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SOME APPLICATIONS OF COMPLEX GEOMETRY TO FIELD THEORY

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SOME APPLICATIONS OF COMPLEX GEOMETRY
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ROBERT POOL

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APPROVED, THESIS COMMITTEE:

[Signatures]

Raymond O. Wells, Jr., Professor of Mathematics
Chairman

[Signatures]

James P. Hannon, Professor of Physics

[Signatures]

Robert J. Stanton, Assistant Professor of Mathematics

HOUSTON, TEXAS

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ABSTRACT

Let $\mathbb{M}$ be compactified complexified Minkowski space, $\mathbb{P}$ and $\mathbb{P}^*$ twistor space and dual twistor space, respectively, and $\bar{A}$ ambitwistor space, a complex hypersurface in $\mathbb{P} \times \mathbb{P}^*$. There is a geometric correspondence between $\mathbb{M}$ and $\bar{A}$ such that the points in $\bar{A}$ correspond to the null lines in $\mathbb{M}$. Let $U$ be an open set in $\mathbb{M}$ and $U''$ the open set in $\bar{A}$ which corresponds to $U$ under this correspondence. There exist canonical isomorphisms between the sets of solutions to generalized zero-rest-mass field equations on $U$ and various cohomology groups on $U''$. There is a one-to-one correspondence between Yang-Mills fields on $U$ and vector bundles on $U''$ satisfying a certain triviality condition. Further, a given Yang-Mills field is homogeneous if and only if the corresponding vector bundle on $U''$ can be extended to third order in $\mathbb{P} \times \mathbb{P}^*$, and the axial Yang-Mills current can be canonically identified with the obstruction to third order extension. The action density of the field corresponds to a certain geometric object on $U''$. 
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I. Introduction

In the past few years a great deal of work has been done on transforming the equations of physics to a geometric setting, following a program initiated by Roger Penrose ([18]). Complete solutions to the zero-rest-mass field equations have been provided in this context, as well as solutions of the Dirac equations for massive particles ([21], [4]). Furthermore, it has been shown how the information describing Yang-Mills fields can be encoded in certain vector bundles over complex manifolds ([20], [11]). The solutions to Einstein's equations have proved to be less tractable although some promising results have been obtained ([15]). There are certainly still many more questions than answers in this program. This thesis examines two of these questions.

The original Penrose transform ([16]) and Ward correspondence ([20]) provide techniques to construct solutions to the zero-rest-mass equations and the self-dual Yang-Mills equations, respectively, using data on projective twistor space, a complex manifold isomorphic to \( \mathbb{P}_3(\mathbb{C}) \). A generalization of these constructions to ambitwistor space, a compact complex manifold generalizing projective twistor space, enables one to construct solutions to more general equations. In particular, general-
izing the Ward correspondence to obtain solutions to the general Yang-Mills equations implies that we have the means to include Yang-Mills currents in the description; self-dual fields are necessarily currentless. In this thesis we shall include a new proof of the generalized Ward correspondence which is more geometric than the original proof ([11],[25]). We shall also include a complete description of the axial Yang-Mills currents as third order obstructions to extensions of certain vector bundles. This result was announced in [9], but details of the proof have not yet appeared. The proof presented here depends upon generalizations of the work in [5], which are derived in this thesis, as well as upon a calculational lemma which is proved in detail.

The other subject which this paper addresses is the interpretation of the action of a Yang-Mills field in the above-mentioned geometric setting. The action arises naturally in field theory and is essential in the modern approaches to quantum field theory. If on Minkowski space $M_0$ we have a Yang-Mills-Higgs field given by a connection $A$ and a Higgs field $\phi$, we let $F_A$ denote the Yang-Mills field and $D_A$ the covariant derivative associated to the connection. Suppose that $F_A$ has values in $\mathfrak{g}$, the Lie algebra of the gauge group $G$, and that $\phi$ takes values in $\mathbb{L}$, a representation space for $G$, and let $(\ , \ )$ denote the vector space inner product on $\mathbb{L}$ or $\mathfrak{g}$. Let $\ast$ denote
the Hodge $*$-operator. Then the action $\mathcal{A}(A, \bar{\phi})$ for the given Yang-Mills-Higgs field is
\[
\mathcal{A}(A, \bar{\phi}) = \frac{1}{2} \int_{M_0} \left\{ (F_A,F_A) + (D_A \bar{\phi}, D_A \bar{\phi}) + \frac{\lambda}{4} \left[ (\bar{\phi}, \bar{\phi}) - 1 \right]^2 \right\}
\]
where $\lambda \geq 0$ is a constant. If we restrict to pure Yang-Mills fields then $\bar{\phi} = 0$, $\lambda = 0$ and the action is just
\[
\mathcal{A}(A) = \frac{1}{2} \int_{M_0} (F_A,F_A) = \frac{1}{2} \int_{M_0} \text{tr} \ F_A \wedge^* F_A.
\]
This is the functional we are interested in characterizing in the geometric setting. In this thesis we develop a characterization of the action density $F_A \wedge^* F_A$ as a geometric object on ambitwistor space.

Besides these two above results this thesis contains a generalization of the results of [5] to the ambitwistor setting; these results are basic to any work done with ambitwistors. The paper [5] provides most of the notation used here as well as the motivation for many of the results.

In Chapter II the basic background and notation will be introduced with references to the relevant literature. In particular the ambitwistor correspondence, relating Minkowski space to ambitwistor space via a correspondence manifold, is introduced. Chapter III contains the basic ambitwistor results which will be used throughout the rest of the paper. These include the relationship between cohomology on ambitwistor space and on the correspondence manifold, a relative deRham sequence on the correspondence
manifold, and direct image results relating cohomology on the correspondence manifold to cohomology on Minkowski space. The generalizations of the Penrose transform and the Ward correspondence appear in Chapter IV with proofs and discussion. Here complete sets of holomorphic solutions to generalized versions of the zero-rest-mass equations on Minkowski space are given by various cohomology groups on ambitwistor space; also, a sheaf-theoretic interpretation of a connection on a vector bundle is introduced and used in proving the generalized Ward correspondence. Chapter V is devoted to the description of the correspondence between the axial Yang-Mills currents and obstructions to extensions of certain vector bundles on ambitwistor space. The action of a Yang-Mills field and its representation on ambitwistor space is discussed in Chapter VI.
II. Preliminaries

The purpose of this chapter is to introduce the basic background necessary to the development of the ideas in this thesis and to fix the notation which shall be used throughout. For the most part we shall follow the notation of [5], which is the basic background reference necessary for an understanding of the concepts presented here. There will also be included a summary of the local coordinates which shall be used on the various complex manifolds which appear.

1. The basic fibrations. The fundamental geometric data which shall concern us here are contained in several double fibrations of (open subsets of) certain complex manifolds. Let \( T \) denote twistor space, a four-dimensional complex vector space equipped with a nondegenerate Hermitian bilinear form \( \bar{\xi} \) of signature +,+,−,− (cf. [22], [23]). For the purposes of this paper, we will choose coordinates \( Z^\alpha = (Z^0, Z^1, Z^2, Z^3) \) on \( T \) such that \( \bar{\xi} \) has the form

\[
\bar{\xi}(Z^\alpha) = Z^0 \bar{Z}^2 + Z^1 \bar{Z}^3 + Z^2 \bar{Z}^0 + Z^3 \bar{Z}^1.
\]

We denote by \( T^* \) the dual to \( T \) and choose dual coordinates \( W_\alpha = (W_0, W_1, W_2, W_3) \) so that the induced form \( \bar{\xi} \)
on $\mathcal{T}^*$ has the form

$$\tilde{s}(\omega) = w_0\bar{w}_2 + w_1\bar{w}_3 + w_2\bar{w}_0 + w_3\bar{w}_1.$$ 

Most of the time the alternate notation

$$({w^A,\pi_A}) = (w_0^0, w_1^1, \pi_0, \pi_1) := (Z^0, Z^1, Z^2, Z^3)$$

$$({\eta_A,\xi_A'}) = (\eta_0, \eta_1, \xi_0', \xi_1') := (w_0, w_1, w_2, w_3)$$

will be used here. We note that this is spinor notation, but the significance of this notation will not be important to the ideas developed here and we shall not discuss it (see, for example, [5]).

A **flag manifold** on a complex vector space $V$ is a compact complex manifold which consists of nested sets of linear subspaces of $V$. That is, if $1 \leq i_1 < i_2 < \cdots < i_n < \dim V$ are integers, then a flag manifold of $V$ is defined by

$$\mathcal{F}_{i_1 i_2 \cdots i_n}(V) := \{ L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_n} : L_{i_j} \text{ is a (complex) linear subspace of } V \text{ of dimension } i_j \}.$$ 

For the case $n=1$ we have that $\mathcal{F}_{i}(V)$ is the Grassmannian of $i$-planes in $V$, and if $i=1$, as well, then $\mathcal{F}_{1}(V) = \mathbb{P}(V)$, the projective space of lines in $V$. See [24] for details and a proof that $\mathcal{F}_{i_1 i_2 \cdots i_n}(V)$ is actually a compact complex manifold.

The **twistor correspondence** is a double fibration of flag manifolds on $\mathcal{T}$:
The fibrations $\mu$ and $\nu$ are defined by

$$\mu(L_1, L_2) = L_1$$
$$\nu(L_1, L_2) = L_2,$$

and the correspondence $\tau$ is defined by

$$\begin{align*}
\tau(Z) &= \nu \circ \mu^{-1}(Z) = \mathbb{P}^2(C), \\
\tau^{-1}(z) &= \mu \circ \nu^{-1}(z) = \mathbb{P}^1(C).
\end{align*}$$

The manifold $\mathbb{P}$, projective twistor space, is isomorphic to $\mathbb{P}^3(C)$, by the comments on flag manifolds above. For details on the twistor correspondence, see [22].

The dual twistor correspondence is

$$\begin{align*}
\mathbb{P}^* := \mathbb{P}^2(\mathbb{C}) &\quad \xymatrix{ & \mathbb{P} & } & \mathbb{P}^* := \mathbb{P}^3(\mathbb{C}) \ar@{~}[rr]_\circ \ar@{~}[rl]_\tau^* & \mathbb{P} = \mathbb{P}^2(\mathbb{C}) \ar[ll]_\nu^* \ar@{~}[ll]_\mu^*}
\end{align*}$$

where $\mu^*$, $\nu^*$, and $\tau^*$ are defined analogously to the twistor case. Note that there is a natural isomorphism $\mathbb{P}^* = \mathbb{P}(\mathbb{C})$.

This thesis will be concerned mainly with the ambitwistor transform:
\[ G := \mathbb{F}_{123}(T) \]
\[ \mathcal{A} := \mathbb{F}_{13}(T) \rightarrow \mathbb{M} = \mathbb{F}_{2}(T) . \]

Here \( \tau(x) \equiv \mathbb{P}_{1}(C) \) and \( \tau^{-1}(x) = \mathbb{P}_{1} \times \mathbb{P}_{1} \). (Normally there will be no chance of confusion arising from the same notation being used for the correspondence mapping in the twistor case as well as in the ambitwistor case because the context will distinguish them. When the need arises to distinguish them explicitly, the ambitwistor correspondence will be written with a subscript \( \mathcal{A}, \tau_{\mathcal{A}} \).) It is important to note that \( \mathcal{A} \) is naturally embedded in \( \mathbb{P} \times \mathbb{P}^* \); that is, we have
\[ \mathcal{A} = \{(L_1, L_3) \in \mathbb{P} \times \mathbb{P}^* : L_1 \subset L_3 \}. \]

In homogeneous twistor coordinates on \( \mathbb{P} \times \mathbb{P}^* \),
\[ \mathcal{A} = \{ (Z^\alpha, \omega_{\alpha}) : Z^\alpha \omega_{\alpha} = 0 \}, \]
as can be easily checked; thus \( \mathcal{A} \) is a hypersurface in \( \mathbb{P} \times \mathbb{P}^* \). We will refer to \( \mathcal{A} \) as \textit{ambitwistor space}. (The reason for this terminology lies in the fact that twistor constructions seem to have an inherent handedness—i.e., right-handed or left-handed, while the more general constructions are ambidextrous, so to speak.) We note too that \( G = \mathbb{F}_{123} \) can be considered to be a codimension-four submanifold of \( \mathbb{F} \times \mathbb{F}^* = \mathbb{F}_{12} \times \mathbb{F}_{23} \) in a natural way.
We may also consider $\mathcal{A}$ as belonging to another double fibration in the following way. If $\rho$, $\hat{\rho}$ are the projections of $\mathbb{P} \times \mathbb{P}^*$ onto $\mathbb{P}$ and $\mathbb{P}^*$, respectively, we define

$$\pi := \rho|_\mathcal{A} : \mathcal{A} \to \mathbb{P}$$
$$\hat{\pi} := \hat{\rho}|_\mathcal{A} : \mathcal{A} \to \mathbb{P}^*.$$

That is, if $L_1 \subset L_3$ defines a point in $\mathcal{A}$, then

$$\pi(L_1, L_3) = L_1,$$
$$\hat{\pi}(L_1, L_3) = L_3.$$

The double fibration appears then as

\[
\begin{array}{ccc}
\mathbb{P} \times \mathbb{P}^* & \xrightarrow{\rho} & \mathcal{A} & \xrightarrow{\pi} & \mathbb{P} \\
\uparrow & \nearrow & \searrow & \nearrow & \downarrow
\end{array}
\]

where the diagram is commutative. A similar relationship holds for the various correspondence spaces,

\[
\begin{array}{ccc}
\mathbb{F} \times \mathbb{F}^* & \xrightarrow{G} & \mathcal{G} & \xrightarrow{\delta} & \mathbb{F} \\
\uparrow & \nearrow & \searrow & \nearrow & \downarrow
\end{array}
\]

but it will not be of importance in our considerations.
2. **Minkowski space.** Each of the first three double fibrations above has in common the manifold $\mathcal{M}$, the Grassmannian of two-planes in $\mathbb{T}$. This manifold can be identified in a natural way with the complexification of the conformal compactification of Minkowski space. That is, if Minkowski space $M_0$ is compactified via its conformal group to a compact four dimensional real manifold $\mathcal{M}$, then $\mathcal{M}$ can be embedded as a totally real submanifold of $\mathcal{M}$. Also $S^4$, the conformal compactification of Euclidean space, can be embedded as a totally real submanifold of $\mathcal{M}$. For a complete explanation the reader is referred to [22].

Various aspects of the conformal geometry of Minkowski space are encoded in the above double fibrations. (From this point forward, "Minkowski space" will refer to $\mathcal{M}$ while $M_0$ will be called real, or physical, Minkowski space.) If $\ast$ denotes the Hodge $\ast$-operator on $\mathcal{M}$ ([24]) there is a corresponding action on the two-forms on $\mathcal{M}$,

$$\ast : \Lambda^2 T^* \mathcal{M} \to \Lambda^2 T^* \mathcal{M}$$

such that $\ast \ast = -1$. We define the eigenspaces

$$\Omega^2_{\pm} := \{ w \in \Lambda^2 T^* \mathcal{M} : \ast w = \pm i w \}$$

and refer to the elements of $\Omega^2_+$ (resp. $\Omega^2_-$) as self-dual (resp. anti-self-dual) two-forms. A two-plane (i.e., a
submanifold isomorphic to $\mathbb{P}_2(\mathbb{C})$ in $\mathbb{M}$ is said to be self-dual if all anti-self-dual two-forms vanish on it, and analogously for an anti-self-dual two-plane. A null two-plane is one all of whose tangent vectors are null with respect to the (complex) Minkowski metric (cf. [23]). A self-dual (resp., anti-self-dual) null two-plane is called an $\alpha$-plane (resp., $\beta$-plane); all null two-planes are either $\alpha$-planes or $\beta$-planes ([7]). All null lines (complexifications of real null lines—light rays—in $\mathbb{M}$) can be expressed uniquely as the intersection of an $\alpha$-plane and a $\beta$-plane ([11]).

If $Z \in \mathbb{P}$, then $\tau(Z)$ is an $\alpha$-plane; if $W \in \mathbb{P}^*$, then $\tau^*(W)$ is a $\beta$-plane. If $(Z,W) \in \mathbb{P} \times \mathbb{P}^*$ where $Z^\alpha W_\alpha = 0$, i.e., where $(Z,W) \in \mathbb{A}$, then $\tau(Z) \cap \tau^*(W) \neq \emptyset$, and, in particular,

$$\tau_\mathbb{A}((Z,W)) = \tau(Z) \cap \tau^*(W);$$

this is a null line in $\mathbb{M}$ by the above comment. Thus we have one-to-one set correspondences

$$\mathbb{P} \rightarrow \{ \alpha\text{-planes in } \mathbb{M} \},$$
$$\mathbb{P}^* \rightarrow \{ \beta\text{-planes in } \mathbb{M} \},$$
$$\mathbb{A} \rightarrow \{ \text{null lines in } \mathbb{M} \}.$$

3. Sheaves on $\mathbb{M}$. We note first that since we shall be dealing for the most part with locally free sheaves
there will be no notational distinction made between such sheaves and the vector bundles corresponding to them, and we shall use the two objects interchangeably.

Considering $\mathbb{M}$ as the Grassmannian of two-planes in $\mathbb{T}$, we define the bundle

$$\mathcal{G}_A' \to \mathbb{M}$$

to be the universal bundle, i.e., the rank two vector bundle whose fibre over a point $x \in \mathbb{M}$ is the two-plane in $\mathbb{T}$ corresponding to $x$. We also define the complement bundle to $\mathcal{G}_A'$ in $\mathbb{T}$ by the exact sequence

$$0 \to \mathcal{G}_A' \to \mathbb{M} \times \mathbb{T} \to \mathcal{G}_A \to 0,$$

and the dual bundles

$$\mathcal{G}^A := (\mathcal{G}_A')^*,$$

$$\mathcal{G}_A := (\mathcal{G}^A)^*.$$

(Cf. [ 5 ].)

We may take various tensor products of these vector bundles, and these will be written, e.g.,

$$\mathcal{G}_{AA'} := \mathcal{G}_A \otimes \mathcal{G}_B$$

$$\mathcal{G}_{(AB)} := \mathcal{G}_A \otimes \mathcal{G}_B \quad \text{(symmetric product)}$$

$$\mathcal{G}_{[A'B']} := \mathcal{G}_A' \wedge \mathcal{G}_B' \quad \text{(antisymmetric product)}.$$

We shall normally use an alternate notation for the antisymmetric products:

$$\mathcal{G} [-1] := \mathcal{G}[AB], \quad \mathcal{G} [1] := \mathcal{G}[AB]$$

$$\mathcal{G} [-1]' := \mathcal{G}[A'B'], \quad \mathcal{G} [1]' := \mathcal{G}[A'B']$$

and

$$\mathcal{G} [k,j] := \mathcal{G} [k]' \otimes \mathcal{G} [j].$$
We may raise and lower indices by noting that, e.g.,

$$\beta_{[A'B']}[1]' \cong \beta$$

so that, for instance, we have

$$\beta A' \cong \beta A' \otimes \beta_{[A'B']}[1]' \cong \beta_{B'}[1]' .$$

This corresponds in local coordinates to multiplying by the matrix $\epsilon_{A'B'}$, which is a local representation of the identity element in $\beta_{[A'B']}[1]'$, in order to lower indices. (Further details appear in the next section.

A complete description of the raising and lowering operations appears in [5].)

4. **Further notation.** An arbitrary open set in $\mathbb{M}$ will normally be denoted by $U$. Given $U$, we define open sets in the other manifolds of the double fibration by

$$U' := \nu^{-1}(U) \text{ or } (\nu^*)^{-1}(U) \text{ or } \rho^{-1}(U),$$

$$U'' := \mu(U') \text{ or } \mu^*(U') \text{ or } \sigma(U')$$

$$= \tau^{-1}(U) \text{ or } (\tau^*)^{-1}(U) \text{ or } \tau^{-1}(U).$$

Thus the three double fibrations will each be written

$$U' \quad U'' \quad U.$$ Alternate notation may be used if there is a chance of confusion arising from the identical notations; in particular $W'$ and $W''$ will be used in Chapter IV for
U' and U'' in the ambitwistor case to distinguish it from the twistor case. For \( x \in M \) we shall often write \( L_x \) for \( \tau^{-1}_M(x) \); for \( \tau^{-1}(x) \) we shall write \( L^T_x \) or simply \( L_x \) if there is no chance for confusion, and \( (\tau^*)^{-1}(x) \) will be denoted by \( L^*_x \).

The complexification of physical Minkowski space gives a noncompact complex manifold, isomorphic to \( \mathbb{C}^4 \), which can be naturally embedded in \( \mathbb{M} \); it will be denoted by \( \mathbb{M}^I \) and called affine Minkowski space. Being isomorphic to \( \mathbb{C}^4 \), \( \mathbb{M}^I \) may be used as a coordinate patch on \( \mathbb{M} \); when it is desirable to work in local coordinates on \( \mathbb{M} \) we shall use this coordinate patch with coordinates \( \{ z^{AA'} \}; A, A' = 0,1 \). These are related to the usual coordinates on Minkowski space by

\[
\begin{bmatrix}
  z^{00'} & z^{01'} \\
  z^{10'} & z^{11'}
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
  z^0 + z^1 & z^2 + iz^3 \\
  z^2 - iz^3 & z^0 - z^1
\end{bmatrix}
\]

where \( z^i = x^i + iy^i \), and \( (x^0, x^1, x^2, x^3) = (ct, x, y, z) \) are the coordinates on \( M_0 \). Differentiation with respect to these variables will be denoted by

\[
\nabla^{AA'} := \frac{\partial}{\partial z^{AA'}}.
\]

Raising and lowering indices, accomplished in the coordinates \( (x^0, x^1, x^2, x^3) \) by multiplication with the
metric tensor $\varepsilon_{ij}$, is done in these coordinates by multiplication with the matrices $\varepsilon_{AB}$ or $\varepsilon_{A'B'}$:

$$
\varepsilon_{AB} = \begin{bmatrix}
\varepsilon_{00} & \varepsilon_{01} \\
\varepsilon_{10} & \varepsilon_{11}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix},
$$

and $\varepsilon_{A'B'}$, $\varepsilon_{AB}$ and $\varepsilon_{A'B'}$ have identical matrix representations. In particular,

$$\varphi^{AA'} = \varepsilon_{AB} \varepsilon_{A'B'} \varphi^{BB'}.$$

For more details see [5] or [23].

5. **Local coordinates.** When we deal with local coordinates it will be on the "affine" parts of the various manifolds. We define the affine parts for the twistor case, the dual twistor case, and the ambitwistor case as

$$
\mathbb{F}^I := \varphi^{-1}(\mathbb{M}^I), \quad \mathbb{P}^I := \mu(\mathbb{F}^I)
$$

$$
\mathbb{F}^{*I} := \varphi^{*I}(\mathbb{M}^I), \quad \mathbb{P}^{*I} := \mu^{*I}(\mathbb{F}^{*I})
$$

$$
\mathbb{G}^I := \rho^{-1}(\mathbb{M}^I), \quad \mathbb{A}^I := \sigma^{I}.
$$

Since $\mathbb{F}^I \cong \mathbb{M} \times \mathbb{P}_I$ we choose local coordinates

$$
\{z^{AA'}, [\pi_A] \} \text{ on } \mathbb{F}^I, \text{ where } z^{AA'} \text{ are coordinates on } \mathbb{M}^I.
$$

Similarly, we have coordinates

$$
\{w^{AA'}, [\eta_A] \} \text{ on } \mathbb{F}^{*I}, \text{ and } \{x^{AA'}, [\pi_A'], [\eta_A] \} \text{ on } \mathbb{G}^I.
$$

Considering $\mathbb{G}^I$ as a submanifold of $\mathbb{F}^I \times \mathbb{F}^{*I}$ and defining

$$
x^{AA'} := z^{AA'} + w^{AA'}
$$

$$
y^{AA'} := z^{AA'} - w^{AA'}$$

we have that \( \mathcal{G}^I \) is given by the set \( \{ y^{AA'} = 0 \} \)
in \( \mathbb{P}^I \times \mathbb{P}^{*I} \), and \( \{ x^{AA'} \} \) are the diagonal coordinates.

