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THE KIRCHHOFF APPROXIMATION FOR THE NEUMANN
PROBLEM.

RICE UNIVERSITY, PH.D., 1979
RICE UNIVERSITY

THE KIRCHHOFF APPROXIMATION FOR THE NEUMANN PROBLEM

by

HOWARD DAVID YINGST

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

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MAY 1979
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ABSTRACT

Let $K$ be a compact, convex obstacle in $\mathbb{R}^n$. We assume $n$ is odd and $M = \partial K$ is smooth with positive Gaussian curvature everywhere. With $\Omega = \mathbb{R}^n \setminus K$, $\omega$ a unit vector in $\mathbb{R}^n$, and $\nu = \text{unit outer normal to } K$ we obtain the solution, $u_s(x,\lambda)$, to the following reduced wave equation

$$(\Delta + \lambda^2) u_s = 0 \text{ in } \Omega \times \mathbb{R} \quad \partial_\nu u_s |_{M} = \partial_\nu e^{-i\nu \cdot x \cdot \omega} |_{M}$$

$$u_s = o(|x|^{\frac{1-n}{2}}) \quad \partial_\nu u_s - i\lambda u_s = o(|x|^{\frac{1-n}{2}}), \quad |x| \to \infty$$

and then define $K$ by $K(x,\lambda)e^{-i\nu \cdot x \cdot \omega} = u_s |_{M}$.

**Theorem 1:** For $\varepsilon > 0$ there exists $C = C_\varepsilon > 0$ such that

1) $|K(x,\lambda)| \leq C_\varepsilon \frac{1}{|x|^{1/6} + \varepsilon}$;

2) $|\langle \nu \cdot \omega \rangle K - \nu \cdot \omega | \leq C_\varepsilon \frac{1}{|\nu \cdot \omega|} (1 + \lambda^2 |\nu \cdot \omega|)^{-5}$.

If $C \cap \{x : \nu(x) \cdot \omega = 0\} = \emptyset$ with $C$ closed a sharper estimate is possible on $C$.

**Proposition 1.4:** $K(x,\lambda) - \frac{\nu \cdot \omega}{|\nu \cdot \omega|} = o(\lambda^{-1})$ uniformly on $C$. 
The Neumann operator, $N$, is now defined by solving

$$(\partial_t^2 - \Delta)u = 0, \quad u_M = u_0 \in \mathcal{C}(\mathbb{M} \times \mathbb{R}) , \quad \text{and} \quad u = 0 \text{ for } t \ll 0$$

and then setting $Nu_0 = \partial_t u_M$. In the course of proving theorem 1 certain results about $N$ were established. Principal among these is the fact that for any $\epsilon > 0$, $N$ is hypoelliptic with loss of $\frac{1}{3} + \epsilon$ derivatives and in fact

$$N^{-1} \in \text{OPS}_{\frac{1}{3}, \frac{2}{3}}^\epsilon (\mathbb{M} \times \mathbb{R}) .$$
ACKNOWLEDGMENTS

With greatest respect, I would like to thank my advisor, Michael Taylor, for nurturing what talent I may have. Not only has he provided excellent guidance and instruction in mathematics but he has also been a good friend. In some sense he demonstrated to me that regardless of the size of an obstacle in one's path one can always see the light. My parents and family also deserve thanks for their patience and understanding. Without their moral support completing this thesis would have been much harder than it already was. And of course the Mathematics Department deserves some credit for providing as stimulating an atmosphere as it did.

I received partial financial support during my graduate stay at Rice from the National Science Foundation.
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0. Introduction

The purpose of this paper is to provide a rigorous justification of the Kirchoff approximation, which arises in the study of the following problem in classical scattering theory.

Let $K$ be a compact convex obstacle in $\mathbb{R}^n$ with smooth boundary. We assume $n$ is odd and $M = \partial K$ is smooth, with positive Gaussian curvature everywhere. Our scattering problem is to analyze the solution $u_s(x, \lambda)$, to the following reduced wave equation.

\[(\Delta + \lambda^2)u_s = 0 \text{ in } \Omega \times \mathbb{R}, \quad \partial_u u_s|_{\partial \Omega \times \mathbb{R}} = \partial_u e^{-i\lambda \cdot x \cdot \omega}, \quad \partial_u u_s|_{M \times \mathbb{R}} \]

\[
(0.1) \quad u_s = 0(|x|^2) \quad \partial_u u_s - i\lambda u_s = o(|x|^2), \quad |x| \to \infty.
\]

Here $\Omega = \mathbb{R}^n \setminus K$, $\omega \in \mathbb{R}^n$ with $|\omega| = 1$, $\lambda \in \mathbb{R}$, and $\nu$ denotes the unit outer normal to $K$. The requirement on the asymptotic behavior of $u_s$ and $\partial_u u_s$ is called the Sommerfeld radiation condition and it guarantees the existence and uniqueness of $u_s$. But since $u_s$ is the $t$-Fourier transform of the solution, $u$, to

\[(\partial^2_t - \Delta)u = 0 \text{ in } \Omega \times \mathbb{R}, \quad \partial_u u|_{\partial \Omega \times \mathbb{R}} = \partial_u \delta(t - x \cdot \omega)|_{M \times \mathbb{R}} \]

\[
(0.2) \quad u = 0 \text{ for } t \ll 0.
\]
we shall not discuss the solvability of (0.1) referring the reader to Scattering Theory by Lax and Phillips.

Now $u_s$ is determined on all of $\Omega$ by $u_s|_M$ and $\partial u_s|_M$. In fact, by the radiation condition Green's theorem implies

\[(0.3)\]

$$u_s(x, \lambda) = \int_M \left[ u_s(y, \lambda) \partial_\nu G_{\lambda}(x-y) - \partial_\nu u_s(y, \lambda) G_{\lambda}(x-y) \right] dS(y)$$

where $dS$ denotes surface measure and $G_{\lambda}$ is the Green's function for $-\Delta + \lambda^2$. (For instance, $G_{\lambda}(x) = |x|^{-1} e^{i\lambda \cdot x}$ when $n = 3$.) Thus a knowledge of $u_s|_M$ is very convenient in calculations involving $u_s$. Classically what has been used is the Kirchhoff approximation --

\[(0.4)\]

$$u_s|_M \approx \frac{\nu \cdot \omega}{|\nu \cdot \omega|} e^{-i\lambda \cdot x} \cdot \omega, \quad \lambda \to \infty .$$

One can derive (0.4) by assuming that the flat wall case should provide an approximate answer. Thus we solve (0.2) with $M$ and $\Omega$ replaced respectively by $T_{x_0}(M)$ and $\{ x \in \mathbb{R}^n : \nu(x_0) \cdot (x-x_0) > 0 \}$ and with the condition $u = 0$, $t << 0$ replaced by the requirement that singularities propagate only forward in time. One easily checks that the solution is $\delta(t-x \cdot \omega)$ for $\omega \cdot \nu(x_0) > 0$ and $-\delta(t-x \cdot \omega + 2(\omega \cdot \nu(x_0))(x-x_0) \cdot \nu(x_0))$ for $\omega \cdot \nu(x_0) < 0$. Taking Fourier transforms and evaluating at $x_0$ then completes
our heuristic derivation of (0.4).

Of course (0.4) is very suspicious looking on
\[ g_\omega = \{ x \in M : \nu(x) \cdot \omega = 0 \} . \] In fact by smoothness up to the boundary for regular elliptic equations with smooth boundary data, \( u_S \mid_M \) must belong to \( C^\omega(M) \) for all \( \lambda \in \mathbb{R} \). Thus (0.4) cannot be taken too seriously and it is our purpose here to obtain an estimate which is valid pointwise on all of \( M \). Writing

\[ u_S(x, \lambda) \mid_M = K(x, \lambda) e^{-i\lambda x \cdot \omega} \]

our major result is

**Theorem 1**: Fix \( \varepsilon > 0 \). Then there exists \( C = C_\varepsilon > 0 \) such that

1) \[ |K(x, \lambda)| \leq C_\lambda \left( \frac{1}{6} + \varepsilon \right) \]

2) \[ |(\nu \cdot \omega) K(x, \lambda) - \nu \cdot \omega| \leq C_\lambda \left( \frac{1}{6} + \varepsilon \right) \left( 1 + \frac{1}{|\nu \cdot \omega|} \right)^{-5} \]

Establishing theorem 1 requires a fairly detailed analysis of the Neumann operator, \( N \), for solutions to the wave equation in \( \Omega \times \mathbb{R} \) with prescribed boundary values. We begin section 1 by relating \( K \) to the Neumann operator and then proceed to a microlocal description of \( N \). On the complement of any conic neighborhood of the so-called grazing set, \( q \subset T^* (M \times \mathbb{R}) \), this description is easily accomplished.
using standard techniques from the theory of Fourier integral operators. In fact in such a region $N$ is microlocally an elliptic operator in $\text{OPS}^{1}_{1,0}$. However, near $\mathfrak{a}$ we must rely on the grazing ray parametrix which is valid since $M$ is assumed to be strictly convex. We shall eventually see that given any point in $\mathfrak{a}$ there exists a conic neighborhood of it in which $N$ can be represented as a hypoelliptic operator in $\text{OPS}^{1}_{1,0}$ conjugated by an elliptic Fourier integral operator. It follows that $N$ is pseudo-local and as a consequence it is easy to establish the classical Kirchoff approximation on the complement of any neighborhood of $\mathfrak{a}$. 

To obtain estimates across $\mathfrak{a}$ we first derive the expression

$$K(x, \lambda) = -i\lambda e^{-i\lambda \psi} N^{-1}(0, (\nu \cdot \omega)) e^{i\lambda \psi}(t, x, \lambda) + O(\lambda^{-\infty})$$

valid for $|t| \leq 1$. Here $\rho(t) \in c_0^\infty(R)$, $\rho = 1$ on $[-2, 2]$, and $\psi = (t - x \cdot \omega)|_M$. For any $a, \frac{1}{2} < a \leq 1$, we can then write $N^{-1}$ as the sum of two terms, $N^{-1} = J(E_1 + E_2)J^{-1}$, where $J$ is the Fourier integral operator mentioned above and

$$E_1 \in \text{OPS}_{a, 1-a}^{\frac{3}{4}} \text{ while } E_2 \in \text{OPS}^{\frac{2}{3}}_{1,0}.$$ Thus we can write

$$K(x, \lambda) = K_1(t, x, \lambda) + K_2(t, x, \lambda) + O(\lambda^{-\infty}) \text{ for } |t| \leq 1 \text{ and in principle } K_1 \text{ can be evaluated since } a > \frac{1}{2}. \text{ But we run into problems with } K_2 \text{ since Egorov's theorem cannot be applied to operators of type } (\frac{1}{3}, 0). \text{ However, its}$$
special form allows one to apply a variant of Egorov's

theorem to $E_2$ and conclude that $JE_2J^{-1} \in \text{OPS}_{\frac{2}{3} + \varepsilon} \cap \frac{2}{3}, \frac{2}{3}$. Un-

fortunately we cannot calculate the principal symbol of $JE_2J^{-1}$ and since $JE_1J^{-1}$ has lower order we will not be able to say anything about $K$ except that it is bounded by some power of $\lambda$. (Recall l) of theorem 1.) The way around this problem is to consider $(\nabla \cdot w)^kK$ instead of $K$. As it turns out $(\nabla \cdot w)^kK_1(t,x,\lambda)$ has a principal term of order 0 essentially arising from the classical Kirchoff approximation. Meanwhile, for $1 \leq k \leq 6$, $(\nabla \cdot w)^kK_2$

\[-\frac{1}{6}k + \varepsilon = 0(\lambda)\] reflecting the fact that the relatively high power of $\lambda$ in $K_2(x,\lambda)$ is concentrated very close to $\partial_w$. Theorem 1 is then established by adding the inequalities corresponding to these observations.

Certain technical devices are needed to carry out the above program. As already remarked we must exploit the special form of $E_2$ to conclude that $JE_2J^{-1}$ is a pseudo-
differential operator. Furthermore we will want to choose a close to $\frac{1}{2}$ because as $a$ increases certain operators related to $E_2$ increase in order. On the other hand, as $a$ gets close to $\frac{1}{2}$ the asymptotic expansion for the symbol of $JE_1J^{-1}$ seems to indicate that the orders of lower order
terms in \((\nu \cdot w)^k K_1\) are approaching that of the top order term. Thus it will be necessary to exploit certain properties of \(E_1\) to avoid this dilemma. Therefore two special classes of pseudo-differential operators are introduced in section 2. A symbol calculus is developed for each and then used to refine the descriptions of \(E_1\) and \(E_2\). Furthermore we give versions of Egorov's theorem appropriate for each class. Finally, motivated by the fact that \(g\) and \(\{d(t-x \cdot w)\}\) have precisely second order contact since \(M\) is strictly convex, we obtain an even more refined description of the operators under consideration which, when combined with the preceding results, yields suitable estimates for \((\nu \cdot w)^k K_1\) and \((\nu \cdot w)^k K_2, 0 \leq k \leq 6\).

We close this introduction by briefly describing some of our terminology. Let \(U \subset \mathbb{R}^n\) be open. As usual, \(S_{\rho, \delta}^m(U), 0 \leq \rho, \delta \leq 1\), denotes the space of symbols of order \(m\) and type \((\rho, \delta)\). Thus, \(p \in S_{\rho, \delta}^m(U)\) if and only if \(p \in C^\infty(U \times \mathbb{R}^n)\) and given \(K \subset U\) and multi-indices \(\alpha\) and \(\beta\) there exists \(C_{K, \alpha, \beta} > 0\) such that \(|\partial_\alpha \partial_\beta p(x, \xi)|\leq C_{K, \alpha, \beta} (1 + |\xi|)^{m-\rho}|\alpha|^{+\delta}|\beta|\) on \(K \times \mathbb{R}^n\). \(S^m(U)\) will denote the space of those symbols having an asymptotic expansion as a sum of homogeneous functions. Given \(p \in S_{\rho, \delta}^m(U), \delta < 1, p(x, D): \mathcal{E}'(U) \to \mathcal{B}'(U)\) is defined by \(p(x, D)u = (2\pi)^{-n} \int p(x, \xi)e^{-ix \cdot \xi} U(\xi) d\xi\) and \(\mathcal{O}PS_{\rho, \delta}^m(U)\) is the space
of all such operators. More generally \( \text{OPS}(U) \) is defined similarly whenever \( S(U) \) is some symbol class. We remark that many of the results stated for all elements in a particular operator class hold only for those which are properly supported. This should not cause any confusion.

