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STRAIN-HARDENING AND RATE EFFECTS IN PLASTICITY

by

LU-TSUEN MEI

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

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HOUSTON, TEXAS

MAY, 1981
ABSTRACT

STRAIN-HARDENING AND RATE EFFECTS IN PLASTICITY

LU-TSUEN MEI

In this study the effects of strain-hardening and rate sensitivity in plasticity theory are investigated. Because the method of characteristics cannot be applied to problems involving rate dependent yield conditions (65) therefore a different procedure is developed. In this investigation of large plastic deformation, the displacement field of the wedge indentation and inverted plane strain extrusion problem can be specified by following the trajectory of each element during deformation processes.

Therefore, the unit diagram, introduced by Hill, et al (50), and the trajectory equations, derived by Hill (51), are utilized to specify the strain, strain rate fields for wedge indentation and extrusion problem, respectively. The plane strain inverted extrusion with 50% reduction in area is employed in this study because a complete solution has been given by Alexander (3) and the relative motion between the die and billet is prevented.

Finite element and finite difference schemes are used to calculate the total work required to produce indentation and extrusion processes; work is then converted to the pressure or force. Atlan and Boulger (4), Adams and Beese (1), and Holzer (54) have presented the empirical equations for describing the strain-hardening and rate dependent characteristics of metals. Due to the simplicity and accuracy power law equations are employed to describe the material properties
presented by Dugdale (34, 35) and the author. These material constitutive equations are then used to calculate the stress field which is the basis for the corrections of strain hardening or rate effects for the solutions based on the perfectly plastic theory.

The agreement between the published experimental results and the theoretical predictions based on the analyses described in this study indicates the method proposed in this investigation is appropriate in modifying the slip line solutions for rigid, perfectly plastic material when strain hardening and rate effects are involved. Aspects of the theory which demand further investigation in future studies are pointed out.
ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation for the advice and assistance of Dr. John B. Cheatham, Jr., in the conduct of this study.

The financial supports from Rice University and the National Science Foundation, through a Rice fellowship and research assistantship, are highly appreciated.

The author also wishes to thank his wife and his family for their encouragement and patience.
NOMENCLATURE

$k$  Yield stress in pure shear
$k_0$  Initial yield stress in pure shear
$k^*$  Dimensionless yield stress
$m$  Strain hardening index
$n$  Strain rate sensitivity index
$p$  Pressure
$q_i$  Heat flux components
$\xi$  Rate of heat addition per unit mass from external sources
$r$  Displacement vector in physical space
$r^*$  Displacement vector on the unit diagram
$S$  Distance
$S_{ij}$  Stress deviator tensor
$t$  Time
$v$  Velocity vector
$v^*$  Velocity vector on the unit diagram
$\partial v/\partial t$  Local acceleration
$v_i$  Velocity component
$x_i$  Spatial coordinates
$\alpha$  Radical shear fan angle; material constant
$\beta$  Semi-wedge angle
$\delta_{ij}$  Kronecker delta
$\varepsilon$  Internal energy per unit mass
$\dot{\varepsilon}$  Strain rate
$\dot{\varepsilon}_e$  Effective strain rate
$\bar{\varepsilon}_e$  Mean value of effective strain
$\varepsilon_{ij}^p$  Plastic strain tensor components

iv
\[ \dot{e}_e \quad \text{Effective strain rate} \]
\[ \eta \quad \text{Natural coordinate; entropy density} \]
\[ d\lambda \quad \text{Scalar in the normality principle} \]
\[ \xi \quad \text{Natural coordinate} \]
\[ \sigma_s, \sigma_o \quad \text{Static yield stress} \]
\[ \sigma_u \quad \text{Ultimate strength} \]
\[ \sigma_1, \sigma_2, \sigma_3 \quad \text{Principal stresses} \]
\[ \sigma_{ij} \quad \text{Stress tensor components} \]
\[ \sigma_e \quad \text{Effective stress} \]
\[ \sigma_x, \sigma_y \quad \text{Normal stresses} \]
\[ \sigma \quad \text{Mean stress} \]
\[ \mu \quad \text{Coefficient of friction} \]
\[ \kappa \quad \text{Work hardening parameter} \]
\[ \phi \quad \text{Internal frictional angle} \]
\[ \psi \quad \text{Function relating and static stress-strain curve} \]
\[ \rho \quad \text{Density} \]
\[ \psi \quad \text{Helmholtz free energy density} \]
\[ \theta \quad \text{Absolute temperature} \]
\[ T_{xy} \quad \text{Shear stress} \]
\[ \lambda \quad \text{Constant} \]
# TABLE OF CONTENTS

<p>| Acknowledgements                          | 1 |
| Nomenclature                              | iv |
| 1. Introduction                           | 1 |
| 2. Theoretical Background and Literature Review | 4 |
| 2.1 Yield Functions for Ductile Metals    | 4 |
| 2.2 Hardening Rules                       | 7 |
| 2.3 Yield Functions for Soils and Rocks   | 9 |
| 2.4 Experimental Results and Related Work | 12 |
| A. Metals                                 | 12 |
| B. Rocks and Soils                       | 15 |
| 2.5 Rate Dependent Deformation Theory     | 21 |
| 2.6 Special Case of Constitutive Equations| 24 |
| A. Application to Metals                  | 25 |
| B. Application to Soils                   | 27 |
| C. Comparison with Experimental Data      | 28 |
| 2.7 Solutions of Strain-Hardening Problems| 29 |
| 2.8 Solutions of Rate Dependent Problems  | 32 |
| 2.9 Solutions of Strain-Hardening and Rate Dependent Problems | 34 |
| 3. Theoretical Development                | 36 |
| 3.1 Plane Strain Plasticity Equations for a Rigid Strain-Hardening Material | 37 |
| 3.2 Plane Strain Plasticity Equations for a Rigid Rate Dependent Material | 40 |
| 3.3 Plane Strain Plasticity Equations for a Rigid Perfectly Plastic Material | 41 |</p>
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4</td>
<td>Perturbation Solutions for Strain-Hardening, Rate Dependent Material</td>
<td>42</td>
</tr>
<tr>
<td>3.5</td>
<td>Limit Analysis</td>
<td>43</td>
</tr>
<tr>
<td>3.6</td>
<td>Field Equations</td>
<td>45</td>
</tr>
<tr>
<td>3.7</td>
<td>Theoretical Solutions to Extrusion and Wedge Indentation Problems</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>for Strain-Hardening, Rate Dependent Materials</td>
<td></td>
</tr>
<tr>
<td>A.</td>
<td>For Rate Sensitive Materials</td>
<td>47</td>
</tr>
<tr>
<td>B.</td>
<td>For Strain-Hardening Materials</td>
<td>50</td>
</tr>
<tr>
<td>C.</td>
<td>For Strain-Hardening and Rate Sensitive Materials</td>
<td>52</td>
</tr>
<tr>
<td>3.8</td>
<td>Displacement, Strain, and Strain Rate Fields</td>
<td>53</td>
</tr>
<tr>
<td>3.9</td>
<td>Deformation for an Element During Indentation</td>
<td>54</td>
</tr>
<tr>
<td>3.10</td>
<td>Inverted Plane Strain Extrusion with 50% Reduction</td>
<td>63</td>
</tr>
<tr>
<td>3.11</td>
<td>Effective Stress and Effective Plastic Strain</td>
<td>69</td>
</tr>
<tr>
<td>3.12</td>
<td>Indentation of a Plastic Half-Space by a Frictional, Rigid Wedge</td>
<td>72</td>
</tr>
<tr>
<td>3.13</td>
<td>The Force and Work Relation for the Case of Frictional Wedge Indentation</td>
<td>73</td>
</tr>
<tr>
<td>3.14</td>
<td>Plane Strain Plasticity Equations for a Rigid Perfectly Plastic Coulomb Material</td>
<td>76</td>
</tr>
<tr>
<td>3.15</td>
<td>Indentation of a Semi-Infinite Mass of Soil by a Smooth, Rigid Wedge</td>
<td>82</td>
</tr>
<tr>
<td>3.16</td>
<td>Indentation of a Semi-Infinite Mass of Soil by a Frictional, Rigid Wedge</td>
<td>86</td>
</tr>
<tr>
<td>3.17</td>
<td>The Force and Work Relation for the Case of Frictional Indentation</td>
<td>88</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (CONT'D)

4. Approximate Method for Axially Symmetric Problems 90
   4. 1 General Equations 91
   4. 2 Approximate Method for Conical Indentation on Metals 93

5. Apparatus and Experimental Procedure 97
   5. 1 Apparatus and Tools 97
   5. 2 Compression Tests on 99.9% Pure Lead 98
   5. 3 Wedge Indentation Tests 100
   5. 4 Plane Strain Inverted Extrusion with 50% Reduction in Area 100

6. Solutions of Some Practical Problems 102
   6. 1 Numerical Methods 103
   6. 2 Calculations of Indentation, Extrusion Pressure for the 103
       Rigid, Strain-Hardening, Rate Dependent, Strain-Harden-
       ing and Rate Dependent Materials
       A. Strain-Hardening Material 104
       B. Rate-Dependent Material 104
       C. Strain-Hardening and Rate Dependent Material 105
   6. 3 False Wedge Solution 105
   6. 4 Axial Symmetric Correction Based on Plane-Strain 107
       Solution
   6. 5 The Mean Value of the Effective Strain for Plane Strain 108
       Problems

7. Theoretical and Experimental Results 111
   7. 1 Results for Metals 111
       A. Numerical Solutions for Perfectly, Rigid Plastic 111
          Material
## TABLE OF CONTENTS (CONT'D)

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B. Numerical Solutions for Rigid, Strain-Hardening Material</td>
<td>112</td>
</tr>
<tr>
<td>C. Numerical Solutions for Rigid, Rate Dependent Material</td>
<td>113</td>
</tr>
<tr>
<td>D. Numerical Solutions for Strain-Hardening and Rate Dependent Material</td>
<td>113</td>
</tr>
<tr>
<td>7. 2 Results for Soils</td>
<td>114</td>
</tr>
<tr>
<td>7. 3 Strain-Hardening Effects Corrected by Using the Mean Value of the Effective Strain</td>
<td>115</td>
</tr>
<tr>
<td>8. Summary and Conclusions</td>
<td>118</td>
</tr>
<tr>
<td>References</td>
<td>119</td>
</tr>
<tr>
<td>Appendix A</td>
<td>130</td>
</tr>
<tr>
<td>Characteristic Directions and Equations Along the Characteristics for a Von Mises Strain-Hardening Material</td>
<td></td>
</tr>
<tr>
<td>Appendix B</td>
<td>134</td>
</tr>
<tr>
<td>Characteristic Directions and Equations Along the Characteristics for Materials Satisfying the Coulomb Yield Condition</td>
<td></td>
</tr>
<tr>
<td>Appendix C</td>
<td>136</td>
</tr>
<tr>
<td>Derivation of Effective Stress and Effective Strain</td>
<td></td>
</tr>
<tr>
<td>Appendix D</td>
<td>141</td>
</tr>
<tr>
<td>Wedge Surface Pressure During Soil Indentation Process</td>
<td></td>
</tr>
<tr>
<td>Appendix E</td>
<td>143</td>
</tr>
<tr>
<td>Flow Chart for Calculating the Total Work in the Plastic Region</td>
<td></td>
</tr>
<tr>
<td>Appendix F</td>
<td>150</td>
</tr>
<tr>
<td>Derivation of Integration Formular and Finite Difference Method in Polar Coordinate System and Difference Equations</td>
<td></td>
</tr>
<tr>
<td>Appendix G</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Elements for Finite Element Method</td>
<td></td>
</tr>
<tr>
<td>Tables</td>
<td>161</td>
</tr>
<tr>
<td>Figures</td>
<td>186</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
</tr>
<tr>
<td>1</td>
<td>Comparison of extrusion pressure based on proposed assumption</td>
</tr>
<tr>
<td>2</td>
<td>Torsion shear stress (ton/in.²) versus shear strain</td>
</tr>
<tr>
<td>3</td>
<td>Constitutive equations of steel, copper and aluminum</td>
</tr>
<tr>
<td>4</td>
<td>Torsion shear stress (ton/in.²) versus shear strain</td>
</tr>
<tr>
<td>5</td>
<td>Results of wedge indentation by finite element method</td>
</tr>
<tr>
<td>6</td>
<td>Results of wedge indentation by finite difference method</td>
</tr>
<tr>
<td>7</td>
<td>The false wedge solution for Von Mises material</td>
</tr>
<tr>
<td>8</td>
<td>Pressure distribution and pressure on the smooth cone surface</td>
</tr>
<tr>
<td>9</td>
<td>Numerical result of the plane-strain inverted extrusion with 50% reduction in area</td>
</tr>
<tr>
<td>10</td>
<td>Calculations of the mean effective strain for smooth wedge indentation</td>
</tr>
<tr>
<td>11</td>
<td>Wedge indentation with strain hardening effect, compared with the published data by D. S. Dugdale</td>
</tr>
<tr>
<td>12</td>
<td>Cone indentation on strain hardening material, compared with published data by D. S. Dugdale</td>
</tr>
<tr>
<td>13</td>
<td>Strain hardening wedge indentation results predicted by using the mean value of the effective strain</td>
</tr>
<tr>
<td>14</td>
<td>Extrusion pressure of plane-strain inverted extrusion process with reduction R=0.5 for strain hardening materials</td>
</tr>
<tr>
<td>15</td>
<td>Inverted extrusion problem with reduction R=0.5 on rate dependent material (lead) without friction</td>
</tr>
<tr>
<td>16</td>
<td>Plane strain inverted extrusion with 50% reduction in area on strain hardening and rate dependent material</td>
</tr>
<tr>
<td>17</td>
<td>Wedge indentation on hardening and rate dependent material</td>
</tr>
<tr>
<td>18</td>
<td>Plane strain wedge indentation on soil with $\phi=20^\circ$</td>
</tr>
<tr>
<td>19</td>
<td>Plane strain wedge indentation on soil with $\phi=30^\circ$</td>
</tr>
<tr>
<td>20</td>
<td>False wedge solution for Coulomb material</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
</tr>
<tr>
<td>21</td>
<td>Pressure distribution and the pressure (p/c) on surface of perfectly smooth conical indenters on Coulomb material with $\phi=30^\circ$</td>
</tr>
<tr>
<td>22</td>
<td>Pressure distribution and the pressure (p/c) on surface of perfectly smooth conical indenters on Coulomb material with $\phi=20^\circ$</td>
</tr>
<tr>
<td>23</td>
<td>Dimensionless indentation pressure $F/(K_o C_w)$ for linearly hardening material</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure

1. The yield surface for the Von Mises and Tresca yield criteria in the principal stress space

2. The isotropic, kinematic hardening rule for Von Mises yield condition and a local hardening rule for Tresca yield condition

3. The yield surface for the extended Von Mises and the extended Tresca yield criteria in the principal stress space

4. Section of the yield surface by the π-plane \(\sigma_1 + \sigma_2 + \sigma_3 = 0\)

5. Load extension diagram for mild steel

6. Effects of strain rate on stress-strain curves for high purity aluminum (after Hauser, Simmons, and Dorm)

7. Effects of strain rate on stress-strain curves for mineral-oil saturated Solenhofen limestone at 20,000 psi confining pressure and room temperature (after Serdengecit and Boozer)

8. Compressive strength of dresser basalt as a function of strain rate and temperature (after Lindholm, Yeakley, and Nagy)

9. Dynamic loading surface and strain-rate vector

10. Dynamic stress-strain curve for work-hardening and strain-rate sensitive material (a) \(\dot{\varepsilon} = \text{const.}\) (b) \(\dot{\varepsilon} = \varepsilon (\varepsilon)\) (after Perzyna)

11. Dynamic stress-strain curve for elastic/(visco-perfectly plastic) material (a) \(\dot{\varepsilon} = \text{const.}\) (b) \(\dot{\varepsilon} = \varepsilon (\varepsilon)\) (after Perzyna)

12. Comparison of the experimental data with the prediction of (a) the power strain-rate law (b) the linear strain-rate law (after Perzyna)

13. Comparison of the experimental data with the prediction of (a) the exponential strain-rate law (b) the series power and the series strain-rate law (after Perzyna)

14. Mohr circle relation and \(S_\alpha, S_\beta\) lines

15. Unperturbed and perturbed slip-line fields for a rate independent Von Mises material (after Spencer)
Figure

16 Average effective-stress effective-strain curves of commercially pure lead in compression for various strain rates

17 Average effective-stress effective-strain curve of commercially pure lead in compression at an effective strain of 1.0 in/in.

18 Comparison of the experimental data with the prediction of the power strain-rate law for plane-strain and axial symmetric extrusion processes

19 Forms of empirical strain-hardening equations

20 The unit diagram

21 Inverted extrusion with 50% reduction in area

22 Grids on the undeformed billet

23 Indentation hodograph of a rigid smooth wedge

24 The motion of an element F' between O* and F on the undis- tributed surface

25 The motion of an element immediately below the apex of the wedge

26 The motion of an element in triangle region III

27 The approximate trajectory of an element in region II

28 Indentation hodograph of a rigid frictional wedge

29 The slip-line field in plastic indentation by a frictional wedge

30 Geometric similarity of the slip-line field during wedge indentation

31 The deformation pattern of the square grids after extrusion

32 Equilibrium relation between the indentation force F and the stress on the flank surface of the wedge

33 Slip-line field for false-wedge solution

34 Coulomb yield condition

35 The direction of failure lines
Figure

36 Geometric relation for smooth wedge indentation on soil
37 Mohr circle for the pressure on the flank of the smooth wedge for Coulomb material
38 Geometric relation for frictional wedge indentation on Coulomb material
39 Hodograph for frictional wedge indentation on soil
40 Mohr circle relation for the frictional wedge indentation on Coulomb material
41 Coulomb yield condition under Harr and Von Karman hypothesis
42 Integration procedure in triangle region I
43 Evaluation of the average value of the normal stress on the smooth cone surface
44 Plane-strain extrusion apparatus
45 Load cell and the deflection transducer
46 Set-up for simple compression test
47 The shear stress versus shear strain curves for mild steel, copper, and aluminum
48 The stress-strain curves for lead and aluminum
49 The stress-strain curves at constant strain rates for pure lead
50 Equilibrium relation for false wedge solution
51 Numerical results for the plane-strain wedge indentation solutions of perfectly plastic material (finite element method)
52 Numerical results for the plane-strain, wedge indentation solutions of perfectly plastic material (finite difference method)
53 False wedge solutions of perfectly plastic material
54 Pressure distribution for cone indentation for perfectly plastic material
55 Wedge indentation pressure for strain-hardening material
56 Conical indentation pressure for strain-hardening material
CHAPTER 1

INTRODUCTION

Since 1864, when Tresca published his results on punching and extrusion experiments and formulated his famous yield criterion, plasticity began to be treated as a science (77). A few years later, based on Tresca's results, Saint-Venant and Levy laid some foundations for modern plasticity. Although Von Mises, Hencky, Prandtl and others made important contributions for the next few decades, the progress of plasticity was slow.

Within the last two decades, the theory of plasticity has been rapidly developing, but much of the scope of its application has been confined to the area of the rigid perfectly plastic material.

The general interest in the dynamic properties of metals is stimulated not only by the complicated character of the dynamic problems, but the difference between the static and dynamic behavior of a metal are of such practical importance that the results of static tests can no longer be used for an appraisal of a dynamic phenomenon (131). Studies in this domain were initiated by Hohenemser and Prager (53) in 1932. This work was not fully appreciated for a long time until Sokolovsky (105) and Malvern (71) showed that the assumption of Hohenemser and Prager may be used as a basis for description of certain dynamic properties of rate sensitive materials for one-dimensional problems. The idea of Hohenemser and Prager is further developed by Perzyna. Perzyna (86, 87) has discussed the constitutive equations for general state of stress for rate sensitive perfectly plastic materials. However, the experimental results obtained by Campbell and Duby (17), Harding, Wood, and Campbell (46), and by
Marsh and Campbell (74), showed clearly that the dynamic tensile or compression stress-strain curves depend on the real change in the value of the rate of strain during impact phenomena. Inclusion of the effects of work-hardening removes an important restriction on the permissible range of load programs and on the application of the constitutive equations to different kinds of metals. Perzyna (83) then has generalized his previous results in describing the dynamic behavior of work-hardening plastic materials.

The stress characteristic equations of a strain-hardening material with Von Mises yield condition was first introduced by Christopherson (25) and Palmer (83) in the analysis of orthogonal cutting process (25, 38, 83). Based upon the photographic records from the experiments, the streamlines of the deforming material are determined. Two work-hardening hypotheses have been proposed (77). The first hypothesis assumes that the amount of hardening depends only on the total plastic work, and the second one depends on the strain history. In this study it is assumed that strain-hardening and work-hardening are interchangeable.

Thus, a conclusion can be drawn that because of strain hardening and rate effects, metals and other types of materials behave quite differently from that of the rigid perfectly plastic model during large plastic deformation and under certain range of deforming speeds.

The purpose of this study is to develop a method which includes both strain hardening and rate effects in the application of plasticity theory to solve technical problems.

To simplify the analysis, elastic strain and anisotropy of the material will be assumed negligible. The rates of deformation will be
confined to the ranges between creep and impact rates, and the temperature range is limited around room temperature throughout this study.

In Chapter 2 a brief review of basic plasticity theory, the related experimental results from tests on metals, soils, and rocks and the literature review of methods used for strain-hardening, rate effects are presented.

In Chapter 3 and Chapter 4 theoretical developments are presented for problems in plane-strain and under conditions of axial symmetry.

In Chapter 5 apparatus and experimental procedure are described and equations for strain, strain-rate calculations for simple compression test are presented.

In Chapter 6 solutions of some practical problems such as plane-strain inverted extrusion and wedge indentation are discussed.

In Chapter 7 the theoretical and experimental results are presented and comparisons between them are made.

The last chapter presents the summary and conclusions of this thesis.

The improved solutions of the demonstrated problems indicate that the theory and methods proposed by the author afford first order approximate solutions to plasticity problems under the influence of strain-hardening and rate effects.
CHAPTER 2
THEORETICAL BACKGROUND AND LITERATURE REVIEW

This section will contain a brief review of basic plasticity theory as well as the experimental results from tests on metals, soils and rocks. Anisotropy and thermal effects in plasticity are not discussed in this review since they are neglected in the materials studied in this work.

In this discussion, it is convenient to refer to the state of stress of a point in the body as a point in a stress space. A loading program will refer to a certain path in the stress space.

A yield criterion may be defined as a hypothesis concerning the limit of elastic deformation due to any possible stress state. By means of a yield function it is then possible to specify the elastic or plastic character of the material behavior. Any proposed yield function should be verified experimentally. The yield function \( f = f (\sigma_{ij}, \varepsilon_{ij}^P) \) may be expressed as a function of the stresses \( \sigma_{ij} \) and the plastic strains \( \varepsilon_{ij}^P \). The yield function is further restricted such that \( f \leq 0 \). When \( f < 0 \) the material behaves elastically, and when \( f = 0 \) the material may or may not deform plastically. Since \( \dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}^P} \dot{\varepsilon}_{ij}^P \), therefore the following conditions may prevail when \( f = 0 \), and when \( \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0 \) the material is going from a plastic state to an elastic one; this condition is called unloading. When \( \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0 \) the loading path is tangent to the surface defined by \( f = 0 \) and no additional plastic strains occur; this condition is called neutral loading. When \( \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0 \) the material is
going from one plastic state to another with accompanying plastic strains; this condition is called loading. The yield surface for a virgin material is called the initial yield surface, and the manner in which the surface changes shape or translates is called the hardening rule. The change of the yield surface during the deformation process is caused by isotropic and anisotropic work-hardening effects and by the influence of the strain rate effects. The yield surface must be at least piecewise continuous.

The yield function, \( f \), is further restricted by Drucker's postulate (28), which may be stated that when a stable, work-hardening body in equilibrium with a given set of loads has another set of self-equilibrium external loads slowly applied and removed, positive work must be done by the external loads during the application of loads and non-negative work must be done during the loading cycle. This statement implies that no useful net energy above any elastic energy may be extracted from a stable, work-hardening body and a system of stresses. An equivalent mathematical statement is:

\[
W_e = \int_{t_1}^{t_2} (\sigma_{ij} - \sigma_{ij}^*) \dot{\varepsilon}_{ij} P dt \geq 0
\]

where \( W_e \) is the work done by the external loads, \( \sigma_{ij} \) are the stresses caused by the external loads, \( \sigma_{ij}^* \) are the equilibrium stresses, and \( \dot{\varepsilon}_{ij} P \) are the plastic strain rates. A Taylor's series expansion of \( W_e \) about \( t = t_1 \) gives \( (\sigma_{ij} - \sigma_{ij}^*) \dot{\varepsilon}_{ij} P \geq 0 \) and if \( \sigma_{ij} = \sigma_{ij}^* \), the second term of the expansion also implies that \( \dot{\varepsilon}_{ij} P \geq 0 \). For a state of stress such that \( f = 0 \), the above equations imply

1. the yield surface is convex

and

2. \( \dot{\varepsilon}_{ij} P \) is normal to a smooth segment of the surface.
Since the gradient of $f$: \( \frac{\partial f}{\partial \sigma_{ij}} \) defines the normal to the yield surface, it follows that the plastic rates will be proportional to the gradient of the yield surface.

\[ \varepsilon_p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \]

This formalizes the concept of using the yield condition as a plastic potential. Thus, according to Drucker's postulate, the plastic strain rate must be derivable from the yield condition for a consistent plasticity theory.

### 2.1 Yield Functions for Ductile Metals

The most common yield functions are the Von Mises and Tresca yield conditions. The Von Mises yield condition suggested by Von Mises in 1913 (119) gives \( f(\sigma_{ij}) = J_2 - K^2 \) where \( J_2 \) is the second invariant of the deviator stress tensor, \( S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma \delta_{ij} \), and \( K \) is the yield stress in pure shear.

\[ J_2 = \frac{1}{2} S_{ij} S_{ij} = \frac{1}{6} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 \]

Hence the material does not yield and the deformation is elastic provided that \( J_2 \) is less than the characteristic value of the material, \( K^2 \).

Tresca (116) suggested, in 1864, that yielding occurs when the maximum value of the extremum shear stress in the material attains a critical value. It follows that the Tresca yield condition requires the maximum and minimum principal stresses to be known in advance. Then the yield condition is

\[ f = \sigma_{\text{max}} - \sigma_{\text{min}} - 2K \]
In the principal stress space the Von Mises yield condition describes a cylinder with the hydrostatic line \((\sigma_1 = \sigma_2 = \sigma_3)\) as its axis and with radius \(K^2\). The Tresca yield condition describes a regular hexagonal prism with the hydrostatic line as its axis (see Figure 1). For a perfectly plastic material, the parameter \(K\) is a constant. That means the material is independent of the stress and strain histories.

2.2 Hardening Rules

For an isotropic work-hardening material, the parameter \(K\) will vary in such a way that only the size of the yield surface will be changed in the stress space during plastic deformation.

Both yield functions stated above represent an isotropic work-hardening material, too, if \(K\) is allowed to change during plastic deformation. Although the isotropic hardening rule provides a reasonable approximation of some loading programs where no load reversals are present, it is still criticized because it does not explain the Bauschinger effect observed in most metals.

Prager (89) has proposed a kinematic hardening rule as a proper generalization of the Bauschinger effect observed in uniaxial tension and compression tests. In this case the yield surface is specified to translate as a rigid body in stress space. The expression for the yield function becomes \(f(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) = F(\sigma_{ij} - \alpha_{ij}) - K^2\) where \(\kappa\) is a work-hardening parameter and the \(\alpha_{ij}\) are components of a tensor that represents translation of the origin enclosed by the original yield surface. The \(\alpha_{ij}\) can be functions of either the stress or strain histories. After Shield and Ziegler (99), the \(\alpha_{ij}\) may be specified by the rule \(\alpha_{ij} = C \varepsilon_{ij}^p\) where \(C\) is a positive constant for linear work-hardening.
Since from normality

\[ \dot{\varepsilon}_{ij} = \frac{\dot{\sigma}_{ij}}{\dot{\sigma}} \]

and from the condition that during the plastic flow \( \dot{f} = 0 \)

\[ \dot{f} = \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial F}{\partial \dot{\sigma}_{ij}} \dot{\sigma}_{ij} \]

and since

\[ \frac{\partial F}{\partial \tau_{ij}} = - \frac{\partial F}{\partial \sigma_{ij}} \]

then it follows that

\[ \lambda = \frac{1}{C} \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \frac{\partial F}{\partial \tau_{kl}} \frac{\partial F}{\partial \sigma_{kl}} \]

Thus the \( \dot{\alpha}_{ij} \) can be found to within an arbitrary constant from the loading function, and the hardening rule is specified, if the \( \dot{\alpha}_{ij} \) are assumed equal to zero for no plastic strain.

It was Prager's intent for the yield surface to translate normal to the loading point of the yield surface. Shield and Ziegler (99) have shown that this is not always the case when some sub-space are chosen to represent the yield surface. Ziegler (126) has proposed a modification of Prager's rule in order to provide a consistent translation of the yield surface for any sub-space representations. Ziegler proposed that the translation tensor be defined by

\[ \dot{\alpha}_{ij} = \mu (\dot{\alpha}_{ij} - \alpha_{ij}) \]

By the same process as above

\[ \mu = \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} / (\sigma_{kl} - \alpha_{kl}) \frac{\partial F}{\partial \sigma_{kl}} \]
where \( \mu \) is some positive quantity that forces the translation tensor increment \( \alpha_{ij} \) to be directed along the line from the instantaneous center of the yield surface to the loading point; and the time derivative is used so that the units will be consistent. Thus the yield surface translates along the direction of the stress vector directed from the instantaneous origin of the translating yield surface. Ziegler has shown this to be invariant under reduction of space dimensions. The isotropic, kinematic hardening rule for Von Mises yield condition as well as a local hardening rule for Tresca yield condition are illustrated in Figure 2. The local hardening rule where one part of a yield surface deforms is the most general form of hardening since several parts of the yield surface could be made to deform such that isotropic or kinematic hardening rule is obtained (52). It should be noted that any combination of hardening rules is allowed. The validity of any hardening rule should only be judged by how accurately it explains the phenomena; however, the simplicity of application of the rule influences its usefulness.

2.3 Yield Function for Soils and Rocks

The yield point of metals is essentially independent of the hydrostatic stress state. However, rocks and soils show an increase in strength with increasing pressures. The three most commonly used yield conditions are the extended Von Mises, extended Tresca and the Mohr-Coulomb yield conditions.

Extended Von Mises:

\[
f = \alpha I_1 + J_2^{\frac{1}{2}} - K
\]
where \( \alpha \) and \( K \) are positive constants at each point of the material; \( I_1 \) is the first stress invariant:

\[ I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z = \sigma_i \]

\( J_2 \) is the second invariant of the stress deviation:

\[ J_2 = \frac{1}{2} S_{ij} S_{ij} \]

Here the tension stresses are considered positive.

Extended Tresca:

\[ f = \sigma_{\text{max}} - \sigma_{\text{min}} - \alpha I_1 - K \]

where \( \sigma_{\text{max}}, \sigma_{\text{min}} \) are principal stresses, \( \alpha > 0, K > 0 \)

Mohr-Coulomb:

\[ f = \sigma_{\text{max}} - \sigma_{\text{min}} - \alpha (\sigma_{\text{max}} + \sigma_{\text{min}}) - K \]

where \( \sigma_{\text{max}}, \sigma_{\text{min}} \) are principal stresses, \( \alpha > 0, K > 0 \).

In the extended Tresca and Mohr-Coulomb yield conditions, compressive stresses are considered positive.

Each of these yield conditions allow a linear increase in strength with increasing hydrostatic pressure, and can be reduced to the well known Coulomb yield condition in the case of plane strain (29, 33):

\[ \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + r_{xy}^2 \right]^{\frac{1}{2}} + \frac{\sigma_x + \sigma_y}{2} \sin \phi - C \cos \phi = 0 \]

where \( C \) is the cohesive strength and \( \phi \) is the angle of internal friction.

In this case \( \alpha \) and \( K \) would be different for each generalized yield condition. The extended Von Mises yield function, as viewed in three-dimensional principal stress space, is a cone with the hydrostatic
line as its axis. The extended Tresca yield function is a pyramid with a
regular hexagonal base and the space diagonal as its axis (see Figure 3).
The Coulomb yield function is a pyramid with an irregular hexagonal base
and the space diagonal as its axis. The intersection of the π-plane,
σ₁ + σ₂ + σ₃ = 0, with these yield surfaces are shown in Figure 4.

