using the frequencies and modes of vibration of the model without the cables as the reference quantities. The effects of the stiffness and location of the guying system are summarized in Fig.7. The cable stiffness has been varied from zero to about two times its actual value, and three different cable locations have been considered. The foundation stiffnesses, $k_{xx}$, $k_{yy}$ and $k_{xy}$, and all the other structural properties in these solutions were taken equal to those for the actual system.

Effect of Cable Stiffness. It is clear from Fig.7 that, within the range of parameters considered, changes in the guying stiffness affect significantly only the fundamental natural period of the system. This effect may be appreciated by noting that the lateral foundation stiffness of $k_{xx} = 5,845$ kips/ft is twelve times greater than the greatest cable stiffness value considered. Accordingly, the structure may in all cases be considered to be fixed against lateral motion at the base. Furthermore, the rocking stiffness of the foundation, $k_{yy} = 2.726 \cdot 10^7$ kips-ft, is small in comparison to the overturning resistance of the guying system. The latter resistance is $kh^2$, where $h$, the elevation of the cables from the base, is equal to 1,480 ft. For $k = 275$ kips/ft, this resistance is twenty-two times larger than $k_{yy}$, and for $k = 100$ kips/ft it is eight times larger.
than $k_{90}$. It follows that the overturning tendency of the structure is resisted mostly by the guying system and that the foundation may be considered to be free to rock.

These considerations, added to the fact that the tower is a relatively stiff structure, lead to a fundamental mode which is not materially different from that of a rigid bar hinged at the base and constrained laterally by the cables near the top. The importance of the guying stiffness is then obvious. That the pin-ended rigid bar is indeed a reasonable model is indicated in Fig.8, where the fundamental period of the model is compared to the exact one. It is seen that the agreement is indeed good. The agreement is further improved when account is taken of the actual rocking stiffness of the foundation, $k_{90}$. With the base considered to be fixed against deflection, the coupling stiffness $k_{x90}$ does not appear in the solution. The comparison with the exact solution for the modified rigid bar model is indeed so good that the differences are not perceptible when plotted to the scale of Fig.8. Representative values are compared below.

<table>
<thead>
<tr>
<th>$k_x$, kips/ft</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>42.7</td>
<td>31.0</td>
<td>25.7</td>
<td>22.3</td>
</tr>
<tr>
<td>Restrained Rigid Bar</td>
<td>45.1</td>
<td>31.9</td>
<td>26.0</td>
<td>22.6</td>
</tr>
<tr>
<td>Modified Rigid Bar</td>
<td>42.5</td>
<td>30.9</td>
<td>25.5</td>
<td>22.2</td>
</tr>
</tbody>
</table>

There are two reasons for the insensitivity of the lower natural periods to changes in cable stiffness:
(1) The second and third lateral modes of the structure have zero displacements near the point of attachment of the cables. This obviously minimizes their influence on these particular modes.

(2) Since the higher modes are associated with large amounts of structural deformation, the strain energy of the guyed system is small compared to that of the tower and its foundation even if the cables were attached to points of large modal displacements.

**Effect of Cable Location.** For the reasons already noted, a change in the elevation of the guyed system would be expected to affect appreciably only the fundamental period. That this is the case is clearly seen in Fig. 7.

**Effect of Foundation Stiffness.** This factor was studied by evaluating the natural frequencies and modes of the system for several values of the foundation flexibility coefficients $d_{xx}$ and $d_{\theta\theta}$. Three groups of solutions were obtained. In the first, $d_{xx}$ was varied keeping $d_{\theta\theta}$ and $d_{x\theta}$ constants; in the second, $d_{xx}$ and $d_{\theta\theta}$ were varied proportionately keeping $d_{x\theta}$ constant; and in the third only $d_{\theta\theta}$ was varied. The guyed system was considered to be attached to node 4 and its stiffness was taken as $k = 275$ kips/ft. Some of these solutions are tabulated below. The foundation flexibility values in this tabulation are expressed in percent of those of the prototype model, for which $d_{xx} = 1.714 \cdot 10^{-4}$ ft/kips, $d_{\theta\theta} = 3.675 \cdot 10^{-8}$ 1/ft·kips, and $d_{x\theta} = d_{\theta x} = 1.072 \cdot 10^{-7}$ 1/kips. These particular solutions were obtained by first solving the eigenvalue problem of the structure without the cables but with the appropriate base flexibilities, and then applying the force method to account for the stiffness of the
guying spring.

<table>
<thead>
<tr>
<th>$d_{xx}$</th>
<th>$d_{00}$</th>
<th>$T_1$, sec</th>
<th>$T_2$, sec</th>
<th>$T_3$, sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>50%</td>
<td>100%</td>
<td>26.7</td>
<td>4.30</td>
<td>1.91</td>
</tr>
<tr>
<td>100%</td>
<td>100%</td>
<td>26.7</td>
<td>4.82</td>
<td>2.30</td>
</tr>
<tr>
<td>125%</td>
<td>100%</td>
<td>26.8</td>
<td>5.10</td>
<td>2.41</td>
</tr>
<tr>
<td>200%</td>
<td>100%</td>
<td>26.8</td>
<td>5.91</td>
<td>2.61</td>
</tr>
<tr>
<td>50%</td>
<td>50%</td>
<td>26.2</td>
<td>4.27</td>
<td>1.91</td>
</tr>
<tr>
<td>100%</td>
<td>100%</td>
<td>26.7</td>
<td>4.82</td>
<td>2.30</td>
</tr>
<tr>
<td>200%</td>
<td>200%</td>
<td>27.0</td>
<td>5.93</td>
<td>2.61</td>
</tr>
<tr>
<td>100%</td>
<td>125%</td>
<td>26.9</td>
<td>4.83</td>
<td>2.30</td>
</tr>
</tbody>
</table>

It is clear from the information tabulated above that the fundamental period is insensitive to changes in the lateral stiffness of the foundation, whereas the remaining periods are sensitive to changes in the horizontal stiffness but not the rotational or rocking stiffness. The reasons for these trends are explained in the following sections.

**Fundamental Mode.** Within the range of the parameters considered, the lateral stiffness of the foundation is much greater than that of the cables. Under these conditions, a rigid body rotation involving no displacement at the base is the "easiest" configuration for the system to assume. Furthermore, since the contribution to the overturning resistance of the system provided by the cables is greater than that provided by the rocking of the foundation, changes in the latter stiffness do not influence significantly the fundamental mode, as long as these changes do not bring the contribution of the foundation resistance to a level comparable
to that of the cables.

**Higher Modes.** The sensitivity of these modes to variations in \( d_{xx} \) and their insensitivity to variations in \( d_{\theta\theta} \) may be appreciated by noting that these modes are associated with large horizontal displacements at the base and small overturning base moments.

Further information on the sensitivity of the higher periods to changes in \( d_{xx} \) is provided in Fig. 9. These results refer to a system without the cables. However, since the presence of the cables has a negligible effect on the second and third modes, the results are believed to be representative of the behavior of the structure with cables as well. In these solutions the ratios \( d_{\theta\theta}/d_{xx} \) and \( d_{x\theta}/d_{xx} \) were considered to be constants. The solutions were obtained by application of the displacement method of analysis using the modes and frequencies of the fixed-base structure without the cables as the reference quantities.

**Effect of Tower Stiffness.** From the results of the previous paragraphs, it should be clear that only the higher deformational modes are affected by changes in the stiffness of the tower. That this is indeed the case is demonstrated by the data summarized below. The structural stiffness in the first column is expressed in percent of that of the prototype. This stiffness value is for the structure without the guy ing spring when completely fixed at the base. The stiffness of the guying system and the foundation flexibilities were taken equal to their actual values in these solutions.
<table>
<thead>
<tr>
<th>Structural Stiffness</th>
<th>$T_1$, sec</th>
<th>$T_2$, sec</th>
<th>$T_3$, sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>80%</td>
<td>26.8</td>
<td>5.15</td>
<td>2.43</td>
</tr>
<tr>
<td>100%</td>
<td>26.7</td>
<td>4.82</td>
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Response in Unsymmetric Direction

This section considers the effects of changing the stiffnesses of the guying system and foundation in the unsymmetric, $y$-direction. Increases in both the lateral and torsional stiffnesses are considered, with each increase presumed to occur independently.

An increase in lateral stiffness may be visualized as being due to the addition of a lateral deflectional spring through the center of the main piles, whereas an increase in rotational stiffness may be due to the addition of a rotational spring about a vertical axis through the same point. These constraints are applied at the base of the structure when the effects of changes in foundation stiffnesses are to be studied, and at the elevation of the guying system when the stiffnesses of the latter system are to be varied.

Since the center of the group of main piles does not, in general, corresponds to the instantaneous center of rotation of the cross section of the structure, the modes of vibration will generally be coupled lateral-torsional modes, and each of the stiffness changes considered will in principle affect both the predominantly lateral and the predominantly torsional modes.

The analyses were performed by application of the force method using
the free vibrational characteristics of the actual structure in the $y$-direction as the reference quantities and applying appropriate deflectional and rotational constraints as noted above.

**Stiffnesses of Guying System.** Fig.10 shows the effects of increasing the lateral stiffness of the guying system by $\Delta k_y$. The increase is expressed in percent of the corresponding stiffness $k_y$ of the actual structure. The latter stiffness is taken as $k_y = 275$ kips/ft.

It is clear that the effects of this increase are similar to those observed earlier for the symmetric $x$-direction. The increase affects significantly only the fundamental, predominantly lateral mode but not any of the higher modes. In particular, the predominantly torsional mode is practically unaffected even though the additional constraint has been considered to be applied through the center of the main piles rather than the instantaneous center of rotation for that particular mode.

The effect of an increase in the torsional resistance of the cable is shown in Fig.11. The normalizing torsional stiffness on the abscissa of this figure is taken as the product of $k_y$ and the distance between the center of the main group of piles and center of mass of the cross section, i.e. $k_\phi = 275 \times 38.76 = 10,659$ kips-ft. It can be seen that a change in $k_\phi$ can affect the lowest, predominantly torsional mode of vibration but not any of the other modes. However, a substantial increase in the torsional resistance of the cable is required to cause any effect on the predominantly torsional mode. Such an increase could be produced by a highly unsymmetrical arrangement of the guying system or by a significant increase in the initial tension of the cables, but neither alternative is practi-
cal.

The insensitivity of the higher modes of vibration to changes in the stiffness of the guying system is due to the fact that for ranges of parameters considered the strain energy of the guying system is a small fraction of that of the tower and its foundation.

**Stiffness of Foundation.** The effects of increasing the lateral and torsional stiffnesses of the foundation are shown in Figs. 12 & 13, respectively, each increase being normalized with respect to the corresponding stiffness of the actual structure. For the purpose of this comparison the lateral foundation stiffness may be taken as \( k_{yy} = k_{xx} = 5,845 \) kips/ft and the rotational one as \( k_{\phi\phi} = 2.25 \times 10^7 \) kips-ft.

It can be seen that a onefold increase in the lateral stiffness of the foundation affects the third mode which is predominantly lateral-deformational and the lowest, predominantly torsional mode, whereas a onefold increase in the torsional stiffness affects mainly the lowest, predominantly torsional mode and, to a minor extent, the fourth mode which is coupled lateral-torsional. The effect on the predominantly torsional mode of a change in lateral foundation stiffness is due to the additional constraint being applied through the center of the main piles rather than the instantaneous center of rotation for the particular mode.

**Relative Effects of Foundation and Guying Resistance.** An increase in torsional foundation stiffness by \( \Delta k_{\phi\phi} = 2.25 \times 10^7 \) kips-ft, corresponding to \( \Delta k_{\phi\phi} / k_{\phi\phi} = 100\% \), results in a predominantly torsional mode with a period of 7.85 sec., whereas an identical increase in torsional guying stiffness, corresponding to \( \Delta k_{\phi\phi} / k_{\phi\phi} = 211,089\% \), results in an identical mode
with period of 6.65 sec. From this comparison, it may be concluded that the structure is more sensitive to changes in the torsional stiffness of the guying system than to changes in the corresponding stiffness of the foundation. Two reasons account for this behavior:

(1) The initial or starting torsional stiffness of the foundation is much larger than that of the guying system. Inasmuch as the typical relationship between a natural period of vibration and the stiffness of a constraint is represented by an asymptotic curve, the increase of an initially small constraint will affect the period of vibration more than the increase of an initially larger constraint.

(2) A constraint applied at a point along a stick-like structure is generally more efficient than when applied at one end.

These considerations suggest that the lowest predominantly torsional mode should be monitored by changing the values of the torsional stiffness of the guying system. As already noted, this is however not possible in practice and the predominantly torsional modes can realistically be monitored only by controlling the torsional stiffness of the foundation.
CONCLUSIONS

An analysis of the effects of the guying and foundation stiffnesses on the free vibrational characteristics of a pile-supported guyed tower has been made. The principal conclusions of the study may be summarized by considering separately the responses in the symmetric and unsymmetric directions.

Response in symmetric direction:

1. The fundamental mode of vibration is essentially a rigid body mode increasing from the base to the top. This mode is affected mainly by the stiffness of the lateral guying system and by its location.

2. A good approximation to the fundamental period may be obtained by considering the tower to be a rigid bar hinged at the base (i.e., fixed against deflection but free against rotation). Improved accuracy may be achieved by incorporating the effect of the rocking stiffness of the foundation.

3. The second and higher modes are deformational modes which are sensitive to changes in the lateral stiffness of the foundation and the stiffness of the tower but are insensitive to changes in the rocking stiffness of the foundation or the stiffness of the guying system.

4. The properties of the fundamental mode of vibration can most efficiently be controlled by monitoring the stiffness of the guying system, whereas those of the second and higher modes can be controlled by monitoring the lateral stiffness of the foundation and/or the stiffness of the tower itself. Within realistic limits, changes in
the latter stiffness will have practically no effect on the fundamental mode. Similarly, realistic variations in the rocking stiffness of the foundation and the lateral guying stiffness will not effect the second and higher modes.

Response in unsymmetric direction:

- The modes of vibration in this direction are generally coupled lateral-torsional modes.

- The predominantly lateral modes and the associated periods are affected by changes in the lateral stiffness of the guying system and the foundation in much the same way as the corresponding quantities for response in the symmetric direction. They are practically unaffected by changes in the torsional stiffness of either the guying system or the foundation.

- The predominantly torsional mode is practically unaffected by realistic changes in the lateral stiffness of the guying system. While it is affected by changes in the torsional resistance of the guying system, unrealistically large changes in stiffness are needed to produce substantial changes in response. This mode is affected by the torsional stiffness of the foundation, and it is this stiffness that should be used to monitor this mode.
Table 1 - Flexibility Matrix $D_{xx}$, $10^{-5}$ ft/kips

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Table 2a - Flexibility Submatrix \( D_{yy} \), \( 10^{-5} \) ft/kips

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$m_{xx}$ in kips sec²/ft  
$m_{yy}$ in kips sec²/ft  
$m_{yt}$ in $10^3$ kips sec²  
$m_{tt}$ in $10^6$ kips ft sec²
Table 4 - Flexibility Matrix $D_{xx}^{wo}$, $10^{-3}$ ft/kips

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Table 5 - Flexibility Matrix $D^f_{xx}$, $10^{-3}$ ft/kips

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Fig. 1. Guyed Tower
Fig. 2. Typical Guying Cable
Fig. 3. Resistance of Guying System
Fig. 4. 9-Nodes Model
(a) Flexible Base Model with Guying System

(b) Flexible Base Model without Guying System

(c) Fixed Base Model without Guying System

Fig. 5. Modes and Periods of Vibration in x-Direction
Fig. 6. Modes and Periods of Vibration in y-Direction

(a) Displacement, y

\( T_1 = 3.49 \text{ sec} \)

\( T_2 = 6.41 \text{ sec} \)

\( T_3 = 8.81 \text{ sec} \)

(b) Rotation x 60 ft

\( T_4 = 27.0 \text{ sec} \)
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CHAPTER

FOUR

LOW-AMPLITUDE DYNAMIC STIFFNESS OF PARABOLIC CABLE

Introduction
System Considered and Fundamental Relations
Statement of Problem
Equations of Motion
Solution for Steady-State Response
Presentation and Analysis of Data
Simple Dynamic Cable Model
Vertical Stiffness
Group Behavior
Conclusion
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Appendices
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INTRODUCTION

A fundamental step in the analysis of the dynamic response of guyed towers and other cable-supported structures is the evaluation of the resistance to deformation of the cable system. For structures for which the response may be considered to be linear about their position of static equilibrium, the resistance of a cable may be expressed by its stiffness, defined as the component of the increment in cable tension in a given direction necessary to induce a unit maximum displacement relative to the ends of the cable in that direction.

The stiffness of a vibrating cable depends on the geometric and physical properties of the cable, the magnitude of the initial cable tension, and the history of the motion itself. Of greatest value in practice is the stiffness of a cable in harmonic motion, for once this stiffness has been established, the corresponding stiffness of an array of cables and the steady-state response of any cable-structure system to a periodic excitation may be determined with relative ease. By application of Fourier transform and Laplace transform techniques, it is also possible to evaluate the transient response of a cable-structure system to an arbitrary excitation. Finally, valuable insight into important aspects of the dynamic response of cables and of cable-structure systems may be obtained from a mere knowledge of their harmonic stiffnesses.

The first solution for the dynamic stiffness of a harmonically excited cable appears to have been contributed by Kolousek (1), who presented a series solution for a uniform, undamped cable deflected in a parabolic profile at its position of static equilibrium. Later, Davenport (2) condensed Kolousek's series solution to a closed form, and Davenport
and Steels (3) generalized the series solution to include the effect of uniform external viscous damping. More recently, Irvine (4) presented a simpler solution for undamped, parabolic cables which is valid over a narrower range of parameters than Davenport's original solution (2). As far as it is known, the closed-form counterpart of the Davenport-Steels solution for damped cables is not available. Implicit in all of these studies is the assumption that the angle of inclination of the cable chord is not very large. Additional studies of the problem have been made by Dean (5) and by McCaffrey and Hartmann (6) who presented approximate series solutions for an undamped cable for which the static deflection configuration is described by catenary. The latter authors and Hartmann and Davenport (7) also have discussed the application of some of these solutions to actual design studies.

Despite the value of these contributions, there is a need for re-examination of the problem with a view of: (a) extending the applicability of existing solutions to parabolic cables of arbitrary chord inclinations; (b) developing a computationally efficient solution for damped cables; and (c) assessing the accuracy of the existing solutions. There is also a need for comprehensive parametric studies which will help assess the effects and relative importance of the various factors which influence the harmonic stiffness value.

This study is intended to be responsive to these needs. It presents a closed-form expression for the steady-state horizontal stiffness of a viscously damped, uniform, inclined cable supported at the lower end and subjected to a harmonically varying horizontal displacement at the upper end. At its position of static equilibrium the cable is considered to
have a parabolic profile, and the amplitudes of the dynamic displacements are presumed to be small. The expression is valid for an arbitrary angle of inclination of the chord in the range between zero and 90 degrees. A simpler solution valid over a narrower range of the parameters is also presented and its accuracy is discussed. For undamped cables, the simplified solution is essentially the same as that presented by Davenport (2).

Comprehensive numerical data are presented and the significance of the results discussed. Special attention is paid to the influence of damping, which to date has not received the attention it deserves. Finally, a simple, single-degree-of-freedom model is proposed which reproduces with good accuracy the more significant aspects of the response of the prototype cable over a wide range of conditions.

In addition to being directly applicable to the particular cable configuration and boundary conditions considered, the information presented is expected to prove of value in design studies of offshore guyed towers for which both the static equilibrium configuration and the boundary conditions at the lower ends of the cables are considerably more complex than those considered herein. Such cables can be analyzed accurately only by application of fairly complex computer programs. The information presented herein should provide a convenient frame of reference for the planning of such solutions and for the analysis and interpretation of the resulting data.

Notation

The letter symbols used in this study are defined when first introduced in the text, and those used most extensively are summarized in Appendix A.
SYSTEM CONSIDERED AND FUNDAMENTAL RELATIONS

A uniform cable suspended from end supports located at different elevations is considered, as shown in Fig. 1. At its position of static equilibrium, the cable is presumed to be deflected in a parabolic profile with its axis normal to the chord joining the support points. Such a configuration would be produced by a lateral load which is normal to, and uniformly distributed along, the chord, as shown in the figure.

Let \( y \) be a Cartesian coordinate system, with the origin taken at the lower end of the cable and the \( x \)-axis taken along the cable chord. The \( x \)-direction will be referred to as the axial direction, and the \( y \)-direction as the normal direction. The deflection of the cable at its position of static equilibrium is then given by the equation

\[
y(x) = \frac{1}{2} \frac{q_y L^2}{T_o} \left[ \frac{x}{L} - \left( \frac{x}{L} \right)^2 \right]
\]

(1)

where \( q_y \) is the intensity of the normal load per unit of chord length; \( L \) is the length of the chord; and \( T_o \) is the axial component of the cable tension, i.e. the component parallel to the chord. This force component is independent of \( x \). The maximum value of \( y \), known as the cable sag, occurs at midspan and is given by

\[
y_{\text{max}} = \frac{1}{8} \frac{q_y L^2}{T_o}
\]

(2)

The cable tension at an arbitrary point, \( T(x) \), is given by

\[
T(x) = T_o \sqrt{1 + \left( \frac{dy}{dx} \right)^2}
\]

(3)

and its maximum value, which occurs at the ends, is given by
\[ T_{\text{max}} = T_0 \sqrt{1 + \frac{1}{4} \left( \frac{q_y L}{T_0} \right)^2} \]  (4a)

When expressed in terms of the sag ratio, \( y_{\text{max}}/L \), Eq. 4a may also be written as

\[ T_{\text{max}} = T_0 \sqrt{1 + 16 \left( \frac{y_{\text{max}}}{L} \right)^2} \]  (4b)

For cables with small sag ratios (values of \( y_{\text{max}}/L \) of the order of 1/8 or less), the parabolic solution summarized above also provides a good approximation to the effects of a distributed vertical load, such as the cable's own weight. If \( q \) is the intensity of the vertical load per unit of cable length, then \( q_y \) should be taken as

\[ q_y = \frac{q S}{L} \cos \theta \]  (5)

where \( S \) is the unstretched length of the cable, and \( \theta \) is the inclination of the cable chord. Implicit in this approach is the assumption that the component of the load in the direction of the chord, namely,

\[ q_x = \frac{q S}{L} \sin \theta \]  (6)

has a negligible effect on the behavior of the cable. This assumption is considered to be quite reasonable for deflections but may not be as satisfactory for the resulting tensions. The effects of \( q_x \) on \( T \) may be accounted for approximately by assuming the cable to act as a column. The effective component of cable tension in the direction of the chord, \( T_{e_0}(x) \), would then be given by the expression

\[ T_{e_0}(x) = T_0 \left[ 1 + \frac{1}{2} \frac{q_x L}{T_0} \left( \frac{2}{L} \frac{x}{L} - 1 \right) \right] \]  (7)

and the total cable tension could be determined from Eq. 3 by replacing \( T_0 \) with \( T_{e_0}(x) \). The quantity \( T_0 \) in this case may be interpreted as the
x-component of the cable tension at midspan (or its average value along the cable length). The values of the cable tension at the two ends are then unequal, as they should be. The maximum value occurs at the upper end and is given by

$$T_{\text{max}} = T_0 \left(1 + \frac{1}{2} \frac{q_x L}{T_0}\right) \sqrt{1 + \frac{1}{4} \left(\frac{q_x L}{T_0}\right)^2}$$

This equation is proposed only for values of $q_x L/(2T_0) << 1$. 
STATEMENT OF PROBLEM

Consider that the upper end of the cable is subjected to a harmonically varying horizontal displacement

\[ X(t) = X e^{i\omega t} \]  \hspace{1cm} (9)

in the plane of the cable, as shown in Fig. 2, where \( X \) is the amplitude of the displacement; \( i = \sqrt{-1} \); \( \omega \) is the circular frequency of the motion; and \( t \) is time. No vertical displacement is permitted at the upper end, and the lower end is considered to be completely fixed. The displacement amplitude \( X \) is presumed to be small, so that the motion of the cable about its position of static equilibrium may be considered to be linear.

Let

\[ \Delta H(t) = \Delta H e^{i\omega t} \]  \hspace{1cm} (10)

be the horizontal component of the steady-state increment at the upper support necessary to maintain the displacement \( X(t) \). This force increment is in excess of the corresponding force associated with the position of static equilibrium of the cable. For an undamped cable, \( \Delta H \) is a real-valued quantity, whereas for a damped cable, it is complex-valued. The real part of \( \Delta H \) represents the force component which is in phase with the imposed displacement, whereas its imaginary part represents the force component 90\(^\circ\) out of phase with the displacement.

Of interest in this study is the relationship between \( \Delta H(t) \) and \( X(t) \), and, more specifically, the horizontal dynamic stiffness of the cable, \( K \), defined as the ratio of the complex-valued force amplitude, \( \Delta H \), to the displacement amplitude, \( X \).
EQUATIONS OF MOTION

Coordinates and Assumptions

The configuration of the vibrating cable will be specified in terms of the axial and normal displacement components, \( u(x,t) \) and \( w(x,t) \), shown in Fig. 2a. These displacements are measured from the position of static equilibrium of the cable in directions parallel to the x-axis and y-axis, respectively.

In evaluating the response, only the normal components of the inertia forces will be considered. As demonstrated at a later section, the effects of the axial components of these forces on the desired response are small compared to those of the normal components, and may be neglected. Damping is presumed to be due to an external viscous source exerting a restraining force in the normal direction only.

Differential Equations

The motion of the cable under these conditions is governed by the differential equation

\[
T_0 \frac{\partial^2 w}{\partial x^2} - \left[ \mu \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} \right] = q_y \frac{\Delta T_0(t)}{T_0}
\]  

(11)

in which \( T_0 \) is the axial or x-component of the cable tension at its position of static equilibrium; \( q_y \) is the intensity of the external normal load producing the static effects; \( \Delta T_0(t) \) is the dynamic increment in \( T_0 \); \( \mu \) is the mass of the cable per unit of chord length; and \( c \) is the coefficient of viscous damping per unit of chord length. Both \( \mu \) and \( c \) are considered to be constants. The force increment, \( \Delta T_0(t) \), like the corresponding force at the position of static equilibrium, \( T_0 \), is independent of the position coordinate, \( x \). This is a consequence of the assumption
that only the normal components of the inertia and damping forces are of importance.

For a cable for which both supports may displace, \( \Delta T_o(t) \) is given by the following equation expressing the compatibility of the resulting strains and displacements:

\[
\Delta T_o(t) = \frac{AE}{L_e} \left\{ [u(L,t) - u(0,t)] - \frac{1}{2} \frac{q^L}{T_o} [w(L,t) + w(0,t)] \right. \\
+ \left. \frac{q^L}{T_o} \int_0^L w(x,t) \, dx \right\}
\]  

(12)

in which \( E \) represents Young's modulus of elasticity for the cable; \( A \) represents its cross sectional area; \( L_e \) is the effective cable length, defined by

\[
L_e = L \left[ 1 + \frac{1}{8} \left( \frac{q^L}{T_o} \right)^2 \right] = L \left[ 1 + 8 \left( \frac{\nu_{\text{max}}}{L} \right)^2 \right]
\]  

(13)

and \( u(0,t), u(L,t), w(0,t) \) and \( w(L,t) \) are the instantaneous values of \( u \) and \( w \) at the lower and upper supports, respectively. Excepting differences in notation, Eqs. 11, 12 and 13 are essentially the same as those employed by Irvine (8,9) and differ from those used by Davenport and Steels (3) only in the use of \( L_e \) in lieu of \( L \). For completeness, the derivation of these equations is given in Appendix B.

It is sometimes convenient to express the equation of motion in terms of the part of the normal displacement component measured from the rotated or instantaneous position of the cable chord, rather than from its position of static equilibrium. If \( v(x,t) \) represents this deformational component (see Fig. 2b), then \( v(x,t) \) and \( w(x,t) \) are interrelated by the equation
\[ w(x,t) = w(0,t) + \left[ w(L,t) - w(0,t) \right] \frac{x}{L} + v(x,t) \]  

(14)

in which the first two terms represent the component of \( w(x,t) \) due to the rigid body motion of the chord line. It should be noted that \( v(x,t) \), like \( w(x,t) \), is measured parallel to the \( y \)-axis.

When expressed in terms of \( v \) and the dimensionless distance coordinate, \( \xi = x/L \), Eqs. 11 and 12 become

\[
\frac{T_o}{L^2} \frac{\partial^2 v}{\partial \xi^2} - \left[ \mu \frac{\partial^2 v}{\partial t^2} + c \frac{\partial v}{\partial t} \right] = q_y \frac{\Delta T_o(t)}{T_o} \\
+ \mu \frac{\partial^2}{\partial t^2} \left[ w(0,t) + [w(1,t) - w(0,t)]\xi \right] \\
+ c \frac{\partial}{\partial t} \left[ w(0,t) + [w(1,t) - w(0,t)]\xi \right] 
\]

(15)

and

\[
\Delta T_o(t) = \frac{AE}{L e} \left[ u(1,t) - u(0,t) + \frac{q_y L}{T_o} \int_0^1 v(\xi,t) \ d\xi \right] 
\]

(16)

**Specialized Form of Equation.** For a harmonically excited cable, the steady-state values of the displacement and tension increment are of the form:

\[ v(\xi,t) = v(\xi)e^{i\omega t} \]  

(17)

and

\[ \Delta T_o(t) = \Delta T_o e^{i\omega t} \]  

(18)

where \( v(\xi) \) and \( \Delta T_o \) are generally complex-valued quantities independent of time, and \( \Delta T_o \) is independent of the position coordinate as well.

For the cable considered in Fig. 2, \( u(0,t) = w(0,t) = 0, u(1,t) = X \cos \theta e^{i\omega t} \) and \( w(1,t) = X \sin \theta e^{i\omega t} \). By making use of Eqs. 17 and 18 and introducing the reference frequency
\[ \omega_0 = \frac{1}{\xi} \sqrt{\frac{T_0}{L}} \]  

(19)

and the dimensionless frequency and damping parameters

\[ \phi = \frac{\omega}{\omega_0} \]  

(20)

and

\[ \zeta = \frac{c}{2\pi \mu \omega_0} \]  

(21)

Eqs. 15 and 16 specialize to:

\[ \frac{d^2 \nu}{d\xi^2} + [\phi^2 - i2\pi\xi\phi] \nu = \frac{q_L}{I_o} \frac{\Delta T_o}{T_o} - \left[ \phi^2 - i2\pi\xi \right] \xi X \sin \theta \]  

(22)

and

\[ \Delta T_o = \frac{AE}{L_e} \left[ X \cos \theta + \frac{q_L}{I_o} \int_0^1 v(\xi) d\xi \right] \]  

(23)

respectively. It should be noted that \( \pi \omega_0 \) represents the fundamental circular natural frequency of a taut or sagless cable supported at both ends, and \( \zeta \) represents the ratio of the actual damping of the cable to the critical damping of the supported-supported taut cable vibrating in its fundamental mode.
SOLUTION FOR STEADY-STATE RESPONSE

The solution of Eq. 23 may be written in the form

\[ v(\xi) = v_h(\xi) + v_p(\xi) \]  \hspace{1cm} (24)

where \( v_h(\xi) \) is the homogeneous solution defined by

\[ v_h(\xi) = C_1 \left[ \cosh \alpha \xi \cos \beta \xi + i \sinh \alpha \xi \sin \beta \xi \right] \]
\[ + C_2 \left[ \sinh \alpha \xi \cos \beta \xi + i \cosh \alpha \xi \sin \beta \xi \right] \]  \hspace{1cm} (25)

and \( v_p(\xi) \) is a particular solution which may be expressed in terms of the unknown tension increment, \( \Delta T_o \), as follows:

\[ v_p(\xi) = C(\phi,\xi) \frac{q L^2 \Delta T_o}{T_o} - \xi X \sin \theta \]  \hspace{1cm} (26)

The quantities \( \alpha \) and \( \beta \) in these expressions are dimensionless factors defined by

\[ \alpha = \frac{\phi}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + (2\pi \zeta / \phi)^2}} = \frac{\pi \zeta \phi}{\beta} \]  \hspace{1cm} (27)

\[ \beta = \frac{\phi}{\sqrt{2}} \sqrt{1 + \sqrt{1 + (2\pi \zeta / \phi)^2}} \]  \hspace{1cm} (28)

\( C_1 \) and \( C_2 \) are constants of integration that must be determined from the boundary conditions on \( v(\xi) \); and \( C(\phi,\xi) \) is a dimensionless amplification factor defined by

\[ C(\phi,\xi) = \frac{1}{\phi^2 - i 2\pi \xi \phi} \]  \hspace{1cm} (29)

The derivation of Eq. 25 is given in Appendix C. Eq. 26 was obtained by inspection and may be verified readily.

The following expressions for \( C_1 \) and \( C_2 \) are obtained upon making use of the boundary conditions \( v(0) = v(1) = 0 \):
\[ C_1 = - C(\phi, \xi) \frac{q_v L^2}{T_o} \frac{\Delta T_o}{T_o} \]  

and

\[ C_2 = \frac{\sinh \alpha \cos \beta - i \cosh \alpha \sin \beta}{\sinh^2 \alpha + \sin^2 \beta} \times \sin \theta \]

\[ + \frac{(\cosh \alpha - \cos \beta)(\sinh \alpha + i \sin \beta)}{\sinh^2 \alpha + \sin^2 \beta} C(\phi, \xi) \frac{q_v L^2}{T_o} \frac{\Delta T_o}{T_o} \]  

(30a)

(30b)

The tension increment, \( \Delta T_o \), may now be evaluated from Eq. 23. On substituting the expression for \( v(\xi) \) just determined, performing the indicated integration, and making use of several trigonometric and hyperbolic identities, the following expression is obtained after a great deal of tedious algebra:

\[ \Delta T_o = \frac{1 + \frac{1}{2} \gamma \sigma}{1 + \frac{1}{12} \gamma \rho} \times \frac{AE}{\ell_e} \cos \theta \]  

(31)

where

\[ \gamma = \frac{2}{\alpha + i \beta} \frac{\cosh \alpha - \cos \beta}{\sinh \alpha - i \sin \beta} - 1 \]  

(32)

\[ \rho = \frac{1}{12} \frac{AE}{T_o} \frac{L}{\ell_e} \left( \frac{q_v L}{T_o} \right)^2 \]  

(33a)

and

\[ \sigma = \frac{q_v L}{T_o} \tan \theta \]  

(34a)

The detailed steps leading from \( v(\xi) \) to Eq. 31 are given in Appendix D.

The quantity \( T_o/(AE) \) in Eq. 33a represents the average axial strain in the cable at its position of static equilibrium. Denoting this by the symbol \( e_o \), and making use of Eq. 2, the parameter \( \rho \) may also be expressed in the form
\[
\rho = \frac{16}{3} \frac{(y_{\text{max}}/L)^2}{e_o} \frac{L}{L_e}
\]  

(33b)

On making use of Eq. 2, the following alternative expression may also be obtained for \( \sigma \):

\[
\sigma = 8 \frac{y_{\text{max}}}{L} \tan \theta
\]  

(34b)

The values of \( \rho \) and \( \sigma \) may range from zero to infinity. A value of \( \rho = 0 \) corresponds either to a sagless, taut cable or to a cable with no axial resistance (\( e_o \) very large); \( \rho = \infty \) corresponds to an inextensional cable; \( \sigma = 0 \) refers either to a sagless cable or to one oriented horizontally; and \( \sigma = \infty \) refers to a vertically oriented cable.

