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EGOROV'S THEOREM FOR A DIFRACTIVE BOUNDARY PROBLEM

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EGOROV'S THEOREM FOR A
DIFFRACTIVE BOUNDARY PROBLEM

by

MARK KELLING FARRIS

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

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HOUSTON, TEXAS

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**ABSTRACT**

Egorov's Theorem for a Diffractive Boundary Value Problem

Mark Kelling Farris

Let $\Delta$ be the Laplacian on $\mathbb{R}^n \setminus K$ with Dirichlet boundary conditions. Assume $K$ is smoothly bounded with strictly convex boundary. By the spectral theorem define $e^{it\sqrt{-\Delta}}$ and extend this operator to $C'(\mathbb{R}^n \setminus K)$.

**Theorem.** Let $P \in OPS^m(\mathbb{R}^n \setminus K)$. Suppose the distribution kernel for $P$ is compactly supported in $\mathbb{R}^n \setminus K \times \mathbb{R}^n \setminus K$, and

$$\{(x, \xi) \in T^*(\mathbb{R}^n \setminus K) : x - t \frac{\xi}{|\xi|} \in \partial K, x - s \frac{\xi}{|\xi|} \notin \partial K \}$$

for $0 < s < t = \emptyset$.

Then modulo a smoothing operator

$$e^{it\sqrt{-\Delta}}P e^{-it\sqrt{-\Delta}} \in OPS^m_{a, 1-a}(\mathbb{R}^n \setminus K) + OPS_{\frac{a-1}{3}, \frac{a}{3}}(\mathbb{R}^n \setminus K)$$

for any $\frac{1}{2} < a < 1$ and any $\varepsilon > 0$. 

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§0. Introduction

This paper consists of a microlocal investigation of $\exp(it\sqrt{-\Delta})$ on the complement of a smoothly bounded, strictly convex obstacle $K \subseteq \mathbb{R}^n$, with Dirichlet boundary conditions. The region of interest is in a conic neighborhood of a grazing direction, i.e. a point $(x, \xi) \in T^*(\mathbb{R}^n \setminus K)$ such that the curve $x - t \frac{\xi}{|\xi|}$ (as $0 \leq t \leq T$) intersects $\partial K$ tangentially. Near such a direction this operator propagates singularities by a merely $C^{\frac{1}{2}}$ mapping. We present two main results. First, near a grazing direction, this operator is essentially represented as Fourier multiplication by a quotient of Airy functions. Second, if $P$ is a classical pseudodifferential operator such that $P$ and $e^{it\sqrt{-\Delta}} P e^{-it\sqrt{-\Delta}}$ are essentially supported away from the boundary then the conjugated operator is again a pseudodifferential operator (of type $(\frac{1}{3}, \frac{2}{3})$). More precise statements appear as theorems 1 and 2 below after establishing notation and presenting some needed results from the theory of Fourier integral operators.

If $M$ is a manifold and $U \subseteq T^*(M)$ is a conic subset, the set of distributions on $M$ with wave front set contained in $U$ will be denoted $\mathcal{E}'_U(M)$. $S^m_{\rho, \delta}(M)$ is the usual symbol class and $S^m_{\rho, \delta}(U)$ is the set of symbols in the class $S^m_{\rho, \delta}$ which are essentially supported in $U$. If $U$ lies
over a fixed coordinate chart we can let $\Sigma = \{(x, \xi) \in U : \xi_n = 0\}$ and define, following Taylor [10], $\eta_\rho^m(\Sigma)$ as the set of smooth $p$ satisfying

$$
1) \quad |D_x^\alpha D_\xi^\beta D_\xi_n^\gamma p(x, \xi)| \leq C_{\alpha \beta \gamma} |\xi'|^m |\xi| (|\xi'|^\rho + |\xi_n|)^{-|\gamma|}
$$

for $|\xi_n| < |\xi'|$ and

$$
2) \quad p \in S_{1,0}^m \text{ on any conic set disjoint from } \Sigma .
$$

Here $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Finally, $S^m(U) \subseteq S_{1,0}^m(U)$ will be those symbols having asymptotic expansions in terms of homogeneous functions. The pseudodifferential operators associated to the class $S(U)$ will be denoted $\text{OPS}(U)$. We remark that $\eta_\rho^*$ and $\text{OPS}_\rho^*$ are closed under composition, taking inverses of elliptic elements, and taking asymptotic expansions and record

**Proposition 0.1.** If $q(x, \xi) \in S_{1,0}^m(R)$ and $m \geq 0$ then

$$
q(x, \xi_1^{-\rho} \xi_n) \in \eta_\rho^{(1-\rho)m}([\xi_n = 0]).
$$

As always in the study of diffraction problems we will use the Airy function $\text{Ai}$ given by the conditionally convergent integral

$$
\text{Ai}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp i(st + t^3/3) dt .
$$
This solves the differential equation

\[ Ai''(s) - sAi(s) = 0 \]

hence extends to be an entire function. We define two more solutions to this equation \( A_{\pm} \) by

\[ A_{\pm}(s) = Ai(e^{\pm 2/3\pi i} s) \, . \]

Letting \( \omega = e^{2/3\pi i} \), notice that

\[ Ai(s) = -\omega A_+(s) + \omega A_-(s) \, , \]

and for \( s \) real,

\[ A_+(s) = \bar{A}_-(s) \, . \]

Finally, \( Ai \) has an asymptotic expansion valid in \( |\arg z| < \pi \) given by \( Ai(z) = \xi(z)e^{-\frac{2}{3}z^{3/2}} \) with

\[ \xi(z) \sim z^{-\frac{2}{3}} \left( \frac{1}{2\sqrt{\pi}} + a_1 z^{-3/2} + \ldots \right) \text{ as } |z| \to \infty . \]

The Fourier multiplier mentioned in the first paragraph is given by \( \chi(\xi) \frac{A_+}{A_-} (\xi_1^{\frac{1}{3}} \xi_2 \xi_3 \ldots) \) where \( \chi \) is a smooth function, homogeneous of degree 0 which is identically 1 in a conic neighborhood \( \Gamma \) of \( \xi = (1, 0, \ldots, 0) \) and vanishes outside a cone \( \tilde{\Gamma} \supseteq \Gamma \). We denote the operator associated
to this multiplier by \( \frac{G_+}{G_-} \). It is well known that 
\( A_\pm(s) \neq 0 \) for \( s \) real.

Now we can state our first result. Let 
\( (x, \xi) \in T^* (\mathbb{R}^n \setminus K) \) be a grazing direction and choose \( T > 0 \) such that \( x - t \frac{\xi}{|\xi|} \in \partial K \) for some \( 0 < t < T \).

**Theorem 1.** There exists conic neighborhoods \( U_1 \) and \( U_2 \) of \( (x, \xi) \) and \( (x - T \frac{\xi}{|\xi|}, \xi) \) respectively, and elliptic Fourier integral operators \( K_i : \mathcal{E}'_{U_1} (\mathbb{R}^n \setminus K) \to C^\infty (\mathbb{R}^n) \) such that modulo a smoothing operator, if \( P \in \mathcal{S}_m (U_1) \) then

\[
\exp(iT\sqrt{-\Delta})P = K_2^{-1} Q \frac{G_+}{G_-} K_1
\]

with \( Q \in \mathcal{O}_m (\Sigma) \). Moreover, if \( P \) is elliptic on \( U_1 \)

then \( Q \) is elliptic on \( \Gamma \).

Before stating theorem 2 we recall a basic result from the calculus of Fourier integral operators. Let \( M \) be a compact manifold (without boundary) and consider the hyperbolic evolution equation

\[
\frac{\partial}{\partial t} u(x,t) = i\lambda(t,x,D_x)u(x,t)
\]

\[ u(x,0) = u_0(x). \]
Here $\lambda$ is a smooth family in $\text{OPS}^1(M)$ with real principal symbol. Denoting the solution operator to this problem by $S(t)$ (so that $u(x,t) = S(t)u_0$) we have

**Egorov's Theorem.** If $P \in \text{OPS}^m_{\rho,1-\rho}(M)$ with $\rho > \frac{1}{2}$ then $Q(t) = S(t)P^{-1}(t) \in \text{OPS}^m_{\rho,1-\rho}(M)$. Moreover, the principal symbols are related by

$$q_m(t,x,\xi) = p_m(\chi(t,x,\xi))$$

where $\chi$ is the flow on $T^*(M)$ generated by the Hamiltonian vector field $H_{\lambda^1}(t,x,\xi)$.

We remark that the condition $M$ is compact can be replaced by the continuity of $S(t)$ on the $L^2$ Sobolev spaces $H^k$ for all $k$. An extension of this to the class $\text{OP}^m_{\rho}$ is given by Taylor [10]:

**Proposition 0.2.** If $P \in \text{OP}^m_{\rho}(\gamma)$ and $J$ is an elliptic Fourier integral operator then

$$JPJ^{-1} \in \text{OPS}^{m+\epsilon}_{\rho,1-\rho}(U).$$

Here, of course, $U$ is the image of a neighborhood of $\text{supp} \ p(x,\xi)$ under the canonical relation associated to $J$. Theorem 2 is an example of such a result for the solution operator to an evolution equation on a region with
(diffractive) boundary. We suppose \( P \in \text{OPS}^m(\mathbb{R}^n \setminus K) \) has distribution kernel compactly supported in \((\mathbb{R}^n \setminus K) \times (\mathbb{R}^n \setminus K)\). If \( ES(P) = U \subseteq T^*(\mathbb{R}^n \setminus K) \), we choose \( T > 0 \) such that

\[(x, \xi) \in U : x - T |\xi|^{-1} \notin \partial K, x - t \frac{\xi}{|\xi|} \notin \partial K \text{ for } 0 < t < T\] = \emptyset.

**Theorem 2.** Under the above conditions, modulo a smoothing operator,

\[e^{iT\sqrt{-\Delta}} P e^{-iT\sqrt{-\Delta}} \in \text{OPS}^m_{a, 1-a}(\mathbb{R}^n \setminus K) + \text{OPS}^{m + \frac{a-1}{2} + \varepsilon}_{\frac{1}{2}, \frac{1}{2}}(\mathbb{R}^n \setminus K)\]

for any \( \varepsilon > 0 \) and any \( \frac{1}{2} < a < 1 \). The second term is a finite sum of operators of the form \( L^{-1}_{PL} \) where \( \tilde{P} \in \text{OPS}^{m + \frac{a-1}{2}}_{\frac{1}{2}, \frac{1}{2}}([\xi_n = 0]) \).

We sketch the arguments leading to these theorems. In section 1 we represent \( e^{iT\sqrt{-\Delta}} \) as \( F(T) - E(T)R \). Here \( F \) is the solution operator for the free space Laplacian, \( R \) is restriction to \( \partial K \times \mathbb{R} \) of this free space wave, and \( E \) produces a forward going wave with boundary values given by \( R \). The operators \( R \) and \( E \) are observed to have folding canonical relations as defined in section 2. The structure of such relations is then examined and sections 2 and 3 show that up to a symplectic change of coordinates the operators can be written in the form \( P_1 i + P_2 i' \) where
the $P_i$'s are pseudodifferential operators and $G_1$ and $G_1'$ are Fourier multiplication by $Ai(\xi_i^{-1/3} \xi_n)$ and $\xi_i^{-1/3} Ai'(\xi_i^{-1/3} \xi_n)$ respectively. Due to the diffractive hypothesis, the $P$'s in these expressions will in general be only of type $\left(\frac{1}{3}, \frac{2}{3}\right)$, however in section 4 things are arranged so that in the special coordinates chosen above, they actually belong to the class $\eta_{1/3}$. Putting all this together leads to expressions of the form

$$P_0 + P_1 G_1^2 + P_2 G_1 G_1' + P_3 G_1'^2$$

where the $P_i$'s are in $\eta_{1/3}^\ast$.