We will write \( \mathbb{P}^I = \mathbb{P}_0^I \cup \mathbb{P}_1^I \) where we define
\[
\mathbb{P}_0^I := \{ [w^A, \pi_A^I] \in \mathbb{P}^I : \pi_0^I \neq 0 \}
\]
\[
\mathbb{P}_1^I := \{ [w^A, \pi_A^I] \in \mathbb{P}^I : \pi_1^I \neq 0 \}
\]
so that we have the following local coordinates:
\[
\mathbb{P}_0^I \leftarrow \{ (\frac{w^0}{\pi_0^I}, \frac{w^1}{\pi_0^I}, \frac{\pi_1^I}{\pi_0^I}) \}
\]
\[
\mathbb{P}_1^I \leftarrow \{ (\frac{w^0}{\pi_1^I}, \frac{w^1}{\pi_1^I}, \frac{\pi_0^I}{\pi_1^I}) \}.
\]

On \( \mathbb{P}^{*I} = \mathbb{P}_0^{*I} \cup \mathbb{P}_1^{*I} \) we have analogous coordinates:
\[
\mathbb{P}_0^{*I} \leftarrow \{ (\frac{s^0}{\eta_0}, \frac{s^1}{\eta_0}, \frac{\eta_1}{\eta_0}) \}
\]
\[
\mathbb{P}_1^{*I} \leftarrow \{ (\frac{s^0}{\eta_1}, \frac{s^1}{\eta_1}, \frac{\eta_0}{\eta_1}) \}.
\]

Combining these coordinates gives coordinates on
\( \mathcal{A}^I \subset \mathbb{P}^I \times \mathbb{P}^{*I} \), and we write \( \mathcal{A}^I = \bigcup_{i=0}^{3} \mathcal{A}_i^I \) where
\[
\mathcal{A}_0^I := (\mathbb{P}_0^I \times \mathbb{P}_0^{*I})|
\]
\[
\mathcal{A}_1^I := (\mathbb{P}_1^I \times \mathbb{P}_0^{*I})|
\]
\[
\mathcal{A}_2^I := (\mathbb{P}_0^I \times \mathbb{P}_1^{*I})|
\]
\[
\mathcal{A}_3^I := (\mathbb{P}_1^I \times \mathbb{P}_1^{*I})|
\]

Of course the six coordinates on \( \mathcal{A}_i^I \) are not independent, since the homogeneous coordinates satisfy
\( \omega^A \eta_A + \xi^A \pi_A = 0. \)

Although the above sets of coordinates are the basic ones which we shall be using, we shall define new sets of coordinates (as linear combinations of the old) when it is useful to do so.
III. Basic Ambitwistor Results

In this chapter we introduce and develop the tools necessary for the work in the remaining chapters. We shall describe the sheaves and differential operators on $\mathbb{M}$ which are involved in the various equations from physics in which we are interested, the geometric objects on $\mathcal{A}$ which will be used to obtain solutions to these equations, and various related objects on the correspondence space $\mathcal{G}$. Then we shall detail results which relate the various data, generalizing the work of [5].

1. Data on $\mathbb{M}$. The deRham sequence on $\mathbb{M}$ may be written as (cf. section 2 of [5]):

\[
\begin{array}{cccccc}
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & \Omega^4 \\
\oplus & \xrightarrow{\oplus} & \oplus & \xrightarrow{\oplus} & \oplus & \xrightarrow{\oplus} & \oplus & \xrightarrow{\oplus} & \oplus \\
\bigoplus_{A'} \bigoplus_{B'} & \xrightarrow{\oplus} & \bigoplus_{(A'B')[-1]} i\bigoplus_{B'} & \xrightarrow{\oplus} & \bigoplus_{(AB)[-1]} i\bigoplus_{B'} & \xrightarrow{\oplus} & \bigoplus_{(-2,-2)} i\bigoplus_{B'} \\
\end{array}
\]

Given a vector bundle $E \rightarrow U \subset \mathbb{M}$ we may tensor each of the above terms with $E$ and replace the exterior derivative with a covariant differentiation operator $D$ to get

\[
\begin{array}{cccccc}
\Omega^0(E) & \xrightarrow{D} & \Omega^1(E) & \xrightarrow{D} & \Omega^2(E) & \xrightarrow{D} & \Omega^3(E) & \xrightarrow{D} & \Omega^4(E) \\
\end{array}
\]
If $f \in \Gamma(V, \Omega^i)$ and $\xi \in \Gamma(U, \Omega^i)$ where $V \subset U$ is open then covariant differentiation is given by (cf., for example, chapter 5 in [14]):

$$D(f \otimes \xi) = df \otimes \xi + f \wedge D\xi.$$ 

Suppose $A$ is the local connection form for $D$; that is, in local coordinates

$$D\xi = d\xi + A \wedge \xi$$

where $A \in \Omega^1 \otimes \text{End } E$, and $\xi \in \Omega^i(E)$. An important related concept is covariant differentiation on the bundle $\text{End } E$. If we define $\widetilde{A} \in \Omega^1 \otimes \text{End}(\text{End } E)$ by (cf. [14])

$$\widetilde{A}(e)(\psi) = A(e(\psi)) - (e \otimes 1)(A\psi)$$

$$= \text{ad } A(e)(\psi)$$

for $\psi, e$ sections of $E$ and $\text{End } E$, respectively, then $\widetilde{D}$ given by

$$\widetilde{D}\xi = d\xi + \widetilde{A} \wedge \xi, \quad \xi \in \Omega^i \otimes \text{End } E$$

defines a connection on $\text{End } E$:

$$\widetilde{D} : \Omega^i \otimes \text{End } E \rightarrow \Omega^{i+1} \otimes \text{End } E.$$ 

Defining a Lie bracket structure on $\mathfrak{g}^*(\text{End } E)$ ([24]), we can rewrite (3.3) as

$$\widetilde{D}\xi = d\xi + [A, \xi].$$

We shall sometimes write $D_A$ and $\widetilde{D}_A$ to make explicit the dependence of the differential operator on the connection.
Suppose \( U \) is an open subset of Minkowski space. We shall be interested in solutions to two sets of equations on \( U \): the zero-rest-mass equations and the Yang-Mills equations. The zero-rest-mass equations of helicity \( n/2 \) are a pair of uncoupled equations
\[
\begin{align*}
\phi^{A\prime}_{\Lambda\Lambda} & = 0 \quad \text{(n indices)} \\
\psi^{A\prime}_{\Lambda\Lambda} & = 0 \quad \text{(n indices)}
\end{align*}
\] (3.4)

where
\[
\phi^{\Lambda\cdots D}_{\Lambda\cdots D} \in \Gamma(U, \wedge^{AB\cdots D}_[-1]) \\
\psi^{\Lambda\prime\cdots D\prime}_{\Lambda\prime\cdots D\prime} \in \Gamma(U, \wedge^{A\prime\cdots D\prime}_[-1\prime]).
\]

For \( n=1 \) these are the Dirac-Weyl equations for a neutrino; for \( n=2 \) they are the homogeneous Maxwell equations; and for \( n=4 \) they are the linearized Einstein equations.

To describe the Yang-Mills equations we let \( A \) be a connection on \( U \) and \( F \) be its induced curvature; in particular, \( A \in \Gamma(U, \Omega^1 \otimes \text{End } E) \) and \( F \in \Gamma(U, \Omega^2 \otimes \text{End } E) \). The Yang-Mills equations with current are then
\[
\begin{align*}
\bar{\nabla}_A F & = 0 \\
\bar{\nabla}_A (\ast F) & = \ast J
\end{align*}
\] (3.5)

where \( \ast J \in \Gamma(U, \Omega^3 \otimes \text{End } E) \) is the axial Yang-Mills current. The homogeneous Yang-Mills equations are
\[
\begin{align*}
\bar{\nabla}_A F & = 0 \\
\bar{\nabla}_A (\ast F) & = 0
\end{align*}
\]
2. **Vector bundles on** \( \mathbb{A} \). We shall be concerned primarily with two types of holomorphic vector bundles over \( U'' \subset \mathbb{A} \). The first class consists of vector bundles over \( \mathbb{A} \) obtained by pulling back vector bundles over \( \mathbb{P} \) and \( \mathbb{P}^* \). Let \( \pi \) and \( \hat{\pi} \) be the projections of \( \mathbb{A} \) onto \( \mathbb{P} \) and \( \mathbb{P}^* \) introduced in Chapter II,

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\pi} & \mathbb{P} \\
\downarrow{\hat{\pi}} & & \downarrow{\pi} \\
\mathbb{P}^* & & \mathbb{P} \\
\end{array}
\quad \quad \quad \quad \quad \quad \begin{array}{ccc}
\{L_1 \subset L_3\} & \xrightarrow{\pi} & \{L_1 \subset L_3\} \\
\downarrow{\hat{\pi}} & & \downarrow{\pi} \\
L_1 & & L_3 \\
\end{array}
\]

Then given vector bundles \( K \to \mathbb{P} \) and \( L \to \mathbb{P}^* \), we form from their pullbacks the vector bundle \( \pi^* K \otimes \hat{\pi}^* L \to \mathbb{A} \).

In the special case of line bundles, every line bundle \( K \) over \( \mathbb{P} \) is equivalent to some power of the hyperplane section bundle \( H : K \cong H^m \to \mathbb{P} \).

We then define \( \mathcal{O}(m,n) := \pi^* H^m \otimes \hat{\pi}^* H^n \), which we may think of intuitively as the sheaf of functions on \( \mathbb{A} \) homogeneous of degree \( m \) in \( Z \) and of degree \( n \) in \( W \). These shall be vital to the construction of solutions to the zero-rest-mass field equations.

The second class of vector bundles over \( U'' \subset \mathbb{A} \)
which will be of interest are those with a certain triviality condition: those vector bundles whose restriction to each \( L_x = \tau^{-1}(x) \cong \mathbb{P}_1 \times \mathbb{P}_1 \) for \( x \in \mathcal{M} \) is trivial. Suppose \( U'' = \tau^{-1}(U) \) for \( U \) an open set in \( \mathcal{M} \). Then, using the terminology of Manin ([14]), we define a vector bundle \( E \rightarrow U'' \) to be \textbf{U-trivial} if \( E \big|_{L_x} \rightarrow L_x \) is trivial for each \( x \in U \).

Such vector bundles will appear in our discussion of the generalized Ward correspondence.

3. The topology of the mapping \( \sigma \). The geometric objects on \( A \) which most concern us are cohomology groups with coefficients in sheaves. The structure of the fibres of \( \sigma: U' \rightarrow U'' \) determines the relationship of these groups to their pullbacks along the fibres. If \( \mathcal{S} \) is a sheaf on \( U'' \), let \( \sigma^{-1}\mathcal{S} \) denote its topological inverse on \( U' \). Suppose that the mapping \( \sigma: U' \rightarrow U'' \) is such that for every \( X \in U'' \) the fibre \( \sigma^{-1}(X) \) is connected and further \( \check{H}^i(\sigma^{-1}(X), \mathcal{C}) = 0 \) for all \( i \leq n \) and for every \( X \in U'' \); then we say that the mapping is \textbf{n-elementary}. Then we have the following result due to Buchdahl ([2],[3]):

\textbf{Theorem III.1 (Buchdahl)}: If \( \sigma: U' \rightarrow U'' \) is n-elementary and \( V \rightarrow U'' \) is a holomorphic vector bundle, then there exists a canonical isomorphism for \( i \leq n \)

\[ \sigma^*: \check{H}^i(U'', \mathcal{O}(V)) \cong \check{H}^i(U', \sigma^{-1}\mathcal{O}(V)). \]
There is nothing special about the projection \( \sigma \) which makes the theorem true; it is true for general projections of complex manifolds. However, we shall need only the above restricted version.

We note that the fibre \( \sigma^{-1}(X) \), for \( X \in U'' \), is holomorphically equivalent to \( \tau(X) \cap U \). Thus any condition on the fibres of \( \sigma \) can be restated as a condition on the intersections of the null lines in \( \mathcal{IM} \) with \( U \), and so the condition that \( \sigma: U' \to U'' \) be \( n \)-elementary is really a restriction on the topology of \( U \).

4. The relative deRham sequence on \( \mathcal{G} \). The counterpart on \( \mathcal{G} \) of the sheaves and differential operators on \( \mathcal{IM} \) discussed above is a relative deRham sequence. In this section we shall describe it first in terms of sheaves on \( \mathcal{G} \) induced from bundles on \( \mathcal{A} \) and then in terms of sheaves and operators induced from \( \mathcal{IM} \). We begin by defining the sheaf of relative 1-forms with respect to the fibration \( \sigma \), written \( \Omega^1_{\sigma} \), by the exact sequence

\[
\sigma^*: \Omega^1_{\mathcal{A}} \to \Omega^1_{\mathcal{G}} \to \Omega^1_{\sigma} \to 0,
\]

and setting \( d_{\sigma} := \pi_{\sigma} \circ d \). We can think of \( \Omega^1_{\sigma} \) as the sheaf of 1-forms "pointing along the fibres." Let \( \sigma^{-1}\mathcal{O}_{\mathcal{A}} \) be the topological inverse image of \( \mathcal{O}_{\mathcal{A}} \), that is, the subsheaf of \( \mathcal{O}_{\mathcal{G}} \) consisting of those functions locally
constant on the fibres of $\sigma$. Then we have the following:

**Proposition III.2** The sequence

$$0 \to \sigma^{-1} \mathfrak{g}_{\mathbb{A}} - \mathfrak{s}_{\mathbb{A}} \mathfrak{d}_{\mathcal{G}} \Omega_{\sigma}^1 \to 0$$

is exact.

Before proving the above proposition we examine some of its immediate consequences. We may tensor (3.6) with $\sigma^{-1} \mathfrak{g}_{\mathbb{A}}(V)$, where $V$ is a holomorphic vector bundle over (an open subset of) $\mathbb{A}$, to get the exact sequence

$$0 \to \sigma^{-1} \mathfrak{g}_{\mathbb{A}}(V) \to \mathfrak{s}_{\mathcal{G}}(\sigma^* V) \mathfrak{d}_{\mathcal{G}} \to \sigma^{-1} \mathfrak{g}_{\mathbb{A}}(V) \mathfrak{s}_{\sigma} - \mathfrak{s}_{\mathbb{A}} \Omega_{\sigma}^1 \to 0$$

or

$$0 \to \sigma^{-1} \mathfrak{g}_{\mathbb{A}}(V) - \Omega_{\sigma}^0 (V) \to \Omega_{\sigma}^1 (V) \to 0,$$

where we define

$$\Omega_{\sigma}^0 (V) := \mathfrak{s}_{\mathcal{G}}(\sigma^* V)$$

$$\Omega_{\sigma}^1 (V) := \sigma^{-1} \mathfrak{g}_{\mathbb{A}}(V) \mathfrak{s}_{\sigma} - \mathfrak{s}_{\mathbb{A}} \Omega_{\sigma}^1.$$

For the special case where $V$ is the sheaf of homogeneous holomorphic polynomials on $\mathcal{A}$ of homogeneity $(m,n)$ we write

$$0 \to \sigma^{-1} \mathfrak{g}_{\mathbb{A}}(m,n) - \Omega_{\sigma}^0 (m,n) \to \Omega_{\sigma}^1 (m,n) \to 0.$$

To show that the sequence (3.6) is exact and to lay the groundwork for future local calculations we introduce local coordinates on $\mathcal{A}$ and $\mathcal{G}$. Recall
that $x^{AA'}$ are coordinates on $\Gamma^I$. Then we have coordinates $(x^{AA'}, [\pi_{A'}], [\eta_{A'}])$ on $\Gamma^I$ and coordinates $([\omega^A = i x^{AA'}, \pi_{A'}], [\eta_{A'}], \xi^{AA'} = -i x^{AA'} \eta_{A'})$ on $\mathbb{P}_x \times \mathbb{P}_x^*$. The equality

$$w^A \eta_A + \xi^{AA'} \pi_{A'} = 0$$

on $\tilde{A}$ will indicate the choice of local coordinates on $\tilde{A}$. We cover $\Gamma^I$ with the following coordinate patches, with local coordinates given on the right:

- $\Gamma^I_0 := \{\pi_0 \neq 0, \eta_0 \neq 0\} \rightarrow \{(x^{AA'}, r_0 := \frac{\pi_1}{\pi_0}, s_0 := \frac{\eta_1}{\eta_0})\}$
- $\Gamma^I_1 := \{\pi_1 \neq 0, \eta_0 \neq 0\} \rightarrow \{(x^{AA'}, r_1 := \frac{\pi_0}{\pi_1}, s_1 := \frac{\eta_1}{\eta_0})\}$
- $\Gamma^I_2 := \{\pi_0 \neq 0, \eta_1 \neq 0\} \rightarrow \{(x^{AA'}, r_2 := \frac{\pi_0}{\pi_1}, s_2 := \frac{\eta_0}{\eta_1})\}$
- $\Gamma^I_3 := \{\pi_1 \neq 0, \eta_1 \neq 0\} \rightarrow \{(x^{AA'}, r_3 := \frac{\pi_0}{\pi_1}, s_3 := \frac{\eta_0}{\eta_1})\}$

Then $\tilde{A}_0^I = \sigma(\Gamma^I_1)$, where $\Gamma^I_1$ was defined in Chapter II.

Condition (3.9) above gives on $\tilde{A}_0^I$, for example,

$$\frac{\omega^A \eta_A}{\pi_0, \eta_0} + \frac{\xi^{AA'} \pi_{A'}}{\pi_0, \eta_0} = \frac{\omega^0 \eta_0}{\pi_0} + \frac{\omega^1 \eta_1}{\pi_0} + \frac{\xi^{AA'} \pi_{A'}}{\pi_0, \eta_0} + \frac{\pi_1 \xi^{AA'}}{\pi_0, \eta_0} = 0.$$ 

We then define local coordinates on $\tilde{A}_0^I$ by:

$$p_0^A := \frac{\omega^A}{\pi_0}, \quad q_0^1 := \frac{\xi^{AA'}}{\eta_0}, \quad r_0 = \frac{\pi_1}{\pi_0}, \quad s_0 = \frac{\eta_1}{\eta_0}$$

corresponding to coordinates

$$p_0 = \omega^A \pi_0, \quad q_0 = \xi^{AA'}, \quad r_0 = \frac{\pi_1}{\pi_0}, \quad s_0 = \frac{\eta_1}{\eta_0}$$

on $\mathbb{P}_0 \times \mathbb{P}_0^*$. Since $\Gamma^I \rightarrow \tilde{A}_0^I$ is a projection with $\xi^1$ fibres, we may use the coordinates on $\tilde{A}_0^I$ along with
one extra coordinate, 

\[ t_0 := i x^{11'} \]

as local coordinates on \( G^I_0 \). We verify that 

\[ \{(p^A_0, q^1_0, t_0, r_0, s_0)\} \]

is actually a coordinate system on \( G^I_0 \) by showing that it is equivalent to the set of coordinates \( \{(x^{AA'}, r_0, s_0)\} \). This is done by checking that the matrix defined by

\[
\begin{pmatrix}
    p^0_0 \\
p^1_0 \\
p^0_0 \\
q^1_0 \\
t^0_0
\end{pmatrix} = i
\begin{pmatrix}
    1 & r_0 & 0 & 0 \\
0 & 0 & 1 & r_0 \\
0 & -1 & 0 & -s_0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^{00'} \\
x^{01'} \\
x^{10'} \\
x^{11'}
\end{pmatrix}
\]

has determinant equal to 1. Proceeding in this manner we obtain local coordinates on all the various coordinate patches:

\[
\begin{align*}
P^I_0 \times P^I_0 & : (p^A_0, q^A_0, r_0, s_0) \\
\mathcal{A}^I_0 & : (p^A_0, q^1_0, r_0, s_0) \\
G^I_0 & : (p^A_0, q^1_0, t_0, r_0, s_0) \sim (x^{AA'}, r_0, s_0) \\
P^I_1 \times P^I_0 & : (p^A_1 := \frac{x^A}{\eta^1_0}, q^A_1 := \frac{s^A}{\eta^1_0}, r_1, s_1) \\
\mathcal{A}^I_1 & : (p^A_1, q^A_1, r_1, s_1) \\
G^I_1 & : (p^A_1, q^1_1, t_1 := i x^{10'}, r_1, s_1) \\
P^I_0 \times P^I_1 & : (p^A_2 := \frac{x^A}{\eta^0_1}, q^A_2 := \frac{s^A}{\eta^1_1}, r_1, s_2) \\
\mathcal{A}^I_2 & : (p^A_2, q^1_2, r_2, s_2)
\end{align*}
\]
Locally, differentiation along the fibres of $\sigma$ is then given by $\partial/\partial t_i$ on $\mathcal{G}^I_1$. As we shall see below, $\eta_{A''A'}^\pi \vartheta^AA'$ is a global operator on $\mathcal{G}$; this operator can be identified as differentiation along the fibres of $\sigma$.

To check this we write

$$
\begin{align*}
\eta_{A''A'}^\pi \vartheta^AA' &= \eta_{0''0'} \frac{\partial}{\partial x} \vartheta^A_1 - \eta_{1''0'} \frac{\partial}{\partial x} \vartheta^A_0 - \eta_{0''1'} \frac{\partial}{\partial x} \vartheta^A_1 \\
& \quad + \eta_{1''1'} \frac{\partial}{\partial x} \vartheta^A_0 \\
&= i \eta_{0''0'} (r_0 \frac{\partial}{\partial p_0} - \frac{\partial}{\partial t_0} - s_0 \frac{\partial}{\partial q_0}) - \eta_{1''0'} \\
& \quad - \quad \eta_{0''1'} \frac{\partial}{\partial x} + \eta_{1''1'} \frac{\partial}{\partial x} \\
&= -i \eta_{0''0'} \frac{\partial}{\partial t_0} \quad \text{on} \quad \mathcal{G}^I_0.
\end{align*}
$$

Similarly, it can be calculated that

$$
\eta_{A''A'}^\pi \vartheta^AA' = \begin{cases} 
-i \eta_{0''1'} \frac{\partial}{\partial t_1} & \text{on} \quad \mathcal{G}^I_1 \\
-i \eta_{1''1'} \frac{\partial}{\partial t_1} & \text{on} \quad \mathcal{G}^I_2 \\
-i \eta_{1''1'} \frac{\partial}{\partial t_2} & \text{on} \quad \mathcal{G}^I_3 
\end{cases}
$$

We now calculate transition functions for $\Omega^1_{\sigma|_I \mathcal{G}^I_1}$:
\[ t_1 = \text{i} x^{10'} = p_0 + r_0 t_0 = d_\sigma t_1 = r_0 \cdot d_\sigma t_0 \]
\[ t_2 = -\text{i} x^{01'} = q_0 + s_0 t_0 = d_\sigma t_2 = s_0 \cdot d_\sigma t_0 \]
\[ t_3 = -\text{i} x^{00'} = q_1 + s_1 t_1 = d_\sigma t_3 = s_1 \cdot d_\sigma t_1 = r_0 s_0 \cdot d_\sigma t_0 . \]

These are the transition functions for the sheaf of homogeneous functions of degree \((1,1)\). Thus

\[ \Omega^1_\sigma|_G I \cong \mathcal{O}_G (1,1)|_G I \]

where \( \mathcal{O}_G (1,1) = \sigma^* \mathcal{O}_A (1,1) \). Note that as in the twistor case the isomorphism cannot be extended to all of \( G \): if \( Y = \mathbb{P}_1 \) is a fibre of \( G \to \mathbb{A} \), then \( \Omega^1_\sigma|_Y = T^*(\mathbb{P}_1) \) is nontrivial since \( \mathbb{P}_1 \) has a nontrivial cotangent bundle, but \( \mathcal{O}_G (1,1)|_Y \) is trivial since \( \mathcal{O}_G (1,1) \) is a pullback along the fibration and thus can be described by transition functions that are constant along the fibres.