**Complete Solution for Stiffness**

Let \( T \) be the cable tension at the upper support when the cable is at its position of static equilibrium, and \( H \) be the horizontal component of this force. Similarly, let \( \Delta T \) be the dynamic increment in \( T \), and \( \Delta H \) be the dynamic increment in \( H \). Finally, let \( y'(L) \) be the derivative of \( y \) with respect to \( x \) evaluated at the top support, and \( w'(L) \) be the corresponding derivative of \( w \). These quantities are shown in their positive directions in Fig. 5, from which it follows that

\[
H = T \cos \left[ \theta - y'(L) \right]
\]  

(35)

and

\[
H + \Delta H = (T + \Delta T) \cos \left[ \theta - y'(L) - w'(L) \right]
\]  

(36)

Limiting now the discussion to cables with small sags and low-amplitude motions, \( T \) and \( \Delta T \) in these expressions may be replaced by \( T_o \) and \( \Delta T_o \), respectively, and the cosine terms may be expanded by considering the cosines of \( y'(L) \) and \( y'(L) + w'(L) \) to be unity, and the sines of these
quantities to be equal to the quantities themselves. With these approximations and the deletion of a higher-order nonlinear term $\Delta T_0 w$, Eqs. 35 and 36 lead to the following linearized expression for $\Delta H$:

$$\Delta H = \Delta T_0 \cos \theta + y'(L) \Delta T_0 \sin \theta + T_0 w'(L) \sin \theta \quad (37)$$

The quantities $y'(L)$ and $w'(L)$ are determined from Eqs. 1 and 14 to be $-q_y L/(2T_0)$ and $[(X/L) \sin \theta + v'(L)]$, respectively.

On substituting these expressions for $y'(L)$ and $w'(L)$ into Eq. 37, evaluating $v'(L)$ by differentiation of Eqs. 24, 25 and 26, and determining $\Delta T_0$ from Eq. 31, the following expression is obtained for the stiffness, $K$, representing the ratio of $\Delta H$ to $X$:

$$K = \frac{1 + \frac{1}{2} \gamma \rho}{1 + \frac{12 \gamma \rho}{\phi^2 - i 2 \pi \zeta \phi}} \left[ 1 + \frac{1}{2} \sigma \left( -1 + \frac{1 + \gamma}{1 - i (2 \pi \zeta / \phi)} \right) \right] \frac{E A}{L_e} \cos^2 \theta$$

$$+ (\alpha + i \beta) \frac{\sinh \alpha \cosh \alpha - i \cos \beta \sin \beta}{\sinh^2 \alpha + \sin^2 \beta} \frac{T_0}{L} \sin^2 \theta \quad (38)$$

Equation 38 is the central equation of the study. Note that, in addition to the frequency ratio, $\phi$, and the damping factor, $\zeta$, the coefficients of the two terms depend on the dimensionless parameters $\rho$ and $\sigma$.

**Simplified Solution for Stiffness**

Excepting cables with large angles of inclination for which $\cos \theta \to 0$ and $\sin \theta \to 1$, the second and third terms on the right hand member of Eq. 37 are small compared to the first term and may be omitted. If only the first term is retained, the following expression for stiffness is obtained from Eq. 31:

$$K' = \frac{1 + \frac{1}{2} \gamma \rho}{1 + \frac{12 \gamma \rho}{\phi^2 - i 2 \pi \zeta \phi}} \frac{AE}{L_e} \cos^2 \theta \quad (39)$$
The prime superscript on $K$ in this and in subsequent expressions identifies the approximate nature of this stiffness.

**Solutions for Special Cases**

It is desirable to identify the solutions for the following special cases:

**Horizontal Cables.** For horizontally oriented cables, both $\theta$ and $\sigma$ are equal to zero, and, except as noted in the following, Eq. 38 reduces to Eq. 39. In other words, there is no difference in this case between the stiffness values determined by the simplified and approximate solutions. The exception occurs for cables without damping, for which the complete solution leads to certain singularities which are not predicted by the simplified solution. This matter is considered further later.

**Undamped Cables.** In the absence of damping, Eq. 32 reduces to

$$\gamma = \frac{2}{\phi} \tan \frac{\phi}{2} - 1$$

(40)

and Eqs. 38 and 39 for the complete and approximate solutions reduce to

$$K = \frac{(1 + \frac{1}{2} \gamma \sigma)^2}{1 + 12 \gamma \rho / \phi^2} \frac{AE}{Le} \cos^2 \theta + \phi \cot \phi \frac{T_o}{L} \sin^2 \theta$$

(41)

and

$$K' = \frac{1 + \frac{1}{2} \gamma \sigma}{1 + 12 \gamma \rho / \phi^2} \frac{AE}{Le} \cos^2 \theta$$

(42)

respectively. Equation 42 is essentially the same as that presented by Davenport (2), except that in Davenport's solution the effective length, $L_e$, is taken equal to the slightly smaller chord length, $L$. Irvine's solution (4) for the same quantity is independent of $\sigma$. 
In the special case of a horizontally oriented undamped cable, Eqs. 41 and 42 reduce to

\[ K_{\theta=0} = K'_{\theta=0} = \frac{1}{1 + 12 \gamma \rho / \phi^2} \frac{AE}{L_e} \]  
(43)

For vertically oriented cables, \( \theta = \pi/2 \) and hence \( \cos \theta = 0 \) and \( \sigma = \infty \). The simplified solution (Eq. 42) in this case leads to zero stiffness value, a result which is obviously incorrect. The complete solution leads to:

\[ K_{\theta=90^\circ} = \left[ \frac{3 \gamma^2 \rho}{1 + 12 \gamma \rho / \phi^2} + \phi \cot \phi \right] \frac{T_0}{L} \]  
(44)

**Static Solutions.** The static value of the cable stiffness may be obtained from the expressions for dynamic stiffness by taking \( \phi = 0 \). The result corresponding to the complete solution is

\[ K_{st} = \frac{1}{1 + \rho} \frac{AE}{L_e} \cos^2 \theta + \frac{T_0}{L} \sin^2 \theta \]  
(45)

and the one corresponding to the simplified solution is given by the first term of Eq. 45, i.e.

\[ K'_{st} = \frac{1}{1 + \rho} \frac{AE}{L_e} \cos^2 \theta \]  
(46)

**Interpretation of Static Solution**

The significance of the various terms in Eq. 45 may be appreciated by introducing the quantities

\[ K_e = \frac{AE}{L_e} \cos^2 \theta \]  
(47)

\[ K_r = \frac{T_0}{L} \sin^2 \theta \]  
(48)

\[ K_i = 12 \frac{T_0}{L} \left( \frac{T_0}{q_y L} \right)^2 \cos^2 \theta \]  
(49)
and rewriting Eq. 45 in the form:

\[ K_{st} = \frac{1}{K_e + \frac{1}{K_i}} + K_r \] (50)

where \( K_e, K_i \), and \( K_r \), like \( K_{st} \), have units of force per unit of length, or stiffness.

To clarify the meanings of \( K_e \) and \( K_r \), consider a weightless, perfectly taut cable of inclination \( \theta \) and length \( L = L_e \) subjected to a unit horizontal displacement at the top, as shown in Fig. 3a. The axial or extensional component of the imposed displacement, \( \cos \theta \), induces a change in cable tension, whereas the normal component, \( \sin \theta \), induces a rotation of the chord with essentially no extension, and hence no change in tension.

The stiffness \( K_e \) represents the change in the horizontal component of the cable tension at the upper end necessary to induce the extensional component of the imposed displacement; and \( K_r \) represents the horizontal force component associated with the normal or rotational displacement component. The change in cable tension corresponding to the axial displacement component is \((AE/L_e)\cos \theta\), and the horizontal component of this force is given by Eq. 47. In an analogous manner, the rotation of the cable is \((\sin \theta)/L\); the normal component of the cable tension in the displaced position is \((T_0 \sin \theta)/L\); and the horizontal component of this force is given by Eq. 48.

The quantity \( K_i \) represents the contribution of the cable sag. Specifically, it defines the horizontal stiffness of the sagging cable computed on the assumption that it is inextensional and that the secon-
dary effect of the rotation of the cable chord is negligible. The imposed displacement in this case is accomodated by a change in the cable geometry, an action involving mainly normal displacements, as shown in Fig. 3b.

To summarize, the static resistance of the cable to the imposed displacement may conveniently be visualized in three parts:

a. The axial, or extensional resistance of the taut cable, represented by the stiffness $K_e$;
b. The rotational, inextensional resistance of the taut cable, represented by the quantity $K_r$; and
c. The inextensional resistance of the sagging cable, which is associated with a change in cable configuration and is represented by the quantity $K_i$.

The same three resistances may also be identified in Eq. 38 which is the dynamic counterpart of Eq. 50.

**Modeling of Cable.** The expression for $K$ defined by Eq. 50 may be recognized to be the same as that governing the effective stiffness of a group of three springs of stiffnesses $K_e$, $K_r$ and $K_i$, arranged in the manner shown in Fig. 4a. The springs in series simulate the resistances referred to under items (a) and (c) above, whereas the third spring simulates the resistance referred to under item (b). The stiffness defined by Eq. 50 is precisely the ratio $F/X$ for the model.

In the simplified solution, $K_r$ is considered to be negligible, and the cable action in this case may be modeled by two springs in series, as shown in Fig. 4b. The static stiffness, $K_{st}'$, may then be expressed in the form
\[
\frac{1}{K_{st}} = \frac{1}{K_e} + \frac{1}{K_i}
\]  

(51)

**Alternative Definitions for \( \rho \) and \( \sigma \)**

With the stiffnesses defined in the preceding section, the dimensionless parameters defined by Eqs. 33 and 34 may be expressed as

\[
\rho = \frac{K_e}{K_i}
\]  

(52)

and

\[
\sigma = \sqrt{\frac{K_r}{12 K_i}}
\]  

(53)

It follows that \( \rho \) is effectively a relative stiffness parameter. In an analogous manner, \( \sigma \) may be viewed either as a relative stiffness parameter (Eq. 53), or as a dimensionless measure of the cable sag (Eqs. 34). More specifically, \( \sigma \) represents the horizontal component of the mid-span deflection due to the static lateral load, \( q_y \), non-dimensionalized with respect to the horizontal projection of the chord length.

Also of interest in the following discussion is the stiffness ratio \( K_r/K_e \). Denoted by the symbol \( \varepsilon \), this ratio may be expressed in either of the following forms:

\[
\varepsilon = \frac{K_r}{K_e} = \frac{T_o L}{A E L} \tan^2 \theta = \frac{L^2}{L} \tan^2 \theta
\]  

(54a)

or, in terms of the dimensionless parameters \( \rho \) and \( \sigma \), as

\[
\varepsilon = \frac{1}{12} \frac{\sigma^2}{\rho}
\]  

(54b)
Realistic Values of ρ and σ

Although the values of these parameters may vary in principle from zero to infinity, in practice they are limited to relatively small ranges. For example, for cables with angles of inclination θ ≤ 60°, sag ratios \( y_{\text{max}} / L \leq 1/8 \), and average axial strains \( e_o \leq 0.002 \), the maximum values of \( σ \) and \( ε \) are \( \sqrt{3} \) and 0.00675, respectively. Significantly greater values of \( σ \) and \( ε \) can be obtained only for very steep cables, with inclination angles, \( θ \), approaching 90 degrees.

Increment in Cable Tension

The analysis so far has been concerned with the dynamic increment of the horizontal component of cable tension at the top support. Also of interest is the dynamic increment of the cable tension itself.

At the position of static equilibrium, the cable tension at an arbitrary point is given by Eq. 3, which may also be expressed approximately as

\[
T(x) \approx T_o \left[ 1 + \frac{1}{2} \left( \frac{dv}{dx} \right)^2 \right] \tag{55}
\]

The corresponding tension for the vibrating cable can be written similarly as

\[
T(x,t) + ΔT(x,t) = [T_o + ΔT_o(t)] \left[ 1 + \frac{1}{2} \left( \frac{dv}{dx} + \frac{∂w}{∂x} \right)^2 \right] \tag{56}
\]

where \( ΔT(x,t) \) is the desired tension increment. On expanding this expression, neglecting higher order terms, and making use of Eq. 55, the following expression is obtained:

\[
ΔT(x,t) = \left[ 1 + \frac{1}{2} \left( \frac{dv}{dx} \right)^2 \right] ΔT_o(t) + T_o \frac{dv}{dx} \frac{∂w}{∂x} \tag{57}
\]
PRESENTATION AND ANALYSIS OF DATA

Results for Undamped Cables

In Figs. 6 and 7 the variation of K with φ is shown for undamped cables having several different values of ρ and two values of σ, of which σ = 0 corresponds to horizontal cables. The dynamic stiffness in each case is normalized with respect to the static stiffness value of the particular cable under consideration. As a result, all curves start with a unit ordinate and depend only on the parameters ρ and σ. All data were obtained from the complete solution.

As is generally true of plots of this type, the stiffness curve in each case decreases with increasing frequency and, after crossing the horizontal axis, it approaches asymptotically minus infinity. The curve then reappears at plus infinity and follows a similar trend with increasing frequency.

The zero crossings of these curves correspond to the natural frequencies of the cable when free to move horizontally at the top, and the infinite ordinates correspond to the natural frequencies of the cable supported at both ends. At the latter frequencies, the cable vibrates in a configuration that involves no movement at the ends; consequently, an infinitely large force is required to induce the unit horizontal displacement implied in the development of the expression for dynamic stiffness. Similarly, at the frequencies corresponding to the zero crossings, the cable vibrates as if it were free to move horizontally at the upper end, and, consequently, no force is required to induce a horizontal displacement at that end. The frequencies corresponding to the first zero cros-
sings are of special interest and are listed in the third and fourth columns of Table 1.

For the horizontally oriented cables considered in Fig. 6, values of $\phi = 2\pi$ and $4\pi$ correspond to the second and fourth natural frequencies of both the supported-free and the supported-supported cables. As a result, the ordinates of the stiffness curves must be simultaneously zero and infinite, and this requires that the curves appear - as they do - as vertical lines. The mathematical origin of this result is that the coefficient of the $\sin^2 \theta$ term in Eq. 41 is infinite at these frequencies, and since $\sin \theta$ is zero, the product is indeterminate. Incidentally, the modes of vibration corresponding to these frequencies are antisymmetric about midspan and involve no extension of the cable axis. The same general trends also are found in the curves for other values of $\sigma$, as is evidenced by the data presented in Fig. 7.

Strictly speaking, the vertical lines in Fig. 6 should be continuous. They are shown as discontinuous in order to clarify the interrelationship of the various branches and to emphasize the similarities of the results presented in Figs 6 and 7.

**Effects of Damping**

The effects of damping are illustrated in Figs. 8 through 10 for cables with values of $\rho = 2$ and $\sigma = 0.1$, which are representative of those employed in designs for deep-water guyed-tower platforms. Fig. 8 shows the real part of $K$ normalized with respect to the static stiffness value, $K_{st}$; Fig. 9 shows the corresponding imaginary part; and Fig. 10 shows the associated modulus or amplitude.
Damping in these solutions is specified by the dimensionless factor \( \zeta \) (Eq. 20) which is expressed in terms of the fundamental natural frequency of the associated supported-supported taut cable. Considering that the fundamental natural frequency of the sagging cable is \( 5.34/\pi \) times greater, the indicated values of \( \zeta \) would have been \( \pi/5.34 \) times as great had they been expressed in terms of the latter frequency.

As would be expected, damping affects significantly the results only at frequencies close to the natural frequencies of the supported-supported cable (i.e. the frequencies for which the undamped stiffness tends to infinity). The effect is greatest at the higher natural frequencies, particularly for values of \( \phi \) close to \( 2\pi \) and \( 4\pi \) which correspond to the antisymmetric, inextensional natural modes of vibration. The presence of even a small amount of damping in these cases eliminates the sharp variations exhibited by the curves for no damping. The same general behavior also is observed close to \( \phi = 9.57 \) which corresponds to the second symmetric mode of vibration of the supported-supported cable. But the reductions are not as great because the associated mode of vibration has a substantial extentional component which is unaffected by damping.

The effects of damping at the remaining frequency values can best be understood by reference to a simple cable model presented in a later section, and further discussion of this matter is deferred to that section.

Comparison of Simplified and Complete Solutions

The stiffness curves for undamped cables with values of \( \rho = 1 \) and
σ = 0.25 presented in Fig. 7 are compared in Fig. 11 with the corresponding curves determined by use of the simplified solution, Eq. 42. The ordinates of the curves for the complete solution are normalized, as before, with respect to $K_{st}$, whereas those for the simplified solution are normalized with respect to the corresponding static stiffness, $K'_{st}$, determined from Eq. 46. For the cable considered, $K'_{st} = 0.990 K_{st}$.

It is clear from Fig. 11 that the results of the two solutions are generally in good agreement, except for the following:

1. The simplified solution fails to predict the singularities, and hence the general behavior of the curves, at frequencies close to $\phi = 2\pi$ and $4\pi$. This is due to the fact that the antisymmetric, inextensional modes of vibration of the supported-supported cable are not excited in the simplified solution.

2. The two solutions also differ slightly over the entire range of frequencies, the difference becoming more pronounced at the higher frequencies.

The discrepancies referred to under item 1 are highly localized, however. Furthermore, since damping significantly reduces the sharp variations in the stiffness curves for no damping, the agreement between the two sets of results would be expected to improve for more realistic, damped cables. That this is indeed the case is demonstrated in Fig. 12, in which the curve representing the stiffness amplitude of cables with $\zeta = 0.10$ presented in Fig. 10 is compared with the corresponding curve determined from the simplified solution.
The factors controlling the broader discrepancy referred to under item 2 may be identified from Eq. 37 by comparing the magnitudes of the second and third terms on the right-hand member relative to the magnitude of the first term. It may be recalled that only the first term is retained in the simplified solution.

The ratio of the second to the first terms in Eq. 37 is $\sigma/2$, and hence $\sigma$ is one of the parameters controlling the applicability of the simplified solution. The ratio of the third and first terms cannot be established as readily because, strictly speaking, it is a function of all the parameters affecting the solution. However, valuable insight into the relative magnitude of these terms may be gained by considering the limiting case of a statically excited taut cable. In this case, $w'(L) = (\lambda \sin \theta)/L$, $\Delta T_o = (AE/L_e) X \cos \theta$, and the ratio of the third to the first terms is precisely the dimensionless stiffness factor $\varepsilon$, defined by Eq. 54.

Of the two parameters, $\sigma$ and $\varepsilon$, the latter has been found to be the more important, and will be used as an approximate index of the applicability of the simplified solution. The smaller the value of $\varepsilon$, the better the agreement between the two solutions. In particular, for horizontally supported cables for which $\theta = 0$ and hence $\varepsilon = 0$, the two solutions lead to identical results, a fact noted previously. For the solutions compared in Fig. 11 $\varepsilon = 0.0052$, whereas for those considered in Fig. 12 $\varepsilon = 0.00042$. The better agreement of the results in the latter figure is due to the smaller value of $\varepsilon$ involved.
Based on the results of comparative studies covering a range of values of \( \rho \) and \( \sigma \), it is suggested that, subject to the qualification noted with respect to the singularities in the results for undamped cables, the simplified solution will be in excellent agreement with the complete solution if \( \varepsilon \) is less than about 0.002. At low frequency values, good agreement may be obtained even when this criterion is violated mildly, as is evidenced by the comparison presented in Fig. 11.

**High-Frequency Behavior of Stiffness Curves**

At large exciting frequencies, the normal component of cable motion tends to zero, and the cable essentially remains at its position of static equilibrium, the imposed displacement being absorbed merely by extensional action. The cable in this case effectively acts as if it were constrained to move within a frictionless tube which permits motion only along its length. Noting that the component of the imposed displacement along the cable is

\[
X \cos [\theta - y'(L)] = X \cos \theta + X y'(L) \sin \theta = (1 - \frac{1}{2} \sigma) X \cos \theta
\]

the resulting increase in cable tension becomes

\[
(1 - \frac{1}{2} \sigma) \frac{AE}{L} X \cos \theta
\]

and the horizontal component of this force, the required stiffness, becomes:

\[
K_{\phi_{\infty}} = (1 - \frac{1}{2} \sigma)^2 \frac{AE}{L} X \cos^2 \theta
\]
This expression can also be deduced from Eq. 41 by deleting the term with the factor $T_0/L$, letting $\phi \to \infty$, and taking the high-frequency limit of $y'$ as $-1$. When normalized with respect to the static stiffness, $K_{st}$, defined approximately by Eq. 46, Eq. 59a becomes

$$\frac{K_{\phi \to \infty}}{K_{st}} = (1 - \frac{1}{2} \sigma)^2 (1 + \rho)$$  \hspace{1cm} (59b)

In the derivation of Eq. 59a, if the angle between the horizontal direction and the direction of the cable tension at the upper end were taken as $\theta$ rather than as $[\theta - y'(L)]$, a reasonable assumption for cables with very small sags, the stiffness expression would become

$$K_{\phi \to \infty} = K_e = \frac{AE}{Le} \lambda \cos^2 \theta$$  \hspace{1cm} (60a)

and its normalized version becomes

$$\frac{K_{\phi \to \infty}}{K_{st}} = 1 + \rho$$  \hspace{1cm} (60b)

The limiting values defined by Eqs. 59b and 60b are in good agreement with the corresponding values of the curves in Figs. 6, 7 and 10, particularly for the damped cable considered in Fig. 10. Since the effect of damping is to reduce the normal component of cable motion, the limiting behavior is attained at a lower frequency value as damping increases.

**Behavior of Steep Cables**

For very steep cables, characterized by values of $\varepsilon$ of the order of unity, the horizontal stiffness dominated by the resistance to rotation of the chord rather than the axial resistance or the resistance to change in cable configuration. Since it does not provide for the rotational
resistance, the simplified solution may lead to serious errors in such cases, and it may be necessary to use the complete solution.

An indication of the errors that may result from the use of the simplified solution is provided in Fig. 13, in which the stiffness curves computed by the simplified and complete solutions are compared for cables without damping having values of $\rho = 1/12$ and $\sigma = 1$ and associated value of $\epsilon = 1$. For a cable with a mean axial static strain of $e_a = 0.002$, the particular combination of parameters used corresponds to an angle of inclination $\theta = 87.4$ degrees.

When determined from the simplified solution, the fundamental natural frequency of the cable in Fig. 13 turns out to be smaller when the top end is fully supported than when it is free to move horizontally, a result which is clearly unacceptable on physical ground. As a result of this error, a drastic change takes place in the behavior of the stiffness curve. Instead of first crossing the zero axis and then reaching infinity, the curve based on the simplified solution first tends to infinity and then crosses the zero axis.

The fundamental natural frequencies of supported-free cables determined by the exact and approximate solutions are compared in Table 1 for several combinations of $\rho$ and $\sigma$ covering a range of values of $\epsilon$ between zero and $\infty$. The latter values are identified in the fifth column of the table. Also listed are the corresponding frequencies determined on the assumption that the cable is weightless or taut. The taut cable solution is presented in the next section, where it is shown
that it depends on the single relative stiffness parameter, $\varepsilon$, rather than separately on $\rho$ and $\sigma$.

Comparison of the data presented reveals that the agreement of the results of the approximate and complete solutions deteriorates when $\varepsilon$ becomes large (greater than about 0.02 for the particular combinations of $\rho$ and $\sigma$ considered). It can further be seen that, for larger values of $\varepsilon$, the results of the complete solution are in good agreement with those of the taut cable solution. The latter observation suggests that the latter solution may also provide a reasonably simple approximation to the associated dynamic stiffness values.

That this is indeed true is demonstrated in Fig. 14 in which the dynamic stiffness curve for undamped taut cables with $\varepsilon = 1$ is compared with the corresponding curves obtained from the complete solution for sagging cables having three different combinations of $\rho$ and $\sigma$ for a value of $\varepsilon = 1$ in all cases. As before, the ordinates of these curves are normalized with respect to the static stiffness value of the particular cable under consideration. The agreement between the four sets of results is indeed excellent.

It should be noted in passing that, had the natural frequencies of the supported-free cables in Table 1 been computed from the expression for dynamic stiffness presented on p.147 of Ref. 4, a value of $\phi = \pi$ would have obtained. This result agrees with the reported values only in the special case of horizontally oriented cables ($\sigma = \varepsilon = 0$).
Taut Cable Solutions

For a taut cable \( y = 0 \) and \( \ell = L \), and Eq. 38 for the dynamic stiffness reduces to

\[
K_{\text{taut}} = \frac{AE}{I_e} \cos^2 \theta + (\alpha + \epsilon)(\frac{\sinh \epsilon \omega d - \cosh \epsilon \omega d}{\sinh \alpha + \sinh \beta}) \frac{T_o}{L} \sin \theta \tag{61}
\]

In the absence of damping, Eq. 56 reduces further to

\[
K^C=0_{\text{taut}} = \frac{AE}{I_e} \cos^2 \theta + \phi \cot \phi \frac{T_o}{L} \sin^2 \theta \tag{62}
\]

from which, on letting \( \phi = 0 \), the following expression is obtained for the static stiffness

\[
(K_{\text{st}})_{\text{taut}} = [1 + \epsilon] \frac{AE}{L} \cos^2 \theta \tag{63}
\]

Had the simplified instead of the complete solution been used, only the first term would have appeared on the right sides of Eqs. 61 to 63.

In Fig. 15 the stiffness of undamped taut cables is plotted as a function of \( \phi \) for several values of \( \epsilon \) in the range between 0.01 and \( \infty \).

In the development of the stiffness expressions presented so far, the effects of the axial inertia forces were neglected. These effects are accounted for in the following paragraphs for an undamped taut cable by assuming that there is no coupling between the transverse and axial motions, but taking due cognizance of the effects that both the transverse and the axial inertia forces have on the dynamic increment of the horizontal force at the top support, \( \Delta H \).

The force increment \( \Delta H \) is expressed by the following specialized form of Eq. 37:

\[
\Delta H = \Delta T_o \cos \theta + T_o \omega^2(L) \sin \theta \tag{64}
\]
where

\[ \Delta T_0 = \frac{AE}{L} u'(L) \]  

(65)

\( u'(L) \) and \( w'(L) \) are determined from the solution of the differential equations governing the motions of the cable in the axial and transverse directions, respectively, making use of the following boundary conditions:

\[ u(0) = w(0) = 0 \]  

(66a)

\[ u(L) = X \cos \theta \]  

(66b)

and

\[ w(L) = X \sin \theta \]  

(66c)

The resulting expression for horizontal stiffness is

\[ K_{\tau=0}^{\text{taut}} = \frac{EA}{L} \cos^2 \theta \left[ \chi \cot \chi + \varepsilon \phi \cot \phi \right] \]  

(67)

where

\[ \chi = e_\varepsilon \phi \]  

(68)

and \( e_\varepsilon \) is the axial cable strain corresponding to the position of static equilibrium. Since \( e_\varepsilon \) is typically less than 0.002, provided the exciting frequency is not very large, \( \chi \cot \chi \) is close to unity, and Eq. 67 effectively reduces to Eq. 67. This demonstration is offered as the justifi-
cation for neglecting the effects of the axial inertia forces in the solutions for both the sagless and sagging cables presented herein.
SIMPLE DYNAMIC CABLE MODEL

Within the ranges of parameters for which the simplified solution may be considered to be adequate, a reasonable approximation to the stiffness curves may be obtained from the simple model shown in Fig. 16, which is the dynamic counterpart of the one presented in Fig. 4 for static conditions of loading. As previously noted, the right spring in this model represents the extensional resistance of the cable, which is associated with predominantly axial displacements, and the left spring represents the inextensional resistance, which is associated primarily with transverse or normal displacements. Since both the inertia and damping forces in the cable are due to the normal component of motion, the mass \( \bar{m} \) and damper \( \bar{c} \) for the model are assigned to the left spring.

Let \( \bar{\rho} \) be the undamped circular natural frequency of the model when its right end is free, and \( \bar{\rho}_s \) be the associated frequency when both ends are supported. Then

\[
\bar{\rho} = \sqrt{\frac{K_i}{\bar{m}}}
\]

(69)

\[
\bar{\rho}_s = \sqrt{\frac{K_i + K_e}{\bar{m}}}
\]

(70)

and

\[
\frac{\rho_s}{\bar{\rho}} = \sqrt{1 + \rho}
\]

(71)

The static stiffness of this model is identical to that of the prototype, and its dynamic stiffness at high frequencies tends to the limiting value defined by Eqs. 60a. With the mass of the model remaining stationary at high frequencies, it should be clear that the dynamic stiffness of the model equals the static stiffness of its right spring, \( K_e \).
A single-degree-of-freedom model cannot, of course, be expected to reproduce all aspects of the behavior of the continuous prototype cable. However, a reasonable approximation to some of the important trends may be obtained by a judicious choice of its parameters. The selection of these parameters is considered after presentation of the expression for the dynamic stiffness of the model.

**Stiffness of Model**

Let \( X(t) = X e^{i\omega t} \) be the displacement at the right end of the model and \( F(t) = F e^{i\omega t} \) be the force required to produce this displacement. The displacement amplitude, \( X \), may be expressed as the sum of two components: the deformation of the right spring, \( F/K_e \); and the deformation of the left spring, which equals the product of \( F/K_1 \) and the amplification factor for a harmonically excited single-degree-of-freedom system. Considering the subsystem composed of the left spring, the mass and the damper, noting that \( \tilde{\beta} \) is its undamped circular natural frequency, and denoting by \( \tilde{\xi} \) its damping factor, \( X \) may be expressed in the form:

\[
X = \frac{1}{1 - \tilde{\phi}^2 + i 2\tilde{\xi} \tilde{\phi}} \frac{F}{K_1} + \frac{F}{K_e}
\]

(72)

where

\[
\tilde{\phi} = \frac{\omega}{\tilde{\beta}} \tag{73}
\]

and

\[
\tilde{\xi} = \frac{\tilde{c}}{2\tilde{m}\tilde{\beta}} \tag{74}
\]

The dynamic stiffness of the model, \( \tilde{K} \), is then given by the ratio \( F/X \), and Eq. 72 may be rewritten as

\[
\frac{1}{\tilde{K}} = 1 + \frac{\rho}{1 - \tilde{\phi}^2 + i 2\tilde{\xi} \tilde{\phi}} \frac{1}{K_e}
\]

(75)
Choice of Model Parameters

Of special interest in practice is the behavior of the cable at small exciting frequencies, usually less than that for which the dynamic stiffness of the cable without damping first becomes zero. If the intent is to provide the best possible representation of this initial, decreasing branch of the stiffness curve, it is reasonable to take \( \bar{\phi} \) equal to \( p_1 \), the fundamental circular frequency of the prototype cable when free to move horizontally at the top. Fixing \( \bar{\phi} \) is equivalent, of course, to fixing \( \bar{m} \). For a cable without damping, this particular choice ensures that the dynamic stiffness of the model and the prototype vanish at the same frequency value. With \( \bar{\phi} \) or its equivalent \( \bar{m} \) determined in this manner, the dimensionless frequency parameter, \( \tilde{\phi} \), in Eq. 73 becomes

\[
\tilde{\phi} = \frac{\omega}{p_1} = \frac{\phi}{\phi_1}
\]

where

\[
\phi_1 = \frac{p_1}{\omega_o}
\]

The damping of the model is determined such that the ratio \( \bar{\xi}/\bar{m} \) is the same as the ratio \( c/\mu \) for the prototype. The damping factor \( \xi \) in Eq. 74 is then given by

\[
\xi = \frac{\pi \omega_o}{p_1} \zeta = \frac{\pi}{\phi_1} \xi
\]

In Figs. 17 through 20 the dynamic stiffness curves determined by use of the model are compared over a range of parameters with the corresponding curves determined by use of the simplified solution for the prototype cable. Fig. 17 refers to horizontally supported undamped cables, whereas the remaining refer to damped cables with \( \zeta = 0.10 \). For
the latter case, the real and imaginary parts of the stiffness are given in addition to the stiffness amplitude.

It can be seen that not only are the initial branches of the two sets of curves in excellent agreement, but also the simple model reproduces with good accuracy the major trends of the more nearly accurate solution over the entire range of frequencies. Naturally, the simple model fails to predict the sharp variations in ordinates at frequencies close to the higher natural frequencies of the cable. These variations are highly localized, however, and become relatively unimportant for cables with reasonable amounts of damping.

**Alternative Choice of Parameters**

In the discussion so far, the parameters of the simple model were determined with a view of representing accurately the initial branch of the stiffness curves. If the intent is to achieve the best possible representation of the behavior of the curves in the vicinity of the first resonant peak, it would be preferable to choose \( \bar{m} \) such that the natural frequency of the model in its supported-supported condition, \( \bar{\omega}_5 \), is the same as the corresponding frequency of the prototype cable. Denoting the latter frequency by \( p_2 \), and making use of Eq.71, the following expression is obtained for the value of \( \bar{c} \) in Eq. 75:

\[
\bar{c} = \sqrt{1 + \rho} \frac{\omega}{p_2} = \sqrt{1 + \rho} \frac{\phi}{\phi_2} \tag{79}
\]

where \( \phi_2 = p_2/\omega_o \). With \( \bar{m} \) selected in this manner, and with the damping of the model determined, as before, by taking \( c/\bar{m} = c/\mu \), the damping factor in Eq. 75 becomes

\[
\xi = \sqrt{1 + \rho} \frac{\pi \omega_o}{p_2} \xi = \sqrt{1 + \rho} \frac{\pi}{\phi_2} \xi \tag{80}
\]
Effect of Damping on Stiffness

In Table 2 are listed the stiffness amplitudes corresponding to the first resonant peak for the cables considered in Fig. 9. In the first two columns are given the values obtained from the complete and simplified cable solutions; and in the third and fourth columns are given the values obtained for the simple model taking $\bar{p} = p_1$ and $\bar{p}_s = p_2$, respectively. As anticipated, the best agreement in this case is obtained when the natural frequencies of the model and the prototype are matched for their supported-supported condition.

A still better agreement between the solutions for the prototype and model in the vicinity of the first resonant peak could have been achieved by selecting $\xi$ so as to yield the exact resonant peak. However, this approach would have required the prior evaluation of the exact peak, and would not have been as convenient to use as the one proposed in the preceding section.

Inasmuch as of special interest in practice is the behavior of the stiffness curves at low frequencies, it is also desirable to examine the effect that damping has on the stiffness value at $\omega = p_1$, namely, the frequency for which the dynamic stiffness of the cable becomes zero in the absence of damping. Using the first of the two approximations considered, and noting that $\phi = 1$ in this case, the following expression is obtained from Eq. 75:

$$
\frac{1}{\hat{\kappa}} = \left[ 1 + \frac{\rho}{i 2 \xi} \right] \frac{1}{\kappa_e}
$$

where $\xi$ is defined by Eq. 74. The corresponding stiffness amplitude, $\hat{\kappa}_o$, is given by

$$
\hat{\kappa}_o = \frac{2 \xi}{\sqrt{4 \xi^2 + \rho^2}} \kappa_e
$$
With this result and the natural frequency of the supported-free cable known, the initial branch of the stiffness curve can be approximated readily for cables having any amount of damping.

In concluding this section, it should be noted that possibly the greatest value of the simple model considered is the insight it provides into the behavior of the cable. It helps to identify the important parameters of the problem and to understand some of the more important trends revealed by the considerably more complex complete solution.
VERTICAL STIFFNESS

The stiffness expressions presented in the preceding sections are strictly valid for the horizontal stiffness of the cable. However, by a proper interpretation of the parameters, they may also be used to evaluate the corresponding vertical stiffness. The latter quantity is defined as the complex-valued amplitude of the harmonic vertical force increment at the top of the cable necessary to induce a steady state displacement amplitude of unit magnitude in the same direction, the top of the cable being held against horizontal motion. This force increment is denoted by $\Delta V$ in Fig. 21a.