Next, if \( W \) is an open subset of \( U \times \mathbb{R} \) and \( p \in S^m_{\rho, \delta}(U) \) then by definition, \( p \) has order \(-\infty\) on \( W \) if given \( K \subset U \), \( \alpha, \beta \), and \( N > 0 \) there exists \( C \) such that

\[
|\partial_x^{\alpha} \partial_{\xi}^{\beta} p(x, \xi)| \leq C(1 + |\xi|^{-N} \text{ on } W \cap (K \times \mathbb{R}^n).}
\]

For a general treatment of pseudo-differential operators see [2] or [10]. Finally, we remark that perhaps our most heinous abuse of notation is the use of \( \ast \) to denote restriction to \( M \times \mathbb{R} \).
1. The Neumann Operator

Our analysis begins with an expression for $K(x, \lambda)$ in terms of the forward Neumann operator, $N$. With $P = \delta_{t}^{-\lambda} - \Delta$ and $\tau : M \subset \mathbb{R}^{n}$, we solve

\begin{equation}
Pv = 0 \quad \text{on } \Omega \times \mathbb{R} \quad v = 0 \quad \text{for } t < 0
\end{equation}

(1.1)

\begin{equation}
\tau^* v = v_0 \in \mathcal{E}'(M \times \mathbb{R})
\end{equation}

(1.1D)

and then define

$$Nv_0 = \delta_{\tau} v$$

Replacing (1.1D) by

\begin{equation}
\delta_{\tau} v = v_0 \in \mathcal{E}'(M \times \mathbb{R})
\end{equation}

(1.1N)

yields the inverse Neumann operator, $N^{-1}v_0 = \tau^* v$. Since (1.1D) and (1.1N) are well-posed, $N$ and $N^{-1}$ are well-defined linear maps taking $\mathcal{E}'(M \times \mathbb{R})$ into $\mathcal{E}'(M \times \mathbb{R})$. Furthermore, $n$ is odd and therefore, by a fundamental result of scattering theory, $v$ decays exponentially on any compact subset, $L$, of $\mathbb{R}^{n}$ as $t \to \infty$. Also, $v$ is smooth for large $t$ and all of its derivatives decay exponentially on $L$, $t \to \infty$. See Majda-Taylor [5]. Therefore, $N$ and $N^{-1}$ extend to linear operators mapping $\mathcal{S}'(M) \otimes \mathcal{S}'(\mathbb{R})$ into $\mathcal{S}'(M) \otimes \mathcal{S}'(\mathbb{R})$. 
Define $u$ by solving (1.1) with the Neumann condition, $\partial_{\nu}u = \partial_{\nu}\delta(t-x \cdot \omega)$ and let $u_0 = \mathcal{N}u$. Thus $Nu_0 = \partial_{\nu}\delta(t-x \cdot \omega)$. By exponential decay, the $t$-Fourier transform of $u, \hat{u}$, is well-defined. Moreover, $\hat{u}$ satisfies the reduced wave equation on $\Omega \times \mathbb{R}$ with the outgoing radiation condition (corresponding to $u = 0, t \ll 0$) and with $\partial_{\nu}\hat{u} = \int_{-\infty}^{\infty} \partial_{\nu}\delta(t-x \cdot \omega)e^{-i\lambda t}dt = \partial_{\nu}e^{-i\lambda x \cdot \omega}$. Therefore $\hat{u}$ satisfies (0.1) and hence $\hat{u} = u_s$. In particular,

(1.3) $\mathcal{N}u_s(x, \lambda) = e^{-i\lambda x \cdot \omega}K(x, \lambda) = \int_{-\infty}^{\infty} u_0(x, t)e^{-i\lambda t}dt$.

Now observe that $N$ is $t$-translation invariant which means it has a Schwarz kernel of the form $n(x, x', t-t')$. Thus, writing $\psi = \mathcal{N}(t-x \cdot \omega)$, we have

$$N(e^{i\lambda \psi}K(\cdot, \lambda)) = \int \int \int [n(x, x', t-t')u_0(x, x')e^{i\lambda(t'-s)}dsdt'dx$$

$$= \int \int [n(x, x', t-t'-s)u_0(s, x')dsdx']e^{i\lambda t' dt'}$$

(1.4)

$$= \int Nu_0(x, t-t')e^{i\lambda t'}dt'$$

$$= \partial_{\nu}e^{i\lambda(t-x \cdot \omega)} = i\lambda(\nu \cdot \omega)e^{i\lambda \psi}.$$ We take (1.4) as our starting point for estimating $K(x, \lambda)$.

Applying (1.4) requires a fairly detailed microlocal description of $N$ and it seems appropriate to fix notation.

Let $X \subset \mathbb{R}^n$ be open and $\Lambda \subset X \times \mathbb{R}^n$ be an open conic subset.
As usual, $\mathcal{A}(X)$ will denote $\{u \in \mathcal{A}(X); WF(u) \subset \Lambda\}$. Now, let $S$ be some symbol class on $X \times \mathbb{R}^n$.

**Definition 1.1:**

a) $P \in OPS(\Lambda)$ if $P: \mathcal{A}(X) \to \mathcal{A}(X)$ is linear and given $\Lambda' \subset \subset \Lambda$ (i.e. $\Lambda' \cap \{|g| = 1\} \subset \subset \Lambda \cap \{|g| = 1\}$), there exists $P_{\Lambda}' \in OPS(X)$ such that $P_{\Lambda}'u = Pu \mod C^\infty$ $\forall u \in \mathcal{E}'(X)$, such that $WF(u) \subset \Lambda'$.

b) $p \in S(\Lambda)$ if $p \in C^\infty(\Lambda)$ and $\forall \lambda \in C^\infty(X \times \mathbb{R}^n)$, homogeneous of degree 1 in $g$ with $\text{supp} \lambda \subset \Lambda$, $\lambda p \in S$.

Next we shall need the decomposition of $T^*(M \times \mathbb{R})$ corresponding to the characteristic set of $P$. Let $\alpha: T(M \times \mathbb{R}) \to T^*(M \times \mathbb{R})$ denote the identification induced by the Riemannian metric on $M \times \mathbb{R}$ where $T(M \times \mathbb{R})$ is naturally embedded in $T(\mathbb{R}^n \times \mathbb{R})$. We then write $T^*(M \times \mathbb{R}) = \mathcal{U} \cup \mathcal{G} \cup \mathcal{E}$, the components being defined by $\mathcal{U} = \{\alpha(x,t,g,\tau): \forall x, \tau^2 > |g|^2\}$, $\mathcal{G} = \{\tau^2 = |g|^2\}$ and $\mathcal{E} = \{\tau^2 < |g|^2\}$. In $\mathcal{E}$, $N$ is the Neumann operator for a regular elliptic boundary value problem and therefore, $N \in OPS_{1,0}(\mathcal{E})$. Since $d_\mathcal{E} = \alpha(\cdot,\cdot,-\omega+(\tau \cdot w)\nu,1) \in \mathcal{G} \cup \mathcal{U}$, this description of $N$ in $\mathcal{E}$ is sufficient for our purposes.

Solutions to (1.1) with $WF(v_0)$ close to $p_0 = \alpha(x_0,t_0,g_0,\tau_0) \in \mathcal{U}$ can be obtained (mod $C^\infty$) by setting

(1.5) \[ v(x,t) = \int a(x,t,\zeta)e^{i\varphi(x,t,\zeta)}\hat{f}(\zeta)d\zeta \]
where \( f \in \mathcal{E}(\mathbb{R}^n) \), \( a \sim \sum_{j=0}^{\infty} a_j \), and \( \varphi, a_j \) are homogeneous of degree 1, \(-j\) in \( \zeta \) respectively. Applying \( P \) to (1.5) and equating terms of equal homogeneity yields the eikonal equation

\[
(1.6) \quad \varphi_\tau^2 = |\varphi_\zeta|^2
\]

and the transport equations (with \( a_\perp = 0 \))

\[
(1.7) \quad (2i\varphi_\tau \partial_\tau - 2i\varphi_\zeta \cdot \nu + i\partial_\zeta^2 \varphi - i_\lambda \varphi)a_j = -Pa_j - 1.
\]

On some conic set, \( U \times \Gamma \subset M \times \mathbb{R}^n \), we can find a smooth \( \varphi_0 \) homogeneous of degree 1 in \( \zeta \) with the properties

\[
(\varphi_\tau \tan \varphi_0, \partial_\zeta \varphi_0) = (\xi_0, \tau_0) \text{ at } (x_0, t_0, \zeta_0) \in U \times \Gamma, \quad (\partial_\zeta \varphi_0)^2 - |\varphi_\tau \tan \varphi_0|^2 > c|\zeta|^2, \quad \text{and} \quad |\det(\zeta|^{-1}\partial_2 z, \zeta_0^\varphi)| > c \text{ where } z = (x, t) \in U \times \Gamma.
\]

It follows that (1.6) is non-characteristic when \( \#\varphi = \varphi_0 \) is prescribed. Thus, ordinary geometric optics provides two solutions with \( \#\varphi = \varphi_0 \) depending on the choice of signs for \( \partial_\nu \varphi \). Choosing one, we can then solve (1.7) on the domain of \( \varphi \) and obtain appropriate \( a_j \) so that \( a \in \mathcal{E}_{1,0} \) is elliptic where \( a \sim \Sigma a_j \).

Next write \( \varphi_\tau(z, \zeta) = \varphi(z+r(\nu,0), \zeta) \) and define \( a_\tau \) similarly. Since \( \det(\zeta|^{-1}\partial_2 z, \zeta^\varphi_0) > \frac{1}{c} \) for \( |r| < \varepsilon, \varepsilon > 0 \),

\[
(1.8) \quad J(r) f = \int a_\tau e^{i\varphi_\tau r(\nu,0)} \varphi_\tau(\zeta) d\zeta
\]

is a smooth l-parameter family of elliptic Fourier integral
operators mapping \( \rho'(U) \) into \( \varepsilon'_\tau(U) \) where \( \Lambda \) is some conic neighborhood of \( p_0, \Lambda \ll \mathcal{M} \). Therefore, by setting \( f = J(0)^{-1}u_0 \), (1.5) provides a solution to \( Pu \in \mathcal{C}^\infty \), \( \iota^\Lambda u - u_0 \in \mathcal{C}^\infty \).

Before choosing the correct sign of \( \varphi \), note that

\[
(1.9) \quad \varphi_{\tau} u = \int (i\omega_{\tau} + \omega_{\tau}) e^{i\varphi} d\zeta
= J'(0)J(0)^{-1}u_0.
\]

Since \( J'(0) \) has the same phase function as \( J(0) \), \( N = J'(0)J(0)^{-1} \in \text{OPS}_{1,0}(\Lambda) \) and \( \tau_N(z, d\varphi(z, \zeta)) = i\varphi_{\tau}(z, \zeta) = i((\omega_{\tau} - \omega_0)^2 - |\tau\tan \varphi_0|^2)^{\frac{1}{2}} \text{mod } S^0_{1,0}(\Lambda) \). Returning to the choice of signs note that \( \varphi_{\tau}(z_0) = \tilde{\varphi}(z_0) \) where \( \tilde{\varphi} \) is obtained from (1.6) with \( M \) replaced by \( \tilde{M} = T_{\tau}^0(M) \). Without loss of generality, \( \tilde{M} = \{(x,0) : x \in \mathbb{R}^{n-1}\} \) so that \( \tilde{\varphi} \) can be chosen with \( \tilde{\varphi}_{\tau}(x, t, \xi, \tau) = \tilde{x} \cdot \xi + t \tau \pm r(\tau^2 - |\xi|^2)^{\frac{1}{2}} \). Now, the condition \( u \in \mathcal{C}^\infty \) for \( t \ll 0 \) means that singularities only propagate forward in time. In view of the relation between \( \text{WF}(u_0) \) and \( \text{WF}(u_\tau) \) this means that \( \varphi_{\tau} \leq 0 \) or equivalently \( \varphi_{\tau} \leq 0 \) for \( \tau \geq 0 \). Therefore, by choosing the right sign, (1.5) yields a parametrix to (1.1) with boundary condition (1.1D). Since \( J(0) \) can be arranged microlocally to be a coordinate change, this establishes
Proposition 1.2: \( N \in \text{OPS}_{1,0}^{1}(\mathcal{H}) \) and for \( \alpha(x,t,\varepsilon,\tau) \in \mathcal{H} \),

(1.10) \( \tau_{N}(\varepsilon(x,t,\varepsilon,\tau)) = n_{1} = \pm i(\tau^{2} - |\varepsilon|^{2})^{1/2} \mod S_{1,0}^{0}(\mathcal{H}), \tau > 0. \)

Without further ado we turn to the behavior of \( N \) in a conic neighborhood of \( \mathcal{H} \). Since \( M \) is strictly convex, solutions \( (\mod c^{a}) \) with \( \text{WF}(u_{0}) \) to \( \mathcal{H} \) can be obtained by the grazing ray parametrix of Taylor [9] and Melrose [5]. More precisely, for any \( p_{0} \in \mathcal{H} \cap \{ \tau > 0 \} \) there exists a conic neighborhood, \( \Lambda \), such that given \( U \subset M \times R \), open and sufficiently small, we can find a neighborhood, \( \tilde{U} \), of \( U \) in \( \Omega \times R \) on which solutions \( (\mod c^{a}) \) to \( Pu = 0, \; \tau u = f \) with \( \text{WF}(f) \subset \varepsilon'(U) \), and \( u = 0 \) for \( t << 0 \) are given by

(1.11) \( u = \int \int \left[ \frac{A(|\varepsilon| - \frac{1}{3} \rho)}{\varepsilon} \right] g + i|\varepsilon| \frac{1}{3} A'(\frac{|\varepsilon| - \frac{1}{3} \rho}{\varepsilon}) h \right] \chi \)

\[ A(-\frac{|\varepsilon|}{3} \eta) \quad A(-\frac{|\varepsilon|}{3} \eta) \quad e^{i\Theta} J^{-1} f(\xi,\eta) d\xi d\eta \]

where

a) \( A(s) = Ai(e^{-\frac{3}{4} s}), \quad Ai \) being the Airy function discussed in Olver [7], so that \( A \) satisfies the "Airy equation"

\[ A''(s) + sA(s) = 0 , \]

\( A \) and \( A' \) have no real zeros, and writing
\[ \psi(s) = \exp(-\frac{2}{3} i s^3)A(s), \]

\[ \psi(s) \sim s^{-\frac{1}{2}} \sum_{j=0}^{\infty} a_j s^{-\frac{3}{2} j}, \quad |s| \rightarrow \infty. \]

(We remark that in the "backward light cone" one chooses

\[ A(s) = \text{Ai}(e^{-s}) \];

b) \( g(x, \xi, \eta), h(x, \xi, \eta) \in S_{1,0}^0(\tilde{U} \times \Gamma) \) where \( \Gamma = \{(\xi, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\xi| < c|\xi| \} \) for some \( c > 0 \) and \( \ast g \) is elliptic while \( \ast h = 0 \) for \( \eta \geq 0 \);

c) \( \phi = t|\xi| + \tilde{\phi}(x, \xi, \eta) \) where \( \tilde{\phi} \) and \( \phi = \phi(x, \xi, \eta) \) are homogeneous of degree 1 in \( (\xi, \eta) \) and for \( \eta \leq 0 \) satisfy

\[ |\nabla \tilde{\phi}|^2 + \frac{1}{|\xi|} |\nabla \phi|^2 = |\xi|^2 \]

\[ \nabla \phi \cdot \nabla \tilde{\phi} = 0; \]

d) For \( \eta > 0, \phi = -\eta \) on \( U \) and \( \phi, \tilde{\phi} \) satisfy the "eikonal equation" in c) to infinite order at \( U \);

e) \( |\xi|^{-1} \frac{\partial \phi}{\partial \nu} \geq a_0 > 0 \) and writing \((z, \zeta) = ((x, t), (\xi, \eta))\), \( \det(|\zeta|^{-1} \frac{\partial^2}{\partial z, \zeta} \phi) \geq a_0; \)

f) \( J \) is microlocally an elliptic Fourier integral operator mapping \( \mathcal{E}^\prime_\Lambda(\mathbb{R}^n) \) to \( \mathcal{E}^\prime_\Lambda(U) \).