It is now an easy matter to prove the above proposition.

**Proof of Proposition III.2** - In the local coordinates which we have introduced above, the sequence (3.6) is just the deRham sequence

\[ 0 \to \mathcal{C} \to \mathcal{O}_C \to \Omega^1_C \to 0, \]

where we have suppressed the five additional parameters along the base space \( \mathbb{A} \). This is obviously exact. □

We now claim that \( \eta_{A'' A', \mathbb{A}''} \) operates naturally on all of \( G \). In particular we claim that
\[ \pi_A \triangledown A' : \mathcal{O} \rightarrow \mathcal{O}(1,1)[-1,-1] \]
on \mathcal{G}. We first note that we can define a bundle homomorphism
\[ \eta_A \pi_A' : \mathcal{O}^{AA'} \rightarrow \mathcal{O}(1,1) \]
by defining the dual homomorphism
\[ t_\eta_A t_\pi_{A'} : \mathcal{O}(-1,-1) \rightarrow \mathcal{O}^{AA'} \]
as follows: we have \( \mathcal{O}(-1,-1) \cong \mathcal{O}(-1,0) \otimes \mathcal{O}(0,-1) \) and \( \mathcal{O}^{AA'} = \mathcal{O}_A \otimes \mathcal{O}_{A'} \), so we need only define the homomorphisms
\[ t_\pi_{A'} : \mathcal{O}(-1,0) \rightarrow \mathcal{O}_{A'} \]
and \[ t_\eta_A : \mathcal{O}(0,-1) \rightarrow \mathcal{O}_A \].
As noted in [5] the fibres over \( (L_1,L_2,L_3) \) of \( \mathcal{O}(-1,0) \) and \( \mathcal{O}_A' \) are \( L_1 \) and \( L_2 \), respectively; then \( t_\pi_{A'} \) is just the inclusion map \( L_1 \subseteq L_2 \). The fibres over \( (L_1,L_2,L_3) \) of \( \mathcal{O}(0,-1) \) and \( \mathcal{O}_A \) are \( \star L_3^* \) and \( \star L_2 \), where the duality pairings are as follows: \( (Z_i)^* = (W_i) \) is the duality pairing between \( T^* \) and \( T^*_\alpha \), and \( \star \) is the duality pairing between 1-planes and 3-planes and between 2-planes and 2-planes. Then \( L_2 \subseteq L_3 \) implies that \( \star L_3^* \subseteq \star L_2 \) and the mapping \( t_\eta_A \) is again just the inclusion map. Then the homomorphism
\[ t_\eta_A t_\pi_{A'} : \mathcal{O}(-1,-1) \rightarrow \mathcal{O}^{AA'} \]
is just the tensor product of the two maps defined above.

On \( \mathcal{M} \) we have the mapping
\[ \triangledown^{AA'} : \mathcal{O} \rightarrow \mathcal{O}^{AA'}[-1,-1] \]
which is just exterior differentiation canonically (from section 2 of [5]). Although $\varphi^{AA'}$ is not defined on $G$ if we restrict our attention to the coordinate patch $G^I$ then $\varphi^{AA'}$ is just differentiation in the base direction. In this case we define $\eta_A^{\Pi_A', \varphi^{AA'}}$ as the composition

$$
\otimes \xrightarrow{\eta_A^{\Pi_A', \varphi^{AA'}}} \Theta_{AA'}(1,1) \xrightarrow{\varphi^{AA'}} \Theta(1,1)[-1,-1].
$$

Since we have seen above that $\eta_A^{\Pi_A', \varphi^{AA'}}$ is differentiation along the fibres of $\sigma$, $d_{\sigma}$, we would expect that it is defined independently of the choice of trivialization. That this is indeed so can be checked easily by moving from one coordinate patch to another. Alternatively, it can be proven by using the results on the twistor and dual twistor cases as follows.

From section 2 of [5] we have that the differential operator on $\mathbb{IF} \times \mathbb{IF}^*$

$$
\eta_A^{\Pi_A', \varphi^{AA'}} : \otimes \rightarrow \Theta_A(1,0)[-1]'
$$

is canonically equivalent to differentiation along the fibres of $\mathbb{IF} \rightarrow \mathbb{IP}$. Raising the index gives

$$
\eta_A^{\Pi_A', \varphi^{AA'}} : \otimes \rightarrow \Theta^A(1,0)[-1,-1]
$$

for differentiation along these fibres, where $(Z^{AA'},[\Pi_A'])$ are coordinates on $\mathbb{IF}$. For the dual case

$$
\eta_A^{\Pi_A', \varphi^{AA'}} : \otimes \rightarrow \Theta^{A'}(0,1)[-1,-1]
$$

is equivalent to differentiation along the fibres of $\mathbb{IF}^* \rightarrow \mathbb{IP}^*$, where $(W^{AA'},[\eta_A])$ are coordinates on $\mathbb{IF}^*$. 
Then, applying the above-mentioned map
\[ \pi_A^* : \mathcal{O}^A \rightarrow \mathcal{O}(1,0) \]
and the corresponding map for the dual situation
\[ \eta_A : \mathcal{O}^A \rightarrow \mathcal{O}(0,1), \]
we obtain the operators on \( \mathbb{P} \times \mathbb{P}^* \)
\[ \eta_A^{\pi_A} \cdot \nabla_{AA'}^z : \mathcal{O} \rightarrow \mathcal{O}(1,1)[-1,-1] \]
\[ \eta_A^{\pi_A} \cdot \nabla_{AA'}^w : \mathcal{O} \rightarrow \mathcal{O}(1,1)[-1,-1]. \]
Since \( \nabla_{AA'} = \frac{1}{2} \left( \nabla_{AA'}^z + \nabla_{AA'}^w \right) \) we have immediately that \( \eta_A^{\pi_A} \cdot \nabla_{AA'} \) is well-defined on all of \( \mathbb{C} \).

Using this result we can express the relative deRham sequence on \( \mathbb{C} \) as
\[ (3.10) \quad 0 \rightarrow \sigma^{-1} \mathcal{O}_A \rightarrow \mathcal{O}_A \rightarrow \mathcal{O}(1,1)[-1,-1] \rightarrow 0. \]
This sequence contains the sheaf \( \sigma^{-1} \mathcal{O}_A \), which reflects the structure of \( \mathcal{A} \), as well as the differential operator \( \eta_A^{\pi_A} \cdot \nabla_{AA'} \) and the sheaf \( \mathcal{O}(1,1)[-1,-1] \) induced from the corresponding objects on \( \mathbb{M} \). It is of fundamental importance in this thesis, providing the means of relating geometric data on \( \mathcal{A} \) to differential data on \( \mathbb{M} \).

5. Direct image sheaves. Taking direct images along \( \rho \) of sheaves on \( \mathbb{C} \) provides us with a way of transferring data from \( \mathbb{C} \) to \( \mathbb{M} \). A spectral sequence allows us to relate cohomology on the two manifolds. In this section we generalize the results of [5] to the ambitwistor case.
The results for the direct image sheaves of the mapping $\mathcal{F} \to \mathcal{M}$ can be summarized as:

$$\nu^q_\ast \mathcal{O}(m) = \mathcal{O}(A' \cdots D') \quad \text{m indices}$$

$$\nu^q_\ast \mathcal{O}(m) = 0 \quad \forall \ q \geq 1$$

$$\nu^q_\ast \mathcal{O}(-1) = 0 \quad \forall \ q \quad m \geq 0$$

$$\nu^q_\ast \mathcal{O}(-m-2) = \mathcal{O}(A' \cdots D')[-1]'$$

The corresponding results on direct images along $\mathcal{F}^\ast \to \mathcal{M}$ are (suppressing the $\ast$ on $\nu^\ast$ for the sake of simplicity):

$$\nu^q_\ast \mathcal{O}(n) = \mathcal{O}(A' \cdots E) \quad \text{n indices}$$

$$\nu^q_\ast \mathcal{O}(n) = 0 \quad \forall \ q \geq 1$$

$$\nu^q_\ast \mathcal{O}(-1) = 0 \quad \forall \ q \quad n \geq 0$$

$$\nu^q_\ast \mathcal{O}(-n-2) = \mathcal{O}(A' \cdots E)[-1]$$

$$\nu^q_\ast \mathcal{O}(-n-2) = 0 \quad \forall \ q \neq 1$$

The analogous results in the ambitwistor case follow easily from the above statements. Using the fact that the fibres of $\rho : \mathcal{G} \to \mathcal{M}$ are $\rho^{-1}(x) = \nu^{-1}(x) \times (\nu^\ast)^{-1}(x)$ $\cong \mathbb{P}_1 \times \mathbb{P}_1$ we can apply the Kunneth formula ([19]) to obtain, for $m, n \geq 0$ and $m$ primed indices and $n$ unprimed indices:

$$\rho^q_\ast \mathcal{O}(m,n) = \mathcal{O}(A' \cdots D')(A' \cdots E)$$

$$\rho^q_\ast \mathcal{O}(m,n) = 0 \quad \forall \ q \geq 1$$

$$\rho^q_\ast \mathcal{O}(m,-n-2) = \mathcal{O}(A' \cdots D')[-1]$$

$$\rho^q_\ast \mathcal{O}(-m-2,n) = \mathcal{O}(A' \cdots D')[-1]'$$
\[ \rho_{\mathbb{P}}^q(k,-1) = \rho_{\mathbb{P}}^q(-1,j) = 0 \quad \forall \, q,k,j \]

\[ (3.13) \quad \rho_{\mathbb{P}}^2(-m-2,-n-2) \cong \mathcal{O}(A' \cdots D')(A \cdots E)[-1,-1] \]

\[ \rho_{\mathbb{P}}^2(k,j) = 0 \quad \text{if } k \text{ or } j \geq -1 \]

Use of the Leray spectral sequence provides a way of relating cohomology on \( \mathbb{P} \) to cohomology on \( \mathbb{M} \). The Leray spectral sequence for a sheaf \( \mathcal{F} \) on \( \mathbb{P} \) is (\([6]\)):

\[ E_2^{p,q} = H^p(U, \rho_{\mathbb{P}}^q\mathcal{F}) = E_\infty^{p,q} = H^p(U', \mathcal{F}) \]

where \( U \subset \mathbb{M} \) is open and \( U' = \rho^{-1}(U) \). For \( m,n \geq 0 \) let

\[ E_2^{p,q} = H^p(U, \rho_{\mathbb{P}}^q\mathcal{F}(m,n)) \]

Then by the above results \( E_2^{p,q} = 0 \) for every \( q \geq 1 \).

This implies that \( E_2^{p,q} = E_\infty^{p,q} \), which in turn implies

\[ H^p(U', \mathcal{O}(m,n)) \cong H^p(U, \rho_{\mathbb{P}}^q\mathcal{F}(m,n)) \]

so that \( H^p(U', \mathcal{O}(m,n)) \cong H^p(U, \rho_{\mathbb{P}}^q\mathcal{F}(m,n)) \)

and \( H^p(U', \mathcal{O}(m,n)) = 0 \quad \forall \, q \geq 1 \).

From this we can conclude

\[ (3.14) \quad H^p(U', \mathcal{O}(m,n)) \cong H^p(U, \mathcal{O}(A' \cdots D')(A \cdots E)). \]

The same reasoning can be used to prove

\[ (3.15) \quad H^p(U', \mathcal{O}(m,n-2)) \cong H^{p-1}(U, \mathcal{O}(A' \cdots D')(A \cdots E)[-1]) \]

\[ (3.16) \quad H^p(U', \mathcal{O}(m-2,n)) \cong H^{p-1}(U, \mathcal{O}(A' \cdots E)[-1][1]) \]

\[ (3.17) \quad H^p(U', \mathcal{O}(m-2,n-2)) \cong \]

\[ H^{p-2}(U, \mathcal{O}(A' \cdots D')(A \cdots E)[-1,-1]). \]

(For all of the above we have \( m,n \geq 0 \) with \( m \) primed indices and \( n \) unprimed indices.) Other cohomology
groups on $U'$ vanish by the vanishing statements on $\rho_A^q(m,n)$. Recalling that $\Omega^1_G = \Theta(1,1)[-1,-1]$, we also have the following isomorphisms:

\begin{align*}
(3.18) \quad & H^p(U', \Omega^1_G(m,n)) \cong H^p(U, \Theta(A' \cdots E')(A' \cdots F)) \\
(3.19) \quad & H^p(U', \Omega^1_G(m,-n-2)) \cong H^{p-1}(U, \Theta(A' \cdots E')[-1]) \\
(3.20) \quad & H^p(U, \Omega^1_G(-m,2,n)) \cong H^{p-1}(U, \Theta(A' \cdots E')[-1]) \\
(3.21) \quad & H^p(U, \Omega^1_G(-m,2,-n-2)) \cong H^{p-2}(U, \Theta(A' \cdots E')(A' \cdots F)[-1,-1])
\end{align*}

where $m,n \geq -1$ and there are $m+1$ primed indices and $n+1$ unprimed indices.

We end the section on a final note. The deRham sequence on $G$

\[ 0 \to \sigma^{-1} \Theta_A \to \Theta_G \xrightarrow{d_G} \Omega^1_G \to 0 \]

can be transformed to $\mathcal{M}$ via the above direct image results. In particular, the operation of differentiation along the fibres of $\sigma$ transforms into the exterior derivative on $\mathcal{M}$:

\begin{align*}
0 \to \sigma^{-1} \Theta_A & \to \Theta_{\mathcal{M}} \xrightarrow{d_{\mathcal{M}}} \Omega^1_{\mathcal{M}} \to 0 \\
0 \to \rho^* (\sigma^{-1} \Theta_A) & \to \Theta_{\mathcal{M}} \xrightarrow{\nabla AA'} \Theta_{\mathcal{M}} \to 0 \\
0 \to \Theta_{\mathcal{M}} & \to \Theta_{\mathcal{M}} \xrightarrow{d} \Omega^1_{\mathcal{M}} \to 0.
\end{align*}

Thus the direct image of $\sigma^{-1} \Theta_A$ naturally gives solutions to $df = 0$, where $f$ is a holomorphic function on $U \subset \mathcal{M}$. 
6. **Direct images sheaves for vector bundles.** In the last section we calculated the direct images of the sheaves \( \mathcal{O}(m,n) \) on \( \mathbb{G} \); now shall examine pullback bundles \( \sigma^*V \) over \( U' \subset \mathbb{G} \) where \( V \) is a \( U \)-trivial vector bundle of rank \( r \) on \( U'' \subset \mathbb{A} \). We recall that in section III.2 we defined a \( U \)-trivial bundle \( V \) to be one such that the restriction \( V|_{\mathbb{P}^1_x} \) is trivial for each \( x \in U \). This implies that \( \sigma^*V \) is trivial on each restriction to \( \rho^{-1}(x) \). This makes calculation of the direct images of \( \mathcal{O}(\sigma^*V) \) rather easy. We first note, using a result from Chapter IV, that the zeroth direct image of \( \mathcal{O}(\sigma^*V) \) is a locally free sheaf and thus can be associated uniquely with a vector bundle \( \tilde{V} - U \) of rank \( r \):

\[
\rho_* \mathcal{O}(\sigma^*V) \cong \mathcal{O}(\tilde{V}).
\]

Also, any direct image \( \rho^q_* \mathcal{O}(\sigma^*V) \) for \( q \geq 1 \) must vanish since \( \sigma^*V \) is trivial along each \( \rho^{-1}(x) \). Using the Kunneth formula for sheaves ([19]) we then have that

\[
\rho^q_\sigma[\mathcal{O}(m,n) \otimes \mathcal{O}(\sigma^*V)] = \mathcal{O}(m,n) \otimes \mathcal{O}(\tilde{V})
\]

for all \( q \) and all \( m,n \).

In this situation the relative deRham sequence on \( U' \) is

\[
0 \to \sigma^{-1}\mathcal{O}(V) \to \mathcal{O}(\sigma^*V) \overset{d_\sigma}{\to} \Omega^1_{\sigma}(V) \overset{\partial}{\to} \mathcal{O}(\sigma^*V) \otimes \mathcal{O}(1,1)[-1,-1]
\]

Differentiation along the fibres \( d_\sigma \) remains well-defined.
since $d_0$ annihilates the transition functions of $\sigma^*V$. Then the Leray spectral sequence of the preceding section can be applied. It reads

\[ E_2^{p,q} = H^p(U, \rho^*_X[S(\xi, m, n) \otimes \sigma(\sigma^*V)]) = H^p(U', \rho(\xi, m, n) \otimes \sigma(\sigma^*V)) \]

Thus the inclusion of $\sigma^*V$ in the direct images adds no complications to the results of the preceding section.

Under the zeroth direct image the relative deRham sequence transforms to

\[
\begin{array}{ccccccc}
0 & \rightarrow & \sigma^{-1}\sigma(V) & \rightarrow & \sigma(\sigma^*V) & \rightarrow & \Omega^1_0(V) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \rho^*_X(\sigma^{-1}\sigma(V)) & \rightarrow & \tilde{\sigma}(\tilde{V}) & \rightarrow & \Omega^1_{AA'}(\tilde{\tilde{V}}) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \rho^*_X(\sigma^{-1}\sigma(V)) & \rightarrow & \tilde{\sigma}(\tilde{V}) & \rightarrow & \Omega^1_{\mathbb{R}^M} \otimes \sigma(\tilde{V}) \rightarrow 0
\end{array}
\]

The induced operator $D$ is the covariant derivative operator on $\tilde{\tilde{V}}$ defined so that the sections of $\rho^*_X(\sigma^{-1}\sigma(V))$ are covariantly constant. In local coordinates on $\mathbb{R}^M$ it may be written

\[ D = (\nabla_{AA'} + A_{AA'}) \, dx^{AA'}, \]

where $A = A_{AA'} \, dx^{AA'}$ is the associated connection form.

Following [10] we describe the connection. Cover $U'$ with the set $\{U'_a\}$ such that $\sigma^*V$ is trivializable on the open sets in the cover. Suppose that $[g_{aa}(x^{AA'}, [\pi_A], [\eta_A])]$ is the set of transition functions for $\sigma^*V$ with
respect to the cover \{U'_\alpha\}. Since \( \sigma^*\mathcal{V} \) is trivial on each \( \rho^{-1}(x) \) we may split \( g_{\alpha\beta} \):
\[
g_{\alpha\beta}(x^{AA'}, [\pi_{A'}], [\eta_A]) = h_\alpha(x^{AA'}, [\pi_{A'}], [\eta_A]) \cdot h_\beta(x^{AA'}, [\pi_{A'}], [\eta_A])
\]
where \( h_\alpha, h_\beta \) are \( n \times n \) matrices on \( U'_\alpha, U'_\beta \), respectively.

Since \( \eta_A, \pi_{A'}, \nu^{AA'} g_{\alpha\beta} = 0 \), it is easy to see that
\[
h_\alpha^{-1}. \eta_A, \pi_{A'}, \nu^{AA'} h_\alpha = h_\beta^{-1}. \eta_A, \pi_{A'}, \nu^{AA'} h_\beta.
\]
We then define \( A_{AA'} \) by the following:
\[
\eta A, \pi_{A'}, AA' = h_\alpha^{-1}. \eta A, \pi_{A'}, AA' h_\alpha
\]
and check that \( A_{AA'} \) is independent of \( \eta_A \) and \( \pi_{A'} \),
and holomorphic in \( x^{AA'} \). This defines D.
IV. The Penrose Transform and the Ward Correspondence

Both the Penrose transform for solutions to the zero-rest-mass equations and the Ward correspondence for solutions to the Yang-Mills field equations can be generalized to the ambitwistor case. In both cases the solutions obtained are in general non-self-dual as opposed to the self-dual or anti-self-dual solutions which arise from the original work ([16],[20]). Further, the ambitwistor work includes as a special case the original results.

1. The ambitwistor transform. Define on $\mathcal{M}$ the sheaf

$$Z_{m,n} := \ker \nabla^{AA'} : \bigotimes (B' \ldots F)_{n} (A' \ldots D')_{-1}'' \rightarrow \bigotimes (A' \ldots F)_{-2} (B' \ldots D')_{-1}.'$$

This generalizes the sheaf $Z'_{m}$ of holomorphic right-handed massless free fields of helicity $m/2$ defined in section 5 of [5]. In particular, for the case $n=0$ we have

$$Z_{m,0} = \ker \nabla^{AA'} : \bigotimes (A' \ldots D')_{-1}'' \rightarrow \bigotimes (B' \ldots D')_{-2}.'$$

Theorem IV.1- For $U$ open in $\mathcal{M}$ such that $\sigma: U' \rightarrow U''$
is 1-elementary and for $m \geq 1$, $n \geq 0$, there is a canonical isomorphism
\[
G : H^1(U'', \mathcal{O}(-m-2, n)) \cong \Gamma(U', \mathcal{O}_m, n).
\]
In the case that $m = 0$, $n > 0$, there is a canonical isomorphism
\[
G : H^1(U'', \mathcal{O}(-2, n)) \cong \Gamma(U', \mathcal{O}(B \cdots F)[-1]'),
\]
where there are $n$ indices.

The above two mappings along with several others which will be introduced later in this section will be referred to as the **ambitwistor transform** for the sheaf $\mathcal{O}_m, n$, $\mathcal{O}(B \cdots F)[-1]'$, etc.; each of them will be written $G$.

**Proof** - Since $U$ is assumed to be 1-elementary the pullback mapping
\[
\sigma^* : H^1(U'', \mathcal{O}(-m-2, n)) \to H^1(U', \mathcal{O}(-m-2, n))
\]
is an isomorphism by Theorem III.1. The relative deRham sequence for this sheaf is
\[
0 \to \mathcal{O}_m(-2, n) \to \mathcal{O}(-m-2, n) \to \Omega^1_\sigma(-m-2, n) \to 0.
\]
The associated long exact sequence is
\[
(4.1) \cdots \to \Gamma(U', \mathcal{O}(-m-1, n+1)[-1, -1]) \to H^1(U', \mathcal{O}(-m-2, n)) \to H^1(U', \mathcal{O}(-m-2, n)) \to \cdots
\]
which can be rewritten by the above work on direct image sheaves as
(4.2) \[ \cdots \to 0 \to H^1(U', \sigma^{-1} \otimes (-m-2, n)) \to \]

\[ \Gamma(U, \sigma^{(B' \cdots F)} \otimes (A' \cdots D')[-1]) \to \Gamma(U, \sigma^{(A' \cdots D')} \otimes (B' \cdots D')[-2, 1]) \]

where we have zero for the last term if \( m = 0 \). Both results follow immediately since (4.2) is exact. \( \Box \)

We now define a second sheaf on \( \mathbb{IM} \),

\[ \mathbb{Z}_{m,n} := \ker \vartheta^{A'} : \sigma^{(B' \cdots E')} \otimes (A' \cdots D')[-1] \to \sigma^{(A' \cdots E')} \otimes (B' \cdots D')[-1, 2]. \]

This sheaf is a generalization of the sheaf of holomorphic left-handed massless free fields of helicity \( n/2 \) from [5]. As before the case \( m=0 \) coincides with the twistor case:

\[ \mathbb{Z}_{0,n} = \ker \vartheta^{A'} : \sigma^{(A' \cdots D')}[-1] \to \sigma^{A'} \otimes (B' \cdots D')[-1, 2]. \]

The same technique used in proving the above theorem suffices to prove the following

**Theorem IV.2** - For \( U \) open in \( \mathbb{IM} \) such that \( \sigma: U' \to U'' \) is \( 1 \)-elementary and for \( n \geq 1, m \geq 0 \) there is a canonical isomorphism

\[ C: H^1(U'', \sigma^{(m, -n-2)}) \cong \Gamma(U, \mathbb{Z}_{m,n}). \]

In the case that \( n = 0, m \geq 0 \), there is a canonical isomorphism

\[ C: H^1(U'', \sigma^{(m, -2)}) \cong \Gamma(U, \sigma^{(B' \cdots E')}[-1]). \]
As discussed in [5] the twistor transform produces solutions to Maxwell's equations both directly—by giving the electromagnetic field—and indirectly—via potentials. The approach which generalizes in the ambitwistor case to give non-self-dual solutions is the latter. Before writing down the ambitwistor transform for electromagnetic fields, we give a brief outline of how these fields are determined by potentials.