If the system in Fig. 21a is viewed from the left side of the page, or, alternatively, of the system is rotated by $90^\circ$ in the counterclockwise direction, the diagram shown in Fig. 21b is obtained. It should be clear that the vertical stiffness is the same as the horizontal stiffness of a cable having an angle of inclination of $\theta = 90^\circ - \phi$ degrees and subjected to a normal pressure of magnitude $q_y = -q_x$. The desired stiffness may, therefore, be obtained from the expressions presented simply by replacing $\theta$ by $90^\circ - \phi$ and $q_y$ by $-q_y$.

As an example, it is noted that the vertical stiffness of an undamped cable corresponding to the complete solution is

$$K_{\text{vertical}} = \frac{(1 + \frac{1}{2} \zeta_\nu \frac{V'}{V})^2}{1 + 2 \zeta_\nu \frac{V'}{V}} \frac{AE}{Ec} \sin^2 \phi + \phi \cos^2 \phi \frac{F}{L} \cos \theta$$

(83)

with

$$\zeta_\nu = -\frac{q_x L}{T_0} \cos \theta$$

(84)

and

$$\rho = \frac{1}{12} \frac{AE}{Ec} \frac{L}{T_0} \left( \frac{q_x L}{T_0} \right)^2$$

(85)
GROUP BEHAVIOR

Lateral Motion

The steady-state harmonic force displacement relationship of a group of cables is obtained as follows.

Consider a system of \( n \) cables connected at their tops to a rigid horizontal frame, and whose horizontal projections meet in a single point. Let the horizontal position of any cable \( i \) be specified by the angle \( \delta_i \) between the direction of top horizontal motion and the vertical cable plane.

The horizontal force amplitudes \( F_I \) and \( F_{II} \) necessary to apply on the frame in direction, respectively normal to the direction of motion are

\[
F_I = \left\{ \sum_{i=1}^{n} K_{i1} \cos \delta_i \right\} + \sum_{i=1}^{n} K_{i2} \sin \delta_i X_I
\]

and

\[
F_{II} = \left\{ \sum_{i=1}^{n} K_{i1} \cos \delta_i \right\} \sin \delta_i - \sum_{i=1}^{n} K_{i2} \cos \delta_i \sin \delta_i \} X_I
\]

where \( X_I \) is the amplitude of horizontal top motion, \( K_{II} \) is the in-plane dynamic stiffness \( K \) of the \( i \)th cable, and \( K_{I2} \) is the out-of-plane dynamic stiffness of the \( i \)th cable. This latter one is given by

\[
K_{Iz} = \frac{F_I}{X_I} \left( \cos \alpha - \cos \beta \right) \frac{\sin \beta \cos \alpha - \cos \beta \sin \alpha}{\sin \beta \cos \alpha + \sin \beta \sin \alpha}
\]

The displacement \( X_I \) also induces a vertical reaction at the top. If \( K_{I3} \) denotes the reaction induced by a unit horizontal displacement in the plane of the \( i \)th cable, then the total reaction \( F_{III} \) is given by

\[
F_{III} = \left\{ \sum_{i=1}^{n} K_{I3} \cos \delta_i \right\} X_I
\]

(86c)
For the important case where more than two cables are arranged symmetrically around the tower and all cables are identical, those expressions reduce to

\[ F_j = \frac{d}{2} \left[ k_1 + k_2 \right] X_1 \]  \hspace{1cm} (88a)

and

\[ F_i = F_{i,1} = 0 \]  \hspace{1cm} (88b, c)

It must be noted that \( k_2 \) is generally smaller than \( k_1 \) and can be neglected.

It must also be noted that a torsional moment acting on the frame is generally associated with the horizontal displacement. This one disappears if the cables are identical and symmetrically arranged around a circular frame.

Torsional Motion

When the distance from the point at which the cables meet to the point of attachment of the \( i \)th cable to the frame is \( r_i \), the amplitude \( \tau \) of harmonic torsional moment to be applied to the frame to induce a steady-state rotation of amplitude \( \Omega \) around the center of projections is given by

\[ \tau = \left\{ \sum_i r_i^2 k_i \right\} \Omega \]  \hspace{1cm} (89)

It must be noted that a horizontal force resultant acting on the frame is generally associated with the rotation, unless the cables are identical and symmetrically arranged around a circular frame.
CONCLUSION

A closed-form expression for the horizontal stiffness of a viscously damped, uniform, inclined cable supported at one end and subjected to a harmonically varying horizontal displacement at the other end has been presented. At its position of static equilibrium the cable has been considered to have a parabolic profile, and the amplitudes of the dynamic displacements have been presumed to be small. The expression is valid for an arbitrary angle of inclination of the cable chord in the range between zero and 90 degrees. A simpler solution valid over a narrower range of the parameters also has been presented and its accuracy discussed. For undamped cables, the simplified solution is essentially the same as that presented previously by Davenport (2).

Comprehensive numerical data have been presented and discussed, and a special effort has been made to interpret the physical significance of the results and to provide insight into the action of the cable and into the parameters that control it. Finally, a simple model has been proposed which reproduces with good accuracy the significant aspects of the response of the prototype cable.
REFERENCES


TABLE 1
VALUES OF $\phi$ CORRESPONDING TO FUNDAMENTAL NATURAL FREQUENCY
OF SUPPORTED-HORIZONTALLY FREE CABLE

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>Complete Solution</th>
<th>Approximate Solution</th>
<th>$\varepsilon = \frac{1}{12} \frac{\sigma^2}{\rho}$</th>
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<td>3.314</td>
<td>$\infty$</td>
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TABLE 2
DIMENSIONLESS STIFFNESS AMPLITUDES CORRESPONDING TO FIRST RESONANT PEAK, FOR \( \rho = 2 \) AND \( \sigma = 0.1 \)

<table>
<thead>
<tr>
<th>( \zeta )</th>
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<th>Simplified Solution</th>
<th>Model, ( \bar{\rho} = p_1 )</th>
<th>Model, ( \bar{\rho}_s = p_2 )</th>
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APPENDIX A

NOTATION

A : cable cross-sectional area;
c : coefficient of viscous damping per unit of chord length;
\check{c} : coefficient of viscous damping of simple model;
C : dimensionless amplification factor defined by Eq. 29;
C_1, C_2 : constants of integration defined by Eqs. 30;
e : base of the system of natural logarithms;
e_0 = \frac{T_o}{AE} : average axial strain in the cable at its position of static equilibrium;
E : cable Young's modulus of elasticity;
H : horizontal component of the cable tension at the upper support when the cable is at its position of static equilibrium, defined by Eq. 35;
\Delta H : dynamic increment in H defined by Eq. 37;
i : = \sqrt{-1};
K : complete horizontal dynamic stiffness defined by Eq. 38;
K_e : extensional stiffness defined by Eq. 47;
K_i : inextensional stiffness defined by Eq. 49;
K_r : rotational stiffness defined by Eq. 48;
K_{st} : complete horizontal static stiffness defined by Eq. 45;
(K_{st})_{taut} : complete horizontal static stiffness of taut cable defined by Eq. 63;
K_{taut} : complete horizontal dynamic stiffness of taut cable defined by Eq. 61;
K_{vertical} : complete vertical dynamic stiffness defined by Eq. 83;
\( K' \): simplified horizontal dynamic stiffness defined by Eq. 39;

\( K'_{st} \): simplified horizontal static stiffness;

\( K \): dynamic stiffness of simple model defined by Eq. 75;

\( K_0 \): stiffness amplitude of simple model at \( \phi = 1 \) defined by Eq. 82;

\( L \): length of the cable chord;

\( L_e \): effective cable length defined by Eq. 13;

\( \bar{m} \): mass of simple model;

\( p_1 \): fundamental circular frequency of the cable when free to move horizontally at the top;

\( p_2 \): fundamental circular natural, symmetric, frequency of the cable when supported at both ends;

\( \bar{p} \): circular natural frequency of the simple model when its right end is free;

\( \bar{p}_s \): circular natural frequency of the simple model when both ends are fixed;

\( q \): intensity of the vertical load per unit of cable length;

\( q_x \): intensity of the axial load per unit of chord length;

\( q_y \): intensity of the normal load per unit of chord length;

\( S \): unstretched length of the cable;

\( t \): time;

\( T \): cable tension at the upper support when the cable is at its position of static equilibrium;

\( \Delta T \): dynamic increment in \( T \);

\( T(x) \): cable tension at an arbitrary point when the cable is at its position of static equilibrium, defined by Eq. 3;

\( T_{\text{max}} \): maximum value of \( T(x) \);
$T_0$ : axial component of the cable tension $T(x)$;

$\Delta T_0$ : dynamic increment in $T_0$;

$T_e^o$ : effective component of cable tension in the direction of the chord defined by Eq. 7;

$T_{e\text{max}}$ : maximum value of the effective cable tension defined by Eq. 8;

$u$ : axial displacement component of vibrating cable;

$v$ : normal deformational displacement component of vibrating cable defined by Eq. 14;

$v_h$ : homogeneous component of $v$;

$v_p$ : particular component of $v$;

$\Delta V$ : dynamic increment in the vertical cable reaction $V$ at the upper support;

$w$ : normal displacement component of vibrating cable;

$x$ : distance coordinate;

$X$ : horizontal displacement amplitude;

$y$ : normal deflection of the cable at its position of static equilibrium;

$y_{\text{max}}$ : maximum value of $y$;

$\alpha, \beta$ : factors defined by Eqs. 27 and 28, respectively;

$\gamma$ : coefficient defined by Eq. 32;

$\varepsilon$ : parameter defined by Eq. 54;

$\zeta$ : damping parameter defined by Eq. 20;

$\xi$ : damping factor of simple model defined by Eq. 74;

$\theta$ : inclination of the cable chord;

$\mu$ : mass of the cable per unit of chord length;

$\xi = \frac{x}{L}$ : dimensionless distance coordinate;
\( \rho, \rho_v \): parameters defined by Eqs. 33 and 85, respectively;
\( \sigma, \sigma_v \): parameters defined by Eqs. 34 and 84, respectively;
\( \phi \): dimensionless frequency defined by Eq. 19
\( \phi_1 \): dimensionless fundamental natural frequency of the cable when free to move horizontally at the top, defined by Eq. 77;
\( \phi_2 = \frac{p_2}{\omega_o} \): dimensionless, symmetric, natural frequency of the cable supported at both ends;
\( \tilde{\phi} \): dimensionless frequency parameter defined by Eq. 76;
\( \chi \): dimensionless parameter defined by Eq. 68;
\( \omega \): circular frequency of the motion; and
\( \omega_o \): circular frequency defined by Eq. 21.
APPENDIX B
EQUATIONS OF EQUILIBRIUM AND MOTION

Equation of Equilibrium

Consider a small cable element at its position of static equilibrium, as shown in part (a) of Fig. B1, where \( dx \) is the length of the element along the x-axis, \( T_o \) is the x-component of the cable tension which is a constant, and \( V \) is the normal or y-component of the tension.

Equilibrium of forces in the y-direction requires that
\[
dV + q_y \, dx = 0 \tag{B1}
\]
and since the cable can resist only tension, the resultant of \( T_o \) and \( V \) must be in the direction of the cable, i.e.
\[
\frac{V}{T_o} = \frac{dy}{dx} \tag{B2}
\]
Differentiating Eq. B2 with respect to \( x \) and substituting the resulting expression into Eq. B1, one obtains
\[
T_o \frac{d^2y}{dx^2} + q_y = 0 \tag{B3}
\]
Integration of the latter equation, subject to the boundary conditions \( y(0) = y(L) \), leads to Eq. 1.

Equation of Motion

The position of the cable element at an arbitrary time and the forces acting on it are shown in part (b) of Fig. B1, where the prefix \( \Delta \) denotes the dynamic increment of the quantity to which it is attached.

Proceeding in a manner analogous to that used in the development of Eqs. B1 and B2, the following expressions are obtained:
\[ dV + d(\Delta V) + q_y \, dx - c \frac{\partial w}{\partial t} \, dx - \mu \frac{\partial^2 w}{\partial t^2} \, dx = 0 \]  

and

\[ \frac{V + \Delta V}{T_o + \Delta T_o} = \frac{dy + dw}{dx} \]  

On making use of Eq. B1, Eq. B4 reduces to

\[ d(\Delta V) - c \frac{\partial w}{\partial t} \, dx - \mu \frac{\partial^2 w}{\partial t^2} \, dx = 0 \]  

and, on making use of Eq. B2 and neglecting a second order term \( \Delta T_o \frac{\partial w}{\partial x} \), Eq. B5 reduces to

\[ \Delta V = T_o \frac{\partial w}{\partial x} + \Delta T_o \frac{dy}{dx} \]  

The desired equation of motion, Eq. 11, may now be obtained by differentiating Eq. B7 with respect to \( x \) and substituting the resulting expression into Eq. B6. It may be recalled that \( \Delta T_o \) in this equation is still unknown.

To determine \( \Delta T_o \), it is first necessary to establish the strain-displacement relationship for the cable. Letting \( e \) be the axial strain measured from the position of static equilibrium, one obtains

\[ e = \frac{ds'}{ds} - \frac{ds}{ds} \]  

where \( ds \) and \( ds' \) are the lengths of the cable element at rest and in motion, respectively. Referring now to part (c) of Fig. B1, it can be seen that

\[ ds^2 = dx^2 + dy^2 \]  

and that

\[ (ds')^2 = (dx + du)^2 + (dy + dw)^2 \]
On expanding the latter expression and making use of Eq. B9, one obtains

\[(ds')^2 = \left\{ 1 + \left[ \left( \frac{\partial u}{\partial s} \right)^2 + 2 \frac{dx}{ds} \frac{\partial u}{\partial s} + \left( \frac{\partial w}{\partial s} \right)^2 + 2 \frac{dy}{ds} \frac{\partial w}{\partial s} \right] \right\} ds^2 \] (B11)

whence

\[ds' = \left\{ 1 + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial s} \right)^2 + 2 \frac{dx}{ds} \frac{\partial u}{\partial s} + \left( \frac{\partial w}{\partial s} \right)^2 + 2 \frac{dy}{ds} \frac{\partial w}{\partial s} \right] \right\} ds \] (B12)

Finally, substituting Eq. B12 into Eq. B8 and deleting the second order terms \( \frac{1}{2} \left( \frac{\partial u}{\partial s} \right)^2 \) and \( \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2 \), the following strain-displacement relationship is obtained

\[e(x,t) = \left( \frac{dx}{ds} \right)^2 \left[ \frac{\partial u}{\partial x} + \frac{dy}{dx} \frac{\partial w}{\partial x} \right] \] (B13)

The component of the tension increment in the direction of the chord, \( \Delta T_0 \), may now be expressed as

\[\Delta T_0 = \Delta T \frac{dx}{ds} \] (B14)

where

\[\Delta T = AE \ e(x,t) \] (B15)

Substituting Eqs. B13 and B14 into Eq. B15, one obtains

\[\Delta T \left( \frac{ds}{dx} \right)^3 = AE \left[ \frac{\partial u}{\partial x} + \frac{dy}{dx} \frac{\partial w}{\partial x} \right] \] (B16)

and on integrating this expression with respect to \( x \) between the limits 0 and L, one obtains Eq. 12, where \( L_e \) is defined by

\[L_e = \int_0^L \left( \frac{ds}{dx} \right)^3 dx \] (B17)

Finally, if \( ds \) is expressed as

\[ds = \left[ 1 + \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \right] dx \] (B18)

and Eq. B17 is integrated making use of Eq. 1 and neglecting the terms in
$y'$ of power higher than the second, Eq. 13 is obtained.

For small sag-to-length ratios, $ds$ may be approximated by $dx$ and $\Delta T$ may be replaced by $\Delta T_0$. Then $L_e$ is equal to the chord length $L_c$.

(a) Configuration at Rest  (b) Configuration during Motion

(c) Elemental Lengths

Fig. B1 - Cable Element
APPENDIX C

SOLUTION OF DIFFERENTIAL EQUATION

The solution of the homogeneous form of Eq. 22 may be taken in the form \( e^{\xi r} \), where \( r \) must satisfy the characteristic equation

\[
    r^2 + \phi^2 - 2 \pi \xi \phi = 0
\]  

(C1)

and the roots of the characteristic equation may be expressed as

\[
    r = \alpha + i\beta
\]  

(C2)

where \( \alpha \) and \( \beta \) are real-valued quantities.

Substituting Eq. C2 into Eq. C1 and equating the real and imaginary parts leads to

\[
    \beta^2 - \alpha^2 = \phi^2
\]  

(C3)

and

\[
    \alpha \beta = \pi \xi \phi
\]  

(C4).

Finally, eliminating \( \alpha \) from Eq. C3 by use of Eq. C4 leads to the following equation in \( \beta \):

\[
    \beta^4 - \phi^2 \beta^2 - (\pi \xi \phi)^2 = 0
\]  

(C5)

This equation has two real-valued roots. One is defined by Eq. 27 and the other is its negative. The corresponding values of \( \alpha \) are determined from Eq. C4 or Eq. 28.

It follows that the two characteristic roots, \( r_1 \) and \( r_2 \), are

\[
    r_1 = \alpha + i\beta
\]  

(C6)

and

\[
    r_2 = -\alpha - i\beta
\]  

(C7)

The desired solution may, therefore, be expressed as

\[
    v_h(\xi) = B_1 e^{\alpha \xi} e^{i\beta \xi} + B_2 e^{-\alpha \xi} e^{-i\beta \xi}
\]  

(C8)
or, on replacing the exponential functions by trigonometric and hyperbolic functions, in the form of Eq. 25

For a cable without damping, $\zeta = 0$ and hence $\alpha = 0$ and $\beta = \phi$. In this case, Eq. 25 reduces to

$$v_h(\xi) = D_1 \cos \phi \xi + D_2 \sin \phi \xi$$  \hspace{1cm} (C9)

where $D_1$ and $D_2$ are new integration constants.
APPENDIX D

EVALUATION OF TENSION INCREMENT

The evaluation of the increment $\Delta T_o$ in component of tension by Eq. 23 requires first the evaluation of $v(\xi)\,d\xi$ where $v(\xi)$ is given by Eqs. 24 to 26. This is

$$\int v(\xi)\,d\xi = C_1 \left[ \frac{\alpha}{\alpha^2 + \rho^2} \sinh \alpha \cos \rho + \frac{\rho}{\alpha^2 + \rho^2} \cosh \alpha \sin \rho + \frac{\rho}{\alpha^2 + \rho^2} \cosh \alpha \sin \rho - \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \cos \rho \right] + C_2 \left[ \frac{\alpha}{\alpha^2 + \rho^2} (\cosh \alpha \cos \rho - 1) + \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \sin \rho + \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \sin \rho - \frac{\rho}{\alpha^2 + \rho^2} (\cosh \alpha \cos \rho - 1) + C(\phi, \xi) \frac{\rho^2}{\alpha^2 + \rho^2} \frac{\alpha}{\alpha^2 + \rho^2} \frac{1}{\cosh \alpha} \right] \, d\xi \quad (D1)$$

Substituting into this expression for the constants of integration $C_1$ and $C_2$ given by Eq. 30 and performing the multiplications involved then gives

$$\int v(\xi)\,d\xi = -C(\phi, \xi) \frac{\rho^2}{\alpha^2 + \rho^2} \frac{\alpha}{\alpha^2 + \rho^2} \left[ \frac{\alpha}{\alpha^2 + \rho^2} \sinh \alpha \cos \rho + \frac{\rho}{\alpha^2 + \rho^2} \cosh \alpha \sin \rho - \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \cos \rho \right] + \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \sin \rho \frac{1}{\cosh \alpha} \left[ \frac{\alpha}{\alpha^2 + \rho^2} \cosh \alpha \sin \rho - \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \cos \rho \right]$$

$$\sinh \alpha \cos \rho + \frac{\rho}{\alpha^2 + \rho^2} \cosh \alpha \sin \rho + \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \sin \rho - \frac{\rho}{\alpha^2 + \rho^2} (\cosh \alpha \cos \rho - 1) + \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \sin \rho + \frac{\rho}{\alpha^2 + \rho^2} \sinh \alpha \sin \rho - \frac{\rho}{\alpha^2 + \rho^2} (\cosh \alpha \cos \rho - 1) + C(\phi, \xi) \frac{\rho^2}{\alpha^2 + \rho^2} \frac{\alpha}{\alpha^2 + \rho^2} \frac{1}{\cosh \alpha} \right] \, d\xi \quad (D1)$$
This expression can be simplified by making use of the fundamental relationships \( \sin^2 \beta + \cos^2 \beta = 1 \) and \( \cosh^2 \alpha - \sinh^2 \alpha = 1 \), and rearranging the terms. By doing so Eq. D2 becomes

\[
\int_{v_{10}}^{v_1} d\xi = \mathcal{C}(\phi, \zeta) \left[ \frac{\eta \xi^2}{2} \frac{\alpha}{\phi} \left[ 1 - \frac{2}{\lambda + c \kappa} \frac{\cosh \alpha - \cosh \beta}{\sinh \alpha - \cosh \beta} \right] + \frac{\lambda}{2} \frac{\kappa}{\phi} \frac{\cosh \alpha - \cosh \beta}{\sinh \alpha - \cosh \beta} \right] (D3)
\]

This integral can now be substituted into Eq. 23. Doing so and furthermore separating the terms in \( \Delta T_0 \) from those in \( X \) and introducing Eq. 29 for \( \mathcal{C}(\phi, \zeta) \) gives

\[
\Delta T_0 \left\{ 1 - \frac{\mu \xi}{\ell c} \left( \frac{\eta \xi^2}{2} \right) \frac{1}{\phi} \frac{1}{\lambda + c \kappa} \left[ 1 - \frac{2}{\lambda + c \kappa} \frac{\cosh \alpha - \cosh \beta}{\sinh \alpha - \cosh \beta} \right] \right\} = \frac{X}{\ell c} \left( \cos \theta + \frac{1}{2} \frac{\eta \xi^2}{\phi} \sin \theta \left[ -1 + \frac{2}{\lambda + c \kappa} \frac{\cosh \alpha - \cosh \beta}{\sinh \alpha - \cosh \beta} \right] \right) (D4)
\]

Introducion of the dimensionless parameters defined by Eqs 32 to 34 into Eq. D4 readily gives expression 31 for the increment \( \Delta T_0 \) in tension component.
Fig. 1. Cable in Equilibrium
Fig. 2. Components of Displacements
(a) Action of Taut Cable

(b) Inextensional Action of Saging Cable

Fig. 3. Action of Taut and Saging Cable
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CHAPTER

FIVE

FREE-VIBRATION CHARACTERISTICS OF A SUPPORTED-SUPPORTED PARABOLIC CABLE

Introduction
System and Assumptions
Background Information
Presentation and Analysis of Data
Approximate Frequency Spectrum
Infinite Series for Integrals of Natural Modes
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INTRODUCTION

The free-vibrational characteristics of sagging cables have been the subject of numerous investigations over the years. A complete account of these studies is available in Ref. 1. Among the more valuable recent contributions are those of Irvine and Caughey (2,1), who presented the correct expressions for the natural frequencies and modes of vibration of a horizontally oriented uniform parabolic cable and contributed extensive numerical data, and of Irvine (3,1), who extended this solution to an inclined cable.

The present study is intended to be complementary to these recent contributions. Its objectives are to elucidate certain aspects of the free vibration of inclined parabolic cables, providing physical interpretations to major trends; and to present simple, approximate expressions with the aid of which the complete spectrum of the natural frequencies can be sketched readily. In addition, simple closed-form expressions are presented for certain infinite series involving integrals of the natural modes of the cable. These expressions, along with similar expressions presented previously (4,1) are expected to prove of value in analyses of the dynamic response of cables to external forces and/or support excitations.

Notation

The letter symbols used are defined when first introduced in the text, and those used extensively are summarized in Appendix A.
SYSTEM AND ASSUMPTIONS

A uniform cable suspended from rigid end supports located at different elevations is considered, as shown in Fig. 1. At its position of static equilibrium, the cable is presumed to be deflected in a parabolic profile with its axis normal to the chord joining the support points. Such a configuration would be produced by a lateral load of intensity \( q_y \) which is normal to, and uniformly distributed along, the chord, as shown in the figure. The configuration of the cable is referred to a Cartesian coordinate system, \( x-y \), with its origin taken at the lower end of the cable, and the \( x \)-axis taken along the chord line, and the positive direction of \( y \) taken in the direction of the applied load.

For cables with small sag ratios, the parabolic profile also provides a good approximation for the effects of a vertical load, such as the cable's own weight. If \( q \) is the intensity of the vertical load per unit of cable length, then \( q_y \) should be interpreted as

\[
q_y = \frac{qS}{L} \cos \theta
\]

where \( S \) is the unstretched length of the cable, and \( L \) and \( \theta \) are the length and inclination of the cable chord, respectively. Application of the information presented herein to vertically loaded cables requires that the axial component of the load be small such that the cable is everywhere under tension and the distribution of the tension along the length is reasonably uniform.

The analysis is based on the assumption that the displacements of the cable about the position of static equilibrium are small, and that the inertia forces associated with the axial displacement components are
negligible in comparison to those due to the transverse or normal displacement components. Only motions in the plane of the cable are considered. The configuration of the vibrating cable will be specified by the normal displacement component, measured from the position of static equilibrium along the y-axis. The amplitude of this displacement at an arbitrary point will be denoted by $\psi(\xi)$, where $\xi$ is the dimensionless distance $x/L$. While the axial displacement components may be determined from $\psi(\xi)$, these will not be examined herein, and the term natural mode will refer to the configuration of normal displacements.
BACKGROUND INFORMATION

For the conditions considered, the mode of vibration, \( \psi(\xi) \), is either symmetric or antisymmetric about midspan. The antisymmetric modes involve no extension of the cable axis and no change in cable tension, whereas the symmetric modes are generally associated with changes both in cable tension and in the length of the cable axis.

Let \( \omega_n \) be the \( n \)th circular natural frequency of the cable corresponding either to the symmetric or the antisymmetric group of vibrational modes. This frequency may be expressed in the form

\[
\omega_n = \frac{\phi_n}{L} \sqrt{\frac{T_o}{\mu}}
\]  

(2)

where \( \phi_n \) is a dimensionless quantity, \( T_o \) is the axial or \( x \)-component of the cable tension at the position of static equilibrium, and \( \mu \) is the mass of the cable per unit of chord length. The force component \( T_o \) is constant along the length of the cable.

Antisymmetric Modes

The values of \( \phi_n \) corresponding to antisymmetric modes of vibration are given by

\[
\phi_n = 2n\pi
\]  

(3)

and the associated modes are given by

\[
\psi_n(\xi) = B_n \sin 2n\pi \xi
\]  

(4)

where \( B_n \) is a normalizing constant. Note that these results are the same as those for a weightless, taut cable.
Symmetric Modes

The values of $\phi_n$ corresponding to symmetric modes of vibration are defined by the frequency equation

$$\phi_n^2 + 12\rho \left[ \frac{2}{\phi_n} \tan \frac{\phi_n}{2} - 1 \right] = 0$$

(5)

where $\rho$ is a dimensionless factor defined by

$$\rho = \frac{1}{12} \frac{AE}{T_o} \frac{L}{L_e} \left( \frac{q_L}{T_o} \right)^2$$

(6)

$A$ and $E$ are the cross sectional area and Young's modulus of elasticity for the cable; and $L_e$ is the effective cable length, defined by

$$L_e = \left[ 1 + \frac{1}{\beta} \left( \frac{q_L}{T_o} \right)^2 \right] L$$

(7)

The associated $n$th symmetric mode is given by

$$\psi_n(\xi) = C_n \left[ \cos \left( \phi_n (\xi - \frac{1}{2}) \right) - \cos \frac{\phi_n}{2} \right]$$

(8)

when $C_n$ is a normalizing constant.

Equations 5 and 8, in somewhat different forms, were first obtained by Irvine and Caughey (2,1) by solving the appropriate characteristic value problem. Eq. 5 could also have been obtained from the expression for the horizontal harmonic stiffness of the cable reported by Davenport et al (5,6) by equating to zero the denominator of the latter expression. However, since no distinction is made in Davenport's solution between $L$ and $L_e$, the ratio of these two quantities in Eq. 6 would be unity in the latter approach. Eq. 5 and Eq. 8 can also be derived from the single expression for harmonic stiffness presented in Chapter IV, and this derivation is given in Appendix B.
The significance of the factor \( \rho \) in Eq. 5 may be appreciated by considering the effect of a static increase in the component of the cable tension along the chord. Let \( \Delta T_0 \) be the increase in \( T_0 \) and \( \Delta_0 \) be the corresponding displacement. The latter displacement may then be expressed in the form

\[
\Delta_0 = \frac{\Delta T_0}{K_1} + \frac{\Delta T_0}{K_2}
\]

(9)

where \( K_1 \) is the extensional stiffness of the cable defined by

\[
K_1 = \frac{AE}{L_e}
\]

(10)

and \( K_2 \) is the inextensional cable stiffness, which is associated with a change in cable configuration but no extension of the cable axis and is defined by

\[
K_2 = 12 \frac{T_0}{L} \left( \frac{T_0}{q_y L} \right)^2
\]

(11)

From Eq. 6 it can now be concluded that \( \rho \) represents the ratio of \( K_1 \) and \( K_2 \), i.e.

\[
\rho = \frac{K_1}{K_2}
\]

(12a)

An alternative interpretation of \( \rho \) is also useful. Recalling that the maximum normal displacement of the cable at the position of static equilibrium, \( y_{\text{max}} \), is given by

\[
y_{\text{max}} = \frac{1}{8} \frac{q_y L^2}{T_0}
\]

(13)
and noting that
\[ e_o = \frac{T_o}{AE} \] (14)
represents the resulting average axial cable strain, one may express Eq. 6 in terms of \( e_o \) and the sag ratio, \( y_{\text{max}}/L \), as follows:
\[ \rho = \frac{16}{3} \frac{(y_{\text{max}}/L)^2}{e_o \frac{L}{L_e}} \] (12b)
The effective cable length, \( L_e \), may similarly be expressed in terms of the sag ratio as follows:
\[ L_e = \left[ 1 + 8 \left( \frac{y_{\text{max}}}{L} \right)^2 \right] L \] (15)
PRESENTATION AND ANALYSIS OF DATA

In Fig. 2 the squares of the frequency coefficients $\phi_n$, normalized with respect to $\pi^2$, are plotted as a function of the relative stiffness factor, $\rho$. The curves corresponding to the symmetric modes of vibration are designated by the symbol $S$, and those corresponding to antisymmetric modes are designated by the symbol $A$, the subscript $n$ being used to identify the order of the mode for the particular modal group under consideration.

A zero value of the abscissa corresponds either to a weightless, taut cable or to a cable with finite sag but negligible extensional rigidity. Considering that the smallest value of $\phi$ in this case is $\pi$, the ordinates in Fig. 2 may be interpreted as the squares of the ratios of the natural frequencies of the actual cable to the fundamental natural frequency of the associated taut cable.

The results for $\rho = 0$ are well-known. The frequency value for the $n$th symmetric mode is given by

$$\phi_n = (2n-1)\pi$$

(16)

and the associated mode may be written as

$$\psi_n(\xi) = C_n \sin[(2n-1)\pi\xi]$$

(17)

These expressions are specialized forms of Eqs. 5 and 8, respectively. The corresponding quantities for the antisymmetric modes are defined by Eqs. 3 and 4, respectively.

To the degree of approximation involved in the linearized theory employed, the vibration for systems with $\rho = 0$ induces no change in cable tension. The same also is true of cables vibrating in an antisymmetric
mode, irrespective of their value of \( \rho \). As a result, the extensional resistance of such cables is not mobilized, and the frequency curves for these modes appear as horizontal lines in the figure, crossing those for the antisymmetric modes.

The modal configuration corresponding to a frequency of a specified order may be symmetric or antisymmetric depending on the value of \( \rho \) involved. For example, the lowest frequency for values of \( \rho \) less than about 3 corresponds to a symmetric mode, whereas for the larger values it corresponds to an antisymmetric mode. Similarly, for very small and very large values of \( \rho \), successive frequency values correspond to alternatingly symmetric and antisymmetric modes, whereas for values of \( \rho \) such as 10 and 20, two symmetric modes are included between the first and second and between the second and third antisymmetric modes, respectively.

The points of intersection of the frequency curves for the symmetric and antisymmetric modes are referred to as cross-over points. The values of \( \rho \) corresponding to these points may be determined from Eq. 5 by letting \( \phi_n = 2\pi m \) and solving for \( \rho \). Denoted by \( \rho^* \), these values are defined by the equation

\[
\rho^* = \frac{n^2 \pi^2}{3}
\]

(18)

Symmetric Modes

The extensional rigidity of cables is generally mobilized for the symmetric modes of vibration, and the frequencies of these modes increase with increasing \( \rho \). Specifically, the frequency curve for the fundamental mode increases first at a rapid rate and then at a progressively reduced rate, approaching asymptotically a limiting value of slightly less than
$3\pi$ at $\rho = \infty$. The curves for the higher modes consist of three segments: a gradually increasing, almost horizontal segment on the left; an intermediate, inclined segment which corresponds to progressively higher values of $\rho$ for the higher modes; and a nearly horizontal segment on the right, for which the value of $\phi_n$ is slightly less than $2\pi$ greater than the starting value on the left.

A value of $\rho = \infty$ corresponds to an inextensional cable with infinite axial rigidity. The values of $\phi_n$ in this case are defined by the following specialized form of Eq. 5:

$$\tan \frac{\phi_n}{2} = \frac{\phi_n}{2} \tag{19}$$

and the corresponding modes are defined by Eq. 8. The latter equation may also be written in the form

$$\psi_n(\xi) = D_n \left[ \frac{\phi_n}{2} \sin \phi_n \xi + \cos \phi_n \xi - 1 \right] \tag{20}$$

where $D_n$ is a new normalizing factor. Eq. 20 is obtained from Eq. 8 by expanding the first term on the right and making use of Eq. 19.

**Natural Modes and Mobilization of Extensional Resistance.** The extent to which the extensional resistance of the cable gets mobilized depends on the value of $\rho$ and the order of the mode under consideration. In Fig. 3 the first symmetric mode of vibration is displayed for five different values of $\rho$. The dashed lines represent the position of static equilibrium of the cable, whereas the solid lines represent the vibrational configuration to an exaggerated scale. In all cases, the sag of the cable is considered to be the same so that the increase in the value of $\rho$ must be interpreted as being due to an increase in the cable's
extensional rigidity. All modes are normalized such that the peak amplitude of the normal displacement component is the same. The configurations of the normal displacement components for four of the modes considered in Fig. 3 are reproduced in the lower part of Fig. 4, in which also are shown the corresponding plots for the second and third symmetric modes.

It is clear from these plots that, with \( \rho \) increasing from zero to infinity, the \( n \)th mode of vibration changes from a configuration involving \( 2n-1 \) half-sine waves to one involving \( 2n+1 \) half-waves. The individual waves for \( \rho = \infty \) appear to be sinusoidal, but in reality they are not, as may be appreciated from the presence of the second and third terms on the right hand member of Eq. 20. The motion at \( \rho = \infty \) is inextensional, and such action is clearly not possible for a sinusoidal configuration involving an odd number of half-waves.

The sinusoidal modes of vibration corresponding to the left hand limit of \( \rho = 0 \) are clearly extensional; however, since the cable has no extensional rigidity in this case, it offers no resistance to extension. The extensional resistance of the cable influences significantly the response only for values of \( \rho \) corresponding to the inclined segments of the frequency curves presented in Fig. 2. This can clearly be seen in Fig. 5 which shows the variation with \( \rho \) of the dynamic increment in \( \Delta T_o \), the component of cable tension in the direction of the chord.