For a detailed discussion of the above see [6], [9] and [11].

It will be convenient later to have an integral
representation of $J$. Set $F = J^{-1}f$. Since $\iota^*u = f$, (1.11) implies that

\begin{equation}
(1.12) \quad f = JF = \int [(\iota^*g)\tilde{B} + i(\iota^*h)q\tilde{C}]e^{i(\Theta + \gamma)\hat{F}(\xi, \eta)}d\xi d\eta
\end{equation}

where $\gamma$, $\tilde{B}$, $\tilde{C}$ and $q$ are defined by

\begin{equation}
\gamma = \iota^*(\frac{2}{3} |\xi| \left[ (\frac{p}{|\xi|})^{\frac{3}{2}} - (\frac{-\eta}{|\xi|})^{\frac{3}{2}} \right])
\end{equation}

\begin{equation}
e^{i\gamma \tilde{B}} = \iota^* \frac{A(\xi)}{A(-|\xi| - \frac{1}{3} \eta)}
\end{equation}

\begin{equation}
e^{i\gamma \tilde{C}} = \iota^* \frac{A'(\xi)}{A(-|\xi| - \frac{1}{3} \eta)}
\end{equation}

\begin{equation}
q = |\xi|^{-\frac{1}{3}} \frac{A'}{A}(-|\xi| - \frac{1}{3} \eta).
\end{equation}

Now observe that by (1.11d), $\frac{\Theta}{|\xi|} = \frac{-\eta}{|\xi|} + O((\frac{|\eta|}{|\xi|})^\infty)$ on $U$. It follows that $\gamma$ is smooth and sufficiently small to insure that $\Theta + \gamma$ is a non-degenerate phase function on $U \times \Gamma$. Next, one can show that $\tilde{B}$, $\tilde{C}$ belong to $S_{1,0}^0(U \times \Gamma)$ and are elliptic. Finally, since $h$ vanishes on $U \times \{ \eta > 0 \}$, it is easy to check that $hq \in S_{1,0}^0$ and therefore, $(\iota^*g)\tilde{B} + i(\iota^*h)q\tilde{C} \in S_{1,0}^0(U \times \Gamma)$ is elliptic since $\iota^*g$ is. For more details see chapter 3 of [11].
We can now describe the Neumann operator in \( g \). By Airy's equation,

\[
(1.14) \quad \mathcal{N} f = a_{\nu} u = \int \left[ \frac{A'(|\xi| - \frac{1}{3} \rho)}{\xi} \right] g - \frac{1}{3} \rho \_\nu g + \frac{A(|\xi| - \frac{1}{3} \rho)}{\xi} \left( g \_\nu^* + i g \_\nu \right) \frac{1}{3} \rho \_\nu (\xi - \frac{1}{3} \eta) A(-|\xi| - \frac{1}{3} \eta) A(-|\xi| - \frac{1}{3} \eta) \]

\[
e^{i\phi \_\nu} J^{-1} f d\xi d\eta
\]

\[
+ \int \left[ -i \frac{A(|\xi| - \frac{1}{3} \rho)}{\xi} \right] g - \frac{1}{3} \rho \_\nu h + \frac{A'(|\xi| - \frac{1}{3} \rho)}{\xi} \left( i g \_\nu - \frac{1}{3} \rho \_\nu h \_\nu - \frac{1}{3} h \_\nu \right) A(-|\xi| - \frac{1}{3} \eta) A(-|\xi| - \frac{1}{3} \eta) \]

\[
|\xi| - \frac{1}{3} \eta \right) e^{i\phi \_\nu} J^{-1} f d\xi d\eta
\]

\[
= \int \left[ \rho \_\nu g + i h \_\nu - h \_\nu \right] e^{i(\phi + \nu)} qJ^{-1} f d\xi d\eta
\]

\[
+ \int [q \_\nu + i g \_\nu - i g \_\nu \_\nu h - i g \_\nu] e^{i(\phi + \nu)} J^{-1} f d\xi d\eta
\]

\[
= K_1 QJ^{-1} f + K_2 J^{-1} f
\]

where \( Q \) is the operator with Fourier multiplier, \( q(\xi, \eta) \), on \( \Gamma \) and \( K_1, K_2 \) are (microlocally defined) Fourier integral operators associated to the same canonical transformation, \( \mathcal{J} \), as that of \( J \). Since \( J \) is elliptic we can write \( K_1 = JA \) and \( K_2 = JB \) with \( A, B \in \text{OPS}_{1,0}^1 \).
Therefore, \( \forall f \in \mathcal{E}'(U) \)

\[
Nf = J(AQ+B)J^{-1}f \mod \mathcal{C}^\infty .
\]

(1.15)

An immediate consequence of (1.15) is that \( N \) is pseudo-local. Indeed, one easily checks that

\[
q = |\xi|^{-\frac{1}{3}} \frac{A'}{A} (-|\xi|^{-\frac{1}{3}} \eta) \in S^0_{\frac{1}{3},0} (V \times \Gamma)
\]

since the asymptotic expansion for \( \psi \) implies \( \frac{A'}{A}(\lambda) \in S^0_{\frac{1}{3},0}(\mathbb{R}). \) Thus, \( (AQ+B)\lambda(z, D) \) belongs to \( O\mathcal{P}S^1_{\frac{1}{3},0}(V) \) if \( \text{supp} \lambda \subset V \times \Gamma. \) In particular, \( J(AQ+B)\lambda(z, D)J^{-1} \) is pseudo-local and because \( N \) belongs to \( O\mathcal{P}S^1_{1,0} \) in \( \mathcal{E} \) and \( \mathcal{H}, \) a microlocal partition of unity shows that \( N \) is pseudo-local as asserted. As a consequence we can localize (1.4).

Lemma 1.3: Let \( \rho(t) \in \mathcal{C}^\infty_0(\mathbb{R}^n) \) be identically 1 on some neighborhood of \([-1, 1]\). Then, for \( |t| \leq 1, \)

\[
e^{-i\lambda^T \psi} N(\rho K(\cdot, \lambda)e^{i\lambda^T \psi}) = -i\lambda(\nu \cdot \omega) + O(\lambda^{-\infty}) .
\]

(1.16)

Proof: Since the kernel of \( N \) is smooth off the diagonal, the kernel of \( N(1-\rho) \) is smooth for \( |t| \leq 1. \) Hence,

\[
N(1-\rho)(K(\cdot, \lambda)e^{i\lambda^T \psi}) = O(\lambda^{-\infty}) \text{ for } |t| \leq 1 \text{ and the lemma then follows immediately from (1.4).} \]
Another consequence of $N$ being pseudo-local is that away from $g_w = \{x \in M : \gamma_x w = 0\}$, the Kirchoff approximation is immediate.

**Proposition 1.4:** Let $U \subset M$ be open with $\tilde{U} \cap g_w = \emptyset$.

Then,

$$K(x, \lambda) - \frac{\gamma^* w}{|\gamma^* w|} = O(\lambda^{-1})$$

uniformly on $\tilde{U}$.

**Proof:** Choose $\phi(x) \in C^\infty(M)$ with $\phi = 1$ on $\tilde{U}$ and $\phi = 0$ on some neighborhood of $g_w$. Just as in lemma 1.3 we have

\begin{equation}
(1.16') \quad e^{-i\lambda \psi} N(\psi \delta K(\cdot, \lambda) e^{i\lambda \psi}) = -i\lambda (\gamma \cdot w) + O(\lambda^{-\alpha})
\end{equation}

for $(x, t) \in \tilde{U} \times [-1, 1]$. Now $d\psi = \alpha(\cdot, \cdot, -w + (\gamma \cdot w) \nu, 1)$ which, by the strict convexity of $M$, implies the existence of an open cone $\Lambda \subset T^*(M \times \mathbb{R})$ such $\Lambda \subset \mu$ and $\{(z, d_x \psi) : z = (x, t) \in \text{supp } \phi \times \mathbb{R}\} \subset \Lambda$. Choose $h$ homogeneous of degree 0 with $h = 1$ in $\Lambda$ and supp $h \subset \mu$. Clearly

\begin{equation}
(1.17) \quad h(z, D)(\psi \delta K(\cdot, \lambda) e^{i\lambda \psi}) = \psi \delta K(x, \lambda) e^{i\lambda \psi} + O(\lambda^{-\alpha})
\end{equation}

Moreover, $Nh(z, D) \in \text{OPS}^1_{1, 0}(M \times \mathbb{R})$ by proposition 1.2.

Therefore, $(1.16')$, (1.17), and the fundamental asymptotic expansion imply that on $\tilde{U} \times [-1, 1]$, 

\[-i\lambda(\nu,\omega) = e^{-i\lambda\psi} = e^{-i\lambda\Phi(z,D)(\varphi_\rho K(\cdot,\lambda) e^{i\lambda\psi}) + O(\lambda^{-\infty})}
\]

\[= n_1(z,\lambda d_z\psi) h(z,\lambda d_z\psi) \varphi(x) \rho(t) K(x,\lambda) + O(1)\]

(1.18) \[= -i\lambda^2 (1-|\omega+\nu,\omega|^{2})^{1/2} \varphi(x) \rho(t) K(x,\lambda) + O(1)\]

\[= -i\lambda|\nu,\omega| K(x,\lambda) + O(1)\]

by (1.10). This completes the proof.

To obtain a finer description of N it will be useful to factor the canonical transformation for J, call it \(\mathcal{G}\), through \(U \times \Gamma\) as follows. Write \(\tilde{\phi} = \chi \phi + \gamma\) and denote \(((x,t),(\xi,\eta)) \in U \times \Gamma\) by \((z,\zeta)\). Possibly shrinking \(\Lambda\) we can find \(V \subset \mathbb{R}^n\) open such that \(\mathcal{G}: V \times \Gamma \subset T^*(V) \rightarrow \Lambda \subset T^*(M \times \mathbb{R})\) is a diffeomorphism given by

(1.19) \([\partial_\zeta \tilde{\phi}(z,\zeta), \zeta] \mathcal{G} (z, d_z \tilde{\phi}(z,\zeta))\).

Now define \(\mathcal{G}_1(z,\zeta) = (\partial_\zeta \tilde{\phi}(z,\zeta), \zeta)\) and \(\mathcal{G}_2(z,\zeta) = (z, d_z \tilde{\phi}(z,\zeta))\). Then, since \(\gamma = 0(\frac{1}{|\xi|^\infty})\) and

\(|\det(\zeta \frac{1}{d_z \tilde{\phi}(z,\zeta)})| \geq a_0 > 0\) by (1.10e) \(\mathcal{G}_1\) and \(\mathcal{G}_2\) are diffeomorphisms (after shrinking appropriate sets if necessary) and the following diagram commutes,

\[
\begin{array}{ccc}
V \times \Gamma & \xrightarrow{\mathcal{G}} & \Lambda \\
\downarrow{\mathcal{G}_1} & & \downarrow{\mathcal{G}_2} \\
U \times \Gamma & & \\
\end{array}
\]
Furthermore, the properties of $\rho, \tilde{\phi}$ asserted in (1.11c,d,e) show that

\begin{align*}
\text{a) On } U \{ \eta < 0 \} : & \quad \rho > 0, \ \phi,_{\gamma} = 0, \ |\phi,_{x,\tilde{\phi}}| = |\phi,_{x,\tan,\tilde{\phi}}| < |\xi| ; \\
\ \ \ \ (1.21) \\
\text{b) On } U \{ \eta > 0 \} : & \quad \rho = -\eta, \ |\xi|^{-1} \phi,_{\gamma} = 0((\frac{\eta}{|\xi|})^{\infty}), \ |\phi,_{x,\tilde{\phi}}| > |\xi|.
\end{align*}

Since $d_{z,\tilde{\phi}} = \phi(z,\gamma,\tilde{\phi},|\xi|) + d_{z,\gamma}$ and since $\gamma$ is small, we therefore have

\begin{align*}
\mathcal{J}_2(U \{ \eta < 0 \}) = \mathcal{H} \cap \wedge \quad \mathcal{J}_2(U \{ \eta > 0 \}) = \varepsilon \cap \wedge \\
\ \ \ (1.22) \\
\mathcal{J}_2(U \{ \eta = 0 \}) = \mathcal{G} \cap \wedge.
\end{align*}

We can now discuss the operator $AQ+B$ appearing in (1.15). First recall that $A = J^{-1}K_1$ and $B = J^{-1}K_2$ so that the integral expressions for $K_1$ and $K_2$ implicit in (1.13) yield

\begin{align*}
\sigma_A = (\tilde{\mathcal{C}} \frac{\phi,_{\gamma}-\eta}{gB+h\phi C}) \circ \mathcal{J}_1^{-1} = a_1 \mod S^0_{1,0}(V \times \Gamma) \\
\ \ \ (1.23) \\
\sigma_B = (\tilde{\mathcal{B}} \frac{i\phi,_{\gamma}-i|\xi|^{-1} \rho,_{\gamma}}{gB+h\phi C}) \circ \mathcal{J}_1^{-1} = b_1 \mod S^0_{1,0}(V \times \Gamma) .
\end{align*}

(More precisely we mean the restriction to $U$ of any term in (1.23) defined on $U$. ) Since $\rho$ and $\phi,_{\gamma}$ vanish on $U \{ \eta = 0 \}$ while $\rho,_{\gamma}$ and $\mathcal{J}_1 g$ are elliptic it follows
that \( a_1 \) is elliptic and \( b_1 = \eta b_0', b_0 \in S_{1,0}^0 \).