We define a **Maxwell potential** on \( U \subseteq M \) to be a section of \( \mathcal{O}_{AA} \) over \( U \). The electromagnetic field tensor (on a local coordinate system)

\[
F_{AA'B'B'} = \varepsilon_{AB} \psi_{A'B'} + \varepsilon_{A'B'} \varphi_{AB}
\]

can be derived from a Maxwell potential \( \mathfrak{g}_{AA'} \in \Gamma(U, \mathcal{O}_{AA'}) \) by

\[
\begin{align*}
\psi_{A'B'} &= \nabla^{A'} (B' \mathfrak{g}_{A'})^A \\
\varphi_{AB} &= \nabla^{A'} (B' \mathfrak{g}_A)^A' .
\end{align*}
\]

This corresponds to exterior differentiation in the following way: if we write \( \mathfrak{g} = \mathfrak{g}_{AA'} dx^{AA'} \) and \( F = F_{AA'B'B'} dx^{AA'} \wedge dx^{BB'} \) for the one- and two-forms giving the potential and the field, then (4.3) is equivalent to \( d\mathfrak{g} = F \).

It is evident that we can add to \( \mathfrak{g} \) a section of the form \( df \), where \( f \in \Gamma(U, \mathcal{O}) \), without changing \( F \):

\[
d(\mathfrak{g} + df) = d\mathfrak{g} = F .
\]
This transformation can be written in spinor notation as

\[ \hat{\xi}_{AA'} = \hat{\xi}_{AA'} + \nabla_{AA'} \hat{f}. \]

This freedom in writing down a potential for an electromagnetic field is termed a **gauge freedom**, an electromagnetic field being given by a Maxwell potential modulo gauge freedom.

**Theorem IV.3** - For \( U \) open in \( \mathcal{M} \) such that \( \sigma: U' \to U'' \) is 1-elementary there is a canonical isomorphism

\[ \sigma_*: H^1(U'', \mathcal{O}) \cong \frac{[\text{Maxwell potentials}]}{[\text{gauge freedom}]} \]

**Proof** - We have as usual the isomorphism

\[ \sigma_*: H^1(U'', \mathcal{O}) \cong H^1(U', \sigma^{-1}\mathcal{O}). \]

The relative deRham sequence has the related long exact sequence

\[ 0 \to \Gamma(U', \sigma^{-1}\mathcal{O}) \to \Gamma(U', \mathcal{O}) \xrightarrow{\text{d}_\sigma} \Gamma(U', \Omega^1_\sigma) \to H^1(U', \sigma^{-1}\mathcal{O}) \to H^1(U', \mathcal{O}) \to \ldots. \]

Using the results on direct images

\[ \Gamma(U', \mathcal{O}) \cong \Gamma(U, \mathcal{O}) \]
\[ \Gamma(U', \Omega^1_\sigma) \cong \Gamma(U, \mathcal{O}_{AA'}) \]
\[ H^1(U', \mathcal{O}) = 0 \]

immediately implies the result

\[ H^1(U', \sigma^{-1}\mathcal{O}) \cong \frac{\Gamma(U, \mathcal{O}_{AA'})}{\nabla_{AA'} \Gamma(U, \mathcal{O})}, \]

which completes the proof. \( \Box \)
The general result is an easy generalization of the above work, although it is not immediately clear what physical significance, if any, it has outside of the cases which arise in the normal twistor construction. There is a canonical isomorphism for \( m,n \geq -1 \),

\[
H^1(U'', \mathcal{O}(m,n)) \cong \frac{\Gamma(U, \mathcal{O}(A' \cdots E')(A' \cdots F)[-1,-1])}{\nu_{\mathfrak{A}}' \Gamma(U, \mathcal{O}(B' \cdots E')(B' \cdots F)])}
\]

where the bottom term is taken to be zero if \( m=-1 \) or \( n=-1 \). The case \( n=0 \) is equivalent to the twistor construction given in [5] and gives potentials for left-handed massless free fields. If \( m=0 \) we get right-handed massless free fields, and the case \( m=n=0 \) is the electromagnetic case discussed above.

It is instructive at this point to see exactly how the above ambitwistor results contain the twistor results of [5]. Suppose \( U \subset \mathcal{M} \) is open and that \( \sigma: U' \to U'' \) and \( \mu: W' \to W'' \) are each 1-elementary, where we write \( W' = \nu^{-1}(U) \) and \( W'' = \mu(W') \). The Penrose transform for right-handed massless fields can be given by the string of isomorphisms (see [5] or [21]) for \( m \geq 1 \):

\[
H^1(W'', \mathcal{O}(-m-2)) \cong H^1(W', \mathcal{O}(-m-2)) \cong \ker \frac{d}{\mathfrak{A}}: H^1(W', \mathcal{O}(-m-2)) \to H^1(W', \mathcal{O}(-m-2)) \cong \ker \frac{\nu_{\mathfrak{A}}'}{\mathfrak{A}}: \Gamma(U, \mathcal{O}(A' \cdots D')[-1]) \cong \Gamma(U, \mathcal{O}(A\cdots D')[-2]).
\]
The last line is just \( \Gamma(U, Z'_m) \), where \( Z'_m \) is the sheaf of holomorphic right-handed massless free fields of helicity \( m/2 \), as discussed above. According to Theorem IV.1 we have for the ambitwistor case

\[ H^1(U'', \mathcal{O}(-m-2, 0)) \cong \Gamma(U, Z_m', 0) \]

where \( Z_m, 0 = Z'_m \) as noted above. We shall now compare the two constructions.

Since \( \tau^{-1}_{\mathcal{A}}(x) \cong \tau^{-1}(x) \times (\tau^*)^{-1}(x) \cong IP_1 \times IP_1 \) for each \( x \in U \), the Kunneth formula implies

\[ H^1(U'', \mathcal{O}(-m-2, 0)) \cong H^1(W'', \mathcal{O}(-m-2)) \]

using

\[ H^0(IP_1, \mathcal{O}) \cong \mathcal{O} \]

and

\[ H^1(IP_1, \mathcal{O}(-m-2)) = 0, \ m \geq 1. \]

The pullbacks to the correspondence spaces of these two cohomology classes are then also isomorphic:

\[ H^1(U'', \sigma^{-1} \mathcal{O}(-m-2, 0)) \cong H^1(W'', \mu^{-1} \mathcal{O}(-m-2)). \]

We then must examine the two relative deRham sequences, on \( IP \):

\[ (4.4) \quad 0 \to \mu^{-1} \mathcal{O}(-m-2) \to \mathcal{O}(-m-2) \overset{d}{\to} \Omega^1_\mu(-m-2) \overset{d}{\to} \Omega^2_\mu(-m-2) \to 0 \]

and on \( C \),

\[ 0 \to \sigma^{-1} \mathcal{O}(-m-2, 0) \to \mathcal{O}(-m-2, 0) \overset{d}{\to} \Omega^1_\sigma(-m-2, 0) \to 0. \]

These sequences may be rewritten as

\[ (4.4') \quad 0 \to \mu^{-1} \mathcal{O}(-m-2) \to \mathcal{O}(-m-2) \overset{\pi_A^A, \nu_A^A}{\to} \mathcal{O}^A(-m-1)[-1, -1] \]

\[ \overset{\pi_A^A, \nu_A^A}{\to} \mathcal{O}(-m)[-2, -1] \to 0 \]

and
0 \rightarrow \sigma^{-1} \odot (-m-2,0) \rightarrow \odot (-m-2,0) \xrightarrow{\eta_A^\pi_A' \nabla^{AA'}} \odot (-m-1,1)[-1,-1] \rightarrow 0.

It is again easy to check that
\[ H^1(U', \odot (-m-2,0)) \cong H^1(W', \odot (-m-2)). \]

To compare the descent to \( M \) in the two cases we must examine the spectral sequences of the twistor case carefully; the ambi-twistor case is simpler since the relative deRham sequence has only three terms.

In order to trace the information obtained from (4.4) by the spectral sequence of a differential resolution, we split it into two short exact sequences
\[
0 \rightarrow \sigma^{-1} \odot (-m-2) \rightarrow \odot (-m-2) \xrightarrow{d_{\mu}} \mathcal{S} \rightarrow 0
\]
\[
0 \rightarrow \Omega_{\mu}^1(-m-2) \xrightarrow{d_{\mu}} \Omega_{\mu}^2(-m-2) \rightarrow 0
\]
where \( \mathcal{S} : = \ker d_{\mu} : \Omega_{\mu}^1(-m-2) \rightarrow \Omega_{\mu}^2(-m-2) \). We then obtain the long exact sequence
\[
(4.5) \quad \cdots \rightarrow \Gamma(W', \mathcal{S}) \rightarrow H^1(W', \sigma^{-1} \odot (-m-2)) \rightarrow H^1(W', \odot (-m-2)) \xrightarrow{d_{\mu}} H^1(W', \mathcal{S}) \rightarrow \cdots.
\]

Let us compare this sequence with (4.1) above. We have
\[ \Gamma(W', \mathcal{S}) = 0 \] since \( \Gamma(W', \Omega_{\mu}^1(-m-2)) \cong \Gamma(W', \sigma^A (-m-1)[-1,-1]) = 0 \) just as we have \( \Gamma(U', \odot (-m-1,1)[-1,-1]) = 0 \) in (4.1) (for \( n = 0 \)). The next two terms in the respective sequences are isomorphic by the above comments. We may then write the two sequences:
\[
0 \rightarrow H^1(W', \sigma^{-1} \odot (-m-2)) \rightarrow H^1(W', \odot (-m-2)) \xrightarrow{\eta_A^\pi_A' \nabla^{AA'}} H^1(W', \mathcal{S})
\]
\[
0 \to H^1(U', \sigma^{-1} \otimes (-m-2,0)) \to H^1(U', \sigma(-m-2,0)) \xrightarrow{\eta_A \pi_A^* v^{AA'}} \\
H^1(U', \sigma(-m-1,1)[-1,-1])
\]
so that
\[
(4.6) \quad H^1(W', \mu^{-1} \otimes (-m-2)) \cong \ker \pi_A^* v^{AA'} : H^1(W', \sigma(-m-2)) \\
\to H^1(W', \sigma)
\]
\[
(4.7) \quad H^1(U', \sigma^{-1} \otimes (-m-2,0)) \cong \ker \eta_A \pi_A^* v^{AA'} : H^1(U', \sigma(-m-2)) \\
\to H^1(U', \sigma(-m-1,1)[-1,-1]).
\]
To see that these are equivalent we first note that we can rewrite (4.6) as
\[
(4.6)' \quad H^1(W', \mu^{-1} \otimes (-m-2)) \cong \ker \pi_A^* v^{AA'} : H^1(W', \sigma(-m-2)) \\
\to H^1(W', \sigma A(-m-1)[-1,-1]).
\]
As above we have an isomorphism
\[
H^1(W', \sigma A(-m-1,1)[-1,-1]) \cong H^1(U', \sigma A(-m-1,0)[-1,-1]).
\]
Finally, using the results of the last chapter, we note that
\[
\eta_A : H^1(U', \sigma A(-m-1,0)[-1,-1]) \to H^1(U', \sigma(-m-1,1)[-1,-1])
\]
is an isomorphism, both of these groups being isomorphic to \(\Gamma(U, \sigma^A_{(B'\ldots D')}[-1])\). Thus (4.6)' and (4.7) contain exactly the same information since the corresponding groups are isomorphic and the mappings \(\pi_A^* v^{AA'}\) and \(\eta_A \pi_A^* v^{AA'}\) agree as mappings on \(H^1(U', \sigma(-m-2,0))\). The final step is to transform to \(\mathbb{M}\) to get, for both the twistor and the ambitwistor case,
\[
\ker v^{AA'} : \Gamma(U, \sigma_{(A'\ldots D')}[-1])' \to \\
\Gamma(U, \sigma^A_{(B'\ldots D')}[-2,-1]).
\]
A similar analysis applies to Theorem IV.2. Theorem IV.3 is a special case of the generalized Ward correspondence discussed in the next section, and we shall discuss its relation with the twistor case after having introduced that correspondence.

To close out the section we shall derive some results which will be used in Chapter V when we discuss extensions of vector bundles on $\mathbb{A}$ to neighborhoods in $\mathbb{PP} \times \mathbb{PP}^*$. 

**Theorem IV.4**- Let $U$ be open in $\mathbb{M}$ such that $\sigma: U' \to U''$ is 2-elementary. Then

(a) $H^1(U'', \mathcal{O}(-1,-1)) \cong \Gamma(U, \mathcal{O}[-1,-1])$

(b) $H^1(U'', \mathcal{O}(-k,-k)) = 0$, $k \geq 1$

(c) $H^2(U'', \mathcal{O}(-1,-1)) = 0$

(d) $H^2(U'', \mathcal{O}(-2,-2)) \cong \Gamma(U, \mathcal{O}[-1,-1])$

(4.8) (e) $H^2(U'', \mathcal{O}(-3,-3)) \cong \Gamma(U, \text{ker } d: \Omega^3_M \to \Omega^4_M)$

**Proof**- The vanishing statements follow easily from the long exact sequences associated to the relative deRham sequences and the vanishing of cohomology groups in those sequences on $\mathbb{G}$. To prove (b) we need only note that $H^1(U', \mathcal{O}(-k,-k)) = 0$ for all $k$ and $\Gamma(U', \mathcal{O}(-k,-k)) = 0$ for all $k > 0$. For (c) use the fact that $H^2(U', \mathcal{O}(-1,-1)) = 0$ and $H^1(U', \mathcal{O}[-1,-1]) = 0$.

We wish to spend more time developing the remaining
isomorphisms since later in the paper we shall have to trace these isomorphisms explicitly. We shall start our discussion for each of the isomorphisms on the correspondence manifold $\mathcal{G}$, assuming that the pullback isomorphism along the fibres has already been performed. We specify the isomorphism (a) by the exact sequence

$$\cdots \to \tau(U', \mathcal{O}(-1,-1)) \to \tau(U', \mathcal{O}^{1}(-1,-1)) \to H^{1}(U', \sigma^{-1} \mathcal{O}(-1,-1))$$

$$\to H^{1}(U', \mathcal{O}(-1,-1)) \to \cdots$$

which can be rewritten as

$$\cdots \to 0 \to \tau(U, \mathcal{O}[-1,-1]) \to H^{1}(U', \sigma^{-1} \mathcal{O}(-1,-1)) \to 0.$$ 

Then (a) is the composition of two isomorphisms: the inverse of the coboundary homomorphism $\delta$, which is an isomorphism in this case, and the isomorphism $\tau(U', \mathcal{O}[-1,-1]) \equiv \tau(U, \mathcal{O}[-1,-1])$, which is just given by evaluation along the fibres.

To prove (d) we use the exact sequence

$$\cdots \to H^{1}(U', \mathcal{O}^{1}(-2,-2)) \to H^{2}(U', \sigma^{-1} \mathcal{O}(-2,-2)) \to$$

$$\tau(U, \mathcal{O}[-1,-1]) \to H^{2}(U', \mathcal{O}(-2,-2))$$

$$\tau(U, \mathcal{O}[-1,-1]) \to H^{2}(U', \mathcal{O}^{1}(-2,-2)) \cdots$$

The right-hand isomorphism is given fibrewise by the canonical isomorphism (cf. (3.21))

$$H^{2}(\sigma^{-1}(x), \mathcal{O}(-2,-2)) \equiv \tau(x, \mathcal{O}[-1,-1]).$$

For (e) the exact sequence reads

$$\cdots \to H^{1}(U', \mathcal{O}^{1}(-3,-3)) \to H^{2}(U', \sigma^{-1} \mathcal{O}(-3,-3)) \to$$

$$H^{2}(U', \mathcal{O}(-3,-3)) \to H^{2}(U', \mathcal{O}^{1}(-3,-3)) \to \cdots$$
This sequence is canonically isomorphic to
\[ 0 \to H^2(U', \sigma^{-1}e(-3,-3)) \to H^0(U, \sigma_{\AA'}^{AA'} [-1,-1]) \to H^0(U, e [-2,-2]). \]
Noting that the last mapping is canonically isomorphic to
\[ d: \Omega^3_M \to \Omega^4_M \]
completes the proof. □

2. The generalized Ward correspondence. The Ward correspondence ([20]) provides a way of encoding the information describing self-dual Yang-Mills fields on Minkowski space geometrically as vector bundles over \( \mathbb{R}^4 \). A generalization due independently to Isenberg-Yasskin-Green ([11]) and Witten ([25]) provides for the general non-self-dual case. In this section we shall interpret and prove this correspondence as a sheaf-theoretic result, as opposed to the vector bundle/transition function approach originally used.

We shall first briefly discuss Yang-Mills fields on \( \mathbb{R}^4 \) and then introduce an equivalent sheaf-theoretic concept; details for the following description can be found, for instance, in [1] or [13]. Suppose \( G \) is a holomorphic Lie subgroup of the Lie group \( GL(n,\mathbb{C}) \); let \( \tilde{P} \) be a principal \( G \)-bundle and \( \tilde{E} \) its associated rank \( n \) vector bundle. Suppose further that \( A' \) is a holomorphic connection on \( \tilde{P} \) and \( A \) the related connection on \( \tilde{E} \). If we denote by \( \mathfrak{g} \) the Lie algebra of \( G \), then \( A \) can be con-
sidered to be a $g$-valued 1-form on $\mathcal{M}$; since $G \subseteq \text{GL}(n, \mathbb{C})$ we consider $g$ to be a matrix group. Associated to the connection is the covariant derivative

$$D_A : \Omega^i(\tilde{E}) \to \Omega^{i+1}(\tilde{E})$$

and the associated curvature $F$, a $g$-valued 2-form on $\mathcal{M}$. This curvature is referred to as the Yang-Mills field and is completely specified by the connection $A$.

Alternatively, we can consider a connection as smoothly defining for each $y \in \tilde{E}$ a subset $H_y$ of $T_y\tilde{E}$, called the set of horizontal tangent vectors at $y$, such that:

(i) $T_y\tilde{E} = H_y + V_y$, where $V_y$ consists of the vectors in $T_y\tilde{E}$ "pointing along the fibres" of $\tilde{E} - \mathcal{M}$; and

(ii) if $g : \tilde{E} \to \tilde{E}$ is the mapping induced from the group action $g : \tilde{\mathbb{P}} \to \tilde{\mathbb{P}}$ for $g \in G$, and $y' = gy$, then

$$g_xH_y = H_{y'}.$$

Given a connection on an associated vector bundle $\tilde{E} - U \subseteq \mathcal{M}$, we shall now introduce an equivalent sheaf-theoretic concept. Suppose that for each null line $l$ in $\mathcal{M}$ the intersection $l \cap U$ is connected and simply connected, where $U$ is assumed to be open. Then we define a subsheaf $\mathcal{F}$ of $\mathcal{G}(\tilde{E})$, the sheaf of horizontal sections of $\tilde{E}$, as follows. Let $s \in \mathcal{G}(V, \tilde{E})$ for $V \subseteq U$ a neighborhood of $x \in U$, and let $\{ X_i, i = 1, 2, 3, 4 \}$ be a set of null vectors spanning $T_x \mathcal{M}$. Then $s$ is horizontal at $x$ if $s^*(X_i) \in H_s(x)$ for each $i$. We say
s \in \mathcal{F}(V) \text{ if } s \text{ is horizontal at each point in } V. \text{ Since we have assumed } U \text{ is connected and simply connected for each } \mathcal{L}, \text{ we may think of } \mathcal{F} \text{ as the sheaf of sections of } \tilde{E} \text{ constant along the null lines. We note that (ii) implies that } \mathcal{F} \text{ is invariant under } g: \tilde{E} \to \tilde{E} \text{ for each } g \in G.

Conversely, a subsheaf \( \mathcal{F} \subset \mathcal{O}(\tilde{E}) \) will be called horizontal if \( \mathcal{F} \) is invariant under \( g: \tilde{E} \to \tilde{E} \) for each \( g \in G \) and if the following condition holds: if \( \Gamma(\mathcal{L} \cap U, \tilde{E}|_{\mathcal{L}}) \) is the subsheaf of sections of \( \tilde{E}|_{\mathcal{L}} \) which are the restrictions of sections of \( \mathcal{F} \), then for any \( x \in \mathcal{L} \) there is a canonical isomorphism

\[
\tilde{E}|_x \cong \Gamma_{\mathcal{F}}(\mathcal{L} \cap U, \tilde{E}|_{\mathcal{L}}),
\]

for every null line \( \mathcal{L} \). Such a sheaf defines a connection on \( \tilde{E} \) in the following way: if \( s \in \mathcal{F}(V) \) where \( V \) contains the point \( x \) and \( s(x) = y \), define

\[ H_y := \text{subset of } T_{\tilde{E}} \text{ spanned by } \{ s_{xX_i}, i=1,2,3,4 \}. \]

Thus the specification of a horizontal subsheaf of the sheaf of sections of a vector bundle is equivalent to the specification of a connection on that vector bundle; it is not difficult to move from one definition of the connection to another, calculating the 1-form \( A \) from the set \( \{ H_y : y \in \tilde{E} \} \) and vice versa.

A given Yang-Mills field is given by choosing a connection \( A \), but the choice of this connection is not unique. For example, in the case of electromagnetic fields, \( F = dA \), so that the addition to \( A \) of any closed
1-form does not change $F$, as discussed in Chapter III. In the general case (cf. [14], for example) we may perform a **gauge transformation** on $A$, 

$$A \rightarrow g^{-1}Ag + g^{-1}dg,$$

where $g \in \Gamma(U, U \times G)$. A Yang-Mills field is then represented by an equivalence class of connections, the equivalence being given by the gauge transformations. The equivalence classes for the sheaf-theoretic approach are given as follows. Two pairs $(\mathcal{F}, \tilde{E})$ and $(\mathcal{F}', \tilde{E}')$ are equivalent, written $(\mathcal{F}, \tilde{E}) \sim (\mathcal{F}', \tilde{E}')$, if:

(i) $\tilde{E} \sim \tilde{E}'$, i.e., $\tilde{E}$ and $\tilde{E}'$ are equivalent as vector bundles; and

(ii) considering $\mathcal{F}'$ as a horizontal subsheaf of $\tilde{E}$ under the equivalence (i), there is a section $g \in \Gamma(U, U \times G)$ such that $g: \mathcal{F} \rightarrow \mathcal{F}'$.

With these preliminaries we can state the generalized Ward correspondence:

**Theorem IV.5**- Let $U$ be an open set of $\mathbb{M}$ such that the intersection of $U$ with each null line is either empty or connected and simply connected. Then there is a one-to-one correspondence between

(a) equivalence classes of holomorphic associated vector bundles of rank $n$, $E - U'' \subset \mathcal{A}$ such that $E|_{\tau-1}(x)$ is trivial for every $x \in U$; and

(b) equivalence classes of pairs $(\mathcal{F}, \tilde{E})$ where $E$
is a holomorphic associated vector bundle of rank \( n \) over \( U \), and \( \mathcal{F} \) is a horizontal subsheaf of \( \mathcal{O}(\hat{E}) \).

**Proof:** Given \( E \to U'' \) as in (a) we form the pair of sheaves over \( U' \), \( \sigma^{-1}\mathcal{O}(E) \subseteq \mathcal{O}(\sigma^*E) \); we then take direct images to get a pair of sheaves \( \rho_*(\sigma^{-1}\mathcal{O}(E)) \subseteq \rho_*\mathcal{O}(\sigma^*E) \) over \( U \).