For the particular boundary conditions considered, the value of \( \Delta T_o \) corresponding to the \( n \)th mode is given by the equation

\[
(\Delta T_o)_n = \frac{AE}{L_e} \frac{q}{L} \int_0^1 \psi_n(\xi) d\xi
\]

(21)
which, on substituting Eq. 8 and integrating, becomes

$$\Delta T_0 = \frac{AE}{L_e} \frac{qL}{T_o} C_n \left[ \frac{2}{\phi_n} \sin \frac{\phi_n}{2} - \cos \frac{\phi_n}{2} \right]$$

(22a)

If use is now made of the frequency equation, (Eq. 5), and of Eqs. 6 and 13, the following simpler expression is obtained for \((\Delta T_o)_n\):

$$\frac{\Delta T_o}{T_o} = -\frac{1}{8} C_n \frac{\phi_n}{y_{\max}} \phi_n \cos \frac{\phi_n}{2}$$

(22b)

The results in Fig. 5 are normalized such that \(C_n = 0.1 \ y_{\max}\).

As would be expected from the information already presented, the curves in Fig. 5 start with zero ordinates at the left, approach small but finite ordinates at the right, and attain their larger ordinates at the values of \(\rho\) corresponding to the inclined segments of the frequency curves in Fig. 2, the absolute maximum ordinates being attained near the values of \(\rho\) corresponding to the cross-over points. The location of the latter points are identified by small vertical lines. It is of interest to note that the higher the order of the mode the greater is the value of \(\Delta T_o/T_o\).

**Extensional Energy.** Further insight into the relative importance of the extensional resistance of the cable may be gained from Fig. 6 which shows the variation with \(\rho\) of the proportion of the potential energy of the cable contributed by extension of its axis while vibrating in each of the first three symmetric modes. The potential energy in each case is denoted by \(V\), and the extensional component is denoted by \(V_e\).
The energies $V$ and $V_e$ may be evaluated conveniently as follows.

Let $\overline{q}(\xi)$ be the intensity of the equivalent static normal load necessary to produce the natural mode $\psi_n(\xi)$. This load and deflection are interrelated by the differential equation (see, for example, p. 107 of Ref. 1)

$$\frac{T_o}{L^2} \psi_n''(\xi) = q_y \frac{\Delta T_o}{T_o} - \overline{q_y}(\xi) \quad (23)$$

where a prime superscript denotes differentiation with respect to $\xi$, and $\Delta T_o$ is defined by Eq. 22. The energy $V$ is then given by the equation

$$V = \frac{1}{2} \int_0^1 \overline{q_y}(\xi) \psi_n(\xi) \, d\xi \quad (24a)$$

which, upon substituting $\overline{q_y}(\xi)$ from Eq. 23, becomes

$$V = \frac{1}{2} q_y L \frac{\Delta T_o}{T_o} \int_0^1 \psi_n(\xi) \, d\xi - \frac{1}{2} \frac{T_o}{L} \int_0^1 \psi_n''(\xi) \, d\xi \quad (24b)$$

The first term on the right hand member of this equation represents the energy component due to the extension of the cable, and the second term represents the inextensional component corresponding to the change in cable configuration. The latter component will be denoted by $V_i$.

On substituting Eq. 8 into Eq. 24b, performing the indicated operations, and utilizing Eq. 22a for $\Delta T_o$ and Eq. 6 for $\rho$, the following expressions are obtained for $V_e$ and $V_i$:

$$V_e = 6\rho \frac{T_o}{L} C_n^2 \left[ \frac{2}{\phi_n} \sin \frac{\phi_n}{2} - \cos \frac{\phi_n}{2} \right]^2 \quad (25a)$$

and

$$V_i = \frac{1}{4} \frac{T_o}{L} C_n^2 \left[ \phi_n^2 - \phi_n \sin \phi_n \right] \quad (25b)$$
The desired energy ratio is then the ratio of \( V_e \) to the sum of \( V_e + V_i \). On making use of the frequency equation (Eq. 5), this ratio simplifies to

\[
\frac{V_e}{V} = \frac{2/3}{1 + \rho \left( \frac{2}{\phi_n} \tan \frac{\phi_n}{2} \right)^2}
\]  
(26)

The latter expression has been reported previously by Irvine (1) using a different approach. It is hoped that the present alternative derivation may prove helpful.

Returning to Fig. 6, it can be seen that \( V_e/V \) is essentially negligible for values of \( \rho \) approaching zero and infinity, and that the larger ordinates are obtained for the range of values of \( \rho \) corresponding to the inclined segments of the curves in Fig. 2, the absolute maximum ordinates occurring at the cross-over points.

At the nth cross-over point, \( \phi_n = 2n\pi \) and Eq. 8 for the symmetric mode of vibration specializes to

\[
\psi^*_n = C_n \left[ \cos[2n\pi(\xi - \frac{1}{2})] - (-1)^n \right]
\]

(27)

where the star superscript emphasizes the fact that the result corresponds to the value of \( \rho \) defined by Eq. 18. The first three of these modes are shown in the left part of Fig. 7, from which the importance of the extensional action can be seen clearly. Included in the right hand part of the figure are the corresponding antisymmetric modes which are inextensional.
APPROXIMATE FREQUENCY SPECTRUM

The salient features of the frequency curves for the symmetric modes of vibration (Fig. 2) may now be summarized as follows:

1. The nearly horizontal segments on the left are defined approximately by Eq. 16, and the associated modes of vibration are defined by Eq. 17. These modes are extensional but, because of the smallness of the cable's extensional rigidity, both the change in the initial cable tension and the extensional component of the strain energy are quite small in this case.

2. The nearly horizontal segments on the right are defined approximately by Eq. 19, and the corresponding modes are defined by Eq. 20. These modes are essentially inextensional and involve small but finite changes in cable tension.

3. The intermediate inclined segments correspond to modes of vibration dominated by extensional cable action. The changes in tension in this case are significant, and the extensional component of the strain energy may be as high as 2/3 of the total potential energy, the peak values being attained at the cross over points.

A good approximation to these inclined segments of the curves may be obtained by application of the Rayleigh principle, making use of the vibration modes corresponding to the cross-over points. The latter modes are defined by Eq. 27, and the corresponding values of $\phi_n^*$ are $\phi_n^* = 2n\pi$.

The following expression for strain energy of the cable corresponding to the $n$th crossing point is obtained from Eq. 25 by letting $\phi_n = \phi_n^*$.
\( \psi^* = c_n^* \left[ \frac{T_o}{L} + n^2 \pi^2 \right] \)  

(28)

The corresponding kinetic energy, \( T^* \), is determined from the equation

\[
T^* = \frac{1}{2} \mu L \omega^2 \int_0^1 [\psi^*(\xi)]^2 d\xi
\]  

(29a)

where \( \omega \) is the circular frequency of vibration. On substituting Eq. 27 into Eq. 29a and integrating, one obtains

\[
T^* = \frac{3}{4} \mu L \omega^2 c_n^2
\]  

(29b)

The desired approximation is finally obtained by equating Eqs. 28 and 29b.

When expressed in terms of the dimensionless factor, \( \phi_n \), the resulting expression is

\[
\phi_n^2 = \left[ \frac{4}{3} n^2 + \frac{8}{\pi^2} \rho \right] \pi^2
\]  

(30)

The symbol \( n \) in this expression should be interpreted as the order of the inclined segment under consideration, a value of unity corresponding to the segment starting at the lower left-hand part of Fig. 2.

The spectrum of natural frequencies obtained by application of the three approximate expressions referred to above are shown in dashed lines in Fig. 8. Each curve consists of two horizontal segments and a linear sloping segment. In using this approximation, the three segments should be connected by smooth transition curves rather than with the discontinuous slopes indicated in the figure. Subject to this qualification, the agreement between the approximate and exact curves is excellent for most practical purposes.
With the frequency spectrum for symmetric modes approximated in the indicated manner, the complete spectrum may be obtained simply by superimposing to it the horizontal lines corresponding to the antisymmetric modes. The frequencies of the latter modes are defined by Eq. 3.

In addition to providing an extremely simple means for estimating the natural frequencies, the approximate spectrum presented in this section enables one to identify the salient features of the corresponding modes of vibration.

**Alternative Derivation of Eq. 30**

The approximate expression for the inclined segments of the frequency curves can also be derived from Eq. 5 by application of Taylor's series. Letting, for convenience, \( z = \phi_n^2 \) and solving for \( \rho \), Eq. 5 may be written as

\[
\rho = \frac{1}{12} \frac{z}{1 - \frac{z}{2} \tan \frac{z}{2}}
\]  

(31)

A Taylor series expansion of \( \rho \) around the value of \( z \) corresponding to the \( n \)th cross-over point, \( z_\circ = (\phi_n^\circ)^2 = (2n\pi)^2 \), yields

\[
\rho(z) = \rho(z_\circ) + \frac{d\rho}{dz} \bigg|_{z = z_\circ} (z - z_\circ)
\]  

(32)

where the derivative on the right hand member can be shown to equal 1/8. Substituting this value into Eq. 32, along with \( z = \phi_n^2 \), \( z_\circ = (2n\pi)^2 \) and \( \rho(z_\circ) = n^2\pi^2/3 \) (see Eq. 18), and solving for \( \rho \), one obtains precisely Eq. 30.
INFINITE SERIES FOR INTEGRALS OF NATURAL MODES

Irvine (4,1) has presented valuable closed-form expressions for several infinite series containing integrals of the natural modes of vibration of the parabolic cables considered. Three additional relations, involving only the symmetric modes and the associated frequencies, are presented in the following. They are

\[ \sum_{n=1}^{\infty} \frac{\left[ \int \psi_n^2 d\xi \right]^2}{\int \psi_n^2 d\xi} = 1 \quad (33) \]

\[ \sum_{n=1}^{\infty} \frac{1}{\phi_n^2} \frac{\left[ \int \psi_n d\xi \right]^2}{\int \psi_n^2 d\xi} = \frac{1}{12} \frac{1}{1+p} \quad (34) \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{\phi_n^4} \frac{\left[ \int \psi_n d\xi \right]^2}{\int \psi_n^2 d\xi} = \frac{1}{120} \frac{1}{(1+p)^2} \quad (35) \]

in which all integrals in these and the following expressions are to be evaluated between \( \xi = 0 \) and \( \xi = 1 \).

Equation 33 is obtained by expanding the unit step function in the range between \( \xi = 0 \) and \( \xi = 1 \) in terms of the natural modes of vibration of the cable, as follows:

\[ 1 = \sum_{n=1}^{\infty} a_n \psi_n(\xi) \quad (36) \]

where the participation factors, \( a_n \), are determined in the usual manner, by first multiplying the two members of Eq. 36 by \( \psi_r(\xi) \), then integrating
them over their range of definition, and making use of the orthogonality of the natural modes. This leads to

$$a_n = \frac{\int \psi_n d\xi}{\int \psi_n^2 d\xi}$$  \hspace{1cm} (37)

Eq. 33 is finally obtained by substituting Eq. 37 into Eq. 36 and integrating its two members between $\xi = 0$ and $\xi = 1$.

Equations 34 and 35 are obtained in a similar manner by expanding the normal displacement function, $w(\xi)$, produced by a lateral normal pressure of unit intensity per unit of chord length in terms of the natural modes of vibration. The exact, closed-form expression for this function is

$$w(\xi) = \frac{1}{2(1+\rho)} \left[ \frac{L}{T_o} \right]^2 [\xi - \xi^2]$$  \hspace{1cm} (38)

and the corresponding expression is

$$w(\xi) = \sum_{n=1}^{\infty} c_n \psi_n(\xi)$$  \hspace{1cm} (39)

Inasmuch as the intensity of the normal load corresponding to the nth mode is $\mu \omega_n^2 \psi(\xi)$, and the coefficient of the nth term in a modal expression of the uniform unit pressure is given by Eq. 37, the participation factor, $c_n$, in Eq. 39 is defined by

$$c_n = \frac{1}{\mu \omega_n^2} \frac{\int \psi_n d\xi}{\int \psi_n^2 d\xi}$$  \hspace{1cm} (40)

Substituting Eq. 40 into Eq. 39 and equating the resulting expression to the corresponding expression on the right hand member of Eq. 38, one
obtains

$$\sum_{n=1}^{\infty} \frac{1}{\mu \omega_n^2} \int_{\psi_n^2} d\xi \psi_n(\xi) = \frac{1}{2(1+\rho)} \frac{L^2}{T_o} [\xi - \xi^2]$$

(41)

Equation 34 is finally obtained by making use of Eq. 2, integrating the two members of Eq. 41 between $\xi = 0$ and $\xi = 1$, and utilizing the orthogonality relationship of the natural modes. Equation 35 is obtained similarly by first squaring the two members of 41 and then following the steps just indicated.
CONCLUSION

With the information presented herein, the spectrum of natural frequencies of simply supported parabolic cables and the salient features of the associated modes of vibration can be determined readily. The closed-form expressions for the various infinite series considered are expected to prove of value in studies of forced vibration, and their derivations illustrate the approach by which additional expressions may be developed.
REFERENCES


APPENDIX A

NOTATION

\( a_n \) : Participation factor defined by Eq. 37;

\( A \) : Cable cross-sectional area;
    Symbol referring to antisymmetric modes;

\( B_n \) : Normalizing constant;

\( c_n \) : Participation factor defined by Eq. 40;

\( C_n \) : Normalizing constant;

\( D_n \) : Normalizing constant;

\( e_o \) : Average axial strain defined by Eq. 14;

\( E \) : Cable Young's modulus of elasticity;

\( K_i \) : Extensional cable stiffness defined by Eq. 10;

\( K_2 \) : Inextensional cable stiffness defined by Eq. 11;

\( L \) : Length of cable chord;

\( L_e \) : Effective cable length defined by Eq. 7;

\( q_y \) : Intensity of load normal to, and uniformly distributed
    along, cable chord;

\( S \) : Symbol referring to symmetric modes;

\( T \) : Kinetic energy defined by Eq. 29a;

\( T_o \) : Axial component of cable tension;

\( V \) : Total strain energy defined by Eqs. 24;

\( V_e \) : Extensional component of strain energy defined by Eq. 25a;

\( V_i \) : Inextensional component of strain energy defined by Eq. 25b;

\( w \) : Normal displacement function defined by Eq. 38;

\( x, y \) : Axes of inclined cartesian coordinate system;

\( \Delta_o \) : Displacement defined by Eq. 9;

\( \Delta T_o \) : Increase in \( T_o \);

\( \theta \) : Inclination of cable chord;

\( \mu \) : Cable mass per unit of chord length;
\[ \xi = \frac{x}{L} : \text{Dimensionless distance}; \]
\[ \rho : \text{Relative stiffness factor defined by Eq. 6}; \]
\[ \phi : \text{Dimensionless frequency}; \]
\[ \psi(\xi) : \text{Amplitude of normal displacement}; \text{ and } \]
\[ \omega : \text{Circular frequency}. \]
APPENDIX B
DERIVATION OF FREQUENCY AND MODAL EQUATIONS

Frequency Equations

The frequency equations for both the symmetric and antisymmetric modes of vibration can be derived from the undamped harmonic stiffness $K$ below (see Chapter Four) by determining the value of $\phi$ for which $K = \infty$.

$$K = \frac{AE}{L} \cos^2 \theta \left[ 1 + \frac{1}{12} \frac{qL}{T_o} \tan \theta \left( \frac{2}{\phi} \tan \frac{\phi}{2} - 1 \right) \right]^2 \frac{1 + 12 \rho \frac{1}{\phi^2} \left( \frac{2}{\phi} \tan \frac{\phi}{2} - 1 \right)}{1 + 12 \rho \frac{1}{\phi^2} \left( \frac{2}{\phi} \tan \frac{\phi}{2} - 1 \right)} + \frac{T_o}{L} \sin^2 \theta \phi \cot \phi \tag{B1}$$

Equation B1 becomes infinite when the denominator of the first term is zero, or when the second term is infinite. The first condition leads to Eq. 5 and the second leads to Eq. 3. While the second term of Eq. B1 is minus infinity at values of $\phi_n = (2n-1)\pi$, the corresponding value of $K$ is finite because the first term also tends to infinity at these frequencies.

Natural Modes

The normal displacement for a damped cable due to the horizontal harmonic displacement $Xe^{i\omega t}$ considered in Chapter Four specializes to the following equation for no damping:

$$\ddot{w}(\xi) = \frac{qL^2}{T_o} \frac{1}{T_o} \frac{1}{\phi^2} \left[ \cos \frac{\phi}{2} - \cos \left( \phi \left( \xi - \frac{1}{2} \right) \right) \right] + X \sin \theta \frac{1}{\sin \phi} \sin \phi \xi \tag{B2}$$

Substituting the boundary condition $X = 0$ into Eq. B2 leads to Eq. 8 for the symmetric modes. For the antisymmetric modes, $\Delta T_o = 0$ as seen from
Eq. 21 and Eq. B2 leads to Eq. 4.
Fig. 1. System
Fig. 2. Frequency Spectrum
Fig. 3. Fundamental Symmetric Mode
Fig. 4. Natural Symmetric Modes

\[ \rho = 50 \]

\[ \rho = 20 \]

Third Symmetric Mode
Second Symmetric Mode
Fundamental Symmetric Mode

\[ \rho = 10 \]

\[ \rho = 1 \]
Fig. 5. Tension Increment for Symmetric Modes
Fig. 6. Extensional Component of Strain Energy for Symmetric Modes
Fig. 7. Natural Modes at Cross-Over Points

antisymmetric

symmetric

n=3

n=2

n=1
Fig. 8. Frequency Spectrum for Symmetric Natural Modes
CHAPTER

SIX

SELECTED STUDIES ON THE LOW-AMPLITUDE BEHAVIOR OF CABLES

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Knowledge of the low amplitude stiffness of a guying cable is of fundamental importance in the evaluation of its constraining effect. This quantity has been seen in Chapter Four to be frequency dependent when the excitation is harmonic. In particular, it is zero when the frequency of excitation corresponds to one of the natural frequencies of the cable free to move in the horizontal direction at the top. These natural frequencies and associated modes are therefore of particular interest by themselves. It has to be noted that in spite of this importance no analytical solutions have apparently been reported in the literature for these quantities.

The objectives of this Chapter are first to obtain analytically the natural frequencies and modes of a parabolic cable fixed at the lower end and free to move horizontally at the upper one, and second to present a finite element method of analysis from which the above informations may be derived and used in establishing the cable dynamic stiffness. For sake of completeness, the cable dynamic stiffness obtained from the modes of vibration of the supported-supported cable will also be presented in a last section.
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NATURAL FREQUENCIES AND MODES

Introduction

The expressions for the natural frequencies and modes of vibration of a parabolic cable fixed at the lower support and free to move horizontally at the upper one are derived in this section. They are given in both a complete and simplified form. These solutions are discussed in detail and compared to the ones valid for a taut cable. The orthogonality relationships of the modes are also presented.

System Considered

The system considered is identical to the one used in Chapter Four where a detailed discussion of its characteristics can be found. It is an inclined uniform cable contained in a vertical plane and subjected to a uniform load normal to the chord of intensity \( q_y \) per unit length of cable chord. It is supported at both ends, its chord length is \( L \) and its inclination \( \theta \), and the component of the tensile force in the direction of the chord is \( T_0 \). Its deflected configuration at the position of static equilibrium is defined by

\[
y(x) = \frac{q_y L^2}{10} \left[ \left( \frac{x}{L} \right)^2 - \left( \frac{x}{L} \right)^2 \right]
\]

when expressed in the inclined coordinate system \( x-y \) originating from the lower support and where the \( x \)-axis is in the direction of the chord and the \( y \)-axis is normal to the chord.

When the cable is subjected to its own weight \( q \) per unit length of unstretched cable, the expression above can still be applied by taking

\[
q_y = q \frac{S}{L} \cos \theta
\]
where $S$ is the unstretched length of the cable. The parabolic solution above then neglects the axial component of the weight $q_x L$, what may be considered acceptable when this component is small in comparison to the corresponding component of tension $T_0$.

The other characteristics of the cable are its cross-sectional area $A$, the Young's modulus $E$, and the mass coefficient $μ$ assumed to be uniformly distributed per unit length of cable chord.

**Fundamental Relationships**

**Governing Differential Equation.** The governing differential equation has been presented in Chapter Four. It was obtained under the assumption that the axial components of inertia and damping forces are negligible. Further disregarding the normal component of damping force, which is of no interest in this study, this equation is

$$\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{A}{E} \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{q_x}{F} \frac{\partial T_0(x,t)}{\partial z}$$

(3)

with

$$\Delta T_0(x) = \frac{AF}{E} \left\{ \left[ u(x,t) - u(0,t) \right] - \frac{1}{2} \frac{q_x L}{E} \left[ w(L,t) + w(0,t) \right] + \frac{q_x}{2} \int_0^L w(x,t) \, dx \right\}$$

(4)

and

$$L_e = L \left[ 1 + \frac{1}{2} \left( \frac{q_x L}{F} \right)^2 \right]$$

(5)

$w(x,t)$ represents the component of cable displacement normal to the chord and measured from the position of static equilibrium, $w(0,t)$ and $w(L,t)$ are the end displacements normal to the chord, and $u(0,t)$ and $u(L,t)$ are the end displacements in the direction of the chord. $ΔT_0$ re-
resents the change in the component of cable tension in the direction of
the chord due to cable motion.

The equations above are essentially identical to those reported by
Davenport et al. (1,2) and Irvine et al. (3 to 7) and are proposed for
small sag to length ratios only.

Equation of Motion. Let the cable be subjected to a harmonic hori-
izontal deflection of frequency $\omega$ and amplitude $X$ at the upper support,
i.e. $X(t) = X e^{i\omega t}$. The associated cable lateral deflection is then
$w(x,t) = w(\xi) e^{i\omega t}$, where $w(\xi)$ is obtained by solving Eqs. (3) and (4)
and is given as

$$w(\xi) = -\frac{9 \pi c \xi}{T_0} \frac{T_0}{T_e} \frac{1}{\phi^2} \left[ \frac{\cos \left( \phi (\xi - \frac{1}{2}) \right)}{\cos \left( \frac{\phi}{2} \right)} - 1 \right] + X \sin \phi \frac{\sin \phi (\xi)}{\sin \phi}$$  \hspace{1cm} (6)

with

$$\omega_T = \frac{\mu c L}{T_0} \left\{ X \cos \phi - \frac{1}{2} \frac{9 \pi c}{T_0} X \sin \phi + \frac{9 \pi c}{T_0} \right\} w(\xi) d\xi$$  \hspace{1cm} (7)

The dimensionless distance $\xi = \frac{X}{L}$ as well as the dimensionless frequency
$\phi = \frac{\omega}{\omega_0}$ have been introduced in Eqs. (6) and (7). $\omega_0$ is defined as

$$\omega_0 = \sqrt{\frac{T_0/L}{\mu c L}}$$  \hspace{1cm} (8)

which when multiplied by $\pi$ represents the fundamental natural frequency
of a supported-supported taut cable in lateral vibration.

Dynamic Stiffness. The change in axial component of tension $\Delta T_0$ can
be related to the top displacement $X$ by combination of Eqs. (6) and (7). This gives
\[ \Delta T_o = X \cos \theta \frac{AE}{Ec} \frac{1 + \frac{1}{2} \sigma \sqrt{\nu}}{1 + \frac{1}{2} \rho \frac{\nu}{\phi^2}} \]  \hspace{1cm} (9)

where

\[ \rho = \frac{1}{12} \frac{AE}{E_o} \frac{E}{Ec} \left( \frac{\alpha L}{E_o} \right)^2 \]  \hspace{1cm} (10)

\[ \sigma = \frac{q \alpha}{E_o} \cot \phi \]  \hspace{1cm} (11)

\[ \nu = 2 \frac{\alpha \sin \phi}{\phi} - 1 \]  \hspace{1cm} (12)

The parameters \( \rho \) and \( \sigma \), both stiffness ratios, have been extensively discussed in Chapter Four which should be referred for explanations on their meaning.

It has been shown in Chapter Four that the change in the horizontal component of the tensile force at the upper support is expressed by

\[ \Delta H(t) = \Delta H e^{i\omega t}, \]  \hspace{1cm} (13a)

The undamped dynamic stiffness, \( K(\phi) = \frac{\Delta H}{\Delta} \), is then obtained by combining Eqs. (6) and (9). Thus

\[ K(\phi) = \frac{AE}{Ec} \cos^2 \theta \left\{ \frac{(1 + \frac{1}{2} \sigma \sqrt{\nu})^2}{1 + \frac{1}{2} \rho \frac{\nu}{\phi^2}} + \frac{1}{12} \frac{\sigma^2}{\rho} \phi \cot \phi \right\} \]  \hspace{1cm} (14a)

It was also shown that \( \Delta H \) can be approximated by

\[ \Delta H \approx \Delta T_o \cos \theta \]  \hspace{1cm} (13b)

for not too steep cables. The associated undamped stiffness is then
\[ K'(\phi) = \frac{\mu e}{2e} \cos^2 \theta \frac{1 + \frac{1}{2} \sigma \sqrt{\epsilon}}{1 + \frac{1}{12} \rho \sqrt{\epsilon}} \]  \hspace{1cm} (14b)

where the prime indicates the approximate nature of the stiffness.

**Frequencies and Modes**

**Frequency Equation.** The characteristic frequency equation of the parabolic cable fixed at the bottom and free to move horizontally at the upper support is obtained by equating to zero the undamped dynamic stiffness of Eq. (14a). This gives

\[ \frac{\left[1 + \frac{1}{2} \sigma \sqrt{\epsilon}\right]^2}{1 + \frac{\sigma^2}{12 \rho}} \phi + \phi \cot \phi = 0 \]  \hspace{1cm} (15a)

When obtaining the corresponding taut cable and simplified frequency equations in forthcoming sections, it will be seen that those depend on a single parameter, either \( \epsilon = \frac{\sigma^2}{12 \rho} \) or \( \sigma \). It is believed that these two parameters represent most adequately the physical features of the system in this particular configuration. Eq. (15a) is consequently rewritten in terms of \( \epsilon \) and \( \sigma \) as

\[ \frac{\left[1 + \frac{1}{2} \sigma \sqrt{\epsilon}\right]^2}{\epsilon + \sigma^2 \sqrt{\epsilon}} \phi + \phi \cot \phi = 0 \]  \hspace{1cm} (15b)

Note that whereas the frequency equation of a cable supported at both ends, as obtained in Chapter Five, depends on the single parameter \( \rho \), the present solution depends on the two parameters \( \rho \) and \( \sigma \) or alternatively \( \sigma \) and \( \epsilon \).

**Natural Modes of Vibration.** The corresponding modes of vibration are obtained by substituting the dimensionless natural frequencies \( \phi \)
into Eq. (6) for $w(\zeta)$. The modes represent the lateral displacements of
the cable measured from the position of static equilibrium. In the form
of Eq. (6) they depend on both the change in component of tension $\Delta T_0$
and on the top horizontal displacement $X$. These two quantities are related
by Eq. (9) and the modes $\psi(\zeta)$ can be written in terms of $X$ only as

$$
\psi(\zeta) = X \sin \Theta \left\{ \frac{\sin \phi}{\sin \phi} + \frac{1}{\phi^2} \frac{1 + 2 \sigma / \phi}{1 + 2 \sigma / \phi} \left[ \frac{\cos \phi (\zeta - \frac{1}{2})}{\cos \frac{\phi}{2}} \right] - 1 \right\} \quad (16a)
$$

Making use of the frequency equation (15), this expression can also be
written as

$$
\psi(\zeta) = X \sin \Theta \left\{ \frac{\sin \phi}{\sin \phi} + \frac{\sigma}{1 + 2 \sigma / \phi} \frac{\cos \phi}{\phi} \left[ \frac{\cos \phi (\zeta - \frac{1}{2})}{\cos \frac{\phi}{2}} \right] - 1 \right\} \quad (16b)
$$

Note that $X \sin \Theta$ is the amplitude of the normal component of displace-
ment at the top and that the remaining part of Eq. (16b) for $\psi(\zeta)$ de-
PENDS ON THE PARAMETER $\sigma$ ONLY, ONCE THE FREQUENCY $\phi$ HAS BEEN SPECIFIED.

**Discussion**

**Frequency Curves.** The frequency equation (15b) is reported in Fig.1
as a function of $\sigma$ for values of $\varepsilon$ equal to 0.01, 0.1, and 1. A value
of $\sigma$ equal to zero can correspond to two different conditions. It can
first correspond to a horizontal cable. The value of parameter $\varepsilon$ associ-
ated with such a condition is then also zero, and the $n^{th}$ natural fre-
quency is

$$
\phi_n = n \pi \quad (17)
$$

It should be noted that these frequencies are independent of both the
sag and the extensional resistance. This latter independence is due to
the presence of an axial component of displacement associated with the vibration. This axial component is such that no strain and consequently no mobilization of extensional resistance is associated with the modal vibration. The axial component is also related to the sag, being more pronounced for cables with large sag than for cables with low sag. Even though the value of the sag relates the normal to the axial component of motion, it does not affect the natural frequencies.

A value of \( \sigma \) equal to zero can also correspond to a taut cable, in which case the value of \( \varepsilon \) is arbitrary. The corresponding frequency equation is given by

\[
1 + \varepsilon \phi \cos \phi = \sigma
\]

(18)

The solutions of this characteristic equation specify the frequencies at which the curves originate at the left of Fig. 1. The taut cable condition will be further investigated in a forthcoming section.

An increase in the value of \( \sigma \) can be interpreted as an increase in sag to length ratio \( \frac{l_0 q L}{\varepsilon L_0} \) for a specific chord inclination \( \theta \), as an increase in cable chord inclination for a specific sag to length ratio, or as a combination of these two cases. It must however be remembered in such an interpretation that the present theory is proposed for small sag to length ratios, of the order of 1/8 or less. A value of \( \sigma \) tending to infinity must then be interpreted as corresponding to a vertically oriented cable with the associated value of \( \varepsilon \) also tending to infinity, unless the expression \( \frac{T_0/L}{AE/L_e} \) is zero which is obviously an unrealistic case. Eq. (15a) is then most conveniently used to obtain the corresponding frequency equation as
\[ \frac{3 \rho y^2}{1 + \frac{12 \rho y}{\phi^2}} + \phi \csc \phi = 0 \]  

(19)

The natural frequencies of the vertically oriented cable therefore depend only on the parameter \( \rho \). When the vertical cable is taut, \( \rho \) is equal to zero and the \( n \)th natural frequency is given by

\[ \phi_n = (2n - 1) \frac{\pi}{2} \]  

(20)

When the cable is totally inextensible, \( \rho \) tends to infinity and the frequency equation becomes

\[ 1 + \frac{1}{4} y \csc \phi = 0 \]  

(21)

The behavior of the frequency curves for intermediate values of \( \sigma \) are seen in Fig.1 to be rather complex. One trend however emerges for the value of \( \varepsilon = 0.1 \). It is observed that for this value of parameter the three frequency curves reported in the figure are schematically analogous to one ascending branch and two descending branches. The increasing branch is made up of the starting part of the fundamental frequency curve, the intermediate part of the second frequency curve and the ending part of the third frequency curve. The first descending branch is made up of the starting part of the second frequency curve and the ending part of the fundamental frequency curve, while the second descending branch is made up of the starting part of the third frequency curve and the ending part of the second frequency curve. Even though increasing and decreasing segments are recognized in the curves corresponding to other values of \( \varepsilon \), no similar branches appear as clearly. Further discussions on the natural frequencies are deferred to further sections where they are presented in
terms of the inclination \( \theta \) and compared to the corresponding taut cable solutions.

**Modes of Vibration.** The modes of vibration \( \psi(\xi) \) for a value of \( \varepsilon = 0.1 \) are reported in Fig. 2 for values of \( \sigma \) equal to 0.5, 2, and 5. The modes are normalized so that the absolute value of the maximum amplitude is equal to 1. One notes that the branches previously indicated are clearly recognizable in the figure. The modes associated with the ascending branch are almost symmetrical with relatively few top displacement, while the modes associated with the descending branches have a relatively large top displacement.

The interpretation of the frequency curves as well as of the corresponding modes is complicated by the fact that the physical systems associated with a specific frequency curve \( (\varepsilon = c^{st}) \) are entirely different from one another. The interpretation can more easily be made with reference to the taut cable solution. This requires presentation of the frequency curves for the saging cable as a function of \( \theta \), and of the taut cable solution.

**Natural Frequencies and Modes as a function of \( \theta \).** The first three natural frequencies of the cable are reported in Fig. 3 as a function of the chord inclination \( \theta \) for a fixed value of \( \frac{I_{0}/L_e}{L_e} \) equal to 0.002 and values of sag to length ratio \( \frac{y_{\text{max}}}{L} \) equal to 1/32, 1/24 and 1/12. The values of parameters above correspond to realistic systems. Similar trends to those observed earlier are noticeable in this figure. Increasing segments up to a value of \( \theta \) equal to approximately 70 degrees appear, with decreasing segments beyond this value. Small increasing seg-
ments near $\theta = 90$ degrees are also observed in the higher frequencies. When comparing Figs. 1 & 3, one recognizes that the sharp changes in frequency curves near $\theta = 70$ to 80 degrees correspond to those near $\sigma = 1$ to 2.

The modes of vibration $\psi(\xi)$ corresponding to values of $\frac{T_0}{L \frac{AE}{Le}} = 0.002$ and $\frac{\gamma_{\max}}{L} = \frac{1}{12}$ are reported in Fig. 4 for angles of inclination $\theta$ equal to 60, 75 and 90 degrees.

**Taut Cable.** The taut cable frequency equation (18) is reported in Fig. 5 as a function of $\epsilon$. One sees that an increase in the value of this parameter corresponds to a decrease in the natural frequencies. This is explained as follows. Assume that a change in $\epsilon$ corresponds to a change in cable inclination $\theta$ only. $\epsilon = 0$ then corresponds to a horizontal taut cable whose natural frequencies are multiple of $\pi$. Increasing the cable inclination corresponds to a relaxation of the top boundary due to the possibility for the cable to move freely in the horizontal direction. This relaxation is evidently maximum at $\theta = 90$ degrees and the corresponding natural frequencies are at values $\pi/2$ below the ones of the horizontal cable. The importance of the relaxation is affected by the term $\frac{T_0}{AE}$ entering in the parameter $\epsilon$ for intermediate values of angle $\theta$. A horizontal motion at the upper support induces a change in cable length equal to the axial component of the top horizontal displacement. It is maximum for a horizontally oriented cable and decreases with increasing cable inclination, and is zero for a vertically oriented cable. These changes in cable length are associated with a mobilization of the extensional resistance $AE$ and when this resistance is very large a change in
cable length would be associated with a large strain energy. Under these conditions the cable will therefore vibrate in a configuration associated with no changes in cable length, i.e. identical to the one of the horizontal cable. When on the contrary the extensional resistance is negligible, no strain energy is associated with cable extension. The corresponding modal configuration is then identical to the one of a vertically oriented cable.

One has to note that the proposed theory for a sagging cable, from which the present taut cable solution is derived, neglects the inertia effects in the axial direction. It is of interest to briefly investigate the frequency equation of the taut cable when these effects are considered. The characteristic equation is

\[ \sqrt{\frac{AE}{T_0}} \phi \cot \gamma \left( \frac{AE}{T_0} \phi \right) + E \phi \cot \gamma \phi = 0 \] (22)

Solutions to this equation are reported in Fig.5 for different values of strain \( \frac{T_0}{AE} \) characterizing the axial inertia. This inertia shows its effect between the two limiting cases of a horizontal respectively vertical cable, however in a very limited manner. The unrealistically high values of initial strain \( \frac{T_0}{AE} \) equal to 1/100 and 1/50 in the figure had in fact to be chosen such that this effect can be visualized. It must be noted that the frequencies depicted in the figure correspond to predominantly normal modes of vibration since \( \sqrt{\frac{AE}{T_0}} \phi \gg \phi \). The predominantly axial modes occur at much higher frequencies and only then is the axial inertia of importance.