Now let \( \tilde{\Gamma} = \{ \tilde{c} \mid \xi < |\eta| < c|\varepsilon| \}, \quad c > \tilde{c} > 0 \). By the asymptotic expansion for \( \varepsilon \),

\[
(1.24) \quad A'(s) \sim i s^{\frac{1}{2}} + \frac{1}{s} \sum_{j=0}^{3} b_j s^{-\frac{3}{2}j}, \quad |s| \to \infty
\]

and therefore, \( q(\xi, \eta) = |\varepsilon|^{-\frac{1}{3}} \frac{A'}{A}(-\frac{1}{3}|\eta| \varepsilon^{\frac{1}{3}}) \in S_{1,0}^0(\tilde{\Gamma}) \),

\( q = i(-|\varepsilon|^{-1}|\eta|^{\frac{1}{2}}) \mod S_{1,0}^{-1}(\tilde{\Gamma}) \). Moreover, \( |\xi|^{-1} \rho = -|\xi|^{-1} \eta \)

\[ + O(|\frac{|\eta|}{|\varepsilon|}|^\infty) \text{ on } U \text{ and hence, } \tilde{C} \tilde{B} q = |\varepsilon|^{-\frac{1}{3}} \frac{A'}{A}(-|\xi|^{-\frac{1}{3}}) \]

\( = i(|\varepsilon|^{-1} \rho)^{\frac{1}{2}} \mod S_{1,0}^{-1}(U \times \tilde{\Gamma}) \) while \( \frac{|\varepsilon|^{-1} \tilde{B}}{\tilde{C}} \)

\[ |\varepsilon|^{-\frac{2}{3}} \frac{A'}{A}(|\varepsilon|^{-\frac{1}{3}} \rho) = -i(|\varepsilon|^{-1} \rho)^{\frac{1}{2}} \mod S_{1,0}^{-1}(U \times \tilde{\Gamma}) \). Thus, \mod S_{1,0}^0(V \times \tilde{\Gamma}), we have

\[
(1.25) \quad \sigma_{AQ+B} = \frac{g_\varepsilon q\tilde{C} - g_\varepsilon qh\tilde{C} + ig_\varepsilon \tilde{B} - i|\varepsilon|^{-1} \rho \tilde{B}}{g\tilde{B} + i h q \tilde{C}} \cdot \theta_1^{-1}
\]

\[
= \frac{i \rho \varepsilon (|\varepsilon|^{-1} \rho)(g\tilde{B}) + i \rho \varepsilon (i h q \tilde{C}) + i \rho \varepsilon (g\tilde{B}) + i \rho \varepsilon (|\varepsilon|^{-1} \rho)^{\frac{1}{2}} (i h q \tilde{C})}{g\tilde{B} + i h q \tilde{C}}
\]

\[
= i(\rho \varepsilon + \rho \varepsilon (|\varepsilon|^{-1} \rho)^{\frac{1}{2}}) \cdot \theta_1^{-1} \mod S_{1,0}^0(V \times \tilde{\Gamma})
\]

That (1.25) must agree with ordinary geometric optics is contained in
Proposition 1.5: If \( M \) is strictly convex then \( \forall \ p_0 \in a \cap \{ \tau > 0 \} \) there exists a conic neighborhood, \( \Lambda \), on which \( N \) has the following microlocal description. Given \( U \subset M \times \mathbb{R}^n \) open and sufficiently small there exists an elliptic Fourier integral operator, \( J: \mathcal{E}'(U) \to \mathcal{E}'(V) \) \( (V \subset \mathbb{R}^n \) open, \( \Gamma = \{ \left| \eta \right| < c \left| \xi \right| \} \) and operators, \( A, B \in \text{OPS}^1_{1,0} \) \( (V \times \Gamma) \) such that \( \forall \ f \in \mathcal{E}'(U), \)

\[
Nf = J(AQ+B)J^{-1}f .
\]

\( Q \) has symbol \( q(\xi, \eta) = \left| \xi \right|^3 \frac{1}{A} \frac{1}{A'} \left( -\left| \xi \right|^3 \eta \right) \in S^0 \left( \frac{1}{3}, 0 \right) \) \( (V \times \Gamma) \)

and hence, \( AQ+B \in \text{OPS}^1_{1,0} \) \( (V \times \Gamma) \). Furthermore \( \sigma_A = a_1 + a_2 \)

and \( \sigma_B = b_1 + b_0 = \eta b'_0 + b_0 \) where \( a_0, b_0, b'_0, b_0 \in S^0_{1,0} \) \( (V \times \Gamma) \), \( a_1, b_1 \in S^1_{1,0} \) \( (V \times \Gamma) \) are given by (1.23), and \( a_1 \) is elliptic. Moreover, if \( a(x,t,\xi, \tau) \in \Lambda \cap \mathcal{U}, \)

\[
(1.26) \quad i(\Theta_{\eta} + \rho_{\eta} (\left| \xi \right|^{-1} \rho)^{1/2}) \circ \mathcal{G}^{-1}_2 (a(x,t,\xi, \tau)) = -i(\tau^2 - \left| \xi \right|^2)^{1/2} .
\]

Proof: We need only verify (1.26). In (1.25) we saw that \( \sigma_N = \sigma = J(AQ+B)J^{-1} = i(\Theta_{\eta} + \rho_{\eta} (\left| \xi \right|^{-1} \rho)^{1/2}) \circ \mathcal{G}^{-1}_2 \mod S^0_{1,0} \) \( (V \times \Gamma) \).

(1.26) follows immediately from (1.10), the expression for the principal symbol of \( N \) in \( \mathcal{U} \), since both sides are homogeneous of degree one.
We conclude this section by discussing a microlocal parametrix for \( J(AQ+B)J^{-1} \). (Abusing notation, we often write \( N = J(AQ+B)J^{-1} \) and let \( N^{-1} \) denote the parametrix.) On \( V \times \Gamma \) we let \( Q^{-1} \) have symbol

\[
\frac{1}{q} = |\xi|^\frac{1}{3} \frac{A}{A'} (|-\xi|^{-\frac{1}{3}} \eta) \quad \text{and write } \ AQ+B = (A+BQ^{-1})Q. \quad \text{Now, in general, if } p(\lambda) \in S_{1,0}^m(\mathbb{R}), \ m \geq 0, \ \text{then } p(|\xi|^{-\rho} \eta) \in S_{n,0}^m(1-\rho)(V \times \Gamma). \quad \text{In particular,}
\]

\[
\eta \cdot \frac{1}{q} = |\xi|^\frac{2}{3} (\eta |\xi|^\frac{1}{3} \frac{A}{A'} (|-\xi|^{-\frac{1}{3}} \eta)) \in S_{1,0}^{\frac{1}{3}}, \quad \text{since } \frac{\lambda A}{A'}(\lambda)
\]

\( \in S_{1,0}^{\frac{1}{3}}(\mathbb{R}). \) Thus, \( BQ^{-1} \in \text{OPS}_{\frac{1}{3}},0 \) \( (V \times \Gamma) \) with \( \sigma_{BQ^{-1}} = \eta b_0 \frac{1}{q} \)

\( \frac{1}{3} \) \mod \( S_{\frac{1}{3}},0 \) and it follows that \( A+BQ^{-1} \in \text{OPS}_{\frac{1}{3}},0 \) \( (V \times \Gamma) \) is elliptic since \( A \) is. Therefore there exists a parametrix

\[
(A+BQ^{-1})^{-1} \in \text{OPS}_{\frac{1}{3}},0 \quad \text{and hence}
\]

\[
(1.27) \quad N^{-1} = JQ^{-1}(A+BQ^{-1})^{-1}J^{-1} \mod \text{OPS}^{-\infty}(\Lambda). \]

Consequently \( N^{-1} \) is pseudo-local since \( N \) is elliptic in \( \psi \) and \( \varepsilon \) and a minor modification of (1.4) establishes

\[
(1.28) \quad K(x,\lambda) = e^{-i \lambda \psi} N^{-1}(-i \lambda \rho (\nu \cdot \omega) e^{i \lambda \psi} + O(\lambda^{-\infty}), \ |t| \leq 1 ,
\]

where \( \rho(t) \in e^{\omega}(\mathbb{R}) \) with \( \rho = 1 \) on \([-2,2]\).
2. Special Algebras of Pseudo-Differential Operators

To improve the description of \( N^{-1} \) in \( \Lambda \), it will be useful to write \( Q^{-1}(A+BQ^{-1})^{-1} = E_1 + E_2 \) where \( \sigma_{E_1} \) and \( \sigma_{E_2} \) are supported respectively on \( \{|\eta| \geq \tilde{c}|\xi|^a\} \) and \( \{|\eta| \leq c|\xi|^a\} \), \( 0 < \tilde{c} < c \). We shall arrange that

\[ E_1 \in \text{OPS}_{a,1-a} \]

so that for \( a > \frac{1}{2} \) the usual geometric optics construction can be used to analyze \( JE_1 J^{-1} \). Meanwhile a variant of Egorov's theorem due to Taylor will allow us to conjugate \( E_2 \) by Fourier integral operators and obtain pseudo-differential operators. Thus we introduce the class \( \text{OP}_{a,1-a} \) to obtain more refined information about \( JE_1 J^{-1} \) than is possible from \( E_1 \in \text{OPS}_{a,1-a} \) and we use \( \text{OP}_{\tilde{a}} \) to handle \( JE_2 J^{-1} \). We begin by defining \( \tilde{\eta} \) and \( \eta \) and listing some elementary properties after which we discuss local properties of the associated operators.

Let \( X \) be an open subset of \( \mathbb{R}^n \). When \( \Sigma \) is a closed submanifold of \( X \times (\mathbb{R}^n \setminus 0) \) we denote \( \text{dist}((z,\zeta),\Sigma) \) by \( d_{\Sigma}(z,\zeta) \), a function homogeneous of degree 1 in \( \zeta \). \( \Sigma \) is a linear variety if \( \Sigma = \{ (x,\zeta) : g_2(x)\zeta = 0 \} \) where \( g_2(x) : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is smooth in \( x \) and linear having rank \( k \leq n \) \( \forall x \). In this case, there exists \( g_1(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k} \) such that \( \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \) is invertible and we use \( (x,\xi = g_1(x)\zeta, \eta = g_2(x)\zeta) \) as...
coordinates.

**Definition 2.1:** Let $\Sigma$ be a closed conic submanifold of $\mathbb{R}^n \setminus 0$ and assume $p \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $0 < a, c < 1$.

a) $p \in \mathfrak{m}^{m,k}_{a,1-a}(\Sigma)$ if and only if $p \in S_{1,0}^{m+k}$ in any open conic set disjoint from $\Sigma$ and there exists a conic neighborhood, $\Gamma$, of $\Sigma$ such that given $\Gamma' \subset \subset \Gamma$ and multi-indices $\alpha, \beta$

\[
|a_x^\alpha \xi^\beta p(x, \xi)| \leq c |\xi|^{m+|\beta|} (|\xi|^a + d_\Sigma(x, \xi))^{k-|\alpha|-|\beta|}
\]

$\forall (x, \xi) \in \Gamma'$, $|\xi| > 1$, where $c = c_{\Gamma', \alpha, \beta} > 0$.

b) When $p$ is a linear variety, $p \in \mathfrak{m}^m_p(\Sigma)$ if and only if $p \in S_{1,0}^m$ away from $\Sigma$ and in $\Gamma' \subset \subset \Gamma$, $\Gamma$ some conic neighborhood of $\Sigma$,

\[
|a_x^\alpha \xi^\beta \eta^\gamma p(x, \xi, \eta)| \leq c |\xi|^{m-|\alpha|} (|\xi|^{a+|\beta|} + |\eta|)^{-|\gamma|}
\]

Clearly $\mathfrak{m}^{m,k}_{a,1-a}(\Sigma)$ depends only on $\Sigma$ and one can easily check that the same is true of $\mathfrak{m}^m_p(\Sigma)$. Usually we delete the reference to $\Sigma$ keeping in mind the fact that $\Sigma$ must be a linear variety to define $\mathfrak{m}^m_p$.

**Proposition 2.2:** a) $\mathfrak{m}_p^m \subset S_{\rho,0}^m$ and for $k \leq 0$, $\mathfrak{m}^{m,k}_{a,1-a} \subset S_{a,1-a}^{m+ak}$.

b) For $\mu > 0$, $\mathfrak{m}_p^m \subset \mathfrak{m}^{m+\mu}_p$ and
\[ \tilde{\eta}_{a,1-a} \cap \tilde{\eta}_{a,1-a} \cap \tilde{\eta}_{a,1-a} \]

c) \[ \tilde{\eta}_{o} \cdot \tilde{\eta}_{o} \subset \tilde{\eta}_{o} \quad \text{and} \quad \tilde{\eta}_{a,1-a} \cdot \tilde{\eta}_{a,1-a} \subset \tilde{\eta}_{a,1-a} \]

d) \[ \Xi \Xi \tilde{\eta}_{o}^{\alpha} (\tilde{\eta}_{o}^{m}) \subset \tilde{\eta}_{o}^{m-|\alpha|-\rho|\gamma|} \quad \text{and} \quad \Xi \Xi \tilde{\eta}_{o}^{\alpha} (\tilde{\eta}_{a,1-a}^{m,k}) \subset \tilde{\eta}_{a,1-a}^{m-|\alpha|-\rho|\beta|} \]

e) \[ \tilde{\eta}_{o}^{m} \subset \tilde{\eta}_{o}^{m'}, \rho' \geq \rho, \quad \text{and for} \quad k \leq 0, \quad \tilde{\eta}_{a',1-a'}^{m,k} \subset \tilde{\eta}_{a,1-a}^{m,k}, \quad a' \geq a. \]

**Proof:** All are straightforward consequences of the definition.

We note in passing that for \( \frac{1}{2} < a \leq 1 \), \( \text{OP} \tilde{\eta}_{a,1-a}^{m,k} (\Sigma) \) can be defined on \( X \), a paracompact manifold, when \( \Sigma \) is a closed conic submanifold of \( T^* X \) and that \( \tilde{\eta}_{o}^{m} \) can be defined on \( X \) when \( \Sigma \) is locally the kernel of a vector bundle map. Moreover one can check that for \( a, \rho > \frac{1}{2} \), \( \text{OP} \tilde{\eta}_{a,1-a}^{m,k} \) and \( \text{OP} \tilde{\eta}_{o}^{m} \) are well-defined operator classes on \( X \). But our problem is local and therefore such generality is unnecessary. In fact, we shall need to use \( \text{OP} \tilde{\eta}_{1}^{m} \) and hence, \( X \) will always mean an open subset of \( \mathbb{R}^n \).