We claim that this pair of sheaves satisfies the conditions for (b). First we show that \( \rho_*\mathcal{O}(\sigma^*E) \) is a locally free sheaf so that we may associate to it a vector bundle. Note that \( \sigma^*E|_{\rho^{-1}(x)} \) is trivial for every \( x \in U \) and that \( \rho^{-1}(x) \cong \mathbb{P}^1 \times \mathbb{P}^1 \) is compact for every \( x \). The direct image sheaf \( \rho_*\mathcal{O}(\sigma^*E) \) is given by the presheaf

\[
\mathcal{V} - \Gamma(\rho^{-1}(\mathcal{V}), \sigma^*E).
\]

For small \( \mathcal{V} \), \( \rho^{-1}(\mathcal{V}) \cong \mathcal{V} \times (\mathbb{P}^1 \times \mathbb{P}^1) \) where \( \sigma^*E \) is trivial over each \( \mathbb{P}^1 \times \mathbb{P}^1 \). This implies that

\[
\Gamma(\rho^{-1}(\mathcal{V}), \sigma^*E) \cong \Gamma(\mathcal{V}', \sigma^*E|_{\mathcal{V}'})
\]

where \( \mathcal{V}' \cong \mathcal{V} \) and \( \mathcal{V}' \subseteq \rho^{-1}(\mathcal{V}) \subseteq U' \). But for small \( \mathcal{V} \),

\[
\Gamma(\mathcal{V}', \sigma^*E|_{\mathcal{V}'}) \cong \Gamma(\mathcal{V}', \mathcal{O}_{\mathcal{V}'} \oplus \cdots \oplus \mathcal{O}_{\mathcal{V}'}) \text{ (n times)}
\]

and so \( \rho_*\mathcal{O}(\sigma^*E) \) is locally free. Thus there is a rank \( n \) vector bundle \( \hat{E} \to U \) such that \( \mathcal{O}(\hat{E}) \cong \rho_*\mathcal{O}(\sigma^*E) \).

To check that \( \hat{E} \) is an associated vector bundle we need only see that its transition functions take values in \( G \); this follows easily from the fact that the transition functions of \( E \) take values in \( G \). Since \( \sigma^{-1}\mathcal{O}(E) \) is constant along the fibres of \( \sigma \) and these fibres project
down to null lines in U it is easy to see that $\rho_*(\sigma^{-1}\Theta(E))$
is a horizontal subsheaf of $\rho_*(\sigma^*E) \cong \Theta(\widetilde{E})$. The
invariance of $\rho_*(\sigma^{-1}\Theta(E))$ under multiplication by
g $\in G$ follows from the fact that the bundle $\sigma^*E$ has
transition functions which take values in G and which
are constant along the fibres of $\sigma$, and that $\sigma^{-1}\Theta(E)$
are those sections of $\sigma^*E$ which are constant along the
fibres; then $\sigma^{-1}\Theta(E)$ is invariant under multiplication
by g $\in G$, which implies the result.

Conversely, suppose we are given data (b). We claim
that

$$\sigma_*(\rho^{-1}\Theta \otimes \sigma^{-1}\Theta_{U''})$$

is a locally free sheaf on $U''$, and in particular that it
provides an inverse for the above construction. It is
defined by the presheaf

$$U'' \to \Gamma(\sigma^{-1}(V'') \cap U', \rho^{-1}\Theta \otimes \sigma^{-1}\Theta_{U''}).$$

It is easy to see, since $\rho^{-1}\Theta \otimes \sigma^{-1}\Theta_{U''}$ is constant along
the fibres of $\sigma$, that this defines a locally free sheaf
on $U''$, and thus a vector bundle of rank n over $U$, which
we shall denote $E$. It is also evident that $E|_{\tau^{-1}(x)}$ is
trivial for every $x \in U$.

It is an easy matter to check that under the above
construction if $E \sim E'$ then $(\mathcal{F}, \widetilde{E}) \sim (\mathcal{F}', \widetilde{E}')$ and vice versa. In fact, it is equivalent to check that $E$
is equivalent to the trivial rank n vector bundle on $U''$
if and only if $(\mathcal{F}, \widetilde{E})$ is equivalent to the pair $(\mathcal{C}, U \times \mathbb{C}^n)$,
where $U \times \mathbb{C}^n$ is the trivial bundle and $\mathbb{C}$ is the sheaf of constant sections. This follows from the fact that a vector bundle is trivial if and only if there are global nowhere-zero sections.

We claim that the above constructions are inverses. We show this in four steps:

(i) Given data (a), we have $\sigma'(\sigma^{-1}\Theta(E)) \cong \Theta(E)$:
\begin{align*}
V'' &\cong \Gamma(\sigma^{-1}(V'') \cap U', \sigma^{-1}\Theta(E)) \\
&= \Gamma(\sigma(\sigma^{-1}(V'') \cap U'), E) \\
&= \Gamma(V'', E)
\end{align*}

(ii) Given $\Theta \subset \Theta(\sigma^*E)$ such that $\Theta$ is constant along $\sigma^{-1}(X) \cap U'$, we have $\rho^{-1}(\rho_*(\Theta)) \otimes \sigma^{-1}\Theta_{U''} \cong \Theta$.

The sheaf $\rho^{-1}(\rho_*(\Theta))$ is defined by the presheaf
\begin{align*}
V' &\rightarrow \Gamma(V', \rho^{-1}(\rho_*(\Theta))) \\
&= \Gamma(\rho(V'), \rho_*(\Theta)) \\
&= \Gamma(\rho^{-1}(\rho(V')), \Theta).
\end{align*}

Thus the sections of $\rho^{-1}(\rho_*(\Theta))$ over $V'$ are obtained by restricting the sections of $\Theta$ over $\rho^{-1}(\rho(V'))$ to $V'$. But $\rho^{-1}(\rho(V'))$ is a family of compact fibres $\rho^{-1}(x)$ where $\Theta$ is trivial along each of these fibres, so these sections are constant along each $\rho^{-1}(x) \cap V'$.

We have thus lost exactly the holomorphic variation along $\rho^{-1}(x) \cap V'$ in $\Theta$. Since $\sigma^{-1}\Theta_{U''}$ varies holomorphically along $\rho^{-1}(x) \cap V'$ and is constant along the fibres of $\sigma$, tensoring with $\sigma^{-1}\Theta_{U''}$ exactly
provides the extra variation needed.

(iii) Given \( \mathcal{H} \subset \sigma(\mathfrak{E}^*) \) such that \( \mathcal{H} \) is locally constant along \( \phi^{-1}(x) \) and along \( \sigma^{-1}(X) \cap U' \), we have

\[
\sigma^{-1}(\sigma^* (\mathcal{H} \otimes \sigma^{-1}\mathfrak{E}_{U''})) \cong \mathcal{H} \otimes \sigma^{-1}\mathfrak{E}_{U''}:
\]

\[
V \rightarrow \Gamma(V', \sigma^{-1}(\sigma^* (\mathcal{H} \otimes \sigma^{-1}\mathfrak{E}_{U''})))
\]

\[
= \Gamma(\sigma(V'), \sigma^* (\mathcal{H} \otimes \sigma^{-1}\mathfrak{E}_{U''}))
\]

\[
= \Gamma(\sigma^{-1}(\sigma(V')) , \mathcal{H} \otimes \sigma^{-1}\mathfrak{E}_{U''})
\]

\[
\cong \Gamma(V', \mathcal{H} \otimes \sigma^{-1}\mathfrak{E}_{U''}).
\]

The isomorphism in the last line follows from the fact that \( \mathcal{H} \) and \( \sigma^{-1}\mathfrak{E}_{U''} \) are constant along the fibres of \( \sigma \), and that \( \mathfrak{E}^* \) is trivial on the fibres of \( \sigma \), so that there is nothing lost in restricting from \( \sigma^{-1}(\sigma(V)) \) to \( V' \).

(iv) Given \( \mathfrak{F} \subset \mathcal{E}(\mathfrak{E}) \) such that \( \mathfrak{F} \) is horizontal, we have \( \phi^* (\rho^{-1}\mathfrak{F} \otimes \sigma^{-1}\mathfrak{E}_{U''}) \cong \mathfrak{F} :\)

\[
V \rightarrow \Gamma(V, \rho^* (\rho^{-1}\mathfrak{F} \otimes \sigma^{-1}\mathfrak{E}_{U''}))
\]

\[
= \Gamma(\rho^{-1}(V), \rho^{-1}\mathfrak{F} \otimes \sigma^{-1}\mathfrak{E}_{U''})
\]

\[
\cong \Gamma(\rho^{-1}(V), \rho^{-1}\mathfrak{F})
\]

\[
= \Gamma(\rho^{-1}(V), \mathfrak{F})
\]

\[
= \Gamma(V, \mathfrak{F}).
\]

The isomorphism follows from the fact that, since the fibres of \( \rho \) are compact and the sections of \( \sigma^{-1}\mathfrak{E}_{U''} \) are analytic, tensoring with \( \sigma^{-1}\mathfrak{E}_{U''} \) just multiplies by a constant.

This completes the proof. If we picture the mappings as indicated below,
we see that (i)-(iv) imply that the constructions are indeed inverses.

Since the generalized Ward correspondence provides for all vector bundles with holomorphic connection over an open set \( U \) of \( IM \) it must certainly include the original Ward correspondence—where the connection is anti-self-dual—as a special case. In the remaining part of this chapter we shall examine the relationship between the two. (To be precise, we shall be examining the dual Ward correspondence in relation to the general Ward correspondence.)

Suppose that \( U \) is an open set in \( IM \) and that \( \mathcal{L} \cap U \) is either empty or connected and simply connected for every null line \( \mathcal{L} \). Suppose further that \( P_a \cap U \) is either empty or connected and simply connected for
every $\alpha$-plane $P_\alpha = \tau(Z)$, $Z \in \mathbb{P}$. Suppose that $E - \mathcal{W}' \subset \mathbb{P}$ is a holomorphic vector bundle such that $E|_{\tau^{-1}(x)}$ is trivial for every $x \in U$; then $E$ satisfies the conditions for the Ward construction. Then also $\pi^*E - U' \subset \mathbb{A}$ satisfies the conditions of Theorem IV.5 (a), where $\pi$ is the projection from $\mathbb{A}$ to $\mathbb{P}$. We claim that the vector bundles with connection on $U$ obtained from $E$ via the Ward construction and from $\pi^*E$ via the generalized Ward construction are the same.

We define the bundles
\begin{align*}
\tilde{E} := \nu_\pi(\mu^*E) & \rightarrow U \\
\tilde{E}' := \rho_\pi(\sigma^*(\pi^*E)) & \rightarrow U.
\end{align*}

In particular,
\begin{align*}
\tilde{E}_x &= \Gamma(\tau^{-1}(x), E) \\
\tilde{E}'_x &= \Gamma(\tau_{\alpha}^{-1}(x), \pi^*E).
\end{align*}

This immediately implies that $\tilde{E}_x \cong \tilde{E}'_x$ from which it follows that $\tilde{E} \cong \tilde{E}'$ since the isomorphism is canonical.

We now wish to check that the connection $A'$ on $\tilde{E}'$ obtained from the generalized Ward construction is anti-self-dual and in particular that it agrees with the connection $A$ obtained from the Ward correspondence. This connection is given in the above sheaf-theoretic interpretation by the horizontal subsheaf $\mathcal{G} := \nu_\pi(\mu^{-1}\sigma(E)) \subset \mathcal{O}(\tilde{E})$; if $\Gamma_{\mathcal{G}}(P_\alpha \cap U, \tilde{E}|_{P_\alpha})$ is the subsheaf of sections of $\tilde{E}|_{P_\alpha}$ which are the restrictions of the sections of
There is a canonical isomorphism for any $x \in P_\alpha$

$$
(4.9) \quad \tilde{E}_x \cong \Gamma_{\tilde{\mathcal{F}}}(P_\alpha \cap U, \tilde{E}|_{P_\alpha})
$$

for each $\alpha$-plane $P_\alpha$. If $\tilde{\mathcal{F}} := \rho_*(\sigma^{-1} \rho(\pi^* E))$, we claim that $\tilde{\mathcal{F}} = \mathcal{F}$; in particular, we need only prove the existence of canonical isomorphisms

$$
(4.10) \quad \Gamma_{\tilde{\mathcal{F}}}(P_\alpha \cap U, \tilde{E}|_{P_\alpha}) \cong \Gamma_{\mathcal{F}}(P_\alpha \cap U, E|_{P_\alpha})
$$

for each $P_\alpha$. This can be done by using the canonical isomorphism

$$
\Gamma_{\tilde{\mathcal{F}}}(\ell \cap U, \tilde{E}|_{\ell}) \cong \tilde{E}_x \cong E_x
$$

for $x \in \ell$ and referring to the picture below:

Two points $y$ and $y'$ in $P_\alpha$ are null-separated if and only if they lie on a common null line $\ell = \tau_A(X)$ for some $X \in \pi^{-1}(Z)$. In this case $L_y = \tau^{-1}_A(y)$ and $L_{y'}$ intersect at the one point $X$. We have by construction

$$
\tilde{E}_y = \Gamma(L_y, E|_{L_y})
$$

$$
\tilde{E}_{y'} = \Gamma(L_{y'}, E|_{L_{y'}}).
$$

The value of a section $s \in \Gamma_{\tilde{\mathcal{F}}}(\ell \cap U, E|_{\ell})$ is determined
by the value at one point, say \( y \). Then the value at \( y' \) is given by going to \( \pi^{-1}(Z) \) and choosing the section in \( \Gamma(L_{y'}, E|_{L_{y'}}) \) which agrees at \( X \) with the section of \( \Gamma(L_y, E|_{L_y}) \) corresponding to \( s(y) \). We define a section \( \tilde{s} \in \Gamma_{\tilde{\alpha}'}(\mathcal{P}_\alpha \cap U, \tilde{E}'|_{\mathcal{P}_\alpha}) \) by choosing \( \tilde{s}(y) \) and then moving along null lines as above to determine the value of \( \tilde{s} \) at all other points. Since \( \pi^*E \) is trivial on \( \pi^{-1}(Z) \) this process gives a unique value for \( \tilde{s} \) at each point of \( \mathcal{P}_\alpha \). That is, sections of \( \pi^*E|_{\pi^{-1}(Z)} \) can be written as global functions on \( \pi^{-1}(Z) \) and the process of comparing sections of \( \pi^*E \) over \( L_y \) to those over \( L_y' \), to those over \( L_y'' \), etc., is well-defined. This implies

\[ \Gamma_{\tilde{\alpha}'}(\mathcal{P}_\alpha \cap U, \tilde{E}'|_{\mathcal{P}_\alpha}) \cong \tilde{E}'_x \cong \tilde{E}_x \]

for any \( x \in \mathcal{P}_\alpha \). This along with equation (4.9) is sufficient to imply (4.10).
V. Extensions of Vector Bundles Over $\mathfrak{A}$

Given that Yang-Mills fields on Minkowski space can be represented by vector bundles on ambitwistor space, it is a natural question to ask which vector bundles correspond to those fields $F$ which satisfy $d \ast F = 0$.

A characterization of these appeared in the original work of [11] and [25]. More recently, in [9] it has been announced how this question can be answered more specifically in the language of extension theory. Namely, the axial Yang-Mills current

$$\ast J = d \ast F$$

can be characterized as an explicit geometric object on $\mathfrak{A}$: the obstruction to extending the bundle $E$ on $U'' \subset \mathfrak{A}$ to the third order neighborhood in $\mathbb{P} \times \mathbb{P}^*$. Then the vector bundles on $U''$ for which the obstruction to third order extension vanishes will correspond to fields which satisfy the Yang-Mills equations. In this chapter we discuss extension theory for bundles on $\mathfrak{A}$ and provide a complete proof of the above result.

1. Preliminaries. Let $U'' = \tau^{-1}(U)$ be open in $\mathfrak{A}$ and let $V''$ be a neighborhood of $U''$ in $\mathbb{P} \times \mathbb{P}^*$ such that $V'' \cap \mathfrak{A} = U''$. Cover $U''$ with coordinate neighborhoods
\{U_\alpha\} and write \{V_\alpha\} for the corresponding cover of 
V'. (We choose V' to be a thickening of U" so its 
cover consists of thickening the sets in the cover of U".)
Let \(t_\alpha\) be the normal coordinate of \(U_\alpha\) in \(V_\alpha\). (Throughout 
this chapter we will be assuming mappings
\[
    h_\alpha : U_\alpha \to \hat{U}_\alpha \subset \mathbb{C}^5 \\
    k_\alpha : V_\alpha \to \hat{V}_\alpha \subset \mathbb{C}^6
\]
but we will not distinguish between \(U_\alpha\) and \(\hat{U}_\alpha\), \(V_\alpha\) and 
\(\hat{V}_\alpha\). Thus we will speak of coordinates on \(U_\alpha\) and \(V_\alpha\).

Let \(E \to U"\) be a holomorphic vector bundle corres-
ponding to a bundle \(\tilde{E} \to U\) with holomorphic connection \(A\) 
as described in the last chapter; the bundles \(E, \tilde{E}\) shall 
be considered to be fixed throughout the discussion in 
this chapter. Let
\[
\{g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \text{GL}(n, \mathbb{C})) : g_{\alpha\beta}g_{\beta\gamma} = 1 \text{ and } \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1\}
\]
be the set of transition functions for the vector bundle 
\(E \to U"\).

Let \(\mathcal{I}\) denote the ideal sheaf of \(\mathbb{A}\) in \(\mathbb{P} \times \mathbb{P}^*\). 
Then write \(N^* = \mathcal{I}/\mathcal{I}^2\) for the conormal bundle, and write 
\(\mathcal{O}(k) = \mathcal{O}_{\mathbb{P} \times \mathbb{P}^*}/\mathcal{I}^{k+1}\). We have the exact sequences
\[
0 \to (N^*)^{k+1} \to \mathcal{O}(k+1) \to \mathcal{O}(k) \to 0
\]
and
\[
0 \to (N^*)^{k+1} \to \mathcal{I}/\mathcal{I}^{k+2} \to (N^*)^k \to 0
\]
where the first term is the \((k+1)\)-symmetric power of 
the conormal bundle, \((N^*)^{k+1} \cong \mathcal{I}/\mathcal{I}^{k+1}\). Let \(U^{(k)}\) be
the $k$th order neighborhood of $U''$ in $V''$ (cf. [8]).

The two cohomology groups which will be important in the following discussion of extensions of the bundle $E$ are $\mathcal{H}^1(U'', \text{End } E \otimes (N^*)^k)$ and $\mathcal{H}^2(U'', \text{End } E \otimes (N^*)^k)$. If $\{\hat{\phi}_{\alpha\beta}\}$ and $\{\hat{\xi}_{\alpha\beta\gamma}\}$ are Čech cocycles determining cohomology classes in the above two groups, respectively, they satisfy the antisymmetry relations

$$\hat{\phi}_{\alpha\beta} = -\hat{\phi}_{\beta\alpha}; \quad \hat{\xi}_{\alpha\beta\gamma} = -\hat{\xi}_{\beta\alpha\gamma} = -\hat{\xi}_{\alpha\gamma\beta},$$

and the coboundary relations

$$\hat{\phi}_{\alpha\beta} + \hat{\phi}_{\beta\gamma} + \hat{\phi}_{\gamma\alpha} = 0,$$

$$\hat{\xi}_{\alpha\beta\gamma} - \hat{\xi}_{\alpha\beta\delta} - \hat{\xi}_{\alpha\gamma\delta} - \hat{\xi}_{\beta\gamma\delta} = 0.$$ 

Let $M_n(C)$ denote the set of $n \times n$ matrices with complex entries. Then, using trivializations of $E$ and $N^*$ we can write the above cocycles as sets of functions $\{\phi_{\alpha\beta} \in \Gamma(U_{\alpha\beta}^{(k)}, M_n(C))\}$ and $\{\xi_{\alpha\beta\gamma} \in \Gamma(U_{\alpha\beta\gamma}^{(k)}, M_n(C))\}$ which satisfy the antisymmetry conditions

$$\phi_{\alpha\beta} = -(t_\beta/t_\alpha)^k g_{\alpha\beta} \phi_{\beta\alpha} g_{\alpha\beta}^{-1};$$

$$\xi_{\alpha\beta\gamma} = -(t_\beta/t_\alpha)^k g_{\alpha\beta} \xi_{\beta\gamma} g_{\alpha\beta} g_{\alpha\gamma}^{-1},$$

and the coboundary conditions

$$\phi_{\alpha\beta} + (t_\beta/t_\alpha)^k g_{\alpha\beta} \phi_{\beta\gamma} g_{\beta\alpha}^{-1} g_{\alpha\beta}^{-1} + (t_\gamma/t_\alpha)^k g_{\alpha\gamma} \phi_{\gamma\alpha} g_{\alpha\gamma}^{-1} = 0,$$

$$\xi_{\alpha\beta\gamma} + (t_\beta/t_\alpha)^k g_{\alpha\beta} \xi_{\beta\gamma} g_{\beta\alpha}^{-1} \xi_{\beta\gamma}^{-1} + (t_\gamma/t_\alpha)^k g_{\alpha\gamma} \xi_{\gamma\beta} \xi_{\gamma\beta}^{-1} \xi_{\gamma\beta}^{-1} + (t_\delta/t_\alpha)^k g_{\alpha\delta} \xi_{\delta\beta} g_{\alpha\delta}^{-1} \xi_{\delta\beta}^{-1} = 0.$$ 

In practice we will usually assume that the functions $\phi_{\alpha\beta}$ and $\xi_{\alpha\beta\gamma}$ are defined on $V_{\alpha\beta\gamma}$ and $V_{\alpha\beta}$ rather than on $U_{\alpha\beta}^{(k)}$ and $U_{\alpha\beta\gamma}^{(k)}$, and in this case we demand that
the above conditions hold up to order \( k+1 \) in an expansion in terms of the normal coordinate. (Of course, the notation \( U_{\alpha \beta} \), for example, means the intersection \( U_\alpha \cap U_\beta \), and so forth.)

2. Extensions of bundles over \( \mathbb{A} \). We shall detail the extension of a vector bundle step by step, but first, as a point of reference, we quote a general result which is a basic theorem in extension theory. (See, for example, [8] or [14].)

**Theorem**—(a) A vector bundle \( E^{(n)} \) over \( U^{(n)} \) has an extension \( E^{(n+1)} \) over \( U^{(n+1)} \) if and only if the obstruction \( \eta(E^{(n)}) \), a cohomology class in \( H^2(U''', \text{End } E \otimes (N^*)^{n+1}) \), vanishes.

(b) If \( \eta(E^{(n)}) = 0 \), the cohomology group \( H^1(U''', \text{End } E \otimes (N^*)^{n+1}) \) acts transitively on the set of extensions.