**Sagging versus Taut Cable.** The frequency curves for the sagging ca-
ble can now be compared to those of the taut cable. One observes in Fig.3 that the natural frequencies of the sagging cable first increase with increasing chord inclination while in Fig.5 those of the taut cable decrease. That the frequencies of the sagging cable are larger than those of the taut cable is due to the sag affecting the cable extension. Remembering that a change in cable length is principally composed of the axial component of the horizontal end displacement and of the integral of the initial cable curvature multiplied by the modal deflection, one sees that this latter component is present only in the case of a sagging cable. The extension mobilizing the extensional resistance is evidently one of the reasons accounting for the natural frequencies of a sagging cable being higher than those of a taut cable. A second reason is as follows. Even though the sagging cable cannot avoid the mobilization of the extensional resistance, it is trying to minimize this mobilization by vibrating in a configuration associated with relatively few extension. Such a configuration is however associated with an increase in deformational energy, a further reason which accounts for the frequencies of the sagging cable being higher than those of the taut cable.

The sharp changes in the frequency curves of Fig.3 are associated with the decreasing importance of any changes in component of cable tension, $\Delta T_o$, as $\theta$ approaches 90 degrees. Changes in end slope are of importance in such cases. This is best visualized in the case of the taut cable. When the cable is horizontal, the free boundary condition requiring no changes in horizontal component of end tension is equivalent to requiring no changes in tension, i.e. $\Delta T_o = 0$. When the cable is verti-
cal, the free boundary condition is satisfied even when \( \Delta T_0 \neq 0 \) as long as the end slope remains vertical. The first effect corresponds to the first term of Eq. (13a) while the latter one corresponds to the two last terms in the same equation.

**Simplified Solution**

**Frequency Equation.** Equating to zero the simplified undamped dynamic stiffness of Eq. (14b) gives the following simplified frequency equation:

\[
1 + \frac{1}{2} \sigma f' = 0
\]

(23)

This equation is an approximation to the frequency equation (15).

In addition to the frequencies obtained from Eq. (23), frequencies associated with antisymmetric modes of vibration also satisfy the simplified boundary condition at the upper support. They are

\[
\phi_n = 2n \pi
\]

(24)

where \( n \) indicates the order of the antisymmetric frequency.

It is interesting to note that the simplified natural frequencies depend on the parameter \( \sigma \) only, which is a function of both the chord inclination and the sag to length ratio. Simplified frequency curves are compared with corresponding complete ones in Figs. 6 & 7. It is especially seen from Fig.7 that the agreement between both solutions is satisfactory for not too large values of \( \theta \), i.e. not too steep cables. This was expected since the simplified theory is indeed intended for not too steep cables. Since the extensional resistance \( AE \) does not enter in the parameter governing the simplified solution one has to conclude that
the increase in values of natural frequencies when compared to those of the taut cable is due to an increase in the deformational energy only. Since the simplified and complete solutions are in good agreement for low values of chord inclination, one may also conclude that an increase in the values of natural frequencies of the complete solution results from predominantly the same source, for not too steep cables.

**Modes of Vibration.** The simplified modes of vibration are simply given by Eq. (6) for \( w(\xi) \) in which it is recognized that equating the simplified dynamic stiffness to zero is equivalent to equating the increment in axial component of cable tension \( \Delta T_0 \) to zero as seen from Eq. (13b). The simplified modes are then

\[
\psi(\xi) = \psi_0 \sin \phi \frac{\sin \phi\xi}{\sin \phi}\]  

(25)

To these modes corresponding to the frequency equation (23), one must add those associated with the frequencies of Eq. (24). They are

\[
\psi_n(\xi) = C_n \sin (2\pi \xi)\]  

(26)

Where \( C_n \) is a coefficient of amplitude for the antisymmetric mode of order \( n \).

**Taut Cable.** It is of interest to note that the natural frequencies of a taut cable obtained from the simplified theory are independent of the cable inclination and are given as

\[
\phi_n = n \pi\]  

(27)

where \( n \) is the order of the frequency. The corresponding modes are

\[
\psi_n(\xi) = C_n \sin (n\pi \xi)\]  

(28)
When comparing these constant frequencies to those of Fig. 5 obtained from the complete theory, one observes large differences, unless the parameter \( \varepsilon \) is very small. Indeed, in most applications \( \varepsilon \) is small enough to allow for the use of the simplified solution. For example, a taut cable with an initial strain \( \frac{T_0}{AE} \) equal to 0.002 and with a chord inclination \( \theta \) equal to 60 degrees gives \( \varepsilon = 0.006 \). The corresponding complete fundamental natural frequency is \( 0.994 \pi \), for which the simplified solution of \( \pi \) is obviously a good approximation.

Orthogonality Relationships

**Complete Modes.** It must be noted that the modes of vibration of Eq. (16b) obtained from the complete solution satisfy the following orthogonality relationship:

\[
\int_0^1 \psi_i(\xi) \psi_j(\xi) \, d\xi = 0 \quad \text{for} \quad i \neq j \tag{29}
\]

The proof is as follows.

Consider two different modes of order 'i' and 'j' satisfying the governing differential equation (3)

\[
\psi_i''(\xi) + \phi_i^2 \psi_i(\xi) = \frac{q_0 \xi^2}{T_0} \left( \frac{\omega T_0}{T} \right)_i \quad \text{and} \tag{30a}
\]

\[
\psi_j''(\xi) + \phi_j^2 \psi_j(\xi) = \frac{q_0 \xi^2}{T_0} \left( \frac{\omega T_0}{T} \right)_j \tag{30b}
\]

where a prime denotes the differentiation with respect to \( \xi \). If we multiply the first of this expression by \( \psi_j(\xi) \) and the second by \( \psi_j(\xi) \) and further integrate them from 0 to 1, we obtain
\[ \int_{\xi}^{\xi'} \psi_{\xi}''(\xi) \psi_{\xi}(\xi) \, d\xi + \phi_{\xi}^{2} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi = \frac{\rho \gamma_{E}^{2}}{\tau_{0}} \frac{(\alpha \tau_{0})_{j}}{\tau_{0}} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \, d\xi \]  

(31a)

and

\[ \int_{\xi}^{\xi'} \psi_{\xi}''(\xi) \psi_{\xi}(\xi) \, d\xi + \phi_{\xi}^{2} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi = \frac{\rho \gamma_{E}^{2}}{\tau_{0}} \frac{(\alpha \tau_{0})_{j}}{\tau_{0}} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \, d\xi \]  

(31b)

Note that the first term of Eq. (31a) can be rewritten as

\[ \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi = \psi_{\xi}(\xi_{1}) \psi_{\xi}(\xi_{1}) - \psi_{\xi}(\xi_{1}) \psi_{\xi}(\xi_{1}) + \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi \]  

(32)

where the boundary condition at the lower support, \( \psi(0) = 0 \), has been introduced. The boundary condition at the upper support is \( \Delta H = 0 \), or from Eq. (13a)

\[ \psi(1) = -\frac{\xi}{\tau_{0}} \frac{1}{\sin \theta} \left[ \alpha \tau_{0} \cos \theta - \frac{1}{\tau_{0}} \frac{\rho \gamma_{E}^{2}}{\tau_{0}} \alpha \tau_{0} \sin \theta \right] \]  

(33)

\( \psi(1) \) is further equal to \( X \sin \theta \) from Eq. (16b). Substituting Eq. (32) into Eq. (31a) and making use of the expressions for \( \psi'(1) \) and \( \psi(1) \), we obtain

\[ \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi + \phi_{\xi}^{2} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi = \frac{\rho \gamma_{E}^{2}}{\tau_{0}} \frac{(\alpha \tau_{0})_{j}}{\tau_{0}} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \, d\xi + \]

\[ + \psi_{\xi}(\xi_{1}) \psi_{\xi}(\xi_{1}) - \psi_{\xi}(\xi_{1}) \psi_{\xi}(\xi_{1}) + \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi \]

\[ + X \frac{\gamma_{E}}{\tau_{0}} \left[ (\alpha \tau_{0})_{j} \cos \theta - \frac{1}{\tau_{0}} \frac{\rho \gamma_{E}^{2}}{\tau_{0}} (\alpha \tau_{0})_{j} \sin \theta \right] - X \frac{\gamma_{E}}{\tau_{0}} \left[ (\alpha \tau_{0})_{j} \cos \theta - \frac{1}{\tau_{0}} \frac{\rho \gamma_{E}^{2}}{\tau_{0}} (\alpha \tau_{0})_{j} \sin \theta \right] \]  

(34)

Obtaining \( \psi(\xi) \) from Eq. (7) and substituting into Eq. (34) gives

\[ \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi + \phi_{\xi}^{2} \int_{\xi}^{\xi'} \psi_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi = \]

\[ = \frac{\xi}{\tau_{0}} \frac{\rho \gamma_{E}}{\tau_{0}} (\alpha \tau_{0})_{j} (\alpha \tau_{0})_{j} + X \frac{\gamma_{E}}{\tau_{0}} \left[ \frac{1}{\tau_{0}} \frac{\rho \gamma_{E}^{2}}{\tau_{0}} \sin \theta - \cos \theta \right] \]  

(35a)

Similarly substituting the expression for \( \psi_{\xi}(\xi) \) obtained from Eq. (7) into Eq. (31b) results in
\[ \int_{0}^{1} \varphi'_{\xi}(\xi) \psi'_{\xi}(\xi) \, d\xi \quad + \quad \phi_{\xi}^{\prime} \int_{0}^{1} \varphi'_{\xi}(\xi) \psi_{\xi}(\xi) \, d\xi = \\
= \frac{\xi}{\tau_0} \frac{\sqrt{\omega}}{\alpha^2} \left( \frac{\omega \tau_0}{\omega \tau_0} \right)_0^{\xi} + X \frac{\omega \tau_0}{\omega \tau_0} \left( \frac{\omega \tau_0}{\omega \tau_0} \right)_0^{\xi} \leq \left[ \frac{1}{2} \frac{\rho_{\xi} L}{\tau_0} \sin \theta \quad - \cos \theta \right] \quad (35b) \]

The orthogonality relationship of Eq. (29) is then readily found by comparing Eqs. (35a) and (35b). The following relationship is then also obtained from Eq. (29):

\[ \int_{0}^{1} \left[ \varphi'_{\xi}(\xi) - \frac{\rho_{\xi} \omega^2}{\tau_0} \left( \frac{\omega \tau_0}{\omega \tau_0} \right)_0^{\xi} \right] \psi_{\xi}(\xi) \, d\xi = 0 \quad \text{for} \quad \xi \neq \xi \quad (36) \]

**Simplified Modes.** The simplified theory is based on an approximate treatment of the boundary at the upper support. One consequence of this approximation is the absence of orthogonality relationship for the simplified modes of vibration.
FINITE ELEMENT METHOD

Introduction

As an alternative to the analytical procedure presented earlier, a finite element method can be used to obtain the low amplitude modes and frequencies of a cable. These can then be used in the evaluation of the cable dynamic stiffness. Such a procedure may be of value when the conditions differ greatly from the ones assumed in the analytical procedures. When limited to small amplitude response, the analysis is performed in two steps. First, the nonlinear position of static equilibrium is obtained. Second, the corresponding mass, stiffness, and eventually damping matrices are constructed and a linear dynamic analysis performed. For large amplitude response, the second step would have to be replaced by a nonlinear dynamic analysis.

The purpose of this section is to present the fundamental steps involved in the derivation of the low amplitude dynamic stiffness by a finite element method. This approach makes use of the modes and frequencies of vibration of a cable supported at the lower end and free to move horizontally at the upper end.

Preliminaries

Static Analysis. Numerous authors have reported numerical procedures and results in the area of nonlinear static cable analysis. These latter are iterative procedures, incremental methods, and numerical solutions of differential equations. Iterative procedures are based either on an assumed initial cable configuration from which imbalanced forces are computed and used to adjust the cable configuration, or on assumed forces acting
at one end of the cable, from which the cable configuration is derived, the end-forces being adjusted until the cable configuration is compatible with the boundary conditions. In incremental methods the external load is applied in successive linear load steps until the desired magnitude of load is reached. Alternatively, incremental deflections are used. Numerical solutions of differential equations are based on a specific type of external loading and cable properties valid within successive segments and governed by a given differential equation which is solved numerically by proper consideration of the boundary and junction conditions. Analytical solutions, generally based on the taut, parabolic, or catenary cable configuration, have also been reported. Some of these numerical and analytical methods can be found in references (8) to (14), with historical and literature review in the publications by Irvine (11), Judd and Wheen (14), and in a state-of-the-art report (13).

**Dynamic Analysis.** Once the nonlinear position of static equilibrium is known, the dynamic analysis is no different than for any other structure. Two basic approaches can be used. The first approach makes use of the tangent stiffness, mass and eventually damping matrices corresponding to the position of static equilibrium, the nonlinear effects being introduced by imaginary reactions. In a linear dynamic analysis, these reactions are zero. In the second approach, the matrices are updated at each time step, if necessary. Such a procedure specialized to cables can be found in reference (15) with further literature given in references (9) and (13).
Fundamental Properties of cable

**Stiffness Matrix.** The discretized cable fixed at the lower end and free to move in an horizontal plane at the upper end is characterized by its tangent stiffness matrix at the position of static equilibrium. When the cable is discretized into straight, uniform segments the elemental stiffness matrices associated with each segment are diagonal when referred to axes in the direction of, and perpendicular to the individual segments. The stiffness coefficient associated with the direction of the axis of the \(i^{th}\) segment is simply \(A_i E_i/L_i\), and the ones associated with directions perpendicular to the axis are \(T_i/L_i\). \(A_i\) is the cross-sectional area of the cable segment, \(E_i\) is its Young's modulus of elasticity, \(L_i\) its length at the position of static equilibrium, and \(T_i\) its tension at the same position. The elemental stiffness matrices expressed in local coordinates need then to be transformed into an arbitrarily chosen global coordinate system and assembled into a single global stiffness matrix \(K\).

**Mass Matrix.** The global mass matrix \(M\) is simply found by lumping the distributed masses at the nodes. A consistent mass matrix can also be used.

**Modes and Frequencies.** Once \(K\) and \(M\) are defined, the natural circular frequencies \(p_j\) and modes of vibration \(\phi_j\) are readily obtained from the proper eigenvalue equation. The subscript 'j' indicates the order of the mode.

One has to note that these quantities refer to the cable free to move horizontally at the top and that a constant horizontal force is applied at this location. This force is the one necessary to maintain sta-
tic equilibrium.

**Dynamic Stiffness**

**Equation of Motion.** Let the global axis system be such that the direction '1' represents a horizontal axis in the vertical plane containing the cable chord, and that direction '2' represents a horizontal axis perpendicular to direction 1. Further denote by \( \phi_{j1} \) and \( \phi_{j2} \) the \( j \)th modal displacements at the upper end in the 1- and 2-directions, respectively.

A harmonic force of amplitude \( F_1 \) and frequency of excitation \( \omega \) applied at the top in the 1-direction induces steady state harmonic displacements of amplitude \( v_{11} \) and \( v_{21} \) at the top. The first of the two subscripts refers to the direction of displacement and the second to the direction of excitation. These amplitudes are

\[
V_{11} = F_1 \sum_j \frac{1}{p_i^2} \frac{1}{1 - \left( \frac{\omega}{\omega_i} \right)^2 + \frac{i \zeta_j}{\omega}} \frac{\phi_{3j}}{\phi_{31}} \frac{\phi_{3i}}{\phi_{31}}
\]

\( (37a) \)

and

\[
V_{21} = F_1 \sum_j \frac{1}{p_i^2} \frac{1}{1 - \left( \frac{\omega}{\omega_i} \right)^2 + \frac{i \zeta_j}{\omega}} \frac{\phi_{3j}}{\phi_{31}} \frac{\phi_{32}}{\phi_{31}}
\]

\( (37b) \)

where \( \zeta_j \) is the damping ratio of the \( j \)th mode and \( i = \sqrt{-1} \). Implied in these expressions is the assumption that the form of the cable damping matrix is such as to allow uncoupling of the modes.

When a harmonic force excitation of amplitude \( F_2 \) and frequency of excitation \( \omega \) is applied in the 2-direction, the amplitude of steady state displacements \( v_{12} \) and \( v_{22} \) are similarly given by
\[ V_{z1} = F_z \frac{1}{\rho_2} \frac{1}{1 - \left(\frac{\omega}{\rho_2}\right)^2 + i 2 \xi_d \frac{\omega}{\rho_2}} \frac{\phi_{z1}}{\xi \phi_1^{\dagger} \Xi M_1 \Xi^{\dagger} \xi_1^{\dagger}} \]  
(38a)

and

\[ V_{z2} = F_z \frac{1}{\rho_2} \frac{1}{1 - \left(\frac{\omega}{\rho_2}\right)^2 + i 2 \xi_d \frac{\omega}{\rho_2}} \frac{\phi_{z2}}{\xi \phi_1^{\dagger} \Xi M_1 \Xi^{\dagger} \xi_1^{\dagger}} \]  
(38b)

**Dynamic Flexibility.** The dynamic flexibility \(D(\omega)\) is defined as the amplitude of steady state displacement at cable top induced by a harmonic force of unit amplitude and frequency of excitation \(\omega\) applied at the same point. The ratio \(v_{11}/F_1 = D_{11}(\omega)\) then represents the dynamic flexibility associated with the 1-direction, \(v_{22}/F_2 = D_{22}(\omega)\) the one associated with the 2-direction, and \(v_{21}/F_1 = v_{12}/F_2 = D_{12}(\omega)\) the coupling dynamic flexibility. This latter one is zero when the cable remains in a vertical plane at its position of static equilibrium.

**Dynamic Stiffness.** The corresponding dynamic stiffness \(K(\omega)\), defined as the amplitudes of harmonic forces necessary to be applied at the top of the cable to induce steady state top displacements of unit amplitude, are

\[ K_{11}(\omega) = \frac{D_{22}(\omega)}{D_{11}(\omega) D_{22}(\omega) - D_{12}^2(\omega)} \]  
(39a)

\[ K_{22}(\omega) = \frac{D_{11}(\omega)}{D_{11}(\omega) D_{22}(\omega) - D_{12}^2(\omega)} \]  
(39b)

\[ K_{12}(\omega) = \frac{-D_{12}(\omega)}{D_{11}(\omega) D_{22}(\omega) - D_{12}^2(\omega)} \]  
(39c)
When $D_{12}(\omega) = 0$, Eqs (39a) to (39c) simply become

\[ k_{11}(\omega) = \frac{1}{D_{11}(\omega)} = \frac{F_1}{\nu_{11}} \]  \hspace{1cm} (40a)

\[ k_{22}(\omega) = \frac{1}{D_{22}(\omega)} = \frac{F_2}{\nu_{22}} \]  \hspace{1cm} (40b)

\[ k_{12}(\omega) = 0 \]  \hspace{1cm} (40c)
DYNAMIC STIFFNESS OBTAINED FROM SUPPORTED-SUPPORTED MODES

Preliminaries

The dynamic stiffness of a cable can alternatively be obtained in a form involving the modes of vibration of the cable at both ends. The dynamic stiffness of the parabolic cable used in the early part of this Chapter is derived in such a manner in this section.

The derivation requires prior knowledge of the natural frequencies and modes of the supported-supported parabolic cable as well as of its deflections, expressed in a modal form, when support-excited. These problems have been investigated by Irvine (3 to 7), and the natural frequencies and modes further discussed in Chapter Five. These solutions are briefly reviewed next.

Frequencies and Modes. The \( j \)\(^{th} \) symmetric mode of vibration of the supported-supported cable, \( \psi_j^s(\xi) \), has been given in Chapter Five as

\[
\psi_j^s(\xi) = C_j \left[ \cos \left( \frac{\phi_j^s}{2} (\xi - L) \right) - \cos \left( \frac{\phi_j^s}{2} \right) \right]
\]  

(41)

where \( C_j \) is a coefficient of amplitude and \( \phi_j^s \) is the dimensionless natural frequency, associated with the \( j \)\(^{th} \) symmetric mode, defined by the frequency equation

\[
1 + 12 \rho \left[ 1 + \frac{2}{\phi_j^s} \cos \left( \frac{\phi_j^s}{2} \right) \right] = 0
\]  

(42)

The \( i \)\(^{th} \) antisymmetric mode, \( \psi_i^a(\xi) \), is

\[
\psi_i^a(\xi) = C_i \sin \left( 2\pi \frac{\phi_i^a}{L} \xi \right)
\]  

(43)

and the corresponding dimensionless frequency \( \phi_i^a \) is
\[ \phi_i = 2 \cdot \pi \]

(44)

Response to Support-Excitations. When external damping expressed by a coefficient \( \zeta \) per unit length of cable-chord is introduced, the governing differential equation is (see Chapter Four):

\[ \frac{\partial^2 \omega(x, t)}{\partial x^2} + \frac{\zeta}{\rho} \frac{\partial \omega(x, t)}{\partial t} - \frac{E}{\rho} \frac{\partial^2 \omega(x, t)}{\partial t^2} = \frac{q_0}{\rho} = \frac{\Delta T_0(t)}{\rho} \]

(45)

where \( \Delta T_0(t) \) is given by Eq. (4).

Let the normal deflection \( w \) be first expressed as a function of the dimensionless distance \( \xi = \frac{x}{L} \) instead of \( x \) and let it further be visualized as being composed of a rigid normal displacement and a deformational displacement \( v(\xi, t) \) as

\[ w(\xi, t) = \omega(\xi, \xi) + \left[ \omega(0, \xi) \right] \xi + v(\xi, t) \]

(46)

\( v(\xi, t) \) satisfies the same boundary conditions as those of the modes \( \psi(\xi) \), and it can be written in terms of the modes as

\[ v(\xi, t) = \sum_j \psi_j^s(\xi) p_j^s(t) + \sum_j \psi_j^a(\xi) p_j^a(t) \]

(47)

where \( p_j^s(t) \) and \( p_j^a(t) \) are time dependent modal coefficients for the symmetric and antisymmetric modes, respectively.

Introducing Eqs. (46) and (47) into Eqs. (45) and (4), multiplying Eq. (45) by a mode of vibration and integrating from 0 to 1 while making use of the orthogonality property of the modes and of Eq. (4), one obtains the following form of the equation of motion:
\[ \ddot{p}_j^{s}(t) + \frac{c}{l} \dot{p}_j^{s}(t) + (\omega_j^{s})^2 p_j^{s}(t) = \left\{ \frac{\rho_s}{l} \frac{1}{\kappa_{sa}} \left[ \psi_{j,1}^s \left( u(t, \xi) - u(t, \varsigma_1) \right) \right] \right\} - \left[ \frac{c}{l} \dot{\psi}_{j,1}^s + \frac{\dot{\psi}_{j,1}^s}{2} \right] \left\{ \int_0^1 \psi_{i,1}^s \, d\xi \right\} \left\{ \int_0^1 (\psi_{i,1}^s)^2 \, d\xi \right\} \]  

(48a)

for the symmetric modes, and

\[ \ddot{p}_j^{a}(t) + \frac{c}{l} \dot{p}_j^{a}(t) + (\omega_j^{a})^2 p_j^{a}(t) = \left\{ \frac{\rho_a}{l} \frac{1}{\kappa_{sa}} \left[ \psi_{j,1}^a \left( \dot{u}(t, \xi) - \dot{u}(t, \varsigma_1) \right) \right] \right\} - \left[ \frac{c}{l} \dot{\psi}_{j,1}^a + \frac{\dot{\psi}_{j,1}^a}{2} \right] \left\{ \int_0^1 \psi_{i,1}^a \, d\xi \right\} \left\{ \int_0^1 (\psi_{i,1}^a)^2 \, d\xi \right\} \]  

(48b)

for the antisymmetric modes. The dot represents the differential with respect to time, while \( \omega_j^{s} \) and \( \omega_j^{a} \) are the \( j \)th symmetric and \( i \)th antisymmetric natural frequencies of the supported-supported cable, respectively. It should be noted that the symmetric component of the support-excitation affects only the symmetric modes, while the antisymmetric component affects only the antisymmetric modes. Also, the component of the support-excitation in the direction of the cable-chord does not affect the antisymmetric modes. This is due to the fact that such a component of motion enters only as boundary condition in the expression for \( \Delta T_0 \) and that an antisymmetric displacement pattern does not affect the value of \( \Delta T_0 \) since no change in cable-length is associated with it.

Once the modal components \( p_j^s \) and \( p_i^a \) are known, the change in the component of the tensile force in the direction of the chord is readily given as
\[ \Delta T_0 (x) = -\frac{\sigma E}{L_e} \left[ u(x, t) - u(0, t) \right] + \frac{\beta}{\pi} \int_0^1 \psi_4^2 (s) \Delta T_0 (s) ds \] (49)

with

\[ (\Delta T_0)_d = -\frac{\sigma E}{L_e} \int_0^1 \psi_4^2 (s) \Delta T_0 (s) ds \] (50)

Notice that only the symmetric modes enter in these expressions.

The modal equations reported above depend upon both the acceleration and velocity of the support excitation. This does not represent any difficulty when solving those equations by direct integration, but prevents the use of existing response spectra. This disadvantage could be overcome if spectra having the appropriate input form were generated. This dependence is due to the fact that the damping is proportional to the velocity of the total lateral displacement \( W(x, t) \) and that it is the relative lateral displacement \( v(x, t) \) which is expressed in terms of modal components.

**Modal Dynamic Stiffness of a Parabolic Cable**

**Fundamentals.** By setting in Eqs. (48) and (49) \( u(0, t) = w(0, t) = 0 \), \( u(1, t) = X(t) \cos \Theta \) and \( w(1, t) = X(t) \sin \Theta \), the conditions of a horizontal displacement \( X(t) \) at the top support are reproduced. The symmetric modal equation (48a) then becomes:

\[
\begin{align*}
\ddot{p}_4^2 (s) + \frac{\sigma}{\pi} \dot{p}_4^2 (s) + (\omega_4^2)^2 p_4^2 (s) = & \left\{ \frac{\sigma E}{L_e} \frac{1}{L_e} \frac{\sigma E}{L_e} (-X(s) \cos \Theta) \right\} - \\
& \left[ \frac{\beta}{\pi} \frac{X}{s} \sin \Theta + \frac{1}{2} X(s) \sin \Theta \right] \int s \frac{\psi_4^2 (s) ds}{\beta (\psi_4^2 (s))^2 ds} 
\end{align*}
\] (51)
and the change in tension is

\[ \Delta T_0(t) = \frac{A E}{L_0} X(t) \cos \theta + \frac{A E}{L_0} \frac{2 \xi}{\tau_0} \sum_i \left[ \int_0^1 \psi_i^2(z) \, dz \right] \]  

(52)

The change in horizontal force at the top, \( \Delta H(t) \), is then approximately given by Eq. (13b).

When the displacement \( X(t) \) is harmonic, so is \( \Delta H(t) \). The ratio of amplitudes \( \Delta H \) to \( X \) is then the simplified dynamic stiffness reported in Chapter Four in a closed-form. It is here equal to:

\[ \tilde{K}(\phi) = \frac{A E}{L_0} \cos^2 \theta \left\{ 1 + \sum_i \left[ \frac{-12 \rho + \frac{1}{2} \xi \left( \phi^2 - \xi \pi \xi \phi \right)}{-\phi^2 + \left( \xi \pi \xi \phi \right)^2} \right] \right\} \]  

(53)

where \( \phi \) is the dimensionless frequency of excitation and \( \xi \) is defined as

\[ \xi = \frac{1}{2\pi} \omega_0 \quad (54) \]

For static conditions, \( \phi = 0 \) has to be substituted in Eq. (53) and the modal expansion of the static stiffness is

\[ \tilde{K}_{\text{st}} = \tilde{K}(\phi) = \frac{A E}{L_0} \cos^2 \theta \left\{ 1 + \sum_i \left[ \frac{-12 \rho}{\left( \phi_i^2 \right)^2} \frac{1}{\int_0^1 \psi_i^2(z) \, dz} \right] \right\} \]  

(55)

Performing the summation over all the modes and using the expression presented in Chapter Five for this infinite series, this latter equation becomes

\[ K_{\text{st}} = \frac{A E}{L_0} \cos^2 \theta \frac{1}{1 + \rho} \]  

(56)

as it should be.
The tilde in Eqs. (53) and (55) refers to the modal form of the dynamic stiffness.

Application. Using only the fundamental mode in Eq. (53) and introducing the values of parameters $\zeta = 0$ (no damping) and $\sigma = 0$ (horizontal cable), one obtains the curves of Fig.8 reported for $\rho = 0.2$, 1, and 5, respectively. $\tilde{K}(\phi)$ in this figure has been normalized with respect to $K_{st}$ of Eq. (56). The curves corresponding to the closed-form solution of Chapter Four are reported in the same figure. One sees that large differences occur, unless $\rho$ is small. The differences are particularly significant in the low frequency region but not in the high frequency region. This is due to the fact that almost no lateral vibration takes place at high frequencies, as explained in Chapter Four, while such vibration does take place at low frequencies.

Focussing attention on the static value $\tilde{K}_{st}$, one can see that it rapidly converges towards the value $K_{st}$ when an increasing number of modes is considered. This is illustrated by the values of $\tilde{K}_{st}/K_{st}$ reported in the Table below for one to three modes and for different values of $\rho$.

<table>
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<th>$\tilde{K}<em>{st}/K</em>{st}$</th>
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CONCLUSIONS

It was shown that the natural frequencies and modes of a parabolic cable fixed at one end and free to move in the horizontal direction at the other end depend on two independent parameters only. Representative results have been discussed and it has been shown that a simplified frequency equation can be used for not too steep cables.

The evaluation of a cable dynamic stiffness by a finite element method was also discussed, making use of the frequencies and modes of the cable free to move horizontally at the top and fixed at the bottom. The same quantity was obtained in an analytical manner for a parabolic cable, making use of the supported-supported modes of vibration of the cable.
REFERENCES


13. Subcommittee on Cable-Suspended Structures of the Task Committee on Special Structures, of the Committee on Metals, of the Structural Division, "Cable Suspended Roof Construction State-Of-The-Art", Journal of the Structural Division, ASCE, June, 1971, pp. 1715-1761.

Fig. 1 - Natural Frequencies as a Function of $\sigma$

Parameter: $\epsilon = \frac{I_n}{\omega L} \tan \theta$

$\sigma = \frac{\theta_n^2}{\omega L} \tan \theta$
Fig. 2 - Modes of Vibration for $\varepsilon = 0.1$

$\varepsilon =$

Third node  Second node  Fundamental node
Fig. 3 - Natural Frequencies as a Function of $\theta$
Fig. 4 - Modes of Vibration for $\frac{\tau_{OL}}{AE/Le} = 0.002$ and $\frac{y_{max}}{L} = 1/12$

- $\theta = 90^\circ$: Third node
- $\theta = 75^\circ$: Second node
- $\theta = 60^\circ$: Fundamental mode
Fig. 5 - Natural Frequencies of a Supported-Horizontally Taut Cable
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Fig. 7 - Complete and Simplified Frequencies as a Function of $\theta$
CHAPTER

SEVEN

STUDIES ON CABLE-STRUCTURE INTERACTION OF GUYED SYSTEMS WITH
APPLICATION TO A PILE-SUPPORTED GUYED TOWER

Introduction
Harmonic Response of Single-Degree of Freedom Tower
Amplification Factor for Multidegree of Freedom Tower
Transient Analysis
Evaluation of Natural Frequencies of a Tower-Cable System
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INTRODUCTION

Objective and Scope

The tower response of a cable-tower system is usually computed under the assumption of massless cables. The primary objective of this chapter is to examine the extent to which such an assumption is valid. After a brief review of the concept of dynamic cable stiffness, a quantity in which cable inertia is in-built, a tower modeled as a single degree of freedom system will be investigated for a harmonic, lateral excitation. This approach will permit the evaluation and interpretation of the fundamental parameters affecting the tower response. Application will then be made to the guyed tower presented in chapter III for each of the lateral tower modes. The same structure will also be analyzed as a multi-degree of freedom system and it will be shown by way of comparison that the single degree of freedom characterization of the tower is indeed adequate. The adaptation of the solutions above to torsional excitations will also be made, and several schemes of accounting for the cable inertia in a transient analysis will be presented. The effects of cable inertia on the natural frequencies of the tower will also be discussed. The latter discussion will make use both of the concept of dynamic cable stiffness and of the Force Method of analysis presented in chapter II.

Fundamentals of Dynamic Cable Stiffness

Definitions. The in-plane dynamic stiffness, $K_{i}(\phi)$, is
defined as the amplitude of the harmonic horizontal force in the vertical cable-plane necessary to apply at the top of the cable to produce a harmonic deflection of unit magnitude in the same direction and location, the cable being assumed to be linear and the amplitudes to be small. The out-of-plane dynamic stiffness, \( K_2(\phi) \), is defined in the same manner for a horizontal direction of force and deflection normal to the cable chord.

**Parameters.** \( K_1(\phi) \) is given by Eqs. 38 and 39 of chapter IV, where it is denoted \( K(\phi) \) and \( K'(\phi) \). \( K_2(\phi) \) is given by Eq. 87 of Chapter IV. When \( K_1(\phi) \) is nondimensionalized with respect to its static counterpart, it depends only on the dimensionless frequency \( \phi = \omega / \omega_0 \) where \( \omega \) is the frequency of excitation and \( \omega_0 \) a characteristic of the cable defined by Eq. 19 of chapter IV, on the dimensionless parameters \( \rho \) and \( \zeta \) characterizing the cable and respectively defined by Eqs. 33 and 34 of chapter IV, and on the damping ratio \( \zeta_c \) representing the viscous action of the surrounding medium and defined by Eq. 21 of chapter IV. For the cables attached to the guyed tower presented in chapter III, the values above are \( \omega_0 = 0.167 \) 1/sec, \( \rho = 2.19 \), and \( \zeta = 0.10 \). \( \zeta_c \) will be taken as 0% or 10%.

\( K_2(\phi) \) nondimensionalized with respect to its static counterpart depends on \( \phi \) and \( \zeta_c \).

**Group Behavior.** The lateral dynamic stiffness \( K(\phi) \) of a group of \( n \) identical cables arranged symmetrically around a tower is given by Eq. 88 of chapter IV as
\[ K(\phi) = \frac{n}{2} [K_1(\phi) + K_2(\phi)] \] (1)

The stiffness \( K_2 \) is generally small compared to \( K_1 \) and will be neglected in the following analysis. Furthermore, \( K_1(\phi) \) will be evaluated from the simplified dynamic stiffness expression presented in Eq. 39 of chapter IV. The stiffness of the cable system considered, normalized with respect to the static value \( K(0) \), is plotted in Fig. 1 as a function of the frequency ratio, \( \phi \). Damping is assumed to be negligible.
HARMONIC RESPONSE OF SINGLE-DEGREE OF FREEDOM TOWER

Model

The simplified system considered is reported in Fig. 2a. It is a discrete mass \(M\) restrained by a spring of stiffness \(K_t\), a viscous damper of constant \(C_t\), and a cable-spring of frequency-dependent stiffness \(K(\phi)\). This system is representative of a cantilever tower constrained by a guying system and vibrating in a specific configuration. \(M, K_t\) and \(C_t\) are then respectively the effective mass, stiffness and damping coefficients of the tower without cables, and \(K(\phi)\) is the guying stiffness.

The guying stiffness \(K(\phi)\) is expressible as a real component \(K_R(\phi)\) and an imaginary component \(K_I(\phi)\) as

\[
K(\phi) = K_R(\phi) + i K_I(\phi)
\]

where \(i = \sqrt{-1}\). This leads to the alternate form of the model shown in Fig. 2b.