Bishop (10) has attempted to correlate all the criteria with
experimental data and has concluded that the Coulomb criterion best pre-
dicts soil behavior. Roscoe et al (95) contend that the available experi-
mental data (particular triaxial extension tests) are not sufficiently
reliable to allow one of the criteria to be favored over the others. They
thus recommend the extended Von Mises yield function because of its
simplicity. Since the above stated yield functions can be reduced to a
Coulomb type expression for the plane strain case, this implies that the
constants of the extended yield conditions can be adjusted such that all
three criteria will give identical limit loads. Therefore the extended
Von Mises yield condition will be utilized in this analytical study on the
basic model for soils and rocks in the remainder of this work.

Application of normality to each of the yield conditions pre-
dicts volumetric increases during plastic deformation for materials that
obey either yield condition. This is illustrated by using the extended
Von Mises yield condition.

According to the concept of plastic potential:

\[ \dot{\varepsilon}_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \]

where \( \dot{\varepsilon}_{ij}^p \) is the plastic strain rate and \( \lambda \) is a positive facotr of pro-
portionality which may assume different values for different particles.
Substitute the expression for \( f \) gives the following

\[
\varepsilon_{ij}^P = \lambda [\alpha \delta_{ij} + \frac{S_{ij}}{2J_2^{\frac{1}{2}}}] 
\]

then

\[
\varepsilon_{ii}^P = 3\alpha \lambda > 0, \text{ if } \alpha \neq 0
\]

which shows that plastic deformation must be accompanied by an increase in volume. This property is known as dilatancy.

Jenike and Shield (56) and Drucker (32) have discussed the inherent instability of a material that expands while deforming. Even though volumetric expansion has been noted by several experimenters (32), (30), (107), this type of deformation does not normally occur in tests on rocks and soils. This inconsistency between any generalized Coulomb yield condition and the strain behavior of rocks and soils have prompted workers to propose end caps on the cone or pyramids of the generalized Coulomb yield conditions so that application of Drucker's postulate predicts volumetric decreases during plastic deformation.

2.4 Experimental Results and Related Work

A. Metals. If a rod of ductile metal is pulled in a testing machine at room temperature and at a strain rate of approximately \( 2 \times 10^{-3} \) per second (59), the straight line portion of the load-extension diagram represents the region in which the law of linear elasticity is expected to hold. As the rod is further deformed beyond its yield point, the load increases with further strain. This kind of phenomenon, named as strain hardening, can be observed for most of the engineering materials in a variety of experiments; a typical load extension diagram for mild steel is shown in Figure 5.
Wedge and cone indentation experiments were carried out by Dugdale (34, 35) with specimens of mild steel, copper, and aluminium, using indenters with a variety of semi-angles. After making comparisons, Lockett (66) concluded that his calculated values of the average indentation pressure were somewhat lower than the measured values. He pointed out the qualitatively similar conclusions resulted from the comparisons of Dugdale's wedge indentation experimental data with the theoretical solution of Hill et al (50). These discrepancies between theory and experiment are not too surprising, since Dugdale has shown that the presence of work-hardening and some friction (the indenters were lubricated) would tend to increase the values of indentation pressure.

Experiments in plane-strain extrusion were carried out by Johnson (60) with specimens of lead and aluminium with dies of semi-angles 30°, 45°, 60°, 75°, and 90° at various reductions. Due to strain-hardening, he proposed to make a rational allowance for the change of the theoretically constant shear yield stress /K by determining a new value of 2K from the stress strain curve.

The test results of Taylor and Whiffin (109, 120), Duwez and Clark (37), Johnson, Wood, and Clark (57) and Campbell (16, 14) have shown that higher stresses are needed for metals to reach plastic state for a sudden than for a slowly acting load. Mild steel and pure iron, having a distinct yield point, behave in a different way during static and dynamic loading. The influence of the strain-rate on the yield limit for mild steel has been examined in detail by Taylor (108), Manjoine (73), Clark and Duwez (23), Hauser, Simmons, and Dorr (67), and Marsh and Campbell (74).
Hauser, Simmons, and Dorn (47) presented a method by which the plastic properties of materials can be investigated at strain-rate up to $1.5 \times 10^4$/sec through impulsive-loading techniques. Some representative stress-strain curves for wide range of strain-rates of high purity aluminium at $295^\circ$ K are shown in Figure 6.

Campbell (14, 18), Campbell and Duby (17), Harding, Wood, and Campbell (46), have shown that the upper yield limit for mild steel during a dynamic loading-process may reach a value of 2.5 to 3 times as high as the static value. These tests show clearly that the yield limit increases with increasing strain-rate. At the same time, the very important phenomenon of a definite reduction of the strain-hardening effect during the process of dynamic loading as compared to the strain-hardening effect observed during the static test is pointed out.

Dynamic stress-strain curves for aluminium were obtained by Johnson, Wood, and Clark (57) and Campbell (16) for mild steel by Campbell (18), and Campbell and Duby (17). The dynamic characteristics for a few metals were determined by Harding, Wood, and Campbell (46), and for pure copper, pure aluminium and an aluminium alloy by Kolsby and Douch (61).

The distribution of permanent strain along a specimen subjected to dynamic loading was investigated by Karman and Duwez (118), Campbell (18), and Kolsky and Douch (61). They found that, in general, the dynamic stress-strain curves lie above the static curves. Also, the amount of plastic strain caused by a given stress applied rapidly is less than that caused by the stress applied statically.

Some materials, when suddenly loaded above their yield point, require a finite, though small, amount of time for yielding to begin. This
time is known as the delay time of yielding. Johnson, Wood, and Clark (58) showed that the delay time in mild steel is a function of the initial impact stress. The logarithm of the delay time was found to decrease linearly with an increase in applied stress, above a value of 41,000 psi.

Campbell and Maiden (15) studied the influence of previous dynamic loadings on the subsequent static properties of a material. The upper static yield stress can be considerably reduced by the application of an impact stress of sufficient magnitude and duration to cause appreciable permanent deformation.

Sokolov (103, 104), Manjoine (73), Alder and Phillips (2), Krafft, Sullivan, and Tipper (62), MacDonald, Carlson, and Lankford (69), Maiden and Campbell (70), and Chidester and Malvern (22) have studied the influence of temperature on the dynamic behavior of metals. It was observed that if the temperature is decreased the rate-sensitivity of the material increases. It has been demonstrated experimentally (91) that at -180°C the low yield point of pure iron is equal for all strain rates. That means the influence of strain rate on the mechanical properties of metals depends markedly on the value of absolute temperature. Krafft, Sullivan, and Tipper (62) have shown that a slight change of the strain rate at high temperatures cause a considerable rise or drop in the yield point.

B. Rocks and Soils. Recent experimental data show clearly that soils exhibit rheological effects and are sensitive to the change of the strain rate. Thus, the characteristic feature of the behavior of soils, especially in their dynamic response, is the time-dependence of the deformation process. Chadwick, Cox, and Hopkins (19) have shown that
as the rate of strain increases from zero to about 1.5/sec, the ultimate strengths of clays and sands increase by factors of about 2 and 1.2, respectively. They also note that, in clays, the resistance to the flow of pore water is probably one reason for the existence of rate effects. In real soils the inelastic strain-rate tensor depends on the time-history as well as on the path-history of the stresses.

Boozer, Hiller, and Serdengecti (13) varied strain rates from 0.06 to 13 percent per second in studying the effects of fluid composition on rock properties. A wider range of strain rates, from $4 \times 10^{-3}$ to 80 percent per second, was achieved by Serdengecti and Boozer (98) in their study of the effects of strain rate, temperature and confining pressure on rock properties. Some typical results for Solenhofen limestone are shown in Figure 7. Their results indicate that the yield points and ultimate strengths of the rocks they tested increased with increasing strain rate and decreased with increasing temperature. This similarity of the effects of temperature and strain rate is like that found for metals by Zener and Holomon (125). Serdengecti and Boozer developed an equation of state for the rock samples based on the equivalence parameter of Zener and Holomon. The equation they developed was

$$\sigma_u = \frac{\Delta p}{K} p_o \left(\varepsilon \varepsilon_0 e^{Q/RT}\right)$$

where

- $\sigma_u$ = ultimate strength
- $\varepsilon$ = strain rate
- $Q$ = heat of activation
- $R$ = universal gas constant
- $T$ = absolute temperature
\[ r = r_o + \Delta p / K_2 p_o \]

\[ \Delta p = p - p_o \]

\( p \) = actual confining pressure

\( p_o \) = reference pressure

\( r_o \) = value of \( r \) at \( p_o \)

\( K_2, K_1, K_2 \) = material properties

Pavlova and Shriner (85) used a pneumatic drop hammer at velocities ranging from static to about 160 ft/sec to strike a 10 mm circular die to study the effects of rate of loading on the plastic deformation of marble. They observed that the plastic zone decreased in size with increased velocity of impact. Slow and rapid loading tests were compared by Maurer (75) for blunt wedges forced into Indiana limestone which was under confining pressure. Craters created by static loading were formed in 0.1 to 5 seconds while craters created by impact loading were formed in 0.001 to 0.005 seconds. Maurer found that under conditions of simulated mud drilling, the transition from brittle to pseudoplastic crater formation occurred at a higher differential pressure on the material for the impact tests than for the static tests. Also, the force versus depth of penetration curves at a given differential pressure for the impact tests lie above those for the static tests. For simulated water drilling conditions, the hydrostatic pressure on the sample had no significant effect on the force required to create craters in the static loading tests. However, in the impact tests, the required force increased as the hydrostatic pressure increased. Under simulated air drilling conditions there was no significant difference between the results from the static and the impact loading tests. In all three situations investigated, the impact test crater was smaller than the crater formed by the
static loading under the same conditions.

Lindholm, Yeakley, and Nagy (64) varied the temperature, confining pressure and strain rate in tests they performed to determine the fracture strength of Dresser basalt. Using hydraulic loading facilities at low strain rates and a Hopkinson pressure bar apparatus at high strain rates, they performed uniaxial compression or extension tests at rates from 10^{-4}/sec to 10^{3}/sec, radial confining pressures from 0 to 100,000 psi and temperatures from 80 to 800\(^{0}\)K. From the data, they deduced a fracture criterion of the form

\[
\frac{\sigma_1}{S_C(0)} + \frac{S_C(0) - S_{BC}(0)}{S_C(0) S_{BC}(0)} \sigma_2 - \frac{\sigma_3}{S_T(0)} = 1 - \beta T (A - \log \dot{\varepsilon}_a)
\]

where \(\sigma_1, \sigma_2, \text{ and } \sigma_3\) are the principal stresses, T is the test temperature and \(\dot{\varepsilon}_a\) is the strain rate. The constants \(S_C(0)\) and \(S_{BC}(0)\) are strength values at zero absolute temperature and B and A are constants involving the activation energy, volume and frequency. Lindholm found that there was a minimum fracture stress, a maximum fracture stress and a range of strain rates over which the fracture stress depends on the strain rate (see Figure 8).

Drucker, Gibson, and Henkel (30) have put a hemisphere end cap to an extended Von Mises cone in order to explain compaction of wet clays at certain stress states. Their work-hardening theory proposed that the end cap grew in diameter as the material yielded at stress states on the end cap. The diameter of the end cap would grow such that it always remained in contact with the yield cone. By Drucker, et al's (30) proposal in an axial compression test in a triaxial apparatus where a sample is
subjected to increasing axial loads at a constant confining pressure and
where the loading path first touches the end cap portion of the yield
surface, the following type of deformation would occur:

1. As the cap increased in size the sample volume would
decrease in smaller and smaller increments in propor-
tion to the axial strain as the axial load increased.

2. Eventually the material would deform incompressibly
   for an instant until,

3. Expansion would take place as the loading path touched
   the yield cone and the material became perfectly plastic.

Even though this work presented a detailed qualitative description of
the growth of a yield surface, it is not complete because no quantita-
tive hardening rule was proposed.

Roscoe and Burland (96) using results of triaxial tests per-
formed by Roscoe and Schofield (94) proposed a yield surface similar to
the surface proposed by Drucker, et al (30). The end cap is limited in
size by a Coulomb envelope and grows as a function of the volumetric
strain such that subsequent yield surfaces are surfaces of uniform volu-
metric strain. Three different hardening rules are proposed, utilizing
different approximations for the amount of dissipative work done on the
material as it deforms plastically. The closer the approximations ap-
proach the exact expression \( W = \int \sigma_{ij} \varepsilon_{ij} \) the better their hardening
rules predict the experimental results. It appears that the approxima-
tions are chosen more for convenience than for accuracy and that using
the exact expression for the plastic work would give just as satisfac-
tory result for the hardening rule.
Borden and Khayatt (8) have concluded from triaxial tests on sands that sand does workharden. They have derived a plastic potential function to describe the incremental stress-strain relations for the sand. However, it is claimed that the concept in plasticity of coincidence of the plastic potential and yield surface is not applicable for the material tested. Concerning the concept of failure, Borden and Khayatt discuss the merits of using certain failure envelopes that coincide with the extended Von Mises and Tresca and Mohr-Coulomb yield surfaces. They conclude that the Mohr-Coulomb criterion is the most applicable failure surface for sands as well as clays.

Swanson (107) has reported results of triaxial tests on hard, graniticlike rocks, generally considered to be brittle materials, and has found them to be ductile at high confining pressures and has proposed a yield condition and work-hardening theory to explain their stress-strain behavior. Swanson's yield condition is a Mohr type yield condition that decays exponentially to a Von Mises yield condition at high hydrostatic pressures; his failure surface, the surface of stress state at brittle failure of the material, is given by the same type of surface as the initial yield surface. Swanson's yield condition predicts volume increases during plastic flow and he proposes a non-associated flow rule to account for this behavior.

Cheatham (21) has reported results of experiments designed to probe the yield surface of a porous limestone. The results definitely indicate the existence of an end cap, and of material work-hardening. Cheatham and Miller (79) have performed triaxial tests on two types of porous limestones in order to find detailed information about the stress-
strain behavior of these materials. Results of these tests show that a yield condition that combines a Coulomb yield condition with an end cap provides an adequate description of the material's stress-strain behavior. The end cap is allowed to grow in a definite manner and is limited in size by Coulomb condition. The yield condition can be expressed as

\[ f_1 = \sigma_{\text{max}} - \sigma_{\text{min}} - A \sigma_{\text{min}} - B \]
\[ f_2 = \sigma_{\text{max}} - \sigma_{\text{min}} + a \sigma_{\text{min}} - \beta \]

where

\[ A \sigma_{\text{min}} + B \leq \beta - a \sigma_{\text{min}} \rightarrow f_1 \text{ is the governing yield function} \]
\[ A \sigma_{\text{min}} + B \geq \beta - a \sigma_{\text{min}} \rightarrow f_2 \text{ is the governing yield function} \]

and where \( A \) and \( B \) are material constants and \( a \) and \( \beta \) are work-hardening parameters associated with the material. It was found that the growth of the yield surface depended upon the amount of volumetric strain or the amount of plastic work per unit volume done on the material. The consistency between the predicted and observed stress-strain behavior indicates that this type yield condition is the proper one needed for some rocks. And this yield condition can be generalized to apply to any material that shows a volume change during permanent deformation processes.

2.5 Rate Dependent Deformation Theory

The general foundations for the study of viscoplastic problems were laid by Hohenemser and Prager (53) in 1932. This work was not fully appreciated until Sokolovsky (105) and Malvern (71) showed that the assumption of Hohenemser and Prager may be used as a basis for description of rate-sensitive materials, though this concern is for one-dimensional problems only.
Owing to the assumption that viscous properties of the materials become manifest only after the passage to the plastic state, so it is assumed that the strain rate can be resolved into an elastic and inelastic part
\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \]
The inelastic part of the strain rate, denoted by \( \dot{\varepsilon}_{ij}^p \), represents combined viscous and plastic effects.

The initial yield condition will not differ from the known condition of the inviscid theory of plasticity, and a generalized static yield condition is expressed in the form
\[ F(\sigma_{ij}, \varepsilon_{kl}^p) = \frac{f(\sigma_{ij}, \varepsilon_{kl}^p)}{\kappa} - 1 \]
where the function \( f(\sigma_{ij}, \varepsilon_{kl}^p) \) depends on the state of stress \( \sigma_{ij} \) and the state of plastic strain \( \varepsilon_{kl}^p \). The parameter \( \kappa \) is defined by the expression
\[ \kappa = \kappa(N_p) = \kappa \left( \int_0^{\varepsilon_{ij}^p} \sigma_{ij} \, d \varepsilon_{ij}^p \right) \]
This quantity is called the strain-hardening parameter.

The flow surface \( F = 0 \), in the nine-dimensional stress space, is assumed regular and convex. The constitutive equations for work-hardening and rate sensitive plastic materials are expressed in the following form (38):
\[ \dot{\varepsilon}_{ij} = \frac{1}{2\mu} S_{ij} + \frac{1-2\nu}{E} S \delta_{ij} + \gamma^o\left(\frac{\partial F}{\partial \sigma_{ij}}\right)^\circ \]
where the symbol $[\Phi(F)]$ is defined as follows:

$$[\Phi(F)] = \begin{cases} 
0 & \text{for } F \leq 0 \\
\Phi(F) & \text{for } F > 0 
\end{cases}$$

The function $\Phi(F)$ may be chosen to represent the results of tests on the behavior of metals under dynamic loading. The constitutive equations can be rewritten in the slightly different form

$$\dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{S}_{ij} + \frac{1-2\nu}{E} S \delta_{ij} + \gamma \left[\Phi(F)\right] \frac{\partial f}{\partial \sigma_{ij}}$$

where $\gamma = \gamma^0/K$ denotes a viscosity constant of the material.

The above relations involves the assumption that the rate of increase of the inelastic components of the strain tensor is a function of the excess stresses above the static yield condition. The function of stresses above the static yield criterion generates the inelastic strain-rate according to a viscosity law of the Maxwell type. The elastic components of the strain tensor are considered to be independent of the strain-rate. The above constitutive equations describe the work-hardening effect of the material. The introduction of the function $F$ makes the study of the anisotropic and isotropic work-hardening possible for a broad class of compressible materials.

Denote the inelastic part of the strain rate by $\varepsilon_{ij}^P$ and

$$I_2 = \frac{1}{2} \varepsilon_{ij}^P \varepsilon_{ij}^P$$

then

$$\varepsilon_{ij}^P = \gamma \Phi(F) \frac{\partial f}{\partial \sigma_{ij}}; \quad (I_2^P)^{1/2} = \gamma \Phi(F) \left(\frac{1}{2} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{kl}} \right)^{1/2}$$
Since

\[ F(\sigma_{ij}, \varepsilon_k^p) = \frac{f(\sigma_{ij}, \varepsilon_k^p)}{\kappa} - 1 \]

so

\[ f(\sigma_{ij}, \varepsilon_k^p) = \kappa (V_p) \left\{ 1 + \phi^{-1} \left( \frac{I_2^p}{\gamma} \left( \frac{1}{2} \frac{\partial f}{\partial \sigma_{pq}} \cdot \frac{\partial f}{\partial \sigma_{pq}} \right)^{-\frac{1}{2}} \right) \right\} \]

This expression implicitly represents the dynamic yield condition for elastic/viscoplastic, work-hardening materials, and describes the dependence of the yield condition on the strain rate. Since the inelastic strain rate tensor is proportional to the gradient of the yield function \( f(\sigma_{ij}, \varepsilon_k^p) \), therefore in the nine-dimensional stress space it is always directed along the normal to the subsequent dynamic loading surface (see Figure 9). The change of the yield surface during the deformation process is caused by isotropic and anisotropic work-hardening effects, and by the influence of the strain-rate effect.

2.6 Special Cases of Constitutive Equations

Assume the function \( F \) of the elastic/viscoplastic material with isotropic work-hardening can be expressed as

\[ F = \frac{f(\sigma_{ij})}{\kappa} - 1 \]

where the function \( f(\sigma_{ij}) \) depends now on the state of stress only.

Thus \( f(\sigma_{ij}) = f(I_1, J_2, J_3) \)

where \( I_1 \) denotes the first invariant of the stress tensor \( \sigma_{ij} \), \( J_2 \) and \( J_3 \) and the second and third invariants of the stress deviator \( S_{ij} \) respectively.
Then
\[ \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1} \delta_{ij} + \frac{\partial f}{\partial J_2} S_{ij} + \frac{\partial f}{\partial J_3} t_{ij} \]

where
\[ t_{ij} = S_{ik} S_{kj} - \frac{2}{3} J_2 \delta_{ij} \]
\[ \dot{\epsilon}^p_{ij} \text{ will then take the form of} \]
\[ \dot{\epsilon}^p_{ij} = A(I_1, J_2, J_3, \kappa) \delta_{ij} + B(I_1, J_2, J_3) S_{ij} + C(I_1, J_2, J_3) t_{ij} \]

where
\[ A(I_1, J_2, J_3, \kappa) = \gamma \Phi \left[ \frac{\varepsilon}{\kappa} - 1 \right] \frac{\partial f}{I_1} \]
\[ B(I_1, J_2, J_3, \kappa) = \gamma \Phi \left[ \frac{\varepsilon}{\kappa} - 1 \right] \frac{\partial f}{J_2} \]
\[ C(I_1, J_2, J_3, \kappa) = \gamma \Phi \left[ \frac{\varepsilon}{\kappa} - 1 \right] \frac{\partial f}{J_3} \]

A. Application to Metals. Assume metals are incompressible, i.e., \( A(I_1, J_2, J_3, \kappa) = 0 \) or \( \partial f / \partial I_1 = 0 \), the general constitutive equations for strain-rate sensitive metals may be written as
\[ \dot{\epsilon}^p_{ij} = B^*(J_2, J_3, \kappa) S_{ij} + C^*(J_2, J_3, \kappa) t_{ij} \]

Assume then the function \( f(\sigma_{ij}) = (J_2)^{\lambda} \), the strain rate tensor will take the form of
\[ \dot{\epsilon}_{ij} = \frac{1}{2\mu} \dot{\sigma}_{ij} + \gamma \left( \sqrt{\frac{J_2}{\kappa}} - 1 \right) \frac{S_{ij}}{(J_2)^{\lambda/2}} \]
\[ \dot{\epsilon}_{ii} = \frac{1}{3K} \dot{\sigma}_{ii} \]
Then, the dynamical yield condition becomes

\[ \sqrt{J_2} = \kappa(W_p) \left[ 1 + \phi^{-1} \left( \frac{\sqrt{J_2 P}}{\gamma} \right) \right] \]

For one-dimensional states, the relation for strain rate is

\[ \dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \gamma^* \left\{ \phi \left[ \frac{\sigma}{\phi(\varepsilon P)} - 1 \right] \right\} \]

where

\[ \gamma^* = \frac{2}{\sqrt{3}} \gamma, \quad \phi(\varepsilon P) = \sqrt{3} \kappa(W_p) \]

This relation was first introduced by Malvern (71).

The dynamical yield condition then has the form of

\[ \sigma = \phi(\varepsilon P) \left[ 1 + \phi^{-1} \left( \frac{\dot{\varepsilon} P}{\gamma^*} \right) \right] \]

where \( \sigma = \phi(\varepsilon P) \) represents the static stress-strain relation for simple tension. The results of the dynamic stress-strain relation in the plastic range for simple tension for \( \dot{\varepsilon} = \) constant and for the realistic case \( \dot{\varepsilon} = \dot{\varepsilon}(\varepsilon) \) are plotted in Figure 10.

For the elastic/visco-perfectly plastic material, the function \( F \) does not depend on the strain, i.e.

\[ F = \frac{f(J_2, J_3) - 1}{c}, \quad c = \text{constant} \]

The constitutive equations then become

\[ \dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{\varepsilon}_{ij} + \gamma \left\{ \phi \left[ \frac{f(J_2, J_3)}{c} - 1 \right] \right\} \frac{\partial f}{\partial \sigma_{ij}} \cdot \dot{\sigma}_{ij} \]

\[ \dot{\varepsilon}_{ii} = \frac{1}{3K} \dot{\sigma}_{ii} \]

where \( \gamma = \gamma^0/C \), which was first introduced by Perzyna (82).
Then the function \( f \) can be written as

\[
f(J_2, J_3) = C \left\{ 1 + \phi^{-1} \left[ \frac{\sqrt{J_2}}{\gamma} \left( \frac{1}{2} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{kl}} \right)^{-\frac{1}{2}} \right] \right\}
\]

Assume

\[
F = \frac{\sqrt{J_2}}{\kappa} - 1
\]

where \( \kappa \) is the yield stress in simple shear, then the constitutive equations become

\[
\dot{\varepsilon}_{ij} = \frac{1}{2\mu} S_{ij} + \gamma \left[ \frac{\sqrt{J_2}}{\kappa} - 1 \right] S_{ij} , \quad \dot{\varepsilon}_{ii} = \frac{1}{3\kappa} \sigma_{ii}
\]

and the dynamical criterion is

\[
\sqrt{J_2} = \kappa \left[ 1 + \phi^{-1} \left( \frac{\sqrt{J_2}}{\gamma} \right) \right]
\]

For one-dimensional states, the relation for strain rate is

\[
\dot{\varepsilon} = \frac{\sigma}{E} + \gamma \left[ \frac{\sigma}{\sigma^*} - 1 \right]
\]

and the dynamical criterion is

\[
\sigma = \sigma^* \left[ 1 + \phi^{-1} \left( \frac{\dot{\varepsilon}^*}{\gamma^*} \right) \right]
\]

The results of the dynamic stress-strain curve for elastic/visco-perfectly plastic material with \( \dot{\varepsilon} = \) constant and \( \dot{\varepsilon} = \dot{\varepsilon}(\varepsilon) \) are plotted in Figure II.

**B. Application to Soils.** Assume the static yield function for the elastic-visco-perfectly plastic soil takes the form of

\[
F = \frac{\alpha T_1 + J_2^{\frac{1}{2}}}{\kappa} - 1
\]

where \( \alpha \) is a constant describing the dilatation rate of the soil. Then
the constitutive equations give

\[ \dot{\varepsilon}_{ij} = \frac{1}{2\mu} \dot{\sigma}_{ij} + \frac{1-2\nu}{E} \delta_{ij} \dot{\sigma} + \gamma \left\{ \phi \left[ \frac{aI_1 + J_2^{1/2}}{\kappa} - 1 \right] (\dot{\sigma}_{ij} + \delta_{ij}) \right\} \]

the rate of cubical dilation takes the following form

\[ \dot{\varepsilon}_{II} = \frac{1-2\nu}{E} + 3\alpha \gamma \left\{ \phi \left[ \frac{aI_1 + J_2^{1/2}}{\kappa} - 1 \right] \right\} \]

If \( \alpha \neq 0 \), then the inelastic deformation is accompanied by an increase of volume, this property is known as dilatancy.

The dynamic yield condition has the form of

\[ a I_1 + J_2^{1/2} = \kappa \left\{ 1 + \phi^{-1} \left[ \frac{(I_2 \Gamma)^{1/2}}{\gamma} \left( \frac{3}{2} a^2 + \frac{1}{4} \right) \right] \right\} \]

C. Comparison with Experimental Data. Perzyna (36, 37) has introduced the following special types of function

\[ \phi(F) \]
\[ \phi(F) = F^\delta, \quad \phi(F) = F, \quad \phi(F) = \exp F - 1 \]

\[ \phi(F) = \sum_{\alpha = 1}^{N} A_{\alpha} [\exp F^\alpha - 1], \quad \phi(F) = \sum_{\alpha = 1}^{N} B_{\alpha} F^\alpha \]

When the elastic deformation are negligible in comparison with the plastic deformations, the combination of \( \phi(F) = F^\delta \) and

\[ \sigma = \sigma_0 \left[ 1 + \phi^{-1} \left( \frac{\dot{\varepsilon}_p}{\gamma} \right) \right] \] gives

\[ \dot{\varepsilon} = \gamma^* \left( \frac{\sigma}{\sigma_0} - 1 \right)^\delta \]

This relation is equivalent to the Cowper-Symonds-Border strain-rate law (11, 24).

Substitute the above proposed functions \( \phi(F) \) into the dynamic stress-strain relation \( \sigma = \sigma_0 \left[ 1 + \phi^{-1} \left( \frac{\dot{\varepsilon}_p}{\gamma} \right) \right] \) will give the power, linear,
exponential and series power strain-rate laws as follows

\[ \sigma = \sigma_o \left[ 1 + \left( \frac{\dot{\varepsilon}}{\gamma^*} \right)^{1/\delta} \right], \quad \sigma = \sigma_o \left( 1 + \frac{\varepsilon^p}{\gamma^*} \right) \]

\[ \sigma = \sigma_o \left[ 1 + \log \left( 1 + \frac{\varepsilon^p}{\gamma^*} \right) \right], \quad \dot{\varepsilon} = \sum_{a=1}^{5} A_a \left[ \frac{\sigma}{\sigma_o} \right]^\alpha \exp\left( \frac{\sigma}{\sigma_o} - 1 \right)^\alpha - 1 \]

\[ \dot{\varepsilon} = \sum_{a=1}^{5} B_a \left( \frac{\sigma}{\sigma_o} - 1 \right)^\alpha \]

and the comparison of the experimental data of Clark and Duwez (37) with the prediction of the above stated strain-rate laws are shown in Figures 12 and 13.

Since all experimental investigations concerning the dynamic behavior of the materials have been performed under one-dimensional conditions, Perzyna (37) has concluded that the constants determined from one-dimensional states of stress may not be used as the same constants in the general constitutive equations. This kind of difficulty can be overcome by relating the constants of one-dimensional experiments to those associated with the equation of the effective stress, which will be explained in the remainder of this study. It is also verified in the light of the experimental results that the influence of the elastic part of the strain rate is very small.

After making comparisons, it can be seen from Figures 12, 13 that the power strain-rate law affords pretty good prediction for the experimental data but with a simple form. Therefore, this kind of power strain-rate law will be adopted in this investigation.

2.7 Solutions of Strain-Hardening Problems

The stress characteristic equations of a strain-hardening material with Von Mises yield condition was first introduced by
Christopherson (25) and Palmer (83) in the analysis of orthogonal cutting process (38, 78, 63). Based upon the photographic records from the experiments, the streamlines of the deforming material are determined. The complete slip-line field could be deduced by making use of the condition that there is no change in velocity along a slip-line. Assuming that work-hardening alone contributed to the variation in yield stress, the equivalent plastic strain increment was integrated along the streamline loading to the point in question to give the equivalent plastic strain. The stress-strain curve was then used to determine the yield stress, which was used in the stress characteristics equation to obtain the tool force.

Avitzur (6, 7) obtained an upper bound solution for flow of strain-hardening material with Von Mises yield condition through conical converging die. The upper bound theorem (31, 90, 102) was modified by using the equivalent plastic work (72, 77). The equivalent strain was obtained from the assumed kinematically admissible velocity field. A linear strain-hardening curve was then used to determine the equivalent stress in the deforming region.

Mendelson (77) presented a general method for solving elasto-plastic problems, where the plastic strain was of the same order of magnitude as the elastic strain. The equivalent stress and equivalent plastic strain were used for Von Mises material to eliminate $d\lambda$ in the flow rate. The loading path was divided into a number of increments. For each increment of load, the plastic strain increments were obtained by a successive elastic approximation.
Farmer and Oxley (40) have presented an experimental method of obtaining flow fields for plane-strain extrusion using printed grids (0.002 in. square). And then a slip-line field is constructed by calculating the directions of the maximum strain-rate from the measured velocity gradients. It is shown that account must be taken in the stress equilibrium equations of variations in flow stress caused by strain-hardening in order to satisfy internal stress consistency and equilibrium of forces.

Duvov and Ivlev (36, 35) applied the perturbation method on the imbedding of a thin rigid body in a plastic medium with hardening. It is assumed that a translation hardening with yield condition in the form

\[
[(\sigma_x - C \varepsilon_x) - (\sigma_y - C \varepsilon_y)]^2 + 4 (\tau_{xy} - C \varepsilon_{xy})^2 = 4K^2
\]

where \(C\) and \(K\) are constants.

In the analysis they perturbed the perfectly plastic solution as a zero'th order solution to obtain a first order solution for the strain-hardening material.

The finite element method has been used to solve the elasto-plastic problems, and a general formulation of problems in matrix form and several iterative processes for solving elastoplastic problems were discussed by Nayak (81) and Zienkiewicz (127, 128, 129).

Turner, Clough, Martin, and Topp (117) proposed a finite element model for two-dimensional problems, with triangular elements and linear displacement field. The method has been extended to include plastic behavior.
Rij and Hodge (97) have proposed a slip model for finite element plasticity. In this method a regular pattern of triangular elements is assumed. The elements are assumed to be rigid and across the line separating any two adjoining elements a discontinuity may exist in the tangential component. Compatibility, equilibrium, and constitutive equations are developed with the aid of the Principle of Virtual Work. Prandtl's punch problem for contained flow is solved under plane strain conditions.