In sections 5, 6 and 7 we analyze such expressions using the added information that $e^{iT\sqrt{-\Delta}}$ propagates singularities by the usual laws of geometrical optics. That is, singularities reflect off the boundary under the rule angle of incidence = angle of reflection, which holds by the results of Melrose [4] and Taylor [11] even in the case of (exactly second order) tangential incidence. In the following this propagation will be known as the broken canonical transformation for $e^{iT\sqrt{-\Delta}}$.

The author respectfully thanks his advisor Michael E. Taylor for advice and guidance.
§1. Spectral theory vs. Fourier integral operators

Let \( \Omega = \mathbb{R}^n \backslash K \) and \( \Delta \) be the Laplacian on \( L^2(\Omega) \) with domain \( \mathcal{D}(\Delta) = H^2(\Omega) \cap H^1(\Omega) \). By the spectral theorem we can define \( \lambda = \sqrt{-\Delta} \) and \( e^{it\lambda} \). Giving \( \mathcal{D}(\lambda^k) \) the graph topology, elliptic regularity yields the inclusion
\[
\mathcal{E}'(\Omega) \subset \bigcup \mathcal{D}(\lambda^k)^{\ast}.
\]
By duality we can now define \( e^{it\lambda} : \mathcal{E}'(\Omega)^{\ast} \to \mathcal{E}'(\Omega) \). The goal of this section is to write this operator locally as a composition of Fourier integral operators with classical symbols along with one pseudodifferential operator, the Neumann operator, of bad type. So fix \( \Theta \subset \subset \Omega \) and consider \( e^{iT\lambda} \) restricted to \( \mathcal{E}_U^\prime(\Theta) \) where \( U \) is any closed cone in \( T^\ast(\Theta) \). We suppose \( U \) and \( T > 0 \) satisfy condition \((\ast)\) of the introduction.

If \( u_0 \in \mathcal{E}_U^\prime(\Theta) \) then \( u = e^{iT\lambda} u_0 \) solves
\[
\frac{\partial^2}{\partial t^2} - \Delta u = \Box u = 0 \quad \text{in} \quad \mathbb{R} \times \Omega
\]
\[
u(0,x) = u_0(x)
\]
\[
\frac{\partial}{\partial t} u(0,x) = i\lambda u_0(x)
\]
and \( u \big|_{\partial K \times \mathbb{R}} = 0 \) in some generalized sense. We will show \( e^{iT\lambda} u_0 = (F(T) - E(T) \circ R) u_0 \) modulo a smooth error where these operators come from the following procedure. Fix \( \psi \in C^\infty_0(\Omega) \) with \( \psi = 1 \) on \( \Theta \). Let \( v \) solve the free space problem
\[ \Box v = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^n \]

\[ v(0,x) = u_0(x) \]

\[ \frac{\partial}{\partial t} v(0,x) = i\psi(x) \Lambda_F u_0(x) \]

where \( \Lambda_F = \sqrt{-\Delta_F} \) and \( \Delta_F \) is the Laplacian on \( \mathbb{R}^n \). The solution to this is given modulo a smoothing operator by

\[ F_0(t)u_0(x) = (2\pi)^{-n} \int e^{i(x \cdot \xi + t|\xi|)} \hat{u}_0(\xi) d\xi. \]

We let \( F(t) \) be a properly supported Fourier integral operator which differs from \( F_0(t) \) by a smoothing operator. By standard propagation of singularities results we can compose \( F(t) \) with restriction to \( \partial K \times \mathbb{R} \) to yield an operator

\[ Ru_0(x,t) = (2\pi)^{-n} \int e^{i(k(x) \cdot \xi + t|\xi|)} \hat{u}_0(\xi) d\xi \]

modulo a smoothing operator. Here \( k: \partial K \to \mathbb{R}^n \) is inclusion. Finally let \( w \) solve

\[ \Box w = 0 \quad \text{on} \quad \mathbb{R} \times \Omega \]

\[ w(0,x) = 0 \]

\[ \frac{\partial}{\partial t} w(0,x) = 0 \]

\[ w|_{\partial K \times \mathbb{R}} = v|_{\partial K \times \mathbb{R}^+}. \]
If we knew what the normal derivative of $w$ should be on $\partial K \times R$, we could use Green's formula to write

$$w(t,x) = c_n \int_R \int_{\partial K} e^{it\lambda} \left[ \frac{\partial}{\partial \nu} G(x-y,\lambda) \tilde{w}(y,\lambda) - G(x-y,\lambda) \frac{\partial \tilde{w}}{\partial \nu}(y,\lambda) \right] \times$$

$$\times ds(y)d\lambda.$$

Here $\tilde{w}$ is the $t$-Fourier transform of $w$, $ds$ is surface measure on $\partial K$, $c_n$ is a dimensional constant, and $G(x,\lambda) = G_n(|x|,\lambda)$ is the fundamental solution for $\Delta + \lambda^2$ on $R^n$ given recursively by

$$G_2(r,\lambda) = H_0^{(1)}(\lambda r) = \text{Hankel function of the first kind}$$

$$G_3(r,\lambda) = \frac{1}{r} e^{-i\lambda r}$$

$$G_{n+2}(r,\lambda) = \frac{1}{r} \frac{\partial}{\partial r} G_n(r,\lambda).$$

The representation of $\frac{\partial w}{\partial \nu}|_{\partial K \times R}$ in terms of $w|_{\partial K \times R}$ is given by the Neumann operator and is discussed in §4. We remark here that $N$ can be represented by a pseudodifferential operator of type $(\frac{1}{3}, \frac{2}{3})$ and hence is pseudolocal. By finite (and non-zero) propagation speed of singularities for solutions to the wave equation, and the restriction (*) on $U$ and $T$, we can represent the solution operator to the above boundary value problem, modulo a smoothing operator,
by a properly supported operator which we denote $E(T)$.

Using an asymptotic expansion (in $\lambda$) for $G$ we can assume

$$E(T) = G_1 + G_2 \circ N$$

where $G_1, G_2$ are Fourier integral operators.

**Proposition 1.1.** $e^{iT\Lambda}u_0 - (F(T) - E(T) \circ R)u_0 \in C^\infty(\tilde{\Omega})$.

**Proof.** Pick a sequence $\{u_{0j}\} \subseteq H^1(\Omega) \cap \mathcal{E}'(\Omega)$ which converges to $u_0$ in the sense of distributions, and let

$$u_j(x,t) = [e^{i\lambda t\Lambda} - (F(t) - E(t) \circ R)]u_{0j}(x).$$

This satisfies

$$\square u_j \in C^\infty(\tilde{\Omega} \times \mathbb{R})$$

$$u_j(0,x) = 0$$

$$\frac{\partial}{\partial t} u_j(0,x) = i(\Lambda - \psi(x)\Lambda_F)u_{0j}$$

$$u_j \big|_{\partial \Omega \times \mathbb{R}^+} \in C^\infty(\partial K \times \mathbb{R}^+)$$

Clearly as distributions, $\{u_j(x,T)\}$ converges to

$$u(x,T) = [e^{iT\Lambda} - F(T) - E(T) \circ R]u_0(x).$$

The proposition now follows from
Lemma 1.1. If \( u \in C'(\emptyset) \) then \((\Lambda - \psi(x) \Lambda_F)u \in C^\infty(\emptyset)\).

Proof of lemma. Let \( B_r \) be the open ball of radius \( r \) about \( 0 \) in \( \mathbb{R}^n \) and \( \varphi \in C_0^\infty(B_r) \). We will show that

\[ \varphi(x)(\Lambda - \psi(x) \Lambda_F)u(x) \in C^\infty(\emptyset) \]

In fact, consider \( v(x,t) = \varphi(x)(e^{-t\Lambda} - \psi(x)e^{-t\Lambda_F})u(x) \). Since \( \frac{\partial}{\partial t} v(x,0) = \varphi(x)(-\Lambda + \psi(x) \Lambda_F)u(x) \) it suffices to show that \( v(x,t) \in C^\infty(\emptyset \times [0,1]) \). Notice that \( v(x,t) \) solves

\[ \frac{\partial^2}{\partial t^2} + \Delta v(x,t) = f \in C^\infty(\emptyset \times [0,1]) \]

\[ v(x,0) = 0 \]

\[ v(x,1) \in C^\infty(\emptyset) \]

\[ v(x,t) = 0 \text{ for } x \in \partial B_r \]

\[ v(x,t) = 0 \text{ for } x \in \partial K \]

If we set \( \Lambda_1 \) = Laplacian on \( B_r \cap \Omega \) with \( \delta(\Lambda_1) = H^2 \cap H^1 \)

and let \( \Lambda_1^2 = -\Lambda_1 \), variation of parameters yields the unique solution to the above boundary value problem

\[
v(x,t) = \int_0^t e^{-s(\Lambda_1 - \Lambda_1 \Lambda_1)} f(x,s) ds + e^{-ts}\int_0^1 f(x,r) dr ds + \frac{e^{-t\Lambda_1} - e^{-s\Lambda_1}}{1 - e^{-\Lambda_1}}
\]

\[
x (v(x,1) - \int_0^1 e^{-s\Lambda_1} f(x,r) dr ds)
\]
This formula makes sense in a bounded domain since there the spectrum of $\Delta$ is bounded away from 0. It is immediate from this formula that $\nu \in C^\infty(\bar{\Omega} \cap B_r \times [0,1])$.

In order to represent $e^{-iT\Delta}$ in a similar fashion, it is necessary to replace $G(x,\lambda)$ by $G(x,-\lambda) = \tilde{G}(x,\lambda)$ and use the appropriate "backwards" Neumann operator.
§2. Folding canonical relations

In this section we develop the symplectic geometry necessary to put our representation for $e^{iTA}$ in a tractable form. The key is the notion of equivalence of glancing hypersurfaces developed in [5]. We restrict this operator to $e^U(\mathcal{O})$ with $\mathcal{O}$ and $U$ as in §1. Assume $U$ contains a direction $(x_0, \xi_0)$ such that $x - t \frac{\xi}{|\xi|} \in \partial K$ for precisely one $t > 0$ and for this $t$ we have $t < T$. Of course, non-grazing points can be handled in an easier fashion.