Amplification Factor for Massless Cables

When the cables are assumed to be massless, the stiffness of the guying system is independent of the exciting frequency and equal to the static value \(K_R(0)\). The steady state displacement \(x(t) = x(\omega)e^{i\omega t}\) of the model subjected to a harmonic force \(F(t) = F e^{i\omega t}\) is then readily obtained. Its amplitude is

\[
x(\omega) = \frac{F}{K_F + K_{R}(0) + i\omega C_{c} - M\omega^2}
\]

Dividing this expression by the static displacement \(x_0\) defined by
\[ x_0 = \frac{F}{K_0 + K_R(0)} \]  

one obtains the amplification factor, \( AF \), as

\[ AF = \frac{1}{1 + i \beta S_x(\frac{\omega}{\rho}) - (\frac{\omega}{\rho})^2} \]  

(5a)

where the circular natural frequency \( \rho \) of the system as well as the damping ratio \( \beta \) have been introduced. They are given by

\[ \rho = \sqrt{\frac{K_p + K_R(0)}{M}} \]  

(6)

and

\[ S_x = \frac{C_x}{2M \rho} \]  

(7)

### Amplification Factor for Cables with Mass

When the inertia of the cables is accounted for, the amplitude of steady state displacement becomes

\[ x(\omega) = \frac{F}{\left[ K_0 + K_R(0) \right] + i \omega \left[ C_x + K_0/\omega \right] - M \omega^2} \]  

(3b)

and the corresponding amplification factor, \( AF^x \)

\[ AF^x = \frac{1}{\left[ \frac{K_p + K_R(0)}{K_0 + K_R(0)} \right] + i \omega \left[ \frac{C_x + K_R(0)/\omega}{K_0 + K_R(0)} \right] - (\frac{\omega}{\rho})^2} \]  

(5b)

One recognizes by inspection of these equations that the imaginary component of the guying stiffness \( K_I(\phi) \) has the same effect as that of a viscous damper of coefficient \( K_I(\phi)/\omega \). This leads to the model of Fig. 2c.

As noted previously, the dynamic stiffness of the cable depends on the dimensionless frequency \( \phi = \omega/\omega_c \). Nondimensional-
nalizing Eq. 5b and further rearranging the terms, one obtains

$$MF^0 = \frac{1}{\frac{\kappa_k(0) + \kappa_k(\phi) / \kappa_k(0)}{1 + \kappa_k / \kappa_k(0)} + \frac{2(\xi_c + \xi_d(0))}{\omega_0^2} \frac{\omega_0}{F} \phi - \left(\frac{\omega_0}{F}\right)^2 \phi^2}$$

with

$$S^k(\phi) = \frac{1}{2} \frac{\kappa_k(\phi) / \kappa_k(0)}{1 + \kappa_k / \kappa_k(0)} \frac{p}{\omega_0} \frac{1}{\phi}$$

Note that the amplification factor depends on the parameters $p/\omega_0$, $\kappa_k / \kappa_k(0)$ and $S^k$, and on the variables $\phi = \omega / \omega_0$, $K_R(\phi) / K_R(0)$ and $K_I(\phi) / K_R(0)$.

**Governing Parameters**

*Frequency Ratio $p/\omega_0$.* $p/\omega_0$ represents the ratio of the circular natural frequency $p$ of the system computed under the assumption of massless cables to the characteristic cable frequency $\omega_0$. From the definition of $\omega_0$ given in Eq. 19 of chapter IV as

$$\omega_0 = \sqrt{\frac{T_0/L}{M/c}}$$

where $T_0$ is the component of the cable tensile force in the direction of the chord, $L$ the length of the cable-chord and $cL$ the total cable mass, one obtains $p/\omega_0$ as

$$p/\omega_0 = \sqrt{\frac{\kappa_k(0)}{T_0/L}} \sqrt{\frac{cL}{M}}$$

or

$$p/\omega_0 = \sqrt{\frac{n/2}{n/2 T_0/L}} \sqrt{\frac{n/2 cL}{M}}$$

The coefficient $n/2$ has been introduced in Eq. 11b for sake of consistency with Eq. 1.
Let us consider first an undamped tower model for which the only constraint is the one provided by an undamped guying system. In this case $\gamma = 0$, $K_t = 0$ and $K_I = 0$. Curves for the numerical values of the amplification factors defined by Eqs. 5a and 8 are given in Figs. 3 and 4 for values of $p/\omega_o$ equal to 1 and 2.5, respectively. The following trends may be noticed on comparing the curves obtained from Eq. 8 to those obtained from Eq. 5a.

First, there are additional peaks. These are due to the additional degrees of freedom introduced by accounting for the inertia of the cables, each peak corresponding to a natural frequency of the cable-tower system. The importance of these peaks may be clarified by the following interpretation of Eq. 11. $\frac{n}{T} \frac{r}{\ell}$ being generally negligible in comparison to $K_R(0)$, the first term of Eq. 11b is generally very large. Values of $p/\omega_o$ equal to 1 and 2.5 are then associated with very small values of $\frac{n}{T} \frac{r^2}{M}$. The set of cables being attached to a mass of comparatively very large inertia, the natural frequencies associated with predominantly cable modes are consequently close to the ones of the supported-supported cable. At these latter frequencies, $K(\phi)$ is however infinite (see chapter IV) and the corresponding amplification factor null. Infinite values of amplification factor close to zero values results to highly localized peaks.

The second trend concerns the effect of cable inertia on the primary peak, namely, the peak which appears when
massless cables are assumed. This peak is being shifted when considering cable inertia, in a more pronounced manner for \( p/\omega_o = 2.5 \) than for \( p/\omega_o = 1 \). This is explained as follows. The new location of the peak, \( p_{\text{new}}/\omega_o \), is obtained from the generalized form of Eq. 6 in which the effective value of the dynamic stiffness at \( p_{\text{new}}/\omega_o \) is introduced. It is

\[
p_{\text{new}} = \sqrt{K_b + K_n(p_{\text{new}}/\omega_o)}
\]

from which \( p_{\text{new}}/p \) is given as

\[
p_{\text{new}}/p = \sqrt{\frac{K_b + K_n(p_{\text{new}}/\omega_o)}{K_b + K_n(0)}}
\]  \hspace{1cm} (13a)

In the present example \( K_b = 0 \), from which

\[
p_{\text{new}}/p = \sqrt{\frac{K_n(p_{\text{new}}/\omega_o)}{K_n(0)}}
\]  \hspace{1cm} (13b)

\( p_{\text{new}} \) must first be estimated, from which \( K_R(p_{\text{new}}/\omega_o)/K_R(0) \) can be evaluated and \( p_{\text{new}} \) iteratively corrected by Eq. 12.

As a first approximation, \( K_R(p_{\text{new}}/\omega_o) \) may be taken equal to \( K_R(p/\omega_o) \). At \( p/\omega_o = 1 \), \( K_R(1)/K_R(0) \) is equal to 0.93 while at \( p/\omega_o = 2.5 \), it is equal to 0.49. Note further that for high values of \( p/\omega_o \), \( K_R(p/\omega_o)/K_R(0) \) can be larger than unity, in which case the main peak is shifted to the right.

Finally, one has to note that the differences observed between the curves obtained from Eqs. 5a and 8 are larger at low frequencies of excitation (left of curves) than at high frequencies (right of curves). This is explained by the fact that at high frequencies the mass \( M \) of the tower-model controls the response. At such frequencies, the amplitude of undamped steady state response of a system both with massless
cables and with cables with mass is given approximately by
\[ x(\omega) \approx -\frac{F}{M\omega^2} \] (14)

The corresponding amplification factor is then
\[ a^p \approx A_f' \approx -\frac{1}{(\omega_p)^2} \] (15)
in which stiffnesses appear only implicitly, in \( p \). This observation is of the utmost importance, as it shows that the large differences between the dynamic stiffness and its static counterpart occurring at high frequencies are of no consequences on the amplification factor computed either way.

**Stiffness Ratio** \( K_t/K_R(0) \). \( K_t/K_R(0) \) represents the ratio of the stiffness of the model without cables to the static stiffness of the guying system. A decreasing value of \( K_t/K_R(0) \) corresponding to an increasing contribution of the guying system to the total stiffness of the system, the effect of cable inertia on the amplification factor is more pronounced for low values of \( K_t/K_R(0) \) than for large values. That this is the case may be appreciated by comparing the solid curve in Fig. 5, obtained for \( K_t/K_R(0) = 5 \), to the corresponding curve in Fig. 4, obtained for \( K_t/K_R(0) = 0 \).

**Damping.** Curves of amplification factors for a system with \( \xi = 5\% \) and \( \xi = 10\% \) are reported in Fig. 6.

The general trends of these curves are the same as those discussed previously, with the introduction of cable-damping further reducing the magnitudes of the primary and additional peaks. The quantity \( \frac{1}{\phi} K_t(\phi)/K_R(0) \), appearing in Eq. 9, is
plotted in Fig. 7 as a function of $\phi$, along with $K_T(\phi)/K_R(0)$. The value of $\lim_{\phi \to 0} K_T(\phi)/K_R(0)$ at $\phi = 0$ is of the type $0/0$. Its limit is finite and is derived in Appendix A.

**Application to a Guyed Tower**

The effect of cable-inertia on single degree of freedom characterizations of the guyed tower described in chapter III is assessed. The modes of vibration in the symmetric, x-direction are utilized at this effect, as follows. The first and second natural circular frequencies obtained under the assumption of massless cables are listed in Fig. 5 of chapter III as $p_1 = 0.235$ l/sec and $p_2 = 1.31$ l/sec, respectively. The same quantities for the tower without cables are $p_1^{WO} = 0.0483$ l/sec and $p_2^{WO} = 1.30$ l/sec, respectively. Defining $p^{WO}$ as

$$p^{WO} = \sqrt{\frac{K_r}{M}}$$

one obtains $K_T/K_R(0)$ by comparison of Eq. 16 with Eq. 6 as

$$K_T/K_R(0) = \left[ \left( \frac{p}{p^{WO}} \right)^2 - 1 \right]$$

This gives $K_T/K_R(0) = 1/23$ for the first mode, and $K_T/K_R(0) = 191$ for the second mode. The cable considered is characterized by a value of $\omega_0 = 0.167$ l/sec, from which $p/\omega_0$ is given as $p_1/\omega_0 = 1.41$ and $p_2/\omega_0 = 7.84$.

**Fundamental Mode.** The curves of amplification factors for an undamped and damped system are reported in absolute terms in Figs. 8 and 9, respectively. It can be seen in these curves that the first of the additional peaks is unimportant. This is due to the fact that the tower response is "mass-
controlled" in this frequency region. The same conclusion applies for the other additional peaks appearing at higher values of $\phi$.

The main peak is affected in the way of a change in its position along the $\omega/p$-axis and of a reduction of its maximum value in the damped system. The new peak position is given by Eq. 13 and is approximately equal to 0.92 in the present case. The reduction in magnitude of the main peak may be very substantial depending upon cable damping, with the additional peaks being even more reduced.

**Second Mode.** The undamped and damped curves of amplification factors are shown in Figs. 10 and 11, respectively. The high value of ratio $K_t/K_R(0)$ means that the guying stiffness provides only a small fraction of the total constraint. This evidently minimizes the effect of cable inertia. Especially, $p_{new}/p$ is practically equal to 1, even though $K_R(p/\omega_0)$ is significantly different from $K_R(0)$ ($K_R(7.84)/K_R(0)=4.5$).

**Higher Modes.** Modes higher than the second are characterized by ratios of $K_t/K_R(0)$ from one to several orders of magnitude larger than the one of the second mode. Cable inertia has then no perceptible effect on the response of the tower models and static cable stiffness may be used.
AMPLIFICATION FACTOR FOR MULTIDEGREE OF FREEDOM TOWER

Model

Consider a tower disconnected from its guyng system and moving laterally in a vertical plane. This structure is characterized by

. Its circular natural frequencies, \( p_j^2 \);
. The corresponding modes of vibration, \( \phi_j \);
. The mass matrix, \([M]\); and
. The number of degrees of freedom, \( n \).

Assume further that the tower is constrained by a lateral guyng system of frequency dependent stiffness \( K(\omega) \), located at a node at a distance \( x_0 \) from the base. The amplitude of the \( j \)th mode at this node is \( \phi_j(x_0) \).

Amplification Factors

Let \( u(t) = u(\omega) e^{i\omega t} \) be the steady state deflection of the tower at \( x_0 \) induced by a harmonic, horizontal force vector \( \{H(t)\} = \{H^i\} e^{i\omega t} \). Representing the effect of the guyng system on the tower by a reaction \( F_R(t) = F_R(\omega) e^{i\omega t} \) with \( F_R = K(\omega) \cdot u(\omega) \), one obtains \( u(\omega) \) for the undamped system as

\[
 u(\omega) = \sum_{j=1}^{n} \left[ \frac{H_j^T \phi_j \phi_j^T - F_R(\omega) \phi_j(x_0)}{\left( \phi_j^T \left[ M \phi_j \right] \phi_j \right)^2 \left[ 1 - \left( \frac{\omega}{\rho_j} \right)^2 \right]} \phi_j(x_0) \right] \frac{1}{1 - \left( \frac{\omega}{\rho_j} \right)^2} \phi_j(x_0)
\]

(18)

The static deflection \( u_o \), obtained from a static application of \( \{H^i\} \), is obtained by substituting \( \omega = 0 \) and \( K(\omega) = K(0) \) in Eq. 18. The amplification factor, defined here as \( A_F = u(\omega)/u_o \), is then equal to
\[ AF^* = \frac{n}{2} \left[ \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} \right)^2 \right] \left( 1 - \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} \right)^2 \right) \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} \right)^2 \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} \right)^2 \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} \right)^2 \right]^{\frac{1}{2}} \]

The amplification factor computed under the assumption of massless cables, \( AF \), is obtained by setting \( K(\omega) = K(0) \) in Eq. 19.

**Application to a Guyed Tower**

The modes of vibration in the symmetric, \( x \)-direction of the guyed tower without cables presented in chapter III were utilized here. The dynamic stiffness of the guying system was obtained from Eq. 1, where \( K_1(\phi) \) and \( K_2(\phi) \) were computed by a finite element method assuming a parabolic cable profile in a vertical plane at the position of static equilibrium. The characteristics of the cables are those reported in chapter III. The description of the finite element method is found in chapter VI.

The differences associated with the utilization of the finite-element method instead of the method leading to the simplified dynamic stiffness are that the in-plane, antisymmetric modes of vibration of the supported-supported cable are reproduced, and that the out-of-plane cable resistance, \( K_2 \), is considered. This means that the out-of-plane cable-modes are reproduced.

The amplification factors were obtained for a force vector \( \{H\} \) composed of a single force acting at \( x_0 \). The curves of
amplification factors are reported in Fig. 12 for massless and massive cables, respectively. The types of supported-supported cable modes associated with the zero values of amplification are also indicated in the figure. It is observed that the curves are similar to those obtained from the single degree of freedom characterization of the guyed tower. The only new element is the appearance of the additional peaks associated with the predominantly antisymmetric, in-plane modes of vibration and the out-of-plane modes of vibration of the cables.

Torsion

Analysis. The analysis previously performed for lateral motion is also valid for torsional motion. The masses have simply to be interpreted as second mass moments of inertia and the stiffnesses as rotational stiffnesses.

Application to a Guyed Tower. For the guyed tower investigated in chapter III, the second mode of vibration in the unsymmetric, y-direction was seen to be predominantly torsional. For that particular mode, the torsional stiffness, $K_\phi(\phi)$, of the guying system is due to its eccentric arrangement with respect to the instantaneous center of rotation of the tower for that mode. Denoting by $e$ the distance from the center of rotation to the point at which the lateral reaction of the guying system acts, $K_\phi(\phi)$ is given by

$$K_\phi(\phi) = e^2 K(\phi)$$

(20)
where $K(\phi)$ is the lateral guying stiffness defined by Eq. 1. For that mode, $K_t/K_{\phi}(0)$ is equal to approximately 40 and $p/\omega_0$ to 4.27. For such values of parameters, cable inertia has no significant effect on the tower response and static values of cable stiffness may be used.
TRANSIENT ANALYSIS

As cable inertia may, through its effect on the dynamic stiffness and consequently on the tower response, be of importance, it is of interest to have a simple way to account for cable inertia in a transient analysis. Different ways of doing so are briefly discussed below.

Modal Superposition Method. If the analysis is to be performed by modal decomposition, one must evaluate the natural modes of vibration of the coupled cable-tower system. This can be performed in a direct manner using a finite element method, or by application of the concepts presented in the next section. In such a method particular care should be given to the following. A modal method of analysis makes use of the orthogonality properties of the modes with respect to the mass matrix. This orthogonality relationship is valid only if all degrees of freedom of the system are considered. It is therefore necessary that all these degrees of freedom, including those of the cables, be accounted for in evaluating the terms \[ \phi_i^T M \phi_i \].

Additional Tower Mass. A second method would be to add a concentrated mass to the tower at the point of attachment of the guying system, and assume the cables to be massless. This mass \( M_{\text{add}} \) can for example be chosen so that the fundamental frequency \( p_1 \) be exact, i.e. identical to the one of the system without additional mass but with massive cables. It is given by

\[
M_{\text{add}} = \frac{k_n(t) - k_n(t)}{p_1^2}
\]  

(21)
\( M_{\text{add}} \) can also be chosen so as to give accurately the zero cable stiffness, i.e.

\[
M_{\text{add}} = \frac{K_n(\omega)}{p_{\text{free}}^2}
\]  

(22)

where \( p_{\text{free}} \) is the fundamental circular natural frequency of the supported-free cable (i.e. the lowest circular frequency at which the cable stiffness is zero). In both cases damping can be considered to be proportional to the added mass, as discussed in chapter IV. In this approach the force-displacement relationship of the guying system is the same as for static conditions of loading, an advantage when the effects of nonlinearity in the guying system must be considered. Such a model however has the following, slight disadvantage. The introduction of \( M_{\text{add}} \) is identical to the introduction of a spring of stiffness \( K_{\text{add}} = -\omega^2 M_{\text{add}} \). From the analysis of chapter IV, one knows that the dynamic cable stiffness tends to a limiting value at high frequencies, what is not reproduced by \( K_{\text{add}} \) as it tends to \( \pm \infty \).

**Guying Model.** Another approach is to represent the guying system by the simple dynamic model shown in Fig. 16 of chapter IV. This approach would lead to results of high accuracy since the cable model has been shown to provide an excellent representation of the dynamic stiffness of the cable over the entire range of frequencies. Nonlinearities in the guying system can then be introduced in either one, or both springs as required by the source of the nonlinearities.
Frequency Domain. Finally, direct application of the concept of cable dynamic stiffness can be made in connection with an analysis in the frequency domain. This, however, presumes a linear behavior of the guying system.
EVALUATION OF NATURAL FREQUENCIES OF A TOWER-CABLE SYSTEM

The evaluation of the natural frequencies of a tower-cables system when accounting for cable inertia can be efficiently performed by application of the Force Method of analysis presented in chapter II and the concept of dynamic stiffness for the guying system. Such an evaluation is performed here for the symmetric, x-direction of the guyed tower of chapter III, with the dynamic stiffness of the guying system defined by the first term of Eq. 1, and $K_1(\phi)$ obtained by the simplified solution of parabolic cables (see chapter IV).

The approach is as follows:

(1) Plot the natural frequencies of the tower as a function of the cable stiffness (such plots were obtained in chapter III);

(2) Plot the effective, dynamic stiffness of the cables as a function of the frequency of vibration; and

(3) Obtain the natural frequencies of the tower-cable system, given by the points of intersection of the two sets of curves.

This approach is reported in Fig. 13. In the figure, the solid curve represents the natural frequencies of the tower as a function of the guying stiffness. This is the curve of Fig. 7 of chapter III, reported at a logarithmic scale. The dashed curve represents the dynamic guying stiffness as a function of the frequency. This is the curve of Fig. 1. The vertical, line -curve at K=275 kips/ft is the static counterpart of the dashed curve.
The intersections of the dashed curve with the solid one give the natural frequencies of the system when considering cable inertia. Those are associated with predominantly tower modes indicated by $T_1'$ to $T_3'$, and with predominantly cable modes indicated by $C_1'$ to $C_3'$. $T_1$ to $T_3$ are the corresponding 'tower modes' when cable inertia is neglected.

This approach clearly shows the effect of cable inertia on the natural frequencies of the tower-cable system. It should be noted that the intersections of the solid lines and the dashed lines can fall, in principle, in the negative range of stiffness, but this is not the case in Fig. 13.
CONCLUSIONS

The effect of cable inertia on the response of a guyed tower has been investigated. The two major parameters needed to evaluate this effect are the ratio of the stiffness of the tower, $K_t$, to the static stiffness of the guying system, $K_R(0)$, and the ratio of the tower frequency, $\nu$, to the characteristic cable frequency, $\omega_c$.

From the studies performed on a single degree of freedom tower, it appears that:

- The fundamental lateral tower-mode may be affected by the cable inertia due to the fact that most of its resistance comes from the cables ($K_t/K_R(0)$ low). For the particular tower investigated the value of $\nu/\omega_c$ is however such that substantial differences between static and dynamic cable stiffness occur at frequencies at which the response is 'mass controlled', i.e. insensitive to the stiffness. The static cable stiffness may therefore, in this particular case, be used;

- The higher lateral modes of vibration are practically not affected due to the fact that the stiffness of the guying system enters for a negligible part into the total modal stiffness. The static cable stiffness may then be used;

- The effect of cable inertia on the predominantly torsional modes of vibration is similar to the one on the higher, lateral modes, and the static cable stiffness may be used;

- By way of comparison, it was shown that the single
degree of freedom tower model is representative of the multi-degree of freedom tower;

. The overall conclusion is that the importance of the interaction problem should first be evaluated by looking at the effect of cable inertia on the modes of vibration, especially the fundamental, lateral one.
APPENDIX A

The quantity $\frac{1}{\phi} \frac{K_1(\phi)}{K_n(10)}$ is of the type 'zero over zero' at $\phi = 0$. Its limit can most conveniently be found by using the series expression for the simplified dynamic stiffness given in chapter V as

$$\frac{K(\phi)}{K_n(10)} = (1+\rho) \left\{ 1 + \sum_{m=1}^{\infty} \left[ \frac{1}{\phi^2} \left( \frac{\phi^2 - \phi_3^2}{\phi^2 - \phi_3^2} \right)^m \left( \int_0^1 \psi_3(\xi) d\xi \right)^2 \right] \right\} (A1)$$

where $\phi_3$ and $\psi_3(\xi)$ are the dimensionless symmetrical natural frequencies and modes of the supported-supported cable, respectively. The imaginary component of this expression divided by $\phi$ can be written as

$$\frac{1}{\phi} \frac{K_1(\phi)}{K_n(10)} = (1+\rho) \left\{ \sum_{m=1}^{\infty} \left[ \frac{1}{\phi^2} \left( \frac{\phi^2 - \phi_3^2}{\phi^2 - \phi_3^2} \right)^m \left( \int_0^1 \psi_3(\xi) d\xi \right)^2 \right] \right\} (A2)$$

At $\phi = 0$, this becomes

$$\frac{1}{\phi} \frac{K_1(\phi)}{K_n(10)} \bigg|_{\phi = 0} = (1+\rho) \left\{ \sum_{m=1}^{\infty} \left[ \frac{1}{\phi^2} \left( \frac{\phi^2 - \phi_3^2}{\phi^2 - \phi_3^2} \right)^m \left( \int_0^1 \psi_3(\xi) d\xi \right)^2 \right] \right\}$$

$$= 2\pi C(1+\rho) \sum_{m=1}^{\infty} \frac{1}{\phi^2} \left( \int_0^1 \psi_3(\xi) d\xi \right)^2 \left( \int_0^1 \psi_3(\xi) d\xi \right)^2$$

$$= 2\pi C(1+\rho) \sum_{m=1}^{\infty} \frac{1}{\phi^2} \left( \int_0^1 \psi_3(\xi) d\xi \right)^2$$

(A3)

The infinite series in Eq. A3 can be replaced by the closed-form expressions given in chapter V. When the latter are substituted into Eq. A3, one obtains

$$\frac{1}{\phi} \frac{K_1(\phi)}{K_n(10)} \bigg|_{\phi = 0} = 2\pi C \left( \frac{\rho}{10} - \frac{\sigma^2}{24} (1+\rho) \right) \frac{(\rho/10) - (\sigma/24)}{1+\rho}$$

(A4)
Fig. 1 - Dynamic Stiffness of an Undamped Cable with $\rho=2.19$ and $\sigma=0.10$
(a) Initial Form

(b) First Alternate Form

(c) Second Alternate Form

Fig. 2 - Model
Fig. 3 - Frequency Response Curves for Undamped System with $K_t/K_R(0)=0$ and $p/\omega_0=1$
Fig. 4 - Frequency Response Curves for Undamped System with $K_t/K_R(0)=0$ and $p/\omega_o=2.5$
Fig. 5 - Frequency Response Curves for Undamped System with $K_t/K_R(0)\approx 5$ and $\omega_0 = 2.5$
Fig. 7 - Imaginary Component of Dynamic Stiffness for Cable with $p=2.19$, $\sigma=0.10$ and $\xi=10\%$
Fig. 3 - Frequency Response Curves for First Mode of Guyed Tower; $K_c/K_R(0)=1/23$, $p/\omega_o=1.41$ and $c_x=0$
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Fig. 11 - Frequency Response Curves for Second Mode of Guyed Tower;
$K_e/K_R(0)=191$, $p/\omega_n=7.84$ and $\xi_e=5\%$
Fig. 12 - Frequency Response Curves for Symmetric, x-Direction of Guyed Tower Considered in Chapter III, cont'd on next page
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Fig. 13 - Evaluation of Natural Frequencies of Tower-Cable System
CHAPTER EIGHT

METHOD FOR TRANSIENT ANALYSIS OF A GUYED TOWER AND CORRESPONDING COMPUTER PROGRAM

Introduction
Method of Analysis
Details of Implementation for Computer Program
References
Appendices
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INTRODUCTION

Objectives

The objectives of this chapter are:
1. To describe an efficient method for evaluating the dynamic response of guyed tower systems subjected to dynamic excitations; and
2. To describe a computer program developed to implement this analysis.

The dynamic excitation may be due to wave, current, wind and/or earthquakes.

Structure Considered

The system considered is sketched in Fig. 1 and further described below.

Tower. The tower is a space frame, constrained at its base by the foundation system and near the top by the guying cables. It supports at its top a nearly rigid platform of large mass, and is subjected to gravity and buoyancy forces, in addition to the lateral forces. The response of the tower itself is presumed to remain within the linearly elastic range of deformations.

The method of analysis is described for a two-dimensional frame. It will later become apparent that the method of analysis developed is equally valid for a three-dimensional structure as well. In the development of the computer program, the tower is represented by a stick-like model.
**Guying System.** Each guying cable exerts a nonlinear and motion-dependent constraint on the tower. In the development of both the method of analysis and the computer program, this constraint is represented by nonlinear elastic relationships between the reaction and the displacement of the top of each cable.

**Foundation System.** The foundation system also exerts a nonlinear and motion-dependent constraint on the tower. It is assumed, in the development of both the method of analysis and the computer program, that the constraint is linearly elastic. The effect of a nonlinear elastic foundation constraint may be accounted for in a manner analogous to the one used to account for the guying system.

**Excitations.** The excitation forces discussed here may be classified into:

1. The dead weight of the structure and the deck, and associated buoyancy forces;
2. The time-independent excitation forces (forces due to wind and current are considered to be of this type);
3. The time- and motion-dependent exciting forces, such as those induced by wave action.

The dependence of the fluid forces on the motion of the structure is not considered in the computer program.
METHOD OF ANALYSIS

Equation of Motion

Let the 2-dimensional discrete model of the guyed tower (Fig. 2a) be defined by its mass matrix \([M]\), its damping matrix \([C]\), and its stiffness matrix \([K]\). The structure is presumed to be subjected to an external force vector \(\{F\}\) arising from wind and fluid excitation. The equation for the displacement vector \(\{x\}\) may then be described as

\[
[M]\ddot{x} + [C]\dot{x} + [K]x = \{F\}
\]

(1)

where a dot indicates differentiation with respect to time. This equation is nonlinear both in the stiffness matrix \([K]\) and the force vector \(\{F\}\).

**Stiffness Matrix.** The stiffness matrix is response-dependent, i.e. nonlinear, because of the nonlinear behavior of the guying system and because of the axial forces acting through displacements (second order effects). Foundation nonlinearities, if considered, would also affect the stiffness matrix.

**Mass and Damping Matrices.** The mass and damping matrices are taken as constants. The mass matrix is obtained by lumping the structural mass, entrained mass of water and hydrodynamic added mass at the individual nodes.

The damping matrix is usually specified indirectly through modal damping factors.

**Force Vector.** The force vector is response-dependent
because the fluid forces depend on the structural velocity \( \ddot{x}(t) \).

The fluid forces \( \{ F_z(t) \} \) acting on the structure are functions of the fluid particle velocities and accelerations, \( \{ u(z,t) \} \) and \( \{ \dot{u}(z,t) \} \), respectively, and of the structural velocities \( \{ \ddot{x}(t) \} \) and accelerations \( \{ \dddot{x}(t) \} \). These quantities further depend on time, \( t \), and on depth, \( z \), below the mean water level, where the forces are computed. The fluid force at a specified node is given by Morison's equation (1), as follows:

\[
F_z(z,t) = (c_i - 1) \rho V \left( \dot{u}(z,t) - \ddot{x}(t) \right) + \rho V \dot{u}(z,t) + \frac{1}{2} c_d \rho A (\dot{u}(z,t) - \ddot{x}(t)) \cdot |u(z,t) - \dot{x}(t)| \quad (2a)
\]

where \( c_i \) is the inertia coefficient, \( c_d \) is the drag coefficient, \( \rho \) is the fluid mass density, and \( A \) and \( V \) denote, respectively, the projected structural area normal to the direction of the fluid flow, and the structural volume. Both the area and the volume are lumped at the node at which the force is computed. If the term in Eq. 2a indicating the structural acceleration is incorporated in the left hand side of the equation of motion, the following expression is obtained for the fluid force:

\[
F_z(z,t) = c_i \rho V \dot{u}(z,t) + \frac{1}{2} c_d \rho A (\dot{u}(z,t) - \ddot{x}(t)) \cdot |u(z,t) - \dot{x}(t)| \quad (2b)
\]

This fluid force, along with any relevant wind force, is introduced on the right hand side of Eq. 1.

The fluid velocity \( u(z,t) \) in Eqs. 2 is the sum of the
time-independent current velocity, \( u_c(z) \), and the time dependent wave-induced fluid velocity, \( u_w(z,t) \). Thus

\[
\begin{align*}
\omega(z,t) &= u_c(z) + u_w(z,t) \\
\text{(3a)}
\end{align*}
\]

The fluid force \( F^*_f(z,t) \) can consequently be decomposed into a time-independent current force

\[
\frac{1}{2} c_d \rho A \omega(z) \cdot |\omega(z)| \\
\text{(3b)}
\]

and a time-dependent component

\[
\begin{align*}
&c' \rho V \omega(z,t) + \frac{1}{2} c_d \rho A \left( \omega(z,t) \cdot |\omega(z)| \right) \\
&\quad - \frac{1}{2} c_d \rho A u_c(z) |\omega(z)| \\
&\quad - \frac{1}{2} c_d \rho A u_c(z) |\omega(z)| \\
\text{(3c)}
\end{align*}
\]

Methods of Solution

The methods available for the analysis of systems of the type under investigation can be classified into three groups:

1. Analytical solutions;
2. Direct integration methods; and
3. Modal analysis methods.

**Analytical Solutions.** Formal analytical solutions are possible only for simple systems with special types of nonlinearities. They are impractical for a complex problem of the type considered here.

**Direct Integration Methods.** Direct integration methods are widely used in the analysis of the response of nonlinear systems. In these methods, the coupled equations of motion are integrated directly without trying to uncouple them by
transformation into modal coordinates.

**Modal Analysis.** Modal methods of analysis are widely used in linear problems and, strictly speaking, they are not valid when nonlinearities are involved. However, they can be effectively employed in combination with the incremental method or the method of pseudo-forces (also known as the method of equivalent forces, or of imaginary reactions). In the incremental method, a modal analysis is carried out over each time-step, the mode shapes and natural frequencies being constantly updated. In the pseudo-force method, a linearized structure, without any nonlinear elements, is considered. The effects of the nonlinear elements are accounted for by a set of equivalent external forces.

**Comparison.** A method of analysis of the modal type is of value here because the major sources of structural nonlinearities are localized at the level of the guying system and possibly at the foundation.

The pseudo-force method has been chosen over the incremental method for the following reasons. For the class of problems considered, the major disadvantage of the incremental method of analysis is the necessity of updating the natural frequencies and mode shapes of vibration during each time step. Further, the nonlinearities arising from the fluid forces are not accounted for and must still be taken care of in an alternate way. In the pseudo-force method, the same natural frequencies and modes of vibration are used during the entire
analysis, and all the structural, geometric and the force nonlinearities are lumped on the right hand side of the equation of motion, i.e. treated as an excitation. Further discussion of this method may be found in references (2) to (4).

**Pseudo-Force Method**

The pseudo-force method requires the decomposition of the response-dependent stiffness matrix $[K(x)]$ into a constant (or linear) part $[K]_e$ and a response-dependent (or non-linear) increment $[\Delta K(x)]$, i.e.

$$[K] = [K]_e + [\Delta K]$$

(4)

Once this operation is performed and Eq. 4 substituted into Eq. 1, the following equation of motion, linear in its left hand side, can be obtained

$$[M]\dddot{x} + [C]\dot{x} + [K]_e x = [F] - [\Delta K] x$$

(5)

The linearization process of the stiffness matrix is explained below.

**Linearization of Stiffness Matrix**

Let $[K_t]$, $[K_c]$ and $[K_g]$ define, respectively, the following:

1. The linear stiffness matrix of the tower without the guying system resting on a linear elastic foundation. (The secondary effects of the axial forces are neglected in formulating this matrix);

2. The nonlinear stiffness matrix accounting for the effect of the cable constraint; and

3. The geometric stiffness matrix accounting for the
second order effects of the axial forces. In an 'exact' approach, \([K_t]\) and \([K_g]\) are computed as a unit, as explained later.

These matrices are related to the stiffness matrix \([K]\) by

\[
[K] = [K_t] + [k_c] + [K_q]
\]

(6)

**Tower Stiffness Matrix.** \([K_t]\) is of size \(n\) by \(n\), where \(n\) is the number of dynamic degrees of freedom of the tower, including its foundation. The matrix is linear and depends only on the properties of the tower frame and the foundation.

**Cable Stiffness Matrix.** \([K_c]\) also is of size \(n\) by \(n\). All its elements are zero, with the exception of those corresponding to the nodes at which a cable is attached to the tower. These latter elements correspond to the nonlinear force-displacement relationships of the cable-top and \([K_c]\) is therefore nonlinear.

This stiffness matrix can be brought into the form

\[
[k_c] = [k_c]_0 + [\Delta K_c]
\]

(7)

where \([k_c]_0\) is linear and \([\Delta K_c]\) is a nonlinear increment. This is done as follows. The force-displacement characteristics of each guying cable, or set of cables, is represented by an arbitrarily chosen, linear component to which a correcting, nonlinear term is added. The first components are then introduced into \([k_c]_0\), while the second ones are introduced into \([\Delta K_c]\).
Geometric Stiffness Matrix. In a first order analysis, the member stiffnesses are evaluated by ignoring the effects of the axial forces acting through the displacements. (In this case \([K_T] + [K_g] = [K_T] \) or \([K_g] = [0] \). In the second order analysis considered herein, the effects of the axial forces are taken into account. For a prismatic member in flexure, the stiffness coefficients appear as trigonometric and hyperbolic expressions and are functions of the value of the axial force in the member. When the axial force is zero, the stiffness coefficients reduce to those used in a first order analysis. The exact second order stiffness coefficients are presented in Appendix A.