**Proposition 2.3:** a) If \( p(x,D) \) and \( p'(x,D) \) are properly supported with \( p \in \tilde{\eta}_{o}^{m} \) and \( p' \in \tilde{\eta}_{o}^{m'} \) then \( p(x,D)p'(x,D) \in \text{OP} \tilde{\eta}_{o}^{m+m} \). Furthermore if \( p \) is elliptic (i.e. \( |p(x,\zeta)| \geq c|\zeta|^m, \quad c > 0 \)) there exists a parametrix for \( p(x,D) \) in...
b) If \( p(x,D), p'(x,D) \) are properly supported with 
\[ p \in \mathfrak{m}^{\mathfrak{a},k}_a, \quad p' \in \mathfrak{m}^{\mathfrak{a},k'}_a, \quad \frac{1}{2} < a \leq 1, \] then \( p(x,D)p'(x,D) \in \mathfrak{m}^{m+m',k+k'}_a \). Moreover, for \( k \leq 0 \), if \( p \) is elliptic (i.e. \( |p(x,\zeta)| \geq c|\zeta|^{m+ak} \), \( c > 0 \)) there exists a parametrix for \( p(x,D) \) in \( \mathfrak{m}^{m+m',k+k'}_a \).

**Proof:** Proposition 2.2 a) and b) show that the desired products and inverses exist in \( \mathfrak{ops}^*_\rho,0 \) and \( \mathfrak{ops}^*_a,1-a \). Moreover we have

\[
(2.3) \quad \sigma p(x,D)p'(x,D) \sim \sum_{\alpha \geq 0} \frac{i^{\frac{|\alpha|}{\alpha}}}{\alpha} \delta_{\zeta}^\alpha \delta_x^\alpha \delta_z^\alpha
\]

and when \( p \) is elliptic, \( \sigma p(x,D) - 1 \sim \sum_{l=0}^s d_0 l \) with \( d_0 = \frac{1}{p} \)

and

\[
(2.4) \quad d_l = -\frac{1}{p} \sum_{\mu=0}^{l-1} \sum_{|\alpha|+u=v} \frac{i^{\frac{|\alpha|}{\alpha}}}{\alpha} \delta_{\zeta}^\alpha \delta_x^\alpha \delta_z^\alpha.
\]

Noting that \( \delta_{\zeta}^\alpha \delta_x^\alpha \delta_z^\alpha \in \mathfrak{m}^{m+m'-\alpha,|\alpha|,k'-2|\alpha|} \) and that inductively \( d_l \in \mathfrak{m}^{m-\alpha l,-m+l,-k-2l} \) (see a) of the next lemma), we see that the proposition follows from

**Lemma 2.4:** a) If \( p \in \mathfrak{m}^m_\rho \) (respectively \( \mathfrak{m}^{m,k}_a \), \( k \leq 0 \)) is elliptic then \( \frac{1}{p} \in \mathfrak{m}^{m,-m,-k}_\rho \).

b) If \( p_j \in \mathfrak{m}^m_j_\rho \) (respectively \( \mathfrak{m}^{m,j}_a \), \( j = 0,1,2, \ldots \), and \( m_j \downarrow -\infty \) \( (m_j+\alpha k_j \downarrow -\infty, m_j \geq m_0) \) then there
exists \( p \in \eta \otimes (\tilde{\eta}^m_{a,1-a}, k_0') \) with \( p \sim \Sigma p_j \).

**Proof:** a) is straightforward. The proof of b) is the same as that of Theorem 2.7 in [2]. The only remark worth making is that for \( \tilde{\eta} \), one can first reduce to the case when

\[ m_j = m_0', \quad \text{all } j, \quad \text{by using the inclusion} \]

\[ \tilde{\eta}^m_{a,1-a} \subseteq \tilde{\eta}^m_{a,1-a} + \frac{1}{a}(m_j - m_0'). \]

This explains the condition \( m_j + ak_j < -\infty \).

**Corollary 2.5:** \([\text{OP}\eta^m_o, \text{OP}\eta^m_p] \subseteq \text{OP}\eta^m_{o,-}\) and

\[ [\text{OP}\eta^m_{a,1-a} \otimes \text{OP}\eta^m_{a,1-a}] \subseteq \text{OP}\eta^m_{a,1-a} + 1, k + k' - 2. \]

Now assume that \( \Sigma \) is a linear variety and \( r \in c^0(\mathbb{R}^n) \). On \( X \times \Gamma \), \( \Gamma = \{(g, \eta) \in \mathbb{R}^{n-k} \times \mathbb{R}^k : |\eta| < c|g| \} \), we define

\[ (2.5) \quad r_u (g, \eta) = r(|g|^{-u} \eta) \]

for \( 0 < u \leq 1 \).

(As with \( \eta^m_o \), whenever \( r_u \) appears, \( \Sigma \), a priori, is a linear variety.) We also fix \( \varphi, \chi \in c^\infty(\mathbb{R}^k) \) such that \( \varphi + \chi = 1 \), \( \chi \in c^\infty_0 \), \( \chi = 1 \) near 0.

**Proposition 2.6:** a) \( \varphi^m_{a,1-o} \subseteq \tilde{\eta}^m_{a,1-a} \) and \( \chi^m_{a,1} \subseteq \eta^m_o \), \( a \geq \rho \).

If \( r(\lambda) \in S^m_{1,0}(\mathbb{R}^k) \) then on \( X \times \Gamma \),

b) \( r_a \in \tilde{\eta}^m_{a,1-a} \) \( \forall m \);
c) For \( m \geq 0 \), \( r^\lambda \in \mathfrak{e}_\rho^{(1-\rho)m} \) and more generally, 
\( r^\lambda \chi_a \in \mathfrak{e}_\rho^{(a-\rho)m} \), \( a \geq 0 \);

d) If \( |a| \geq m \geq 0 \), \( \mathfrak{e}_\rho^{\alpha} r^\lambda \in \mathfrak{e}_\rho^{-\rho|a|} \).

**Proof:** a) follows immediately from the definitions. Next we differentiate \( r^\lambda \) obtaining

\[
\mathfrak{e}_\rho^\alpha \mathfrak{e}_\rho^\gamma r^\lambda = \sum_{\lambda=0}^{\mathfrak{e}_\rho^0 \cdots \mathfrak{e}_\rho^\lambda} \mathfrak{e}_\rho^\alpha \mathfrak{e}_\rho^\gamma (|z|^{-\rho} \gamma \times |\gamma| = |(\gamma_1, \ldots, \gamma_k)| = \lambda

\]

\[
\mathfrak{e}_\rho^1(|z|^{-\rho}) \cdots \mathfrak{e}_\rho^\lambda(|z|^{-\rho}) \mathfrak{e}_\rho^{\alpha+\gamma} r^\lambda(|z|^{-\rho} \eta) .
\]

The \( \lambda \)th summand in (2.6), \( I_\lambda \), is clearly estimated by

\[
I_\lambda \leq c |\gamma| |\gamma|^{-\rho|\gamma|\sigma|\gamma| \mu} (1+|z|^{-\rho} \eta)^m - |\gamma| - \rho
\]

\[
\leq c |z|^{-\rho m} | \| z \|^{\rho} \| \eta \|^{m-|\gamma|} .
\]

Thus, b) is immediate since \( \mathfrak{e}_\Sigma = |\eta| \) and \( |z| \) can be replaced by \( |z| \) in \( \Gamma \) and c) follows from the observation that on \( \text{supp} \chi_a \), \( (|z|^{\rho} + |\eta|)^m \leq c |z|^{am} \) for \( m \geq 0 \). For d), we write \( \mathfrak{e}_\rho^\alpha = \mathfrak{e}_\rho^\alpha \mathfrak{e}_\rho^\gamma \) and observe that

\[
\mathfrak{e}_\rho^\alpha(|z|^{-\rho|\gamma|}) \mathfrak{e}_\rho^1(|z|^{-\rho}) \cdots \mathfrak{e}_\rho^\lambda(|z|^{-\rho}) \in \mathfrak{e}_\rho^{\alpha|\gamma| - \rho |\gamma| - \rho} .
\]

Also, \( \mathfrak{e}_\rho^{\alpha+\gamma} r^\lambda(|z|^{-\rho} \eta) \) belongs to \( \mathfrak{e}_\rho^{(m-|\gamma|)(1-\rho)} \) for

\( m-|\gamma| \geq 0 \) and \( \mathfrak{e}_\rho^0 \) for \( m-|\gamma| \leq 0 \). It follows that for \( m-|\gamma| \leq 0 \), \( I_\lambda \in \mathfrak{e}_\rho^{-\rho|\gamma| - |\gamma| - \rho |\gamma| - \rho |\gamma|} \) while for
\[ m - |\gamma| \geq 0, \quad I_\rho \in \eta \left( m - |\gamma| \right) (1 - \theta) - c |\gamma| - |\theta| - \rho \subseteq \eta \left( m - |\gamma| - |\theta| \right) (1 - \rho) - c (|\gamma| + |\theta|) \] which establishes d) since

\[ |\alpha| = |\theta| + |\gamma| \geq m. \]

We can now refine the description of the operators

\[ AQ+B \] and \[ (A+BQ^{-1})^{-1} \] appearing in the Neumann operator and its inverse. Recalling that \[ \sigma_Q = |\xi|^{-\frac{1}{3}} A' (\xi) \eta \]

and that \[ A'(s) \in S_{\frac{1}{2},0} \] it is easy to check that \( Q \in \) \[ \text{OP} \eta_{\frac{1}{3}}(\Sigma_1) \cap \text{OP} \eta_{\frac{2}{3},0} \] where \( \Sigma_1 = \{(x,\xi,\eta) \in V \times \mathbb{R}_{\eta=0} \} \).

Furthermore, \( \sigma_B = \eta b_0' + b_0 \) with \( b_0', b_0 \in S_{1,0}^{\frac{1}{3}} \) and hence

\[ \sigma_B \cdot \frac{1}{q} = b_0' |\xi|^{-\frac{1}{3}} A' (\xi) - \frac{1}{3} \eta \] + \( b_0 q^{-1} = \tilde{b}_0 r(|\xi|^{-\frac{1}{3}} \eta) + b_0 q^{-1} \)

\[ \in \eta_{\frac{1}{3}} \cap \eta_{\frac{2}{3},0} \] where \( r(s) = s A'(s) \in S_{\frac{1}{2},0} \) and \( \tilde{b}_0 = b_0' |\xi|^{-\frac{1}{3}} \)

\[ \in S_{1,0} (V \times \Gamma). \] It follows that

\[ (2.7) \quad A+BQ^{-1} \in \text{OP} \eta_{\frac{1}{3}}(\Sigma_1) \cap \text{OP} \eta_{\frac{2}{3},0} \]

Since \( A \) is elliptic proposition 2.3 a) implies \( (A+BQ^{-1})^{-1} \)

\[ \in \text{OP} \eta_{\frac{1}{3}}^{-1}. \] Next we write \( (A+BQ^{-1})^{-1} = (I+CQ^{-1})A^{-1} \) where

\( C = A^{-1}B. \) By the proof of proposition 2.3,
where the \( d_{t} \) are given by (2.4) with \( p = 1 + \sigma_{c} q^{-1} \). Clearly \( \sigma_{c} q^{-1} = b_{0} \eta_{A0}^{-1} q^{-1} + c_{0} q^{-1} = \tilde{c}_{0} r (\eta_{2}^{-1} \eta) + c_{0} q^{-1} \), \( \tilde{c}_{0} \in S_{1,0} \), and hence \( \sigma_{c} q^{-1} \in \eta_{0,0}^{-1} \). It follows trivially that \( p \in \eta_{0,0}^{-1} \) while for \( |\alpha| \geq 1 \), \( \tilde{\alpha}_{x} p \in \eta_{\frac{1}{3}, \frac{2}{3}}^{-1} \) and \( \tilde{\alpha}_{\alpha} p \in \eta_{\frac{1}{3}, \frac{2}{3}}^{-1} \). Therefore

\[ 0_{\alpha}(l) p = \tilde{\alpha}_{\alpha} p \in \eta_{\frac{1}{3}, \frac{3}{3}}^{-1} \] and an easy induction shows that

\[ d_{l} = -\frac{1}{p} \sum_{|\alpha|=l} \alpha_{\alpha} \tilde{\alpha}_{x} p \tilde{\alpha}_{\alpha} p - \frac{1}{p} \sum_{u=1}^{l-1} \sum_{u+l} \alpha_{\alpha} \tilde{\alpha}_{x} d \cdot \tilde{\alpha}_{x} p \in \eta_{\frac{1}{3}, \frac{2}{3}}^{-1, 1-l} . \]

Thus, by lemma 2.4b), \( (I+GQ^{-1})^{-1} \in \eta_{0,0}^{-1} \) which clearly implies

\[ (2.8) \quad (A+BQ^{-1})^{-1} \in \text{OG}\eta_{\frac{1}{3}, \frac{2}{3}}^{-1, 0}(\eta_{1}) \cap \text{OG}\eta_{\frac{1}{3}, \frac{3}{3}}^{-1}(\eta_{1}) \]

\[ \sigma(A+BQ^{-1})^{-1} = \frac{1}{\sigma_{A}^{-1} + \sigma_{B}^{-1}} \mod \eta_{\frac{1}{3}, \frac{2}{3}}^{-1, 1} \cap \eta_{\frac{3}{3}}^{-1} . \]

To expose the relevance of (2.7) and (2.8) for \( N = J(AQ+B)J^{-1} \) and \( N^{-1} = JQ^{-1} (A+BQ^{-1})^{-1} J^{-1} \) we have...
Proposition 2.7: Let $F$ be an elliptic Fourier integral operator (i.e. $F u(x) = \int a(x, \zeta) e^{i \phi(x, \zeta)} \hat{f}(\zeta) d\zeta$ where $a$ is elliptic, $d_x \phi \neq 0$, and $|\zeta|^{-1} d_{\zeta, \phi}$ is invertible.)

a) If $P \in \text{OPS}^m_0(\Sigma)$, $0 < \varphi \leq 1$, then $\text{FPF}^{-1} \in \text{OPS}^{m+1}_{\rho, 1-\rho}$ for all $\varepsilon > 0$.

b) If $P \in \text{OPS}^m_0(\Sigma)$, $1/2 < a \leq 1$, then $\text{FPF}^{-1} \in \text{OPS}^{m,1}_a(\Sigma)$ where $\Sigma$ is the canonical relation associated to $F$ and $\sigma_{\text{FPF}^{-1}} = \sigma_{\rho} \circ \pi^{-1} \mod \text{OPS}^{m+1,1}_a(\Sigma)$.

