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Non-splitting of H-Space Sequences

by

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I. Introduction

In the category of connected, simply connected compact Lie groups, any short exact sequence

$$1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$$

splits; that is, there exists a homomorphism $\gamma: C \rightarrow B$ such that $\beta \circ \gamma$ is the identity on C .

There is a program in modern algebraic topology, generally considered to have begun with the work of H. Hopf, which attempts to understand Lie groups from the homotopy point of view. Certain homotopy categories are defined which share a number of formal properties, such as cohomology structure, with Lie groups. H-Spaces are a category of this kind. Another is the category of FD groups, characterized by Rector [7], consisting of homotopy types whose spaces of loops are homotopy equivalent to finite CW complexes. Notice that the classifying spaces of compact Lie groups are examples of this category.

The main application of the results of this paper is to construct a short exact sequence of FD groups which does not split. In particular, we have a sequence

$$* \rightarrow S^3 \rightarrow S^3 \times S^3 \rightarrow S^3 \rightarrow *$$

in which $S^3 \times S^3$ is of the homotopy type of the loop space of an FD group other than the classifying space of

its Lie group, $B_{S^3} \times B_{S^3}$.

The Lie structure of S^3 , the multiplication of quaternions of unit length, is classified by \mathbf{HP}^∞ , infinite dimensional quaternionic projective space. Let $\mathbf{HP}^\infty(p)$ be the localization (in the sense of Sullivan [10]) of this space at the natural prime p . The principal result of this paper will be a characterization of possible degrees of maps

$$\mathbf{HP}^\infty(p) \rightarrow \mathbf{HP}^\infty(p).$$

It will be shown that the degree of such a map is a rational number with denominator prime to p which has a square root in the ring of p -adic integers.

In Section II, we will state Theorem 1, which explicitly describes the action of a self-map of \mathbf{HP}^∞ on complex K -theory. Theorem 2 gives possible degrees of such maps. The bound on allowable self-maps is then used to construct a sequence of FD groups which does not split. The construction utilizes the technique of homotopy mixing, due originally to Zabrodsky [11].

Section III contains some notes on the interpretation and application of these results.

In Section IV, a proof of Theorem 1 is given. Complex K -theory and the theory of ordinary differential equations are used.

In Section V, a proof of Theorem 2 is given, using

only arithmetic and elementary number theory in addition to Theorem 1.

II. Principal Results

The classifying space of the Lie group S^3 is the direct limit of the inclusions

$$\mathbb{H}P^1 \subset \mathbb{H}P^2 \subset \mathbb{H}P^3 \subset \dots \subset \mathbb{H}P^n \subset \dots$$

of quaternionic projective spaces, usually denoted $\mathbb{H}P^\infty$. We will consistently denote it by B (short for BS^3). As a CW complex, it has one cell in each dimension $4n$, $n = 1, 2, 3, \dots$.

Complex K -theory is a $\mathbb{Z}/2\mathbb{Z}$ -graded cohomology theory defined on CW complexes. It has a descending filtration induced by the inclusion of skeleta. Associated to the theory is the Atiyah-Hirzebruch spectral sequence, whose E_∞ -term is related to the K -theory of a space and whose E_1 -term is related to the ordinary cohomology of the space. The complex K -theory functor will be denoted K^* . If f is a map of spaces, then K^*f will be denoted $f^!$.

We have a path fibration

$$\begin{array}{ccccc} \Omega B & \rightarrow & PB & \rightarrow & B \\ \downarrow & & \downarrow & & \\ S^3 & & * & & \end{array}$$

where P is the path space functor and Ω is the loop space functor. $H^*(S^3) = \wedge(\mathbb{Z})$, the exterior algebra on one generator of dimension three with integral coefficients.

Applying the Serre Spectral Sequence, we get $H^*(B) = \mathbb{Z}[y]$, a polynomial ring on one generator where the dimension of y is four. The Atiyah-Hirzebruch Spectral Sequence then gives

$$K^*(B) = \mathbb{Z}[[x]] ,$$

a power series ring on one generator in which x has filtration degree four.

The "degree" of a map $B \xrightarrow{f} B$ is the integer a such that $f^*(y) = ay$ where y is defined above.

Theorem 1: There exists a generator x of $K^*(B)$ such that if $f: B \rightarrow B$ has degree $a \in \mathbb{Z}$ then

$$f^!(x) = \sum_{n=1}^{\infty} \left[\frac{2}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) \right] x^n .$$

This gives us an easy proof of the well-known but non-trivial

Corollary 1: (Berstein [4]) The degree of a map $f: B \rightarrow B$ is a square-integer.