Linear Approximation. The exact second order coefficients have relatively complex form and thus are not easy to handle. A first simplification can be made by expanding the expressions for these coefficients by a Taylor Series and retaining only the first two non-zero terms. Of these terms, the first is constant and equal to the first order coefficient whereas the second term is linear in \(N\), the magnitude of the axial force in the member. The stiffness coefficients obtained by this approach are identical to those obtained by the energy procedure using a cubic function for the lateral displacements of the member. The solution involving these simplified coefficients would be termed as 'linear approximation'. These coefficients are also presented in Appendix A.

P-A Approximation. It has been recognized for a long
time that the major contribution to the second order effects in unbraced frames comes from the axial reactions at the ends of the members acting through the end lateral displacements of these members. This recognition has given rise to an approximate method for accounting for these second order effects termed 'P-∆ Approximation'. In this method it is assumed that the axial forces only alter the values of the end-shear forces acting on the member. This method results in stiffness coefficients which are also characterized by the first order term and a linear expression in N. These coefficients are also presented in Appendix A.

**Comparison.** The complexity of the expressions for the stiffness coefficients is the major disadvantage of the exact procedure. Specifically, these expressions do not lend themselves to easy uncoupling of \([K_e]+[K_g]\), as required by the pseudo force method.

The accuracy of the linear and P-∆ approximations has been evaluated by studying a cantilever beam subjected to lateral and axial forces. Results are reported in Appendix B. When computing displacements, both approximations lead to satisfactory results. It is important to point out that the search of critical load of the beam, or of any other structure, would not be as accurate.

When the tower is modelled in form of a stick (Fig. 2b), only lateral degrees of freedom are retained. The first-order stiffness matrix is found by inversion of the flexibility
matrix obtained by application of unit, lateral forces at
selected nodes of the initial system (Fig. 2a). Only the
P-Δ approximation is compatible with this approach, and had
to be chosen in the development of the computer program.

**Decomposition of the Geometric Stiffness Matrix.** The
geometric stiffness matrix \([K_g]\) is nonlinear because it de-
pends on the value of the axial forces \([N]\). These axial forces
are induced by the dead weight and buoyancy forces, cable
reactions, and exciting forces. Dead weight and buoyancy
forces are evidently constant, and arbitrary component of the
vertical cable reaction (for example the one associated with
the position of equilibrium) can also be taken as constant.
The corresponding, constant axial forces may then be denoted
by \([N_0]\) and the remainders by \([\Delta N]\). Thus

\[
[N] = [N_0] + [\Delta N]
\]

\([K_g]\) can then be written as

\[
[K_1] = [K_0] + [\Delta K_g]
\]

where \([K_0]\) is the linear geometric stiffness matrix asso-
ciated with \([N_0]\) and \([\Delta K_g]\) is the nonlinear one associated
with \([\Delta N]\).

**Linearization.** The linearization of the equation of mo-
tion (Eq. 5) is then performed by defining \([K_1]\) as

\[
[K_1] = [K_0] + [K_1]_0 + [K_2]_0
\]

and \([\Delta K]\) as
\[ [\Delta \mathbf{X}] = [\Delta \mathbf{K}_e] + [\Delta \mathbf{K}_d] \]  \hspace{1cm} (10b)

**Combination of Static and Dynamic Analysis, Final Equations**

Eqs. 3 suggest that static, time- and motion-independent forces, \( \{F_{st}\} \), and dynamic, time- and possibly motion-dependent forces, \( \{F_d\} \), excite the system simultaneously. The response to the combined excitation can then best be obtained in the following manner. Let us first rewrite Eq. 5 as

\[ [M] \mathbf{\ddot{x}} + [C] \mathbf{\dot{x}} + [K] \mathbf{x} = \{ F_{st} \} + \{ F_d \} - [\Delta \mathbf{K}] \mathbf{x} \]  \hspace{1cm} (11)

Let us further express the total displacement vector \( \mathbf{x} \) as

\[ \mathbf{x} = \mathbf{x}_{st} + \mathbf{x}_d \]  \hspace{1cm} (12)

where \( \mathbf{x}_{st} \) is the displacement due to the static forces \( \{F_{st}\} \), and \( \mathbf{x}_d \) is the difference between \( \mathbf{x} \) and \( \mathbf{x}_{st} \).

From Eq. 11, \( \mathbf{x}_{st} \) is simply obtained from

\[ [K] \mathbf{x}_{st} = \{ F_{st} \} - [\Delta \mathbf{K} (\mathbf{x}_{st})] \mathbf{x}_{st} \]  \hspace{1cm} (13)

while \( \mathbf{x}_d \) is given by

\[ [M] \mathbf{\ddot{x}} + [C] \mathbf{\dot{x}} + [K] \mathbf{x} = \{ F_d \} + [\Delta \mathbf{K} (\mathbf{x}_{st})] \mathbf{x}_{st} - [\Delta \mathbf{K}] \mathbf{x} \]  \hspace{1cm} (14)

Eqs. 13 and 14 are the equations of motion in their final form, before modal decomposition. The right-hand side of Eq. 14 will be denoted by \( \{F^*\} \), i.e.

\[ \{ F^* \} = \{ F_d \} + [\Delta \mathbf{K} (\mathbf{x}_{st})] \mathbf{x}_{st} - [\Delta \mathbf{K}] \mathbf{x} \]  \hspace{1cm} (15)

One notes that for a linear system, \( \{F^*\} = \{F_d\} \) and \( \mathbf{x}_d \) is then associated with the dynamic forces \( \{F_d\} \) only.
Modal Decomposition

Let \( \{ \phi_j \} \) be the \( j^{th} \) mode of vibration of the linearized structure defined by \([M]\) and \([K]\), and \( p_j \) be the corresponding circular natural frequency. The component of the displacement vector \( \{ x_d \} \) can then be expressed in terms of the modes of vibration \( \{ \phi_j \} \) and the unknown, time-dependent modal coordinates \( q_j(t) \) as

\[
\{ x_d(t) \} = \sum_j q_j(t) \{ \phi_j \}
\]

(16)

The \( j^{th} \) modal equation is then, after introduction of the \( j^{th} \) damping ratio \( \xi_j \)

\[
\ddot{q}_j + 2 \xi_j \omega_j \dot{q}_j + \omega_j^2 q_j = \frac{\{ \phi_j^T \} \{ F^* \}}{\{ \phi_j^T \} [M] \{ \phi_j \}}
\]

(17)

These latter equations, along with Eqs. 13 to 15, are the central equations of the proposed method of analysis.

Integration Procedure

It must be recalled that the force vector \( \{ F^* \} \) on the right-hand side of Eq. 17 depends on the displacements \( \{ x \} \) and the velocities \( \{ \dot{x} \} \), if fluid-structure interaction effects are accounted for. In order to obtain the solution to the equation of motion (Eq. 17) at time \( t+\Delta t \), it is first necessary to predict the values of the displacements at this time and to evaluate the magnitudes of the corresponding forces \( \{ F^* \} \). The equation of motion can then be solved, the forces may be corrected, and the displacements recomputed as necessary. This iterative scheme is as follows:
1. Assume a displacement vector \( \{ x(t+\Delta t) \}^o \) and a velocity vector \( \{ \dot{x}(t+\Delta t) \}^o \) (usually \( \{ x(t+\Delta t) \}^o = \{ x(t) \} \) and \( \{ \dot{x}(t+\Delta t) \}^o = \{ \dot{x}(t) \} \));

2. Obtain the corresponding force vector \( \{ F^*(t+\Delta t) \}^o \);

3. Perform the integration of the modal equations and substitute into Eq. 16;

4. Obtain the total displacement vector \( \{ x(t+\Delta t) \}' \) from Eq. 12, and the corresponding velocity vector \( \{ \dot{x}(t+\Delta t) \}' \);

5. Repeat steps 1 to 3 by replacing \( \{ x \}^o \) by \( \{ x \}' \) and \( \{ \dot{x} \}^o \) by \( \{ \dot{x} \}' \), until convergence is attained.

Two sources of potential difficulties may occur in such a procedure. They relate to the convergence of the iterative scheme, and to the stability of the integration method.

**Convergence.** The aspect of convergence has been discussed by Stricklin and Haisler (5), with reference to a massless, nonlinear spring. Such a system is in fact representative of the type of nonlinearities associated with the guying and the foundation systems. It has been shown by the above authors that convergence of spring deflection is guaranteed if its stiffness (tangent modulus) is less than twice the value of the arbitrarily chosen linear stiffness, \( k_s \), when operating in the normal range of displacements. This condition is required to be met with, even when the first assumed displacement \( x^o \) is close to the exact displacement \( x \). For a softening spring for example, convergence is assured if the initial
stiffness is chosen as \( k_s \).

Geometric nonlinearities also affect the stiffness matrix of the structure. It should be expected that convergence would be achieved if a criterion similar to the one discussed above is applied to the overall stiffness matrix. As geometric nonlinearities are of softening type for compressive forces, convergence should be assured with the method of solution chosen.

Insofar as the force nonlinearities are concerned (fluid structure interaction), it is believed that convergence is achieved when the integration time steps are taken sufficiently small.

**Stability.** The integration method used to predict the displacement is generally not required to fulfill any special requirements since the derived displacements are corrected in the next step. The integration method used to correct the displacements should be stable. As might be difficult, in a nonlinear problem, to make sure that stability criteria be satisfied, the method should be unconditionally stable.

The exact piecewise linear method of integration, i.e. a method based on the exact solution of a single degree of freedom system subjected to a load which varies linearly with respect to time, is considered appropriate in this study and chosen as the method of integration. The resulting expressions are reported in Appendix C. For simplicity of computation this method is chosen both as predictor and
correxter.
DETAILS OF IMPLEMENTATION FOR COMPUTER PROGRAM

Model

The computer program has been written for a stick-like structure resting on a linear elastic foundation. Such a model is shown in Fig. 2b. Only lateral degrees of freedom are considered, i.e. one per node.

The constraint of the guying system is represented in terms of its horizontal and vertical resistance to horizontal displacements. These resistances are shown in Figs. 3a and 3b, respectively. In Fig. 3a, \( R_H \) is the horizontal guying reaction, \( d \) the lateral guying displacement, and \( k_s \) is an arbitrarily chosen stiffness value which is needed to evaluate \( [K_c] \). \( k_s \) must satisfy the criterion for convergence indicated earlier. In Fig. 3b, \( R_V \) is the vertical guying reaction and \( R_{V_0} \) the reaction at rest (i.e. zero displacement).

Exciting forces are assumed to be motion-independent (i.e. the fluid-structure interaction effects are ignored).

Method of Solution

The method of solution is the one described earlier. One has to note that, in this particular case, the treatment of the nonlinear component of the geometric stiffness matrix can be simplified in the following manner.

**Geometric Stiffness Matrix.** The nonconstant axial forces, denoted by \( N \) earlier, are in this particular case equal to \( (R_V - R_{V_0}) \) for all members located below the level of the guying system, and zero for all members located above the
level of the guying system. \[ \Delta K_g \] is then proportional to \( (R_v - R_{v0}) \) and can be rewritten as

\[
[\Delta K_g] = (R_v - R_{v0}) \left[ \Delta K_g(R_v=1) \right]
\]

where \( \Delta K_g(R_v=1) \) has been obtained for an unit vertical cable reaction. It is constant and needs to be computed only once.

**Member Forces**

The member forces may be obtained from equilibrium of all external, inertia and constraining forces acting on the system. As seen in Fig. 4, the moments, shears, and axial forces at node \( i-1 \) and member \( i \) above the guying system are (see notation in Fig. 4):

\[
M_{i-1} = \sum_{j=i}^{top} \left[ (F_j - m_j \ddot{x}_j) (h_j - h_{i-1}) + P_j (x_j - x_{i-1}) \right]
\]  \hspace{1cm} (19a)

\[
Q_i = \sum_{j=i}^{top} \left[ (F_j - m_j \ddot{x}_j) \right] + \frac{x_i - x_{i-1}}{h_i - h_{i-1}} \sum_{j=i}^{top} P_j
\]  \hspace{1cm} (19b)

\[
N_i = \sum_{j=i}^{top} P_j - \frac{x_i - x_{i-1}}{h_i - h_{i-1}} \sum_{j=i}^{top} \left[ (F_j - m_j \ddot{x}_j) \right]
\]  \hspace{1cm} (19c)

The moments are evidently related to the shears by

\[
M_{i-1} = M_i + (h_i - h_{i-1}) Q_i
\]  \hspace{1cm} (19d)

For nodes and members below the guying system, say located at the node \( k \), the horizontal reaction \( R_H \) is to be added to the horizontal forces \( (F_k - m_k \ddot{x}_k) \) and the vertical reaction \( R_V \) is to be added to the vertical force \( P_k \).
COMPUTER PROGRAM

Organization of Program

The program consists of three major parts:

a. Part 1, which defines the system to be considered;

b. Part 2, which evaluates the response of the system to the static forces; and

c. Part 3, which evaluates the response of the system to the combined static and dynamic forces.

System Definition (Part 1). This part performs all operations necessary to obtain the linearized stiffness matrix \([K_o]\). The matrix \([\Delta K_g(R_V=1)]\) is also evaluated here.

The principal informations needed in input are:

- The stiffness matrix \([K_t]\);
- The linear stiffness \(k_s\) for the guying system (needed to obtain \([K_c]\));
- The dead weight and buoyancy forces; and
- The vertical guying reaction at rest, \(R_V\). \([K_g]\) can be obtained from the latter informations.

The following flow chart summarizes the steps involved:

```
[\[K_t\]]
|      |
[\[K_c\]]
|      |
[\[K_s\]]
|      |
[\[K\]]
|      |
[\[\Delta K_g(R_V=1)\]]
```
Static (Part 2). This part solves Eq. 13 for $\{x_{\text{st}}^i\}$. The major informations needed in input are:

- The static exciting forces $\{F_{\text{st}}^i\}$; and
- The nonlinear relationships $R_H$ versus $d$ and $R_V$ versus $d$.

The convergence of the iteration is measured on the lateral displacement $d$ of the guyed system. This is sufficient since all nonlinearities in $\{F^i_{\text{st}}\}$ are generated at the location of the guyed system.

The flow chart is as follows, where $\{F^i_{\text{st}}\}$ represents the right-hand member of Eq. 13:

```
\begin{align*}
\{F^i_{\text{st}}\} &= \{F_{\text{st}}\} \\
\{x_{\text{st}}^i\} &= [K]_{\text{st}}^{-1}\{F^i_{\text{st}}\} \\
[\Delta K(\{x_{\text{st}}^i\})] &= [\Delta K_0(\{x_{\text{st}}^i\}) + (R_V(\{x_{\text{st}}^i\} - R_{\text{st}}) \{\Delta K_0(R_{\text{st}})\}] \\
\{F^i_{\text{st}}\} &= \{F_{\text{st}}\} - [\Delta K(\{x_{\text{st}}^i\})] \{x_{\text{st}}^i\} \\
\{M_{\text{st}}^i\}, \{Q_{\text{st}}\}, \{N_{\text{st}}\}
\end{align*}
```

Dynamic (Part 3). This part first solves the eigenvalue problem necessary for the evaluation of the modes $\{\phi_j^{i}\}$ and the associated frequencies $p_j$. The modal equation, Eq. 17, is then solved and the displacements $\{x_d^i\}$ are obtained from Eq. 16. $\{x_d^i\}$ is then combined with $\{x_{\text{st}}^i\}$ to give the total
displacement vector \( \{x_i\} \) (eq. 12).

The major informations needed in input are:

- The mass matrix \( [M] \);
- The damping ratios \( \zeta_j \); and
- The dynamic, exciting forces \( \{F_d\} \).

The flow chart is as follows:

\[
[K]_m \{\phi_i\} = \omega^2 [M] \{\phi_i\}
\]

\[
\zeta_j
\]

\[
E = E_{star} +
\]

\[
\{ \phi_i \} = \{ x_{s+1} \}
\]

\[
R_V = (R_V)_{s+1}
\]

\[
R_H = (R_H)_{s+1}
\]

\[
\{F_d\} = \{F_d\} + [K]_{x} \{x_{s+1}\} - [K]_{x} \{x_{s+1}\}
\]

\[
\varepsilon = \varepsilon + \Delta \varepsilon
\]
Program Capabilities

A stick model with a maximum of 10 nodes can be analyzed by the program. The maxima of the response quantities are automatically printed, and the time histories of selected response quantities (displacements, moments, shears and cable reactions) may be plotted as desired.

The program accepts input force histories of arbitrary length, which may be specified at unequal time intervals. As the analysis is performed at equal time intervals, $\Delta t$, the forces at these times are computed from the specified values by linear interpolation.

This program has been used to perform the analyses reported in chapter IX.
REFERENCES


APPENDIX A - ELEMENTAL STIFFNESS MATRIX

The stiffness coefficients, defined as the member end forces induced by unit end displacements, are presented for a two-dimensional prismatic beam element. They are reported as a stiffness matrix $K$ with the element $K_{ij}$ being the force in the $i$-direction induced by an unit displacement in the $j$-direction. The detailed derivation of these expressions can be found in numerous publications (6 to 8) and will be only partially reproduced here.

System Definition

Let the symbols $y(x)$ and $v(x)$ denote the lateral and axial displacement of the beam element, respectively. $x$ is the coordinate in the axial direction, measured from the left-hand side of the beam. The characteristics of the beam are its flexural rigidity, $EI$, axial rigidity, $EA$, and length, $L$. The external, compressive axial force, $N$, acts on both ends of the beam. The system is shown in the figure below.

Let also the symbols $u_1$ to $u_5$ denote the axial, lateral and rotational displacements at the left-hand side and right-hand side. Their positive directions are shown below.
The symbols $F_1$ to $F_6$ will denote the corresponding end-forces.

As the stiffness coefficients for the axial direction are not affected by the axial force $N$, only the stiffness coefficients related to lateral and rotational deflections will be presented. Those depend on $y(x)$ only.

**Stiffness Coefficients**

**Members Without Axial Force (N=0).** The governing differential equation for the beam element on which acts no axial force is simply

$$y^{IV} = 0$$

(A1)

where the superscript IV denotes the 4th derivative with respect to the dimensionless distance $\xi = x/l$. Its general solution is

$$y(\xi) = A \xi^4 + B \xi^3 + C \xi^2 + D$$

(A2)

The coefficients A to D are found by substitution of the appropriate boundary conditions. For example, for a lateral displacement $u_2$ only, Eq. A2 becomes

$$y(\xi) = u_2 \cdot [2 \xi^3 - 3 \xi^2 + 1]$$

The end-forces are then found from the relationships below:

$$F_2 = Q(0) = \frac{E}{I} \frac{x}{l^3} u^{''}(0)$$

(A3)

$$F_3 = M(0) = -\frac{E}{I} \frac{x}{l^2} u^{''}(0)$$

(A4)

$$F_4 = Q(1) = -\frac{E}{I} \frac{x}{l^2} u^{'''}(1)$$

(A5)
\[ F_6 = M(1) = \frac{EI}{L^2} \psi''(1) \]  

(A6)

The end-force \( F_3 \) due to \( u_2 \) is then, for example,

\[ F_3 = 6 \frac{EI}{L^2} u_2 \]

The coefficient \( 6 \frac{EI}{L^2} \) is the stiffness coefficient \( k_{32} \). The other coefficients are found in a similar manner.

The first order stiffness matrix is then

\[
K^\nu_w = \begin{bmatrix}
\frac{EA}{L} & 12 \frac{EI}{L^2} & 6 \frac{EI}{L^2} & -12 \frac{EI}{L^2} & 6 \frac{EI}{L^2} & 1 \\
6 \frac{EI}{L^2} & 6 \frac{EI}{L^2} & -6 \frac{EI}{L^2} & 2 \frac{EI}{L} & - \frac{EA}{L} & 2 \\
& 6 \frac{EI}{L^2} & 4 \frac{EI}{L} & -6 \frac{EI}{L^2} & 2 \frac{EI}{L} & - \frac{EA}{L} & 4 \\
& & 6 \frac{EI}{L^2} & 2 \frac{EI}{L} & -6 \frac{EI}{L^2} & 4 \frac{EI}{L} & 1 \\
\end{bmatrix}
\]

Members With Axial Force, Exact. When the axial force \( N \) is present, the governing differential equation is

\[ u^{(4)} + \frac{NL^2}{EI} \psi'' = 0 \]  

(A7)

The general solution of this equation depends on the sign of \( N \).

When \( N \) is positive (compression) the solution is

\[ u(x) = A \sin \lambda x + B \cos \lambda x + C x + D \]  

(A9a)
where
\[ \lambda = \sqrt{\frac{NL^2}{EI}} \quad (A9b) \]

The stiffness coefficients are then obtained as in the previous case. The only difference resides in a new formulation of Eqs. a3 and A5 which become, respectively,
\[ F_2 = Q(o) = \frac{F_T}{L^3} \left[ q''(0) + \lambda^2 q'(0) \right] \quad (A10) \]
\[ F_2 = Q(1) = -\frac{F_T}{L^3} \left[ q''(1) + \lambda^2 q'(1) \right] \quad (A11) \]

The stiffness matrix is then, with the introduction of \( \xi = \sin \lambda \) and \( \zeta = \cos \lambda \):

\[
K^N =
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\frac{EA}{L} & \frac{\lambda^2 s}{L^3 (2-zc-Ls)} & \frac{\lambda^2 - \lambda s}{L^2 (2-zc-Ls)} & \frac{\lambda s}{L (2-zc-Ls)} & \frac{\lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\lambda^2 - \lambda s}{L (2-zc-Ls)} \\
\frac{\frac{F_T}{L^3} \lambda^2 s}{L^3 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L^2 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} \\
-\frac{\frac{F_T}{L^3} \lambda^2 s}{L^3 (2-zc-Ls)} & -\frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L^2 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} \\
\frac{\frac{F_T}{L^3} \lambda^2 s}{L^3 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L^2 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} \\
\frac{\frac{F_T}{L^3} \lambda^2 s}{L^3 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L^2 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} \\
\frac{\frac{F_T}{L^3} \lambda^2 s}{L^3 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L^2 (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} & \frac{\frac{F_T}{L^3} \lambda^2 - \lambda s}{L (2-zc-Ls)} \\
\end{array}
\]

When \( N \) is negative (traction), the solution is
\[ u(x) = A \sinh \nu x + B \cosh \nu x + C x + D \quad (A13a) \]

where
\[ \nu = \sqrt{-\frac{NL^2}{EI}} \quad (A13b) \]
The stiffness matrix is then obtained as, with the introduction of $S = \sin \lambda$ and $C = \cos \lambda$:

$$K^{-N} = \begin{pmatrix}
\frac{E_A}{l} & \frac{E_A}{l} & -\frac{E_A}{l} & & & \\
\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & -\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & & \\
-\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & -\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \\
\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & -\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \\
\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & -\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \\
\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & -\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \\
\frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2} & \frac{E_I}{l^3} \frac{L^2}{2-2d+\lambda^2}
\end{pmatrix}$$

Members With Axial Forces, Linear Approximation. The coefficients of matrices A12 and A14 may be linearized by expanding them into a Taylor Series. This approach is illustrated for the coefficient $K_{16}$ of matrix A12. This coefficient is

$$K_{16} = \frac{E_I}{l} \frac{\lambda^2 - \lambda \sin \lambda}{2 - 2 \cos \lambda - \lambda \sin \lambda} \quad (A15)$$

The Taylor expansions of $\sin \lambda$ and $\cos \lambda$ are well known and can be substituted into Eq. A15, which becomes

$$K_{16} = \frac{E_I}{l} \frac{\lambda^2 - \lambda \left[ 1 - \frac{\lambda^3}{2!} + \frac{\lambda^5}{5!} + \cdots \right]}{2 - 2 \left[ 1 - \frac{\lambda^3}{2!} + \frac{\lambda^5}{5!} + \cdots \right] - \lambda \left[ 1 - \frac{\lambda^3}{2!} + \frac{\lambda^5}{5!} + \cdots \right]} \quad (A16)$$

Retaining only the lower order terms, one obtains
\[ K_{36} \simeq \frac{EI}{L} \left[ \frac{1}{12} - \frac{L^2}{120} \right] \simeq \frac{EI}{L} \left[ 2 + \frac{L^2}{20} \right] \] (A17)

One can note that the first term of the final expression is the first order coefficient. The second term forms, along with its counterpart of the other stiffness coefficients, the geometric stiffness matrix below

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\frac{6}{5} & \frac{1}{2} & \frac{6}{5} & \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} \\
\frac{1}{10} & \frac{2}{15} & \frac{1}{10} & \frac{1}{30} & \frac{2}{15} & \frac{2}{15} \\
\frac{6}{5} & \frac{1}{2} & \frac{1}{10} & -\frac{6}{5} & \frac{1}{2} & \frac{1}{10} \\
-\frac{1}{10} & \frac{1}{30} & \frac{1}{10} & \frac{1}{30} & \frac{2}{15} & \frac{2}{15} \\
\end{array}
\]

The same stiffness matrix is obtained when using matrix A14 instead of A12.

The same coefficients can also be obtained by application of energy principles, as follows. Let us approximate the displacement \( y(z) \) by the cubic displacement function below which satisfies all geometric boundary conditions:

\[ y(z) = u_z \left[ 2z^2 - 3z^2 + 1 \right] + u_1 z \left[ 2z^2 - 2z^2 + z \right] \\
+ u_2 z \left[ 2z^2 - z^2 + z \right] + u_3 z \left[ z^2 - z^2 \right] \] (A19)

The strain energy \( U \) associated with this displacement func-
tion is found from

\[ \Psi = \frac{1}{2} \frac{E I}{L^2} \int_0^L (\ddot{u}^2) \, ds - \frac{1}{2} \frac{N}{L} \int_0^L (\dot{u}^2) \, ds \]  

(A20)

Application of the principle of minimum of potential energy gives then the end forces as

\[ F_i = \frac{\partial \Psi}{\partial u_i} \quad i = 2, 3, 5, 6 \]  

(A21)

Equation A21 represents a relationship between the end forces \( F_i \) and the end displacements \( u_i \), i.e. the desired stiffness matrix.

**Members With Axial Force, P-\( \Delta \) Approximation.** This approximation is obtained by assuming that only shear forces are affected by the axial force, and requiring overall equilibrium of the element. Only the coefficients related to lateral end displacements are then affected, and this as illustrated in the figure below.

The corresponding geometric stiffness matrix \( K_g \) is then

\[ K_g = N. \]  

(A22)
APPENDIX B - CANTILEVER BEAM

The uniform, prismatic beam shown below has been subjected to end lateral and axial forces F and N, respectively. N has been taken equal to 48.25% the critical Euler load, and F as 0.1·N. The end deflection and fixed-end moment have been computed based on the linear approximation, P-Δ approximation and first-order approach, varying the number n of elements in which the beam is subdivided. The results, compared to the exact ones, are reported in the tables below.

\[ F = N \] /10

\[ N = 48.25\% \] \( N_{cr} \) with \( N_{cr} = \frac{
abla^2 x}{(2\pi)^2} \)

**Free End Deflection:**

<table>
<thead>
<tr>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
<th>n=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Approx Error in %</td>
<td>0.3</td>
<td>0.02</td>
<td>0.005</td>
<td>0.002</td>
</tr>
<tr>
<td>P-Δ Approx Error in %</td>
<td>13.6</td>
<td>4.4</td>
<td>2.0</td>
<td>1.2</td>
</tr>
<tr>
<td>1st Order Error in %</td>
<td>47.9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Fixed End Moment:**

<table>
<thead>
<tr>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
<th>n=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Approx Error in %</td>
<td>0.1</td>
<td>0.007</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>P-Δ Approx Error in %</td>
<td>5.9</td>
<td>1.9</td>
<td>0.9</td>
<td>0.5</td>
</tr>
<tr>
<td>1st Order Error in %</td>
<td>43.2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX C - INTEGRATION PROCEDURE

The following description of the procedure used to integrate the governing equations of motion follows the one presented in Ref. 9.

The equation of motion for a SDOF system subjected to a linearly varying exciting force shown below is

\[ \ddot{u}(t) + 2 \zeta \rho \dot{u}(t) + \rho^2 u(t) = f(t) = f_0 + \frac{a_t}{2} t \]  

(C1)

where \( u \) is the displacement, \( \rho \) the circular natural frequency, \( \zeta \) the critical damping ratio, and \( f(t) \) the exciting force at an arbitrary time \( t \) between \( t_0 \) and \( t_1 \).

The displacement of the system at a time \( t = t_0 + \tau \) is given by

\[ u(t) = A_0 + A_1 \tau + A_2 e^{-\zeta \rho \tau} \cos \rho \tau + A_3 e^{-\zeta \rho \tau} \sin \rho \tau \]  

(C2)

where

\[ \rho^* = \rho \sqrt{1 - \zeta^2} \]  

(C3a)

\[ A_0 = \frac{1}{\rho^*} \left[ f_0 - 2 \zeta \frac{A_1}{\rho^*} \right] \]  

(C3b)

\[ A_1 = \frac{1}{\rho^*} \frac{\alpha_t}{\rho^*} \]  

(C3c)

\[ A_2 = u(t_0) - A_1 \]  

(C3d)

\[ A_3 = \frac{1}{\rho^*} \left[ \dot{u}(t_0) + \zeta \rho A_2 - A_1 \right] \]  

(C3e)

The velocity of the system at time \( t \) is then
\[ \dot{\rho} \dot{z} = A_1 + \left[ \rho^2 A_3 - 5\rho A_2 \right] e^{-5\rho z} \cos \rho z - \left[ \rho^2 A_2 + 5\rho A_3 \right] e^{-5\rho z} \sin \rho z \] (C4)

and the acceleration is determined from Eq. C1 as

\[ \ddot{\rho} \dot{z} = \rho_0 + \frac{\partial \rho}{\partial \rho} \dot{z} - 25 \rho \dot{\rho} \dot{z} - \rho^2 \dot{\rho} \dot{z} \] (C5)
Fig. 1 - Grued Tower
Fig. 2 - Models of Guyed Tower
Fig. 3a - Lateral Reaction of Guying System

Fig. 3b - Vertical Reaction of Guying System
Fig. 4 - Member Forces
CHAPTER NINE

RESPONSE OF GUYED TOWER TO LATERAL EXCITATIONS

Introduction
Model and Studies of Free Vibration
Studies of Forced Vibration
Conclusions
Tables
References
Figures
INTRODUCTION

A thorough understanding of the behavior of guyed tower systems requires studies of their response to wind, current and waves. The objectives of this Chapter are (1) to present the results of some exploratory studies of this type, emphasizing the effects of selected nonlinearities of the system, and (2) to assess the relative contributions of the various modes of vibration to the overall response.

The response quantities evaluated include displacements, moments, shearing and axial forces, and the reactions of the cable system. Both static and dynamic excitations are considered. The static excitations include forces due to wind and current, while the dynamic excitations are wave-induced forces, represented either by a single harmonic function or a combination of a large number of harmonic functions of varying amplitudes and random phase angles.

It is shown that the deflections of the upper part of the structure are essentially controlled by the fundamental mode of vibration, while that of the base is also affected by the higher modes, especially the second one. Member forces are also affected by the second mode. It is further shown that the nonlinearity of the horizontal resistance provided by the guying system can affect the response of the structure. On the other hand, the nonlinearity of the vertical resistance of the cables is of less practical importance, as it can be assumed to be constant and equal to that for the system at rest without inducing significant errors.

The system considered in the analysis is identified first. Then the results of additional studies of the free vibrational characteristics of the tower are presented. The objective of these studies has been to
assess the importance of the so-called P-Δ effect. This material is followed by the presentation and discussion of the response of the tower to static forces, harmonic forces, and transient forces. The data presented were obtained by application of the computer program described in Chapter VIII.
MODEL AND STUDIES OF FREE VIBRATION

System Considered

The characteristics of the model associated with the x-direction of the guyed tower presented in Chapter III are used in this study. The tower and the foundation are assumed to be linearly elastic, their assembly being characterized by the flexibility matrix \( D_{\text{xx}}^{\text{wo}} \) of Chapter III. However, the designation of the nodes has been changed so as to correspond to that used in the computer program described in Chapter VIII. Specifically, node 1 is at the sea bed, node 9 at the top of the structure, and node 6 at the point of attachment of the guying system. A sketch of the model is presented in Fig.1. The horizontal and vertical resistances of the guying system are taken in the form shown in Fig.3 of Chapter III.

For the purpose of the present analysis, additional properties had to be specified. They include the vertical forces due to the structural dead weight and the buoyancy forces. These forces, discretized at the nodes, are listed in Table 1 where a positive sign indicates that the force acts downwards and a negative sign indicates that it acts upwards. The negative forces are caused by the presence of buoyancy tanks along the height of the structure. Also needed for the analyses presented in this Chapter are volumes and projected areas of the structure normal to the x-direction. These quantities are needed for the computation of the fluid forces. The values of these quantities, also discretized at the nodes, are given in the third and fourth columns of Table 1, respectively.

Studies of Free Vibration

Periods. The natural periods of vibration of the model obtained
with due consideration of the P-Δ effects are listed in the second column of Table 2. They were computed by taking a value of 275 kips/ft for the lateral stiffness of the guyng system. The vertical forces due to dead weight and buoyancy were considered, as was a vertical force of 10,330 kips at node 6 corresponding to the reaction of the guyng system at rest.

The periods obtained by disregarding the vertical reaction of the guyng system are listed in columns 3 of Table 2, and the corresponding errors are listed in column 4. The errors are defined as

$$\frac{T_{\text{exact}} - T_{\text{approx}}}{T_{\text{exact}}} \times 100$$

(1)

where $T_{\text{exact}}$ represents a period computed considering all P-Δ effects, and $T_{\text{approx}}$ represents the corresponding period computed with partial or complete disregard of the P-Δ effects.

Similar results are listed in columns 5 and 6 for the model in which no P-Δ effects are considered. The periods are the same as those presented in Fig.5a of Chapter III.

It is clear from the informations presented that the effect of the vertical forces on the fundamental period of vibration is substantial. Disregarding all vertical forces underestimates the fundamental period by 8 percent. It is also clear that the effect of the dead weight and buoyancy forces are more important than those of the vertical reaction of the guyng system.

That the effect of the vertical forces is important is further confirmed by the following analysis of the instability of the system.
Instability. The fundamental critical mode of the system can be approximated by considering the tower to be a rigid bar fixed against deflection at the base and elastically constrained against rotation by a rocking spring of stiffness \( k_{\theta} = 2.73 \times 10^6 \text{kip-ft} \). The bar is further constrained by a lateral spring of stiffness \( k = 275 \text{ kips/ft} \) at the junction of the cables and the tower. For a rotation \( \theta \) of the base, the overturning moment is given by the expression

\[
M_{\text{over}} = \sum_j P_j h_j \theta
\]  

(2a)

where \( P_j \) is the total vertical force, including vertical cable reaction at node \( j \), acting on the \( j^{\text{th}} \) node and \( h_j \) is the height of the \( j^{\text{th}} \) node measured from the base. For the system considered,

\[
M_{\text{over}} = 1.01 \times 10^6 \theta \quad \text{kip-ft}
\]  

(2b)

The corresponding resisting moment, \( M_{\text{res}} \), is given by

\[
M_{\text{res}} = k_{\theta} h^2_6 \theta + k_{\theta \phi} \phi \theta
\]  

(3a)

where the first term represents the contribution of the resistance of the guying system and the second term the contribution of the rocking constraint at the base. Substituting the appropriate values, one obtains

\[
M_{\text{res}} = 6.30 \times 10^6 \theta \quad \text{kip-ft}
\]  

(3b)

The ratio \( M_{\text{res}} / M_{\text{over}} \) represents the factor of safety against instability, and is equal to 6.24. It should be emphasized that this particular ratio is valid only for small deflections. Large deflections would be associated with a decrease in lateral guying stiffness as well as with a slight increase in the vertical reaction of the guying system, and both changes would tend to reduce the factor of safety.
Model for Fundamental Period. The approximate model used here to evaluate the fundamental mode of instability is identical to the one used in Chapter III to approximate the fundamental mode of vibration. The relationship between the fundamental period $T$ computed by disregarding the $P-\Delta$ effects and $\tilde{T}$ obtained by considering these effects is then simply approximated by

$$\tilde{T} = \frac{T}{\sqrt{1 - M_{s\text{ar}}/M_{s\text{eq}}}}$$ (4a)

or

$$\tilde{T} = 1.09 \ T$$ (4b)

The error as defined by Eq. 1 is then

$$\frac{\tilde{T} - T}{T} = 8.23\%$$

a good approximation to the exact value of 8.23% listed in Table 2.