Proof: a) We omit the proof referring the reader to chapter 4 of [11].

b) We can assume that $F = F(t_0)$ where $(\mathcal{A}_t - i \lambda(t,x,D)) F(t) \in \text{OPS}^{-m}$, $F(0) = I$, and $\lambda = \lambda_1 + \lambda_0$, $\lambda_0 \in S_{0,1,0}$ while $\lambda_1$ is homogeneous of degree 1 in $\zeta$. Set $Q(t) = F(t)PF(t)^{-1}$. Then $Q'(t) = i[\lambda(t),Q(t)]$ and we look for a pseudo-differential operator valued solution to this equation, $Q(t)$, with $Q(0) = I$ and $\sigma_{Q(t)} = q(t) \sim \sum_{j=0}^{n} q_j(t)$. We obtain $q_0$ by solving

$$(2.9) \quad (\mathcal{A}_t - H_{\lambda_1}) q_0 = 0 \quad q_0(0,x,\zeta) = p(x,\zeta)$$

and set $r_1 = \sigma_{Q_0} - i[\lambda, Q_0]$, $Q_0(t) = q_0(t,x,D)$. Since $p \in \text{OPS}^m_a$ so is $q_0$ and hence, $r_1 \in \text{OPS}^{m+1,1}_a$. Now assume that $q_0, \ldots, q_{j-1}$ have been obtained with $q_j \in \text{OPS}^{m+1,1}_a$ and such that for $0 < i \leq j-1$, ...
\[(\mathbf{e}_L - H_L) q_L = -r_L, \quad q_L(0,x,t) = 0\]

(2.10)

\[r_L = \sum_{u=0}^{L-1} \sigma_{\lambda_1} - i[\lambda, Q_u] \in \mathfrak{a}_{\lambda_1}^{m+L, k-2L}\]

where \(Q_u(t) = q_u(t,x,D)\). Then, \(r_j \in \mathfrak{a}_{\lambda_1}^{m+j, k-2j}\) since

\[\sigma[\lambda, Q_{j-1}] \sim \sum_{|\alpha| \geq 1} \frac{i^{-|\alpha|}}{\alpha!} (e^{\sigma(\lambda_1 + \lambda_0)} e_{\alpha} q^{j-1} - e_{\alpha} q_j q^{j-1} e^{\sigma(\lambda_1 + \lambda_0)})\]

while \(r_j = r_{j-1} + \sigma Q_{j-1} - i[\lambda, Q_{j-1}] = H_L q_{j-1} - i[\lambda, Q_{j-1}]\).

Hence, we can solve (2.10) to obtain \(q_j \in \mathfrak{a}_{\lambda_1}^{m+j, k-2j}\) and it follows that choosing \(q \sim \mathfrak{a}_{\lambda_1}^{m, k-2}\) yields \(Q(t)\) with \(Q'(t) = i[\lambda(t), Q(t)] \in \text{OPS}^\infty\). Since \(\text{FPF}^{-1} = Q(t_0)\) this establishes the first assertion in b) and also shows that \(\sigma_{\text{FPF}}^{-1} = q_0(t_0) \mod \mathfrak{a}_{\lambda_1}^{m+1, k-2}\). The second assertion then follows since (2.9) requires \(q_0\) to be constant along the integral curves is precisely \(\mathfrak{a}_{\lambda_1}\).

We can now show that \(N\) and \(N^{-1}\) are pseudo-differential operators. We write

\[(AQ+B) = (AQ+B)\varphi_a(D) + (AQ+B)\chi_a(D) = F_1 + F_2\]

(2.11)

\[Q^{-1}(A+BQ^{-1})^{-1} = Q^{-1}(A+BQ^{-1})^{-1} \varphi_a(D) + Q^{-1}(A+BQ^{-1})^{-1} \chi_a(D)\]

\[= E_1 + E_2\]
Using (2.7) and proposition 2.6 a) we see that \( F_1 \)

\[ \varepsilon \Omega_{a, 1-a}^{\frac{1}{3}} (\Sigma_1) \]. Furthermore, \( q_x = \frac{1}{2} \frac{A'}{A} (|x| - \frac{1}{3} \eta) \)

\[ - \frac{1}{3} + \frac{1}{2} (a - \frac{1}{3}) \]

\[ \in \frac{1}{3} \] while \( B_x (D) \in \Omega_{a}^{a} \) since the principal symbol of \( B \) vanishes on \( \eta = 0 \). It follows that that \( F_2 \)

\[ \varepsilon \Omega_{1}^{\frac{1}{3}} (a+1) \]. Choosing \( a > \frac{1}{2} \) but close we conclude that

\[ N = JF_1 J^{-1} + JF_2 J^{-1} \in \Omega_{1}^{1} (\Lambda), \ N = JF_1 J^{-1} \mod \Omega_{1}^{\frac{3}{3}}, \frac{3}{3}, \frac{2}{3} \]

and \( JF_1 J^{-1} \in \Omega_{a, 1-a}^{\frac{1}{3}, \frac{1}{2}} (\Omega) \). Next we recall that \( Q^{-1} \in \Omega_{1}^{1} \)

\[ \cap \Omega_{1}^{\frac{1}{3}, \frac{1}{2}} \] and therefore by (2.8), \( JE_1 J^{-1} \in \Omega_{a, 1-a}^{\frac{1}{3}, \frac{1}{2}} (\Omega) \)

while \( JE_2 J^{-1} \in \Omega_{1, 2}^{\frac{1}{3}, \frac{2}{3}} \). Hence, \( N^{-1} \in \Omega_{1}^{\frac{3}{3}, \frac{2}{3}} (\Lambda) \). We

note in passing that

\[ (2.12) \quad \sigma_{E_1} = \sigma_{-1} \cdot \frac{\phi_a}{\sigma_{a+q}^{-1} \sigma_B} \mod \Omega_{a, 1-a}^{\frac{1}{2}, \frac{5}{2}} (\Sigma_1) \] .

It will be convenient later to obtain a more explicit formula for \( \sigma_{E_1} \). First we write
$$\sigma_A + q^{-1} \sigma_B = a_1 + q^{-1} b_1 + a_0 + q^{-1} b_0 \quad \text{where} \quad (a_1 + q^{-1} b_1)$$
is elliptic in \(\eta^{1,0}_{1,3}, \frac{2}{3}, \frac{2}{3}\) and \(a_0 + q^{-1} b_0 \in \eta^{\frac{1}{2}, -\frac{1}{2}}_{1, \frac{3}{2}}\). It follows that \(\phi_a \frac{(a_1 + q^{-1} b_0)^j}{(a_1 + q^{-1} b_1)} \in \eta_{a, 1-a}^{\frac{1}{2}, -\frac{1}{2}}\) and therefore

(2.13) \quad \sigma_{E_1} = q^{-1} \frac{\phi_a}{a_1 + q^{-1} b_1} \mod \eta_{a, 1-a}^{\frac{1}{2}, -\frac{5}{2}}.

Now choose \(\tilde{\phi} \in C(\mathbb{R})\) with \(\tilde{\phi} = 1\) on \(\text{supp} \phi\), \(\tilde{\phi} = 0\) near 0. Recalling (1.23) we have

$$\phi_a q^{-1} (a_1 + q^{-1} b_1)^{-1} = \frac{\phi_a q^{-1} (g\tilde{B} + ih\tilde{C})}{(\rho \sqrt{g-h\tilde{v}})\tilde{C} + (i g \tilde{v} - i |\tilde{v}|^{-1} \rho \sqrt{g}) q^{-1} B^{-1}} \phi_a q^{-1} (g\tilde{B} + ih\tilde{C}) \tilde{a} [ (\rho \sqrt{s + i \tilde{v}}) g \tilde{B} + (\rho \sqrt{s + i \tilde{v}}) g \tilde{B} + (- |\tilde{v}|^{-1} \rho \sqrt{s + i \tilde{v}}) ih \tilde{C} ]$$

(2.14) \quad \phi_a (g\tilde{B} + ih\tilde{C}) \tilde{a} [ (\rho \sqrt{s + i \tilde{v}}) g \tilde{B} + (\rho \sqrt{s + i \tilde{v}}) g \tilde{B} + (\rho \sqrt{s + i \tilde{v}}) ih \tilde{C} ]$$

where \(s = \frac{Cq}{B} = |\tilde{v}|^{-1} \frac{1}{3} A \frac{1}{A} (|\tilde{v}|^{-1} \frac{1}{3} \rho)\). Now \(|\tilde{v}|^{-1} = - |\tilde{v}|^{-1} \eta\) + \(O(|\tilde{v}|^{-1} |\tilde{v}|)^{\infty}\) and an easy modification of proposition 2.6

b) shows that \(\frac{A'}{A} (|\tilde{v}|^{-1} \frac{1}{3} \rho) \in \eta^{\frac{1}{6}, \frac{1}{2}}_{1, \frac{3}{2}}\). Furthermore,

\(|\tilde{v}|^{-1} \frac{1}{3} \rho \geq c |\tilde{v}| a^{-1} \frac{1}{3}\) on \(\text{supp} \tilde{\phi}_a\) and therefore the asymptotic expansion for \(\frac{A'}{A}\), (1.24), can be used for \(|\tilde{v}|\) large.
In particular, since 
\[ \tilde{\phi}_a |z|^{-\frac{1}{3}} \left( |z|^{-\frac{1}{3}} \rho \right)^{-\frac{1}{2}} \left( |z|^{-\frac{1}{3}} \rho \right)^{-\frac{3}{2}} \epsilon_{a,1-a} \]
we have

\[ (2.15) \quad \tilde{\phi}_a^s = i \tilde{\phi}_a |z|^{-\frac{1}{3}} \left( |z|^{-\frac{1}{3}} \rho \right)^{-\frac{1}{2}} \mod \epsilon_{0,1-a} \]

Similarly

\[ (2.16) \quad \tilde{\phi}_a^\omega = \frac{1}{i} \tilde{\phi}_a \rho \left( |z|^{-\frac{1}{3}} \rho \right)^{-\frac{1}{2}} \mod \epsilon_{1,1-a} \]

(Note that the factor \( \tilde{\phi}_a \) removes the singularities at \( \eta = 0 \).)

We now denote by \( d \) the bracketed term in the denominator of the last expression in (2.14). Using (2.15) and (2.16) we have

\[ (2.17) \quad \tilde{\phi}_a^d = \tilde{\phi}_a \left( i_\omega \rho \left( |z|^{-\frac{1}{3}} \rho \right)^{\frac{1}{2}} + i_\omega \rho \right) (g\tilde{B}+ihq\tilde{C}) \mod \epsilon_{1,1-a} \]

since \( hq\tilde{C} \in S^0_{1,0} \). Moreover, \( g\tilde{B}+ihq\tilde{C} \) is elliptic and hence

\[ (2.18) \quad \tilde{\phi}_a^d = \tilde{\phi}_a (m_1+m_2) (g\tilde{B}+ihq\tilde{C}) \]

for some \( m_0 \in \epsilon_{1,1-a} \) where \( m_1 = i_\omega \rho \left( |z|^{-\frac{1}{3}} \rho \right)^{\frac{1}{2}} + i_\omega \rho \). Now, \( \omega_\rho \) vanishes precisely on \( \rho = 0 \) while \( |z|^{-\frac{1}{3}} \rho \geq a_0 > 0 \). Thus, we can write \( m_1 = (|z|^{-\frac{1}{3}} \rho)^{\frac{1}{2}} [i_\omega \rho + |z|^{-\frac{1}{3}} \rho)^{\frac{1}{2}} k] \) for some \( k \in S^1_{1,0} \). Since \( \tilde{\phi}_a = 1 \) on \( \text{supp} \ \phi_a \),
\[ \varphi_a \equiv \frac{\varphi_a}{m_1} \left( \frac{1}{|\xi|^{\frac{1}{2}} \rho} \right)^{\frac{1}{2}} \left[ (\varphi_a + \varphi_a(|\xi|^{-1} \rho)^{\frac{1}{2}} \right] \] and by the ellipticity

of \( \varphi_a \), the second factor belongs to \( \tilde{\eta}_{a,1}^{\frac{1}{2}} \). On the other hand, \( \varphi_a \cdot (|\xi|^{-\frac{1}{2}} \rho)^{\frac{1}{2}} = |\xi|^{-\frac{1}{2}} \rho \varphi_a \cdot (|\xi|^{-\frac{1}{2}} \rho)^{\frac{1}{2}} \in \tilde{\eta}_{a,1}^{\frac{1}{2}} \)

and hence \( \frac{\varphi_a}{m_1} \in \tilde{\eta}_{a,1}^{\frac{1}{2}} \). Therefore, \( \frac{m_0}{m_1} \in \tilde{\eta}_{a,1}^{\frac{1}{2}} \) and this yields

\[ \frac{\varphi_a}{\varphi_a^d} \left( g\tilde{g} + \eta q\tilde{q} \right) = \frac{\varphi_a}{m_1} \left( 1 + \frac{\varphi_a}{m_1} m_0 \right)^{-1} \]

\[ = \frac{\varphi_a}{m_1} \bmod \tilde{\eta}_{a,1}^{0,-2} \]

In view of (2.13), (2.14) and (2.19) we have established

\[ \varphi_{E_1} = \frac{\varphi_a}{i\rho \nu (|\xi|^{-\frac{1}{2}} \rho)^{\frac{1}{2}} + i\rho} \varphi_a^{-1} \bmod \tilde{\eta}_{a,1}^{\frac{1}{2},-\frac{5}{2}} (\Sigma_1) \]

Corresponding to the decomposition, \( N^{-1} = J E_1 J^{-1} + J E_2 J^{-1} \) we have for \( |t| \leq 1 \),
\[
K(x, \lambda) = e^{-i\lambda \psi_N^{-1}}(-i\lambda_0(\nu \cdot \omega)e^{i\lambda \psi})(x, t) + O(\lambda^{-\alpha})
\]

\[
= e^{-i\lambda \psi}JE_1J^{-1}(-i\lambda_0(\nu \cdot \omega)e^{i\lambda \psi}) + O(\lambda^{-\alpha})
\]

\[
= K_1(t, x, \lambda) + K_2(t, x, \lambda) + O(\lambda^{-\alpha})
\]

where \( \rho(t) \in \mathcal{C}_0([a, b]) \), \( \rho = 1 \) on \([-2, 2]\) (see (1.28)). To establish theorem 1 we shall need to estimate the remainder terms \((\nu \cdot \omega)^kK_2\) and \((\nu \cdot \omega)^kK_3\) where

\[
K_3(x, \lambda) = e^{-i\lambda \psi}JRJ^{-1}(-i\lambda_0(\nu \cdot \omega)e^{i\lambda \psi})
\]

with \( \sigma_R = \sigma_{E_1} - \frac{\varphi_0}{m_1} \eta_1 \eta_1^{-1} \in \mathcal{C}_{a, 1-a} (\mathbb{R}) \) by (2.20). Naively, all we can say is \((\nu \cdot \omega)^kK_2 = O(\lambda^{-\alpha})\) and \((\nu \cdot \omega)^kK_3 = O(\lambda^{-\alpha})\).