Proof: By the "ratio test" from freshman calculus

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\frac{2}{(2n+2)!} \prod_{i=0}^n (a-i^2) / \frac{2}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{a-n^2}{(2n+1)(2n+2)} \right] \\ = -\frac{1}{4} . \end{aligned}$$

Therefore the power series for $f^!(x)$ "converges" in the

real number field at $x = 1$, and in particular, the limit of the coefficients of the power series is zero. But by definition these coefficients are integers

($f'(x) \in \mathbb{Z}[[x]]$), so the coefficients must be zero for all $n \geq N$, for some N fixed. Therefore $a = m^2$ for some integer m . Q.E.D.

In order to construct nonstandard examples of finite dimensional topological groups, we apply the technique of homotopy mixing due to Zabrodsky. This approach involves "fracturing" a homotopy type into mod- p components by localizing the space at each prime p , then reassembling the space with a "twist". For this we need

Corollary 2: B_p stands for B localized at a prime p . There exists a generator x of $K^*(B_p)$ such that if $f: B_p \rightarrow B_p$ is any map of degree $a \in \mathbb{Z}_{(p)}$ then

$$f'(x) = \sum_{n=1}^{\infty} \left[\frac{2}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) \right] x^n .$$

Since $K^*(B_p) = \mathbb{Z}_{(p)}[[x]]$, each coefficient of this power series must be an element of $\mathbb{Z}_{(p)}$.

Proof: The proof of Theorem 1 and the process of localization are natural.

Theorem 2: The degree of a map $f: B_p \rightarrow B_p$ is a square in the field of p -adic numbers.

The p -adic integers $\hat{\mathbb{Z}}_p$ are the $\lim_{n \in \mathbb{Z}} \{ \mathbb{Z}/p^n \mathbb{Z} \}$.

$\frac{r}{s} \in \mathbb{Z}_{(p)}$ can be projected into each $\mathbb{Z}/p^n\mathbb{Z}$ since $p \nmid s$, so $\mathbb{Z}_{(p)}$ has a canonical inclusion in $\hat{\mathbb{Z}}_p$, and Theorem 2 makes sense. If $p \nmid a$, the result is true by a well-known argument with Steenrod reduced powers. The proof given in Section V does not need this restriction.

The concept of a short exact sequence in the category of FD groups corresponds to a "quasifibration" of classifying spaces; i.e., a sequence of spaces and maps to which a long exact sequence of homotopy groups may be associated. To construct nonsplitting examples, we will require the following formulation of Zabrodsky mixing, due to Hilton, Mislin, and Roitberg [6]:

Given a space X of finite type (i.e., its homotopy groups are finitely generated), then X represents the functor

$$W \rightarrow \text{pullback}\{[W, X_p] \rightarrow [W, X_0] / p \text{ prime}\}$$

from finite CW complexes to pointed sets. That this functor is representable follows from a variant of the Brown Representability Theorem due to Adams [1]. The maps on the right-hand side are induced by the localizations of X . A mix of X is the type that represents the corresponding functor where the maps are induced by homotopy localizations other than the canonical ones.

Theorem 3: There exists a mix of the principal fibration $B \times B \rightarrow B$ which is a quasifibration $X \rightarrow B$

which has no section.

Proof: We must find a set of diagrams

$$\left\{ \begin{array}{c} \begin{array}{ccccc} & & S_p & & \\ & \swarrow & \pi_2 & \searrow & \\ B_p & \times & B_p & & B_p \\ & \downarrow & & & \downarrow \iota \\ & & & & B_0 \\ & \swarrow & S_0 & \searrow & \\ B_0 & \times & B_0 & & B_0 \\ & \downarrow & \pi_2 & & \\ & & & & \end{array} \\ \\ \begin{array}{l} p \text{ a prime} \\ \\ T_p \text{ a homotopy} \\ \text{localization} \end{array} \end{array} \right\}$$

(π_2 = projection on the second factor , ι = the canonical localization), which has no compatible set of sections $\{S_0, S_p \mid p \text{ a prime}\}$ of the horizontal maps π_2 . Note that $B_0 = K(Q,4)$, so we have a free choice of maps $S_0 \in Q^2$ and $T_p \in M^2(\mathbb{Z}(p))$.