2.3 Solutions of Rate Dependent Problems

Spencer (106) considered the effects of local acceleration terms on the problem of a flat punch indenting a half-space. His results indicate that the pressure under the punch and the slip line fields are both different from their counterparts in the classical, rate independent solution.

Wierzbicki (121, 122) studied the problem of a thick-walled spherical container of an elastic/viscoplastic material subjected to either a sudden increase in internal pressure or a sudden constant displacement of the inner surface. He found a closed solution to the problem with the pressure boundary condition. The viscoplastic flow of a circular plate that is simply supported along its edge and subjected to a uniformly distributed transverse load was studied by Appleby and Prager (5). Their solution was based upon a material obeying the Tresca yield condition. A similar problem based on the Huber-Mises yield condition was examined by Wierzbicki (123).
A discussion of the strain rate effect on impulsive loading was given by Bonder and Symonds (11, 12) and Ting and Symonds (114). They found if the ratio of the kinetic energy input to the maximum possible elastic energy is greater than 3, then the rigid-perfectly plastic theory of beams gives a reasonably good first order theory. By considering a rate dependent yield condition of the form

\[ \dot{\varepsilon} = \dot{\gamma} \left( \frac{\sigma}{\sigma_0} - 1 \right)^\delta \]

where \( \dot{\gamma} \) = viscosity
\( \sigma_0 \) = initial yield stress
\( \delta \) = material property

they were able to obtain satisfactory agreement between theory and experimental results.

Chandra and Jonas (20) have shown that the force required to extrude a strain rate sensitive material under conditions of homogeneous deformation can be calculated by the uniform work method. A new definition of the mean strain rate during extrusion is proposed, which is based on one-dimensional analysis. For extrusion ratios of 40 to 160 and a power law strain rate sensitivity of 0.2, this method leads to extrusion pressures 30 to 60 pct higher than when no account is taken of the rate sensitivity of the flow stress in the die zone.

Ragab and Duncan (92) have studied the effects of rate sensitivity on the back extrusion of time-dependent alloys. Plane strain slip line field solutions are employed to determine average strain and pressure near the corner of the punch. A simple geometric factor is then derived from the slip line field and then is used to convert the
solution for perfectly plastic material to one for a rate-sensitive material. In this analysis the derivation of the mean effective strain rate is quite rough, and these solutions are frequently based on an experimentally determined velocity field, so the real cost of numerical solutions can be high. The theoretical pressures are in many cases slightly above the experimental ones although the process is assumed to be frictionless.

Tomita and Sowerby (115) have presented a numerical method for analyzing the plane strain deformation of rate sensitive materials. A rate of energy functional is introduced which is thought to take adequate account of the strain rate sensitivity of the material. In the numerical technique the functional is minimized with respect to a kinematically admissible velocity field and used in a discretized form in a finite element analysis. The frictionless, plane-strain, side extrusion process was simulated and the theoretically predicted forming pressure showed reasonably good agreement with the experimental values.

2.9 Solutions of Strain-Hardening and Rate Dependent Problems

Norman Jones (92) has presented a simple method for estimating the combined influence of strain-hardening and strain-rate sensitivity on the permanent deformation of rigid-plastic structures loaded dynamically. A study is made of the particular case of a beam supported at the ends by immovable frictionless pins and loaded with a uniform impulse. The results of his work indicates that considering strain-hardening alone when appropriate or strain-rate sensitivity along gives permanent deformations which are similar to those predicted by an analysis retained both effects simultaneously.
Chosh (47) has proposed a simplified constitutive description of the true stress, strain, and strain rate in the form of

$$\sigma = K \varepsilon^n \dot{\varepsilon}^m$$

where

- $\sigma$ = the true stress
- $n$ = strain hardening exponent
- $m$ = strain-rate hardening exponent
- $K$ = material constant

in his investigation of sheet metal stamping process. It has been shown that $m$ increases with strain at low-strain rates and decreases with strain at high strain rates for Cu-Aleutectics at 450°C. A simultaneous drop in $m$ and rise in $n$ are observed with increasing deformation temperature in many metals (76, 93), and a change in $m$ as a function of strain is a common feature in such cases. It has been shown that the terminal value of $m$ for such materials must be considered in order to relate to ductility. An overall constitutive description: $\sigma(\varepsilon, \dot{\varepsilon}, T)$ is suggested to be more desirable in that case than isolated measurement of $n$ and $m$.

In this section a brief review of the development of strain-hardening and rate dependent plasticity theory was carried out. It will be the purpose of the next section to propose a new method to solve problems of unrestricted plastic flow, such as wedge indentation and extrusion problems.
CHAPTER 3

THEORETICAL DEVELOPMENT

In this chapter the balance equations of continuum mechanics and the basic concepts will be presented first. Then the formulation of a two dimensional, plane strain, isotropic strain hardening and rate sensitive plasticity theory will be derived.

Any physically realistic problem must satisfy the balance equations of continuum mechanics. The components form of these equations can be written as follows:

Balance of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \mathbf{v}_i)}{\partial x_i} = 0 \quad (3-1)$$

Balance of linear momentum:

$$\frac{\partial \rho \mathbf{a}_i}{\partial x_j} + \rho \mathbf{b}_i = \rho \mathbf{a}_i \quad (3-2)$$

Balance of internal energy:

$$\sigma_{ij} \frac{\partial \mathbf{v}_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho r = \rho \dot{\varepsilon} \quad (3-3)$$

Entropy inequality:

$$- \rho (\dot{\psi} + \dot{\eta}) + \sigma_{ij} \frac{\partial \mathbf{v}_i}{\partial x_j} - \frac{1}{\delta} q_i \frac{\partial \delta}{\partial x_i} \geq 0 \quad (3-4)$$

Where

- \(\rho\) = density
- \(\mathbf{v}_i\) = the velocity in the \(i^{th}\) coordinate direction
- \(t\) = time
- \(\sigma_{ij}\) = the \(i, j\) component of the stress tensor
- \(X_i\) = spatial coordinates
\[ b_i = \text{body forces per unit mass in the } i^{th} \text{ coordinate direction} \]
\[ a_i = \text{acceleration in the } i^{th} \text{ coordinate direction} \]
\[ q_i = \text{heat flux in the } i^{th} \text{ coordinate direction} \]
\[ r = \text{rate of heat addition per unit mass from external sources} \]
\[ \varepsilon = \text{internal energy per unit mass} \]
\[ \psi = \text{Helmholtz free energy density} \]
\[ \eta = \text{entropy density} \]
\[ \theta = \text{absolute temperature} \]

In this study the thermodynamic effects represented by the last two equations will be ignored, so that a purely mechanical theory can be developed. The deviation between theoretical predictions and experimental observations might be caused by this omission of thermodynamic effects. Then, the inclusion of thermodynamic effects would be a necessary way to follow in improving this theory. Perzyna's paper (130) would be a good starting point to the study of thermodynamic modifications.

3.1 Plane Strain Plasticity Equations for a Rigid Strain Hardening Material

The isotropic hardening yield condition of Von Mises in rectangular Cartesian coordinates is

\[ J_2 = K^2 \left( \varepsilon_p \right) \quad (3-5) \]

where \( J_2 = \frac{1}{2} S_{ij} S_{ij} \), \( S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \) and \( K \) is the yield stress in pure shear.

According to the flow rule, the plastic strain increment is

\[ d \varepsilon_{ij} = d \lambda S_{ij} \]

- 37 -
which can be written as

\[
\begin{align*}
\frac{d\varepsilon_x}{d\lambda} &= \frac{1}{3} (2\sigma_x - \sigma_y - \sigma_z) ; \\
\frac{d\varepsilon_y}{d\lambda} &= \frac{1}{3} (2\sigma_y - \sigma_z - \sigma_x) ; \\
\frac{d\varepsilon_z}{d\lambda} &= \frac{1}{3} (2\sigma_z - \sigma_x - \sigma_y) ; \\
\frac{d\varepsilon_{xy}}{d\lambda} &= d\lambda \tau_{xy} \\
\frac{d\varepsilon_{yz}}{d\lambda} &= d\lambda \tau_{yz} \\
\frac{d\varepsilon_{zx}}{d\lambda} &= d\lambda \tau_{zx}
\end{align*}
\]

(3-6)

For plane plastic flow with \( \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_z = 0 \) the flow indicates that \( \tau_{xz} = \tau_{yz} = 0 \) and

\[
\sigma_z = \frac{1}{2} (\sigma_x + \sigma_y)
\]

which reduces the yield condition (3-5) to

\[
\frac{\sigma_x - \sigma_y}{2} + \tau_{xy}^2 = \kappa^2
\]

(3-7)

\( \sigma_x, \sigma_y \) and \( \tau_{xy} \) can be expressed in terms of \( \sigma \) and \( \psi \) using Mohr's circle relations (see Figure 14).

\[
\begin{align*}
\sigma_x &= \sigma - K \sin 2\psi \\
\sigma_y &= \sigma + K \sin 2\psi \\
\tau_{xy} &= K \cos 2\psi
\end{align*}
\]

(3-8)

where \( \sigma = (\sigma_x + \sigma_y)/2 \) is the mean stress and \( \psi \) is the angle between the \( x \)-axis and the direction of the maximum shear stress.

The equilibrium equations for plane strain are

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0
\end{align*}
\]

(3-9)
Substitution of equation (3-8) into (3-9) yields

\[
\frac{\partial \sigma}{\partial x} - 2K \cos 2\psi \frac{\partial \psi}{\partial x} - \sin 2\psi \frac{\partial^2 \psi}{\partial x^2} + \cos 2\psi \frac{\partial K}{\partial y} - 2K \sin 2\psi \frac{\partial \psi}{\partial y} = 0
\]

\[
\frac{\partial \sigma}{\partial y} + 2K \cos 2\psi \frac{\partial \psi}{\partial y} + \sin 2\psi \frac{\partial^2 \psi}{\partial y^2} + \cos 2\psi \frac{\partial K}{\partial x} - 2K \sin 2\psi \frac{\partial \psi}{\partial x} = 0
\]

The characteristic directions are

\[
\frac{dy}{dx} = \tan \psi \quad \text{along } S_\alpha \quad (3-10)
\]

\[
\frac{dy}{dx} = \tan \left( \psi + \frac{\pi}{2} \right) \quad \text{along } S_\beta
\]

Equations along the characteristics take the form

\[
\frac{\partial \sigma}{\partial S_\alpha} - 2K \frac{\partial \psi}{\partial S_\alpha} = -\frac{\partial K}{\partial S_\beta} \quad \text{along } S_\alpha
\]

\[
\frac{\partial \sigma}{\partial S_\beta} + 2K \frac{\partial \psi}{\partial S_\beta} = -\frac{\partial K}{\partial S_\alpha} \quad \text{along } S_\beta \quad (3-11)
\]

and

\[
\frac{\partial \nu}{\partial S_\alpha} - \nu_\beta \frac{\partial \psi}{\partial S_\alpha} = 0 \quad \text{along } S_\alpha
\]

\[
\frac{\partial \nu}{\partial S_\beta} + \nu_\alpha \frac{\partial \psi}{\partial S_\beta} = 0 \quad \text{along } S_\beta \quad (3-12)
\]

The derivation of the characteristic equations are shown in Appendix A.

Cox, et al (131), Shield (132), and Lockett (66) have converted the characteristic equations into finite difference schemes to solve axially symmetric soil, metal problems, but the calculation procedures are not trivial. For strain hardening material Oxley (25) has shown that even if the characteristics are known analytically, integration along the characteristics will be tedious.
3.2 Plane Strain Plasticity Equations for a Rigid Rate Dependent Material

The yield condition for a rate dependent Von Mises material can be written in the form

\[ J_2 = K^2 \left( \varepsilon_p^2 \right) \]

(3-13)

where \( J_2 = \frac{1}{2} S_{ij} S_{ij} \), \( S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \) and \( K \) is the yield stress in pure shear.

A plane strain problem, in rectangular Cartesian coordinates, is a problem in which

\[ \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \frac{\partial}{\partial z} = b = a = V_x = 0 \]

(3-14)

Then the balance of linear momentum and the yield condition can be written in the form

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho b_x = \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) \]

(3-15)

\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho b_y = \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) \]

(3-16)

\[ F = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 - K \left[ (\varepsilon_{ij})^2 \right] = 0 \]

(3-17)

The Mohr's circle relations give

\[ \sigma_x = \sigma - K \sin 2\psi \]

(3-18)

\[ \sigma_y = \sigma + K \sin 2\psi \]

\[ \tau_{xy} = K \cos 2\psi \]

where \( \sigma = (\sigma_x + \sigma_y)/2 \) is the mean stress and \( \psi \) is the angle between the x-axis and the direction of the maximum shear stress. Substitution of Equations (3-18) into Equations (3-15) and (3-16) yields
\[ \frac{3\sigma}{\partial x} - 2K \cos 2\psi \frac{\partial \psi}{\partial x} + \sin 2\psi \frac{\partial K}{\partial x} + \cos 2\psi \frac{2K}{\partial y} - 2K \sin 2\psi \frac{\partial \psi}{\partial y} \]  
\[ = \rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}_x \frac{\partial \mathbf{V}}{\partial x} + \mathbf{V}_y \frac{\partial \mathbf{V}}{\partial y} \right) \]  
\[ \frac{\partial \sigma}{\partial y} + 2K \cos 2\psi \frac{\partial \psi}{\partial y} + \sin 2\psi \frac{\partial K}{\partial y} - 2K \sin 2\psi \frac{\partial \psi}{\partial x} + \cos 2\psi \frac{2K}{\partial x} \]  
\[ = \rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}_x \frac{\partial \mathbf{V}}{\partial x} + \mathbf{V}_y \frac{\partial \mathbf{V}}{\partial y} \right) \]  

When the indicated differentiations of \( K \) with respect to \( x \) and \( y \) are carried out, second order derivations of \( \mathbf{V}_x \) and \( \mathbf{V}_y \) with respect to \( x \) and \( y \) result. Linton (65) has shown that for a system of equations with second derivatives of some dependent variables and only first derivatives of other dependent variables the characteristic determinant is identical zero for all surfaces. Since Equations (3-19), (3-20) contain second order derivatives of \( \mathbf{V}_x \) and \( \mathbf{V}_y \) and no second order derivatives of \( \sigma \) and \( \psi \), its characteristic determinant is identical zero and no information on the values of \( \sigma \), \( \psi \), \( \mathbf{V}_x \), and \( \mathbf{V}_y \) can be obtained from the characteristic theory of solution. Thus, a deformation problem in a material with a rate dependent yield condition cannot be solved by the method of characteristics.

3.3 Plane Strain Plasticity Equations for a Rigid Perfectly Plastic Material

The Von Mises yield condition in rectangular Cartesian coordinates is

\[ J_2 = K^2 \]  

(3-21)

where \( J_2 = \frac{1}{2} S_{ij} S_{ij} \), \( S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \) and the pure shear yield stress \( K \) is constant in this type of material.

Then the characteristic directions and the Equations along the
characteristics can be reduced to the following forms provided that inertia effects are neglected.

\[
\frac{dy}{dx} = \tan \psi \quad \text{along } S_c
\]  

\[
\frac{dy}{dx} = \tan (\psi + \frac{\pi}{2}) \quad \text{along } S_{\beta}
\]  

\[
\frac{\partial \sigma}{\partial S_\alpha} - 2K \frac{\partial \psi}{\partial S_\alpha} = 0 \quad \text{along } S_\alpha
\]  

\[
\frac{\partial \sigma}{\partial S_\beta} + 2K \frac{\partial \psi}{\partial S_\beta} = 0 \quad \text{along } S_\beta
\]  

\[
\frac{\partial V_\alpha}{\partial S_\alpha} - V_\beta \frac{\partial \psi}{\partial S_\alpha} = 0 \quad \text{along } S_\alpha
\]  

\[
\frac{\partial V_\beta}{\partial S_\beta} + V_\alpha \frac{\partial \psi}{\partial S_\beta} = 0 \quad \text{along } S_\beta
\]  

3.4 Perturbation Solutions for Strain Hardening, Rate Dependent Materials

Duvov and Ivlev (36, 55) applied the perturbation method on the imbedding of a thin rigid body in a plastic medium with hardening. A translation hardening rule is employed and they perturbed the perfectly plastic solution as a zero'th order solution to obtain a first order solution for the strain hardening material. Spencer (106) solved the problem of a flat punch indenting a half-space of von Mises material by perturbing the classical punch solution about a zero velocity state. The zero'th order solution to the problem is the classical rate independent solution. The stress under the punch for the zeroth order solution is given by

\[
\sigma = K (2 + \pi) + \sigma_f
\]  

where \( \sigma_f \) is the uniform pressure on the surface not under the punch. Stress under the punch for the first order solution is given by
\[ \sigma = \sigma_f + k (2 + \pi) + \rho (2 + \frac{\pi}{2}) \frac{a^2 - x^2}{2a} \frac{\partial V(x, t)}{\partial t} \]  

(3-26)

where \(2a\) is the width of the punch. Slip-lines for the zero'th and first order are shown as dashed and solid lines in Figure 15. Linton (65) has shown that, for small velocities, the contribution of the local acceleration is of first order. Velocity gradient effects would become first order in problems having non-zero velocities in the zero'th order solution.

Spencer (106) has presented an approximate solution for an annular, flat punch by using perturbations on plane strain characteristics, but this technique appears limited to cases where there is a fixed radius whose dimensions are large with respect to other quantities.

It can be realized that perturbation solutions for strain hardening and rate dependent materials are limited to geometric boundary conditions and small velocities.

3.5 Limit Analysis

Limit analysis theorems were first introduced for continuous media by Drucker, Prager, and Greenberg (133). Two of these theorems establish criteria for determining upper and lower bounds for yielding of perfectly plastic materials. The theorems are stated as follows:

1. If a safe statically admissible state of stress which does not violate the yield condition can be found at each stage of loading, collapse will not occur under the given load schedule.
2. If a kinematically admissible collapse state can be found at any stage of loading, collapse must impend or have taken place previously.

A statically admissible state is one which satisfies the equilibrium equations

\[ \frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \]

throughout the material and the stress boundary conditions on those sections of the boundary of the material where surface tractions are specified. A velocity field is called kinematically admissible if it is finite and satisfies the velocity boundary conditions. Such a velocity field describes a kinematically admissible collapse state if the rate at which the surface tractions and body forces do work on the velocities equals or exceeds the rate of dissipation of energy computed from the strain rates treated as purely plastic strain rates. The state of stress denoted by the first theorem defines a lower bound load for a given problem while the second theorem defines an upper bound load.

Since there is no flow in the lower bound solution, there will be no strain hardening, no rate effects, and the solution will be identical to the perfectly plastic case. Pan (84) and Linton (65) have used limit analysis to obtain upper bound solutions for strain hardening and rate dependent materials, respectively.

In this study, it is attempted to solve large deformation problems for strain hardening and rate dependent materials independently,
then solve for strain hardening and rate dependent material. A different approach is proposed in the remainder of this chapter.

3.6 Field Equations

The Lagrangian strain tensor $E_{ij}$ can be expressed in terms of the displacements in the form of

$$E_{ij} = \frac{1}{2} \left( u_{i;j} + u_{j;i} + u_{m;i} u_{m;j} \right)$$  \hspace{1cm} (3-27)

where $u_i$ is the displacement component in the $i^{th}$ coordinate direction, the semicolon ($;i$) represents the covariant differentiation with respect to the material coordinates.

The strain components in polar coordinates take the form

$$E_{rr} = \frac{\partial u}{\partial r} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial \theta} \right)^2 \right]$$  \hspace{1cm} (3-28)

$$E_{r\theta} = \frac{1}{2} \left[ (\frac{\partial u}{\partial \theta} - v) / r + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial r} (\frac{\partial u}{\partial \theta} - v) / r + \frac{\partial v}{\partial \theta} (u + \frac{\partial v}{\partial \theta}) / r \right]$$

$$E_{\theta\theta} = \left( u + \frac{\partial v}{\partial \theta} \right) / r + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial \theta} - v \right)^2 / r^2 + \left( u + \frac{\partial v}{\partial \theta} \right)^2 / r \right]$$

where $u$, $v$ are the displacement component in $r$, $\theta$ direction, respectively.

The lagrangian strain rate tensor $\dot{E}_{ij}$ can be represented in the form

$$\dot{E}_{ij} = \frac{1}{2} \left( \dot{u}_{i;j} + \dot{u}_{j;i} \right) + \frac{1}{2} \left( \dot{u}_{m;i} u_{m;j} + u_{m;i} \dot{u}_{m;j} \right)$$  \hspace{1cm} (3-29)

The strain rate components in polar coordinates are

$$\dot{E}_{rr} = \frac{\partial \dot{u}}{\partial r} (1 + \frac{\partial u}{\partial r}) + \left( \frac{\partial \dot{v}}{\partial r} \right) \left( \frac{\partial v}{\partial r} \right)$$

$$\dot{E}_{r\theta} = \frac{1}{2} \left( \frac{\partial \dot{v}}{\partial r} \right) + \frac{1}{2} \left[ \frac{\partial \dot{u}}{\partial \theta} - \dot{v} \right] \left( \frac{\partial u}{\partial \theta} - v \right) + \left( \frac{\partial \dot{u}}{\partial r} \right) \left( \frac{\partial \dot{v}}{\partial \theta} \right) - \dot{v} \right) \frac{\partial \dot{u}}{\partial \theta}$$

$$+ \frac{\partial \dot{v}}{\partial r} \left( \frac{\partial \dot{v}}{\partial \theta} + u \right) + \left( \frac{\partial \dot{v}}{\partial \theta} \right) \left( \frac{\partial \dot{v}}{\partial r} \right) - \dot{u} \right] / r$$  \hspace{1cm} (3-30)
\[
\dot{E}_{\theta \theta} = \left( \frac{\partial \dot{v}}{\partial \theta} + \dot{u} \right) / r + \left[ \left( \frac{\partial \dot{u}}{\partial \theta} - \dot{v} \right) \left( \frac{\partial \dot{u}}{\partial \theta} - \dot{v} \right) + \left( \frac{\partial \dot{v}}{\partial \theta} + \dot{u} \right) \left( \frac{\partial \dot{v}}{\partial \theta} + \dot{u} \right) \right] / r^2
\]

Since large plastic deformation will be experienced during extrusion and wedge indentation problems, therefore the higher order terms of the strain, strain rate tensor will be significant.

3.7 Theoretical Solutions to Extrusion and Wedge Indentation Problems For Strain Hardening, Rate Dependent Materials

For simplicity the materials under investigation are assumed to be homogeneous and isotropic. Body forces as well as inertia effects are neglected in this study. Since the plastic deformation we are going to encounter is very large compared with the elastic strain, the latter is assumed to be negligible; i.e. the materials under consideration will be rigid hardening and rate sensitive ones.

In order to use the slip-line theory of rigid perfectly plastic material as the foundation of this new theory, the materials under investigation are also assumed to be incompressible. This assumption holds true for metals under fairly large plastic deformation.

The most important assumptions in this theory are stated as follows: (1) The plastic deformation patterns based on the slip-line solution to the large deformation problems, such as wedge indentation and extrusion, of the rigid perfectly plastic material are unique provided that the geometric boundary conditions are the same. (2) Strain hardening and rate effects have little influence over the deformation patterns predicted by the slip-line solution to the large deformation problems of the rigid perfectly plastic material.

Based on these assumptions, the strain hardening and rate effects of the real world materials can be corrected by finding the true
strain, strain rate fields and the corresponding stress distribution in the entire plastic region from the displacement and velocity fields of the associated slip-line solutions.

These assumptions are based on published experimental observations.

A. For rate sensitive materials:

1. Extrusion pressures and velocity fields, which were determined from slip-line solutions, were in qualitative agreement with experimental observations (111).

2. The stress trajectories calculated from the deformation of grid lines in a stepwise extrusion process agree approximately with those obtained from the slip-line method (41).

3. Plane-strain extrusions of lead at low friction have velocity fields which show good agreement with those obtained from slip-line solutions (112).

4. In plane strain extrusions, the experimental and theoretical direction and magnitudes of velocity vectors in the main portion of the deformed metal were substantially identical (124).

5. If the metal were an ideal metal with a constant-flow stress, then the flow pattern for plane strain should be identical for all strain rates as predicted by the Hencky's theory. Even if the flow stress differs, as for other metals, the flow pattern is unique and only depends on the geometry (113).
6. Strain rates and the properties of the metal do not appear to affect the velocity patterns. This seems to indicate that these patterns are unique and depend on geometry and boundary conditions only, except perhaps at high strain rates (113).

7. Lead and Aluminium have identical velocity-vector pattern in an inverted extrusion process, for the same geometrical shape and the same degree of boundary lubrication; this indicates that the flow pattern may be independent of the mechanical properties of the metal (110).

8. The general shapes of the stress curves of lead and 25-0 aluminium were similar, except that the stresses in the aluminium billet near the orifice were small tensile stresses (110).

9. The flow patterns on the meridian plane of a lead billet extruded into a 1.5-in-diameter solid bar with an 3.2/1 reduction ratio at room temperature were identical for extrusion speeds of 0.1, 0.74, and 5.15 in/min. The particle flow directions for these extrusion speeds were determined from the stepwise extrusion process (113).

The constitutive equation for commercial pure lead (99.9% pure) used by Thomsen and Frisch (112) in an inverted extrusion process under conditions of plane strain and axial symmetry is shown in Figures 16 and 17 or:

\[ \sigma_e = 2500 (\varepsilon_e)^{0.12} \text{ psi} \]
where $\sigma_e$ = The effective stress
$\varepsilon$ = The effective strain rate

Velocity = Extrusion die velocity, in/min
Pressure = Average extrusion pressure, psi

Since Frisch and Thomsen (41) reported that the extrusion velocities and the properties of the metal do not appear to affect the velocity patterns in the plastic region during the extrusion process, it seems to indicate that these patterns are unique and depend on geometry and boundary conditions only.

Because the strain rate tensor is related to the extrusion velocity, so the ratio of two effective strain rates can be related by the ratio of the two corresponding extrusion velocities under the assumption of the unique velocity pattern stated above.

Therefore, it is assumed in this study that the strain rate effects in large plastic deformation can be corrected by using the proper effective stress at each point through the experimentally obtained constitutive equation instead of using the constant effective stress of the rigid, perfectly plastic material. In other words, if the extrusion pressure $P_o$ for a certain extrusion velocity $V_o$ is specified, then the extrusion pressure $P$ for arbitrary velocity $V$ can be obtained by

$$(P/P_o) = (V/V_o)^{0.12}$$

For this published experimental case.

For Example:

If the plane-strain extrusion pressure with extrusion ratio 4.3 and velocity 0.029 in/min. is known to be 4625 psi. then the extrusion
pressure associated with velocities 0.067, 0.092, 0.240 in/min. and the same extrusion ratio can be calculated as follows:

when \( V = 0.067 \), \( P_{cal} = 4625 \) 
\[
\frac{0.067}{0.029} \div 0.12 = 5114
\]

when \( V = 0.092 \), \( P_{cal} = 4625 \) 
\[
\frac{0.092}{0.029} \div 0.12 = 5312
\]

when \( V = 0.240 \), \( P_{cal} = 4625 \) 
\[
\frac{0.240}{0.029} \div 0.12 = 5960
\]

In a similar manner, for a fixed extrusion ratio all the extrusion pressures can be specified provided that the extrusion pressure for the slowest extrusion velocity is known (see Table 1 and Figure 18).

From Table 1 it can be seen that the maximum deviation of the calculated extrusion pressure from experimental extrusion pressure is within 7%, this fact indicates that the assumption of correcting the effective stress within the plastic region through the experimentally obtained constitutive equation is an appropriate one provided that the extrusion speed is below the impact extrusion range.

The only requirement for these correction observations is the assumption that the extrusion pressure for the slowest extrusion speed for each extrusion ratio is known.

For rate sensitive materials, it has been shown that an equation of the type \( \sigma_e = K \varepsilon_e^n \) provides a reasonable approximation to the actual behavior (87, 92). In this equation \( \sigma_e \), \( \varepsilon_e \) and \( n \) are the effective stress, strain rate, and the strain rate sensitivity index, respectively.

B. For strain hardening materials: When the deformed material work-hardens, there is no simple relations for the stress distribution; the current yield stress of such materials depends on their strain history.
The only reasonably practical, but rather less accurate, way of solving theoretically, extrusion and related processes, is to modify the results obtained from the slip-line field solution by using a mean yield stress $Y_m$. But the necessary assumption associated with the above stated method is that the mean value of the equivalent total strain of any work-hardening material does not differ from that of the nonhardening material, the extrusion being carried out under identical geometrical and frictional conditions.

The deformation patterns in aluminium and lead are reported to be in close agreement by Thomsen and Frisch for extrusion (112) and by Kunogi for piercing (133). This supports the above stated method (indicated by Hill), although this agreement seems to be due partly to the strain-rate dependence of the yield stress of lead (60).

W. Johnson (60) assumed that the form of the strain-hardening characteristic of the material does not seriously affect the true strain imparted. Based on the experimental results of plane strain extrusion he made the following conclusions:

1. The method of allowing for strain-hardening by using an averaged value of the yield stress (this determined mean value of $2k$ is read from the stress-strain curves) is quite satisfactory for practical purpose.

2. Distorted grids as predicted by the theory, though not exact, certainly provide very good first approximations.

3. Dead metal zones are nearly always associated with the 90° (square) and 75° dies (these semiangles are measured with respect to the extrusion direction and directed away from it).
There are varieties of empirical strain-hardening equations proposed by different researchers (1), the typical are in the forms of

(a) \( \sigma_e = A(B + \varepsilon_e)^n \)

(b) \( \sigma_e = Y + C \varepsilon_e^n \)

(c) \( \sigma_e = \sigma_\infty - (\sigma_\infty - Y) e^{-\varepsilon_e/n} \) \hspace{1cm} (3-31)

(d) \( \sigma_e = Y + \tanh \left[ \left( \frac{E \varepsilon_e}{Y} \right) \right] \)

Parameters \( A, B, C, Y, E, \sigma_\infty \) are shown in Figure 19.

For simplicity, forms (a) and (b) will be adopted in this study.

**C. For strain-hardening and rate sensitive materials:** It had been proposed earlier by several research workers (43, 134) that yield stress at a constant temperature might be taken as a function of strain \( \varepsilon \) and strain rate \( \dot{\varepsilon} \) such that \( Y = f(\varepsilon, \dot{\varepsilon}) \).

Symonds and Perrone (134) suggested that \( \sigma_e = \sigma_0 f(\varepsilon) g(\dot{\varepsilon}) \) could be used to analyze structures loaded dynamically, where \( f(\varepsilon) \) and \( g(\dot{\varepsilon}) \) are strain-rate-sensitivity and strain-hardening relations, respectively. It is well known that strain-hardening of some materials decreased with increase in strain-rate (82, 93, 130).

Strictly speaking, a stress-strain-rate constitutive equation cannot be written in the product form with \( f(\varepsilon) \) and \( g(\dot{\varepsilon}) \) uncoupled. However, in order to retain mathematical simplicity, it is assumed that this combined effect of strain-hardening and strain-rate sensitivity could be considered an acceptable way for corrections. Actually, Norman Jones (82) found out that this simple method is good for estimating the combined influence of strain-hardening and strain-rate sensitivity on the permanent
deformation of rigid-plastic structures loaded dynamically.

The result of his work indicates that considering strain-hardening along when appropriate or strain-rate sensitivity alone gives permanent deformations which are similar to those predicted by analysis retaining both effects simultaneously. Ghosh (43) proposed a simplified constitutive equation in the form of \( \sigma = K \varepsilon^m \) in his study on sheet metal forming, where \( K \) = constant, \( n \) = strain hardening exponent, and \( m \) = strain-rate exponent.

Since we assume that the slip-line field will not be influenced by strain rate or strain hardening effects, we will retain this assumption when the two effects are combined together.

In this study, the uncoupled stress-strain, stress-strain-rate constitutive equation will be employed to compare the theoretical and experimental results of wedge indentation and inverted plane strain extrusion problems.

3.3 Displacement, Strain and Strain Rate Fields

In this investigation of large plastic deformation, the displacement fields of the wedge indentation and inverted plane strain extrusion problems can be specified by following the trajectories of each particle to its final position with respect to a given coordinate system.