To get around this, we shall eventually construct \(P_b \in \text{OPS}_{b, 1-b}^0(M \times \mathbb{R})\) such that \(P_b\) has order \(-\infty\) on \(\{(z, \zeta) \in T^*(M \times \mathbb{R}) : d_{S_2}^2(z, \zeta) \geq c|\zeta|^b\}\) where \(S_2 = \{(z, \lambda) : z \in M \times \mathbb{R}, \lambda > 0\}\). Thus, to handle \((\nu \cdot \omega)^kK_3\) we should hope that \((\nu \cdot \omega)^{k+1}JRJ^{-1}P_b\) has relatively low order. Similarly, since \(E_2 = Q^{-1}(A+BQ^{-1})^{-1}\chi_a(D)\)

\[
= Q^{-1}(A+BQ^{-1})^{-1}\chi_a(D)\chi_b(D) \mod \text{OPS}^{-\infty}, \quad b > a,
\]

we should hope that \((\nu \cdot \omega)^{k+1}J\chi_b(D)J^{-1}P_b\) has low order. That these
assertions are true follows from the strict convexity of \( M \) as will be seen in the next proposition. Note that by strict convexity, \( S_2 \) and \( S_1 = a \) have precisely second order contact along \( S_1 \cap S_2 \subset \mathcal{g}_m = \{ \nu \cdot w = 0 \} \) and \( |\nu \cdot w| \) is comparable to \( d_{\mathcal{g}_m} \).

Returning to previous notation, if \( \Sigma \subset X \times \mathbb{R}^n \) is a smooth conic submanifold, \( u_b(\Sigma, c) \) (often written \( u_b(\Sigma) \)) will denote \( \{ (x, \zeta) : d_{\Sigma}(x, \zeta) \leq c |\zeta|^b \} \).

**Proposition 2.8:** Let \( \Sigma_1 \) and \( \Sigma_2 \) be closed conic submanifolds of \( X \times \mathbb{R}^n \) having precisely second order contact along their intersection with \( \Sigma_1 = \{ \eta = 0 \} \). If \( p \in S^0_{b, 1-b} \) has order \( -\infty \) on \( u_b(\Sigma_2) \) and \( s \in S^0 \) has principal symbol \( s_0 \) satisfying \( |s_0| \leq |\zeta|^{-1} d_{\Sigma_1 \cap \Sigma_2} \) then

a) \( r \in \mathcal{H}^m, k \) of \( \Sigma_1 \), \( k \leq 0 = s_0^{\mu \nu} \rho \in S^m_{b, 1-b} \)

provided \( \epsilon \leq -2k \);

b) \( s^\mu \lambda_{\nu \rho} \rho \in S_{b, 1-b}^{\mu \nu} \) provided \( \frac{\lambda_{\nu \rho}}{\nu \rho} (1-b) \leq 1. \)

Furthermore, if \( \Sigma_2 = \bigcap_{j=1}^l \{ \lambda_j = 0 \} \) where \( \lambda_j \) are smooth, homogeneous of degree 1 in \( \zeta \), and \( d_{\lambda_j} \) are linearly independent and if \( t \in S^m \) has principal symbol vanishing to second order on \( \Sigma_1 \cap \Sigma_2 \) then

c) we can write \( t = \sum_{j=1}^k \eta_j a_j + \sum_{j=1}^l \lambda_j b_j + v \) where
\[ a_j, b_j, v \in \mathbb{S}^{m-1}. \]

**Proof:** We first observe that if \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \) are closed compact submanifolds of \( \mathbb{R}^n \) having precisely second order contact on their intersection then there exists \( C > 0 \) such that

\[
\text{dist}(\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2, ([d_\perp \leq c] \cap [d_\perp \leq d])) \leq C(c+d)^{\frac{1}{2}}
\]

for all \( (c,d) \in [0, C_0] \times [0, D_0] \). Considering \( \tilde{\Sigma}_1 = \Sigma_1 \cap \mathbb{S}^*, \mathbb{S}^* = \{(x, \zeta) : |\zeta| = 1\} \), we see that the hypothesis on \( s_0 \) guarantees the inequality

\[
|s_0|^{\mu} \leq c(|\zeta|^{-1}d_\perp \Sigma_1 + |\zeta|^{b-1})^{\frac{1}{2}}
\]

for all \( \mu \geq 0 \) on \( u_b(\Sigma_1) \cap u_b(\Sigma_2) \). Also we note that \( s'_o = s'_o + s' \)

with \( s' \in S^{-1} \) and trivially, \( s'x_p \in S_{b,1-b} \) since \( \frac{b}{2}(1-b) \leq 1 \).

a) Because of the hypothesis on \( p \) we need only show that \( s'_o + r_p \in S_{b,1-b}^{m+bk-\frac{b}{2}l(1-b)} \) on \( u_b(\Sigma_2) \). On \( u_b(\Sigma_1) \) one easily sees that \( r \in S_{b,1-b}^{m+bk} \) so that in this set, \( s'_0 \circ r \in S_{b,1-b}^{m+bk} \) is determined since \( k + \frac{b}{2}l \leq 0 \). Applying \( \alpha_j, \zeta \) to \( s'_o \circ r \) shows that it suffices to estimate terms like

\[
I = (\alpha_1, \alpha_1 \circ r)(\alpha_2, \alpha_2 \circ r)(\alpha_3, \alpha_3 \circ r)\ldots(\alpha_v, \alpha_v \circ r) 
\]

on \( u_b(\Sigma_1) \cap u_b(\Sigma_2) \) where \( \beta = \Sigma \beta, \alpha = \Sigma \alpha, \mu \leq \nu - 2 \), and

with \( \alpha' = \Sigma \alpha, \beta' = \Sigma \beta, \nu - 2 \leq |\alpha'| + |\beta'|. \) But on

\( u_b(\Sigma_1) \cap u_b(\Sigma_2), \ |\zeta|^{b} \leq (d_\perp + |\zeta|) \leq c|\zeta|^b \) and therefore
we can estimate

\[ |I| \leq c |\zeta| \left( (1-b) |\beta_1| - b |\alpha_1| + m + |\beta_2| - |\alpha'| - \frac{1}{2}(t-v+2) \right) \]

\[ \left( d_{\Sigma_1} + \frac{1}{2} |\zeta|^b \right)^k |\alpha_2| - |\beta_2| + \frac{1}{2}(t-v+2) \]

\[ \leq c |\zeta|^{m+bk-\frac{1}{2}t(1-b)-b|\alpha|+(1-b)|\beta|} \]

establishing a).

b) Since \( \chi_b \) has order \(-\infty\) on \( u_b(\Sigma_1) \) while \( p \) has order \(-\infty\) on \( u_b(\Sigma_2) \) the opening observation shows that \( \chi_b p \) has order \(-\infty\) on \( \frac{u_{1+b}(\Sigma_1 \cap \Sigma_2)}{2} \). Thus it suffices to establish estimates on \( \frac{u_{1+b}(\Sigma_1 \cap \Sigma_2)}{2} \) where

\[ |s_0|^u \leq c |\zeta|^{-\frac{1}{2}u(1-b)}. \]

Now, \( \sum_{x, \zeta} \left( \chi_b p \right) (s_{x, \zeta} s_0 \alpha_b) \) is a sum of terms of the form

\[ I = \sum_{x, \zeta} (\chi_b p)(s_{x, \zeta} s_0) \cdots (s_{x, \zeta} s_0)^{s_{x, \zeta} s_0} \]

where \( \Sigma_{x, \zeta} \) is \( \delta \), \( \Sigma_{x, \zeta} = \alpha \), \( \nu-1 \leq \tau \), and \( \nu-1 \leq |\alpha'| + |\beta'| = \sum_{i=2}^{\nu} \alpha_i | + \sum_{i=2}^{\nu} \beta_i |. \) On \( \frac{u_{1+b}(\Sigma_1 \cap \Sigma_2)}{2} \) we can therefore estimate

\[ |I| \leq c |\zeta|^{-b|\alpha_1|+(1-b)|\beta_1| - |\alpha'| - \frac{1}{2}(1-b)(t-v+1)} \]

\[ \left( d_{\Sigma_1} + \frac{1}{2} |\zeta|^b \right)^k |\alpha_2| - |\beta_2| + \frac{1}{2}(t-v+2) \]

\[ \leq c |\zeta|^{m+bk-\frac{1}{2}t(1-b)-b|\alpha|+(1-b)|\beta|} \]
which establishes b).

c) For completeness we repeat the argument which appears in Majda-Taylor [4]. It suffices to show that the ideal in $C^\infty(S^*(X\times \Gamma))$ generated by $\eta_1, \ldots, \eta_k$ and $\lambda_1, \ldots, \lambda_u$ contains all functions vanishing to second order on $\Sigma_1 \cap \Sigma_2 \cap S^*(X\times \Gamma)$, $\Gamma = \{d_{\Sigma_1 \cap \Sigma_2} < \epsilon | \zeta | \}$. We can find coordinates $u_1, \ldots, u_{2n-1}$ on $S^*$ such that $\Sigma_1 \cap S^* = \{u_{2n-k} = \ldots = u_{2n-1} = 0\}$ and $\Sigma_2 \cap S^* = \{u_{2n-1} - u_1^2 = u_2 = \ldots = u_u = 0\}$. Thus it is clear that the ideal generated by $\{\eta_1\}$ and $\{\lambda_1\}$ contains all functions vanishing to second order as desired.
3. The Estimates for Theorem 1

In this section we obtain the estimate asserted in theorem 1 using the decomposition, (2.21), of $K(x,\lambda)$. Thus, for $j = 1,2$ we need to estimate with $T = J^{-1}(\nabla \cdot \mu)J$

\begin{equation}
(\nabla \cdot \omega)^k K_j(t, x, \lambda) = -i \lambda e^{-i \lambda \psi} T^k \mathcal{E}_j T J^{-1}(\rho e^{i \lambda \psi}) \ .
\end{equation}

We first examine the remainder, $K_2$, and then turn to the "principal" part, $K_1$. 

Some preliminary remarks are in order. First, as already noted, for $c \leq 1$ there exists $P_c \in \text{OPS}^{0}_{c, 1-c}(M \times \mathbb{R})$ such that $P_c$ has order $-\infty$ on $(U_c(S_2))$ and $e^{-i \lambda \psi} P_c(\rho e^{i \lambda \psi}) = 1 \mod S^{-m}$ on $|t| \leq 1$. In fact, we can choose $\sigma$ homogeneous of degree 1 in $\zeta$, smooth on $\Lambda \setminus S_2$ such that $\sigma = d_{S_2}$ on a small conic neighborhood, $\tilde{\Lambda}$, of $S_2$ and $\sigma \geq d_{S_2}$ on $\Lambda \setminus \tilde{\Lambda}$. Thus we can obtain the desired $P_c$ by letting $P_c = \chi(|\zeta|^{-c})$ where $\chi \in \mathcal{C}_0^\infty$ with $\chi = 1$ near 0. Set $\tilde{P}_c = J^{-1} P_c J$. Next observe that (locally) there exist $n-1$ linearly independent vector fields, $X_j$, on $M \times \mathbb{R}$ such that $X_j \psi = 0$ and $S_2 = \bigcap_{j=1}^{n-1} \{ X_j = 0 \}$. Proposition 2.8c) applies with $\lambda_j = \sigma X_j \circ \mu$ and $t$ replaced by the principal symbol of $T^2$ where $T = J^{-1}(\nabla \cdot \omega)J$. Since $|\nabla \cdot \omega|$ is comparable to $|\zeta|^{-1} d_{S_1 \cap S_2}$ near $S_{\omega}$ we have
\[ T^2 = pB_0 + S + V \]

where \( \tau_p = \gamma, B_0, V \in \text{OPS}^{-1}, S \in \text{OPS}^0, \text{ and } SJ^{-1}(e^{i\lambda \psi}) = 0(\lambda^{-\infty}) \text{ on } |t| \leq 1. \) Turning to \((\nu \cdot \omega)^K_k\), we first establish the

|K_2| estimating machine: Let \( \tilde{E} = \text{OPS}_{1, c, 1-c}^m \) and \( R \in \text{OPS}_{c, 1-c}^m \), \( c > \frac{2}{3} \). Then \( \forall \varepsilon > 0, \)

\[ e^{-i\lambda \psi} JERJ^{-1}(e^{i\lambda \psi}) = O(\lambda^{1+\frac{1}{2}}(1+2+\varepsilon)) \]

**Proof:** We first note that since \( c > \frac{2}{3} \), \( JERJ^{-1} = JERJ^{-1} = JERJ^{-1} \) \( \in \text{OPS}_{\frac{1}{3}, \frac{2}{3} + \varepsilon}^m \). We cannot appeal to the fundamental asymptotic expansion lemma since it does not apply to operators of type \( \frac{1}{3}, \frac{2}{3} \). But we can find a coordinate change, \( \xi \), such that \( \tilde{\psi} = \psi \xi \) is linear (\( d\psi \neq 0 \) near \( \nu \cdot \omega = 0 \)). With \( Ku = u\xi \) we have \( JERJ^{-1} = K^{-1}[(KJ)\tilde{E}(KJ)^{-1}][(KJ)R(KJ)^{-1}]K = K^{-1}FK \) where \( F \in \text{OPS}_{\frac{1}{3}, \frac{2}{3} + \varepsilon}^m \). The linearity of \( \tilde{\psi} \) completes the proof --

\[ \lambda JERJ^{-1}(e^{i\lambda \psi}) = \lambda F((\psi \xi) e^{i\lambda \tilde{\psi}}) \xi^{-1} = O(\lambda^{1+\frac{1}{2}}(1+2+\varepsilon)) \]

We also need a trivial technical result.

**Lemma 3.1:** If \( A \in \text{OPS}_{\rho, \delta}^m \), has order \(-\infty\) on \( u_\alpha(\Sigma_1) \), \( \Sigma_1 = \{ \eta = 0 \} \), and \( b + \rho > 1, b > a \), then \( A\chi_b(D) = A \mod \text{OPS}^{-\infty} \).
**Proof:** Merely observe that $A(I - \chi_b(D))$ has order $-\infty$ on $\nu_b(\Sigma_1)$ since $1 - \chi_b(\xi, \eta)$ vanishes there while it has order $-\infty$ on $\nu_a(\Sigma_1) \supset \nu_b(\Sigma_1)$ since $A$ has order $-\infty$ there.