Denote conic neighborhoods of the image of $(x_0, \xi_0)$ under the canonical relations for $R$ and $E \circ R$ by

$$V \in T^*(\partial K \times R)$$

and

$$W \in T^*(\Omega)$$

respectively. In what follows, these neighborhoods will shrink as necessary without comment. In coordinates the canonical relations for $R$ and $E (= E(T))$ are

$$C(R) \subseteq V \times U$$

$$= \{(y, t, \eta, \tau), (x, \xi) : x - t \frac{\xi}{|\xi|} = k(y), k_y^\xi = \eta, |\xi| = \tau\}$$

and
\[ C(E) = \{(z, \zeta, (y, t, \eta, \tau) : z + (T - t) \frac{\zeta}{|\zeta|} = k(y), k_y^*(\zeta) = \eta, \tau = |\zeta| \} \]

(Recall \( k \) is the inclusion \( \partial K \subseteq \mathbb{R}^n \).) To any such relation \( C \) we associate two mappings \( \pi_1 \) and \( \pi_2 \) which are simply projections of \( C \) onto the first or second factors of the product space in which \( C \) lives. Such a projection is said to be a simple fold if there exists a hypersurface in \( C \) on which the rank of the projection map drops by 1. Since we are assuming that \( \partial K \) is convex, it follows that both projections \( \pi_1 \) and \( \pi_2 \) are simple folds on each of the relations \( C(R) \) and \( C(E) \). A canonical relation with this property is said to be a folding canonical relation. Following [6] we associate several objects to such a relation \( C \). First is a folded symplectic structure on \( C \) obtained by pulling back the usual symplectic form on either factor. Next, there are smooth involutions on \( C \) denoted \( J_1 \) and \( J_2 \) obtained by switching the preimages of the projections \( \pi_1 \) and \( \pi_2 \) respectively. Finally, the image of \( C \) under \( \pi_1 \) is a symplectic manifold with boundary, on which are so-called boundary maps \( \delta_{1}^{\pm} \) given by \( \delta_{1}^{\pm} = \pi_1 \circ J_2 \circ \pi_1^{-1} \) where \( \pm \) depends on the choice of continuous inverse of \( \pi_1 \). A similar definition holds for \( \delta_{2}^{\pm} \) on \( \pi_2(C) \). For example, \( \pi_1(C(R)) = \pi_2(C(E)) = \{(y, t, \eta, \tau) : \inf_{\eta=k_y^*\xi} |\xi| \leq \tau \} \). Here the
maps $\delta^\pm$ are obtained by finding $\xi$ such that $k^*_y \xi = \eta$, $|\xi| = \tau$ and flowing along the Hamiltonian vector field $H_{|\xi|^2 - \tau}$ in $T^*(\mathbb{R}^{n+1})$ until you again lie over $\partial K$.

We say two such folding canonical relations are equivalent if there exist homogeneous symplectic maps of the base spaces such that the product map intertwines the canonical relations. It is proved in [6] that any two folding canonical relations of a fixed dimension are equivalent. We need something slightly stronger than this since we must keep track of the composite relation $C(R) \circ C(E)$. We will use the following as a standard model.

$$C(A) \subseteq \mathbb{R}^{2n} \times \mathbb{R}^{2n}$$

with coordinates $(x, \xi), (\bar{x}, \bar{\xi})$

given by

$$x_1 = \bar{x}_1 \pm \frac{1}{3} \left( \frac{-\xi_n}{\xi_1} \right)^{3/2}$$

$$x_2 = \bar{x}_2$$

$$\vdots$$

$$x_{n-1} = \bar{x}_{n-1}$$

$$x_n = \bar{x}_n \pm \left( \frac{-\xi_n}{\xi_1} \right)^{1/2}$$

$$\xi = \bar{\xi}.$$
We will see in the next section that this is the relation for the operator $G_i$. Using $(x, \xi', \bar{x}_n)$ as coordinates on $C(A)$ with $\xi' = (\xi_1', \ldots, \xi_{n-1}')$ it is easy to compute the associated objects

$$\pi_1(x, \xi', \bar{x}_n) = (x, \xi', -\xi_1(x_n - \bar{x}_n)^2)$$

$$\pi_2(x, \xi', \bar{x}_n) = (x_1 - \frac{1}{3}(x_n - \bar{x}_n)^3, x'', \bar{x}_n, \xi', -\xi_1(x_n - \bar{x}_n)^2),$$

so

$$J_1(x, \xi', \bar{x}_n) = (x, \xi', 2x_n - \bar{x}_n)$$

$$J_2(x, \xi', \bar{x}_n) = (x_1 - \frac{2}{3}(x_n - \bar{x}_n)^3, x'', 2\bar{x}_n - x_n, \xi', \bar{x}_n)$$

$$\delta_1^{\pm}(x, \xi) = (x_1 \pm \frac{2}{3}\left(\frac{-\xi_n}{\xi_1}\right)^{3/2}, x'', x_n \pm 2\left(\frac{-\xi_n}{\xi_1}\right)^{1/2}, \xi', \bar{x}_n),$$

and finally, the folded symplectic form on $C(A)$

$$\omega_A = \pi_1^*(\gamma d\xi_1 \wedge dx_1) = \pi_2^*(\Sigma d\xi_1 \wedge d\bar{x}_1)$$

$$= \sum_{1}^{n-1} d\xi_1 \wedge dx_1 - d(\xi_1(x_n - \bar{x}_n)^2) \wedge dx_n$$

$$= \sum_{2}^{n-1} d\xi_1 \wedge dx_1 + d\xi_1 \wedge d(x_1 - \frac{1}{3}(x_n - \bar{x}_n)^3) - d(\xi_1(x_n - \bar{x}_n)^2) \wedge d\bar{x}_n.$$
symplectic coordinates \((y, \eta)\) on \(\pi_1(C(R))\) with \(\eta_n \leq 0\), \(\exists \pi_1(C(R)) = \{ \eta_n = 0 \}\), and such that in these coordinates \(\delta^+\) takes the standard form given above. As a consequence, letting \(x_1 = y_1 \circ \pi_1, \xi_1 = \eta_1 \circ \pi_1\), yields smooth functions on \(C(R)\) satisfying

\[
J^*_2(x_1, \ldots, \xi_n)(p) = (x_1(p) \mp \text{sgn } p) \frac{2}{3} \left( \frac{-\xi_n(p)}{\xi_1(p)} \right)^{3/2},
\]

\[
x''(p), x_n \pm \text{sgn } p 2 \left( \frac{-\xi_n(p)}{\xi_1(p)} \right)^{1/2}, \xi(p),
\]

where \(\nu\) is the continuous inverse of \(\pi_1|_{C(R)}\) associated to \(\delta^+\), \(\text{sgn } p = 1\) if \(p \in \text{Im } \nu\), \(\text{sgn } p = -1\) if \(p \notin \text{Im } \nu\). Now we define maps \(\chi^\pm : C(R) \to C(A)\) given in the coordinates of \(C(A)\) used above by

\[
\chi^\pm(p) = \left( x_1(p), \ldots, x_n(p), \xi'(p), x_n(p) \pm \text{sgn } p \left( \frac{-\xi_n(p)}{\xi_1(p)} \right)^{1/2} \right)
\]

\[
= (x(p), \xi'(p), \bar{x}_n(p))
\]

Now

\[
\pi_1 \circ \chi^\pm(p) = (x_1(p), \ldots, x_n(p), \xi_1(p), \ldots, \xi_n(p))
\]

\[
= (y_1 \circ \pi_1(p), \eta_1 \circ \pi_1(p))
\]

so
\[ \chi^\pm \omega = \chi^\pm \left( \pi_1 \Sigma \xi \wedge dx \right) \]

\[ = \omega \mathcal{C}(R) \]

Also \( \chi^\pm J_1 \chi^\mp^{-1} (x, \xi', \bar{x}_n) = J_1 (x, \xi', \bar{x}_n) \) since \( x_1, \ldots, \xi_n \) are clearly \( J_1 \) invariant on \( \mathcal{C}(R) \) and

\[ J_1 \left( x_n (p) + \text{sgn} \left( \frac{-\xi_n (p)}{\xi_1 (p)} \right) \right)^{1/2} = x_n (p) - \text{sgn} \left( \frac{-\xi_n (p)}{\xi_1 (p)} \right)^{1/2} \]

Similarly \( \chi^\pm J_2 \chi^\mp^{-1} (x, \xi', \bar{x}_n) = J_2 (x, \xi', \bar{x}_n) \).

Notice that \( J_1 \circ \chi^\pm = \chi^\mp \). This says that a \( \chi \) can be chosen so that a distinguished component of the fold \( \mathcal{C}(R) \) can be mapped to such in \( \mathcal{C}(A) \).

Now we can extend \( \chi \) to all of \( V \times U \) as follows.

Since \( \chi \) preserves the folded symplectic form and intertwines \( J_1 \), the functions \( (x, \xi', -\xi_1 (x_1 - \bar{x}_n)^2) \) on \( \mathcal{C}(R) \) are \( J_1 \) invariant and project to homogeneous symplectic coordinates \( (x, \xi) \) on \( V \cap \pi_1 (\mathcal{C}(R)) \). These clearly extend to coordinates on all of \( V \). Notice that \( (x, \xi)|\pi_1 (\mathcal{C}(R)) = (y, \eta) \). Similarly we have that \( (x_1 - \frac{1}{3}(x_n - \bar{x}_n)^3, x'', -\xi_1 (x_n - \bar{x}_n)^2, \xi' \bar{x}_n) \) are \( J_2 \) invariant and hence project to homogeneous symplectic coordinates on \( U \cap \pi_2 (\mathcal{C}(R)) \). Before extending these coordinates, we notice that a similar construction for \( \mathcal{C}(E) \), starting with the same coordinates \( (y, \eta) \) on \( \pi_2 (\mathcal{C}(E)) = \pi_1 (\mathcal{C}(R)) \) yields
Proposition 2.1. There exist homogeneous symplectic coordinates

\[ \chi_1 : U \cap \pi_2(X(E)) \to \mathbb{R}^{2n} \]

\[ \chi_J : V \to \mathbb{R}^{2n} \]

\[ \chi_2 : W \cap \pi_1(C(E)) \to \mathbb{R}^{2n} \]

such that

\[ \chi_J \times \chi_1(C(R)) = \chi_2 \times \chi_J(C(E)) \subseteq C(A) \subseteq \mathbb{R}^{4n} \]

Let us examine the composite relation \( C(R) \circ C(E) \subseteq W \times V \). This can be thought of as a triple valued mapping. One branch corresponds to an external reflection, a second corresponds to passing through the boundary once and then being internally reflected, and a third which completely ignores the boundary.
Meanwhile, on the domain of $\chi_1 \times \chi_2$, this gets mapped into the relation $C(A) \circ C(A)$, which corresponds to

$$
(x, \xi) \mapsto \{x_1 \pm \frac{2}{3} \left(\frac{-\xi}{\xi_1}\right)^{3/2}, x', x_n \pm \frac{2}{3} \left(\frac{-\xi}{\xi_n}\right)^{1/2}, \xi\}, (x, \xi)\}.
$$

Now we can go back to the proof of the last proposition and notice that we can pick $\chi^\pm$ so that the first branch in $C(R) \circ C(E)$ above corresponds to $(x, \xi) \mapsto (x_1 + \frac{2}{3} \left(\frac{-\xi}{\xi_1}\right)^{3/2}, \ldots, \xi_n)$ and the third branch corresponds to the identity map. Finally, notice that this third branch is the map $(x, \xi) \mapsto (x - T \frac{\xi}{\xi_1}, \xi)$ restricted to $\pi(U)$.

**Proposition 2.2.** $\chi_1$ and $\chi_2$ can be extended to homogeneous symplectic coordinates on $U$ and $W$ respectively so that $\chi_1 \times \chi_2$ maps the relation in $U \times W$ corresponding to $(x, \xi) \mapsto (x - T \frac{\xi}{\xi_1}, \xi)$ into the identity relation.