For wedge indentation the unit diagram, introduced by Hill et al (59), is employed to specify the displacement field within the plastic boundary predicted by the slip-line solution. The details of using the unit diagram are presented in Figure 20.
For inverted plane extrusion the trajectory equations are derived using the slip-line solution which satisfies the velocity boundary conditions. Then the displacement field is specified by finding the final positions of the corners of the undeformed grids using the trajectory equations. The illustration is presented in Figure 21 and Figure 22.

For large plastic deformations, the higher order terms of the strain tensor cannot be neglected. And these higher order terms show their significant roles in this analysis, as was expected.

The strain rate field can be determined by using the field equations for strain rate. The strain rate tensor is different from the rate of deformation tensor used by R. Linton (65) in his analysis for rate effects in plasticity.

3.9 Deformation of an Element During Indentation

Indentation of a plastic half-space by a rigid wedge is the problem of pseudosteady plastic flow which Hill, Lee and Tupper (50) first solved and published in 1947. In this problem the slip-line field expands uniformly while its image in the stress and hodograph planes remain fixed. Hill, Lee and Tupper solved the indentation problem for a smooth, rigid wedge. In 1953, Grunzweig, Longman and Petch (44) solved the same indentation problem, but with frictional interfaces and perfectly rough, rigid wedges.

The determination of the motion of a particular element of the plastic material appears to be rather complicated since the velocity of the element is influenced by its varying position in space and also by the continual expansion of the velocity field. However, the use of the unit
diagram introduced by Hill, Lee and Tupper greatly facilitates the solution. The unit diagram is obtained by transforming the velocity field into a geometrically similar field in which the penetration is always unity. That means we follow the motion of an element in an associated field, which is the actual field scaled so that the penetration is always unity. The indentation hodograph of a rigid smooth wedge is shown in Figure 23, and the unit diagram is shown in Figure 20.

At a generic stage of the indentation process the depth of penetration \( t \) is the only characteristic length. Due to the geometrical similarity, the slip-line pattern remains fixed while the size of the slip-line field increases in proportion to \( t \). But for \( t = 0 \), the slip-line field reduced to a point \( O \), named the center of perturbation.

The vector \( \mathbf{r}_* \) denotes the actual position vector of the element \( P \) with reference to \( O \) when the penetration is \( t \), define \( \mathbf{r} = t \mathbf{r}_* \), so that \( \mathbf{r}_* \) is the corresponding position vector on the unit diagram.

Since the monotonically increasing depth of penetration \( t \) may be used as time, the velocity of the element \( P \) in the physical plane can be defined as \( \mathbf{v} = \mathbf{v}_* \). Thus, the velocity with which the wedge penetrates into the material has unit magnitude; in the unit diagram, this velocity is represented by the vector from \( O^* \) to the apex \( E \) of the wedge.

\[
\mathbf{v}_* = \mathbf{v} = \frac{d(t \mathbf{r}_*)}{dt} = \mathbf{r}_* + t \frac{dr*}{dt} = \mathbf{r}_* + t \mathbf{v}_* \quad (3-32)
\]

Thus

\[
t \frac{d\mathbf{r}_*}{dt} = \mathbf{v} - \mathbf{r}_* \quad (3-33)
\]
Where \( \mathbf{\tilde{v}} \) is the velocity vector of the corresponding point \( P^* \) in the unit diagram.

For simplicity we only need to look at the right half of the symmetric slip-line field of the smooth indentation problem shown in Figure 24.

The lip pushed up by the wedge is assumed to have a rectilinear boundary AB. There is a constant state of stress in the isosceles right triangle ABC, and the fan ACD provides the transition to a second isosceles right triangle of constant stress ADE, the base of which lies on the right flank of the smooth wedge.

To determine the length of the lip AB and its inclination against the undisturbed surface of the plastic half-space, assume that the depth of penetration of the wedge is unity, so that the slip-line field instantaneously coincides with its image in the unit diagram.

It can be shown from the indentation hodograph, Figure 23, that \( \mathbf{\tilde{v}} = \mathbf{0}^* \) is a vector of constant length \( \sqrt{2}\sin \beta \) for material in plastic flow; but the velocity \( \mathbf{v} \) for the rigid material below the slip-line BCDE is zero. This velocity discontinuity does not involve a discontinuity of diplacement because the line of discontinuity penetrates into the material so that a given particle is affected only momentarily by the velocity discontinuity.

The point with position vector \( \mathbf{\tilde{v}} \) lies on the arc B"E" of a circle, centered at \( O^* \), radius \( \sqrt{2}\sin \beta \) in the unit diagram, see Figure 20.

Vector \( \mathbf{\tilde{v}}^B \) corresponds to the velocity vector of all elements in region I, and \( \mathbf{\tilde{v}}^B \) is thus parallel to CB. Vector \( \mathbf{\tilde{v}}^E \) corresponds to elements in region III and \( \mathbf{\tilde{v}}^E \) is parallel to ED. An intermediate vector
O"P" corresponds to elements II lying on the radius AQ perpendicular to O"P". Hill named this circle as the focal circle and the points B", P", E" as foci.

From \( \frac{dr^*}{dt} = \gamma - \xi^* \), it can be visualized that the velocity of an element on the unit diagram is directed toward the focus corresponding to its momentary position. This face can be illustrated as follows:

When the element is still rigid, its velocity \( \gamma = 0 \), then \( \frac{dr^*}{dt} = \gamma - \xi^* = -\xi^* \), so its motion on the unit diagram is directed toward O", as shown on the unit diagram. The trajectories in I and III are simply segments of straight lines directed toward B" and E" respectively, but the trajectories in region II are curved.

Thus, it is only necessary to magnify the diagram at each stage by the appropriate value of \( t \) to find the actual trajectory of a particular element in space.

From the dotted representative trajectories on the unit diagram it can be seen that as the indentation proceeds an element crossing CB passes through regions I first, and then regions II and III. Thus, an element on the original surface after becoming plastic moved in the unit diagram along the free surface EA of the lip towards A and then swings round at A, continuing to move toward E" to form part of the inner lip.

It is clear that an element which at a certain stage is in region I can never have been in II or III. An element first becoming plastic at a point on DC passes through II and then into III for sufficiently large penetrations. An element crossing DD always remains in III, so that if
an element is in III to the left of \( E''D \) at a certain point, it can never previously have been in regions I or II.

Suppose now that the indentation at a given moment is \( t \) and that \( AEDCB \) is the plastic region. The material in triangle \( ACB \) at this stage has either been rigid or moving with uniform velocity \( \sqrt{2} \) as a point in region I. Initially, this material occupied triangle \( FCB \), where \( F \) is the intersection with \( O^* \) of the parallel to \( CB \) through \( A \), as shown in Figure 24. Similarly, the material in triangle \( E''ED \) initially occupied triangle \( O^*ED \). The final position \( E''ED \) can be generated by a process of pure shear from the initial position \( O^*ED \).

From Figure 24 the motion of an element \( F' \) formerly between \( O^* \) and \( F \) on the undisturbed surface can be specified in two steps. The element first moves as a point of region I, parallel to \( CB \) until it reaches \( O^*A \), when it is overtaken by the wedge; then it moves relative to but remains in contact with the wedge and moves as a point of region III, parallel to \( E'D \), to \( A' \).

However, there is no simple way to describe the motion of material initially inside region \( O^*FCD \) except using numerical techniques.

In order to evaluate the strain field in the plastic region, when the penetration is \( t \), it is necessary to calculate the final position of each specified point \( P \) whose initial position vector is \( \xi_0 \) relative to \( O^* \). It can be illustrated as follows:

Suppose the plastic boundary first reaches \( P \) when the penetration is \( t_0 \) \( (t_0 < t) \), and let \( \xi_0^* = \xi_0 / t_0 \) be the position vector on the unit diagram at this stage. The point on the unit diagram now moves along the corresponding trajectory through \( \xi_0^* \).
If \( S \) is the distance moved along the trajectory from \( \overline{r}_0^* \) in this period of time \((t - t_0)\), when the wedge goes down from \( t_0 \) to \( t \), and \( f(s) \) is the focal distance, then from \( t \frac{dr^*}{dt} = \frac{\overline{y} - \overline{r}^*}{|\overline{y} - \overline{r}^*|} = f(s) \) we will have:

\[
\ln\left(\frac{t}{t_0}\right) = \int_0^S \frac{ds}{f(s)} \quad \text{where} \quad \langle \overline{y} - \overline{r}^* \rangle = f(s) \quad (3-34)
\]

Since

\[
t \frac{dr^*}{dt} = \frac{\overline{y} - \overline{r}^*}{|\overline{y} - \overline{r}^*|} = \frac{dt}{t}, \quad |d \overline{r}^*| = ds
\]

Then

\[
\ln t \bigg|_{t_0}^t = \int_0^S \frac{ds}{f(s)}
\]

\[
\ln \frac{t}{t_0} = \int_0^S \frac{ds}{f(s)} \quad (3-35)
\]

In regions I and III (not having crossed II), \( f(s) = d - s \), where \( d \) is the distance from \( \overline{r}_0^* \) to the focus \( B'' \) and \( E'' \) respectively.

Thus,

\[
\ln \frac{t}{t_0} = \int_0^S \frac{ds}{d-s} = \ln \frac{d}{d-s}
\]

Then

\[
\frac{t}{t_0} = \frac{d}{d-s}, \quad d-s = d \frac{t_0}{t}, \quad S = d \left(1 - \frac{t_0}{t}\right)
\]  

Since \( \overline{r}_0, t_0 \) are given, so \( \overline{r}_0^* \), and \( d \) are fixed. Once \( t \) is specified, then \( S \) can be fixed by \( S = d \left(1 - \frac{t_0}{t}\right) \). From the calculated \( S \), we are able to find out the new position vector of point \( P, \overline{r}^* \), with respect to the origin \( O^* \). Then, by definition, \( \overline{r} = t \overline{r}^* \), we, therefore, can locate the final position vector of point \( P, \overline{r} \) in the physical space. These procedures are illustrated in Figures 25 and 26.
From Figure 25 it is clear that a particle P, immediately below the apex of the wedge, will remain stationary before it is reached by the apex of the wedge. As the wedge proceeds down, this particle moves so as to keep in contact with the flank of the wedge.

After connecting the eight different final positions of the particle associated with the eight different depths of penetration of the wedge (from 1/3 to 1), it can be seen that they fit on a straight line which is parallel to the velocity vector $O_x E''$. The fact that the eight final positions are evenly spaced on the straight line implies that all of the particles located beneath the apex of the wedge will start to move with the velocity $O_x E''$ once it is in contact with the wedge apex. The distance each particle can go is equal to the product of $|O_x E''|$ and the total moving time available, $(t - t_0)$. It can be confirmed that the particle originally located at $O_x$ will move to $E''$ when the wedge goes down unity.

From Figure 26 it can be visualized that the particle P has the same properties owned by the particles beneath the wedge apex. The only difference is that the moving period is not the same because the required initial time $t_0$, when it is reached by the plastic boundary, will be different.

Thus, it is clear that the using of a unit diagram affords us a definite way to find out the final position of each particle in the simple triangle region III. Since each element of $O_x E$ is deformed linearly to $E'' E$, and $O_x D$ to $E'' D$, this shows that triangle region III is actually formed by a process of pure shear from the initial position $A 0_x E D$. By the same token, it can be seen that the triangle region I
is also formed by a process of pure shear from the initial region $\Delta TCB$ (see Figure 24).

However, the integration $\int_0^s \frac{ds}{f(s)}$ has to be evaluated numerically in region II. There are some interesting facts that need our attention. For all points initially lie on a radius through $0^a$ will follow the same trajectory during deformation. Since the plastic boundary first reaches a point $\lambda \xi_0$ when the penetration is $\lambda \, t_0$, so $\xi_0^{\lambda} = (\lambda \xi_0)/(\lambda \, t_0) = (\xi_0/t_0)$ as before. Therefore, the same process of all numerical evaluation serves for all points $\lambda \xi_0$. The details of calculation are presented as follows:

The radial shear region II (fan ADC) can be viewed as a limit of infinitely many small triangles.

In Figure 27 the fan ADC is divided into eight equal smaller fans, and the arc $E''E'$ is divided in the same way. Then the center positions of each small arc are labeled as 1', 2', ..., 3'. An approximate method is used to specify the final position of each particle that its initial position vector $r_0^{\ast}$ on the unit diagram is point C. After connecting radius $A1$, $A2$, ..., $A3$, the fan region ADC is then replaced by eight small fan regions, and each of them can be viewed as a small triangle region. Assume points 1', 2', 3', ..., 8' as the velocity focus for triangles $A12$, $A23$, $A34$, ..., $A89$. Then the particle originally located at C will first move in the direction of $C1'$, so a line through C, 1', which intersects line $A2$ at point $a$ is drawn. Then line $a2'$ which intersects line $A3$ at point $b$ is drawn. Follow the same procedure until line $g3'$ intersects line $AD$ at point $h$. Since point $h$ is located in region III, so this
particle \( P \) will move in the direction of \( kE'' \), toward \( E'' \). Compared with the curved trajectories in region II, given by Hill, the approximate curved trajectories in the eight smaller fan areas bear close resemblance to the former. It is clear then, the accuracy of this approximate method will increase more with the smaller fan regions in region II. The criterion to stop the repeated procedures is the available moving period.

Since the initial time \( t_0 \) and the final time \( t \) are given, so the available moving period is equal to \( (t - t_0) \). In fan A12, the focal distance is segment \((1'C)\), and \( s \) is equal to segment \((ac)\), according to 
\[
S = d \left(1 - \frac{t_0}{t}\right);
\]
t' can then be specified as the particle moves to point a. If \( t' \) is less than \( t \), make another movement in fan A23, otherwise go back to specify \( s \) by \( s = d \left(1 - t_0/t\right) \). For simplicity \( l \) is chosen for the final \( t \), so \( \tau = \tau^* = (1) \tau^* = \tau^* \). Similarly, it is possible to find out the final positions of particles with \( \tau_0^* \) in region I when they have to cross boundary AC to enter region II, and even into region III.

It is clear now that for each undeformed particle located within the plastic boundary line \( \partial ABCDE \), once the initial and final time \( t_0, t \) are specified, its deformed final position can be specified. Then the displacement field in the entire region can be found. The strain field, and effective strain field can be evaluated by using the field equations of strain tensor and the effective strain shown in equation (3-41).

The last step in calculating the total plastic work for wedge indentation will be the integration of the plastic work over the whole region. The techniques, arguments used for smooth wedge indentation problem can be used for frictional and rough cases with Figures 28, 29, 30.
There is an interesting fact that can be used to specify the initial time \( t_0 \) for arbitrary point within the region ODCGF in the physical space shown in Figure 30.

Due to geometrical similarity, it can be found that points D, C, F will move along lines AD, AC, AG, AF linearly with point O along line AO. In Figure 30, \( t' = \frac{1}{t} t \) then points O, D, C, G, F move to points O', D', C', G' and F' where O'D' \( \parallel \) OD, C'G' \( \parallel \) CG, F' are located at the center of line AF, AG. This can easily be checked by constructing new plastic boundary from O' by keeping all the corresponding angles the same.

For arbitrary point within the undeformed region ABCD the associated time \( t_0 \), when it is first reached by the plastic boundary ODCG, can be specified in the following way. Let the prime denote the corresponding points of O, D, F, C, G when \( t' \) is arbitrary \( (t' < t) \).

For points along line AD, \( t_0 = (AD') / ACL \). For points along AC, \( t_0 = (AC') / ACL \). For points within region ACS, they are always reached by triangle AF'C'G'. If the coordinates of the undeformed point P' are known, then find out the coordinates of point G' by drawing a line parallel to line CG through P', then \( t_0 = (AG') / ACL \). For points along line BC, use the fact that triangle A DCG moves to triangle A FCG through a process of simple shear.

3.10 Inverted Extrusion with 50% Reduction

As a determination of the method a detailed description is given for the steady state inverted extrusion with a square die and 50% reduction in area. The slip-line solution will be studied first, and
then a process is developed to find the displacement field within the plastic region as extrusion proceeds. In Figure 21, OAB is one-half of the plastic region. The \( \beta \) slip-lines are radii through the die corner \( \gamma \), and the \( \alpha \) slip-lines are concentric circular arcs. The dead metal in the corner is bounded by OA, it takes no part in the steady flow even though part of it may be stressed to the plastic limit. Inverted extrusion is distinguished from direct extrusion by the absence of friction between the container wall and the undeformed billet, since they do not move relative to each other. Therefore, the slip-line at A will make an angle of \( \frac{\pi}{4} \) with the wall. Due to symmetry they must also intersect the central axis at B at \( \frac{\pi}{4} \).

The velocity boundary conditions require the normal component of velocity on OA must be zero; across AB and OB the normal components of velocity must be compatible with the rigid motions of the undeformed billet and extruded material respectively.

For rigid, perfectly plastic material, there is a tangential discontinuity in velocity in moving across the plastic boundary, because the elastic strain and work hardening are neglected.

For convenience, assume the metal and container together move with unit speed towards the stationary die, so that the plastic region is fixed space. Let \( \theta \) be the angle between a radius from the corner and the axis; the sense of \( \theta \) is taken such that \( \theta \) is \( + \frac{\pi}{4} \) on OA and \( - \frac{\pi}{4} \) on OB.
The equations of Seiringa are

\[ du - Vd\phi = 0 \quad \text{Along an } \alpha \text{ slip-line} \]
\[ dV + ud\phi = 0 \quad \text{Along a } \beta \text{ slip-line} \]

Thus, along the radius \( dV + ud\phi = 0 \), since \( d\phi = 0 \), \( dV = 0 \) or \( V = \text{const.} \). By the continuity condition across \( AB \), then \( V = \cos \theta \).

Since \( du - Vd\phi = 0 \) along an \( \alpha \) slip-line, we integrate from \( OA \) along an \( \alpha \) slip-line. Since \( du = Vd\phi = \cos \theta \, d\phi \) and \( d\phi = -d\theta \), then

\[ u = \int du = \int Vd\phi = - \int \cos \theta \, d\theta = - \sin \theta + C \]

Since \( u = 0 \) at \( \theta = \frac{\pi}{4} \), thus \( C = \sin \frac{\pi}{4} \).

So \( u = \frac{1}{\sqrt{2}} - \sin \theta \)

\[ V = \cos \theta \]

Hence on \( OB \), \( u \) is the constant and equal to \( \sqrt{2} \), this is compatible with the rigid motion of the extruded metal, the speed of the extrusion being 2. All conditions on the plastic boundary have been satisfied.

The stresses can be found from the equations of Hencly.

\[ P + 2k\phi = \text{constant} \quad \text{along an } \alpha \text{ slip-line} \]
\[ P - 2k\phi = \text{constant} \quad \text{along a } \beta \text{ slip-line} \]

Along each radius, \( \phi \) is constant, so \( P \) is a constant. Considering the equilibrium of the extruded metal, \( P \) is equal to \( k \) on \( OB \). Integrate along an \( \alpha \) slip-line, we find \( P = k \left( 1 + \pi^2 \right) \) on \( OA \). The total force on the die per unit width normal to the plane is \( k \left( 2 + \pi^2 \right) \times \text{die width} \) (neglecting possible friction between the wall and the dead metal).
The extrusion pressure, \( P \), is defined as extrusion force divided by the area of the original billet.

\[
P = k \left(2 + \pi \right)/2 = k \left(1 + \frac{\pi}{2}\right) = 2.5708k
\]

The trajectory of a particular element which crosses \( AB \) (\( r = a \)) when \( \theta = \theta_0 \) has the equation

\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{V}{u} = \frac{\cos \theta}{\sqrt{2}} - \sin \theta
\]

and therefore

\[
\frac{r}{a} = \frac{\left(\frac{1}{\sqrt{2}} - \sin \theta_0\right)}{\left(\frac{1}{\sqrt{2}} - \sin \theta\right)}
\] (3-37)

When \( \theta_0 = \frac{\pi}{4} \), the element simply moves along \( AO \) with speed \( 1/\sqrt{2} \). The time taken to move along the trajectory (3-37) between two points lying on the radii \( \theta_1 \) and \( \theta_2 \) \((\theta_1 > \theta_2)\) is given by

\[
t = \int_{\theta_2}^{\theta_1} \frac{rd\theta}{u} = a \left(\frac{1}{\sqrt{2}} - \sin \theta_0\right) \left[\mathcal{F}(\theta_1) - \mathcal{F}(\theta_2)\right]
\] (3-33)

where

\[
\mathcal{F}(\theta) = \frac{2 \cos \theta}{1 - \sin \theta} - 4 \coth^{-1} \left[\left(\sqrt{2} + 1\right) \tan \left(-\frac{\pi}{4} - \frac{\theta}{2}\right)\right]
\]

From equations (37) and (33) we are able to calculate the deformation of a square grid rules on a cross-section of the billet (see Figure 22).

Once the deformation of a square grid is calculated, we then obtain the displacement field. From the calculated displacement field,
the strain field and effective strain field can be evaluated, and the
total work required for this large plastic deformation can be found.
Then the total work is converted to extrusion pressure.

Because the extrusion speed is assumed to be unity for con-
venience, so the total time required for stroke \( A' \) is equal to \( A' \).
In steady state, the deformation pattern at a certain time should be
equal to that constructed by locating the trajectories of the unde-
formed grids, shown in the range \( A' \), throughout the extrusion process.

Let \( t = 0 \) designate the instant when the central particle \( \gamma \)
of the grid line \( EF \) enters the fan \( ADD \) with \( \theta = \theta^0 \). The other points
of this line continue to move with unit velocity towards the right until
they reach the circular arc \( ADB \). Thus, these points enter the fan at

\[
t = a \left( \frac{1 - \cos \theta}{1} \right) = a \left( \frac{1 - \cos \theta}{1} \right) ; \quad \theta = \sin^{-1} \left( \frac{y}{a} \right)
\]

In the same manner, the time, \( t_n \), needed to reach the plastic
arc for any point within the range \( A' \) can be found if its coordinates
are specified. So the rest of time \((A' - t_n)\) will be available for each
particle to transverse the plastic region or even go out of the plastic
region through the radius \( AB \).

Since the time taken to move along the trajectory \( \frac{r}{a} = \)

\[
\frac{\left( \frac{1}{\sqrt{2}} - \sin \frac{\theta_1}{2} \right)}{\left( \frac{1}{\sqrt{2}} - \sin \frac{\theta_0}{1} \right)} \quad \text{between two points lying on the radii } \theta_1 \text{ and } \theta_2 \quad (\theta_1 > \theta_2)
\]
is given by

\[
t = a \left( \frac{1}{\sqrt{2}} \sin \frac{\theta_0}{2} \right) \left[ F (\theta_1) - F (\theta_2) \right]
\]
so for given \( \phi_0 \), let \( \phi_1 = \phi_0 \), \( \phi_2 = -\frac{\pi}{4} \) the time, \( t_{BD} \), needed to go through the plastic region can be specified. If \((A' - t_R) < t_{BD}\), then we can fix \( t = t_{BD}, \phi_1 = \phi_0 \), to find out \( \phi_2 \), then find out the corresponding radius. If \((A' - t_R) > t_{BD}\), then the particle will go out of the plastic region with horizontal velocity only, \( 2 \) units, towards the right, and its \( y \) coordinate can be obtained by setting \( \phi = -\frac{\pi}{4} \) in the trajectory equation.

Then \( \phi_0 = -\pi/4 \), the element simply moves along the center line, once it goes across point 2, it will move to the right with the velocity of two units instead of one unit. With all the information stated above, the displacement, strain-effective strain field over the undeformed configuration can be calculated.

The final step is to evaluate the work required to proceed this extrusion process, and then convert the work back to force and the extrusion pressure.

Finite difference scheme is used to solve this extrusion problem, the integration formula over small grids is listed in Appendix 7. The deformation pattern of the undeformed grids is shown in Figure 31. The calculated extrusion pressure \( P_{cal} \) can be specified by

\[
P_{cal} = \frac{W}{(M \times A')}
\]

\( W = \) The total plastic work required for pushing the undeformed billet \((A' \times M \times 1)\) through the whole extrusion stroke \( A' \) with unit depth perpendicular to the plane \((A' \times M)\)

\( M = \) Half of the thickness of the billet.
3.11 Effective Stress and Effective Plastic Strain

In plasticity analysis the true stress-strain curve is more useful than the conventional stress curve. In this analysis it is essential to have the true stress-strain, stress-strain rate, stress-strain and strain rate curves to specify the material constitutive equations through curve fittings. The actual stress distribution in the interested plastic region can then be found as the basis for the corrections of strain hardening, rate effects and even the combined effect of strain hardening and rate effects.

The true stress-strain, stress-strain rate curves of the interested material can be obtained through a series of simple compression tests with different compression speeds. The connections between the simple compression test and the triaxial test is through the unique experimentally verified generalized stress-strain curve; named as the effective stress-effective strain curve. The importance of the generalized stress-strain curve is that under this curve the total plastic work done per unit volume of unstressed material up to a given final equivalent deformation is always the same.

Because of the effective stress-effective strain relationship, it is very convenient to convert the specified strain, strain rate fields into the effective strain, strain rate fields. And then find out the corresponding stress distribution in the plastic region through the constitutive equation of the material under investigation.
Once the stress distribution, strain and strain rate fields are specified in the plastic region, then the required plastic work for the plastic deformation can be found through numerical integrations over the entire volume.

For the comparison of different testing methods it is important to be able to compare the amount of deformation to which a material is subjected in different tests. In other words, a generalized function of the stresses, named the effective stress, and a generalized function of strains, named the effective strain, are required so that results obtained by different loading programs can all be correlated by means of a single curve of effective stress versus effective strain.

According to Hill, Lee and Tupper, the generalized stress \( \sigma_e \) is defined by

\[
\sigma_e = \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + \varepsilon (\tau_{xy}^2 + \tau_{yx}^2 + \tau_{xz}^2) \right]
\]

(3-58)

and the generalized plastic strain \( \varepsilon_e \) is defined by

\[
\varepsilon_e = \frac{\sqrt{2}}{3} \left[ (\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \gamma (\varepsilon_{xy}^2 + \varepsilon_{yx}^2 + \varepsilon_{xz}^2) \right]^{\frac{1}{2}}
\]

(3-61)

It has been shown experimentally that a reasonable approximation to one generalized stress-strain curve is obtained for all types of loading under combined stresses though most of the tests have been restricted to cases in which the principal stresses increase proportionately. The special
property of the $\sigma_e$, $\varepsilon_e$ curve is that the total plastic work done per unit volume of unstressed material up to a given final equivalent deformation $\varepsilon_e$ is always the same, and is equal to the area $\int_0^{\varepsilon_e} \sigma_e \, d\varepsilon_e$ under the generalized stress-strain curve.

This can be seen from the equation for the increment of plastic work per unit volume, see Appendix C.

$$d\mathcal{W} = \tau_{xx} d\varepsilon_{xx} + \tau_{yy} d\varepsilon_{yy} + \tau_{zz} d\varepsilon_{zz} + 2 (\tau_{xy} d\varepsilon_{xy} + \tau_{yz} d\varepsilon_{yz} + \tau_{zx} d\varepsilon_{zx})$$

$$= \sigma_e \varepsilon_e$$

(3-42)

Since the plastic volume change is zero, $\int_0^{\varepsilon_e} \sigma_e d\varepsilon_e$ is the total work per unit volume of unstressed material.

Therefore, the total plastic work in the plastic flow region of the slip line solution to the smooth wedge indentation problem can be evaluated through the following steps.

Specify the total depth of penetration first, then find out the coordinates of point A, B, C, etc. in a fixed coordinate system.

Find out the displacement field in the plastic flow region, and then evaluate the associated strain field. Once the strain field is specified, we can calculate the effective strain field by definition. For rigid perfect plastic material satisfying Von Mises yield condition, the effective stress is $\sqrt{3} k$, $k$ is the shear stress by definition. Then the integration of the plastic work $\int_V \sigma_e \varepsilon_e \, dV$ through the entire plastic volume will yield the total plastic work required in the whole indentation process.
Based on the above stated assumptions and concepts, this new method of corrections for strain hardening, strain-rate effects and even the combined effects of the two is briefed as follows:

The plane strain slip line solutions of the wedge indentation extrusion problems will be used as the case studies for all corrections. First of all, find out the strain, strain rate fields in the plastic region. Then use these fields as the first approximation of the real fields for the strain hardening and rate sensitive materials. Specify the corresponding stress distribution through the calculated effective strain, strain rate fields and the material constitutive equations. Then the required plastic work can be obtained by using numerical integrations. The last step is then to convert the total work back to the force or extrusion pressure.

Once the plane strain corrections are worked out well, then it is possible to extend these solutions to axial symmetric problems.

3.12 Indentation of a Plastic Half-Space by a Frictional, Rigid Wedge

Grunzweig, Longman and Petch (44) follow Hill, Lee and Tupper (50) have made calculations for the frictional wedge. When the wedge is rough, the corresponding slip-line field differs only in that the slip lines no longer meet the wedge face at 45°, provided FG remains straight (Figure 29).

From the height of F (Figure 29) above the original surface,

\[ H \cos \beta - C = \sqrt{2} H \cos \lambda \sin \left( \frac{\pi}{4} + \beta - \lambda - \psi \right) \]  \hspace{1cm} (3-43)

where \( H \) is the length \( FG \) of the contact surface, \( t \) is the depth of penetration beneath the original surface, \( \beta \) is the semi-angle of the wedge and \( \lambda \), \( \psi \) are the angles \( \angle FO \) and \( \angle TD \).
From the displacement velocities and the condition for geometric similarity,

\[ K \cos (\psi + \lambda - \pi/4) - C \cos (\beta - \psi + \pi/4 - \lambda) = \frac{C \sin \beta}{\sqrt{2}} \cos \lambda \]  

(3-44)

At the contact surface the normal stress on the wedge is \( p \cdot \sin 2\lambda \) and the frictional stress in the direction of \( \theta \) is \( K \cos 2\lambda \), where \( K \) is the shear yield stress of the indented material and \( P \) is the hydrostatic pressure in the zone \( \psi \theta \). By Hanchys' theorem, \( P = K (1 + 2 \psi) \). Hence, the pressure \( P \) on the wedge is \( P = K (1 + 2 \psi + \sin 2\lambda) \)  

(3-45)

and

\[ \cos 2\lambda = \mu (1 + 2 \psi + \sin 2\lambda) \]

where \( \mu \) is the coefficient of friction.

The method of determining the trajectory of an element is very similar to that used by Hill, Lee and Tupper for the smooth-rigid wedge indentation problem and for brevity the details are omitted (see Figure 30).

3.13 The Force and Work Relation for the Case of Frictional Wedge Indentation

Based on equilibrium condition (see Figure 32):

\[ F_e = 2 (\sigma \sin \beta) \cdot h + (\mu \cos \beta) \cdot h \]

so

\[ \frac{F_e}{2} = \sigma h [\sin \beta + \mu \cos \beta] \]

Due to symmetry only half of the total deformation need to be studied.

Then the external work \( W \) required for that should be equal to

\[ W = \int_0^c \left( \frac{F_e}{2} \right) \, dc = \int_0^c \sigma h [\sin \beta' + \mu \cos \beta'] \, dc \]
Since, \( \sigma \), the normal stress on the wedge is independent of the depth of penetration \( c \) and \( h \) is linearly related to \( c \)

\[
h = C / [\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)]
\]

Thus

\[
W = \sigma [\sin \beta + \cos \beta] \int_0^c \frac{cd}{[\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)]}
\]

\[
= \sigma [\sin \beta + \mu \cos \beta] c^2 / 2 [\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)]
\]

\[
= \frac{1}{2} \sigma [\sin \beta + \mu \cos \beta] \frac{F_e}{4}
\]

\[(3-46)\]

The external work \( W \) consists of plastic work and frictional work. The plastic work, \( W_p \), is contributed by plastic deformation in the plastic region associated with the slip-line field. The frictional work, \( W_f \), is the amount of work done by the frictional force along the flank of the wedge during indentation. Therefore

\[
W = W_p + W_f
\]

\[(3-47)\]

The velocity \( V_f \) with which an element moves along the flank of the wedge equals to \( V_0 \cos (\beta - \lambda) / \cos \lambda \) (see Figure 20).

Thus,

\[
W_f = \int_0^{\Delta T} (\mu \sigma h) V_f \, dt ; \quad dt = dc/V_o
\]

where \( \Delta T = C/V_o \) is the period of time throughout the indentation process, \( V_o \) is the velocity with which the wedge is pushed down and \( t \) is time.