We begin the extension process by choosing a set \( \{g^{(1)}_{\alpha \beta} \in \Gamma(U^{(1)}_{\alpha \beta}, \text{GL}(n, \mathbb{C}))\} \) such that \( g^{(1)}_{\alpha \beta} \) and \( g^{(1)}_{\alpha \beta} g^{(1)}_{\beta \alpha} = I \) where multiplication takes place over \( g^{(1)} \). Now let the set of functions \( \{h^{(1)}_{\alpha \beta} \in \Gamma(U_{\alpha \beta}, M_n(\mathbb{C}))\} \) act on \( \{g^{(1)}_{\alpha \beta}\} \) in the following way:
\[ g_{\alpha \beta}^{(1)} \rightarrow \tilde{g}_{\alpha \beta}^{(1)} := (I + t_{\alpha} h_{\alpha \beta}^{(1)}) g_{\alpha \beta}^{(1)}. \]

Demanding that \( \tilde{g}_{\alpha \beta}^{(1)} \tilde{g}_{\alpha \beta}^{(1)} = I \) gives the requirement on \([h_{\alpha \beta}^{(1)}]\) that
\[
h_{\alpha \beta}^{(1)} = -(t_{\beta} / t_{\alpha}) g_{\alpha \beta} h_{\beta \alpha}^{(1)} g_{\alpha \beta}^{-1}.\]

The obstruction to the set \([\tilde{g}_{\alpha \beta}^{(1)}]\) providing a first extension is
\[
I - \tilde{g}_{\alpha \beta}^{(1)} \tilde{g}_{\gamma \alpha}^{(1)} \tilde{g}_{\gamma \alpha}^{(1)} = -(t_{\alpha} h_{\alpha \beta}^{(1)} + t_{\beta} g_{\alpha \beta} h_{\beta \gamma}^{(1)} g_{\alpha \beta}^{-1} + t_{\gamma} g_{\alpha \gamma} h_{\alpha \beta}^{(1)} g_{\alpha \gamma}^{-1} + (I - g_{\alpha \beta} h_{\alpha \beta}^{(1)} g_{\gamma \alpha}^{(1)}).)
\]

(5.1)

If we define a cohomology class \([\xi^{(1)}]\) by the following:
\[
t_{\alpha} \xi_{\alpha \beta}^{(1)} := I - \tilde{g}_{\alpha \beta}^{(1)} \tilde{g}_{\gamma \alpha}^{(1)} g_{\gamma \alpha}^{(1)},
\]

this defines a cohomology class in \( H^{2}(U'', \text{End } E \otimes (N^{*})) \), which is defined invariantly with respect to the choice of \([g_{\alpha \beta}^{(1)}]\), depending only upon the transition functions \([g_{\alpha \beta}]\). If this class is nonzero there is certainly no choice of \([h_{\alpha \beta}^{(1)}]\) which will make (5.1) vanish. If \([\xi^{(1)}] \sim 0\), then we have
\[
t_{\alpha} \xi_{\alpha \beta}^{(1)} = t_{\alpha} x_{\alpha \beta} + t_{\beta} g_{\alpha \beta} x_{\beta \gamma} g_{\alpha \beta}^{-1} + t_{\gamma} g_{\alpha \gamma} x_{\gamma \alpha} g_{\alpha \gamma}^{-1}
\]
for some cochain \([\chi_{\alpha \beta} \in \Gamma(U_{\alpha \beta}, M_{n}(\mathfrak{c}))\]). Setting \( h_{\alpha \beta}^{(1)} = \chi_{\alpha \beta} + \varphi_{\alpha \beta} \) where \([\varphi_{\alpha \beta}]\) gives a cocycle in \( H^{1}(U'', \text{End } E \otimes (N^{*})) \) then guarantees that (5.1) vanishes. Since we know from the last chapter that
\[
H^{2}(U'', \otimes (-1,-1)) = 0
\]
we use this to see that
\[
H^{2}(U'', \text{End } E \otimes (N^{*})) \cong H^{2}(U'', \text{End } E(-1,-1)) = 0,
\]
using the easily calculated isomorphism \( N^{*} \cong \otimes (-1,-1) \).
Thus all the obstructions to first extension vanish, and 
$H^1(U''', \text{End } E(-1,-1))$ operates transitively on the set of 
first extensions. It is easy to verify that this action 
is also effective.

Now we choose a first extension $E^{(1)}$ of $E$ given 
by $\{g^{(1)}_{\alpha \beta}\}$ where 
g^{(1)}_{\alpha \beta}g^{(1)}_{\beta \alpha} = I \text{ and } g^{(1)}_{\alpha \beta}g^{(1)}_{\gamma \alpha}g^{(1)}_{\gamma \alpha} = I,$
where multiplication is in $\wp(1)$. Pick a set of functions 
$\{g^{(2)}_{\alpha \beta} \in \Gamma(U^{(2)}, \text{GL}(n,\mathbb{C}))\}$ such that 
g^{(2)}_{\alpha \beta} = g^{(1)}_{\alpha \beta} + \beta^2 \text{ and }
g^{(2)}_{\alpha \beta}g^{(2)}_{\beta \alpha} = I.

As above, the set 
$\{t^{(2)}_{\alpha \beta} \delta^{(2)}_{\gamma \alpha} := I - g^{(2)}_{\alpha \beta}g^{(2)}_{\beta \alpha}g^{(2)}_{\gamma \alpha}\}$
defines the obstruction to second extension 
$[\xi^{(2)}] = \eta(E^{(1)}) \in H^2(U''', \text{End } E \otimes (\mathbb{N}^*)^2).$

It is easy to check that $\xi^{(2)}$ is indeed a cocycle and 
that the cohomology class which it determines is independent of the choice of extension $g^{(2)}_{\alpha \beta}$ of $g^{(1)}_{\alpha \beta}$.

The action of $H^1(U''', \text{End } E \otimes (\mathbb{N}^*)^2)$ on the set of 
first extensions induces an action on $H^2(U''', \text{End } E \otimes (\mathbb{N}^*)^2)$, the group containing the obstructions to second extension, in the following way. Choosing a cocycle 
$\{\phi_{\alpha \beta}\} \in H^1(U''', \text{End } E \otimes (\mathbb{N}^*)^2)$, we define an action on 
$\{g^{(2)}_{\alpha \beta}\}$ by 
g^{(2)}_{\alpha \beta} - \tilde{\gamma}^{(2)}_{\alpha \beta} := (I + t_{\alpha \beta} - \phi_{\alpha \beta} + t_{\alpha \beta}^2 - \rho_{\alpha \beta})g^{(2)}_{\alpha \beta},$
where the functions $\rho_{\alpha \beta}$ are chosen to ensure that 
$\tilde{\gamma}^{(2)}_{\alpha \beta}g^{(2)}_{\beta \alpha} = I.$
The corresponding action on $H^2(U'', \text{End } E \otimes (N^k)^2)$ is
\[ t_{\alpha^a \beta^a \gamma}^2(2) - t_{\alpha^a \beta^a \gamma}^2(2) := I - \tilde{g}_{\alpha^a \beta^a \gamma}^2(2) \tilde{g}_{\beta^a \gamma}^2(2). \]

It can be checked that the cohomology class of $\{ \varphi_{\alpha^a \beta^a} \}$ depends only on the first extension
\[ \tilde{g}_{\alpha^a \beta^a}^{(1)} := (I + t_{\alpha^a \beta^a} \varphi_{\alpha^a \beta^a}) \tilde{g}_{\alpha^a \beta^a}^{(1)} \]
Ignoring the terms which involve $\varphi_{\alpha^a \beta^a}$--which contribute only a coboundary--the new obstruction term is
\[ t_{\alpha^a \beta^a \gamma}^2(2) = t_{\alpha^a \beta^a \gamma}^2(2) - (t_{\alpha^a \beta^a} + t_{\beta^a \gamma} \varphi_{\beta^a \gamma} \tilde{g}_{\beta^a \gamma}^{(1)} + t_{\gamma^a \alpha^a} \varphi_{\gamma^a \alpha^a} \tilde{g}_{\gamma^a \alpha^a}^{(1)}) \]
\[ - (t_{\alpha^a \beta^a} \varphi_{\alpha^a \beta^a} \tilde{g}_{\alpha^a \beta^a}^{(1)} + t_{\beta^a \gamma} \varphi_{\beta^a \gamma} \tilde{g}_{\beta^a \gamma}^{(1)} + t_{\gamma^a \alpha^a} \varphi_{\gamma^a \alpha^a} \tilde{g}_{\gamma^a \alpha^a}^{(1)}). \]

If we define
\[ t_{\alpha^a \beta^a \gamma}^2 := t_{\alpha^a \beta^a} + t_{\beta^a \gamma} \tilde{g}_{\beta^a \gamma}^{(1)} + t_{\gamma^a \alpha^a} \tilde{g}_{\gamma^a \alpha^a}^{(1)}, \]
then we can write the above equation as simply
\[ \tilde{g}_{\alpha^a \beta^a \gamma}^2 = \tilde{g}_{\alpha^a \beta^a \gamma}^2 - (\hat{\varphi}_{\alpha^a \beta^a} + \hat{\varphi}_{\beta^a \gamma} + \hat{\varphi}_{\gamma^a \alpha^a}). \]

We have not yet used the fact that we are working with ambitwistor space in analyzing the obstructions to second order extension. Recall that under the ambitwistor transform we have the isomorphisms
\[ G: H^1(U'', \text{End } E(-1,-1)) \cong H^0(U, \text{End } \tilde{E}[-1,-1]) \]
\[ G: H^2(U'', \text{End } E(-2,-2)) \cong H^0(U, \text{End } \tilde{E}[-1,-1]). \]
Thus we may transform the action of the group of first extensions on the group of obstructions to second order extension to an endomorphism of $H^0(U, \text{End } \tilde{E}[-1,-1])$. It is not hard to see that the third term in the
above expression for $\hat{\xi}'(2)$ makes no contribution to the cohomology class in $H^0(U, \text{End } \bar{E}[-1,-1])$ obtained from $\hat{\xi}'(2)$ under the mapping $\mathcal{G}$. First we pull back to $U'$ the cocycle $\{\hat{\phi}_{\alpha\beta}\}$ to get

$$\{\sigma^*\hat{\phi}_{\alpha\beta}\} \in H^1(U', \sigma^{-1} \mathcal{O}(-1,-1)).$$

Since $H^1(U', \mathcal{O}(-1,-1)) = 0$ there is a cochain $\{\hat{\lambda}_\alpha \in \Gamma(U'_\alpha, \mathcal{O}(-1,-1))\}$ where $U'_\alpha = \sigma^{-1}(U_\alpha)$ such that

$$\sigma^*\hat{\phi}_{\alpha\beta} = \hat{\lambda}_\alpha - \hat{\lambda}_\beta \text{ on } U'_\alpha.$$

Then on $H^2(U', \mathcal{O}(-1,-1)) \cong H^2(U', \mathcal{O}(-1,-1)),$

$$\sigma^*(\hat{\phi}_{\alpha\beta} \hat{\phi}_{\beta\gamma} + \hat{\phi}_{\alpha\gamma} \hat{\phi}_{\gamma\alpha} + \hat{\phi}_{\beta\gamma} \hat{\phi}_{\gamma\alpha}) = (\hat{\lambda}_\alpha - \hat{\lambda}_\beta)(\hat{\lambda}_\beta - \hat{\lambda}_\gamma) + (\hat{\lambda}_\alpha - \hat{\lambda}_\gamma)(\hat{\lambda}_\gamma - \hat{\lambda}_\alpha).$$

Thus there is a cochain $\hat{\delta}_{\alpha\beta} := \hat{\lambda}_\alpha \hat{\lambda}_\beta - \frac{1}{2}(\hat{\lambda}_\alpha \hat{\lambda}_\gamma + \hat{\lambda}_\beta \hat{\lambda}_\gamma)$ such that

$$\hat{\delta}_{\alpha\beta} \in \Gamma(U'_\alpha, \mathcal{O}(-2,-2))$$

and

$$\sigma^*(\hat{\phi}_{\alpha\beta} \hat{\phi}_{\beta\gamma} + \hat{\phi}_{\alpha\gamma} \hat{\phi}_{\gamma\alpha} + \hat{\phi}_{\beta\gamma} \hat{\phi}_{\gamma\alpha}) = \hat{\delta}_{\alpha\beta} + \hat{\delta}_{\beta\gamma} + \hat{\delta}_{\gamma\alpha}.$$

The fact that this term is a coboundary implies that it has no effect on the value of $G([[\hat{\xi}'(2)]]).$ In particular by examining the form of $\hat{\tau}_{\alpha\beta\gamma}$ we see that the induced endomorphism on $H^0(U, \text{End } E[-1,-1])$ is linear.

It is not difficult to check that the cohomology class defined by the cocycle $\{\hat{\tau}_{\alpha\beta\gamma}\}$ is the image of $[[\hat{\omega}_{\alpha\beta}]]$ under the connecting homomorphism of the short exact sequence

$$0 \to \mathcal{O}^2/\mathcal{O}^3 \to \mathcal{O}/\mathcal{O}^3 \to \mathcal{O}/\mathcal{O}^2 \to 0$$

$$\oplus(-2,-2) \quad \oplus(-1,-1),$$

$$\oplus(-1,-1),$$
that is,
\[ \delta : H^1(U'', \text{End } E(-1,-1)) \to H^2(U'', \text{End } E(-2,-2)) \]
\[ \delta : \{ \varphi_{\alpha\beta} \} \to \{ \tau_{\alpha\beta\gamma} \}. \]

The connecting homomorphism thus induces an endomorphism on
\[ H^0(U, \text{End } \tilde{E}[-1,-1]) : \]
\[ \Delta : G([\varphi_{\alpha\beta}]) \to G([\tau_{\alpha\beta\gamma}]), \]
and the action of \( H^1(U'', \text{End } E(-1,-1)) \) on \( H^2(U'', \text{End } E(-1,-1)) \) translated to \( M \) is just
\[ G([\hat{\varphi}]) : G([\hat{\xi}(2)]) \to G([\hat{\xi}(2)]) - \Delta(G([\hat{\varphi}])). \]

**Lemma V.1**- The mapping \( \Delta \) is an isomorphism.

Before proving this lemma we shall use it to prove the following:

**Theorem V.2**- There is a unique second order extension of the bundle \( E - U'' \). The obstruction to third order extension can be canonically identified with the axial Yang-Mills current \( \ast J = d(\ast F) \), where \( F \) is the curvat re associated to the connection \( A \) obtained from \( E \) via the generalized Ward correspondence.

**Proof**- Given a first extension \( E^{(1)} \) of \( E \) with transition functions \( \{ g_{\alpha\beta}^{(1)} \} \) we calculate the obstruction to first extension \( [\xi(2)] \in H^1(U'', \text{End } E(-1,-1)) \) and
then find its image under the ambitwistor transform in $H^0(U, \text{End } \bar{E}[-1,-1])$. Since $\Delta$ is an isomorphism there is exactly one cohomology class $[\varphi] \in H^1(U'', \text{End } E(-1,-1))$ such that

$$\Delta([\varphi]) = [\varphi^{(2)}].$$

This implies that there is exactly one first extension

$$\tilde{g}^{(1)}_{\alpha \beta} = (I + t_{\alpha \beta} \varphi) g^{(1)}_{\alpha \beta}$$

for which the obstruction to second extension vanishes. Furthermore, since $H^1(U'', \text{End } E(-2,-2)) = 0$, there is a unique second order extension $\{g^{(2)}_{\alpha \beta}\}$ which extends $\{\tilde{g}^{(1)}_{\alpha \beta}\}$.

Having found a unique second order extension we attempt to extend to third order. The obstruction to third order extension lies in $H^2(U'', \text{End } E(-3,-3))$, which by the work in the last chapter is isomorphic to $\Gamma(U, \ker \nabla : \text{End } \bar{E} \otimes \Omega^3_{\mathcal{M}} \to \text{End } \bar{E} \otimes \Omega^4_{\mathcal{M}})$, the set of axial Yang-Mills currents. It can be seen that the image under this isomorphism of the third order obstruction is exactly the current

$$*J = d(*F)$$

associated to the Yang-Mills field of the construction. (See [14].) In particular the Yang-Mills field equations are satisfied if and only if the corresponding bundle on ambitwistor space can be extended to third order in $\mathbb{P} \times \mathbb{P}^*$. ⊙
3. Proof of Lemma V.1. If the fibres of $\sigma: U' \to U''$ are $1$-elementary we have isomorphisms and an induced mapping

$$\hat{\delta}: H^1(U', \sigma^{-1}\Theta(-1,-1)) \to H^2(U', \sigma^{-1}\Theta(-2,-2))$$

$$\delta: H^1(U'', \Theta(-1,-1)) \to H^2(U'', \Theta(-2,-2))$$

Then $\delta$ is an isomorphism if and only if $\hat{\delta}$ is. The proof that $\hat{\delta}$ is an isomorphism consists of three parts: 1) representation of the cohomology groups $H^1(U', \sigma^{-1}\Theta(-1,-1))$ and $H^2(U', \sigma^{-1}\Theta(-2,-2))$ via Čech cohomology; 2) canonical identification of these two groups with $H^0(U, \Theta[-1,-1])$; and 3) representation of the homomorphism $\delta$ and examination of the endomorphism on $H^0(U, \Theta[-1,-1])$ induced by $\hat{\delta}$.

For the proof we shall restrict ourselves to the "affine" parts of the manifolds involved; that is, we shall assume $U \subseteq M^I$. We shall avoid the necessity of dealing with an unwieldy number of open sets in the cover of $U'$ in this manner, and the result on the entire manifold follows from results in the general abstract theory. Specifically, we know from the abstract theory that we have isomorphisms and induced mappings

$$H^1(U'', \Theta(-1,-1)) \overset{\delta}{\to} H^2(U'', \Theta(-2,-2))$$

$$H^1(U', \sigma^{-1}\Theta(-1,-1)) \overset{\delta}{\to} H^2(U', \sigma^{-1}\Theta(-2,-2))$$

$$H^0(U, \Theta[-1,-1]) \overset{\Delta}{\to} H^0(U, \Theta[-1,-1])$$

where $U$ is an open set in $M$. If $\Delta$ is shown to be
an isomorphism on open sets in $\mathcal{M}_I^I$ it follows that
$\Delta$ is an isomorphism on all of $U$.

Local coordinates on $G^I \subset \mathcal{M}_I^I \times \mathcal{M}_I^{*I}$ can be given
as follows: let $\{z^{AA}', [w^{AA}']\}$ be two sets of coordi-
nates on $\mathcal{M}_I^I$; then $\{z^{AA}', [\pi_{A'}]\}$ and $\{w^{AA}', [\eta_{A'}]\}$
will serve as coordinates on $\mathcal{M}_I^I$ and $\mathcal{M}_I^{*I}$, respectively,
where $[\pi_{A'}]$ and $[\eta_{A'}]$ are homogeneous coordinates on
$\nu^{-1}(z^{AA}') \simeq \mathcal{P}_1$ and $(\nu^*)^{-1}(w^{AA}') \simeq \mathcal{P}_1$. We cover $\mathcal{M}_I^I$
and $\mathcal{M}_I^{*I}$ with two coordinate patches apiece and cover
$\mathcal{M}_I^I \times \mathcal{M}_I^{*I}$ with four local coordinate systems (where the
local coordinates are listed to the right):

$$V'_0 := \{\pi_0 \neq 0, \eta_0 \neq 0\} \rightarrow \{z^{AA}', w^{AA}', \pi_0 = \frac{\eta_1}{\eta_0}, \eta_0 = \eta_1\}$$

$$V'_1 := \{\pi_0 \neq 0, \eta_0 \neq 0\} \rightarrow \{z^{AA}', w^{AA}', \pi_0 = \frac{\eta_1}{\eta_0}, \eta_0 = \eta_1\}$$

$$V'_2 := \{\pi_0 \neq 0, \eta_1 \neq 0\} \rightarrow \{z^{AA}', w^{AA}', \pi_0 = \frac{\eta_1}{\eta_0}, \eta_0 = \eta_1\}$$

$$V'_3 := \{\pi_1 \neq 0, \eta_1 \neq 0\} \rightarrow \{z^{AA}', w^{AA}', \pi_1 = \frac{\eta_1}{\eta_0}, \eta_0 = \eta_1\}$$

Define

$$x^{AA'} := \frac{1}{\sqrt{2}}(z^{AA'} + w^{AA'}), \ y^{AA'} := \frac{1}{\sqrt{2}}(z^{AA'} - w^{AA'})$$

We then consider $G^I$ as the "diagonal" in $\mathcal{M}_I^I \times \mathcal{M}_I^{*I}$,
and, setting $U'_i = V'_i | G^I$, we cover $G^I$ with the set
$\{U'_i, i = 0,1,2,3\}$ where the $U'_i$ have local coordinates
$\{x^{AA'}, \pi_i, \eta_i\}$.

Suppose that we have a cocycle $\hat{f}_{A} \in \Gamma(U'_i, \sigma^{-1} \otimes (-1,-1))$ representing a cohomology class $[\hat{f}] \in$
$H^1(U', \sigma^{-1} \otimes (-1,-1))$. Let $f_{A}^i$ denote the representation
of $\hat{f}_{A}$ under the $i$th trivialization. Expanding $f_{A}^i$ in
powers of \( r_i \) and \( s_i \) we obtain

\[
\begin{align*}
 f_{01}^0 &= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} F_{n,m}^{01,1} (xAA') r_0^n s_0^m \\
f_{02}^0 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{n,m}^{02,0} (xAA') r_0^n s_0^m \\
f_{03}^0 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}^{03,0} (xAA') r_0^n s_0^m \\
f_{12}^1 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}^{12,1} (xAA') r_1^n s_1^m \\
f_{13}^1 &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}^{13,1} (xAA') r_1^n s_1^m \\
f_{23}^2 &= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} F_{n,m}^{23,2} (xAA') r_2^n s_2^m .
\end{align*}
\]

We use the relationships

\[
 f_{\alpha\beta}^1 = r_0 f_{\alpha\beta}^0 , \quad f_{\alpha\beta}^2 = s_0 f_{\alpha\beta}^0
\]

(which arise from the fact that the \( \hat{f}_{\alpha\beta} \) have coefficients in \( \sigma^{-1}\Theta(-1,-1) \)) to trivialize all of the \( f_{\alpha\beta} \) over \( U_0' \):

\[
\begin{align*}
 f_{12}^0 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}^{12,0} (xAA') r_0^n s_0^m \\
f_{13}^0 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}^{13,0} (xAA') r_0^n s_0^m \\
f_{23}^0 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{n,m}^{23,0} (xAA') r_0^n s_0^m \\
\end{align*}
\]

where

\[
\begin{align*}
 F_{n,m}^{12,0} &= F_{n-1,m}, \quad F_{n,m}^{13,0} = F_{-n-1,m}, \quad F_{n,m}^{23,0} = F_{n,-m-1}.
\end{align*}
\]

Since \( f_{\alpha\beta}^i \) describes a cocycle we must have

\[
 f_{\alpha\beta}^0 + f_{\beta\gamma}^0 - f_{\alpha\gamma}^0 = 0.
\]

This implies that we can completely specify \( \{f_{\alpha\beta}\} \) by using a doubly indexed set of holomorphic functions

\[
 A_{n,m}(xAA') , \quad n,m \in \mathbb{Z},
\]

in the following manner. We write


\[ F_{n,m}^{0,1} = A_{n,m} \quad \text{all } n, \text{ and } m \geq 0 \]

\[ F_{n,m}^{0,2} = \begin{cases} A_{n,m} & n,m \geq 0 \\ -A_{n,m} & n \geq 0, \ m < 0 \end{cases} \]

\[ F_{n,m}^{0,3} = A_{n,m} \quad n,m \geq 0 \text{ or } n,m < 0 \]

\[ F_{n,m}^{1,2} = -A_{n,m} \quad n < 0, \ m \geq 0 \text{ or } n \geq 0, \ m < 0 \]

\[ F_{n,m}^{1,3} = \begin{cases} A_{n,m} & n,m < 0 \\ -A_{n,m} & n < 0, \ m \geq 0 \end{cases} \]

\[ F_{n,m}^{2,3} = A_{n,m} \quad \text{all } n, \text{ and } m < 0 \]

where \( F_{n,m}^{0,0} = 0 \) otherwise. To guarantee that each \( f_{\alpha \beta}^{0} \)

is constant along the fibres we demand that

\[
0 = d^{0} f^{0}_{\alpha \beta} = \frac{\pi_{A^{'}}^{\eta_{A}}}{\eta_{0}^{\pi_{A}}} \bigwedge A^{'} f^{0}_{\alpha \beta}
\]

\[
= \left( \frac{\partial}{\partial x_{11}} - s_{0} \frac{\partial}{\partial x_{01}} - r_{0} \frac{\partial}{\partial x_{10}} + r_{0}s_{0} \frac{\partial}{\partial x_{00}} \right) \cdot f_{\alpha \beta}^{0}.
\]

This implies that the functions \( A_{n,m} \) must satisfy

certain differential equations:

\[
\frac{\partial A_{n,m}}{\partial x_{11}} - \frac{\partial A_{n-1,m}}{\partial x_{01}} - \frac{\partial A_{n-1,m-1}}{\partial x_{10}} + \frac{\partial A_{n-1,m-1}}{\partial x_{00}} = 0 \quad \text{all } n,m
\]

\[
\frac{\partial A_{n,0}}{\partial x_{11}} - \frac{\partial A_{n-1,0}}{\partial x_{10}} = 0 \quad \text{all } n
\]

\[
\frac{\partial A_{n,-1}}{\partial x_{01}} - \frac{\partial A_{n-1,-1}}{\partial x_{00}} = 0 \quad n \neq 0
\]

\[
\frac{\partial A_{0,m}}{\partial x_{11}} - \frac{\partial A_{0,m-1}}{\partial x_{01}} = \frac{\partial A_{-1,m}}{\partial x_{10}} - \frac{\partial A_{-1,m-1}}{\partial x_{00}} = 0 \quad m \neq 0
\]

\[
\frac{\partial A_{0,-1}}{\partial x_{01}} + \frac{\partial A_{-1,0}}{\partial x_{10}} = \frac{\partial A_{0,-1}}{\partial x_{11}} + \frac{\partial A_{-1,0}}{\partial x_{00}} = 0
\].
Such a set \( \{A_{n,m}\} \) completely specifies a cocycle. By recourse to the general theory we avoid the questions of the existence of such a set and the convergence of the series: if \( H^1(U', \sigma^{-1} \otimes (-1,-1)) \neq 0 \), then such cocycles exist and the above work merely gives an explicit representation. Of course, if \( H^1(U', \sigma^{-1} \otimes (-1,-1)) = 0 \) the result follows trivially.