**Proof.** In fact, pick any extension of $\chi_1$ to $U$. We know that if $(x, \xi) \in \pi_2(C(R))$ then $(x - T \frac{\xi}{\xi_1}, \xi) \in \pi_1(C(E))$, and $\chi_2(x - T \frac{\xi}{\xi_1}, \xi) = \chi_1(x, \xi)$. So on $\{ (x, \xi) \mid (x + T \frac{\xi}{\xi_1}, \xi) \in U \}$ let $\chi_2(x, \xi) = \chi_1(x + T \frac{\xi}{\xi_1}, \xi)$. This clearly does the trick.
§3. Fourier integral operators with folding canonical relations

Adjust $U, V$ and $W$ so that their images under $\chi_1$, $\chi_2$, and $\chi_J$ respectively are a fixed conic neighborhood $\Gamma \subseteq \mathbb{R}^{2n}$ about $(x, \xi) = (0, (1, 0, \ldots, 0))$. Here conic means conic in the $\xi$ directions. Also assume $|\xi_i| \leq \frac{1}{c_1} |\xi_1|$ on $\Gamma$ for $i < n$. Choose elliptic Fourier integral operators $K, J,$ and $L$ corresponding to the maps $\chi_1, \chi_J,$ and $\chi_2$ respectively. Then modulo smoothing operators

$$e^{iT^\Lambda} = F(T) - E(T)R$$

$$= (L^{-1}L)(F(T) - E(T)\circ R)K^{-1}K$$

$$= L^{-1}(\tilde{F} - \tilde{E})K$$

where $\tilde{E} = LEJ^{-1}$ and $\tilde{R} = JRK^{-1}$ have canonical relation $C(A)$ of the previous section. Another such operator is given by $G_1$ which is defined as follows.

Recall $A_i$ is given by the integral, for $|\xi_n| \leq c_1$,

$$A_i(\xi_1^{1/3} \xi_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp i (t\xi_1^{1/3} \xi_n + t^3) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp i (\xi_1^{1/3} t + \xi_n^{-2} t^3) \xi_1^{-2/3} dt$$

Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ on $[-2, 2]$. A simple stationary phase argument yields
\[ \text{Ai}(\xi_1^{-1/3} \xi_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\frac{t}{\xi_1}) \exp i (\xi_1^{-1/3} \xi_n t + \frac{1}{3} \xi_1^{-2/3} t^3) \, dt \]

\[ + O(\xi_1^{-N}) \quad \text{for all } N \quad \text{as } \quad |\xi| \to \infty. \]

So, with \( \varphi(x,y,t,\xi) = (x-y) \cdot \xi + \xi_1^{-1/3} \xi_n t + \xi_1^{-2/3} t^3 \), for \( u \in \mathcal{E}_{\Gamma}(\mathbb{R}^n) \) we have, modulo a smoothing operator,

\[ \partial^\mu u(x) = (2\pi)^{-n-1} \int e^{i\varphi(x,y,t,\xi)} \chi(\frac{t+\xi_1^{-1/3}}{\xi_1}) \frac{2/3}{\xi_1^{-1/3}} u(y) \, dy \, dt \, d\xi. \]

Here \( c \) is chosen so that on \( \Gamma \) we have \( |\xi_n| \leq \frac{1}{c} \xi_1 \).

Similarly, \( \partial^\mu u \) is given by the same expression with the integrand multiplied by \( i \frac{t}{\xi_1^{-1/3}} \). Notice that these are operators of order \( -\frac{1}{6} \) and modulo smoothing operators are the same as Fourier multiplication by \( \text{Ai}(\xi_1^{-1/3} \xi_n) \) and \( \xi_1^{-1/3} \text{Ai}'(\xi_1^{-1/3} \xi_n) \). In this section we show that these are essentially the only examples.

**Proposition 3.1.** Let \( B \) be a Fourier integral operator of order \( m \) acting on \( \mathcal{E}_{\Gamma} \) with canonical relation \( \mathcal{C}(A) \).

Then there exist \( P_1, P_2 \in \text{OPS}^{m+1/6}(\Gamma) \) such that

\[ B = P_1 \partial^\mu + P_2 \partial^\mu \]

modulo a smoothing operator.

**Proof.** We may assume \( B \) is given by an operator of the form
\[ B_u(x) = (2\pi)^{-n-1} \int e^{i\phi} a(x, y, t, \xi) \xi_1^{-2/3} u(y) dy dt d\xi \]

where \( a \in S^{m+1/6} \). We will show that asymptotic expansions for \( p_1 \) and \( p_2 \) can be chosen so that the symbol of \( B - p_1 G_1 - p_2 G_1' \) vanishes to arbitrarily negative order. We begin by considering a point \((x_0, \zeta_0, y_0, \sigma_0) \in T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^n)\). Choose a smooth function \( \psi(x, y) \) such that \( \nabla_{x, y}\psi(x_0, y_0) = (\zeta_0, \sigma_0) \) and if \( \nabla^{(\omega-\psi)}(x_0, y_0, t_0, \xi_0) = 0 \) then \( \nabla^{(\omega-\psi)}(x_0, y_0, t_0, \xi_0) \) is non-singular.

If these conditions are satisfied we must have

\[ \xi_0 = \zeta_0 = -\sigma_0 \]
\[ t = \pm(\xi_{01} \xi_{0n})^{1/2} \]

and

\[ y_0 = x_0 \pm (\frac{1}{3}(\frac{-\xi_{0n}}{\xi_{01}})^{3/2}, 0, \ldots, 0, (\frac{-\xi_{0n}}{\xi_{01}})^{1/2}) \]

\[ = x_0 \pm (\eta(\xi_0)) \cdot \]

Consequently, we have, if \( u \in C_0^0(\mathbb{R}^n \times \mathbb{R}^n) \) is supported near \((x_0, y_0)\), and \( K \) is the distribution kernel for \( B - p_1 G_1 - p_2 G_1' \), the leading term in the asymptotic expansion for \( \langle K, u e^{-i\lambda \psi} \rangle \) is
\[ e^{-i\lambda \psi(x_0,y_0)} \frac{1-n}{2} \left| \det \nabla^2 (\varphi_0^\psi)(x_0,y_0,t_0,\xi_0) \right|^{-1/2} \]
\[ = e^{i \frac{\pi}{2} \text{sgn} \nabla^2 \varphi_0^\psi \xi_0^{-2/3}} \times \]
\[ \times [a(x_0,y_0,\lambda t_0,\lambda \xi_0)-p_1(x_0,\lambda \xi_0)-i \frac{t_0}{\xi_0} p_2(x_0,\lambda \xi_0)] \]
\[ p_{20}(x, \xi) = \frac{-i}{2} [a_0(x, x+\eta, (-\xi_1 \xi_n)^{\frac{1}{2}}, \xi) - a_0(x, x-\eta(\xi), (-\xi_1 \xi_n)^{\frac{1}{2}}, \xi)] \cdot \frac{-\xi_1}{\xi_n} \]

on \( \xi_n \leq 0 \) and extend to \( \xi_n > 0 \) as before. The above argument establishes that \( B - P_1 G_i - P_2 G_i' \) is an operator of order \( m-1 \). An inductive argument establishes the proposition.

As an example of this, and also for later use we notice

**Lemma 3.1.** Suppose \( G_i P = P_1 G_i + P_2 G_i' \)

\[ G_i' P = P_3 G_i + P_4 G_i' \]

where \( P \in \text{OPS}^m(\Gamma) \). Then the principal symbols satisfy

\[ p_{1m}(x, \xi) = \frac{1}{2} [p_{m}(x+\eta(\xi), \xi) + p_{m}(x-\eta(\xi), \xi)] \]

\[ p_{2m}(x, \xi) = \frac{-\xi_1}{\xi_n} \frac{-\xi_1}{\xi_n} \frac{-1}{2} [p_{m}(x+\eta(\xi), \xi) - p_{m}(x-\eta(\xi), \xi)] \]

\[ p_{3m}(x, \xi) = \frac{-\xi_1}{\xi_n} \frac{+\xi_1}{\xi_n} \frac{i}{2} [p_{m}(x+\eta(\xi), \xi) - p_{m}(x-\eta(\xi), \xi)] \]

\[ p_{4m}(x, \xi) = \frac{1}{2} [p_{m}(x+\eta(\xi), \xi) + p_{m}(x-\eta(\xi), \xi)] \]

Recall \( \eta(\xi) = \left( \frac{1}{3} \left( \frac{-\xi_1}{\xi_n} \right)^{3/2}, 0, \ldots, \left( \frac{-\xi_1}{\xi_n} \right)^{1/2} \right) \).
§4. The Neumann operator

The last proposition does not apply to the operator $\tilde{E}$ since $E$ does not have a classical symbol. However, from section 1 we can write $E(T) = G_1 + G_2 N$ where $G_1$ and $G_2$ do have classical symbols. So $\tilde{E} = LE(T)J^{-1} = \tilde{G}_1 + \tilde{G}_2 JN^{-1}$ and proposition 3.1 applies to $\tilde{G}_1$. Next let us turn to $JN^{-1}$.

**Proposition 4.1.** $J$ can be chosen so that, on $\xi^\prime_T$, $JN^{-1} = AQ + B$ where $A, B \in \text{OPS}^1(\Gamma)$ and $Q \in \text{OP}h_0^{1/3}$ has symbol $q(\xi) = \xi_1^{1/3} \frac{A^\prime}{A_-} (\xi_1^{1/3} \xi_n)$.

**Proof.** This result is contained in proposition 3.20 of [8], which uses the results of [7]. The choice of $A_-$ here comes from the observations that points in $C(R)$ have positive $\tau$ component, and that the extension map $E$ must propagate singularities forward in time as they leave the boundary. Inspection of the parametrices in [4], [10] or [11] shows that $A_-$ has the correct oscillatory behavior for this.
§ 5. Proof of theorem 1

Recall that with $P \in \text{OPS}^m(U)$, $(F(T) - E(T)R)P = L^{-1}SK$

where

$$S = (\tilde{F} - (\tilde{G}_1 + G_2(AQ + B))\tilde{R})\tilde{P}$$

$$= (\tilde{F} - (\tilde{G}_3 + \tilde{G}_4 Q)\tilde{R})\tilde{P}.$$  

Here $\tilde{G}_3 = \tilde{G}_1 + G_2B$, $\tilde{G}_4 = \tilde{G}_2A$, and $\tilde{R}$ are Fourier integral operators of order zero with canonical relation $C(A)$, $\tilde{F}$ is an elliptic operator in $\text{OPS}^0(\Gamma)$, and $\tilde{P} \in \text{OPS}^m(\Gamma)$ has principal symbol $\tilde{p}_m(x, \xi) = p_m(\gamma_1^{-1}(x, \xi))$. Repeated use of proposition 3.1 now yields

$$S = [P_0 + (P_1 G_1 + P_2 G_1') + (P_3 G_1 + P_4 G_1')Q)(P_5 G_1 + P_6 G_1')] \tilde{P}$$

$$= [P_0 + P_7 G_1^2 + P_8 G_1 G_1' + P_9 G_1'^2] \tilde{P}$$

$$= [P_{10} + P_{11} G_1^2 + P_{12} G_1 G_1' + P_{13} G_1'^2]$$

where $P_{10} \in \text{OPS}^m(\Gamma)$ and for $i = 11, 12, 13$, $P_i \in \text{OPS}^{m + \frac{1}{3}}$, is of the form $P_\alpha + P_\beta Q_\gamma$ with $P_\alpha \in \text{OPS}^{m + \frac{1}{3}}$, $P_\beta \in \text{OPS}^{1/6}$, and $P_\gamma \in \text{OPS}^{m + \frac{1}{6}}$. Hence each symbol $P_i$ has an asymptotic expansion of the form

$$p_i(x, \xi) \sim p_i^0 + \sum_{\alpha \geq 0} p_i^\alpha(x, \xi) D_\xi^\alpha q(\xi).$$
where each \( p^\alpha_1 \in S^{m+1/3} \). For notational convenience we rewrite \( S \) as

\[
S = p_0 + p_1 G i^2 + p_2 G i G i' + p_3 G i^2.
\]