Then

\[
W_f = \int_0^C \frac{V_0 \cos (\beta - \lambda)}{\cos \lambda} \frac{dc}{V_o}
\]
Let \( \xi = \frac{c}{v_o} \); \( \xi \sim \frac{c}{v_o} \) then \( d\xi = \frac{dc}{v_o} \)

Therefore

\[
W_e = \mu \sigma \int_0^\xi \frac{v_o \cos (\beta - \lambda)}{\cos \lambda} \ d\xi
\]

But

\[
h = \frac{c}{[\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)]} = \frac{\xi v_o}{[\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)]}
\]

So

\[
W_e = \mu \sigma \int_0^\xi \frac{v_o^2 \cos (\beta - \lambda)}{[\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)] \cos \lambda} \ d\xi
\]

\[
= \frac{1}{2} \mu \sigma \frac{v_o^2 \cos (\beta - \lambda) (c^2/v_o^2)}{[\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \pi/4 - \lambda)] \cos \lambda} = \frac{\mu c \cos (\beta - \lambda)}{2 \cos \lambda}
\]

(3-48)

The substitution of \( W_p \) and \( W_f \) into \( W \) yields

\[
\frac{HC}{2} \sigma [\sin \beta + \mu \cos \beta] = W_p + \frac{HC}{2} \sigma [\frac{\cos (\beta - \lambda) \mu}{\cos \lambda}]
\]

So

\[
\sigma = \frac{2W_p}{HC [\sin \beta + \mu \cos \beta - \mu \cos (\beta - \lambda)/\cos \lambda]} \quad (3-49)
\]

Therefore once the plastic work \( W_p \) is obtained from the plastic deformation region, then the normal stress \( \sigma \) on the flank of the wedge can be specified.

Then the required external indentation force \( \sigma \) can be evaluated as

\[
F_e = 2\sigma H [\sin \beta + \mu \cos \beta] \quad (3-50)
\]

When the wedge is smooth, then the slip lines meet the wedge face at 45°, i.e., \( \lambda = 45° \). Then \( \mu = 0 \) and the pressure \( P \) on the wedge reduces to

\[
P = K [1 + 2\psi + \sin (90°)] = 2K (1 + \psi) \quad (3-51)
\]
which is identical to the solution given by Hill et al.

For wedges with semi-angle $\beta$ less than $45^\circ$, $\lambda = 0^\circ$ if $\mu$ is sufficiently large to produce a shearing stress $\sigma$ on the wedge. If $\beta > 45^\circ$ the solution indicated in Figure 29 is valid only if angle $\angle DOD' = \pi/2'$, otherwise a statically admissible extension of the stress field into the non-deforming region immediately below $O$ is not possible (134), where $D'$ is the corresponding point of $D$ on the left slip-line field. If $\beta > \pi/4$ and is greater than the value required for the solution indicated in Figure 29 with $\angle DOD' = \pi/2'$, a solution can be found that involves a dead metal cap covering the end of the wedge. The slip line field for this solution is shown in Figure 33. A statically admissible extension of the stress field into the dead metal cap is indicated by the dotted extensions of the slip lines.

3.14 Plane Strain Plasticity Equations for a Rigid Perfectly Plastic Coulomb Material

The theory of plasticity can be utilized to solve certain types of soil mechanics problems such as stability of slopes, bearing capacity of foundation slabs and pressure on retaining walls provided that the soil is replaced by an idealized model. In this study soil and rock are treated as rigid perfectly plastic material which satisfy the Coulomb's yield condition.

The plane strain Coulomb yield condition can be represented as (see Figure 34):

$$R = C \cos \phi - \frac{\sigma_x + \sigma_y}{2} \sin \phi$$  (3-52)
where \( R \) is the radius of Mohr's circle at slip, the maximum shearing stress

\[
\sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} ;
\]

\( C \) is the cohesion, \( C \cos \phi \) is the radius of Mohr's circle at slip when the mean normal stress, \((\sigma_x + \sigma_y)/2\), in the plane is zero and \( \phi \) is the angle between the tangent to the Mohr's circles at slip and the negative \( \sigma \) axis.

It can be seen from Figure 34 that the shearing stress on the slip surface is not the maximum shearing stress \( R \) but is \( R \cos \phi \).

It has been shown in Chapter 2 that the extended Von Mises yield function is utilized as a proper generalization of the Coulomb yield condition so that the former can be used as a basic model for soils and rocks. The extended Von Mises yield condition is

\[
f = a I_1 + J_2^{1/2} = k \quad (3-53)
\]

where \( a \) and \( k \) are positive constants at each point of the material; \( I_1 \) is the sum of the principal stresses and \( J_2 \) is the second invariant of the stress deviataion.

In order to solve plane strain problems, parameters \( a \) and \( k \) must be specified when the yield function (3-53) is reduced to the Coulomb yield condition in the case of plane strain. In plane strain case \( \varepsilon_3^p, \varepsilon_{13}^p \) and \( \varepsilon_{23}^p \) vanish. Since \( \varepsilon_{1j}^p \left[ \alpha \delta_{i3} + S_{ij}/2 J_2^{1/2} \right] \lambda \) so \( S_{13} \) and \( S_{23} \) \((\tau_{x3}, \tau_{y3})\) are zero, and

\[
\varepsilon_{33}^p = 0 = \lambda \left[ a \delta_{33} + S_{33}/2 J_2^{1/2} \right].
\]

so

\[
S_{33} = -2 a J_2^{1/2} \quad (3-54)
\]
By definition, \( S_{ij} = \sigma_{ij} - \left( \frac{I_1}{3} \right) \delta_{ij} \), so
\[
S_{33} = -2a J_2^{\frac{1}{2}} = \sigma_{33} - \left( \frac{I_1}{3} \right) (1) = \frac{2I_1}{3} - (\sigma_1 + \sigma_2)
\]

Thus
\[
I_1 = \frac{3}{2} (\sigma_{11} + \sigma_{22}) - 3a J_2^{\frac{1}{2}} \tag{3-55}
\]

\[
J_2 = \frac{1}{6} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2
\]

In plane strain case, \( \tau_{yz} = 0, \tau_{zx} = 0 \), so

\[
J_2 = \frac{1}{6} \left[ 2 \sigma_x^2 + 2 \sigma_y^2 + 2 \sigma_z^2 - 2 \sigma_x \sigma_y - 2 \sigma_y \sigma_z - 2 \sigma_z \sigma_x \right] + \tau_{xy}^2
\]
\[
= \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x \right] + \tau_{xy}^2
\]
\[
= \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z \right] + \tau_{xy}^2
\]
\[
= \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 + 2 \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - I_1 \right] + \tau_{xy}^2
\]

But
\[
\sigma_e = S_{33} + \frac{I_1}{3} = \frac{I_1}{3} - 2a J_2^{\frac{1}{2}}
\]
\[
2\sigma_e^2 = \frac{2I_1}{9} - \frac{8}{3} a I_1 J_2^{\frac{1}{2}} + 8a^2 J_2
\]
\[
\sigma_e I_1 = \frac{I_1^2}{3} - 2a I_1 J_2^{\frac{1}{2}}
\]
\[
\frac{I_1}{3} + a J_2^{\frac{1}{2}} = \frac{1}{2} (\sigma_{11} + \sigma_{22}) = \frac{1}{2} (\sigma_x + \sigma_y)
\]
Thus

\[ J_2 = \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 + \frac{2}{9} \frac{I_1^2}{\sigma} - \frac{8}{3} \alpha I_1 J_2^{1/2} + 8 \alpha^2 J_2 - 8 \sigma_x \sigma_y - \frac{I_1^2}{3} + 2 \alpha I_1 J_2^{1/2} \right] + \tau_{xy}^2 \]

\[ = \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y - \frac{1}{9} I_1^2 - \frac{2}{3} \alpha I_1 J_2^{1/2} - \sigma_x^2 J_2 + 9 \alpha^2 J_2 \right] + \tau_{xy}^2 \]

\[ J_2 = \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y - \left( \frac{I_1}{3} + \alpha J_2^{1/2} \right)^2 \right] + 3 \alpha^2 J_2 + \tau_{xy}^2 \]

\[ J_2 (1 - 3 \alpha^2) = \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y - \left( \frac{\sigma_x}{2} + \frac{\sigma_y}{2} \right) \right] + \tau_{xy}^2 \]

\[ = \frac{1}{3} \left[ \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y - \frac{1}{4} \sigma_x^2 - \frac{1}{4} \sigma_y^2 - \frac{1}{2} \sigma_x \sigma_y \right] + \tau_{xy}^2 \]

\[ = \frac{1}{3} \left[ \frac{3}{4} \sigma_x^2 + \frac{3}{4} \sigma_y^2 - \frac{3}{2} \sigma_x \sigma_y \right] + \tau_{xy}^2 = (\sigma_x^2 + \sigma_y^2 - 2 \sigma_x \sigma_y) / 4 \]

\[ + \tau_{xy}^2 \]

So

\[ J_2 = \left[ (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2 \right] / (1 - 3 \alpha^2) \]  \hspace{1cm} (3-56)

The substitution of (3-55) and (3-56) into the yield function (3-51) yields

\[ f = \alpha I_1 + J_2^{1/2} = \alpha \left[ \frac{3}{2} \left( \sigma_{11} + \sigma_{22} \right) - 3 \alpha J_2^{1/2} \right] + J_2^{1/2} \]

\[ = \frac{3}{2} \alpha \left( \sigma_x + \sigma_y \right) - 3 \alpha^2 J_2^{1/2} + J_2^{1/2} = 3 \alpha \left( \frac{\sigma_x + \sigma_y}{2} + (1 - 3 \alpha^2)^{1/2} \frac{\sigma_x - \sigma_y}{2} \right) \]

\[ \sqrt{\frac{\sigma_x - \sigma_y}{2} + \tau_{xy}^2} \]

\[ = 3 \alpha \left( \frac{\sigma_x + \sigma_y}{2} \right) + (1 - 3 \alpha^2)^{1/2} \sqrt{\frac{\sigma_x - \sigma_y}{2} + \tau_{xy}^2} = k \]
or
\[
\frac{k}{(1-3\alpha^2)^{1/2}} = \frac{3\alpha}{(1-3\alpha^2)^{1/2}} \sigma_x + \sigma_y + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]  
(3-57)

Equation (3-57) becomes identical with equation (3-52) provided that
\[
C = \frac{k}{(1-12\alpha^2)^{1/2}}
\]
\[
\sin \phi = \frac{3\alpha}{(1-3\alpha^2)^{1/2}}
\]
\[
\cos \phi = \frac{(1-12\alpha^2)^{1/2}}{(1-3\alpha^2)^{1/2}}
\]

If the weight of the soil and the inertia effects are neglected, the equilibrium equations in the case of the plane strain reduce to
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0
\]
\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]
(3-53)

The equations (3-53) are hyperbolic and the two characteristic directions are inclined at an angle \(\frac{\pi}{4} + \frac{\phi}{2}\) to the direction of the algebraically greater principal stress. In Figure 35, the lines 1, 2 are the directions of the principal stresses \(\sigma_1, \sigma_2\) (\(\sigma_1 < \sigma_2\)) at a point and the lines \(\alpha, \beta\) are the characteristic lines passing through the point. The characteristic line which lies between 1 and 2 directions is named as the first failure line and the angle of inclination of this line to the x-axis is denoted by \(\theta\) (100).
The equations along the characteristics are

\[
\frac{1}{2} \cot \phi \ln P + \Theta = \text{const.} \quad \text{along a first failure line}
\]

\[
\frac{1}{2} \cot \phi \ln P - \Theta = \text{const.} \quad \text{along a second failure line}
\]

which were first obtained by Kotter, the derivation is shown in Appendix B.

The equations of the velocity field referred to the characteristic lines are

\[
dV_1 - (V_1 \tan \phi + V_2 \sec \phi) \, d \Theta = 0 \quad \text{along a first failure line}
\]

\[
dV_2 + (V_1 \sec \phi + V_2 \tan \phi) \, d \Theta = 0 \quad \text{along a second failure line}
\]

The simplest pattern of failure lines consists of two families of straight lines at an angle \( \frac{\pi}{2} + \phi \) and this pattern corresponds to a region of constant stress. Since \( \Theta \) is constant along the failure lines, thus velocity along each failure line is constant.

Then one of the families of failure lines consists of concurrent straight lines, the other family is a system of logarithmic spirals which have the point of intersection as center. This stress distribution is called a zone of radial shear. Since \( \Theta \) is constant along the straight line so the velocity associated with it is constant. From the characteristic velocity field equations it can be shown that the velocity associated with the logarithmic spiral will vary exponentially along the logarithmic spiral.

Shield has shown that a line separating a region of plastic flow from a region which remains at rest must be a failure line. It is shown further that the change in velocity across the line is inclined at an angle \( \phi \) to the line of discontinuity. The straight line and the logarith-
mic spiral of angle $\phi$ are the only lines of discontinuity which permit rigid motions, translation and rotation respectively, of the regions separated by the line.

3.15 Indentation of a Semi-infinite Mass of Soil by a Smooth Rigid Wedge

The problem of the indentation of a semi-infinite mass of soil by a smooth, rigid wedge under conditions of plane strain has been solved by Shield (197). Shield's solution is obtained by following the solution of the same problem which is solved by Mill, Lee and Tupper.

Since the configuration is geometrically similar at each stage of the penetration, it is possible to obtain the complete history of the motion without following the deformation step by step.

As the wedge is pressed into the soil, the displaced soil will form a raised lip at each side of the wedge, and the shape of the lips must be determined as part of the solution to the problem. Shield has assumed that the surfaces of the lips are straight and shown that a solution exists which satisfies this assumption. The right hand side portion of the pattern of failure lines is indicated in Figure 35. FO is the right flank of the wedge, which is of angle $2\beta$, GK is the as yet undeformed surface of the soil and FG is the lip. The regions FOC, FOC are regions of constant stress while region FDC is a zone of radial shear of angle. Denote the length of the lip FG by $r$, the distance of A from FG by $b$, and the elevation of $\gamma$ above AG by $h$. The depth of penetration of the wedge is denoted by $t$ and if the downward velocity of the wedge is taken as the unit of velocity, $t$ may also be taken as the time variable. The lip FG makes an angle $\beta - \alpha$ with the undisturbed level AG, and the expressions for $r$, $h$, $b$, $h$ in terms of $t$, $\alpha$, $\beta$
are shown as follows (see Figure 36):

\[ t = H \cos \beta - h = H \cos \beta - H_3 \sin (\beta - \alpha) \]

\[ H_3 = 2H_2 \cos \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \]

\[ H_1 = H_2 \tan \phi \]

\[ H = 2H_1 \cos \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \]

\[ H = 2H_2 e^{-\alpha \tan \phi} \cos \left( \frac{\pi}{4} + \frac{\phi}{2} \right) = H_3 e^{-\alpha \tan \phi} \cos \left( \frac{\pi}{4} + \frac{\phi}{2} \right) / \cos \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \]

\[ t = H_3 e^{-\alpha \tan \phi} \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \beta - H_3 \sin (\beta - \alpha) \]

Therefore

\[ H_3 = \frac{t}{\left\{ e^{-\alpha \tan \phi} \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \beta - \sin (\beta - \alpha) \right\}} \]

\[ h = t \sin (\beta - \alpha) / \left\{ e^{-\alpha \tan \phi} \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \beta - \sin (\beta - \alpha) \right\} \]

\[ b = H \sin \beta \sin (\beta - \alpha) + h \cos (\beta - \alpha) \]

\[ b = H_3 e^{-\alpha \tan \phi} \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \sin \beta \sin (\beta - \alpha) + H_3 \sin (\beta - \alpha) \cos (\beta - \alpha) \]

\[ = H_3 \sin (\beta - \alpha) \left\{ e^{-\alpha \tan \phi} \tan \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \sin \beta + \cos (\beta - \alpha) \right\} \]
Thus

\[ b = \frac{t \sin(\beta - \alpha) \left[ e^{-\alpha \tan \phi} \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \sin \beta + \cos(\beta - \alpha) \right]}{\left( e^{-\alpha \tan \phi} \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \cos \beta - \sin(\beta - \alpha) \right)} \]

(3-61)

The \( \alpha \) line GCD0 is a line of discontinuity in the velocity field so that the velocity vector along this line must make an angle \( \phi \) with this line. It follows that \( V_2 = 0 \) along GCD0 and therefore \( V_2 \) is zero throughout the plastic region FGDO, since the \( \beta \) lines are straight. In the region FDO, \( V_1 = V \cos \phi \) is constant, and \( V \) is the magnitude of the velocity vector in FDO. The boundary condition along FO required that the velocity of the wedge and that of the soil in contact with it must have the same projection on the normal to FO, and therefore

\[ (1) \sin \beta = V \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right), \quad V = \sin \beta \sec\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \]

In the zone of radial shear the velocity increases exponentially along each line, and along FC the velocity vector has the constant magnitude

\[ V = \sin \beta \sec\left(\frac{\pi}{4} + \frac{\phi}{2}\right) e^{\alpha \tan \phi} \]

At a given instant, the region FGC moves as a rigid body with velocity \( V \) in the direction perpendicular to FC.

The projection of the velocity of the lip FG on the normal to FG is

\[ V \cos\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \sin \beta \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) e^{\alpha \tan \phi} \]
while the projection of the velocity of the vertex 0, which is moving downward with unit velocity, on the normal to FG is \( \cos (\beta - \alpha) \). Thus, the distance of 0 from FG increases at the constant rate which is the sum of these two projections. At a time \( t \), the distance of 0 from FG can be related as

\[
t [\sin (\beta) \tan (\frac{\pi}{4} + \frac{\phi}{2}) e^{\tan \phi} + \cos (\beta - \alpha)] = b + t \cos (\beta - \alpha)
\]

Thus

\[
\gamma = t \sin \beta \tan (\frac{\pi}{4} + \frac{\phi}{2}) e^{\tan \phi}
\]

(3-62)

The substitution of equation (3-61) into (3-62) yields a relation between the angle \( \alpha \), \( \beta \), and after some reduction, gives

\[
\cos (2\beta - \alpha) = \frac{\cos \alpha [e^{\tan \phi \tan (\frac{\pi}{4} + \frac{\phi}{2})} + e^{-\tan \phi \tan (\frac{\pi}{4} - \frac{\phi}{2})}]}{2 \sin \alpha + e^{\tan \phi \tan (\frac{\pi}{4} + \frac{\phi}{2})} e^{-\tan \phi \tan (\frac{\pi}{4} - \frac{\phi}{2})}}
\]

(3-63)

For each fixed semi-angle \( \beta \), the angle \( \alpha \) can be found numerically by using Newton-Raphson method.

The pressure \( P \) on the flank of the wedge is obtained by using Voelz's circle which is shown in Figure 37, and Appendix D. \( P \) is found to be given by

\[
P = C \cot \phi \left\{ e^{2 \tan \phi \tan^{2}(\frac{\pi}{4} + \frac{\phi}{2})} - 1 \right\}
\]

(3-64)

where \( C \) is the cohesion of the soil.
The total downward force $F$ necessary to drive the wedge into the soil is given by

$$F = (2F \sin \beta) H = (2F \sin \beta) \frac{H}{H} \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) e^{-\alpha \tan \phi} \quad (3-65)$$

The determination of the motion of a particular element of the soil appears to be rather complicated since the velocity of the element is influenced by its varying position in space and also by the continual expansion of the velocity field. However, the use of the unit diagram introduced by Hill et al. greatly facilitates the solution. The method of determining the trajectory of an element is very similar to that used for the perfectly plastic material which satisfies the Von Mises yield condition and for brevity the details of the solution are omitted.

3.16 Indentation of a Semi-infinite Mass of Soil by a Frictional, Rigid Wedge

Shields (100) has pointed out that it is possible to solve the problem when there is a moderate amount of sliding friction between the soil and the wedge. In this case the lines of failure do not meet the flank of the wedge at an angle $\frac{\pi}{4} + \frac{\phi}{2}$ but the pressure on the wedge is uniformly distributed. The lip FG is still straight although it is inclined at a smaller angle to the undistributed surface.

The expressions for $H_3$, $H$, $b$, $h$ in terms of $t$, $a$, $\beta$ can be found by following the same procedures used by Shield for the smooth case. All the expressions can be derived from Figure 33, which give

$$H_3 = F / \left\{ \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \cos \beta e^{-\alpha \tan \phi} \right\}$$

$$= \sin\left(\frac{\pi}{4} + \beta - \lambda + \frac{\phi}{2} - \alpha\right) \cos (\lambda - \phi)$$
\[ R = \frac{R_3 \sin \left( \frac{\pi}{4} - \frac{\phi}{2} \right) e^{-\alpha \tan \phi}}{\cos (\lambda - \phi)} \]

\[ h = R_3 \sin \left( \frac{\pi}{4} + \beta + \frac{\phi}{2} - \lambda - \alpha \right) \]

\[ b = \frac{t \sin \left( \frac{\pi}{4} + \beta + \frac{\phi}{2} - \lambda - \alpha \right)}{\cos (\lambda - \phi)} + \cos \left( \frac{\pi}{4} + \beta + \frac{\phi}{2} - \lambda - \alpha \right) \]

\[
\begin{bmatrix}
    \sin \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \beta e^{-\alpha \tan \phi} \\
    \cos (\lambda - \phi)
\end{bmatrix}
- \sin \left( \frac{\pi}{4} + \beta - \lambda - \alpha + \frac{\phi}{2} \right)
\]

The relation between the angles \( \alpha, \beta \) can be obtained by using the velocity boundary conditions. After some reduction on the angle relation becomes

\[ \sin \beta \cos \phi \sin \left( \frac{\pi}{4} + \frac{\phi}{2} \right) e^{\alpha \tan \phi} / \cos \phi \]

\[ = \sin \left( \frac{\pi}{4} + \beta + \frac{\phi}{2} - \lambda - \alpha \right) \left[ \frac{\sin \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \sin \beta e^{-\alpha \tan \phi}}{\cos (\lambda - \phi)} + \cos \left( \frac{\pi}{4} + \beta + \frac{\phi}{2} - \lambda - \alpha \right) \right] \]

\[
\begin{bmatrix}
    \sin \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \cos \beta e^{-\alpha \tan \phi} \\
    \cos (\lambda - \phi)
\end{bmatrix}
- \sin \left( \frac{\pi}{4} + \beta - \lambda - \alpha + \frac{\phi}{2} \right)
\]

(7-66)

For each fixed semi-angle \( \beta \) and angle \( \lambda \), the angle \( \alpha \) can be found numerically by using Newton-Raphson method.

The normal stress, \( \sigma \), and the shear stress, \( \tau \), on the flank of the wedge can be obtained by using Mohr's circle which is shown in Figure 40 and Appendix D. They are found to be
\[ \sigma = \frac{R_1}{1 + \csc \phi \left( e^{2 \alpha \tan \phi} - 1 \right) + e^{2 \alpha \tan \phi} \sin (2 \lambda - \phi) } \]  
\[ \tau = \frac{R_1}{e^{2 \alpha \tan \phi} \cos (2 \lambda - \phi) } \]  

where \( R_1 = C \cos \phi / (1 - \sin \phi) \), is the radius of the Mohr's circle at slip when one of the principal stress equals to zero.

The coefficient of friction \( \mu \) is then given by

\[ \mu = \frac{\tau}{\sigma} = \frac{e^{2 \alpha \tan \phi} \cos (2 \lambda - \phi)}{1 + \csc \phi \left( e^{2 \alpha \tan \phi} - 1 \right) + \sin (2 \lambda - \phi) e^{2 \alpha \tan \phi} } \]

The total downward force \( F \) necessary to drive the wedge into the soil is then

\[ F = 2M (\sigma \sin \beta + \tau \cos \beta ) \]  

(3-69)

The method of determining the trajectory of an element is similar to that used for the smooth case and for brevity the details are omitted.

3.17 The Force and Work Relation for the Case of Frictional Indentation

The velocity \( V_f \) with which an element moves along the flank of the wedge can be obtained from Figure 39 which is the vector \( \vec{a} \)

\[ V_f = \frac{V_o \cos (\beta - \lambda)}{\cos \lambda} \]

where \( V_o \) is the velocity with which the wedge is pushed down into the soil. Follow the same procedure used in deriving the total work \( W \) and the frictional work \( W_f \). They can be represented as
\[ W = \frac{1}{2} \sigma H T \left[ \sin \beta + \mu \cos \beta \right] \]

\[ u = \frac{\mu \sigma \cos (\beta - \lambda) \cdot T}{2 \cos \lambda} \quad \text{(3-7a)} \]

Thus

\[ \sigma = \frac{2 u}{H T \left[ \sin \beta + \mu \cos \beta - \frac{\mu \cos (\beta - \lambda)}{\cos \lambda} \right]} \quad \text{(3-71)} \]

Then the total downward force \( F \) necessary to drive the wedge into the soil is given by

\[ T = 2\sigma H \left[ \sin \beta + \mu \cos \beta \right] \quad \text{(3-72)} \]

For frictional wedge indentation cases when the friction factor reaches a value such that the angle \( \lambda \) is equal to the non-negative angle \( \beta - \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \), a solution can be found that involves a dead material cap covering the end of the wedge. The way of calculating the false wedge solution for soil problems is very similar to that used for the perfectly plastic Von Mises material.

The derivations of the effective stress and effective strain for Coulomb material are presented in Appendix C.

Several problems involving strain hardening and rate effects will be examined in Chapter 6 and details of the procedure will be presented there.
CHAPTER 4

APPROXIMATE METHOD FOR AXIALLY SYMMETRIC PROBLEMS

Axially symmetric extrusion and conical indentation problems are axially symmetric and therefore cannot be properly described by a two dimensional, plane strain analysis. The mathematical analyses of axisymmetric problems are, in general, difficult even under idealized conditions, therefore only few exact plasticity solutions exist.

Lockett (66) has solved problems for the indentation of metals by perfectly smooth, conical indenters with half-cone angles between 52.5° and 77° by using finite difference equations. Berezanov (9) has presented values of average pressure on cones with half-cone angles of 15° and 30° for indentation of both metals and soils. Miller (80) has presented an approximate method to calculate the pressure distributions on a conical indenter.

Problems for axially symmetric extrusion and conical indentation with strain hardening and rate effects have not yet been solved to the author's knowledge. Therefore, a numerically oriented method is utilized to find the extrusion pressure and the pressure distributions on a conical indenter to analyze experimental data.

Shield and Cox et al. (135) have shown that, for metals and soils obeying the Tresca or Coulomb yield conditions and the associated flow rules, the stress and velocity relations become hyperbolic if the Parr and Von Karman hypothesis (σθ equals to either the maximum or minimum principal stress) is obeyed. Then both the stress and velocity characteristics
are real. The equilibrium equations along the characteristics are similar to the plane strain relations except for terms arising from the increasing volume of the material as the distance from the axis of symmetry increases.

4.1 General Equations

If body forces and the inertia effects are neglected, the equilibrium equations are

\[
\frac{\partial \tau_r}{\partial r} + \frac{\partial \tau_{rs}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0
\]

\[
\frac{\partial \tau_{rs}}{\partial r} + \frac{\partial \sigma_\theta}{\partial z} + \frac{\tau_{rs}}{r} = 0
\]  \hspace{1cm} (4-1)

The Coulomb yield condition, which includes the Tresca condition as a special case, will be utilized for the analysis for stress equations. Coulomb's yield condition may be expressed parametrically as in Figure 41 where

\[
P = \frac{1}{2} (\sigma_r + \sigma_z)
\]

\[
R = P \sin \phi + C \cos \phi
\]

= radius of Mohr's circle at \( P \)

\[
\sigma_r = P + R \cos 2 \theta
\]

\[
\sigma_z = P - R \cos 2 \theta
\]

\[
\sigma_\theta = P - R
\]

\[
\tau_{rs} = R \sin 2 \theta
\]

\( C = \) cohesive strength

\( \phi = \) angle of internal friction
The characteristic relations can be obtained by substituting
equations (4-2) into equations (4-1)

The slope of the characteristics in the \( r, \phi \) plane are given by

\[
\frac{d\phi}{dr} = \tan \left[ \theta \pm \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \right] \quad (4-3)
\]

The equilibrium equations along the characteristics become

\[
\cos \phi \frac{dP}{d\theta} + 2r \alpha + \frac{R}{r} \left[ d\phi (1 - \sin \phi) + \cos \phi dr \right] = 0
\]

\[
- \cos \phi \frac{dP}{d\theta} + 2r \beta + \frac{R}{r} \left[ dr \cos \phi - d\phi (1 - \sin \phi) \right] = 0 \quad (4-4)
\]

where \( \alpha = \theta + \frac{\pi}{4} - \frac{\phi}{2} \) defines the first slip line

\( \beta = \theta - \frac{\pi}{4} + \frac{\phi}{2} \) defines the second slip line

Since \( dR = \sin \phi dP \)

Define \( dN = \frac{\cos \phi}{R} dP \) then \( \frac{dR}{R} = \frac{\sin \phi}{R} dP = \frac{\cos \phi}{R} dP \tan \phi \)

Thus \( \frac{dR}{R} \cot \phi = dN \quad N = \cot \phi \ln \frac{R}{C} \)

Divide equations (4-4) by \( R \) yields

\[
dN + 2d \alpha + \frac{1}{r} \left[ d\phi (1 - \sin \phi) + \cos \phi dr \right] = 0 \text{ along } \alpha \text{ line}
\]

\[
dN - 2d \beta + \frac{1}{r} \left[ - dr \cos \phi + d\phi (1 - \sin \phi) \right] = 0 \text{ along } \beta \text{ line}
\]

(4-5)

If the terms multiplied by \( \frac{1}{r} \) are zero, then the above equations become the plane-strain equilibrium equations along slip lines.

When \( \phi = 0 \), then the above relation can be reduced down to that

for Tresca's material. When \( \phi = 0 \), then
\[ dN + 2d \alpha + \frac{1}{r} [dz + dr] = 0 \quad \text{along } \alpha \text{ line} \]

\[ dN - 2d \beta + \frac{1}{r} [dz - dr] = 0 \quad \text{along } \beta \text{ line} \] (4-6)

The method described here used the plane-strain characteristics to compute the correction factors. The plane-strain characteristics were chosen since it appears that they would be a good first approximation to the axially symmetric characteristics. Thomsen (113) has shown that the plane-strain and axially symmetric extrusion pressures for the same extrusion ratio differ by only a small amount, and the plane-strain extrusion pressure curves lie below the axially symmetric curves. Thomsen (113) has concluded that for inverted extrusions of lead and for the range of diameter ratios 4.3, 2.15, 1.43, the two dimensional analysis can be applied to the axial-symmetric extrusions. From equations (4-5), (4-6), one would expect that the actual characteristics close to the edge of a cone would not be very different from the approximate plane-strain characteristics.

To find the average pressure on a cone and the pressure distribution, one needs to know only the characteristics for the analogous plane-strain problem. The correction factor, determined by integrating along these characteristics, is added to the plane-strain solution (50).

The average pressure and the pressure distribution on a cone are calculated for perfectly plastic material first and then a comparison with the known theoretical solution. The method described in Chapter 3 is then utilized to correct strain hardening and rate effects.

4.2 Approximate Method for Conical Indentation on Metals

The slip line boundary associated with the plane strain solution of wedge indentation problem can be specified provided that the
depth of penetration $t$, semi-wedge angle $\beta$, frictional angle $\lambda$ and the radial fan angle $\alpha$ are given.

Values of pressure on the face of the cone are found by integrating along the $\beta$ characteristic, thus the proper correction factor from equation (4-6) is

$$\frac{1}{r} \left[ ds - dr \right]$$

(4-7)

The integration of expression (4-7) in the triangle region I of the solution to wedge indentation problem will yield (see Figure 42)

$$\int \frac{1}{r} \left( ds - dr \right) = \int \frac{1}{r} \left( \frac{ds}{dr} - 1 \right) dr$$

$$= - \int_{C_2}^{C_1} \left[ 1 + \tan (\beta \theta) \right] \frac{dr}{r}$$

$$= [1 + \tan (\beta \theta)] \ln \frac{C_1}{C_2}$$

(4-8)

Because the radial shear region II can be viewed as the limit of the sum of infinite many small triangles, therefore the integration of expression (4-7) in region II can be simplified by employing relation (4-8) in each small triangle individually.

Since the integration procedure in region III is similar to that used in region I so the correction factor required for conical indentation can be found.

Since the stresses on the cone surface area near the edge act over a larger surface area than the stresses near the tip, the average value of $\sigma$ must be computed on a weighted basis (see Figure 43). The characteristic net chosen by Lockett and Miller was such that the interval
\( A_1 \) along the flank of the cone were at one-tenth of \( R \), the maximum cone radius in contact with the material.