To specify a cochain \( \{\tilde{b}_\alpha\} \in C^0(U', \sigma^{-1} \otimes (-1,-1)) \) we again need a doubly indexed set \( \{B_{n,m}\} \), \( n,m \in \mathbb{Z} \). We set

\[
\begin{align*}
b_{0} &= \frac{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{n,m} (x^{AA'})}{r_0^n s_0^m} \\
b_{1} &= \frac{\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} -B_{n,m} (x^{AA'})}{r_0^n s_0^m} \\
b_{2} &= \frac{\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} B_{n,m} (x^{AA'})}{r_0^n s_0^m} \\
b_{3} &= \frac{\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} -B_{n,m} (x^{AA'})}{r_0^n s_0^m}.
\end{align*}
\]

The requirement that \( d^0_{\tilde{b}_\alpha} = 0 \) implies the following conditions on \( \{B_{n,m}\} \):

\[
\begin{align*}
\frac{\partial B_{n,m}}{\partial x^{ll'}} - \frac{\partial B_{n-1,m}}{\partial x^{01'}} - \frac{\partial B_{n-1,m-1}}{\partial x^{00'}} + \frac{\partial B_{n-1,m-1}}{\partial x^{00'}} &= 0 \quad \text{all } n,m \\
\frac{\partial B_{n,0}}{\partial x^{ll'}} - \frac{\partial B_{n-1,0}}{\partial x^{01'}} &= 0 \quad \text{all } n \\
\frac{\partial B_{n,-1}}{\partial x^{ll'}} - \frac{\partial B_{n-1,-1}}{\partial x^{00'}} &= 0 \quad n \neq 0 \\
\frac{\partial B_{0,m}}{\partial x^{ll'}} - \frac{\partial B_{0,m-1}}{\partial x^{01'}} &= \frac{\partial B_{-1,m}}{\partial x^{ll'}} - \frac{\partial B_{-1,m-1}}{\partial x^{00'}} = 0 \quad m \neq 0 \\
\frac{\partial B_{0,-1}}{\partial x^{01'}} &= \frac{\partial B_{-1,0}}{\partial x^{ll'}} = \frac{\partial B_{0,0}}{\partial x^{ll'}} = \frac{\partial B_{-1,-1}}{\partial x^{00'}} = 0.
\end{align*}
\]
Then the coboundary of \( \{ b^0_\alpha \} \) has the components

\[
\begin{align*}
&b^0_0 - b^0_1 = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} B_{n,m} r^n s^m \\
&b^0_0 - b^0_2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{n,m} r^n s^m - \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} B_{n,m} r^n s^m \\
&b^0_0 - b^0_3 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B_{n,m} r^n s^m + \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_{n,m} r^n s^m
\end{align*}
\]

with \( b^0_1 - b^0_2, b^0_1 - b^0_3, b^0_2 - b^0_3 \) being the obvious combinations of the above. Recall that

\[
\begin{align*}
f^0_{01} &= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_{n,m} r^n s^m \\
f^0_{02} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} r^n s^m - \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} A_{n,m} r^n s^m \\
f^0_{03} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} r^n s^m + \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{n,m} r^n s^m.
\end{align*}
\]

Examination of the differential equations on the sets \( \{ A_{n,m} \} \) and \( \{ B_{n,m} \} \) reveals that the only difference is that

\[
\frac{\partial A_{0,0}}{\partial x^{01'}} + \frac{\partial A_{-1,0}}{\partial x^{10'}} = \frac{\partial A_{0,0}}{\partial x^{11'}} + \frac{\partial A_{-1,-1}}{\partial x^{00'}} = 0
\]

\[
\frac{\partial A_{0,0}}{\partial x^{11'}} - \frac{\partial A_{-1,0}}{\partial x^{10'}} = 0,
\]

but

\[
\frac{\partial B_{0,-1}}{\partial x^{01'}} = \frac{\partial B_{-1,0}}{\partial x^{10'}} = \frac{\partial B_{0,0}}{\partial x^{11'}} = \frac{\partial B_{-1,-1}}{\partial x^{00'}} = 0.
\]

The general solution to

\[
\begin{align*}
&\frac{\partial A_{0,0}}{\partial x^{11'}} = \frac{\partial A_{-1,0}}{\partial x^{10'}} = - \frac{\partial A_{0,-1}}{\partial x^{01'}} = - \frac{\partial A_{-1,-1}}{\partial x^{00'}}
\end{align*}
\]

is

\[
A_{0,0} = \frac{\partial^3 H}{\partial x^{00'} \partial x^{01'} \partial x^{10'}} + B^1(x^{00'}, x^{01'}, x^{10'})
\]
\[ A_{0,-1} = - \frac{\partial^3 H}{\partial x^{00'} \partial x^{10'} \partial x^{11'}} + B^2(x^{00'}, x^{10'}, x^{11'}) \]
\[ A_{-1,0} = \frac{\partial^3 H}{\partial x^{00'} \partial x^{10'} \partial x^{11'}} + B^3(x^{00'}, x^{01'}, x^{11'}) \]
\[ A_{-1,-1} = - \frac{\partial^3 H}{\partial x^{01'} \partial x^{10'} \partial x^{11'}} + B^4(x^{01'}, x^{10'}, x^{11'}) \]

where \( H = H(x^{00'}, x^{01'}, x^{10'}, x^{11'}) \) and the \( B \) are holomorphic. Thus a general cocycle \( \xi_{\alpha \beta} \) can be specified by setting

\[ A_{0,0} = \frac{\partial^3 H}{\partial x^{00'} \partial x^{01'} \partial x^{10'}} + B_{0,0} \]
\[ A_{0,-1} = - \frac{\partial^3 H}{\partial x^{00'} \partial x^{10'} \partial x^{11'}} + B_{0,-1} \]
\[ A_{-1,0} = \frac{\partial^3 H}{\partial x^{00'} \partial x^{01'} \partial x^{11'}} + B_{-1,0} \]
\[ A_{-1,-1} = - \frac{\partial^3 H}{\partial x^{01'} \partial x^{10'} \partial x^{11'}} + B_{-1,-1} \]

\[ A_{n,m} = B_{n,m} \quad \text{for all other } m,n \]

where \( H \) is a holomorphic function on \( U \subset M \), and the \( \{B_{n,m}\} \) is as specified above.

The next step is to characterize the image of this cohomology class under the isomorphism

\[ H^1(U', \sigma^{-1} \circ (-1,-1)) \cong H^0(U, \sigma[-1,1]) \]
\[ \cong H^0(U, \sigma[-1,1]). \]

The first isomorphism is just the inverse of the
connecting homomorphism for the relative deRham sequence. Under Čech cohomology the mapping is given by first choosing a cochain \( \{ \tilde{c}_\alpha \} \in C^0(U', \Theta(-1,-1)) \) such that \( f^0_{a\beta} = c^0_{a\beta} - c^0_{b\beta} \), and then the cocycle \( \{ d^0_{a\alpha} \} \in C^0(U', \Omega^1_{c'(-1,-1)}) \) gives the desired cohomology class in \( H^0(U', \Theta[-1,-1]) \). This construction is independent of the choices involved, so we choose

\[
\begin{align*}
    c^0_0 &= b^0_0 + \frac{\partial^3 H}{\partial x^{00'} x^{01'} x^{10'}} \\
    c^0_1 &= b^0_1 - \frac{\partial^3 H}{\partial x^{00'} x^{01'} x^{11'}} r_0^{-1} \\
    c^0_2 &= b^0_2 - \frac{\partial^3 H}{\partial x^{00'} x^{10'} x^{11'}} s_0^{-1} \\
    c^0_3 &= b^0_3 + \frac{\partial^3 H}{\partial x^{01'} x^{10'} x^{11'}} r_0^{-1}s_0^{-1}.
\end{align*}
\]

It is then easy to calculate that

\[
d^0_{a\alpha} = \frac{\partial^4 H}{\partial x^{00'} x^{01'} x^{10'} x^{11'}}
\]

which determines the cohomology class in \( H^0(U', \Theta[-1,-1]) \).

To pass to Minkowski space is easier: since \( \rho^{-1}(x^{AA'}) \) is compact, any global analytic function on \( \rho^{-1}(x^{AA'}) \) is constant, and the value of a global function in \( H^0(U', \Theta[-1,-1]) \) on \( \rho^{-1}(x^{AA'}) \) depends only on \( x^{AA'} \).

Then the image of \( \hat{f} \) under the above isomorphism is just

\[
\frac{\partial^4 H}{\partial x^{00'} x^{01'} x^{10'} x^{11'}}.
\]
We next wish to develop the same kind of understanding for the isomorphic cohomology groups
\[ H^2(U', \omega^{-1}(-2, -2)) \cong H^0(U, \omega[-1, -1]) \].

A cocycle \( \{ f_{\alpha \beta \gamma} \} \) representing a class in the first group satisfies
\[ f_{\alpha \beta \gamma} - f_{\alpha \beta \delta} + f_{\alpha \gamma \delta} - f_{\beta \gamma \delta} = 0 \quad \text{on } U'_{\alpha \beta \gamma \delta}. \]

Since each of the sections has the same domain, specifying three of the sections completely determines the fourth. This implies that we need three sets of holomorphic functions \( \{ f_{i_{n,m}} \}, i = 1, 2, 3 \) to specify the cocycle:
\[
\begin{align*}
\phi_{012} &= \frac{z^8}{n, m} F_{n,m}(x^{AA'}) r_0 s_0 \\
\phi_{013} &= \frac{z^8}{n, m} F_{n,m}(x^{AA'}) r_0 s_0 \\
\phi_{023} &= \frac{z^8}{n, m} F_{n,m}(x^{AA'}) r_0 s_0 .
\end{align*}
\]

The condition \( d_0^0 = 0 \) gives conditions on \( \{ F_{i_{n,m}} \} \) similar to those in the above work. Also, if \( \{ e_{\alpha \beta} \} \) is a cochain in \( C^1(U', \omega^{-1}(-2, -2)) \), such that
\[
\begin{align*}
e_{01} &= \frac{1}{n, m} E_{n,m}(x^{AA'}) r_0 s_0 \\
e_{02} &= \frac{1}{n, m} E_{n,m}(x^{AA'}) r_0 s_0 \\
e_{03} &= \frac{1}{n, m} E_{n,m}(x^{AA'}) r_0 s_0 \\
e_{12} &= \frac{1}{n, m} E_{n,m}(x^{AA'}) r_0 s_0 \\
e_{13} &= \frac{1}{n, m} E_{n,m}(x^{AA'}) r_0 s_0 \\
e_{23} &= \frac{1}{n, m} E_{n,m}(x^{AA'}) r_0 s_0 ,
\end{align*}
\]
then \( d_0^0 e_{\alpha \beta} = 0 \) gives conditions on \( \{ F_{\alpha \beta} \} \).
We now form the coboundary of \( \hat{\epsilon}_{\alpha \beta} \):
\[
\hat{\epsilon}_{\alpha \beta \gamma} := \hat{\epsilon}_{\alpha \beta} + \hat{\epsilon}_{\beta \gamma} - \hat{\epsilon}_{\alpha \gamma}.
\]
If we expand
\[
E_{\alpha \beta \gamma}^0 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E_{n,m}^{\alpha \beta \gamma} (x^{AA'}) r_0 n_s 0
\]
we can calculate that
\[
E_{012}^{n,m} = \begin{cases} 
E_{01}^{n,m} + E_{12}^{n,m} - E_{02}^{n,m} & n, m \geq 0 \\
E_{01}^{n,m} + E_{12}^{n,m} & n < 0, m \geq 0 \\
E_{12}^{n,m} - E_{02}^{n,m} & n \geq 0, m < 0 \\
E_{12}^{n,m} & n, m < 0 
\end{cases}
\]

\[
E_{013}^{n,m} = \begin{cases} 
E_{01}^{n,m} + E_{13}^{n,m} - E_{03}^{n,m} & n \leq -2, m \geq 0 \\
E_{01}^{n,m} - E_{03}^{n,m} & n > -2, m \geq 0 \\
E_{13}^{n,m} - E_{03}^{n,m} & n \leq -2, m < 0 \\
-E_{03}^{n,m} & n > -2, m < 0 
\end{cases}
\]

\[
E_{023}^{n,m} = \begin{cases} 
E_{02}^{n,m} + E_{23}^{n,m} - E_{03}^{n,m} & n \geq 0, m \leq -2 \\
E_{23}^{n,m} - E_{03}^{n,m} & n < 0, m \leq -2 \\
E_{02}^{n,m} - E_{03}^{n,m} & n \geq 0, m > -2 \\
-E_{03}^{n,m} & n < 0, m > -2 
\end{cases}
\]

Demanding that \( d_0^{\alpha \beta} = 0 \) implies differential equation conditions on \( \{E_{n,m}^{\alpha \beta \gamma}\} \) which are the same as those on \( \{F_{n,m}^{\alpha \beta \gamma}\} \); because of this we shall not worry explicitly about these conditions. It should be noted that this state of affairs reflects the fact that there is an isomorphism
\[ H^2(U', \sigma^{-1}\Theta(-2,-2)) \cong H^2(U', \Theta(-2,-2)). \]

Now we have three doubly indexed sets of functions: \( \{E_{n,m}^{012}\}, \{E_{n,m}^{013}\}, \) and \( \{E_{n,m}^{023}\} \). Calculations show that the only constraint on these functions is that
\[ E_{-1,-1}^{013} = E_{-1,-1}^{023}. \]

Thus we specify a cocycle by
\[ F_{-1,-1}^{013} = E_{-1,-1}^{013} + G(xA' A) \]
\[ F_{-1,-1}^{023} = E_{-1,-1}^{023} - G(xA' A) \]
\[ F_{n,m}^{\alpha \beta \gamma} = E_{n,m}^{\alpha \beta \gamma} \quad \text{for all other } n,m \]
so that \( F_{-1,-1}^{013} - F_{-1,-1}^{023} = 2G \) determines the value of the image of this cocycle under the isomorphism
\[ H^2(U', \sigma^{-1}\Theta(-2,-2)) \cong H^2(U', \Theta(-2,-2)) \]
\[ \cong H^0(U, \Theta[-1,-1]). \]

It can be checked that the image of the above \( \{F_{n,m}^{\alpha \beta \gamma}\} \) under the isomorphism in the second line is the global function on \( U, G(xA' A) \).

It now remains only to characterize the connecting homomorphism and to check its action on the representations developed above. Recall the short exact sequence on \( \mathbb{A} \):
\[ 0 \rightarrow \mathbb{A}^2 / \mathbb{A}^3 \rightarrow \mathbb{A} / \mathbb{A}^3 \rightarrow \mathbb{A} / \mathbb{A}^2 \rightarrow 0. \]
The topological inverse image gives a short exact sequence on \( \mathbb{G} \):
The connecting homomorphism can be characterized as follows. For \( \{ \hat{f}_{\alpha \beta} \} \in \Gamma(U^\prime_{\alpha \beta}, \sigma^{-1}e(-1,-1)) \) defining a cocycle in \( H^1(U^\prime, \sigma^{-1}e(-1,-1)) \) let \( \{ \hat{f}'_{\alpha \beta} \} \) be the image of \( \{ \hat{f}_{\alpha \beta} \} \) under the isomorphism \( \sigma^{-1}e(-1,-1) \cong \sigma^{-1}g/\sigma^2 \).

Choose \( \{ \hat{F}_{\alpha \beta} \in \Gamma(U^\prime_{\alpha \beta}, \sigma^{-1}g) \} \) such that we have
\[
\hat{F}_{\alpha \beta} - \hat{f}'_{\alpha \beta} = \hat{F}_{\alpha \beta} + \sigma^{-1}g^2
\]
under the mapping
\[
\sigma^{-1}g - \sigma^{-1}g/\sigma^2.
\]
That is, \( \hat{F}_{\alpha \beta} \) is a representative for \( \hat{f}'_{\alpha \beta} \). Then
\[
\hat{G}_{\alpha \beta} := \hat{F}_{\alpha \beta} + \hat{F}_{\gamma} - \hat{F}_{\alpha \gamma} \in \sigma^{-1}g^2
\]
since \( \{ \hat{f}'_{\alpha \beta} \} \) is a cocycle. We define \( \hat{g}'_{\alpha \beta} \) as the image of \( \hat{G}_{\alpha \beta} \) under the mapping
\[
\hat{G}_{\alpha \beta} - \hat{g}'_{\alpha \beta} := \hat{G}_{\alpha \beta} + \sigma^{-1}g^3
\]
\[
\sigma^{-1}g^2 - \sigma^{-1}g^2/\sigma^3.
\]
Then, letting \( \{ \hat{g}_{\alpha \beta} \} \) be the image of \( \{ \hat{g}'_{\alpha \beta} \} \) under the isomorphism
\[
H^2(U^\prime, \sigma^{-1}g^2/\sigma^3) \cong H^2(U^\prime, \sigma^{-1}e(-2,-2))
\]
we have that \( \{ \hat{g}_{\alpha \beta} \} = \{ (\delta \hat{f})_{\alpha \beta} \} \) where \( \delta \) is the homomorphism
\[
\delta: H^1(U^\prime, \sigma^{-1}e(-1,-1)) \to H^2(U^\prime, \sigma^{-1}e(-2,-2)).
\]
Let \( I := \pi^A_{\gamma} \eta_{\gamma} AA' \) be the defining function of
$\sigma^{-1} \beta \subset \sigma^{-1}\Theta \mathbb{P} \times \mathbb{P}^*$. Then the mappings

$\{\hat{\alpha}_{\beta}\} \rightarrow \{\hat{\alpha}_{\beta} := \hat{\alpha}_{\beta} \cdot I\}$

$\{\hat{\beta}_{\alpha\gamma}\} \rightarrow \{\hat{\beta}_{\alpha\gamma} := \hat{\beta}_{\alpha\gamma} \cdot I^2\}$

define the isomorphisms

$H^1(u', \sigma^{-1}\Theta(-1,-1)) \cong H^1(u', \sigma^{-1}\beta/\beta^2)$

$H^2(u', \sigma^{-1}\Theta(-2,-2)) \cong H^2(u', \sigma^{-1}\beta^2/\beta^3)$.

A cochain $\{\hat{\alpha}_{\beta} \in \Gamma(u'_\alpha, \sigma^{-1}\beta)\}$ can be written as

$\hat{\alpha}_{\beta} = \hat{\alpha}_{\beta} \cdot I$, $\hat{\alpha}_{\beta} \in \Gamma(u'_\alpha, \sigma^{-1}\Theta \mathbb{P} \times \mathbb{P}^*)$.

We will assume a representative for the $\hat{\alpha}_{\beta}$ which will be a function on an $\varepsilon$-neighborhood of $U'_\alpha$ in $V'_\alpha$ which is constant along the fibres, so we will write

$\hat{d}_{\alpha\beta} = \hat{d}_{\alpha\beta}(x'A', y'A', [\pi_{A'}], [\eta_A]).$

This is a notational convenience only, for we will only be interested in the Laurent expansion of $\hat{d}_{\alpha\beta}$ about the set $\{rA' = 0\}$. Since $\hat{d}_{\alpha\beta}$ is to be constant along the fibres of $\mathbb{P} \times \mathbb{P}^* \rightarrow \mathbb{P} \times \mathbb{P}^*$ we demand

$\pi_{A'}(x'A' + y'A') \hat{d}_{\alpha\beta} = \eta_A(x'A' - y'A') \hat{d}_{\alpha\beta} = 0$.