Since \( L^{-1} SK = \exp(iT\sqrt{-\Delta}) P \) modulo a smoothing operator, propagation of singularities results for solutions to the wave equation imply that for \( u \in \mathcal{E}^\prime_T(X) \) we have

\[
WF(Su) \subset \left\{ \begin{array}{l}
\xi_n \geq 0 \text{ and } (x, \xi) \in WF(U)
\\
(x, \xi) : \text{ or }
\\
\xi_n \leq 0 \text{ and } (x - \frac{4}{3}\frac{\xi_n}{\xi_1^\alpha}, x'', x_n - 4\frac{\xi_n}{\xi_1^\alpha}, \xi) \in WF(U)
\end{array} \right\}
\]

This observation allows the proof of

**Proposition 5.1.** \( S \frac{G}{G_+} \in \text{Op}\mathcal{M}^{m}_{1/3} \).

**Proof.** Let \( \zeta = \zeta(\xi) = \xi_1^{-1/3} \xi_n^\alpha \). We will show

\[
a(x, \xi) = (p_0(x, \xi) + \ldots + p_3(x, \xi)(\xi_1^{-1/3} Ai'(\zeta)) 2) \frac{A_-}{A_+}(\zeta) \in \mathcal{M}^{m}_{1/3}.
\]

The idea is to use the identity \( Ai = -\omega A_+ - \tilde{\omega} A_- \) to remove all the \( Ai \) Airy functions from \( a \) and then break this expression up into 3 terms according to oscillatory behavior. Due to the exponential behavior of \( A_\pm(\zeta) \) as \( \zeta \to +\infty \), we
first introduce a cutoff in the class \( \eta_{1/3} \). Choose \( d > 0 \) and let \( q_1 \in C^\infty(\mathbb{R}) \) satisfy \( q_1(\zeta) = 1 \) for \( \zeta \leq -2d \) and \( q_1(\zeta) = 0 \) for \( \zeta \geq -d \). That \( q_1(\zeta(\xi)) \in \eta_{1/3}^0 \) follows from proposition 0.1. Now, with \( \zeta = \zeta(\xi) \),

\[
a(x, \xi)q_1(\zeta) = \left( w^2 [p_1A_+^2 + p_2\xi_1^{-1/3}A_+A_+ + p_3(\xi_1^{-1/3}A_+^2)]
+ [p_0 + 2p_1A_+A_- + p_2\xi_1^{-1/3}(A_+A_- + A_-^2) + 2p_3(\xi_1^{-2/3}A_+A_-)]
+ \omega^2 [p_1A_-^2 + p_2\xi_1^{-1/3}A_-A_- + p_3(\xi_1^{-1/3}A_-^2)] \right) \frac{A_-}{A_+} q_1.
\]

Making the substitution \( A_\pm(\zeta) = \frac{2}{3} i(-\zeta)^{\frac{3}{2}} \) coming from the asymptotic expansion given in the introduction, and repeatedly using proposition 0.1 shows that

\[
aq_1 = r_1 + r_2e^{-\frac{4}{3} i(-\zeta)^{\frac{3}{2}}} + r_3e^{-\frac{8}{3} i(-\zeta)^{\frac{3}{2}}}
\]

with each \( r_i \in \eta_{1/3}^{m+1/3} \), and is supported on \( \zeta \leq -d \).

Consequently, if \( \Gamma' \subseteq \Gamma \cap \{ \xi_n < 0 \} \) is any closed cone, as operators acting on \( c_1(\Gamma', X) \),

\[
S_{G_\Gamma}^{-1} q_1(D) = R_1 + R_2 + R_3
\]

modulo a smoothing operator. Here \( R_j \) is the Fourier integral operator given by
\[ R_j u = 2^n \int r_j(x, \xi) e^{i(x \cdot \xi - (j-1) \frac{4}{3} i(-\xi)^T) \frac{3}{2}} u(\xi) d\xi. \]

As such, the operators \( R_1, R_2, \) and \( R_3 \) have mutually distinct canonical relations, and furthermore only the relation for \( R_1 \) matches that of \( \mathcal{S} \frac{G}{G^+} q_1 \) anywhere on \( \Gamma' \). Therefore \( R_2 \) and \( R_3 \), when acting on \( \mathcal{E} \frac{G}{G^+} (R^n) \) must be smoothing operators. This in turn implies that when \( r_2 \) and \( r_3 \) are restricted to \( \Gamma' \) they become symbols in \( \mathcal{S}^{\infty}_{1,0}(\Gamma') \).

Also, since \( \zeta \to -\infty \) in \( \Gamma' \), substitution of the asymptotic expansions for \( \psi_\pm \) shows

\[ r_1 \sim q_1(\zeta) \sum_{j \geq 0} r_{ij}(x, \xi) \]

with \( r_{ij} \) homogeneous of degree \( m-j \) in \( \xi \). Notice that this expansion is valid in \( C^\infty \) on any subset of \( \Gamma \) for which \( |\zeta| \to \infty \) as \( |\xi| \to \infty \). Since \( R_2 \) and \( R_3 \) are smoothing operators, we must have \( r_{ij} = 0 \) on \( \Gamma' \). Since the choice of \( \Gamma' \) is arbitrary, we also have \( r_{ij} = 0 \) whenever \( \zeta \leq -d \). Finally, since the choice of \( d \) was arbitrary, we have \( r_{ij} = 0 \) on \( \zeta < 0 \), for \( i = 2,3 \). Similarly \( D^\alpha_x D^\beta_\xi r_{ij}(x, \xi) = 0 \) on \( \zeta < 0 \) for all multi-indices \( \alpha, \beta \).

Let us examine the condition that \( r_{30} = 0 \). Write
\[ p_i = p_{i0} + p_{i1}q + \eta_{1/3}^{-1/3} \] where \( p_{ij} \) are homogeneous of degree \( m + \frac{1}{3} \) in \( \xi \). Here we have used that \( D_{\xi}q \in \eta_{1/3}^{-2/3} \) for \( i = 1, \ldots, n \). A computation yields

\[
2\pi e^{-\frac{1}{3} \pi i} r_{30} = (p_{10} + ip_{11}(\xi_{1})^{-\frac{1}{2}})(-\zeta)^{-\frac{1}{2}} + \frac{1}{3} (p_{20} + ip_{21}(\xi_{1})^{-\frac{1}{2}})
- \frac{2}{3} (p_{30} + ip_{31}(\xi_{1})^{-\frac{1}{2}})(-\zeta)^{-\frac{1}{2}} = 0.

Since this must vanish on \( \zeta < 0 \) and each \( p_{ij} \) is smooth across \( \zeta = 0 \) we have

\[ p_{10}(x, \xi', 0) = 0. \]

This observation allows us to show that each symbol \( r_i \) above for \( i = 1, 2, 3 \) is actually in \( \eta_{1/3}^m \). For instance,

\[
r_1 = \omega^2 [p_1q_{-1} + p_2\xi_1^{-1/3}q_{-1} + p_3\xi_1^{-2/3}q_{-1}] \frac{A_{-1}q_{-1}}{A_{+1}}.
\]

Now \( q_{-1} = \xi_1^{-1/3}A_{-1}q_{-1} \in \eta_{1/3}^{-1/3} \) by proposition 0.1 so

\[ r_1 = \omega^2 p_{10}A_{-1}q_{-1} + \eta_{1/3}^m. \] But \( p_{10}(x, \xi) = \tilde{p}_{10}(x, \xi) \zeta \) where \( \tilde{p}_{10} \) is homogeneous of degree \( m - \frac{1}{3} \) in \( \xi \), so \( p_{10}A_{-1}q_{-1} = \tilde{p}_{10}(x, \xi)(\zeta A_{+1}q_{-1}(\zeta)) \in \eta_{1/3}^m \). The terms \( r_2 \) and \( r_3 \) are handled by a slight modification of this argument. For example, \( r_2 \) is a sum of four terms such as
\[(p_0) \left( \frac{A}{A^+} q_1 e^{\frac{4}{3} i \zeta^2} \right) \in S^m \eta_1^{0 \frac{3}{2}}.\]

The next step is to show that \( r_2 e^{\frac{3}{2} \frac{4}{3} i (-\zeta)^2} \) and \( r_3 e^{\frac{3}{2} \frac{8}{3} i (-\zeta)^2} \) are in \( \eta_1^m \). In fact, since \( r_2 \) is supported on \( \zeta \leq -d \), the vanishing of its classical expansion on any cone of the form \( r' \) above implies

\[
|D_{x, x, x}^{\alpha} D_{\xi, \xi, \xi}^{\delta} r_2(x, \xi)| \leq C_{k, \alpha, \beta, \delta, \xi, \eta}^m |\beta| - \frac{1}{3} \delta \ (1-\zeta)^{-\delta+k}
\]

for any \( k \). So, with \( C \) depending on all indices,

\[
|D_{x, x, x}^{\alpha} D_{\xi, \xi, \xi}^{\delta} r_2(x, \xi)e^{-\frac{4}{3} i (-\zeta)^{3/2}}| \leq \sum_{\gamma_1+\gamma_2=\gamma, \delta_1+\delta_2=\delta} C |D_{x, x, x}^{\alpha} D_{\xi, \xi, \xi}^{\delta} r_2||D_{\xi, \xi, \xi}^{\gamma_1} D_{\xi, \xi, \xi}^{\gamma_1} e^{-\frac{4}{3} i (-\zeta)^{3/2}}|
\]

\[
\leq \sum_{\xi_1} C \xi_1^m |\beta| - \frac{1}{3} \delta_1 (1-\zeta)^{-\delta_1+k} (-\zeta_1)^{-\delta_2} (-\zeta_2)^{\frac{3}{2} (\delta_2+\gamma_2)}
\]

\[
= \sum_{\xi_1} C \xi_1^m |\beta| - \frac{1}{3} \delta (1-\zeta)^{-\delta_1+k} (-\zeta)^{\frac{3}{2} (\delta_2+\gamma_2)}
\]

and since \( \zeta \leq -d \) on \( \text{supp } r_2 \), with \( k = -\frac{3}{2} (\delta_2+\gamma_2) \).
\[ \leq C \xi_1 \left( 1 - \zeta \right)^{-\delta} \frac{4}{3} i(-\zeta)^{3/2} \]

Therefore \[ r_2 e^{-\frac{4}{3} i(-\zeta)^{3/2}} \in \eta_1^{\delta} \] Similarly \[ r_3 e^{-\frac{8}{3} i(-\zeta)^{3/2}} \in \eta_1^{\delta} \]. Hence \( a q_1 \in \eta_1^{\delta} \).

To finish off the proposition we show \( a q_2 \in \eta_1^{\delta} \) where \( q_2 = 1 - q_1 \). For this we notice that \( A_i \) and \( A_i' \) are rapidly decreasing as \( \zeta \to \infty \) so that \( \frac{A}{A_+} q_2 \), \( \frac{A_i^2 A - A_i^2}{A_+} q_2 \), \( \frac{A_i A_i'}{A_+} q_2 \), and \( \frac{A_i'A}{A_+} q_2 \) are all in \( \eta_1^{\delta} \). So

\[ a q_2 = (p_0 + \tilde{p}_1 0 \zeta A_i^2 + (p_1 - \tilde{p}_1 0 \zeta) A_i^2 + \xi_1^{-1/3} p_2 A_i A_i' + \xi_1^{-2/3} p_3 A_i' A_i'^2) \frac{A}{A_+} q_2 \in \eta_1^{\delta} \]

where \( p_1 - \tilde{p}_1 0 \zeta = p_1 0 q + \eta_1^{\delta} \). This establishes the proposition.