Since the total surface area of the cone is given by

\[
\frac{\pi R^2}{\sin \beta} = \frac{\pi C^2 \tan^2 \beta}{\sin \beta} = \frac{\pi C^2 \tan \beta}{\cos \beta}
\]

The effective surface area that \( \sigma_1 \) acts over is

\[
\frac{\pi R^2}{\sin \beta} \left[ (A_1 + 0.05)^2 - (A_1 - 0.05)^2 \right]
\]

Because close to the tip, the total surface area is

\[
\frac{\pi (0.05R)^2}{\sin \beta} = \frac{0.0025 \pi R^2}{\sin \beta}
\]

and close to the edge, the effective surface area is

\[
\frac{\pi [1^2 - 0.95^2] R^2}{\sin \beta} = \frac{0.0975 \pi R^2}{\sin \beta}
\]

so the rest effective areas can be specified by

\[
\frac{\pi R^2}{\sin \beta} \left[ (A_1 + 0.05)^2 - (A_1 - 0.05)^2 \right]
\]

So the total force \( F \) on the conical indenter can be calculated by

\[
F = \sum_{i=0}^{10} \sigma_i A_i \sin \beta, \text{ so } F = \pi R^2 \left\{ 0.0025 \sigma_0 + 0.0975 \sigma_{10} + \sum_{i=1}^{9} \sigma_i \left[ (A_1 + 0.05)^2 - (A_1 - 0.05)^2 \right] \right\}
\]
where $\sigma_o$, $\sigma_{10}$ are the normal stress close to the tip and edge, where

$$\pi R^2 = \pi (C \tan \beta)^2$$

The pressure $P$ on the conical indenter is

$$P = \frac{F}{\pi R^2} = 0.0025 \sigma_o + 0.0975 \sigma_{10} + \sum_{i=1}^{9} \sigma_i [A_i + 0.05]^2 - (A_i - 0.05)^2$$

It is clear that the pressure $P$ on the cone is independent of the dimension of $R$ or the depth of indentation, it is constant. This fact is observed from the experimentally obtained linear relationship between indentation pressure and the depth of indentation.

Therefore, the force on the conical indenter $\frac{F}{\pi R}$ when $R = 1$ used in Lockett's and Miller's paper can be viewed as the pressure on the conical indenter.

The approximate method used for conical indentation on soils is very similar to that for metals and for brevity the details are omitted.

The approximate method used for axially symmetric extrusion problems is derived very closely to that for conical indentation on metals the details of derivation are neglected but the results will be presented in Chapter 6.
CHAPTER 5

APPARATUS AND EXPERIMENTAL PROCEDURE

The object of this experimental program was to determine the constitutive equation of 99.9% pure lead from the true stress-true strain curves and the plane strain extrusion and indentation pressures required for the same pure lead. Since it is necessary to know the yield stress for every material over a considerable range of strain, strain-rates and temperatures to calculate the forces and pressures involved in a number of metal-forming and indentation operations. Such fabricating methods as rolling, forging, extrusion and wedge indentation involve, principally, compressive stresses. It was thus necessary to conduct a series of compression tests on pure lead at room temperature to use compressive stress-strain results in forming indentation calculations. The reasons pure lead was chosen are that the recrystallization temperature of pure lead is below room temperature; it work hardens, and it is rate-sensitive and ductile even under atmosphere pressure. Therefore, pure lead can be utilized as a proper material for the analysis of hot extrusion and indentation processes. Thus the experimental results can be used to compare with the theoretical predictions proposed by the method described in Chapters 3 and 4.

5.1 Apparatus and Tools

A drawing of the plane-strain extrusion apparatus which was made in alloy steel is shown in Figure 44. It was designed to perform tests for inverted extrusion with 50% reduction in area. The drawing of the load cell and the deflection transducer is shown in Figure 45.
Initially the load cell was calibrated against a Riehle Testing Machine and the deflection transducer was calibrated against two standard dial gauges. Output of the load cell and deflection transducer was displayed by an X-Y recorder calibrated such that the curves were axial load – displacement curves for the tests. All the compression tests were performed at room temperature and under atmospheric pressure.

5.2 Compression Tests on 99.9% Pure Lead

A cylindrical specimen of 1.000 in ± 0.010 in. diameter and 1.5000 in. ± 0.015 in. long was placed in the Riehle testing machine between two flat steel plates and loaded in compression. The samples were cast in cylindrical moulds in 1 1/8 in. in diameter and 2.2 in. long. The cast bars were then faced in a lathe to the final dimensions and left to anneal at room temperature. Some specimens 0.50 in. long and 0.75 in. long were used to investigate size effect. Grooves were turned in the ends of the specimens to facilitate retention of the lubricant; 16 grooves, 0.01 in. deep were used following the practice of Loizou and Sims (67) and Johnson (60). The grooving had negligible effect on the length during compression. This method was capable of overcoming end friction; therefore, barrelling was eliminated after compression. Therefore, the ends of the specimen were plane and parallel after compression. However, if too many grooves were machined on the ends of the specimen then the ends of the specimen were splayed outwards due to too much lubricant.
In the conventional stress-strain curve the nominal stress \( \sigma \) is defined as the load divided by the original cross-sectional area is plotted against the engineering strain which is defined as the change in length per unit original length. It is evident that this nominal stress is not the true stress acting in the specimen since the cross-sectional area of the specimen is increasing with load.

The true stress \( \sigma \) is defined as the load \( P \) divided by the current cross-sectional area \( A \). Since incompressibility is assumed for metals, then

\[
A_0 l_0 = A l
\]

where \( A_0 \) and \( l_0 \) are the original cross-sectional area and the gage length and \( A \) and \( l \) are the current values. Therefore, the true stress \( \sigma \) is

\[
\sigma = \frac{P}{A} = \frac{Pl}{A_0 l_0}
\]

The increment of strain \( d\varepsilon \) for a given length is defined as

\[
d\varepsilon = \frac{dl}{l}
\]

and the total strain in going from an initial length \( l_0 \) to the length \( l \) is

\[
\varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln \frac{l}{l_0}
\]

where \( \varepsilon \), the true strain, is also called the natural or logarithmic strain.

The strain rate, \( \dot{\varepsilon} \), is the derivation of strain, \( \varepsilon \), with respect to time or

\[
\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{dl}{l dt} = \frac{v}{l}
\]

where \( v \) is the compression speed of the Riehle testing machine.
A series of compression tests were performed at different compression speeds; therefore, the true stress, true strain and true strain rate can be calculated based on the above stated relations. Therefore, the constitutive equation for the true stress versus true strain, strain rate can be obtained (see Figure 46).

5.3 Wedge Indentation Tests

The indentation test ingot was cast in a rectangular shaped aluminium container without top cover. The cast bar was then machined in such a way that the top and bottom surfaces were plane and parallel. Then, a steel wedge with semi-angle 22.5° was forced vertically into the flat pure lead surface without lubricant. A series of tests were performed with very low speed and speeds of 0.2, 1.0, 6.0 in./min. Three tests were performed for every different speed. The average indentation pressure was then used to make comparison with theoretical prediction. All results will be presented in Chapter 5.

5.4 Plane Strain Inverted Extrusion with 50% Reduction in Area

The specimens were cast in a rectangular shaped steel mould without top and bottom ends. The cast bar was then machined to the final dimensions and every surface remained flat. An end cover was tightly fitted to the die by threaded bolts. When the lubricated specimen was inserted it just fitted the container. Then the lubricated punch was brought into contact with the specimen and the die. The autographic diagram, which is a record of the variation of punch load with punch travel, was then plotted by the X-Y recorder plot associated with the load cell. A series of
tests were performed with different extrusion speeds and three tests were repeated for each extrusion speed. All the experimental results and theoretical predictions will be presented in Chapter 7.

Compression tests were performed for two different kinds of paraffin and sand-paraffin mixtures. All of them are rate sensitive but become brittle even before the compression speeds reaches to 1.0 in./min. However, paraffin can be utilized in investigating the modes of deformation for complicated forming process at low deforming speed.
CHAPTER 6

SOLUTIONS OF SOME PRACTICAL PROBLEMS

The purpose of this chapter is to present the procedures needed to solve extrusion and indentation problems as a means of illustrating the methods described in Chapter 3 and Chapter 4.

The basic assumptions in this study are

(1) The plastic deformation patterns based on the slip-line solution to large deformation problems for rigid perfectly plastic material are unique provided that the geometric boundary conditions are the same.

(2) Strain hardening and rate effects have little influence over the deformation patterns predicted by the slip-line solution to large deformation problems for rigid perfectly plastic material.

(3) The stress field of the deformed region can be specified numerically provided that the material constitutive equation is obtained from simple compression test.

Therefore, the extrusion, indentation pressures for strain-hardening, rate dependent, strain-hardening and rate dependent materials can be calculated provided that the effective strain and strain rate fields associated with the slip-line solutions of the perfectly plastic material are specified numerically.
6.1 Numerical Methods

Finite element and finite difference methods are used to solve the plane-strain indentation problem for the rigid, perfectly plastic material. The rest of the problems illustrated in this study are solved by finite difference method only.

The trajectory of every corner of the grids plotted in the undeformed region has been described in Chapter 3 for both extrusion and indentation problems so the associated displacement field can be specified. Therefore, the strain field can be calculated by using either finite element or finite difference method. The undeformed grids plotted on the billet to be extruded are shown in Figure 22.

Triangular and quadrilateral elements are used in the finite element scheme and the description of these elements are presented in Appendix G. The second order forward, backward and central difference schemes are used in the finite difference method and the formulas are listed in Appendix F. An integration formula for the finite difference method is needed to calculate the plastic work in the deformed region and the derivation in polar coordinate system is presented in Appendix F.

The flow charts used to calculate the total work for the wedges indentation and the inverted extrusion problems are presented in Appendix E.

6.2 Calculations of Indentation, Extrusion Pressures for the Rigid, Strain-Hardening, Rate Dependent, Strain-Hardening and Rate Dependent Materials

Compared with the theoretical slip-line solutions for both indentation and extrusion the corresponding numerical results show good agreement.
with the former so the corrections for strain-hardening and rate effects can be done provided that the material constitutive equations are specified.

The material constitutive equations used in this study are listed as follows:

A. Strain-hardening materials. Wedge-shaped indenting tools have been used for producing plane-strain indentations in three cold-worked metals: mild steel, copper and aluminium. The material properties reported by Dugdale (34) are listed in Table 2 and the constitutive equations obtained through curve fittings are presented in Table 3. The shear stress versus shear strain curves are shown in Figure 47.

Cone indentation experiments have been performed by Dugdale (35) in cold-worked mild steel, copper, and aluminium. The material properties are listed in Table 4 which bear a close resemblance to those used in wedge indentation experiments. Therefore, the constitutive equations used for wedge indentation problems can be used for conical indentation problems.

Experiments in the plane-strain extrusion of lead and aluminium have been performed by Johnson (60). The stress-strain curves and the material constitutive equations are shown in Figure 48.

B. Rate dependent material. Experiments for the inverted extrusion of commercially pure lead (99.9 percent) extruded in plane strain and under conditions of axial symmetry have been performed by Thomsen (113). The effective stress versus effective strain curves at constant strain rates and the material constitutive equation are plotted in Figure 16 and 17, respectively.
C. Strain-hardening and rate dependent material. Experiments for the inverted extrusion with 50\% reduction in area and wedge indentation with semi-wedge angle equals to 22.5° were performed by the author. The stress strain curves at constant strain rate and the constitutive equation are shown in Figure 49.

6.3 False Wedge Solution

The load required for blunt rough wedge during indentation can be found by using the equilibrium condition between the blunt wedge and the false wedge formed below the blunt wedge (see Figure 50). In order to satisfy the equilibrium condition, the load \( P \) required for the blunt wedge when the depth of indentation equals to \( C_\beta \) must be equal to that for the 45° false wedge when the depth of indentation equals to 1 unit.

It has been shown that \( N \) is related to \( C \) by

\[
R = \frac{C}{\cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \frac{\pi}{4} - \lambda)} = \frac{C}{D(\beta, \lambda, \psi)}
\]

where \( D(\beta, \lambda, \psi) = \cos \beta - \sqrt{2} \cos \lambda \sin (\beta - \psi + \frac{\pi}{4} - \lambda) \) is a function of angle \( \beta, \lambda, \psi \) and is constant when the semi-wedge angle \( \beta \) is fixed. Therefore \( N/C = 1/D(\beta, \lambda, \psi) \) is constant. From Figure 50 \( \overline{H} \) is found to be

\[
\overline{H} = \frac{H}{\sin \beta}
\]

\( W = H \sin \left( \frac{\pi}{4} \right) \)

\( \overline{C}_\beta = \overline{H} D(\beta, \lambda, \psi) \)
Thus
\[ \overline{C}_\beta = \frac{W}{\sin \beta \left( \frac{H}{\overline{C}_\beta} \right)} \]

The indentation force \( F \) can be calculated by

\[ F = 2H \left( \sigma \sin \beta + \mu \sigma \cos \beta \right) \]

where \( \sigma \) is the normal stress and \( \mu \) is the coefficient of friction. Because \( \sigma \) and \( \mu \) are uniform along the flank of the wedge so

\[ \frac{F}{C} = 2 \left( \sigma \sin \beta + \mu \sigma \cos \beta \right) \left( \frac{H}{C} \right) \]

\[ = 2\sigma \left( \sin \beta + \mu \cos \beta \right) / \left( \beta , \lambda , \psi \right) \]

is constant provided \( \beta \) is fixed.

From the equilibrium condition it can be seen that the indentation force \( F \) is the force required for the blunt wedge when its penetration depth is \( \overline{C}_\beta \) and is also the force required for the 45° false wedge when the depth of penetration equals to unity. Therefore

\[ \frac{F_\beta}{\overline{C}_\beta} k = \frac{F_\beta}{\overline{C}_\beta} \frac{\pi}{4} \]

The theoretical and numerically calculated value of \( \frac{F}{Ck} \) for the perfectly rough wedge with semi-wedge angle \( \frac{\pi}{4} \) are 13.075 and 12.503, respectively.

The calculations can be illustrated as follows:
When \( \beta = 60^\circ \) the ratio \( \frac{H}{\overline{C}_\beta} \), calculated from geometric relation of the slip line field, equals to 2.73, then
\[
\overline{C_\beta} = \frac{\overline{v}}{\sin \beta \left( \frac{H}{C_\beta} \right)}
= \frac{1.33728}{\sin (60^\circ) (2.73)} \approx 0.55545
\]

where \( \overline{v} = \left( \frac{H}{C} \right) \beta = \pi/4 \times \sin \left( \frac{\pi}{4} \right) = 1.33728 \), thus

\[
\frac{\overline{F}}{\overline{C_\beta} \ k_{(theo.)}} = \frac{13.075 \ k}{0.55545 \ k} = 23.539
\]

\[
\frac{\overline{F}}{\overline{C_\beta} \ k_{(num.)}} = \frac{12.503 \ k}{0.55545 \ k} = 22.510
\]

5.4 Axial Symmetric Correction Based on Plane-Strain Solution

The pressure distribution in the cone surface and the force required for cone indentation can be calculated by the method described in Chapter 5. The pressure distribution can be obtained by adding the correcting terms to the normal stress on the flank of the wedge along each \( \beta \) characteristics. The force required for cone indentation can be found by integrating the normal component of the pressure distribution over the entire conical surface. A finite difference oriented computer program has been written to calculate the pressure distribution and the indentation force associated with conical indentation problem. For strain-hardening, rate dependent materials, the correction factors calculated along \( \beta \) characteristics must be modified in such a way that the stress distribution along each \( \beta \) characteristics should be specified by using the corresponding material constitutive equation.
The procedures used for the correction to the plane strain extrusion problem are very similar to those used for the correction to the plane strain wedge indentation problem. A finite difference oriented computer program has been written to calculate the extrusion pressures for rate dependent material under conditions of axial symmetry.

6.5 The Mean Value of the Effective Strain for Plane Strain Problems

If the mean value of the effective strain, $\bar{\varepsilon}_e$, is calculated from the known slip-line solution of each plane-strain problem then the corresponding effective stress, $\sigma_e$, can be specified by reading the stress-strain curve. Therefore, the slip-line solution of the rigid, perfectly plastic material can be used as an approximation for the solution for the strain-hardening material provided that the yield stress of the former is replaced by the effective stress, $\sigma_e$.

The mean value of the effective strain, $\bar{\varepsilon}_e$, for wedge indentation problem can be calculated as follows:

The effective stress of the rigid, perfectly plastic material satisfying Von Mises yield condition equals to $\sqrt{3} \ k$, which is constant, where $k$ is the shear yield stress. $W$, the work done by the indentation force $F$ when the depth of penetration equals to $C$, can be specified by

$$W = \int_0^C Fdc$$
$$= \int_0^C k (ck) \ dc$$
$$= Kk \frac{c^2}{2}$$

where $\frac{1}{k} \frac{ck}{F}$ is constant.
When the wedge is smooth, the external work, \( W \), done by the indentation force \( F \) equals to the plastic work, \( W_p \), which is caused by plastic deformation. Thus, for unit depth

\[
W_p = \int_V \sigma_e \varepsilon_e \, dV
= \int_A \sqrt{3} k \varepsilon_e \, dA
= \sqrt{3} k \varepsilon_e A
\]

Where \( A \) is the total area of the plastic region.

Since \( W = W_p \), therefore \( K k C^2 / 2 = \sqrt{3} k \varepsilon_e A \). Then

\[
\varepsilon_e = \frac{KC^2}{2 \sqrt{3}A}
\]

For perfectly smooth case, the total area \( A = 2 \) \([2 \times \text{Area}_I + \text{Area}_II]\)

\[
= \pi^2 (1 + \psi/2); \quad (\lambda = \pi/4) \quad (\text{see Figure 29}), \text{ where}
\]

\[
\text{Area}_I = \frac{1}{2} \left( \frac{H}{2} \times \frac{H}{2} \right) = \pi^2/4
\]

\[
\text{Area}_II = [\pi \left( \frac{H}{\sqrt{2}} \right)^2] \left( \frac{\psi}{2\pi} \right) = \frac{H^2 \psi}{4}
\]

Therefore, the mean value of the effective strain \( \varepsilon_e \) can be calculated if the semi-wedge angle is specified.

The mean value of effective strain, \( \varepsilon_e \), in plane strain inverted extrusion with 50% reduction can be specified by as follows:

\[
W = \int_V \sigma_e \varepsilon_e \, dV = \int_A (\sqrt{3} k) \varepsilon_e \, (1) \, dA = \sqrt{3} k \varepsilon_e A
\]

-109-
Extrusion pressure $P_{\text{ext.}}$ is defined as: Extrusion force $F$/original billet area

$$P_{\text{ext.}} = \frac{W}{A_{\text{ext}}} = \frac{W}{A}$$

$$A_{\text{ext.}} A = \frac{\sqrt{3}}{2} k \cdot \varepsilon_e A$$

$$\varepsilon_e = \frac{P_{\text{ext.}} k}{\sqrt{3} k} = \frac{(1 + \pi/2) k}{\sqrt{3} k} = 1.48425$$

$\varepsilon_e$ is much larger than the strain associated with uniform deformation which is $\ln (A_0/A_e) = \ln 2 = 0.69315$ where $A_0$, $A_e$ are the original and exit area in extrusion problem. That is the reason why people (136) tried to introduce "the redundant work" as a correction for the assumption of uniform deformation. Actually, via the step wise method used in extrusion problem, the real deformation pattern is far from uniform deformation (137). Therefore, it is necessary to make further corrections for both the nonuniform deformation as well as strain-hardening and rate effects.

The procedure required to solve the practical problems for Coulomb material are very similar to those described in this Chapter for metals. Finite difference method is used to solve indentation problems and all the results are presented in Chapter 7.
CHAPTER 7

THEORETICAL AND EXPERIMENTAL RESULTS

The numerically calculated theoretical results of the plane strain, axial symmetric problems based on the theory described in Chapters 3 and 4 and the procedures illustrated in Chapter 6 as well as the experimental results from tests performed by Dugdale (34, 35), Johnson (60), Thomsen (113) and the tests described in Chapter 5 are presented in this Chapter.

7.1 Results for Metals

A. Numerical solutions for perfectly, rigid plastic material. The numerically calculated results for the plane-strain wedge indentation problem based on the finite element, finite difference method are listed in Table 5, Figure 51, and Table 6, Figure 52, respectively. The deviations of the numerical solutions from the theoretical slip-line solutions given by Grunzweig, et al. (44) are within 10% for both finite element and finite difference calculations except the smooth wedge with semi-angle 30° and 40° solved by finite difference method. These deviations could be resulted from the inaccuracy of the calculation of the displacement field, the approximation of the irregular boundary of the slip-line field or the found off errors.

The false wedge solutions for plane strain wedge indentation problem are listed in Table 7 and Figure 53. Comparisons have been made among the numerical solutions and the theoretical solutions given by Grunzweig (44) and Haddow (138) and good agreement has been shown among them.

The numerically calculated pressure and pressure distribution for the cone indentation problem are listed in Table 3 and Figure 54. After
comparison it shows that the numerical solutions are closer to the theoretical solutions predicted by Lockett (66) than Miller's (80).

The numerical result of the plane-strain inverted extrusion with 50% reduction in area is shown in Table 9 which is quite close to the slip-line solution given by Hill (51).

The agreement between the numerical results and the theoretical slip-line solutions seems to indicate the strain fields calculated by using the deformation patterns will afford a proper way to correct the solutions of the plasticity problem which are under the influences of strain-hardening and rate effects.

The mean values of the effective strain, $\bar{\varepsilon}_e$, for the smooth wedge indentation case are listed in Table 10.

3. Numerical solutions for rigid, strain-hardening materials. The numerical results of the smooth wedge, conical indentation problems are presented in Table 11, Figure 55 and Table 12, Figure 56, respectively. Lockett (65) has indicated that his calculated values of the average conical indentation pressure were somewhat lower than the experimental values reported by Dugdale (35). Lockett also pointed out that the qualitatively similar conclusions results from the comparisons of Dugdale's wedge indentation experimental data (34) with the theoretical solution of Hill, et al. (50). The agreement between the numerical solutions obtained by following the method described in this study and the experimental results published by Dugdale implies that the theory proposed by the author is a proper one in correcting the strain-hardening effect.

-112-
The strain-hardening wedge indentation results predicted by the mean value of the effective strain, $\bar{\varepsilon}_e$, are presented in Table 13. Compared with the numerical results predicted by the finite difference program, the solutions predicted by utilizing the mean value of the effective strain the latter is approximately 10% lower than the former.

Johnson (60) has not presented the numerical values of the extrusion pressure for plane-strain extrusion problem with 50% reduction in area. If the extrusion pressure obtained by using the mean value of the effective strain, $\bar{\varepsilon}_e$, is increased by 10% then it will be close to the extrusion pressure specified by using the material constitutive equation for the correction of strain-hardening effect. All the results described above are listed in Table 14.

C. Numerical solutions for rigid, rate dependent material. The numerical results of the inverted extrusion process under conditions of plane-strain and axial symmetry are listed in Table 15. Compared with the published experimental results reported by Thomsen (113) the numerical solutions are close to the former. This agreement between the theoretically predicted and experimentally obtained values of the extrusion pressures indicates the theory proposed by the author is a proper one in correcting the rate effect.

D. Numerical solutions for strain-hardening and rate dependent material. The numerical results of the plane-strain inverted extrusion with 50% reduction in area and the indentation of perfectly rough wedge with semi-angle 22.5° are listed in Table 16 and 17. The agreement between the theoretical and experimental results of the extrusion, indentation pressure
indicates that the using of the combined effect of strain-hardening and strain-rate sensitivity in a product form with the two effects remain uncoupled yields a first order solution to plasticity problem which is under the influences of strain-hardening and rate effect.

7.2 Results for Soils

In this study the material constitutive equation is needed to modify the plasticity solution for the rigid, perfectly plastic material so that a better solution could be found when strain-hardening and rate effects are involved. Because the experimental data collected by the author are not sufficient in making comparison between the theoretical and experimental result for indentation problem, therefore the numerical solutions will be compared with the slip-line solutions for the rigid, perfectly plastic material which satisfies the Coulomb yield condition.

The problem of the indentation for a semi-infinite mass of soil by a smooth, rigid wedge under conditions of plane strain has been solved by Shield (190), but the generalization of this problem which is the indentation by a frictional, rigid wedge has not been found in the literature known to the author. Following the work of Shield (190), Ho (138) has generated the solution for the frictional wedge indentation and the author derived, confirmed Ho's solution.

The numerically calculated results for the plane strain wedge indentation problem are listed in Table 13 and 19.
The deviations of the numerical results from the slip-line solutions could be derived from the inaccuracy of the calculation of the displacement field, the approximation of the irregular boundary of the slip-line field or the round off errors.

The false wedge solutions are listed in Table 20.

The numerically calculated pressure and pressure distribution for the cone indentation problem are listed in Table 21 and 22.

7.3 Strain-Hardening Effects Corrected by Using the Mean Value of the Effective Strain

Although the results obtained by using the mean value of the effective strain, $\bar{\varepsilon}_e$, are less accurate than those calculated from the finite difference oriented computer program when the strain-hardening effect can not be neglected yet the former can be obtained more easily. For the rigid, linearly strain-hardening material the average pressures on the wedge are presented in Table 23, 24 and 25.

For rigid, linear strain-hardening material, the stress-strain relation can be written as

$$\kappa = \kappa_0 + H' \varepsilon_p$$

where $\kappa_0$ is the initial yield stress in shear, $H'$ is the slope of the stress-strain curve, and $\kappa$ is the subsequent yield stress in shear at effective strain $\varepsilon_p$. The stress-strain relation can be converted into dimensionless form as

$$\kappa^* = 1 + H^* \varepsilon_p$$
where $K^* = K/K_o$ and $H^* = H'/K_o$

For strain-hardening material with $K_o = 14$ ksi, $H^* = 0.1$ and the semi-wedge angle $\beta = 30^\circ$, the average wedge surface pressure calculated by Pan (84) equals to 37.33 ksi, where calculated by the author equals to 37.95 ksi. This indicates the method proposed by the author is an appropriate one for the correction of strain-hardening effect.

Summary and conclusions will be presented in the next chapter.
CHAPTER 3
SUMMARY AND CONCLUSIONS

In the previous chapters the strain hardening, rate effect, the combined effect of strain hardening and rate sensitivity on the plasticity analysis of Von Mises material were studied. Because deformation problems involving materials with rate dependent yield conditions cannot be solved by the method of characteristics therefore a different method was proposed.

Altan and Doulger (4) have studied domestic and foreign metal-forming articles and presented the yield stress data for a wide range of metals. They have concluded that it was empirically found that the strain dependency of the yield stress can be expressed as \( \sigma = K \varepsilon^n \) and the strain rate dependency of the yield stress can be expressed as \( \sigma = C \dot{\varepsilon}^m \), where \( K \) and \( C \) are material constants, \( n \) and \( m \) are index of coefficient for strain and strain rate dependency. Dwinn (39) has presented a series-method for constructing plastic slip-line fields. Therefore, more metal forming problems, such as upset forging, rolling, extrusion, can be solved by using the method proposed in this study provided that the material constitutive equations are specified.

An approximate method was described to solve axially symmetric problems by adding the correction factors, obtained by integrating the correction terms along the characteristics of the corresponding plane strain problem, to the plane strain solution.

The material constitutive equation for pure lead was obtained through a series of simple compression tests and the compressed specimens were uniformly deformed by machining grooves on the ends of the specimens.
Inverted plane strain extrusion and wedge indentation tests were then performed on pure lead and the experimentally determined constitutive equation was employed for the calculations of the extrusion, indentation pressures.

The agreement between the theoretical predictions and the experimental results indicates that the method proposed in this study is appropriate to the corrections for strain hardening and rate effects. The mean values of the effective strain for extrusion and indentation problem can be utilized to find the approximate solutions when strain hardening is involved.

The same method used to solve metal forming problems were employed to solve the indentation problem for the rigid, perfectly plastic material which satisfies the Coulomb yield condition.

Facets of the theory that need more attention in future studies are

(a) the inclusion of inertia effects
(b) the inclusion of thermodynamic effects
(c) the consideration of elastic strains
(d) the inclusion of the anisotropy of the material
(e) the consideration of all effects in a simple problem

Although this study of strain hardening and rate effects in plastic deformation is not definite, it does furnish a method to solve some practical problems. Hopefully, this study will point the way for a more complete development of strain hardening and rate dependent plasticity theory.
REFERENCES


-122-


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APPENDIX A

CHARACTERISTIC DIRECTIONS AND EQUATIONS ALONG THE CHARACTERISTICS FOR A

VON MISES STRAIN HARDENING MATERIAL

Stress Characteristics

The equilibrium equations for plane strain are

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \]

\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \]  \hspace{1cm} (1)

The Mohr circle relation for \( \sigma_x, \sigma_y, \tau_{xy} \) are (see Figure 14):

\[ \sigma_x = \sigma - K \sin 2 \psi \]

\[ \sigma_y = \sigma + K \sin 2 \psi \]  \hspace{1cm} (2)

\[ \tau_{xy} = K \cos 2 \psi \]

Substitute equations (1) into (2) yields

\[ \frac{\partial \sigma}{\partial x} - 2K \cos 2\psi \frac{\partial \psi}{\partial x} - \sin 2\psi \frac{\partial K}{\partial x} + \cos 2\psi \frac{\partial K}{\partial y} - 2K \sin 2\psi \frac{\partial \psi}{\partial y} = 0 \]

\[ \frac{\partial \sigma}{\partial y} + 2K \cos 2\psi \frac{\partial \psi}{\partial y} + \sin 2\psi \frac{\partial K}{\partial y} + \cos 2\psi \frac{\partial K}{\partial x} - 2K \sin 2\psi \frac{\partial \psi}{\partial x} = 0 \]  \hspace{1cm} (3)

Assume that function \( \sigma(x,y), \psi(x,y) \) are defined along a curve

\[ S = S(x,y), \text{ then} \]

\[ \frac{d\sigma}{ds} = \frac{\partial \sigma}{\partial x} \frac{dx}{ds} + \frac{\partial \sigma}{\partial y} \frac{dy}{ds} \]

\[ \frac{d\psi}{ds} = \frac{\partial \psi}{\partial x} \frac{dx}{ds} + \frac{\partial \psi}{\partial y} \frac{dy}{ds} \]  \hspace{1cm} (4)
So

\[
\begin{vmatrix}
1 - 2K \cos 2\psi & 0 & 2K \sin 2\psi \\
0 & -2K \sin 2\psi & 1 - 2K \cos 2\psi \\
\frac{dx}{dS} & 0 & \frac{dy}{dS} \\
0 & \frac{dx}{dS} & 0 & \frac{dy}{dS}
\end{vmatrix}
\begin{bmatrix}
\frac{\partial \sigma}{\partial x} \\
\frac{\partial \psi}{\partial x} \\
\frac{\partial \sigma}{\partial y} \\
\frac{\partial \psi}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
\sin 2\psi & \frac{3K}{\partial x} - \cos 2\psi \\
-\sin 2\psi & \frac{3K}{\partial x} - \cos 2\psi \\
\frac{dx}{dS} \\
\frac{dy}{dS}
\end{bmatrix}
\]

(5)

The characteristic directions are given by

\[
\begin{vmatrix}
1 - 2K \cos 2\psi & 0 & 2K \sin 2\psi \\
0 & -2K \sin 2\psi & 1 - 2K \cos 2\psi \\
\frac{dx}{dS} & 0 & \frac{dy}{dS} \\
0 & \frac{dx}{dS} & 0 & \frac{dy}{dS}
\end{vmatrix}
= 0
\]

(6)

This determinant can be expanded as

\[-2K \sin 2\psi \left(\frac{dy}{dS}\right)^2 + \frac{dx}{dS} \left(-2K \cos 2\psi \frac{dy}{dS}\right) + \frac{dx}{dS} \left[-2K \cos 2\psi \frac{dy}{dS}\right]

+ \frac{dx}{dS} 2K \sin 2\psi = 0\]

After rearrangement will yield

\[
\left(\frac{dy}{dx}\right)^2 + 2 \cot 2\psi \frac{dy}{dx} - 1 = 0
\]

So

\[
\frac{dy}{dx} = \tan \psi ; \quad \frac{dy}{dx} = \tan (\psi + \pi/2)
\]

(7)
Let $S_{\alpha}$ be at the angle $\psi$ from the $x$-axis and $S_{\beta}$ at angle $\psi + \frac{\pi}{2}$ (see Figure 14).