Writing the expansions as

$d_{01}^{01} = \sum_{n,m} d_{n,m}(x'A', y'A') r_0^n s_0^m$

$d_{02}^{02} = \sum_{n,m} d_{n,m}(x'A', y'A') r_0^n s_0^m$

$d_{03}^{03} = \sum_{n,m} d_{n,m}(x'A', y'A') r_0^n s_0^m$

$d_{12}^{12} = \sum_{n,m} d_{n,m}(x'A', y'A') r_0^n s_0^m$

$d_{13}^{13} = \sum_{n,m} d_{n,m}(x'A', y'A') r_0^n s_0^m$

$d_{23}^{23} = \sum_{n,m} d_{n,m}(x'A', y'A') r_0^n s_0^m$
we apply the above differential equation conditions
and get conditions on the components \( d_{n,m}^{a,b} \). We note
here only the ones which will be important in this
application:

\[
\begin{align*}
\frac{\partial d_{n,0}}{\partial x} - \frac{\partial d_{n,0}}{\partial y} &= \frac{\partial d_{n,0}}{\partial x} - \frac{\partial d_{n,0}}{\partial y} = 0, \quad d = d^{01} \\
\frac{\partial d_{0,m}}{\partial x} + \frac{\partial d_{0,m}}{\partial y} &= \frac{\partial d_{0,m}}{\partial x} + \frac{\partial d_{0,m}}{\partial y} = 0, \quad d = d^{02} \\
\frac{\partial d_{n,m}}{\partial x} + \frac{\partial d_{n,m}}{\partial y} &= \frac{\partial d_{n-1,m}}{\partial x} - \frac{\partial d_{n-1,m}}{\partial y} = 0 \\
\frac{\partial d_{n,m}}{\partial x} - \frac{\partial d_{n,m}}{\partial y} &= \frac{\partial d_{n-1,m}}{\partial x} + \frac{\partial d_{n-1,m}}{\partial y} = 0 \\
\frac{\partial d_{n,m}}{\partial x} + \frac{\partial d_{n,m}}{\partial y} &= \frac{\partial d_{n-1,m}}{\partial x} - \frac{\partial d_{n-1,m}}{\partial y} = 0 \\
\frac{\partial d_{n,m}}{\partial x} - \frac{\partial d_{n,m}}{\partial y} &= \frac{\partial d_{n-1,m}}{\partial x} + \frac{\partial d_{n-1,m}}{\partial y} = 0 \\
\frac{\partial d_{n,m}}{\partial x} + \frac{\partial d_{n,m}}{\partial y} &= \frac{\partial d_{n-1,m}}{\partial x} - \frac{\partial d_{n-1,m}}{\partial y} = 0, \quad d = d^{13} \\
\frac{\partial d_{n,m}}{\partial x} - \frac{\partial d_{n,m}}{\partial y} &= \frac{\partial d_{n-1,m}}{\partial x} + \frac{\partial d_{n-1,m}}{\partial y} = 0, \quad d = d^{23}
\end{align*}
\]

Suppose now that \( \hat{D}_{a,b} \) is a representative for \( \hat{f}_{a,b} \)
where \( \{ \hat{f}_{a,b} \} \) is a cocycle as above. Then

\[
\begin{align*}
d^{01}_{n,m}(x^{AA'},0) &= A_{n,m} \\
d^{02}_{n,m}(x^{AA'},0) &= \begin{cases} A_{n,m} & m \geq 0 \\ -A_{n,m} & n,m \geq 0 \\ A_{n,m} & n \geq 0, m < 0 \\ -A_{n,m} & n,m \geq 0 \text{ or } n,m < 0 \\ A_{n,m} & n < 0, m \geq 0 \text{ or } n \geq 0, m < 0
\end{cases}
\end{align*}
\]
\[ d_{n,m}^{13}(x^{AA'}, 0) = \begin{cases} A_{n,m} & n, m < 0 \\ -A_{n,m} & n < 0, m \geq 0 \end{cases} \]
\[ d_{n,m}^{23}(x^{AA'}, 0) = A_{n,m} \quad m < 0 \]

and all other coefficients vanish. Since \( \hat{\epsilon}_{\alpha \beta} \) is a cocycle,
\[ \hat{D}_{\alpha \beta} + \hat{D}_{\beta \gamma} - \hat{D}_{\alpha \gamma} = \hat{\epsilon}_{\alpha \beta \gamma} \cdot I^2 \]
where \( \{\hat{\epsilon}_{\alpha \beta}(x^{AA'}, 0)\} \) defines a cohomology class in \( H^2(U', \sigma^{-1}\mathbf{S}(-2, -2)) \). Or we can write
\[ \hat{k}_{\alpha \beta \gamma} := \hat{d}_{\alpha \beta} + \hat{d}_{\beta \gamma} - \hat{d}_{\alpha \gamma} = \hat{\epsilon}_{\alpha \beta \gamma} \cdot I. \]

We note here that
\[ k_{n,m}^{013} = \begin{cases} d_{n,m}^{01} - d_{n,m}^{03} & n, m \geq 0 \\ \quad -d_{n,m}^{03} & n > 0, m < 0 \\ d_{n,m}^{01} + d_{n,m}^{13} - d_{n,m}^{03} & n < 0, m \geq 0 \\ d_{n,m}^{13} - d_{n,m}^{03} & n < 0, m < 0 \end{cases} \]
\[ k_{n,m}^{023} = \begin{cases} d_{n,m}^{02} - d_{n,m}^{03} & n, m \geq 0 \\ d_{n,m}^{02} + d_{n,m}^{23} - d_{n,m}^{03} & n > 0, m < 0 \\ -d_{n,m}^{03} & n < 0, m \geq 0 \\ d_{n,m}^{23} - d_{n,m}^{03} & n, m < 0 \end{cases} \]

and similar formulae hold for \( k_{n,m}^{012} \) and \( k_{n,m}^{123} \). Here \( \hat{\epsilon}_{\alpha \beta \gamma} \) is the cocycle which we wish to characterize, and from the above work we know that only the \(-l,-l\) term in the expansion in terms of \( r_0 \) and \( s_0 \) affects the cohomology class. Thus we want to calculate
\[ 013_{e,0,0,0}^{e,-1,-1} - 023_{e,0,0,0}^{e,-1,-1} \]

where

\[ e^{0}_{\alpha \beta \gamma} = \sum_{n,m} \sum_{i,j,k,l=0}^{\infty} \alpha \beta \gamma e_{i,j,k,l}^{\alpha \beta \gamma} (x^{AA'}) \cdot (y^{00'})^{i} \cdot (y^{01'})^{j} (y^{10'})^{k} (y^{11'})^{l} r_{0}^{n} s_{0}^{m} \]

is the expansion for \( e^{0}_{\alpha \beta \gamma} \); in particular,

\[ e^{0}_{\alpha \beta \gamma} (x^{AA'},0) = \sum_{n,m}^{\alpha \beta \gamma} e_{0,0,0}^{\alpha \beta \gamma} (x^{AA'}) r_{0}^{n} s_{0}^{m}. \]

Since \( l^{0} = y^{00'} + r_{0}^{0} y^{01'} + s_{0}^{0} y^{10'} + r_{0}^{0} s_{0}^{0} y^{11'} \), we define

\[ k^{\alpha \beta \gamma} := d^{0}_{\alpha \beta \gamma} + d^{0}_{\beta \gamma} - d^{0}_{\alpha \gamma} = e^{0}_{\alpha \beta \gamma} \cdot l^{0}, \]

and then

\[ k^{\alpha \beta \gamma}_{n,m} = e_{n,m}^{\alpha \beta \gamma} y^{00'} + e_{n-1,m}^{\alpha \beta \gamma} y^{01'} + e_{n,m-1}^{\alpha \beta \gamma} y^{10'} + e_{n-1,m-1}^{\alpha \beta \gamma} y^{11'}. \]

This implies that

\[ \alpha \beta \gamma_{k,0,0,0}^{1,0,0,0} = \alpha \beta \gamma_{n,m}^{0,0,0,0} \]

\[ \alpha \beta \gamma_{k,0,1,0,0}^{1,0,0,0} = \alpha \beta \gamma_{n,m}^{0,0,0,0} \]

\[ \alpha \beta \gamma_{k,0,0,1,0}^{1,0,0,0} = \alpha \beta \gamma_{n,m}^{0,0,0,0} \]

\[ \alpha \beta \gamma_{k,0,0,0,1}^{1,0,0,0} = \alpha \beta \gamma_{n,m}^{0,0,0,0} \]

by comparing terms. We note that \( \alpha \beta \gamma_{k,0,0,0}^{0,0,0,0} = 0 \) necessarily. Define

\[ G := \frac{e_{H}^{4}}{\partial x^{00'} \partial x^{01'} \partial x^{10'} \partial x^{11'}} \]

as above. Then, using

\[ 013_{e,0,0,0}^{e,-1,-1} = 013_{l,0,0,0}^{d,-1,-1} = 13_{d,-1,-1}^{d,-1,-1} - 03_{d,-1,-1}^{d,-1,-1} = 013_{k,0,0,0}^{d,0,-1} = 03_{d,0,1,0,0} \]
\[ \begin{aligned}
&= 013_0,0,0,1,0 = 01d_{-1,0} + 13d_{0,0,1,0} - 03d_{0,0,1,0} \\
&= 013_0,0,0,0,1 = 01d_{0,0,0,1,0} - 03d_{0,0,0,1,0}
\end{aligned} \]

we apply the above differential relations to obtain the necessary information. For example,

\[ \frac{\partial d_{01}}{\partial x_{11}'} = \frac{\partial d_{01}}{\partial y_{11}'} = \frac{\partial}{\partial x_{11}'} 01d_{0,0,0,0,0} = 01d_{0,0,0,0,0} = \frac{\partial}{\partial x_{11}'} A_{0,0} \]

\[ = \frac{\partial}{\partial x_{00}'} \frac{\partial}{\partial x_{01}'} \frac{\partial}{\partial x_{10}'} \frac{\partial}{\partial x_{11}'} G. \]

The other equations give

\[ 01d_{-1,0} = \frac{\partial}{\partial x_{10}'} A_{-1,0} = G \]

\[ 13d_{-1,0} = \frac{\partial}{\partial x_{00}'} A_{-1,0} = G \]

\[ 13d_{-1,-1} = - \frac{\partial}{\partial x_{00}'} A_{-1,-1} = G \]

\[ 03d_{0,0,0,1,0} = 03d_{1,0,0,0,0} + G = 03d_{-1,-1,0} - G. \]

Combining these and using any of the formulae for 013 to obtain

\[ 013_{e-1,-1} \] gives

\[ 013_{e-1,-1} = G - 03d_{-1,-1}. \]

Repeating this process for 023, we obtain

\[ 023_{e-1,-1} = - G - 03d_{-1,-1}. \]
Thus, finally we have
\[ 013e_{-1,0,0,0} - 023e_{-1,0,0,0} = 2G. \]

Thus we have characterized the cohomology class of the cocycle \[ \hat{e}_{\alpha\beta\gamma}(x^{AA'},0) \] according to the work above, and the value of the image of this class under the isomorphism with \( H^0(U, \Theta[-1,-1]) \) is just \( G(x^{AA'}) \). But \( G(x^{AA'}) \) is the value of the image of the cohomology class \( [\hat{e}] \) by assumption, so the induced map
\[
\Delta : H^0(U, \Theta[-1,-1]) \rightarrow H^0(U, \Theta[-1,-1])
\]
\[
H^1(U', \sigma^{-1}\Theta(-1,-1)) \rightarrow H^2(U', \sigma^{-1}\Theta(-2,-2))
\]
is just the identity map. This is what we wished to prove.
VI. The Action of Yang-Mills Fields

One of the invariant quantities associated to a Yang-Mills field \( F \) on a manifold \( X \) is the action

\[
\mathcal{A}(F) := \int_X \text{tr} \ F \wedge *F.
\]

The action is of particular value in the path integral approach to quantum field theory ([12]). Since we have a method of encoding the information describing a Yang-Mills field as a vector bundle on \( \mathcal{A} \), as seen in Chapter IV, we would expect that we can associate an invariant quantity to each of the vector bundles arising from this correspondence which is equal to the action of the corresponding Yang-Mills field on \( \mathcal{A} \). The search for this quantity splits naturally into four parts:

1. Expressing \( F \) as a quantity on \( \mathcal{A} \), done in [14];
2. transforming the Hodge \( * \)-operator to an operator on \( \mathcal{A} \);
3. transforming the antisymmetric tensor product to an operation on \( \mathcal{A} \); and
4. "integrating" over \( \mathcal{A} \) in some sense, to obtain an invariant for the bundle.

1. Cohomological representation of two-forms. Suppose that \( E \to U'' \) is a \( U \)-trivial vector bundle corresponding under the generalized Ward correspondence to a vector bundle \( \tilde{E} \to U \) with connection \( \tilde{A} \). The work of [14] shows how to express the curvature \( F(x) \) cohomologically on \( U'' \): For \( x \in U \) restrict the bundle \( E \) to \( L_x \); by
construction the restriction gives a trivial bundle. Using extension theory and the results of the ambitwistor transform as in the last chapter, it is easy to check that \( E|_{L_x} \) has a unique first extension to the trivial rank \( n \) vector bundle on \( L_x^{(1)} \), the first order neighborhood of \( L_x \). The cohomology group \( H^1(L_x, \text{End } E|_{L_x} \otimes (N_x^*)^2) \) acts transitively and effectively on the set of second extensions, where \( N_x^* \) is the conormal bundle of \( L_x \) in \( \mathcal{A} \). The element of this group which represents the difference between the trivial bundle on \( L_x^{(2)} \) and the restriction of \( E \) to \( L_x^{(2)} \) can be identified with \( F(x) \) under the isomorphism

\[
H^1(L_x, \text{End } E|_{L_x} \otimes (N_x^*)^2) \cong \Omega^2_M(x) \otimes \text{End } \tilde{E}(x).
\]

This isomorphism will be examined more closely in the following paragraphs.

We desire to construct an operator \( *_L \) on the group (actually a vector space) \( H^1(L_x, \text{End } E|_{L_x} \otimes (N_x^*)^2) \) which mirrors the action of the Hodge \( * \)-operator on \( \Omega^2_M(x) \otimes \text{End } \tilde{E}(x) \). This will follow rather easily from an examination of the isomorphism between the groups. We first note that \( E|_{L_x} \) is trivial we may write thus

\[
H^1(L_x, \text{End } E|_{L_x} \otimes (N_x^*)^2) \cong H^1(L_x, (N_x^*)^2) \otimes \text{End } \tilde{E}(x),
\]

so that we need only define the operator \( *_L \) on \( H^1(L_x, (N_x^*)^2) \).
Following [14], we define on \( \mathcal{G} \) the bundle
\[
N^* := \ker(\rho^* \Omega^1_M \to \Omega^1_\mathcal{O})\).
\]

It is easy to check that there is a bundle isomorphism
\[
N^*|_{\rho^{-1}(x)} \cong N^*_x \\
\rho^{-1}(x) \cong L^*_x.
\]

The exact sequence on \( \mathcal{G} \)
\[
(6.1) \quad 0 \to \mathcal{O}(-1,-1) \to \mathcal{O}_A(-1,0) \oplus \mathcal{O}_A'(0,-1) \to N^* \to 0
\]
corresponds to the exact sequence on \( L^*_x \)
\[
(6.2) \quad 0 \to N^*_x|_{\mathbb{P} \times \mathbb{P}^* L^*_x} \to L^*_x|_{\mathbb{P} \times \mathbb{P}^*} \to N^*_x \to 0
\]
in the following way: if we restrict (6.1) to \( \rho^{-1}(x) \), then the restricted sequence is isomorphic to the sequence (6.2) on \( L^*_x \). The symmetric tensor product of (6.1) gives the exact sequence
\[
(6.3) \quad 0 \to \mathcal{O}_A(-2,-1) \oplus \mathcal{O}_A'(1,-2) \to \mathcal{O}^{\otimes 2}(\mathcal{O}_A(-1,0) \oplus \mathcal{O}_A'(0,-1)) \to \mathcal{O}^{\otimes 2}N^* \to 0.
\]

From the direct image results of Chapter III we have
\[
\rho^*\mathcal{O}_A(-2,-1) \oplus \mathcal{O}_A'(1,-2)) = 0, \quad i = 0,1,2,
\]
which implies the isomorphism
\[
(6.4) \quad \rho^*\mathcal{O}^{\otimes 2}(\mathcal{O}_A(-1,0) \oplus \mathcal{O}_A'(0,-1))) \cong \rho^*\mathcal{O}^{\otimes 2}N^*, \quad i=0,1,2.
\]

Again using the various results on direct images from Chapter III we obtain
\[
(6.5) \quad \rho^*[\mathcal{O}^{\otimes 2}(\mathcal{O}_A(-1,0) \oplus \mathcal{O}_A'(0,-1))]
\]
\[
\cong \rho^*[(\mathcal{O}_{(AB)}(-2,0) \oplus \mathcal{O}_{(A'B')})(0,-2)]
\]
\[
\cong \Omega^2_{-} \oplus \Omega^2_{+} = \Omega^2_M.
\]
Since we have that \( \star w = \pm iw \) for \( w \in \Omega_x^2 \), we define operators on \( \mathcal{O}_{(AB)}(-2,0) \) and \( \mathcal{O}_{(A'B')}^\perp(0,-2) \) by:
\[
\star_\xi = -i\xi, \quad \xi \in \mathcal{O}_{(AB)}(-2,0) \\
\star_\varphi = i\varphi, \quad \varphi \in \mathcal{O}_{(A'B')}(0,-2).
\]
We then define the operator \( \star \) on \( \mathcal{O}_{(AB)}(-2,0) \oplus \mathcal{O}_{(A'B')}^\perp(0,-2) \) by
\[
(6.6) \quad \star(\xi, \varphi) = (-i\xi, i\varphi).
\]

Using the below chain of isomorphisms,
\[
H^1(\rho^{-1}(x), \mathcal{O}_{(AB)}(-2,0) \oplus \mathcal{O}_{(A'B')}^\perp(0,-2)) \\
\cong H^1(\rho^{-1}(x), \mathcal{O}^2(\mathcal{O}_A(-1,0) \oplus \mathcal{O}_A(0,-1))) \\
\cong H^1(\rho^{-1}(x), \mathcal{O}^2(\mathcal{N}^*|_{\rho^{-1}(x)})) \\
\cong H^1(L_x, \mathcal{O}^2_{N_x^*})
\]
and the operator defined in (6.6) we define the induced operator
\[
(6.7) \quad \star_L : H^1(L_x, \mathcal{O}^2_{N_x^*}) \to H^1(L_x, \mathcal{O}^2_{N_x^*}).
\]

An equivalent way of defining \( \star_L \) involves defining \( \star_L \) on \( \mathbb{P} \times \mathbb{P}^* \) and restricting. That is, we have the exact sequence relating the conormal bundles of \( L_x \) in \( \mathcal{A} \) and \( \mathbb{P} \times \mathbb{P}^* \),
\[
0 \to \mathcal{N}^*_{\mathcal{A}|\mathbb{P} \times \mathbb{P}^*|L_x} \to \mathcal{N}^*_{L_x} \| \mathbb{P} \times \mathbb{P}^* \to \mathcal{N}^*_{\mathbb{P} \times \mathbb{P}} \to 0 \\
\oplus(-1,-1) \to \oplus(-1,0) \oplus \mathcal{O}(0,-1).
\]
The symmetric product gives the exact sequence
\[
0 \to \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2) \to \mathcal{O}^2(\mathcal{N}^*_{L_x} \| \mathbb{P} \times \mathbb{P}^*) \oplus \mathcal{O}^2_{N_x^*} \to 0
\]
which implies, using the normal vanishing theorems

on $L_X \cong IP_1 \times IP_1$,

$$H^1(L_X, O^2(N_{L_X}^* | IP \times IP^*)) \cong H^1(L_X, O^2(N_{L_X}^*)).$$

We thus need only to define $*_L$ on $H^1(L_X, O^2(N_{L_X}^* | IP \times IP^*)$.

Using

$$N_{L_X}^* | IP \times IP^* \cong N_{\tau^{-1}(x)}^* | IP \oplus N_{(\tau^*)^{-1}(x)}^* | IP^*,$$

we calculate (writing $L_{x*}, L_{x}^*$ for $\tau^{-1}(x)$ and $(\tau^*)^{-1}(x)$)

$$H^1(L_{x*}, O^2(N_{L_{x*}}^* | IP \times IP^*)) \cong H^1(L_{x}, O^2(N_{L_{x}}^* | IP^*))$$

(6.8)

$$\oplus H^1(L_{x*}, O^2(N_{L_{x*}}^* | IP^*)).$$

We define the operator

$$*_L(\xi \oplus \varphi) := -i\xi \oplus i\varphi$$

where

$$\xi \in H^1(L_{x}, O^2(N_{L_{x}}^* | IP^*))$$

$$\varphi \in H^1(L_{x}, O^2(N_{L_{x}}^* | IP^*)).$$

This construction mirrors the above work because in relating the two we are using the bundle isomorphism

$$\Theta_A(-1,0) \mid \nu^{-1}(x) \cong N_{L_X}^* | IP$$

$$\downarrow$$

$$\nu^{-1}(x) \cong L_X$$

and the corresponding isomorphism for the dual twistors.

2. Duality pairings on $\mathcal{M}$ and $\mathcal{A}$. For this section we restrict ourselves to the case of electromagnetic fields $F$. We wish to express the action density $F \wedge *F$ as an object on $\mathcal{A}$. To this end we split $F$
into self-dual and anti-self-dual parts:

\[ F = F^+ + F^- \]

\[ F_{AA'BB'} = \epsilon_{A'B'} \psi_\mathit{AB} + \epsilon_{AB} \psi_{A'B'} \]

where

\[ F^+ \leftrightarrow \epsilon_{AB} \psi_{A'B'} \]

\[ F^- \leftrightarrow \epsilon_{A'B'} \psi_{AB} \]

(Cf. [23].) It is then easy to calculate that

\[ F \wedge \ast F = i(F^+ \wedge F^+ - F^- \wedge F^-) \]

\[ \leftrightarrow i(\psi_{A'B'} \psi_{A'B'} \ast \mathit{AB}) \].

Because the density splits this way we shall first deal with the anti-self-dual part, that is, that part which is dealt with by the twistor transform.

As discussed above, the two-form \( F^- \) corresponds to a cohomology class \( \mathit{AB} \) on \( \mathcal{A} \) for each \( x \in U \):

\[ \mathit{AB}(x) \in \Omega^2_-(x) \leftrightarrow \mathit{AB} \in H^2(L_x, \mathcal{O}^2(N^*_L_x \mid \mathbb{P})) \].

The application of the Hodge \( \ast \)-operator gives multiplication by \( i \). The spinor notation suggests a natural way of viewing the antisymmetric tensor product \( \wedge \) acting on two-forms: if we take the image of \( \mathit{AB} \) under the duality mapping

\[ \mathcal{O}(AB)[-1] \rightarrow \mathcal{O}(AB)[-1,-2] \]

then the action of the antisymmetric tensor product can be written as the duality pairing, given by contraction of indices in local coordinates

\[ \phi \wedge \hat{\phi} \leftrightarrow \phi_{AB} \hat{\phi}_{AB} \]
where $\varphi = \epsilon_{\alpha_1\beta_1} \cdot \omega_{\alpha_2\beta_2} \wedge \omega_{\alpha_3\beta_3}$. This approach transfers easily to a product on ambitwistor space.

Let $\varphi^*_{\text{dual}}$ denote the dual of $\varphi_{\text{dual}}$ under the dual mapping

$$H^1(L_x, \mathcal{O}^2(N^*_{L_x} | IP)) \cong H^0(L_x, \Omega^1_{L_x} [\mathcal{O}^2(N^*_{L_x} | IP)]^*)$$

$$\cong H^0(L_x, \Omega^1_{L_x} [\mathcal{O}^2(N^*_{L_x} | IP)])$$

where Serre duality gives the first line. We claim that the duality pairing between these two cohomology classes given by the cup product

$$\varphi_{\text{dual}} \cdot \varphi^*_{\text{dual}} \in H^1(L_x, \Omega^1_{L_x})$$

corresponds to $\omega_{\alpha\beta} \varphi_{\alpha\beta}$ under the Penrose transform. Further, the value of $\omega_{\alpha\beta} \varphi_{\alpha\beta}$ at $x \in U$ can be obtained from Serre duality

$$H^1(L_x, \Omega^1_{L_x}) \cong H^0(L_x, \mathcal{O}) \cong \mathbb{C},$$

i.e., $\varphi_{\text{dual}} \cdot \varphi^*_{\text{dual}}$ is a (1,1)-form on $L_x$, and we integrate over $L_x$ to obtain a number equal to $\omega_{\alpha\beta} \varphi_{\alpha\beta}(x)$.

To see that this is so we must examine the dual of the diagram

$$H^1(\nu^{-1}(x), \mathcal{O}_{(AB)}(-2))$$

$$H^1(L_x, \mathcal{O}^2(N^*_{L_x} | IP)) \quad H^0(x, \mathcal{O}_{(AB)}[-1])'$$

Taking the dual of $H^1(\nu^{-1}(x), \mathcal{O}_{(AB)}(-2))$ involves the sheaf of one-forms along the fibres of $\nu$, which we shall denote by $\Omega^1_{\nu}$. It can be shown that

$$\Omega^1_{\nu} \cong \mathcal{O}(-2)[-1,-2].$$
With this result we calculate the dual isomorphism
\[ H^1(\nu^{-1}(x), \mathcal{O}_{(AB)}(-2)) \cong H^0(\nu^{-1}(x), \mathcal{O}_{(AB)}[-1,-2]), \]
and this gives the dual of the above diagram:
\[ H^0(\nu^{-1}(x), \mathcal{O}_{(AB)}[-1,-2]) \]
\[ \xrightarrow{\sim} H^0(L_x, \mathcal{O}_{L_x}^1 (\mathcal{O}^2(N_{L_x} | \mathbb{IP}))) \]
\[ \cong H^0(x, \mathcal{O}_{(AB)}[-1,-2]). \]
Using the duality pairing on each of these groups gives the desired result.

The same result can be obtained in the dual twistor case to give a representation for the self-dual part of the action density. The ambitwistor case can then be worked out using the isomorphism (6.8)
\[ H^1(L_x^\Lambda, \mathcal{O}^2(N_{L_x}^\Lambda | \mathbb{A})) \cong H^1(L_x^\Lambda, \mathcal{O}^2(N_{L_x}^\Lambda | \mathbb{IP})) \]
\[ \oplus H^1(L_x^*, \mathcal{O}^2(N_{L_x}^* | \mathbb{IP}^*)) \]
from the last section. If we write $\Psi_x$ for the cohomology class on $\mathbb{IP}^*$ obtained from the dual twistor case, then
\[ \pi^*(\omega_{\mathbb{C}P^3_{\mathbb{R}}} \otimes \Psi_x) \oplus \pi^*(\Psi_x \nu_{\mathbb{C}P^3_{\mathbb{R}}}^*) \in H^2(I_x^\Lambda, \mathcal{O}_{L_x}^2) \]
gives a (2,2)-form which can be integrated over $L_x^\Lambda \cong \mathbb{IP}_1 \times \mathbb{IP}_1$ to obtain the action density at $x$, $F(x)^\Lambda \wedge F(x)$.
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