To prove theorem 1 we simply let \( \tilde{p} = S \frac{G}{G_+} \in \Omega \eta_1^{\delta} \) so that, modulo smoothing operators,

\[ e^{iT \tilde{p}} = L^{-1} S K = L^{-1} \frac{G}{G_+} K \]

It remains to be shown that \( \tilde{p} \) is elliptic whenever \( p \) is. Recall that the vanishing of \( r_{30} \) in the proof of the last proposition led to an identity among the leading terms in the \( p_1 \)'s. The vanishing of \( r_{20} \) yields another identity which we record.
\[ 2\pi p_{00} + (p_{10} + ip_{11} \left( \frac{-\xi n}{\xi_1} \right)^{1/2} \right) (-\zeta)^{-1/2} + (p_{30} + ip_{31} \left( \frac{-\xi n}{\xi_1} \right)^{1/2} \right) (-\zeta)^{1/2} = 0. \]

Here \( p_{00} \) is homogeneous of degree \( m \) in \( \xi \) and \( p_0 = p_{00} + s^{m-1} \). Recalling that \( p_{10}(x, \xi', 0) = 0 \) yields

\[ 2\pi p_{00}(x, \xi', 0) + i\xi_1^{-1/3} p_{11}(x, \xi', 0) = 0, \]

and from the vanishing of \( r_{30} \),

\[ p_{11}(x, \xi', 0) = -p_{20}(x, \xi', 0). \]

Now \( \frac{A_+}{A_-} \) is bounded and bounded away from zero so it suffices to show

\[ |p \frac{A_+}{A_-}| \geq C|\xi|^m \]

for large \( \xi \). With \( q_1 \) and \( q_2 \) as in the proof of the previous proposition we have

\[
\frac{A_+}{A_-} q_1 = (p_0 + p_1 A_+^2 + \ldots + p_{3} \xi_1^{-2/3} A_+^2) q_1
\]

\[
\begin{align*}
&= \tilde{\omega}(p_1 A_+^2 + p_2 A_+^{-1/3} A_+ A_- + p_3 \xi_1^{-2/3} A_+^2) q_1 + O(\zeta^{-N})
\end{align*}
\]

for any \( N \) as \( \zeta \to \infty \)

\[
= \frac{-1}{4\pi} \left( (p_{10} + ip_{11} \left( \frac{-\xi n}{\xi_1} \right)^{1/2} \right) - i\xi_1^{-1/3} (p_{20} + ip_{21} \left( \frac{-\xi n}{\xi_1} \right)^{1/2} \right) - \xi_1^{-2/3} (p_{30} + ip_{31} \left( \frac{-\xi n}{\xi_1} \right)^{1/2} \right) (-\zeta)^{1/2}) q_1 + O(\zeta^{m-1}).
\]
So by the vanishing of $r_{30}$ and $r_{20}$

$$\frac{1}{2\pi} \left( (p_{10} + ip_{11}(-\xi_1^{1/2}))(-\zeta)^{1/2} - \xi_1^{-2/3} (p_{30} + ip_{31}(-\xi_1^{1/2}))(-\zeta)^{1/2} \right) \times q_1 + O(\zeta^{-m-1})$$

$$= - \frac{1}{2\pi} \left( 2\pi p_{00} + 2(p_{10} + ip_{11}(-\xi_1^{1/2}))(-\zeta)^{1/2} \right) q_1 + O(\zeta^{-m-1}) .$$

This is a homogeneous function multiplied by $q_1$, which when evaluated at $\xi_n = 0$ yields

$$\frac{1}{2\pi} \left( i\xi_1^{-1/3} p_{11}(x,\xi',0) \right) = p_{00}(x,\xi',0) .$$

If the operator $P$ has symbol $p$ then we also have

$p_{00}(x,\xi',0) = p(\xi_1^{-1}(x,\xi',0)) + O(\xi_1^{-m-1})$ . Now $p_{00}$ is a homogeneous function which is nonvanishing on $\xi_n = 0$ and hence in a conic neighborhood of $\xi_n = 0$ which we may take to define $\Gamma$ . The above computation now shows there exists a $d > 0$ such that $\tilde{p}$ is elliptic on $\zeta < -d$ . Next consider

$\tilde{p}_{A+}^A q_2 = (p_0 + p_1 A_1^2 + p_2 \xi_1^{-1/3} A_1 A_1') q_2 + O(\xi_1^{-m-1/3})$

$$= (p_{00} + p_{11} q A_1^2 + \xi_1^{-1/3} p_{20} A_1 A_1') q_2 + O(\xi_1^{-m-1/3}) .$$

Notice

$p_{11}(x,\xi) = \xi_1^{1/3} 2\pi i p_{00}(x,\xi) + a \text{ term vanishing on } \xi_n = 0$
and
\[ p_{20}(x, \xi) = -\xi_{1}^{1/3} 2\pi i p_{00}(x, \xi) + \text{a term vanishing on } \xi_{n} = 0 \]

so
\[ p_{A_{-}}^{A_{+}} q_{2} = p_{00}(1 + 2\pi i (\frac{A'}{A_{-}} \text{Ai}^2 - \text{AiAi}')) q_{2} + O(\xi_{1}^{m - 1/3}) \]

\[ = -\omega p_{00} \frac{A_{+}}{A_{-}} q_{2} + O(\xi_{1}^{m - 1/3}) . \]

Hence \( \frac{A_{+}}{A_{-}} \) is elliptic on \( \zeta > -d \) for any \( d > 0 \). This finishes the proof of theorem 1.

We remark that the identity
\[ -\omega \frac{A_{+}}{A_{-}} = 1 + 2\pi i (\frac{A'}{A_{-}} \text{Ai}^2 - \text{AiAi}') \]
provides a means for straightforwardly extending \( \frac{G_{+}}{G_{-}} \) to distributions.
§6. A weak version of Theorem 2

Suppose $P \in \text{OPS}^m(\Omega)$ satisfies the conditions for Theorem 2 of the introduction. That is, the distribution kernel for $P$ has compact support, and the pair $(T, ES(P))$ satisfies condition (*). In order to exploit the above representation of $e^{it\Lambda}$ we decompose $P$ by a microlocal partition of unity. Choose smooth partitions of unity $\{\varphi_i\}$ of $\Omega$ and $\{\psi_j\}$ of the sphere $S^{n-1}$. By considering $\psi_j$ as a function on $n$ homogeneous of degree 0 we write

$$P = \sum_{i,j} \varphi_i(x)P(x,D)\psi_j(D) = \sum_{i,j} P_{ij}(x,D)$$

so that

$$e^{it\Lambda}Pe^{-it\Lambda} = \sum_{i,j} e^{it\Lambda}P_{ij}e^{-it\Lambda}$$

We can choose things such that this is a finite sum and each symbol $p_{ij}$ is supported in an open cone of one of the following three types.

**Type 1.** A cone of the type $U_1$ for which theorem 1 is valid.

**Type 2.** A cone whose closure contains no grazing directions and whose image under the canonical relation $C(R)$ is non-empty and lies over a coordinate chart in $\mathbb{R}K \times \mathbb{R}$.

**Type 3.** A cone whose closure contains no grazing directions
and whose image under $C(R)$ is empty.

First suppose $P$ has symbol $p$ supported in a cone $U_1$ of type 1. The corresponding cone $U_2$ from theorem 1 is by construction a neighborhood of the image of $U_1$ under the broken canonical relation for $e^{it\Lambda}$. When acting on $\epsilon_{U_1}$, we write

$$e^{it\Lambda} = L^{-1}Q_0 \frac{G_+}{G_-} K$$

so

$$e^{it\Lambda}P = L^{-1}PQ_0^{-1}Q_0^{G_+}K = (L^{-1}PQ_0^{-1}L)Q_0^{G_+}K$$

$$\equiv e^{it\Lambda}$$

modulo smoothing operators where, by proposition 0.2,

$$P \in \text{OPS}^{m+\epsilon}_{\frac{1}{3}, \frac{2}{3}}(U_2)$$

for any $\epsilon > 0$.

Next we handle the case corresponding to a cone $U$ of type 2. This time we notice that the image of $U$ under $C(R)$ consists of two distinct components. The first being contained in
\[(x, y, t, \eta, \tau) \in T^*(\mathfrak{g}K \times \mathbb{R}) : k(y) = x - t \frac{x}{|x|} \text{ for some } (x, \xi) \in U \text{ but } x - s \frac{x}{|x|} \not\in \mathfrak{g}K \text{ for } 0 < s < t\]

and the second contained in a similar set corresponding to the second time that rays starting in \( U \) intersect \( \mathfrak{g}K \). Of course, this second set may be empty if \( T \) is sufficiently small. So we can write \( R = R_1 + R_2 \) in such a way that when restricted to \( \mathcal{C}'(U) \), \( R(T) - E(T)R = -E(T)R_1 + F(T) - E(T)R_2 \) and here \( F(T) - E(T)R_2 \) is a smoothing operator. Next we show that in a conic neighborhood of the image of \( U \) under \( C(R_1) \), the Neumann operator is given by an element of the class \( \text{OPS}^1_{1,0} \). Consider the problem

\[
\begin{align*}
\Box u &= 0 \text{ on } \Omega \times \mathbb{R} \\
u &= 0 \text{ for } t < 0 \\
\big|_{\mathfrak{g}K \times \mathbb{R}} &= f \in \mathcal{C}'(\mathfrak{g}K \times \mathbb{R})
\end{align*}
\]

with \( f \) supported in a coordinate chart on \( \mathfrak{g}K \times \mathbb{R} \). We write \( u \) near \( \text{supp } f \) modulo a smooth error as

\[
u(x, t) = (2\pi)^{-n} \int a(x, t, \eta, \tau) e^{i\varphi(x, t, \eta, \tau)} \hat{f} (\eta, \tau) d\eta d\tau
\]

where
a \big|_{\partial K \times \mathbb{R} \times \mathbb{R}^n} = 1

\varphi \big|_{\partial K \times \mathbb{R} \times \mathbb{R}^n} = y \cdot \eta + \tau \quad (\text{Here } y \text{ is coordinates on } \partial K)

\varphi_t^2 = |\nabla_x \varphi|^2

and \ a \in S^0 \text{ is given by } a = \sum a_j \text{ with } a_j \text{ homogeneous functions solving certain transport equations. One can easily verify that the above non-linear equation for } \varphi \text{ is non-characteristic as long as } (y, t, \eta, \tau) \in W^P \text{ and } W^P \subseteq \text{Image of } U \text{ under } C(R_1). \text{ Hence the Hamilton-Jacobi theory yields two smooth solutions, one of which corresponds to singularities going forward in time in the expression for } u \text{ above. Choosing this solution for } \varphi \text{ and solving the corresponding transport equations defines the above operator. Now with } \nu(x) \text{ a normal vector field yields, in these coordinates,}

Nf = \frac{\partial}{\partial \nu} u(x, t) \big|_{\partial K \times \mathbb{R}} = \int \left( \nu + i a \nu \right) e^{i \nu \varphi(x, t, \eta, \tau)} \big|_{\partial K \times \mathbb{R}} \hat{f}(\eta, \tau) d\eta d\tau.