Now fix $\varepsilon > 0$ and choose $a > \frac{1}{2}$, $b > \frac{2}{3}$ such that $5(b - \frac{2}{3}) + 2(a - \frac{1}{2}) < \varepsilon$. Writing $ad T(E) = [T, E]$ we have

$$(\nu \cdot w)^k_{K_2} = -i \lambda e^{-i \lambda \psi} \sum_{t=0}^{k} \frac{k!}{t!} J((ad T)^t E_2) T^{k-t+1} J^{-1}(\rho e^{i \lambda \psi})$$

(3.4)  

$$= -i \lambda e^{-i \lambda \psi} \sum_{t=0}^{k} \frac{k!}{t!} J((ad T)^t E_2) T^{k-t+1} \chi_b(D) \tilde{P}_b J^{-1}$$

$$= \sum_{t=0}^{k} I_{t, k} \rho e^{i \lambda \psi} + O(\lambda^{-\infty})$$

$$= \sum_{t=0}^{k} I_{t, k} \rho + O(\lambda^{-\infty})$$

| $t$ | $t \leq 1$ |

Proposition 2.8b) guarantees that for $k \leq 6$, $1 \leq t \leq k$, $T^{k-t+1} \chi_b(D) \tilde{P} \in \text{OPS}_{b, 1-b}^{-\frac{1}{3}(k-t+1)(1-b)}$. Since $(ad T)^t E_2 \in \text{OPS}_{b, 1-b}^{-\frac{2}{3} - \frac{1}{3} t}$, (3.3) shows that on $|t| \leq 1$,

(3.5)  

$$I_{t, k} = O(\lambda^{1 - \frac{1}{2}(k-t+1)(1-b) - \frac{2}{3} - \frac{1}{3} t + \frac{1}{6} \varepsilon})$$

$$= O(\lambda^{-\frac{1}{6} + \varepsilon})$$

$$= O(\lambda^{-\frac{1}{6} + \varepsilon})$$

Similarly one easily checks that on $|t| \leq 1$,

(3.6)  

$$I_{0, 0} = O(\lambda^{-\frac{1}{6} + \varepsilon})$$

.
To estimate $I_{0,k}$, $1 \leq k \leq 6$, we use the decomposition of $T^2$, (3.2) and write

\begin{equation}
(3.7) \quad I_{0,k} = i\lambda e^{-i\lambda \psi} E_2(PB_0 + S + V)T^{k-1}\chi_b(D)P_{a}J^{-1}(\sigma e^{i\lambda \psi})
= I'_k + I''_k + I'''_k
\end{equation}

with $I'_k$, $I''_k$, $I'''_k$ corresponding to $PB_0$, $S$, $V$ respectively.

Noting that $\chi_a \cdot \eta = |\xi|^a (|\xi|^{-a}\chi(|\xi|^{-a})) \in \eta_a^a$ we see that

\begin{equation}
E_2PB_0 = Q^{-1}(A+BQ^{-1})^{-1}\chi_a(D)PB_0 \in \mathcal{OP}_{\frac{1}{3}}. \text{ Thus, (3.3) yields}
\end{equation}

\begin{equation}
(3.8) \quad I'_k = O(\lambda^{-\frac{2}{3} + a - \frac{1}{2}(k-1)(1-b) + \frac{1}{2} \varepsilon - \frac{1}{6} k + \varepsilon}) = O(\lambda^{-\frac{2}{3}}). \quad \text{Since } V \in \mathcal{OPS}^{-1}, \text{ } I'''_k \text{ is easy to estimate.}
\end{equation}

\begin{equation}
(3.9) \quad I'''_k = O(\lambda^{-\frac{2}{3} - \frac{1}{2}(k-1)(1-b) + \varepsilon - \frac{1}{2} - \frac{1}{6} k + \varepsilon}) = O(\lambda^{-\frac{2}{3}}). \quad \text{To estimate } I''_k \text{ we first observe that } I''_k = O(\lambda^{-\infty}) \text{ since}
\end{equation}

$SJ^{-1}(\rho e^{-i\lambda \psi}) = O(\lambda^{-\infty})$, $|t| \leq 1$. Furthermore,

$\quad I''_k = O(\lambda^{-\frac{2}{3} + \varepsilon})$ since $ST = TS + [S,T]$ and $[S,T] \in \mathcal{OPS}^{-1}$.

For $k \geq 3$, we write
\[ E_2 S^{k-1} = E_2 T^2 S^{k-3} + E_2 [S, T^2] T^{k-3} \]
\[ = E_2 P_0 S^{k-3} + E_2 S^2 T^{k-3} + E_2 V T^{k-3} + E_2 [S, T^2] T^{k-3} \]
\[ = E_2 P_0 T^{k-3} S + E_2 S^{k-3} S + E_2 [S, T^{k-3}] S \]
\[ + E_2 [S, [S, T^{k-3}]] + E_2 P_0 [S, T^{k-3}] + E_2 (V + [S, T^2]) T^{k-3} \]
\[ = A_1 S + A_2 . \]

Since \( A_2 \) has order \(-\infty\) on \( \mu_a(\Sigma_1) \), \( A_2 = A_2 \chi_b(D) \mod \text{OPS}^- \). Thus,

\[ (3.10) \quad I_k'' = -i \lambda e^{-i \lambda \psi} J E_2 S^{k-1} J^{-1} (p e^{i \lambda \psi}) + O(\lambda^{-\infty}) \]
\[ = -i \lambda e^{-i \lambda \psi} J A_2 J^{-1} (p e^{i \lambda \psi}) + O(\lambda^{-\infty}) \]
\[ - \frac{2}{3} + a - 1 + \varepsilon \quad \frac{1}{3} - 2 + \varepsilon \quad - \frac{2}{3} - \frac{1}{2} (k-3) (1-b) + \frac{1}{2} \varepsilon \]
\[ = 0(\lambda_{\frac{1}{3} - 2 + \varepsilon}) + O(\lambda_{\frac{1}{3} - 2 + \varepsilon}) + O(\lambda_{\frac{1}{2} (k-3) (1-b) + \frac{1}{2} \varepsilon}) \]
\[ = 0(\lambda_{\frac{1}{3} - 2 + \varepsilon}) + O(\lambda_{\frac{1}{2} (k-3) (1-b) + \frac{1}{2} \varepsilon}) \]

by (3.3). In view of (3.4) - (3.10) we have thus established

\[ (3.11) \quad (\psi \cdot \omega)^k K_2(x, \lambda) = \begin{cases} 
\frac{1}{6} + \varepsilon & k = 0 \\
O(\lambda_{\frac{1}{3} - 2 + \varepsilon}) & 1 \leq k \leq 6
\end{cases} \]

where \( K_2(x, \lambda) = \int \rho_1(t) K_2(t, x, \lambda) dt \) with \( \rho_1 \in C_0^\infty (-1, 1) \).
\[ \lambda^t R(z, \lambda d_\psi) = O(\lambda^{1+\frac{1}{2}a_\mu-\frac{1}{2}t(1-a)}) \]

**Proof:** With \( P_a \) chosen as earlier, we have

\[ (\nu \cdot w)^k R(z, \lambda d\psi) = (\nu \cdot w)^t R(z, \lambda d\psi) \sigma_{P_a}(z, \lambda d\psi) \]

since \( \sigma_{P_a} = 1 \) on \( S_2 \). But \( P_a \) has order \( -\infty \) on \( u_a(S_2) \) so that by proposition 2.8 c), \( (\nu \cdot w)^t R_{P_a} \in S_{a, 1-a}^{m-\frac{1}{2}a_\mu-\frac{1}{2}t}(1-a) \),

\[ 0 \leq t \leq u. \]

Now define \( E_1 = JE_1J^{-1} \) so that \( JE_1TJ^{-1} = E_1(\nu \cdot w) \).

Then,

\[ K_1 = -i\lambda e^{-i\lambda \psi} E_1(\nu \cdot w)(\rho e^{i\lambda \psi}) \]

and hence, by the fundamental asymptotic expansion lemma (\( a > \frac{1}{2} \)),

\[ (\nu \cdot w)^k K_1 \sim -i\lambda (\nu \cdot w)^k \sum_{|\alpha| \geq 0} \frac{1}{\alpha !} \sigma_{\alpha} E_1(\nu \cdot w) (a, \lambda d\psi) D^\alpha_y \bigg|_{y=z} (\rho e^{i\lambda \gamma}) \]

where \( \gamma(y, z) = \psi(y) = \psi(z) = (y-z) \cdot \nabla \psi \). Since
\[
\sigma_{E_1(v,w)} = (v \cdot w)_{\sigma} \mod \eta_{a,1-a}^{1-\frac{3}{2}}, \quad \sigma_a = (v \cdot w)_{\sigma}^a \sigma_{E_1(v,w)} \mod \eta_{a,1-a}^{0,-2} |a| .
\]
Furthermore, \(\sigma_a \in \eta_{a,1-a}^{1-\frac{3}{2},-\frac{3}{2}}\), which we use for \(|a| \geq 2\) and for \(|a| = 1\), \(\sigma_a \in \eta_{a,1-a}^{1-\frac{5}{2},-\frac{5}{2}}\).

It follows from (3.12) that

\[
(3.15) \quad \lambda(v \cdot w)^{k+1} \sigma_{E_1(v,w)} (z, \lambda d_z \psi) = \begin{cases} 
\lambda(v \cdot w)^{1-2a-\frac{3}{2} k(1-a)} + O(\lambda^{-\frac{3}{2} k(1-a)}), & |a| = 1 \\
0(\lambda^{-\frac{5}{2} k(1-a)}) + O(\lambda^{-\frac{5}{2} k(1-a)-\frac{3}{2} |a|}) & |a| \geq 2
\end{cases}
\]

for \(0 \leq k \leq 4\). Finally, \(D_y^a (\sigma e^{i \lambda \gamma}) = (\lambda^{\frac{|a|}{2}})\) and therefore, for \(|t| \leq 1\),

\[
(3.16) \quad (v \cdot w)^k K_1(t,x,\lambda) = -i \lambda \phi(t) (v \cdot w)^{k+1} \sigma_{E_1(v,w)} (t,x,\lambda d_z \psi) + O(\lambda^{-\frac{3}{2} k}),
\]

\(k \leq 4\).

Now using (2.20) and proposition 2.7b) we see that

\[
(3.17) \quad \sigma_{E_1} = \frac{\sigma_a}{i(\rho \gamma (|z|^{-1} + \phi \gamma))} \circ \rho^{-1} + m
\]
where \( m \in \mathbb{N}^{\frac{1}{2}, -\frac{5}{2}} \). By (3.12),

\[
(3.18) \quad \lambda (\gamma \cdot \omega)^{k+1} m(x, t, d_\psi) = O(\lambda^{-\frac{k}{2}}), \quad k \leq 4.
\]

Next, recalling (1.26) which essentially states that off of ordinary geometric optics is valid, we see that

\[
(3.19) \quad (\rho \gamma |z|^{-1})^{\frac{1}{2}+\eta} \varphi_2^{-1}(x, t, \lambda d_\psi) = -\lambda |\gamma \cdot m| \quad \text{on}
\]

\[
\{(x, t) : d_x, t^\psi \in \mathcal{G}\}.
\]

But both sides are continuous and therefore, (3.19) must hold across \( \mathcal{G} \). It now follows from (3.16)-(3.19) that for \( k \leq 4 \),

\[
(3.20) \quad |\gamma \cdot \omega| (\gamma \cdot \omega)_{\gamma} = \rho (\gamma \cdot m)_{\gamma}^{k+1}(\varphi_\omega \varphi_2^{-1}(x, t, \lambda d_\psi)) + |\gamma \cdot m| O(\lambda^{-\frac{k}{4}}).
\]

Finally, to eliminate the factor \( \varphi_\omega \varphi_2^{-1} \) in (3.20) we observe that since \( S_1 \) and \( S_2 \) have precisely second order contact along \( S_1 \cap S_2 \subset \{\gamma \cdot \omega = 0\} \), \( \varphi_\omega \varphi_2^{-1}(x, t, \lambda d_\psi) = 1 \) on

\[
|\gamma \cdot m| \geq c_\lambda \frac{a-1}{2}.
\]

(See the argument in proposition 2.8.) The conclusion is immediate:
\[
\begin{align*}
|v\cdot w| (v\cdot w) K_1(x, \lambda) &= (v\cdot w)^{k+1} |v\cdot w| (O(\lambda^{-\frac{k}{2}k(1-a)}) + O(\lambda^{-\frac{k}{4}})) \\
&= (v\cdot w)^{k+1} |v\cdot w| O(\lambda^{-\frac{k}{2}k + \epsilon}), \quad k \leq 4.
\end{align*}
\]

Establishing theorem 1 is now straightforward. By (3.16), \( K_1 = -i\lambda \rho (v\cdot w) \sigma_\nu (x, t, \lambda d\eta) + O(1) = O(1) \) since (3.12) can be applied to \( \sigma_\nu \in \eta_{a,1-a}^{-\frac{1}{2},-\frac{1}{2}} (S_1) \). In view of (3.11) this establishes
\[
(3.22) \quad K(x, \lambda) = K_1(x, \lambda) + K_2(x, \lambda) = O(\lambda^{\frac{1}{6} + \epsilon}).
\]

By the second part of (3.11), \( (\lambda \cdot v \cdot w)^{k} \frac{1}{6} v \cdot w |K_2 = O(\lambda^{\epsilon}) \), \( 0 \leq k \leq 5 \), and therefore,
\[
|v\cdot w| K_2(x, \lambda) \leq c_\lambda^{-\frac{1}{6}k+\epsilon} \frac{1}{6} (1+\lambda |v\cdot w|)^{-5}.
\]

Next, since \( O(\lambda^{-\frac{1}{6}k+\epsilon}) = O(\lambda^{-\frac{1}{6}k+\epsilon}), \quad 0 \leq k \leq 6 \), (3.21) implies that
\[
\frac{1}{6} |v\cdot w| (\lambda \cdot v \cdot w)^{k} |v\cdot w| K_2 - v\cdot w = |v\cdot w| O(\lambda^{\epsilon}), \quad 0 \leq k \leq 5,
\]
and hence
\[
(3.23) \quad |\cdot v\cdot w| K_1 - v\cdot w| \leq c_\lambda^{-\frac{1}{6} + \epsilon} \frac{1}{6} (1+\lambda |v\cdot w|)^{-5}.
\]

Since \( K = K_1 + K_2 \) this completes the proof of theorem 1.
REFERENCES