From this simple model it should be clear that the fundamental mode of vibration is the most sensitive one to $P-\Delta$ effects, as confirmed by the values of Table 2.
STUDIES OF FORCED VIBRATION

Excitations

The exciting forces considered in this study are those due to wind, current and wave. The wind and current forces are represented by static forces, whereas the wave forces are time-dependent.

Wind. The wind action is represented by a static force of 200 kips acting at the top of the tower.

Current. The current is represented by a fluid particle velocity, \( u_c \), varying linearly from a maximum of 4 ft/sec at the mean water level to zero at the sea bed. These velocities and the corresponding forces, \( F_c \), for drag coefficients \( c_d = 0.6 \) and \( c_d = 1.0 \), are listed in Table 3. The forces were obtained from Morison's Equation as

\[
F_c = \frac{1}{2} c_d \rho A u_c |u_c|
\]

where \( \rho \) is the fluid mass density equal to \( 1.94 \cdot 10^{-3} \) kips-sec\(^2\)/ft. The forces for \( c_d = 1.0 \) are also shown in Fig.2, along with the wind force.

Wave. A linear Airy wave is considered with its elevation, \( \eta(t) \), represented by

\[
\eta(t) = \frac{2}{\pi} a_i \omega_i \cos(\omega_i t - \theta_i)
\]

where \( a_i \) is the amplitude of the \( i^{th} \) harmonic component, \( \omega_i \) the circular frequency of that component, and \( \theta_i \) is the associated random phase angle.

For deep waters, the wave induced fluid velocity, \( u_w(z,t) \), is given by

\[
u_w (z,t) = \frac{2}{\pi} a_i \omega_i c \exp \left[ -\frac{\omega_i z}{q} \right] \cos(\omega_i t - \theta_i)
\]
where $g$ is the gravitational acceleration equal to 32.17 ft/sec$^2$ and $z$ is the depth measured from the mean water level.

The $i^{th}$ wave amplitude, $a_i$, is obtained from a one-sided wave energy spectrum $S_{nn}(\omega)$ as

$$a_i = \sqrt{2 \cdot S_{nn}(\omega_i) \Delta \omega_i}$$  \hspace{1cm} (8)

where $S_{nn}(\omega)$ is the value of the spectrum at $\omega_i$ and $\Delta \omega_i$ is the frequency increment about $\omega_i$. The spectrum utilized in this study is a Pierson-Moskowitz Spectrum (Ref. 1) expressed in terms of the significant wave height, $H_s$, and the average wave period, $T_o$, as (Ref. 2):

$$S_{nn} = 4\pi^3 \frac{H_s^2}{T_o^2 \omega^5} \exp\left[-\frac{i \pi^3}{T_o^2 \omega^5}\right]$$  \hspace{1cm} (9)

The values of $H_s = 35$ ft and $T_o = 12$ sec were utilized. The spectrum was considered to extend in the range between $\omega = 0.1$ rad/sec and $\omega = 1.90$ rad/sec, and it was discretized using a constant frequency increment of $\Delta \omega = 0.02$ rad/sec. The truncated spectrum is reported in Fig. 3. The first two natural frequencies of the guyed tower investigated are identified in the figure.

The $i^{th}$ phase angle, $\theta_i$, was selected in a random manner. The values of $a_i$ and $\theta_i$ corresponding to the frequencies considered are listed in Table 4.

A harmonic wave of amplitude $a = 35$ ft and period $T = 12.8$ sec ($\omega = 0.49$ rad/sec) is also utilized in the present study. These values of $a$ and $T$ have been obtained from a publication by The American Petroleum Institute (3) for a deep water site in the Gulf of Mexico.
The total fluid force, $F_f$, is obtained from Morison's Equation (Eq. 2b of Chapter VIII) in which interaction with structural velocity is neglected, as

$$F_f = c_i \rho V \omega + \frac{1}{2} c_d \rho A (u_e + u_w) |u_e + u_w|$$  \hspace{1cm} (10)

where $c_i$ is the inertia coefficient taken equal to 2. The wave induced force, $F_w$, is defined as the difference between $F_f$ and $F_c$ as

$$F_w = c_i \rho V \omega + \frac{1}{2} c_d \rho A (u_e + u_w) |u_e + u_w| - \frac{1}{2} c_d \rho A u_e |u_e|$$  \hspace{1cm} (11)

Ramp. Both the transient and the harmonic waves have been computed for 340 sec and defined at increments of 0.75 sec. In the analysis, the wave forces are considered to vary linearly within each increment. A progressive application of the wave forces has been simulated by the introduction of a ramp, defined as follows, in the first 40 sec of the excitation.

The value of the force vector $\{F_w(t)\}$ was computed at $t=40$ sec. The variation of this vector between $t=0$ and $t=40$ sec was considered to be linear, starting from $\{F_w(0)\} = \{0\}$.

Static Behavior

Units. All reported quantities, in this and forthcoming sections, are in consistent units of foot, kip and second.

The reported shear is defined as the component of the resulting member force normal to the displaced member axis. This is the quantity used in design. The axial force is similarly defined as the component in the direction of the axis, while the additional vertical cable-force is the increment beyond the value of the reaction at zero deflection.
Wind Induced Response. Response quantities induced by a static wind force of 200 kips are listed in Table 5. The displacements are also reported in Fig. 4.

It can be seen that the resulting deflections are small compared to the height of the structure, of 1.10 ft at the top. Also, the displacement pattern is for all purposes identical to a straight line. For these deflections, the lateral guying system operates in its linear range while the increment in vertical cable reaction is only 4 kips. This corresponds to an increment of only 0.04% of the value at zero deflection.

Current Induced Response. Similar results have been obtained for the same system subjected to static current forces of Table 3, for $c_d = 1.0$. They are listed in Table 6, the displacements being also shown in Fig. 4. It is observed that the deflections are significantly larger than those induced by wind. In particular, it is 13.7 ft at the top of the tower, or 12 times larger.

Under the action of the current, the lateral guying system also operates in its linear range, and the increment in vertical cable reaction is 2.1% of the vertical cable reaction at zero deflection.

Combined Response. Combined effect of the static wind and current forces ($c_d = 1.0$) are given in Table 7, the deflections being also shown in Fig. 4. One recognizes by comparison with Tables 5 and 6 that the deflections, moments, shearing forces and lateral guying reaction are principally given by superposition of the corresponding quantities obtained for wind alone and current alone. This is due to the fact that the lateral guying stiffness is still in its linear range, and that the increment in vertical cable reaction is only 2.5%.
One notes also that the axial forces remain practically unchanged from one case to another, as they originate primarily from the vertical forces, i.e. weights, buoyancies and vertical cable reaction.

**Vertical Cable Reaction.** That the increment in vertical guying reaction is only 2.5% in the combined excitation suggests that no significant errors would be induced if disregarding this increment. The response to the combined excitation obtained by assuming a constant vertical cable reaction of 10,330 kips, the value at zero deflection, are listed in Table 8. One sees that no significant discrepancies are induced by such an approximation, except for a small difference in the value of the base shear, and, of course, for the vertical cable reaction.

**P-A Effect.** The vertical forces greatly affect the response, as illustrated by the quantities listed in Table 9. They were obtained for the combined excitation under the assumption of no vertical forces being present. One sees in particular that the deflection at the top is 12.4 ft, compared to 14.8 ft when all vertical forces are considered. One can also note the differences in shearing forces originating from this disregard.

**Rigid Bar Model.** For all three cases investigated, the displacement pattern is given approximately by a straight line. One can indeed use the elastically constrained rigid bar model referred to previously to estimate the deflections.

The overturning moments of the external forces are

\[ (M_{\text{ext}})_{\text{wind}} = 3.39 \times 10^5 \text{ kips-ft} \]

\[ (M_{\text{ext}})_{\text{current}} = 4.29 \times 10^5 \text{ kips-ft} \]
\[(M_{ext})_{Combined} = 4.63 \times 10^6 \text{kip} \cdot \text{ft}^2\]

Equating the external overturning moment \(M_{ext}\) to the resisting moment \(M_{res}\) of Eq. 3b leads to the first order rigid rotation from which the first order top displacement is obtained as

\[d_{top}^\circ = h_{top} \cdot \frac{M_{ext}}{6.30 \times 10^3}\]  \hspace{1cm} (12)

where \(h_{top} = 1.695 \text{ ft}\) is the distance from the base to the top. Introducing the appropriate values, one obtains

\[(d_{top}^\circ)_{wind} = 0.91 \text{ ft}\]

\[(d_{top}^\circ)_{current} = 11.55 \text{ ft}\]

\[(d_{top}^\circ)_{combined} = 12.46 \text{ ft}\]

Multiplying these values by the quantity

\[\frac{1}{1 - \frac{M_{over}}{M_{res}}} = \frac{1}{1 - 1.01 \times 10^3/6.30 \times 10^3} = 1.19\]  \hspace{1cm} (13)

gives the following deflections in which effect of the vertical forces is included

\[(d_{top})_{wind} = 1.09 \text{ ft}\]

\[(d_{top})_{current} = 13.75 \text{ ft}\]

\[(d_{top})_{combined} = 14.84 \text{ ft}\]

These values are for all practical purposes identical to their counterpart of Tables 5 to 7, and one sees that the second order deflections are about 19% larger than their first order counterpart.
Response to Single Harmonic Wave and Wind

The analysis of offshore platforms subjected to wave loadings may be performed in a static manner provided the water depth is not too large nor the structure too flexible, in which case an appropriate dynamic analysis has to be performed (Ref. 3). The guyed tower certainly falls in this latter category.

As a first step into an understanding of the transient behavior of the guyed tower, it is of interest to obtain and briefly discuss the dynamic response obtained by subjecting the tower to the wave loading recommended in a static analysis. This is the single harmonic wave introduced earlier, to which the static wind force of 200 kips is added.

\[ c_d = 0.6 \text{ (System 1)} \]. The response of the tower was computed by accounting for all nonlinearities and vertical forces, and using 5% critical damping in all modes. The drag coefficient was taken as \( c_d = 0.6 \), and the response was evaluated for 340 sec at time interval of 0.1 sec.

The time histories of base and top deflections are reported in Figs. 5a and 5b, respectively. One sees that the steady-state top displacement is essentially sinusoidal of period equal to the one of the exciting wave. The steady-state base displacement is also essentially harmonic, but contains several sinusoidal components.

The maximum absolute values of these two deflections are 41.9 ft and 0.86 ft, respectively. Both of them occur in the early part of the response \((t = 48.3 \text{ sec and } t = 55.2 \text{ sec}, \text{ respectively})\), i.e. where the
imposed initial conditions still affect the response. The maximum absolute values of all displacements are reported in Table 10.

\[ c_d = 1.0 \text{ (System 2)} \]. The same analysis was performed using a drag coefficient of \( c_d = 1.0 \). The time histories of base and top displacements are reported in Fig.6a and 6b, respectively. One sees that they differ from those of Figs.5a and 5b only in magnitude. The maximum absolute top displacement is 56.7 ft at \( t = 49.8 \) sec, while the maximum absolute base displacement is 1.16 ft at \( t = 61.8 \) sec. All maximum absolute displacements are reported in Table 11. One may note in this Table that the maximum deflections for the nodes 2 to 9 occur essentially at the same time, while the one of the base (node 1) occurs at a totally different time. This shows that the same modes (shown later to be the fundamental one) essentially control the deflections of nodes 2 to 9, while other modes also contribute to the deflection at the base.

For sake of completeness, additional time histories of deflections and member forces are shown in Figs.6c to 6l. Node 3 is taken as the mid-span node in Figs.6c and 6f.

Comparison. Let introduce a normalizing moment \( M_{\text{norm}} \) so as to facilitate the interpretation of the results. It is defined as the maximum absolute value of the overturning moment of the wave-induced forces \( \{F_w(t)\} \) with respect to the base. If \( \{h\} \) is the elevation of each node measured from the base, it is

\[ M_{\text{norm}} = \text{Maximum} \left| \sum h_i^T \bar{F}_w(t) \right| \]  \hspace{1cm} (14)

This quantity has been chosen for normalization purposes on the basis of
the studies performed in the early part of this Chapter, where the importance of the overturning moment was recognized. The values of \( M_{\text{norm}} \) for the harmonic wave are 1.79 \( \cdot 10^7 \) kips-ft and 2.56 \( \cdot 10^7 \) kips-ft for values of \( c_d = 0.6 \) and \( c_d = 1.0 \), respectively. The depth-wise variation of the forces at the time of \( M_{\text{norm}} \) is reported in Fig. 7. One may note from the increase in \( M_{\text{norm}} \) when changing the drag coefficient that the drag forces are, in term of magnitude, more important than the inertia forces. This is of the approximate order of 2 to 1 for \( c_d = 0.6 \), and 3 to 1 for \( c_d = 1.0 \).

Applying \( M_{\text{norm}} \) statically on the rigid bar model introduced earlier, one obtains the static, second order deflections at the top of

\[
(d_{\text{top}}^{5-5})_{c_d = 0.6} = 57.3^2 \quad \text{Ft}
\]

\[
(d_{\text{top}}^{5-5})_{c_d = 1.0} = 82.03 \quad \text{Ft}
\]

Assuming that the wave induced forces are sinusoidal of period \( T_w = 12.8 \) sec, the steady-state top deflections are obtained by multiplying the deflections above by the amplification factor

\[
A.F. = \frac{1}{\sqrt{[1 - (\frac{T_1}{T_w})^2]^2 + 4\zeta^2 (\frac{T_1}{T_w})^2}} = 0.24
\]  \hspace{1cm} (15)

where \( T_1 = 29.1 \) sec is the fundamental period of the structure, accounting for second order effects, and \( \zeta = 5\% \) is the ratio of critical damping. The deflections are then

\[
(d_{\text{top}}^{5-5})_{c_d = 0.6} = 13.74 \quad \text{Ft}
\]

\[
(d_{\text{top}}^{5-5})_{c_d = 1.0} = 19.65 \quad \text{Ft}
\]
The total deflection, for steady-state condition, is then found by adding to the quantities above the corresponding wind induced static deflection of 1.09 ft. This gives

\[
(d_{\text{top}})_{c_d=0.6} = 14.83 \text{ ft} \\
(d_{\text{tip}})_{c_d=1.0} = 20.74 \text{ ft}
\]

These deflections compare extremely well with the peak magnitudes on the right of Figs. 5b and 6b, respectively.

**Response to Transient Wave and Wind**

The tower response has been evaluated for the structure subjected to the transient wave excitation introduced earlier and to the static wind force of 200 kips. A value of \(c_d=1.0\) has been chosen in the computation of the fluid forces. The corresponding value of \(M_{\text{norm}}\) (Eq. 14) is

\[
M_{\text{norm}} = 1.98 \cdot 10^7 \text{ kips-ft}
\]

occurring at \(t=298.75\) sec. The depth-wise distribution of the wave-induced forces, at that particular time, is shown in Fig.8.

**Systems.** The response of the tower was evaluated for 340 sec at time intervals of 0.1 sec, using 5% critical damping in all modes. Different systems were investigated, each one being characterized by a specific change in system property as summarized below.

<table>
<thead>
<tr>
<th>System:</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Effects</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 Mode</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Modes</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
System: 3 4 5 6 7 8

\[ P_{\text{vert}} = 0 \]
\[ P_{\text{vert,cables}} = Cst \]
Linear

All Effects (System 3). Time histories of response for the tower in which all nonlinearities and P-\( \Delta \) effect are accounted for are presented in Figs. 9a to 9l. Node 3 is considered to be the mid-span node in Figs. 9b and 9f. Maximum values of all response quantities are also listed in Table 12, and their depth-wise distribution further shown in Fig. 10.

One may conclude, from these results, that the deflections in all but the base node are essentially controlled by the fundamental mode of vibration. Higher modes obviously participate to the deflection of the base. Moments and shearing forces are affected by the higher modes to a certain extent, with the exception of the base moment which appears to be essentially controlled by the fundamental mode.

Modal Contributions (Systems 4 and 5). Because of the presence of nonlinearities in the system, contribution of the individual modes of vibration has been evaluated as follows. The modes utilized have been those of the system at zero deflection, accounting for the effect of the vertical forces at that position. Deflections have then been computed in a traditional manner, while member forces have been computed from equilibrium of external and inertia forces. This corresponds to accounting in a 'static' manner for the component of the external forces not mobilizing the modes considered in the analysis.
Time histories obtained by considering the fundamental mode only and both the fundamental and second modes are shown in Figs.11a to 11l and 12a to 12l, respectively. Maximum values are listed in Tables 13 and 14.

These solutions confirm that the deflections of all but the base node are controlled by the fundamental mode and that the base deflection is further significantly affected by the second mode. They also show that, besides the fundamental mode, the second mode also contribute to the member forces. This point is more clearly illustrated in Fig.13 where the depth-wise distribution of the maximum values is shown. Selected maximum values are also compared in the Table below.

<table>
<thead>
<tr>
<th></th>
<th>1 Mode</th>
<th>2 Modes</th>
<th>All Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Deflections, ft</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>0.0744</td>
<td>0.8688</td>
<td>0.8118</td>
</tr>
<tr>
<td>Top</td>
<td>24.13</td>
<td>23.74</td>
<td>23.75</td>
</tr>
<tr>
<td><strong>Moments, 10^6 kips-ft</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>1.163</td>
<td>1.183</td>
<td>1.181</td>
</tr>
<tr>
<td>Node 3</td>
<td>2.289</td>
<td>2.153</td>
<td>2.233</td>
</tr>
<tr>
<td><strong>Shears, 10^3 kips</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>2.497</td>
<td>5.029</td>
<td>5.117</td>
</tr>
<tr>
<td>Below Cables</td>
<td>6.547</td>
<td>5.667</td>
<td>5.876</td>
</tr>
<tr>
<td>Top</td>
<td>2.383</td>
<td>2.787</td>
<td>2.831</td>
</tr>
<tr>
<td><strong>Guying Reactions, 10^3 kips</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal</td>
<td>5.440</td>
<td>5.410</td>
<td>5.409</td>
</tr>
<tr>
<td>Vertical</td>
<td>10.888</td>
<td>10.886</td>
<td>10.886</td>
</tr>
</tbody>
</table>
Vertical Forces (Systems 6 and 7). Effect of the vertical forces has been evaluated by computing the response under the assumption of the absence of any vertical forces. The maximum values obtained in that manner are listed in Table 15, and compared to the exact values in Fig. 14 and in the Table below. One sees that differences occur primarily in the shearing forces, especially at the base and near the cables. Deflections and base moment are also affected, though to a lesser extent.

If one accounts for the effect of the vertical forces but assumes that the vertical cable reaction remains constant, one obtains the maximum values listed in Table 16 and compared below. One sees that they are practically identical to the exact values.

<table>
<thead>
<tr>
<th></th>
<th>P&lt;sub&gt;vert&lt;/sub&gt; = 0</th>
<th>P&lt;sub&gt;v, cable&lt;/sub&gt; = C&lt;sub&gt;t&lt;/sub&gt;</th>
<th>All Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Deflections, ft</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>1.0236</td>
<td>0.8126</td>
<td>0.8118</td>
</tr>
<tr>
<td>Top</td>
<td>24.23</td>
<td>23.74</td>
<td>23.75</td>
</tr>
<tr>
<td><strong>Moments, 10&lt;sup&gt;6&lt;/sup&gt; kips-ft</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>1.335</td>
<td>1.182</td>
<td>1.181</td>
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<tr>
<td>Node 3</td>
<td>2.232</td>
<td>2.233</td>
<td>2.233</td>
</tr>
<tr>
<td><strong>Shears, 10&lt;sup&gt;3&lt;/sup&gt; kips</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>6.140</td>
<td>5.122</td>
<td>5.117</td>
</tr>
<tr>
<td>Below Cables</td>
<td>6.381</td>
<td>5.876</td>
<td>5.876</td>
</tr>
<tr>
<td>Top</td>
<td>2.754</td>
<td>2.832</td>
<td>2.831</td>
</tr>
<tr>
<td><strong>Guying Reactions, 10&lt;sup&gt;3&lt;/sup&gt; kips</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal</td>
<td>5.508</td>
<td>5.408</td>
<td>5.409</td>
</tr>
<tr>
<td>Vertical</td>
<td>10.330</td>
<td>10.886</td>
<td></td>
</tr>
</tbody>
</table>
Linear Guying System (System 8). The structure is linear if the vertical cable reaction remains constant and if the horizontal cable resistance is linear. This is because the vertical forces at zero deflection have already been included in the linearized stiffness matrix.

Assuming the above conditions to be true, one obtains the maximum values listed in Table 17 and further compared to the exact values in the Table below. One sees that practically no errors are induced by such a linearization. This is due to the fact that, for the operative range of cable deflection, the guying system differs only slightly from its linearized counterpart.

<table>
<thead>
<tr>
<th></th>
<th>Linear</th>
<th>All Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Deflections, ft</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>0.8139</td>
<td>0.8118</td>
</tr>
<tr>
<td>Top</td>
<td>23.57</td>
<td>23.75</td>
</tr>
<tr>
<td><strong>Moments, 10^6 kips-ft</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>1.199</td>
<td>1.181</td>
</tr>
<tr>
<td>Node 3</td>
<td>2.234</td>
<td>2.233</td>
</tr>
<tr>
<td><strong>Shears, 10^3 kips</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Base</td>
<td>5.129</td>
<td>5.117</td>
</tr>
<tr>
<td>Below Cables</td>
<td>5.868</td>
<td>5.876</td>
</tr>
<tr>
<td>Top</td>
<td>2.874</td>
<td>2.831</td>
</tr>
<tr>
<td><strong>Guying Reactions, 10^3 kips</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal</td>
<td>5.741</td>
<td>5.409</td>
</tr>
<tr>
<td>Vertical</td>
<td>10.330</td>
<td>10.886</td>
</tr>
</tbody>
</table>
Response to Transient Wave, Current and Wind

Tower response has been similarly evaluated for the structure subjected to the same transient wave, and to the static wind and current forces introduced earlier. A value of \( c_d = 1.0 \) was chosen in all but one solution (system 14). The characteristics of each system are summarized below.

System:  9  10  11  12  13  14
All Effects  X
2 Modes  X
\( P_{\text{vert}} = 0 \)  X
\( P_{\text{vert, cables}} = C_{\text{st}} \)  X
Linear  X
\( c_d = 0.6 \)  X

All Effects (System 9). Time histories of response and maximum values computed by accounting for all nonlinearities and \( P-\Delta \) effect are presented in Figs.15a to 151 and Table 18, respectively. The depth-wise distribution of the maxima is shown in Fig.16.

Characteristics of response are similar to those of system 3, obtained for wave and wind excitation only. The only substantial difference is a change in magnitudes, as illustrated by the maxima reported in the Table below.

The increase in magnitudes originates from a change in fluid forces. First the static current forces are present, second the wave induced forces are modified due to the presence of current.
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<tr>
<td><strong>Shears, 10^3 kips</strong></td>
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<tr>
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<td>5.868</td>
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<tr>
<td>Top</td>
<td>2.831</td>
<td>3.198</td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>Horizontal</td>
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</tr>
<tr>
<td>Vertical</td>
<td>10.886 *</td>
<td>11.025 *</td>
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</tbody>
</table>

*: Plateau

\[
M_{\text{norm}} \quad \text{for the wave induced forces is now}
\]

\[
M_{\text{norm}} = 3.07 \times 10^2 \text{ kips-ft}
\]

The associated depth-wise distribution of wave induced forces \( \{F_w\} \), at time of \( M_{\text{norm}} \), is shown in Fig. 8.

Also contributing to the change in magnitudes is the fact that the guying reactions reach their plateau of resistance during the motion.

*Modal Contributions (System 10)*. Considering the first two modes of vibration only leads to conclusions identical to those drawn earlier. The depth-wise distribution of the maxima is compared in Fig. 17.
Vertical Forces (Systems 11 and 12). Depth-wise distribution of maxima for the system in which all vertical forces are disregarded is shown in Fig. 18. Selected values are also compared in the Table below, along with those obtained under the assumption of a constant vertical cable reaction. One sees that the results are similar to those observed earlier for what relative accuracy is concerned.

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<th>( P_{v,\text{cable}} = C^\ell )</th>
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<td>8.600 *</td>
<td>8.600 *</td>
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<tr>
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<td>10.330</td>
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</table>

* : Plateau

**Linear Guying System (System 13).** Assuming a constant vertical cable reaction and a linear horizontal cable resistance gives the maximum values of Table 19 and their depth-wise distribution of Fig. 19. As illustrated by this figure and the Table below, the largest differences
occur in the magnitude of deflections in the upper part of the structure.

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<td>Vertical</td>
<td>10.330</td>
<td>11.025 *</td>
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<tr>
<td>* : Plateau</td>
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\[ c_d = 0.6 \text{ (System 14)} \]. The maximum values of response obtained by computing the fluid forces with \( c_d = 0.6 \) and accounting for all nonlinearities and P-\( \Delta \) effect are listed in Table 20. These results are characterized by a decrease in magnitude of response as illustrated in Fig. 20. These changes are first due to a decrease in magnitudes of fluid forces. \( M_{\text{norm}} \) is now

\[ M_{\text{norm}} = 2.13 \times 10^7 \text{ kips-ft} \]

A second reason for the change in magnitudes is that the guying system
operates now mostly in its 'nearly' linear range.

From the variation in $M_{\text{norm}}$, one may conclude that the drag component of the wave induced forces dominates the inertia component by an approximate ratio of 3 to 1 for $c_d = 1.0$ and 2 to 1 for $c_d = 0.6$, as had been found for the single harmonic excitation.
CONCLUSIONS

The following conclusions can be drawn from the investigations performed in this Chapter.

. When performing a modal analysis as indicated in the text, it is necessary to consider at least the first two modes of vibration. The fundamental mode typically controls the deflections in the upper part of the tower, while the second mode affects the deflection of the base and the member forces;

. The vertical forces affect the fundamental period of vibration and the response of the tower;

. The assumption of a constant vertical cable reaction introduces unsignificant errors; and

. The nonlinearities of the guying system affect the response to various extents depending upon its operating mostly in the 'nearly' linear range or not.
REFERENCES


Table 1 - Additional System Properties

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* : Not applicable

Table 2 - Comparison of Periods of Vibration

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<th>No Vertical Forces Period sec</th>
<th>Error %</th>
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Table 5 - Wind Induced Response (ft, kips)

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| HORIZONTAL CABLES=FORCE | 2.6086180E+02 |
| VERTICAL ADDITIONAL CABLES=FORCE | 4.2159480E+00 |
### Table 6 - Current Induced Response (ft, kips)

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Table 7 - Wind & Current Induced Response (ft, kips)

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**Horizontal Cables=Force**: 3.5712720E+03

**Vertical Additional Cables=Force**: 2.5276350E+02
Table 8 - wind & Current Induced Response by Constant Vertical Cable Reaction (ft, kips)

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<tr>
<td>9</td>
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**HORIZONTAL CARLES=FORCE:** 3.56873500+03

**VERTICAL ADDITIONAL CARLES=FORCE:** 0.0
Table 9 - Wind & Current Induced Response by No Vertical Forces (ft, kips)

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<td>0.000000D+00</td>
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<td>9</td>
<td>2.000000D+02</td>
<td>1.381350D+00</td>
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<td>9</td>
<td>1.381350D+00</td>
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</table>

| HORIZONTAL CABLES = FORCE : | 2.999314D+03         |

| VERTICAL ADDITIONAL CABLES = FORCE : | 0.000000D+00         |
Table 10 - Deflections of System 1 (Single Harmonic Wave and Wind, \(c_d=0.6\)) in ft

**MAXIMUM ABSOLUTE TOTAL DISPLACEMENT:**

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<th>NODE</th>
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<th>TIME</th>
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Table 11 - Deflections of System 2 (Single Harmonic Wave and Wind, \(c_d=1.0\)) in ft

**MAXIMUM ABSOLUTE TOTAL DISPLACEMENT:**

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<td>9</td>
<td>-5.673339 +01</td>
<td>4.980000 +01</td>
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<td>NODE</td>
<td>DISPLACEMENT</td>
<td>TIME</td>
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Table 12 - System 3 (Transient Wave and Wind, c_d=1.0) (ft, kips, sec)
### Maximum Absolute Total Design Shear:

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### Maximum Absolute Total Design Axial Force:

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<td>5.760000D+01</td>
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<td>5.740000D+01</td>
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### Maximum Absolute Horizontal Cables Force:

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### Maximum Absolute Additional Vertical Cables Force:

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Table 12 - Concluded
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<th>NODE</th>
<th>MOMENT</th>
<th>TIME</th>
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Table 13 - System 4 (Transient Wave and Wind, c_d=1.0, 1 Mode) (ft, kips, sec)

cont'd on next page
### Maximum Absolute Total Design Shear:

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### Maximum Absolute Total Design Axial Force:

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<td>5.7900000 +01</td>
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</tr>
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<td>5.7700000 +01</td>
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<td>2.9870000 +02</td>
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### Maximum Absolute Horizontal Cables Force:

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### Maximum Absolute Additional Vertical Cables Force:

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Table 13 - Concluded
### Maximum Absolute Total Displacement:

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<tr>
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### Maximum Absolute Total Moment:

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Table 14 - System 5 (Transient Wave and Wind, c_d = 1.0, 2 Modes) (ft, kips, sec)

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**Maximum Absolute Total Design Shear:**

**Maximum Absolute Total Design Axial Force:**

**Maximum Absolute Horizontal Cables Force:**

**Force:** -5.409801D+03
**Time:** 5.790000D+01

**Maximum Absolute Additional Vertical Cables Force:**

**Force:** 5.558136D+02
**Time:** 5.790000D+01

Table 14 - Concluded
### MAXIMUM ABSOLUTE TOTAL DISPLACEMENT:

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### MAXIMUM ABSOLUTE TOTAL MOMENT:

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Table 15 - System 6 (Transient Wave and Wind, c_d=1.0, No Vertical Forces)
(ft, kips, sec)

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<th>TIME</th>
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Maximum Absolute Total Design Shear:

Maximum Absolute Total Design Axial Force:

Maximum Absolute Horizontal Caries Force:

Maximum Absolute Additional Vertical Caries Force:

Table 15 - Concluded
Table 16 - System 7 (Transient Wave and Wind, \( c_d=1.0 \), Constant Vertical Cable Reaction) (ft, kips, sec)

cont'd on next page
### Maximum Absolute Total Design Shear:

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### Maximum Absolute Additional Vertical Cables Force:

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Table 15 - Continued
**MAXIMUM ABSOLUTE TOTAL DISPLACEMENT:**

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**MAXIMUM ABSOLUTE TOTAL MOMENT:**

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Table 17 - System 8 (Transient Wave and Wind, c_d=1.0, Linear Guying System) (ft, kips, sec)

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Table 17 - Concluded
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Table 18 - System 9 (Transient Wave, Current and Wind, c_d=1.0) (ft, kips, sec)

cont'd on next page
### Maximum Absolute Total Design Shear:

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### Maximum Absolute Total Design Axial Force:

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### Maximum Absolute Horizontal Carles Force:

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### Maximum Absolute Additional Vertical Carles Force:

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Table 18 - Concluded
| NODE | DISPLACEMENT  | TIME  |  | NODE | MOMENT    | TIME  |
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| 1    | -1.077225D+00 | 3.006000D+02 |  | 1    | 1.990614D+06 | 1.483000D+02 |
| 2    | 1.473394D+01  | 1.510000D+02 |  | 2    | 3.210729D+06 | 3.003000D+02 |
| 3    | 2.471764D+01  | 1.514000D+02 |  | 3    | 3.390423D+06 | 3.001000D+02 |
| 4    | 3.453574D+01  | 1.519000D+02 |  | 4    | 2.946384D+06 | 1.468000D+02 |
| 5    | 4.182539D+01  | 1.521000D+02 |  | 5    | 2.011572D+06 | 1.468000D+02 |
| 6    | 4.664874D+01  | 1.522000D+02 |  | 6    | 9.588941D+05 | 1.469000D+02 |
| 7    | 4.910441D+01  | 1.522000D+02 |  | 7    | 4.770720D+05 | 1.523000D+02 |
| 8    | 5.033222D+01  | 1.522000D+02 |  | 8    | 3.449999D+05 | 1.518000D+02 |
| 9    | 5.325713D+01  | 1.522000D+02 |  | 9    | 0.0       | 0.0   |

Table 19 - System 13 (Transient Wave, Current and Wind, c_d=1.0, Linear Guying System) (ft, kips, sec) cont'd on next page
### Maximum Absolute Total Design Shear

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### Maximum Absolute Total Design Axial Force

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### Maximum Absolute Additional Vertical Cables Force

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Table 19 - Concluded
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Table 20 - System 14 (Transient Wave, Current and Wind, c_d=0.6) (ft, kips, sec) cont'd on next page
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**Maximum Absolute Total Design Shear:**

**Maximum Absolute Total Design Axial Force:**

**Maximum Absolute Total Design Horizontal Cables Force:**

**Maximum Absolute Total Design Vertical Cables Force:**

**Table 20 - Concluded**
Fig. 1 - Model
Fig. 3 - Wave Energy Spectrum
Fig. 4 - Static Deflections
Fig. 5a - System 1 (Single Harmonic Wave and Wind, $c_d=0.6$)
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Fig. 6h - System 2
Fig. 8 - Transient Wave Excitation
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Fig. 9c - System 3

DISPLACEMENT AT CABLES
Fig. 9f - System 3
Fig. 10 - Maxima of System 3 (Transient Wave and Wind, $c_d=1.0$)
Fig. 11a - System 4 (Transient Wave and Wind, $c_d=1.0$, 1 Mode)
Fig. 11b - System 4

Displacement at mid-span vs. time
Fig. 111 - System 4
Fig. 12a - System 5 (Transient Wave and Wind, $c_d=1.0$, 2 Modes)
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Fig. 12e - System 5
Fig. 121 - System 5
Fig. 13 - Maxima of Systems 3, 4 and 5 (Transient Wave and Wind, $c_d = 1.0$, resp. all Modes, 1 Mode, and 2 Modes)
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Fig. 16 - Maxima of System 9 (Transient Wave, Current and Wind, $c_d=1.0$)
Fig. 17 - Maxima of Systems 9 and 10 (Transient Wave, Current and wind, $c_d=1.0$, resp. all Modes, and 2 Modes)
Fig. 18 - Maxima of Systems 9 and 11 (Transient Wave, Current and Wind, $c_d = 0.0$, resp. all Vertical Forces, and No Vertical Forces)
Fig. 19 - Maxima of Systems 9 and 13 (Transient Wave, Current and Wind, $C_d=1.0$, resp. Nonlinear, and Linear System)
Fig. 20 - Maxima of Systems 9 and 14 (Transient Wave, Current and Wind, resp. $c_d=1.0$ and $c_d=0.6$)