So \ N \text{ has symbol, in these coordinates, } a + i a \nu \in S^1. \text{ Hence, when acting on } \mathcal{E}^\prime_u(\Omega), e^{i T^1} \text{ is given by the Fourier integral operator } E(T)R_1 \text{ so, modulo a smoothing operator, } e^{i T^1} e^{-i T^1} = E(T)R_1 \mathcal{P}(E(T)R_1)^{-1} \text{ which is a composition of}
operators whose composite canonical relation is the identity map and hence is in $\text{OPS}_{1,0}^m(\Omega)$.

Finally, if the cone $U$ is of type 3, then $R|_{\varepsilon'_{U'}(\Omega)}$ is a smoothing operator, so

$$e^{iT\Lambda}p e^{-iT\Lambda} = F(T)PF(-T) \in \text{OPS}^m(\Omega).$$

Putting this all together yields a weak form of theorem 2.

**Theorem 3.** With $P,T$ as in the hypothesis of theorem 2,

$$e^{iT\Lambda}p e^{-iT\Lambda} \in \text{OPS}_{1,0}^{m+\varepsilon}(\mathbb{R}^n)$$

for any $\varepsilon > 0$. 
§7. A principal symbol and the proof of theorem 2

Suppose near grazing we write $e^{iT\Lambda} P = L^{-1}Q_0 G_+^+ KP = L^{-1}Q_0 G_+^+ (KP)^{-1} G_-^+ \kappa$ rather than $L^{-1}QQ_0^{-1}Q_0 G_+^+ \kappa$. A description of $e^{iT\Lambda} P e^{-iT\Lambda}$ would follow from sufficient information about operators of the form $\frac{G_+}{G_-} P \frac{G_+}{G_-}$ where $P \in S^m_{1,0}(\Gamma)$. We will obtain such information by direct consideration of an oscillatory kernel for $\frac{G_+}{G_-} P \frac{G_+}{G_-}$. This will result in an improvement of theorem 32. The main technique is the method of stationary phase, as in [2], with slight modifications since the phase function will be singular, and the symbols are of bad type. We assume without loss of generality that $P$ has symbol $p$ with $p(x, \xi) = 0$ for $|\xi| \leq 1$.

With $q_1$ and $q_2$ as in §5, recall that $\frac{A_+}{A_-} q_2(\zeta) \in \gamma^m_{1/3}$ so that $\frac{G_+}{G_-} q_2(D_{\xi}) P \frac{G_+}{G_-} \in \text{OP} \gamma^m_{1/3}$ with symbol given by $p(x, \xi) q_2(\zeta) + \gamma^m_{1/3}$. So we will only consider

$\frac{G_+}{G_-} q_1(D_{\xi}) P \frac{G_+}{G_-}$. Standard results yield that this operator is given by

$$u \rightarrow \int e^{ix \cdot \xi} b(x, \xi) u(\xi) d\xi$$

modulo a smoothing operator with
\[ b(x, \xi) = 2\pi^{-n} \int A_+ \left( \zeta(\eta) \right) q_1(\zeta(\eta)) p(y, \xi) \frac{A_-}{A_+} \left( \zeta(\xi) \right) \times \]
\[ \times e^{i(x-y) \cdot (\eta-\xi)} d\eta. \]

Writing \( 2\pi^{-n} A_+ q_1(\zeta) = B(\zeta)e^{\frac{4}{3}i(-\zeta)^{3/2}} \) with \( B \in \mathcal{C}_0^{1/3} \)

supported on \( \zeta(\eta) = \eta_1^{-1/3} \eta_n \leq -d < 0 \), we will derive an

asymptotic expansion for

\[ \tilde{b}(x, \xi) = \int B(\zeta(\eta))p(y, \xi)e^{i(x-y) \cdot (\eta-\xi)} + \frac{4}{3}i(-\zeta)^{3/2} \]
\[ \times \eta\eta d\eta. \]

First let \( \chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) satisfy \( \chi(\xi) = 1 \) for \( |\xi| \leq \frac{\epsilon}{2} \) and \( \chi(\xi) = 0 \) for \( |\xi| \geq \epsilon \). Then

\[ \tilde{b}(x, \xi) = \int (1 - \chi(|\xi|)^{-\frac{1}{3}}(\eta-\xi))B \ldots d\eta + \int \chi(|\xi|)^{-\frac{1}{3}}(\eta-\xi))B \ldots d\eta \]
\[ = b_1 + b_2. \]

The reader should not be too surprised at the following:

**Lemma 7.1.** \( b_1 \in S^{-\infty}(\Gamma) \) and hence \( b_1(x, \xi) \frac{A_-}{A_+} (\zeta(\xi)) \in S^{-\infty}(\Gamma) \).

**Proof.** Let \( L = \frac{1}{4} \frac{\eta_1}{|\eta-\xi|^{2}} \cdot \nabla_y \) so that

\[ b_1(x, \xi) = \int (1 - \chi(|\xi|)^{-\frac{1}{3}}(\eta-\xi))B(\zeta)e^{i(x-y) \cdot (\eta-\xi)} + \frac{4}{3}i(-\zeta(\eta))^{3/2} \]
\[ \times (t_L)^k p(y, \xi) d\eta \]

for any \( k \). Since \( |\eta-\xi| > \frac{\epsilon}{2} |\xi|^{1/3} \) on \( \text{supp}(1-\chi) \) this is an
absolutely convergent integral for \( k \) sufficiently large, and the lemma follows by direct estimation.

An affine change of variables now leaves

\[
b_2(x, \xi) = \int \chi(|\xi|^{-\frac{1}{3}} \eta) B(\zeta(\eta+\xi)) p(y+x-f(\xi), \xi) \times
\]

\[
e^{i(-y \cdot \eta + f(\xi) \cdot \eta + \frac{4}{3} (-\zeta(\eta+\xi))^3/2)}
\]

\[
x e^{-\frac{2}{3} \frac{1}{\eta_1} \epsilon_1, 0, \ldots, 0, 2\frac{1}{\eta_1} \epsilon_1} \] dyd\eta
\]

with \( f(\xi) = \left( \frac{2}{3} \frac{-\epsilon_1}{\xi_1}, 0, \ldots, 0, 2\frac{-\epsilon_1}{\xi_1} \right) \). \( f(\xi) \) makes sense since, on the support of the integrand, we have

\[-\epsilon_1 \geq d(\eta_1+\xi_1) + \eta_1 \geq d(\eta_1 - \epsilon |\xi|) \geq \epsilon |\xi| \geq c_0 \]

with \( c_0 > 0 \) if \( \epsilon \) is sufficiently small. With \( \xi = |\xi| \xi_0 \)

\[
b_2(x, \xi) = \int g(x, \xi, y, \eta) e^{-i|\xi|y \cdot \eta} dyd\eta
\]

where

\[
g(x, \xi, y, \eta) = |\xi|^{2/3} \chi(|\xi|^{-\frac{1}{3}} \eta) B(\zeta(|\xi|(|\eta+\xi_0|))) p(y+x-f(\xi), \xi) \times
\]

\[
e^{i|\xi| (\eta \cdot f(\xi) + \frac{4}{3} (-\zeta(\eta+\xi_0))^3/2)}
\]

As a function of \((y, \eta)\), \( g \) is smooth with compact support, so letting \( \hat{g} \) denote the Fourier transform in \((y, \eta)\) with
dual variables \((\rho, \sigma)\) we have

\[
b_2(x, \xi) = \int \overline{\mathbb{H}}^{-1}(g)(\rho, \sigma) \mathcal{B}(e^{-i|\xi|y \cdot \eta})(\rho, \sigma) d\rho d\sigma
\]

\[
= \left(\frac{2\pi}{|\xi|}\right)^n \int \overline{\mathbb{H}}^{-1}(g)(\rho, \sigma) e^{i\rho \cdot \sigma} / |\xi| d\rho d\sigma
\]

\[
= \left(\frac{2\pi}{|\xi|}\right)^n \sum_{k=0}^{\infty} \frac{|\xi|^{-k}}{k!} (i\rho \cdot \sigma)^k \overline{\mathbb{H}}^{-1}(g)(\rho, \sigma) d\rho d\sigma
\].

The usual argument here is to write down an asymptotic expansion using

\[
\int (\rho \cdot \sigma)^k \overline{\mathbb{H}}^{-1}(g)(\rho, \sigma) = (-1)^k (\nabla_y \cdot \nabla_\eta)^k g(x, \xi, 0, 0)
\]

which is justified by an estimate like

\[
|D_{y, \eta}^\alpha g| \leq c(1 + |\xi|)^{m+\delta} |\alpha| \quad \text{for some } \delta < \frac{1}{2}.
\]

This is false for our \(g\). However \(g\) does satisfy

\textbf{Lemma 7.2.} \(|(\nabla_y \cdot \nabla_\eta)^k g(x, y, \xi, \eta)| \leq C_k (1 + |\xi|)^{m+n+2\frac{2}{3}k}\)

and if \(|\alpha| + |\beta| \leq 2n+2\) then

\[
|D_{y, \eta}^\alpha D_{\eta}^\beta (\nabla_y \cdot \nabla_\eta)^k g| \leq C_k (1 + |\xi|)^{m+n+\frac{4}{3}(n+1)+2\frac{2}{3}k}.
\]

\textbf{Proof.} Notice that on supp \(g\), \(|\eta| \leq \epsilon |\xi|^{-2/3}\) and \(-\xi \cdot 0_n - \eta_n \geq -\tilde{d} |\xi|^{-2/3}\) for \(\epsilon\) sufficiently small. This leads to

\text{a) } |D_{\eta}^\beta |_{\eta} |\chi| |\frac{2}{3}^\eta| \leq C_\beta (1 + |\xi|)^{n+\frac{2}{3} |\gamma|}
b) \[ |D^g B(\zeta(\xi|\xi|\xi_0))| \]
\[ \leq C_B |\xi| |\xi| (|\xi| (n_1+\xi_0))^{1/3} + |\xi| |n_n+\xi_0|)^{-1/3} |\]
\[ \leq C_B |\xi| |\xi| \frac{2}{3} |\xi| \text{ for } |\xi| \geq 1 \]

since \( B \in \mathcal{N}_{1/3}^0 \).

c) \[ |D^\pi_y p(y+x-f(\xi), \xi| \leq C_\alpha (1+|\xi|)^m \]

d) \[ |D^j D^k e_{\eta_1 \eta_n} | \]
\[ \leq C_{j,k} (1+|\xi|)^{2/3} (j+k) \]
on the support of \( g \). To see d) notice that the left hand side is dominated by
\[ \sum_{j_1+\ldots+j_\psi=j_{1} \ldots k_{\psi}} C_{j_1,\ldots,k_{\psi}} |\xi| |D^{j_1 k_1}_{\eta_1 \eta_n} h(\eta,\xi_0)| \ldots \]
\[ |D^{j_\psi k_\psi}_{\eta_1 \eta_n} h(\eta,\xi_0)| \]
where \( h(\eta,\xi_0) = \eta \cdot f(\xi) + \frac{4}{3} (-\zeta(\eta+\xi_0))^{3/2} \). Now
\[ D^\eta h = \frac{2}{3} \left( \left( \frac{-\xi_0}{\xi_0} \right)^{3/2} - \left( \frac{-\xi_0 n - \eta_n}{\xi_0 + \eta_1} \right)^{3/2} \right) \]
\[ D_{n_1} h = 2 \left( \left( \frac{-\varepsilon_0 n}{\xi_0 l} \right)^{1/2} - \left( \frac{-\varepsilon_0 n - \eta n}{\xi_0 + \eta l} \right)^{1/2} \right) \]

so on \ supp g

\[ \frac{1}{c} |D_{n_1} h| \leq |D_{n_2} h| \]

\[ \leq \sup_{0 < t < 1} \left| (-\varepsilon_0 n - t \eta n)^{-1/2} \left[ \frac{-\eta n}{(\xi_0 + t \eta l)^1} + \frac{\eta n}{(\xi_0 + t \eta l)^{3/2}} \right] \right| \]

\[ \leq C|\xi|^{-1/3} \]

if \( \varepsilon \) is sufficiently small. On the other hand, if \( j+k \geq 2 \)
then

\[ |D_{n_1} D_{n_2} h(\eta, \xi_0)| = C_{j,k} \left( -\varepsilon_0 n - \eta n \right)^{3/2} \left( \xi_0 + \eta l \right)^{-1/2} \]

\[ \leq C_{j,k} \max(|\varepsilon|^{2/3}, 1) \]

The estimate d) follows easily from this. Finally, the lemma follows from Leibnitz formula.