Then
\[
dS_{\alpha} = dx\cos\psi + dy\sin\psi
\]
\[
dS_{\beta} = -dx\sin\psi + dy\cos\psi
\]

And
\[
\frac{\partial}{\partial S_{\alpha}} = \cos\psi \frac{\partial}{\partial x} + \sin\psi \frac{\partial}{\partial y}
\]
\[
\frac{\partial}{\partial S_{\beta}} = -\sin\psi \frac{\partial}{\partial x} + \cos\psi \frac{\partial}{\partial y}
\]

Thus
\[
\begin{bmatrix}
1 & -2K \cos 2\psi & 0 & \sin 2\psi \frac{\partial K}{\partial x} & -\cos 2\psi \frac{\partial K}{\partial y} \\
0 & 2K \sin 2\psi & 1 & -\sin 2\psi \frac{\partial K}{\partial y} & -\cos 2\psi \frac{\partial K}{\partial x}
\end{bmatrix}
\begin{bmatrix}
\frac{dx}{ds} \\
\frac{dy}{ds}
\end{bmatrix}
= 0
\]
\[
\begin{bmatrix}
\frac{d\sigma}{ds} \\
\frac{dx}{ds}
\end{bmatrix}
\]

After expansion and rearrangement which gives
\[
\frac{\partial}{\partial S_{\alpha}} - 2K \frac{\partial}{\partial S_{\alpha}} = -\frac{\partial K}{\partial S_{\beta}} \quad \text{along } S_{\alpha}
\]
\[
\frac{\partial}{\partial S_{\beta}} + 2K \frac{\partial}{\partial S_{\beta}} = -\frac{\partial K}{\partial S_{\alpha}} \quad \text{along } S_{\beta}
\]

Velocity Characteristics

Following the same procedure used to derive the stress characteristics the velocity characteristic directions obtained are exactly the same as equations (7). The equations along the characteristics are
\[
\frac{\partial V_{\alpha}}{\partial S_{\alpha}} - V_{\beta} \frac{\partial}{\partial S_{\alpha}} = 0 \quad \text{along } S_{\alpha}
\]
\[
\frac{\partial V_{\beta}}{\partial S_{\beta}} + V_{\alpha} \frac{\partial}{\partial S_{\beta}} = 0 \quad \text{along } S_{\beta}
\]
when the relations below have been used

\[ V_x = V_\alpha \cos \psi - V_\beta \sin \psi \]

\[ V_y = V_\alpha \sin \psi + V_\beta \cos \psi \]

where \( V_\alpha \) and \( V_\beta \) are the velocities along \( S_\alpha \) and \( S_\beta \).
APPENDIX B

CHARACTERISTIC DIRECTIONS AND EQUATIONS ALONG THE CHARACTERISTICS FOR
MATERIALS SATISFYING THE COULOMB YIELD CONDITION

The Mohr circle relation for $\sigma_x$, $\sigma_y$, $\tau_{xy}$ are

$$
\sigma_x = P \left(1 + \sin \phi \cos 2\theta \right) - c \cot \phi
$$

$$
\sigma_y = P \left(1 - \sin \phi \cos 2\theta \right) - c \cot \phi
$$

$$
\tau_{xy} = P \sin \phi \sin 2\theta
$$

(1)

The equilibrium equations for plane strain are

$$
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0
$$

$$
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
$$

(2)

Substitute (2) into (1) yields:

$$
\frac{\partial P}{\partial x} \left(1 + \sin \phi \cos 2\theta \right) - \frac{\partial \theta}{\partial x} \left(2P \sin \phi \sin 2\theta \right) + \frac{\partial P}{\partial y} \left(\sin \phi \sin 2\theta \right)
$$

$$
+ \frac{\partial \theta}{\partial y} \left(2 \sin \phi \cos 2\theta \right) = 0
$$

$$
\frac{\partial P}{\partial x} \left(\sin \phi \sin 2\theta \right) + \frac{\partial \theta}{\partial x} \left(2 \sin \phi \cos 2\theta \right) + \frac{\partial P}{\partial y} \left(1 - \sin \phi \cos 2\theta \right)
$$

$$
+ \frac{\partial \theta}{\partial y} \left(2 \sin \phi \sin 2\theta \right) = 0
$$

(3)

Since

$$
\frac{d\sigma}{dS} = \frac{d\sigma}{dx} \frac{dx}{dS} + \frac{d\sigma}{dy} \frac{dy}{dS}
$$

$$
\frac{d\theta}{dS} = \frac{d\theta}{dx} \frac{dx}{dS} + \frac{d\theta}{dy} \frac{dy}{dS}
$$

(4)

-134-
Follow the same procedure used for Von Mises material will obtain

\[ \frac{dv}{dx} = \tan [\theta + (\pi/4 - \phi/2)] \quad ; \quad \frac{dv}{dx} = \tan [\theta - (\pi/4 - \phi/2)] \quad (5) \]

And

\[ \frac{1}{2} \cot \phi \ln P + \theta = \text{const.} \quad \text{along } \alpha \text{ line} \]

\[ \frac{1}{2} \cot \phi \ln P - \theta = \text{const.} \quad \text{along } \beta \text{ line} \quad (6) \]

\[ dV_1 - (V_1 \tan \phi + V_2 \sec \phi) \, d\theta = 0 \]

\[ dV_2 + (V_1 \sec \phi + V_2 \tan \phi) \, d\theta = 0 \quad (7) \]
APPENDIX C

DERIVATION OF EFFECTIVE STRESS AND EFFECTIVE STRAIN

I. Von Mises Material

The Von Mises yield condition is

\[ f (\sigma_{ij}) = J^\frac{1}{2} = k \]

where

\[ J^\frac{1}{2} = \left\{ \frac{1}{6} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right\}^{\frac{1}{2}} \]

Under uniaxial compression test, the only nonzero stress is the axial stress, \( \sigma_x \). Assume the effective stress takes the form of

\[ \sigma_e = C J^\frac{1}{2} \]

Then

\[ \sigma_e = C \left\{ \frac{1}{6} (\sigma_x - 0)^2 + 0 + (\sigma_x)^2 + 0 + 0 + 0 \right\}^{\frac{1}{2}} = C \left( \frac{(\sigma_x)^2}{3} \right)^{\frac{1}{2}} \]

Since all the results obtained by different loading program can be correlated by means of the effective stress versus effective strain curve, therefore

\[ \sigma_e = \sigma_x = C \frac{\sigma_x}{\sqrt{3}} \]

So \( C = \sqrt{3} \)

\[ \sigma_e = \sqrt{3} \left\{ \frac{1}{6} (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right\}^{\frac{1}{2}} \]

\[ = \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6 (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{\frac{1}{2}} \]

which is the form of the effective stress proposed by Hill (50).
If the only nonzero stress is the shear stress $\tau_{xy}$, then

$$\sigma_e = \frac{1}{\sqrt{2}} \left[ 0 + 6 \tau_{xy} \right]^{\frac{1}{2}} = \sqrt{3} \tau_{xy}$$

where $\tau_{xy}$ is the maximum shear stress.

According to the flow rule:

$$d \varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \varepsilon_{ij}} = d\lambda \frac{\partial J_2^{\frac{1}{2}}}{\partial \varepsilon_{ij}} = d\lambda \frac{S_{ij}}{2J_2^{\frac{1}{2}}}$$

For the rigid, perfectly plastic material, the total strain is the plastic strain

$$\varepsilon_{ij}^p = \int d \varepsilon_{ij}^p$$

The increment plastic work is $dW = \sigma_{ij} d \varepsilon_{ij}^p$

$$dW = \sigma_{ij} d\lambda \frac{S_{ij}}{2J_2^{\frac{1}{2}}} = \frac{d\lambda}{2J_2^{\frac{1}{2}}} \left( S_{ij} + \frac{I_1}{3} \delta_{ij} \right) S_{ij} = d\lambda J_2^{\frac{1}{2}} = d\lambda \dot{\varepsilon}$$

But

$$d \varepsilon_{ij}^p \cdot d \varepsilon_{ij}^p = d\lambda^2 \frac{2J}{4J_2} = \frac{1}{2} d\lambda^2$$

$$d\lambda = \sqrt{2} d \varepsilon_{ij}^p \cdot d \varepsilon_{ij}^p$$

$$\sigma_e \cdot d \varepsilon_e = dW = d\lambda \dot{\varepsilon}$$

So

$$d \varepsilon_e = \frac{\Delta f}{\sigma_e} = \frac{d\lambda J_2^{\frac{1}{2}}}{\sqrt{3} J_2^{\frac{1}{2}}} = \frac{d\lambda}{\sqrt{3} d \varepsilon_{ij}^p \cdot d \varepsilon_{ij}^p}$$

Due to incompressibility $d \varepsilon_e$ can be rearranged as

$$d \varepsilon_e = \sqrt{\frac{2}{3}} \left\{ \left( d\varepsilon_x - d\varepsilon_y \right)^2 + \left( d\varepsilon_y - d\varepsilon_z \right)^2 + \left( d\varepsilon_z - d\varepsilon_x \right)^2 + \frac{3}{2} \right\}$$

$$\left( d\gamma_{xy}^2 + d\gamma_{yz}^2 + d\gamma_{zx}^2 \right)$$

which is the form of $d \varepsilon_e$ proposed by Hill (50), then
\[ \varepsilon = \frac{2}{3} \left\{ \left( \varepsilon_x - \varepsilon_y \right)^2 + \left( \varepsilon_y - \varepsilon_z \right)^2 + \left( \varepsilon_z - \varepsilon_x \right)^2 + \frac{3}{2} \left( \gamma_{xy}^2 + \gamma_{yz}^2 \right) + \gamma_{zx}^2 \right\}^{\frac{1}{2}} \]

II. Coulomb Material

The Extended Coulomb yield condition is

\[ f = \alpha I_1 + J_2 = k \]

where \( \alpha \) and \( k \) are positive constants at each point of the material; \( I_1 \) is the sum of the principal stresses; \( J_2 \) is

\[ I_1 = \sigma_{ii} \]

\[ J_2 = \frac{1}{2} S_{ij} S_{ij} \]

According to Drucker and Prager, the extended yield function can be reduced to the Coulomb yield condition in the case of plane strain, provided that

\[ C = \frac{k}{(1 - 12\alpha^2)^{\frac{1}{2}}} ; \quad \sin \phi = \frac{3\alpha}{(1 - 3\alpha^2)^{\frac{1}{2}}} ; \quad \cos \phi = \frac{(1 - 12\alpha^2)^{\frac{1}{2}}}{(1 - 3\alpha^2)^{\frac{1}{2}}} \]

and when tension is positive.

Assume the effective stress takes the form of

\[ \sigma_e = \bar{C} \left\{ \alpha I_1 + J_2^{\frac{1}{2}} \right\} \]

In the uniaxial compression test, let the only nonzero stress be

\[ \sigma_x = -\sigma, \sigma > 0. \]

Then

\[ \sigma_e = -\sigma = \bar{C} \left\{ \alpha(-\sigma + 0 + 0) + \frac{1}{\sqrt{6}} \left[ (-\sigma)^2 + 0 + (\sigma)^2 + 0 + 0 + 0 \right]^{\frac{1}{2}} \right\} \]

\[ = \bar{C} \left\{ -\alpha \sigma + \frac{1}{\sqrt{3}} \sigma \right\} \bar{C} \sigma \left\{ -\frac{\sqrt{3} \alpha + 1}{3} \right\} \]

\[-138-\]
So
\[
\sigma_e = \frac{\sqrt{3}}{\sqrt{3} \alpha - 1}
\]
Thus
\[
\sigma_e = \frac{\sqrt{3}}{\sqrt{3} \alpha - 1} \left\{ \alpha (\sigma_x + \sigma_y + \sigma_z) + \frac{1}{6} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \right] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right\}^{\frac{1}{2}} = \frac{(\frac{\sqrt{3}}{\sqrt{3} \alpha - 1}) f}{\sqrt{3}}
\]
If the only nonzero stress is \( \tau_{xy} \) in the plane strain case, then
\[
\sigma_e = \frac{\sqrt{3}}{\sqrt{3} \alpha - 1} \left\{ 0 + [0 + \tau_{xy}^2 + 0 + 0]\right\}^{\frac{1}{2}} = \sqrt{3} \frac{\tau_{xy}}{\sqrt{3} \alpha - 1}
\]
According to the flow rule
\[
d\varepsilon_{ij}^p = d\lambda \left[ \frac{\partial f}{\partial \sigma_{ij}} \right] = d\lambda \left[ \alpha \delta_{ij} + \frac{S_{ij}}{2J_2^{1/2}} \right]
\]
For the rigid, perfectly plastic material,
\[
\varepsilon_{ij}^p = \int d\varepsilon_{ij}^p
\]
The incremental work \( dW = \sigma_{ij} d\varepsilon_{ij}^p \)
\[
dW = \sigma_{ij} d\varepsilon_{ij}^p = \sigma_{ij} d\lambda \left[ \alpha \delta_{ij} + \frac{S_{ij}}{2J_2^{1/2}} \right] = d\lambda \left[ \alpha \delta_{ij} + \frac{S_{ij}}{2J_2^{1/2}} \right] = d\lambda f
\]
\[
d\varepsilon_{ij}^p d\varepsilon_{ij}^p = d\lambda^2 \left[ \frac{\partial f}{\partial \sigma_{ij}} \right]^2 = d\lambda^2 \left[ \alpha \delta_{ij} + \frac{S_{ij}}{2J_2^{1/2}} \right]^2
\]
\[
= d\lambda^2 \left[ 3\alpha^2 + \frac{1}{2} \right]
\]
So
\[
d\lambda = \sqrt{\frac{2}{6\alpha^2 + 1}} d\varepsilon_{ij}^p d\varepsilon_{ij}^p
\]
But
\[
dW = \sigma_e d\varepsilon_e = d\lambda f
\]
Thus
\[ \frac{d\varepsilon_e}{\sigma_e} = \frac{d\lambda f}{\sqrt{3} \sqrt{\sqrt{3\alpha - 1}} f} = \frac{\sqrt{3\alpha - 1}}{\sqrt{3}} \frac{d\lambda}{\sqrt{3\alpha - 1}} \]
\[ = \frac{\sqrt{3\alpha - 1}}{\sqrt{3}} \sqrt{\frac{2}{6\alpha^2 + 1}} \frac{d\varepsilon_{ij}^p}{\varepsilon_{ij}^p} \frac{d\varepsilon_{ij}^p}{\varepsilon_{ij}^p} = \frac{\sqrt{2}}{3} \frac{\sqrt{3\alpha - 1}}{\sqrt{6\alpha^2 + 1}} \frac{d\varepsilon_{ij}^p}{\varepsilon_{ij}^p} \frac{d\varepsilon_{ij}^p}{\varepsilon_{ij}^p} \]

Then, define the effective strain as
\[ \varepsilon_e = \frac{\sqrt{\frac{2}{3} \left( \frac{\sqrt{3\alpha - 1}}{\sqrt{6\alpha^2 + 1}} \right)}}{\sqrt{\frac{\varepsilon_{ij}^p}{\varepsilon_{ij}^p}} \cdot \sqrt{\frac{\varepsilon_{ij}^p}{\varepsilon_{ij}^p}}} \]

If \( \alpha = 0 \)
\[ \sigma_e = - \sqrt{3} J_2 \]
\[ \varepsilon_e = - \sqrt{\frac{2}{3} \frac{\varepsilon_{ij}^p}{\varepsilon_{ij}^p} \frac{\varepsilon_{ij}^p}{\varepsilon_{ij}^p}} \]

which reduced to the forms for Von Mises Material as expected.
APPENDIX D

WEDGE SURFACE PRESSURE DURING SOIL INDENTATION PROCESS

I. Perfectly smooth case (Figure 37):

\[ R_1 = (c \cot \phi + R_1) \sin \phi \quad , \quad R_2 = \frac{c \cos \phi}{1 - \sin \phi} \]

\[ P_1 = R_1 + c \cot \phi \quad \text{(on free surface)} \]

\[ P_2 = P_1 e^{2 \cot \phi} = (R_1 + c \cot \phi) e^{2 \cot \phi} \]

\[ R_2 = P_2 \sin \phi \]

Pressure \( P \) on the wedge surface:

\[ \sigma = P_2 - c \cot \phi + R_2 = P_2 (1 + \sin \phi) - c \cot \phi \]

\[ = (c \cot \phi + \frac{c \cos \phi}{1 - \sin \phi}) e^{2 \cot \phi} (1 + \sin \phi) - c \cot \phi \]

\[ = c \cot \phi \frac{1 + \sin \phi}{1 - \sin \phi} e^{2 \cot \phi} - c \cot \phi \]

\[ = c \cot \phi [\tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) e^{2 \cot \phi} - 1] \]

II. Frictional case

\[ P_1 = R_1 + c \cot \phi = \frac{c \cos \phi}{1 - \sin \phi} + c \cot \phi \]

\[ P_2 = P_1 e^{2 \cot \phi} = \left( \cot \phi + \frac{c \cos \phi}{1 - \sin \phi} \right) e^{2 \cot \phi} ; \quad \frac{R_2}{P_2} = \frac{P_2}{P_1} = e^{2 \cot \phi} \]

\[ \sigma = P_2 - c \cot \phi + R_2 \cos (\frac{\pi}{2} + \phi - 2 \lambda) \]

\[ = (c \cot \phi + \frac{\cos \phi}{1 - \sin \phi}) e^{2 \cot \phi} - c \cot \phi + \frac{c \cos \phi}{1 - \sin \phi} e^{2 \cot \phi} \sin (2 \lambda - \phi) \]

\[ = R_1 \left( 1 + \csc \phi (e^{2 \cot \phi - 1} + e^{2 \cot \phi} \sin (2 \lambda - \phi) \right) \]
\[ \tau = R_2 \sin \left( \frac{\pi}{2} + \phi - 2\lambda \right) = R_2 \cos (2\lambda - \phi) = R_1 e^{2\arctan\phi} \cos (2\lambda - \phi) \]

\[ \mu = \frac{I}{R} = \frac{e^{2\arctan\phi} \cos (2\lambda - \phi)}{1 + \csc\phi (e^{2\arctan\phi} - 1) + e^{2\arctan\phi} \sin (2\lambda - \phi)} \]
APPENDIX E

FLOW CHART FOR CALCULATING THE TOTAL WORK IN THE PLASTIC REGION OF

THE SLIP-LINE SOLUTION OF THE WEDGE INDENTATION PROBLEM

Methods of solution:
A. Finite Element Method
B. Finite Difference Method
(A)

START

READ \( \beta, \psi, \lambda \)

Calculate outline of slip-line field by evaluating the coordinates \( A, O, D, C, G, F, E, F \)

Calculate the theoretical force work and coeff. of friction

Generate the corner coordinates of the undeformed grids in Region II

Calculate the displacement field of the grid corners as the basis to specify the displacement field within the elements

Calculate the Matrix \([J]\) and the inverse Matrix of \([J]\), \([J]\)^{-1}

Calculate the derivative of shape function \( N_i \)

\[
\left\{ \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{array} \right\} = [J] \left\{ \begin{array}{c}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{array} \right\}
\]

1
Calculate the displacement derivative
\[ \frac{\partial u}{\partial x} = \sum_{i=1}^{N} u_i \frac{\partial N_i}{\partial x} ; \quad \frac{\partial v}{\partial x} = \sum_{i=1}^{N} v_i \frac{\partial N_i}{\partial x} \]
\[ \frac{\partial u}{\partial y} = \sum_{i=1}^{N} u_i \frac{\partial N_i}{\partial y} ; \quad \frac{\partial v}{\partial y} = \sum_{i=1}^{N} v_i \frac{\partial N_i}{\partial y} \]

Calculate the strain, effective strain at the sampling points with each element

Calculate the total plastic work in Region II by Gauss-Legendre integration

Calculate the plastic work in Region I, II and the frictional work along the flank of the wedge, then sum together with work in Region II. Get the total work then convert the work back to the required indentation force

Calculate the deviation of the calculated force from the theoretical force, then print out results

END
START

READ \( \beta, \psi, \lambda \)

Calculate the outline of slip-line field by evaluating the coordinates of pts. A, O, D, C, G, F, E, B

Calculate the theoretical force, work and coef. of friction

Generate the corner coordinates of the undeformed grids in Region II

Generate the final corner coordinates of the deformed grids in Region II

Transform the cartesian displacement field into polar coordinate

Evaluate the strain and effective strain field in Region II

Calculate the plastic work in Region II through numerical integration

1
1

Calculate the strain and the effective strain in triangle Region I and III, then evaluate the plastic work in Region I, III

Calculate the frictional work along the flank of the wedge, then find out the total work by adding plastic work in Region I, II, III and frictional work

Convert the total work back to the required indentation force

Calculate the "computed force" deviation from the theoretical one, then print out results

END
FLOW CHART FOR CALCULATING THE TOTAL WORK IN THE PLASTIC REGION OF
THE SLIP-LINE SOLUTION OF THE PLANE STRAIN INVERT EXTRUSION PROBLEM

Method of Solution

Finite Difference Method
START

Read H, A, A' and extrusion speed V

Generate the corner coordinates of the undeformed grids w.r.t. the X, Y axes

Calculate TP, the available time for each corner once it passes the plastic circular fan arc ADB; compute TB, time required to travel from the fan arc to the lower boundary. Radius OB.

\[ TP \leq 0 \]

Out of fan region OADB move with 2V towards right horizontally

\[ (TP-TB) > 0 \]

Evaluate TP and (TP-TB)

\[ (TP-TB) \leq 0 \]

From time TP, calculate the radius and angle to specify the corner final position in the fan OADB

Calculate the displacement field, strain field and effective strain field for the three different regions

Calculate the total plastic work by numerical integration

Convert the total plastic work back to the extrusion pressure

Compare the calculated extrusion pressure with the theoretical one, print out the results

END
APPENDIX F

DERIVATION OF INTEGRATION FORMULA FOR FINITE DIFFERENCE METHOD IN POLAR COORDINATE SYSTEM AND DIFFERENCE EQUATIONS

According to Simpson's rule (in Cartesian Coordinate System):

\[
\int_{x-h}^{x+h} f(z) \, dz = \frac{h}{3} [f(x+h) + 4f(x) + f(x-h)] - \frac{h^5}{90} f^4(\xi)
\]

So

\[
\int_{\theta-\Delta\theta}^{\theta+\Delta\theta} f(r, \theta) \, d\theta = \frac{\Delta\theta}{3} [f(r, \theta + \theta) + 4f(r, \theta) + f(r, \theta - \Delta\theta)] - O(\Delta\theta^5)
\]

\[
\int_{r-\Delta r}^{r+\Delta r} f(r, \theta) \, r \, dr \, d\theta = \int_{r-\Delta r}^{r+\Delta r} \frac{\Delta\theta}{3} \left\{ f(r, \theta + \Delta\theta) + 4f(r, \theta) + f(r, \theta - \Delta\theta) \right\} \, rdr
\]

Since

\[
\int_{r-\Delta r}^{r+\Delta r} f(r, \theta + \Delta\theta) \, rdr = \int_{r-\Delta r}^{r+\Delta r} g(r, \theta + \Delta\theta) \, dr \text{ where } g (r, \theta + \Delta\theta) = rf(r, \theta + \Delta\theta)
\]

So

\[
\int_{r-\Delta r}^{r+\Delta r} f(r, \theta + \Delta\theta) \, rdr = \frac{\Delta r}{3} \left[ ((r+\Delta r) f(r+\Delta r, \theta+\Delta\theta) + 4rf(r, \theta+\Delta\theta)
\]

\[
+ (r-\Delta r) f(r-\Delta r, \theta+\Delta\theta) \right]
\]

Thus

\[
\int_{\theta-\Delta\theta}^{\theta+\Delta\theta} \int_{r-\Delta r}^{r+\Delta r} f(r, \theta) \, rdr \, d\theta
\]

\[
= \frac{\Delta\theta \Delta r}{9} \left\{ (r+\Delta r) \left[ f(r+\Delta r, \theta+\Delta\theta) + 4f(r+\Delta r, \theta) + f(r+\Delta r, \theta-\Delta\theta) \right]
\]

\[
+ (4r) \left[ f(r, \theta + \Delta\theta) + 4f(r, \theta) + f(r, \theta - \Delta\theta) \right] + (r-\Delta r) \left[ f(r-\Delta r, \theta + \Delta\theta) + 4f(r-\Delta r, \theta) + f(r-\Delta r, \theta - \Delta\theta) \right] \}
\]
Difference Equations

Second order Forward difference

\[ Df (x_n) = \frac{1}{2h} [-3f (x_n) + 4f (x_{n+1}) - f (x_{n+2})] + \frac{h^2}{3} f^3 (x_n) \]

Second order Backward difference

\[ Df (x_n) = \frac{1}{2h} [f (x_{n-2}) - 4f (x_{n-1}) + 3f (x_n)] + \frac{h^2}{3} f^3 (x_n) \]

Second order Central difference

\[ Df (x_n) = \frac{1}{2h} [f (x_{n+1}) - f (x_{n-1})] - \frac{h^2}{6} f^3 (x_n) \]
APPENDIX G

Triangular Element

The typical two dimensional triangular element, with nodes i, j, k numbered in anti-clockwise direction is shown in figure A. The displacement vector at each node can be decomposed into two independent components, say in x, y directions. Thus, the displacement vector of this typical triangular element can be represented as:

\[
\begin{align*}
\{U\} &= \begin{bmatrix} U_i \\ U_j \\ U_k \end{bmatrix} = \begin{bmatrix} U_i \\ V_i \\ U_j \\ V_j \\ U_k \\ V_k \end{bmatrix}
\end{align*}
\]

Once the six components of displacement are specified, then the displacements of arbitrary point within this element can be uniquely determined through linear interpolations. The simplest way is to use two linear polynomials.

\[
\begin{align*}
U &= \alpha_1 + \alpha_2 x + \alpha_3 y \\
V &= \alpha_4 + \alpha_5 x + \alpha_6 y
\end{align*}
\]

Then there are six equations, two at each node, and six unknowns \(\alpha_1\) through \(\alpha_6\). After substituting the known nodal displacement into the above equations, then we solve \(\alpha_1\) through \(\alpha_6\) in terms of the nodal displacements \(U_i, U_j, V_i, V_j\) and \(U_k, V_k\).

Then substitute \(\alpha_1\) through \(\alpha_6\) back to equations for U, V and rearrange them, we, then, come up with

\[
\begin{align*}
U &= \frac{1}{2\Delta}(a_i + b_i x + c_i y) U_i + (a_j + b_j x + c_j y) U_j + (a_k + b_k x + c_k y) U_k \\
V &= \frac{1}{2\Delta}(a_i + b_i x + c_i y) V_i + (a_j + b_j x + c_j y) V_j + (a_k + b_k x + c_k y) V_k
\end{align*}
\]
FIGURE A

A two dimensional triangular element
where

\[
2 \Delta = \begin{vmatrix}
1 & x_i & y_i \\
1 & x_j & y_j \\
1 & x_k & y_k \\
\end{vmatrix} = 2 \times \text{Area of the triangle } ijk
\]

\[a_i = x_j y_k - x_k y_j\]
\[b_i = y_j - y_k\]
\[c_i = x_k - x_j\]

\(U, V\) can be rewritten in matrix form as

\[
\begin{bmatrix}
U_i \\
V_i \\
\end{bmatrix} = \begin{bmatrix}
N_i & 0 & N_j & 0 & N_k & 0 \\
0 & N_i & 0 & N_j & 0 & N_k \\
\end{bmatrix} \begin{bmatrix}
U_j \\
V_j \\
U_k \\
\end{bmatrix} = [N] \{u\}_e
\]

where

\[N_i = \frac{1}{2 \Delta} (a_i + b_i x + c_i y)\]
\[N_j = \frac{1}{2 \Delta} (a_j + b_j x + c_j y)\]
\[N_k = \frac{1}{2 \Delta} (a_k + b_k x + c_k y)\]

Since the strain tensor for large plastic deformation is

\[\varepsilon_{ij} = \frac{1}{2} (U_{i;j} + U_{j;i} + U_{k;i} U_{k;j})\]

In cartesian coordinate system, the covariant derivative is simply partial derivative, so

\[\varepsilon_{11} = \frac{1}{2} (U_{1,1} + U_{1,1} + U_{1,1}^2 + U_{2,1}^2)\]
\[ \varepsilon_{12} = \frac{1}{2} \left( U_{1,2} + U_{2,1} + U_{1,1} U_{1,2} + U_{2,1} U_{2,2} \right) \]
\[ \varepsilon_{22} = \frac{1}{2} \left( U_{2,2} + U_{2,2} + U_{1,2}^2 + U_{2,2}^2 \right) \]

or
\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right] \]
\[ \varepsilon_{xy} = \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) \right] \]
\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] \]

but
\[ \frac{\partial u}{\partial x} = \sum_{i=1}^{3} N_i U_i \]
\[ \frac{\partial u}{\partial y} = \sum_{i=1}^{3} U_i \frac{\partial N_i}{\partial y} \]
\[ \frac{\partial v}{\partial x} = \sum_{i=1}^{3} V_i \frac{\partial N_i}{\partial x} \]
\[ \frac{\partial v}{\partial y} = \sum_{i=1}^{3} V_i \frac{\partial N_i}{\partial y} \]

Therefore, the only information needed to calculate strain tensors should be \( \frac{\partial N_i}{\partial x} \) and \( \frac{\partial N_i}{\partial y} \), and they should be constant for triangular element because \( N_i \) are linear functions of \( x \) and \( y \).

Since \( \frac{\partial N_i}{\partial x} = \frac{b_i}{2\Delta} \) and \( \frac{\partial N_i}{\partial y} = \frac{c_i}{2\Delta} \), so the strain tensors can be represented in a matrix form such as
\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{xy} \\
\varepsilon_{yy}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right] \\
\left( \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial y} \right) \left( \frac{\partial v}{\partial y} \right) \\
\frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right]
\end{bmatrix}
\]

But for wedge indentation problem, the triangle regions I, III are formed under a process of simple shear. It is clear, then, that the
base of the two triangles, located on the plastic boundary, are fixed, so the associated displacements along the boundary should be equal to zero. So the only nodal displacement for triangular element at region III and I is the top nodal displacement, i.e. node i.

Then

\[ \frac{\partial u}{\partial x} = \frac{b_i U_i}{2\Delta} \]

\[ \frac{\partial u}{\partial y} = \frac{c_i V_i}{2\Delta} \]

\[ \frac{\partial V}{\partial x} = \frac{b_i V_i}{2\Delta} \quad \text{i is not summed} \]

\[ \frac{\partial V}{\partial y} = \frac{c_i V_i}{2\Delta} \]

\[ \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & 0 & 0 & 0 \\ b_i & 0 & 0 & 0 & 0 \\ 0 & c_i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_i \\ V_i \\ U_j \\ V_j \\ U_k \end{bmatrix} + \frac{1}{(2\Delta)^2} \begin{bmatrix} \frac{b_i^2}{2} & \frac{b_i^2}{2} & 0 & 0 & 0 \\ \frac{b_i C_i}{2} & \frac{b_i C_i}{2} & 0 & 0 & 0 \\ \frac{c_i^2}{2} & \frac{c_i^2}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_i^2 \\ V_i^2 \\ U_j^2 \\ V_j^2 \\ U_k^2 \end{bmatrix} \]

Quadrilateral Element

The typical two dimensional arbitrary quadrilateral element, with nodes i, j, k, l numbered in anti-clockwise direction, is shown in Figure B. A new coordinate system \((\xi, \eta)\) is located at the center of the element so that the displacement \(U, V\) will vary linearly along the element sides. The displacement vector \(U\) of any point within this element can be determined by the known nodal displacements and the new coordinates \(\xi, \eta\).
FIGURE B

A general quadrilateral element and coordinate system
\[ u = \frac{1}{4} \left[ (1-\xi)(1-n)U_i + (1+\xi)(1-n)U_j + (1+\xi)(1+n)U_k + (1-\xi)(1+n)U_l \right] \]
\[ v = \frac{1}{4} \left[ (1-\xi)(1-n)V_i + (1+\xi)(1-n)V_j + (1+\xi)(1+n)V_k + (1-\xi)(1+n)V_l \right] \]

The coordinate transformation formula takes the similar form:
\[ x = \frac{1}{4} \left[ (1-\xi)(1-n)X_i + (1+\xi)(1-n)X_j + (1+\xi)(1+n)X_k + (1-\xi)(1+n)X_l \right] \]
\[ y = \frac{1}{4} \left[ (1-\xi)(1-n)Y_i + (1+\xi)(1-n)Y_j + (1+\xi)(1+n)Y_k + (1-\xi)(1+n)Y_l \right] \]

Since \[ \frac{\partial N_i}{\partial x} = \sum_{i=1}^{4} \frac{\partial}{\partial x} (N_i U_i) = \sum_{i=1}^{4} U_i \frac{\partial N_i}{\partial x}, \] so

\[ \left\{ \begin{array}{c}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{array} \right\} = \left[ \begin{array}{cc}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{array} \right] \left[ \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{array} \right] = \left[ \begin{array}{c}
J
\end{array} \right] \left[ \begin{array}{c}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{array} \right] \]

\[ \frac{\partial x}{\partial \xi} = \frac{1}{4} \left[ - (1-n)X_i + (1-n)X_j + (1+n)X_k - (1+n)X_l \right] \]
\[ = \frac{1}{4} \left[ - (1-\xi)X_i - (1+\xi)X_j + (1+\xi)X_k + (1-\xi)X_l \right] \]

\[ [J] = \frac{1}{4} \left[ \begin{array}{cccc}
- (1-n) & (1-n) & (1+n) & (1+n) \\
- (1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi)
\end{array} \right] \left[ \begin{array}{c}
X_i \\
Y_i \\
X_j \\
Y_j \\
X_k \\
Y_k \\
X_l \\
Y_l
\end{array} \right] \]
\[
\begin{align*}
\frac{\partial N_i}{\partial x} &= [J]^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \\
\frac{\partial N_j}{\partial \xi} &= \delta \left[ \frac{(1 - \xi)(1 - \eta)}{\partial \xi} \right] / 4 = -(1 - \eta) / 4 \\
\frac{\partial N_j}{\partial \eta} &= \delta \left[ \frac{(1 + \xi)(1 - \eta)}{\partial \eta} \right] / 4 = (1 - \eta) / 4 \\
\frac{\partial N_k}{\partial \xi} &= \delta \left[ \frac{(1 + \xi)(1 + \eta)}{\partial \xi} \right] / 4 = (1 + \eta) / 4 \\
\frac{\partial N_k}{\partial \eta} &= \delta \left[ \frac{(1 - \xi)(1 + \eta)}{\partial \eta} \right] / 4 = -(1 + \eta) / 4 \\
\frac{\partial N_l}{\partial \xi} &= -(1 - \xi) / 4 \\
\frac{\partial N_l}{\partial \eta} &= -(1 + \xi) / 4 \\
\frac{\partial N_m}{\partial \xi} &= (1 + \xi) / 4 \\
\frac{\partial N_m}{\partial \eta} &= (1 - \xi) / 4
\end{align*}
\]

It is clear that \( \partial u \), \( \partial v \), \( \partial x \), \( \partial y \) are functions of \( \xi \) and \( \eta \) only provided that all of the nodal displacements are specified.