Using this lemma we justify the asymptotic expansion which appears below by noting
\[
| \sum_{k=\ell}^{\infty} \frac{1}{k!} \xi^{-k} (i\rho \cdot \sigma)^k \delta^{-1} g(\rho, \sigma) d\rho d\sigma |
\]

\[
= | \xi|^{-\ell} \sum_{k=\ell}^{\infty} \frac{1}{k!} |\xi|^{k-\ell} (i\nabla_y \cdot \nabla_\eta)^{\ell} g(\rho, \sigma) d\rho d\sigma |
\]

\[
\leq C_\ell |\xi|^{-\ell} \int |(1+|\rho, \sigma|^2)^{-1/2} (1-\Delta)^{n+1} (i\nabla_y \cdot \nabla_\eta)^{\ell} g(\rho, \sigma)| d\rho d\sigma ,
\]

where \( \Delta = \text{Laplacian in } y \) and \( \eta \),

\[
\leq C_\ell' |\xi|^{-\ell} \sup_{\rho, \sigma} |(1-\Delta)^{n+1} (i\nabla_y \cdot \nabla_\eta)^{\ell} g| 
\]

\[
\leq C_\ell' |\xi|^{-\ell} \|(1-\Delta)^{n+1} (i\nabla_y \cdot \nabla_\eta)^{\ell} g\|_{L^1(y, \eta)} 
\]

\[
\leq C_\ell'' |\text{supp } g| \ t + m + n + \frac{4}{3}(n+1) + \frac{2}{3} \ell 
\]

\[
\leq C_\ell'' |\xi|^{-\ell} \frac{m'-\frac{1}{3} \ell}{3} .
\]

Here \( |\text{supp } g| \) is the measure of that set. Hence, as \( |\xi| \to \)

\[
\tilde{b}(x, \xi) \sim 2\pi^n \sum_{k=0}^{\infty} \frac{1}{k!} (-i\nabla_y \cdot \nabla_\eta)^k (B(\eta+\xi)p(y-x-f(\xi), \xi) 
\]

\[
i(\eta \cdot f(\xi) + \frac{4}{3}(-\zeta(\eta+\xi))^{3/2}) \bigg|_{y=\eta=0} .
\]

Recall that \( b(x, \xi) = \tilde{b}(x, \xi) \frac{\mathbb{A}}{A^+} (\zeta(\xi)) \) so
\( b(x, \xi) \sim \sum_{K=0}^{\infty} \frac{1}{K!} (-i\nabla_x \cdot \eta)^K [p(x-f(\xi), \xi) \frac{A_+}{A_-} q_1(\xi(\eta))] \)

\[ \frac{A_-}{A_+} (\xi(\eta)) e^{i(\eta-\xi) \cdot f(\xi)} \bigg|_{\eta = \xi} . \]

From the estimates of Lemma 7.2, the kth term in this expansion is in \( \mathcal{H}_{1/3}^{m-1/3k} \). We sum this up as

**Proposition 7.1.** If \( P \in \text{OPS}_{1,0}(\Gamma) \) then \( \frac{G_+}{G_-} P \frac{G_-}{G_+} \in \mathcal{H}_{1/3}^m(\Gamma) \).

**Proof.** The conjugated operator has symbol given by \( b(x, \xi) + \mathcal{H}_{1/3}^m \). The above expansion yields \( b \in \mathcal{H}_{1/3}^m \).

More to the point is

**Proposition 7.2.** \( \frac{G_+}{G_-} P \frac{G_-}{G_+} \in \text{OPS}_{a-1/2,1-a}^m + \text{OP}_{1/3}^m \) for any \( \frac{1}{2} < a < 1 \).

**Proof.** Let \( q_{1a}(\xi) = q_1(\xi^{-a} \eta) \) and \( q_{2a}(\xi) = 1 - q_{1a}(\xi) \).

Denoting the symbol of this conjugated operator by \( \tilde{p} \), we have, modulo a symbol in \( \mathcal{H}_{1/3}^{m-1/3} \),

\[
\tilde{p}(x, \xi) = p(x, \xi) q_2 + p(x-f(\xi), \xi) q_1
\]

\[ = p(x, \xi) + (p(x-f(\xi), \xi) - p(x, \xi)) q_{1a} \]

\[ + (p(x-f(\xi), \xi) - p(x, \xi)) q_{2a} q_{1a} . \]

Now \( f(\xi) \) is homogeneous of degree 0 and is smooth on \( \text{supp } q_1 \), so it suffices to show

\[ (p(x-f(\xi), \xi) - p(x, \xi)) q_{2a} q_{1a} \in \mathcal{H}_{1/3}^{m+a-1/2} . \]
Now \( f(\xi) \) is smooth and homogeneous of degree 0 on the support of any term in which it appears so it suffices to show
\[
(p(x - f(\xi), \xi) - p(x, \xi)) q_{2a} q_1 \leq \frac{m + \frac{a-1}{2}}{3}.
\]

But notice on the support of \( q_{2a} q_1 \) we have
\[
|D_{\xi_1}^j D_{\xi_n}^k \left( \frac{-\xi_n}{\xi_1} \right)^{1/2} | = C_{j, k} \left( \frac{-\xi_n}{\xi_1} \right)^{1/2-k} \left( \xi_1 + |\xi_n| \right)^{-k - j/3} \leq C_{j, k} \left( \xi_1 + |\xi_n| \right)^{-k},
\]
so by Taylor's formula
\[
(p(x - f(\xi), \xi) - p(x, \xi)) q_{2a} q_1
= \sum_{|\alpha| \leq M} \frac{1}{\alpha!} D_\alpha^x p(x, \xi) \cdot (-f(\xi))^{\alpha} + O \left( \frac{-\xi_n}{\xi_1} \right)^M (1 + |\xi|)^m
\]

\[
= \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} D_\alpha^x p(x, \xi) \cdot (-f(\xi))^{\alpha} + O(1 + |\xi|)^m + \frac{a-1}{4} M
\]

where the \( \alpha \)th term in this sum is in \( \eta_{1/3}^{m + |\alpha| \frac{a-1}{2}} \). This establishes the proposition.

Combining this result with the construction of section 6 results in the proof of theorem 2.
As a final note we investigate the existence of a well-defined principal symbol for the conjugated operator in theorem 2. For this purpose we restrict our attention to the case where the original operator \( P \) has order \( 0 \). Inspection of the proof shows that we can define a symbol in the class \( S^0_{a,1-a} \) which differs from the symbol of \( e^{iT\Lambda} Pe^{-iT\Lambda} \) by something of lower order, but on several ad hoc choices such as \( a, q_1 \) and \( d \). Instead, consider the \( \mathcal{C}^* \) algebra \( \mathcal{G} \) of operators on \( L^2(\mathbb{R}^n) \) whose symbols have compact \( x \)-support. If \( \mathcal{C} \) is the ideal of compact operators contained in \( \mathcal{G} \) then \( \mathcal{G}/\mathcal{C} \) is a commutative \( \mathcal{C}^* \) algebra. The maximal ideal space of this algebra can be identified with the continuous functions on the cosphere bundle \( S^*(\mathbb{R}^n) \) which vanishes at infinity, and there is an isomorphism \( \sigma : \mathcal{G}/\mathcal{C} \rightarrow C(S^*(\mathbb{R}^n)) \) given on a generator \( P \in \text{OPS}^0 \) by

\[
\sigma(P) = \lim_{t \to \infty} p(x, t^2) .
\]

With an abuse of notation, we consider an operator on \( L^2(\Omega) \) whose distribution kernel is supported away from the boundary as being an operator on \( L^2(\mathbb{R}^n) \). With \( P, T \) as in theorem 2, modulo a smoothing operator \( e^{iT\Lambda} Pe^{-iT\Lambda} \)
is given by an operator $\tilde{P}$ which is a sum of something in $\mathcal{O} \chi \mathcal{P}^0$, along with a finite number of terms of the form $L^{-1}\mathcal{P}L$, with $\tilde{P}$ as in proposition 7.2. Letting $\chi$ denote the broken canonical relation for $e^{-i\mathcal{T}A}$ and also the induced mapping on $S^\chi(\Omega)$ we have

**Proposition 7.3.** $\tilde{P} \in \mathcal{G}/\mathcal{C}$ and

$$\sigma(\tilde{P})(x,\xi) = \sigma(P)(\chi(x,\xi)) .$$

**Proof.** Since $\sigma$ is a homomorphism, by the construction in section 6, we may assume $\tilde{P} = L^{-1}P_L$. From the proof of proposition 7.2, $\tilde{P}$ has symbol

$$\tilde{p}(x-f(\xi),\xi)-\bar{p}(x,\xi))q_{1a}(\xi)+\bar{p}(x,\xi)+\frac{\eta_1}{3}$$

for some $\tilde{p} \in S^0(\Gamma)$. Let $R_N \in \mathcal{O} \chi \mathcal{P}^0(\Gamma)$ have symbol

$$r_N(x,\xi) = (\tilde{p}(x-f(\xi),\xi)-\bar{p}(x,\xi))q_{1}(N_{\xi_1}^{-1}\xi_n)+\bar{p}(x,\xi) .$$

Modulo a compact operator on $L^2$, $\tilde{P}-R_N \in \mathcal{O} \chi \mathcal{P}^0_{a,1-a}(\Gamma)$ and by theorem 3.3 of [3], the norm modulo compacts of $\tilde{P}-R_N$ on $L^2$ is given by
\[ \limsup_{\xi \to \infty} \sup_{x} |(p(x - f(\xi), \xi) - p(x, \xi))(q_1(\xi^{a - 1} n) - q_1(N\xi^{a - 1} n))| \]

\[ \leq \limsup_{\xi \to \infty} C_\pi \cdot |f(\xi)| \cdot (q_1(\xi^{a - 1} n) - q_1(N\xi^{a - 1} n)) | \]

\[ \leq C N^{-\frac{1}{2}}. \]

Hence \( \tilde{\pi} \in G/C \). This also implies, by Egorov's theorem that \( L^{-1} \tilde{\pi} L \in G/C \). Finally, recalling the canonical relation for \( L \) shows

\[ \sigma(\tilde{\pi}) = \lim_{N \to \infty} \sigma(L^{-1} R_N L) = \sigma(P) \circ \chi. \]
References