Because it is always possible to find out the displacement field in the plastic region of the slip-line solution to the wedge indentation problem, so the nodal displacements of the quadrilaterals chosen in region II can always be specified. Thus, it is possible to
evaluate the strain field over the entire interested region.

Once the strain field is specified, then calculate the effective strain field and do integration through the entire volume (unit depth thickness) to find out the total work required to cause the large plastic deformation.
TABLE 1
COMPARISON OF EXTRUSION PRESSURE BASED ON PROPOSED ASSUMPTION

(A) PLANE-STRAIN CASE

<table>
<thead>
<tr>
<th>Extrusion Ratio</th>
<th>Velocity</th>
<th>Pexp</th>
<th>Pcal</th>
<th>( \frac{P_{cal} - P_{exp}}{P_{exp}} \times 100% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3</td>
<td>0.067</td>
<td>4,830</td>
<td>5,114</td>
<td>5.88</td>
</tr>
<tr>
<td></td>
<td>0.092</td>
<td>4,990</td>
<td>5,312</td>
<td>6.46</td>
</tr>
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<td></td>
<td>0.240</td>
<td>5,750</td>
<td>5,960</td>
<td>3.65</td>
</tr>
<tr>
<td></td>
<td>0.035</td>
<td>2,785</td>
<td>2,846</td>
<td>2.17</td>
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<tr>
<td>2.15</td>
<td>0.078</td>
<td>3,060</td>
<td>3,133</td>
<td>2.38</td>
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<td></td>
<td>0.154</td>
<td>3,310</td>
<td>3,399</td>
<td>2.70</td>
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<td>0.050</td>
<td>2,000</td>
<td>2,046</td>
<td>2.30</td>
</tr>
<tr>
<td>1.43</td>
<td>0.098</td>
<td>2,100</td>
<td>2,218</td>
<td>5.63</td>
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<tr>
<td></td>
<td>0.192</td>
<td>2,300</td>
<td>2,404</td>
<td>4.55</td>
</tr>
</tbody>
</table>

(B) AXIAL SYMMETRY CASE

<table>
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<th>Extrusion Ratio</th>
<th>Velocity</th>
<th>Pexp</th>
<th>Pcal</th>
<th>( \frac{P_{cal} - P_{exp}}{P_{exp}} \times 100% )</th>
</tr>
</thead>
<tbody>
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<td>10,050</td>
<td>10,457</td>
<td>4.05</td>
</tr>
<tr>
<td></td>
<td>0.150</td>
<td>11,090</td>
<td>11,672</td>
<td>5.25</td>
</tr>
<tr>
<td></td>
<td>0.240</td>
<td>11,560</td>
<td>12,342</td>
<td>6.83</td>
</tr>
<tr>
<td></td>
<td>0.048</td>
<td>5,910</td>
<td>5,957</td>
<td>0.79</td>
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<td>4.6</td>
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<td>6,500</td>
<td>6,649</td>
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<tr>
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<td></td>
<td>0.240</td>
<td>3,870</td>
<td>3,799</td>
<td>-1.84</td>
</tr>
</tbody>
</table>
**TABLE 2**

**TORSION SHEAR STRESS (T/IN²) VERSUS SHEAR STRAIN**

![Diagram showing axes x, y, z]

Definition of axes in copper and aluminum slabs, and convention for specifying relation of wedge axes to slab axes. The wedge orientation is quoted $x_1$, $x_2$ where $x_1$ and $x_2$ represent $x$, $y$ or $z$.

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<th>Specimen Axis</th>
<th>Direction of Specimen Axis</th>
<th>0.2</th>
<th>0.5</th>
<th>0.7</th>
<th>1.0</th>
<th>1.5</th>
</tr>
</thead>
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<td>20.6</td>
<td>21.0</td>
<td>21.6</td>
<td>22.3</td>
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<tr>
<td></td>
<td></td>
<td>19.5</td>
<td>20.3</td>
<td>20.8</td>
<td>21.2</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td></td>
<td>19.6</td>
<td>20.3</td>
<td>21.0</td>
<td>21.8</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>Average</td>
<td>19.6</td>
<td>20.4</td>
<td>21.0</td>
<td>21.5</td>
<td>22.4</td>
</tr>
<tr>
<td>Copper</td>
<td></td>
<td>10.05</td>
<td>10.40</td>
<td>10.65</td>
<td>10.90</td>
<td>11.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10.03</td>
<td>10.65</td>
<td>10.85</td>
<td>10.95</td>
<td>11.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.90</td>
<td>10.20</td>
<td>10.30</td>
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<td>--</td>
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<td></td>
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<td>10.45</td>
<td>10.55</td>
<td>10.75</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>Average</td>
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<td>10.33</td>
<td>10.47</td>
<td>10.63</td>
<td>10.75</td>
</tr>
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<td>3.65</td>
<td>3.68</td>
<td>3.68</td>
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<td>3.63</td>
<td>3.71</td>
<td>3.72</td>
<td>3.74</td>
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<td></td>
<td></td>
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Since the effective strain is defined as

$$
\varepsilon_e = \frac{1}{3}\left[ (\varepsilon_x \varepsilon_y - \varepsilon_y \varepsilon_y)^2 + (\varepsilon_y \varepsilon_y - \varepsilon_z \varepsilon_x)^2 + (\varepsilon_z \varepsilon_x - \varepsilon_x \varepsilon_x)^2 + \frac{3}{2} (\gamma_{xy} + \gamma_{yz} + \gamma_{xz}) \right]^{1/2}
$$

For torsion tests, the only nonzero strain is \( \gamma_{xy} \), so

$$
\varepsilon_e = \frac{\sqrt{3}}{3} \left( 0 + 0 + 0 + \frac{3}{2} \gamma_{xy} + 0 + 0 \right)^{\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{2}} \gamma_{xy} \Rightarrow \gamma_{xy} = \frac{\sqrt{3}}{\sqrt{2}}
$$

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<th>Material</th>
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Constitutive Equations:

Steel \( \sigma_0 = \sqrt{3} \left[ 18.0 (1.0 + 0.25444 \varepsilon_e^{0.61}) \right] \)

Correlation Factor = 0.99

Copper \( \sigma_0 = \sqrt{3} \left[ 9.65 (1.0 + 0.140 \varepsilon_e^{0.48}) \right] \)

Correlation Factor = 0.96

Aluminum \( \sigma_0 = \sqrt{3} \left[ 3.3 (1.0 + 0.1424242 \varepsilon_e^{0.66}) \right] \)

Correlation Factor = 0.93
### TABLE 4

**TORSION SHEAR STRESS (TON/IN$^2$) VERSUS SHEAR STRAIN**

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### TABLE 7

**The False Wedge Solution for Von Mises Material**

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<th>$\frac{F}{ck}$ Gruzweig</th>
<th>$\frac{F}{ck}$ Haddow</th>
<th>$\frac{F}{ck}$ Num.</th>
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</table>

| Pressure on Cone | 3.88 | 4.45 | 4.17 | 4.28 | 4.66 | 4.94 | 4.76 | 4.97 | 4.94 | 5.20 | 5.33 | 5.32 |

| \( \frac{P - P_i}{P_i} \cdot 100\% \) | -- | 7.47 | -- | 9 | 6.07 | -- | 5 | 3.78 | -- | 2.5 | 2.31 |
### TABLE 9
NUMERICAL RESULT OF THE PLANE-STRAIN INVERTED EXTRUSION
WITH 50% REDUCTION IN AREA (EXTRUSION PRESSURE P)

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<th>$\frac{P}{K}$</th>
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<th>$\frac{P - P_{\text{num}}}{P} \times 100%$</th>
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<td>$\mu$</td>
<td>$\psi$</td>
<td>$(Ck/F) = \frac{1}{K}$</td>
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The calculated mean value of effective strain, $\bar{\varepsilon}_e$, for smooth wedge $\beta$: semi-wedge angle; $\mu$: coef. of friction; $h$: contacted wedge flank length; $C$: depth of penetration; $A$: total plastic area associated with slip-line solution.

For example $\beta = 40^\circ$, let $C = 1$

\[ A = h^2 \left(1 + \frac{\psi}{2}\right) = (1.82)^2 \left[1 + 27^\circ 25'\right] = (1.82)^2 \left[1 + \frac{0.47851}{2}\right] = 4.105 \]

\[ \bar{\varepsilon}_e = \frac{K}{2 \sqrt{3} A} = \frac{1}{2 \sqrt{3} (4.105)} = 0.4884 \]

$\beta = 60^\circ$, $C = 1$

\[ A = h^2 \left(1 + \frac{\psi}{2}\right) = (2.93)^2 \left[1 + \frac{50^\circ 50'}{2}\right] = 12.3932 = 12.393 \]

\[ \bar{\varepsilon}_e = \frac{K}{2 \sqrt{3} (A)} = \frac{0.0521}{2 \sqrt{3} (12.393)} = 0.44708 = 0.4471 \]

-170-
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<th>Beta (Degree)</th>
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<th>PRE.(CAL.)</th>
<th>( \frac{(P_{EXP} - P_{CAL})}{P_{EXP}} \times 100% )</th>
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B. Copper

| 30 | 30.75 | 31.05 | 0.97 |
| 60 | 42.62 | 42.13 | -1.14 |

C. Aluminium

<p>| 30 | 10.81 | 10.57 | -2.19 |</p>
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**B. Copper**

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**TABLE 14**

*EXTRUSION PRESSURE OF PLANE-STRAIN INVERTED EXTRUSION PROCESS WITH REDUCTION R = 0.5 FOR STRAIN-HARDENING MATERIALS (SMOOTH CASE)*

<table>
<thead>
<tr>
<th>Material</th>
<th>$P_{\text{cal}}$</th>
<th>$P_{\text{mean}}$</th>
<th>$P_{\text{theo.}}$</th>
<th>$P_{\text{calexp}}$</th>
<th>$\frac{P_{\text{cal}} - P_{\text{calexp}}}{P_{\text{calexp}}} \times 100%$</th>
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</thead>
<tbody>
<tr>
<td>Pure Lead</td>
<td>1.870</td>
<td>1.732</td>
<td>1.598</td>
<td>1.899</td>
<td>-2.21</td>
</tr>
<tr>
<td>Tellurium Lead</td>
<td>2.844</td>
<td>2.690</td>
<td>2.512</td>
<td>2.949</td>
<td>-3.56</td>
</tr>
<tr>
<td>Super-pure</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Aluminium</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>99.7% pure</td>
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</tr>
<tr>
<td>Aluminium</td>
<td>19.298</td>
<td>16.486</td>
<td>3.711</td>
<td>18.075</td>
<td>6.77</td>
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</table>

$P_{\text{cal}}$: Extrusion pressure calculated by using material constitutive equation

$P_{\text{mean}}$: Extrusion pressure calculated by using $\bar{e} = 1.5$ to find the mean stress from the stress-strain curve

$P_{\text{theo}}$: Extrusion pressure for rigid, perfectly plastic material

$P_{\text{calexp}}$: Extrusion pressure which is 9.64% higher than $P_{\text{mean}}$
TABLE 15
INVERTED EXTRUSION PROBLEM WITH REDUCTION R = 0.5 ON RATE DEPENDENT MATERIAL (LEAD) WITHOUT FRICTION. PLANE STRAIN CASE:

<table>
<thead>
<tr>
<th>Velocity (in./min.)</th>
<th>P_{exp.} (ksi)</th>
<th>P_{cal.} (ksi)</th>
<th>\left(\frac{P_{cal.}-P_{exp.}}{P_{exp.}}\right) \times 100%</th>
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<td>2600</td>
<td>2740</td>
<td>5.38</td>
</tr>
<tr>
<td>0.20</td>
<td>3300</td>
<td>3613</td>
<td>9.48</td>
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</table>

Axial Symmetrical Case:

<table>
<thead>
<tr>
<th>Velocity (in./min.)</th>
<th>P_{exp.} (ksi)</th>
<th>P_{cal.} (ksi)</th>
<th>\left(\frac{P_{cal.}-P_{exp.}}{P_{exp.}}\right) \times 100%</th>
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<tr>
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<td>2720</td>
<td>3038</td>
<td>11.69</td>
</tr>
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<td>0.0600</td>
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<td>11.53</td>
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<tr>
<td>0.1200</td>
<td>3610</td>
<td>3817</td>
<td>5.75</td>
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<td>0.2400</td>
<td>3870</td>
<td>4145</td>
<td>7.11</td>
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Note: The slip-line solution of inverted extrusion P = 2.57 k0, the numerical calculation is close to that P = 2.48 k0. The error is -3.5%

k0: The yield shear stress of perfectly plastic material
TABLE 16

PLANE STRAIN INVERTED EXTRUSION WITH 50% REDUCTION IN AREA ON

STRAIN-HARDENING AND RATE DEPENDENT MATERIAL

<table>
<thead>
<tr>
<th>Extrusion Velocity (in./min.)</th>
<th>Extrusion Pressure (psi)</th>
<th>Error % ( \left( \frac{P_{\text{exp}} - P_{\text{cal}}}{P_{\text{exp}}} \right) \times 100 )</th>
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<tbody>
<tr>
<td>0.017</td>
<td>3826</td>
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<tr>
<td>0.070</td>
<td>5021</td>
<td>14.42</td>
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<td>0.224</td>
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<tr>
<td>0.670</td>
<td>6380</td>
<td>19.23</td>
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</table>

Constitutive Equation of Lead

\[ \sigma_e = 882.25 \left( 1 + 23.33 \varepsilon_e \right)^{0.33} \left( 1 + 100.0 \varepsilon_e \right)^{0.09} \]

Pressure

\( x \times 10^3 \) Psi.
**TABLE 17**

***WEDGE INDENTATION ON HARDENING AND RATE DEPENDENT MATERIAL***

*(PERFECT ROUGH CASE, SEMI-WEDGE ANGLE = 22.5 DEGREE)*

<table>
<thead>
<tr>
<th>Velocity (in/min.)</th>
<th>$(F/FO)_{exp}$</th>
<th>$(F/FO)_{cal}$</th>
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</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.26</td>
<td>1.22</td>
</tr>
<tr>
<td>1.0</td>
<td>1.33</td>
<td>1.37</td>
</tr>
<tr>
<td>6.0</td>
<td>1.51</td>
<td>1.58</td>
</tr>
</tbody>
</table>

$FO$: Wedge indentation force with velocity of 0.00833 in./min.

Material: Pure lead

Constitutive equation of lead:

$$\sigma_e = 882.25 \times \left[ (1.0 + 23.33 \varepsilon_e)^{0.33} \right] \times \left[ (1.0 + 100.0 \varepsilon_e)^{0.09} \right]$$
**TABLE 18**

**PLANE STRAIN WEDGE INDENTATION ON SOIL**

**INTERNAL FRICTIONAL ANGLE $\phi = 20^\circ$**

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<th>$\beta$</th>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$\mu$</th>
<th>$\frac{H}{T}$</th>
<th>$\sigma$</th>
<th>$\frac{F_{\text{theo.}}}{TC}$</th>
<th>$\frac{F_{\text{num.}}}{TC}$</th>
<th>$\frac{F_{\text{num.}} - F_{\text{theo.}}}{F_{\text{theo.}}} \times 100%$</th>
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</table>

β: Semi-wedge angle; λ = Frictional angle; α = Spiral fan angle
H: Contacted wedge flank length; T = Depth of penetration;
C: Normal stress on the flank; F_{theo.} = slip-line solution; F_{num.} = Num. result; C = Cohesion
<table>
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<tr>
<th>$\beta$</th>
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<th>$\alpha$</th>
<th>$H/T$</th>
<th>$\sigma$</th>
<th>$F_{theo.}/Tc$</th>
<th>$F_{num.}/Tc$</th>
<th>$F_{num.} - F_{theo.}$/$F_{theo.}$ x 100%</th>
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<th>C</th>
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<th>F_{num.}</th>
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β: Semi-wedge angle; λ: Frictional angle; α: Spiral fan angle
H: Contacted wedge flank length; T: Depth of penetration;
σ: Normal stress on the flank; F_{theo.}: Slip-line solution; F_{num.}: Num. result; C: Cohesion
### Table 20

**FAISE WEDGE SOLUTION FOR COULOMB MATERIAL**

**INTERNAL FRICTIONAL ANGLE $\phi = 20^\circ$**

<table>
<thead>
<tr>
<th>Semi-wedge Angle $\beta$</th>
<th>Frictional Angle $\lambda$</th>
<th>Frictional Coefficient</th>
<th>$F_{\text{theo}}$</th>
<th>$F_{\text{num}}$</th>
<th>$\frac{F_{\text{num}} - F_{\text{theo}}}{F_{\text{theo}}} \times 100%$</th>
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**INTERNAL FRICTIONAL ANGLE $\phi = 30^\circ$**

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<th>Frictional Angle $\lambda$</th>
<th>Frictional Coefficient</th>
<th>$F_{\text{theo}}$</th>
<th>$F_{\text{num}}$</th>
<th>$\frac{F_{\text{num}} - F_{\text{theo}}}{F_{\text{theo}}} \times 100%$</th>
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<td>496.463</td>
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</table>

$T =$ Depth of penetration

$C =$ Cohesion of soil
### TABLE 21

PRESSURE DISTRIBUTION AND THE PRESSURE (P/C) ON SURFACE OF PERFECTLY SMOOTH CONICAL INDENTERS ON COULOMB'S MATERIAL ANGLE OF INTERNAL FRICTION $\phi = 30^\circ$

<table>
<thead>
<tr>
<th>$\beta$ (semi cone angle)</th>
<th>10°</th>
<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
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</thead>
<tbody>
<tr>
<td>$\frac{\Sigma}{R}$</td>
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<td></td>
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<tr>
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<td>4.8896</td>
<td>6.1110</td>
<td>7.9228</td>
<td>10.4212</td>
<td>13.7415</td>
<td>18.0438</td>
<td>23.5111</td>
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<table>
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<th>Pressure on wedge (P/C)</th>
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### Table 22

Pressure Distribution and the Pressure (P/c) on Surface of Perfectly Smooth Conical Indenters

On Coulomb's Material Angle of Internal Friction $\phi = 20^\circ$

<table>
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<th>20°</th>
<th>30°</th>
<th>40°</th>
<th>50°</th>
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| Force on cone             |     |     |     |     |     |     |     |     |
| when $R = 1$              |     |     |     |     |     |     |     |     |
| $(F/\pi R^2 \alpha)$      | 4.9369 | 4.8739 | 5.3689 | 6.22242 | 7.4013 | 8.9018 | 10.7406 | 12.9361 |

<p>| Pressure on wedge         |     |     |     |     |     |     |     |     |</p>
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<th>Slope $H'$</th>
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<th>Indentation Pressure $F/(K₀CW)$</th>
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$K = K₀ + H'\varepsilon_P$; \hspace{1cm} K = 14\text{KSI}$

$C = \text{Depth of indentation}; \hspace{1cm} K₀ = 14\text{KSI}$
Notations for curve readings:

Solid curves: Results based on the known theoretical slip-line solutions.

\( x \): Experimental data or numerical results based on finite element or finite difference schemes.
FIGURE 1
The Yield Surface for the Von Mises and Tresca Yield Criteria in the Principal Stress Space.
FIGURE 2
The isotropic (a), kinematic (b) hardening rule for Von Mises yield condition and a local hardening rule for Tresca yield condition (c).
The yield surface for the extended Von Mises and the extended Tresca yield criteria in the principal stress space.
FIGURE 4
Section of the yield surface by the $\Pi$-plane ($\sigma_1 + \sigma_2 + \sigma_3 = 0$).
FIGURE 5
Load extension diagram for mild steel.
FIGURE 6
Effects of strain rate on stress-strain curves for high purity aluminum (after Hauser, Simmons and Dorn).

FIGURE 7
Effects of strain rate on stress-curves for mineral-oil saturated Solenhofen limestone at 20,000 psi confining pressure and room temperature (after Serdngecti and Boozer).
Compressive strength of dresser basalt as a function of strain rate and temperature (after Lindholm, Yeakley and Nagy).

Strain rate, 1/sec

FIGURE 8
Subsequent

dynamic loading surface

$F=0$

(Initial yield surface)

$\sigma_{ij}$

$\dot{\varepsilon}_{ij}$

FIGURE 9
Dynamic loading surface and strain-rate vector.
FIGURE 10
Dynamic stress-strain curve for work-hardening and strain-rate sensitive material (a) \( \dot{\varepsilon} = \text{const.} \), (b) \( \varepsilon = \dot{\varepsilon} (\varepsilon) \) (after Perzyna).
FIGURE 11
Dynamic stress-strain curve for elastic/(visco-perfectly plastic) material (a) $\varepsilon = $ const. (b) $\varepsilon = \varepsilon(\varepsilon)$ (after Perzyna).
FIGURE 12
Comparison of the experimental data with the prediction of (a) the power strain-rate law (b) the linear strain-rate law (after Perzyna).
Comparison of the experimental data with the prediction of (a) the exponential strain-rate law (b) the series power and the series exponential strain-rate law (after Perzyna).
FIGURE 14
Mohr circle relation and $S_\alpha$, $S_\beta$ lines

FIGURE 15
Unperturbed and perturbed slip-line fields for a rate independent Von Mises material (after Spencer).
FIGURE 16
Average effective-stress effective-strain curves of commercially pure lead in compression for various strain rates.
Constitutive Equation of Commercially Pure Lead

\[ \sigma_e = 2500 \left( \varepsilon_e \right)^{0.12} \text{ (psi)} \]

Effective Stress (1000 Psi)

Effective Strain Rate (in/in/min)

FIGURE 17

Average effective-stress effective-strain-rate curve of commercially pure lead in compression at an effective strain of 1.0 in/in
Comparison of the experimental data with the prediction of the power strain-rate law for plane-strain and axial symmetric extrusion processes.
Forms of empirical strain-hardening equations

(a) $\sigma_e = A (\epsilon + \varepsilon_e)^n$

(b) $\sigma_e = Y + C \varepsilon_e^n$

(c) $\sigma_e = \sigma_\infty - (\sigma_\infty - Y) e^{-\varepsilon_e/n}$

(d) $\sigma_e = Y + \tanh\left(\frac{E \varepsilon_e}{Y}\right)$
THE UNIT DIAGRAM

FIGURE 20
FIGURE 21
Inverted extrusion with 50% reduction in area
$\beta = \text{Semi-wedge angle}$

$V = \text{The downward wedge velocity}$

$O''H'' = \text{The normal component of the wedge velocity to the right flank of the wedge}$

$O''E'' = \text{The velocity of all elements in region III}$

$O''E'' = \text{The velocity of all elements in region I}$

FIGURE 23
Indentation hodograph of a rigid smooth wedge
FIGURE 24
The motion of an element $F'$ between $O^*$ and $F$ on the undisturbed surface.
\[ S = d \left[ 1 - \frac{t_0}{t} \right] \]
\[ r_0 = 0^*P = \left[ \frac{1}{8} i, 0 j \right] \]

when \( t_0 \leq \frac{1}{8} \), \( P \) remain still

when \( t = \frac{2}{8} \), \( S = d \left[ 1 - \frac{1}{2} \right] = \frac{d}{2} \)

\( S = 0.68, \ x_2 = tr^* = \frac{2}{8} r^* = 2 \)

\( t - \frac{3}{8}, S = d \left[ 1 - \frac{1}{3} \right] = 0.91 \)

\( r_3 = tr^* = \frac{3}{8} r^* \)

etc.

**FIGURE 25**

The motion of an element immediately below the apex of the wedge.
If $\beta = 30^0$, and the coordinates of point $P$ is $\frac{1}{8}, \frac{1}{2}$, then $r_0 = (\cos \frac{\pi}{4} j + \sin \frac{\pi}{4} j)$

$t_0 = \frac{1}{8}(1.0 + \tan 15^0) = 0.1585$, $l_{r_0} = \sqrt{2} (\frac{1}{8}) = 0.17678$ $l_{r_0} = \frac{0.1585}{t_0} = 1.115355$

Then we want to know $r = \_ ?$ when $t = 1$

\[ t = \frac{d}{t_0} \quad S = d (1.0 - \frac{t_0}{t}) \quad S = 0.97 (1.0 - \frac{0.1585}{1}) = 0.82 \]

From both measurement and calculations, $r = o_{p^1} x(t) = x_{p^1}(1) = o_{p^1}^1$

$r = [0.028 \text{ i}, 0.72 \text{ j}]$

**FIGURE 26**

The motion of an element in triangle region III.
FIGURE 27
The approximate trajectory of an element in region II.
\[ 10^*H'' \sim \frac{v \sin \beta}{\cos \lambda} \]
\[ 10^*B'' \sim \frac{v_0 \sin \beta}{\cos \lambda} = 10^*C'' \sim \]
\[ 10^*H'' \sim \frac{v_0}{\sin \left[ \frac{\pi}{2} - (\beta - \lambda) \right] \sin \left( \frac{\pi}{2} - \lambda \right)} \]
\[ 10^*B'' \sim \frac{v_0 \cos (\beta - \lambda)}{\cos (\lambda)} \]

\( \beta \) = Semi-wedge angle  
\( \lambda \) = Frictional angle  
\( V_0 \) = The downward wedge velocity  
\( O^*H \) = The normal component of the wedge velocity to the right flank of the wedge  
\( O^*B'' \) = The velocity of all elements in region III  
\( O^*C \) = The velocity of all elements in region I  
\( EB'' \) = The velocity with which elements move along the right flank of the wedge

FIGURE 28  
Indentation hodograph of a rigid frictional wedge  
(material satisfying Von Mises yield condition)
\( \beta = \text{Semi-wedge angle } 30^\circ \)
\( t = \text{Depth of penetration} \)
\( H = \text{Length "OF", the contact length of the wedge; } \frac{H}{t} = 1.50 \)
\( H_1 = \text{Length of CG; } \frac{H_1}{t} = 1.28 \)
\( H_2 = \text{Length of the lip free surface FG} \)
\( \lambda = \text{The frictional angle } 31^\circ 31'; \mu = 0.15 \)
\( \psi = \text{The fan angle of the slip-line solution for wedge indentation} \)

FIGURE 29

The slip-line field in plastic indentation by a frictional wedge.
t, t': Depth of wedge indentation
ADL: The distance between point A and point D
ACL: The distance between point A and point C
BCL: The distance between point B and point C
AGL: The distance between point A and point G
AEL: The distance between point A and point E on circular arc DC
The slip-line boundary configuration of wedge indentation in physical space

FIGURE 30
Geometric similarity of the slip-line field during wedge indentation.
FIGURE 31
The deformation pattern of the square grids after extrusion.
FIGURE 32
Equilibrium relation between the indentation force $F$ and the stress on the flank surface of the wedge.
The dotted extensions of the slip-lines is a statically admissible extension of the stress field into the dead metal cap.

\[ \beta = \text{The semi-wedge angle} \]
\[ \psi = \text{The fan angle of the false wedge} \]
\[ C = \text{The depth of penetration} \]

FIGURE 33
Slip-line field for false-wedge solution.
FIGURE 34
Coulomb yield condition.
FIGURE 35
The direction of failure lines
FIGURE 36
Geometric relation for smooth wedge indentation on soil.
FIGURE 37
Mohr circle for the pressure on the flank of the smooth wedge for Coulomb material
FIGURE 38
Geometric relation for frictional wedge indentation on Coulomb material
\[ OA^* = V_0 \]
\[ OH'' = V_0 \sin \beta \]
\[ A^*E'' = V_0 \sin \beta / \cos \lambda \]
\[ OE'' = \frac{V_0}{\sin \left( \frac{\pi}{2} - (\beta - \lambda) \right) \sin \left( \frac{\pi}{2} - \lambda \right)} \]
\[ OE'' = -\frac{V_0 \cos (\beta - \lambda)}{\cos (\lambda)} \]

**FIGURE 39**  
Hodograph for frictional wedge indentation on soil.
FIGURE 40
Mohr circle relation for the frictional wedge indentation on Coulomb material.
FIGURE 41
Coulomb yield condition under Harr and Von Karman hypothesis
FIGURE 42
Integration procedure in triangle region I

FIGURE 43
Evaluation of the average value of the normal stress on the smooth cone surface.
FIGURE 44
Plain-strain extrusion apparatus

FIGURE 45
Load cell and the deflection transducer.

FIGURE 46
Set-up for simple compression test.
FIGURE 47
The shear stress versus shear strain curves for mild steel, copper and aluminum.
Log. Strain

True stress-true strain curves for (1) pure lead
(2) 0.065% tellurium lead (3) super-pure aluminum
(4) 99.7% pure aluminum

Constitutive equations

(1) Pure Lead
   Correlation Factor = 1.0
   \( \sigma = 1.077 (1.0 + 0.05571 \varepsilon) \)

(2) 0.065% Tellurium Lead
   Correlation Factor = 1.0
   \( \sigma = 1.6923 (1.0 + 0.04723 \varepsilon) \)

(3) Super-pure Aluminum
   Correlation Factor = 0.99
   \( \sigma = 2.31 (1.0 + 1.71 \varepsilon^{0.4}) \)

(4) 99.7% Pure Aluminum
   Correlation Factor = 1.0
   \( \sigma = 2.50 (1.0 + 2.892 \varepsilon^{0.43}) \)

FIGURE 48
FIGURE 49 The stress-strain curves $\varepsilon$ as constant strain rates for pure lead
F = The load required to make this indentation

$\beta$ = Semi-angle of any blunt wedge (ie. $\beta < 45^\circ$)

$H$ = The length of the contact surface of the false wedge

$\bar{H}$ = The length of the contact surface of the blunt wedge

$W$ = The horizontal distance from point A to the axis of the wedge

$C$ = The depth of penetration of the false wedge, let $C = 1$

$C_\beta$ = The depth of penetration of the blunt wedge, such that the penetration of the false wedge formed beneath the blunt wedge equal to 1 unit.

The horizontal dotted line is the undeformed plane before indentation.

FIGURE 50
FIGURE 51
Numerical results for the wedge indentation solutions of perfectly plastic material (Finite element method)
FIGURE 52
Numerical results for the wedge indentation solutions of perfectly plastic material (Finite difference method)
False wedge solutions of perfectly plastic material
FIGURE 54
Pressure distribution for cone indentation for perfectly plastic material
Figure 55: Wedge indentation pressure for strain-hardening material.
FIGURE 56
Conical indentation pressure for strain-hardening material