Let T_p be represented by the action $\begin{pmatrix} 1 & c_p \\ 0 & 1 \end{pmatrix}$ on generators of four dimensional homology, $c_p \in \mathbb{Z}(p)$. S_p must be represented by $\begin{pmatrix} a_p \\ b_p \end{pmatrix}$ where $a_p, b_p \in \mathbb{Z}(p)$ are p - adic squares by Theorem 2 and $S_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in Q$. By definition of splitting, $\pi_2 \circ S_p = b_p$ is a homotopy equivalence, so b_p is a unit in $\mathbb{Z}(p)$.

Because the diagram above commutes,

$$\begin{pmatrix} a \\ b \end{pmatrix} = S_0 \circ \iota = T_p \circ S_p = \begin{pmatrix} 1 & c_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_p \\ b_p \end{pmatrix} = \begin{pmatrix} a_p + c_p b_p \\ b_p \end{pmatrix}$$

so $b = b_p$ is a p - adic square unit for all primes p ,
so $b = 1$.

We have $a_p + c_p = a$ for all p . Since $a \in \mathbb{Z}_{(p)}$ for all p , a is an integer. In order for a section to exist, we must find a set $\{a, a_p \mid a \in \mathbb{Z}, a_p \text{ a } p\text{-adic square}\}$ such that the equation is true for each p . We here use two important facts from number theory.

(1) If p is a prime of the form $4K + 3$, then -1 is not a square in $\hat{\mathbb{Z}}_p$.

(2) (Dirichlet) Every arithmetic sequence $\{nK + m\}$ where n and m are coprime has an infinite number of primes in it. (In particular, an infinity of primes are of the form $4K + 3$.)

We now make a particular choice of $\{c_p\}$ in the construction of the mix. Let p_1, p_2, \dots be an infinite list of primes of the form $4K + 3$. Set

$$c_{p_1} = 0, \quad c_{p_2} = 1, \quad c_{p_3} = -1, \quad c_{p_4} = 2, \quad c_{p_5} = -2,$$

and so forth. Thus, any integer value is some c_{p_k} . For other primes not in the list, choose c_p in any manner whatever.

For any possible choice of $a \in \mathbb{Z}$, $c_p - 1 = a$ for some prime p in our list, so $a_p = a - c_p = -1$. But -1 is not a p -adic square, and a_p must be by Theorem 2, which is a contradiction.

Therefore we have constructed a homotopy mix

$$X \rightarrow B$$

which does not have a section. By construction

$$X_p \simeq B_p \times B_p$$

for every p prime, so

$$\Omega X \simeq S^3 \times S^3 .$$

Thus, we have found an FD structure on

$$* \rightarrow S^3 \rightarrow S^3 \times S^3 \rightarrow S^3 \rightarrow *$$

which does not have a section.

Q.E.D.

III. Notes

1. Theorem 3 is a somewhat discouraging result for the program to understand Lie Groups from the homotopy point of view. The category of FD groups is shown to lack a very desirable combinatorial property possessed by Lie Groups. It would be nice to find some further homotopy condition that would guarantee split extensions of the FD groups meeting it.

2. Rector [8] has noted that at most two of the enormous number of FD structures on S^3 which are obtained by mixing $\mathbb{H}\mathbb{P}^\infty$ have "maximal tori" in the FD sense. The proof of Theorem 1 has to do with the canonical torus of S^3 in a crucial way. Perhaps the extra homotopy condition mentioned above should have to do with the existence of a homotopy torus for FD structures.

3. Theorem 2 seems to correspond to an argument using Atiyah integrality and Adams operations in complex K - theory. In fact, the result of Theorem 2 can be obtained in this manner for low p - divisibilities of the map-degree, although the argument rapidly increases in complexity as the power of p increases. A complete proof of Theorem 2 using these methods would be very nice, because they do not depend on the particular generator of $\mathbb{H}\mathbb{P}^\infty$ chosen; and so the result could be generalized to other FD structures. One observation that points in this

direction is that the proof of Theorem 2 does not depend on our choice of generator of $K^*(\mathbb{H}\mathbb{P}^\infty)$. In fact, it is easy to see that the coefficient of lowest degree in the image of a generator under $f^!$ which fails to be integral is always of the same degree.

4. Conjecture: Maps $B_p \times B_p \rightarrow B_p \times B_p$ are always diagonal or antidiagonal up to homotopy.

A proof of this would lead to a partial classification of FD extensions $S^3 \rightarrow S^3 \times S^3 \rightarrow S^3$. It may well be tractable using methods similar to those of this paper.

5. Sullivan [10] has constructed self-maps of B_p , p odd, of degree any p -adic square in \mathbb{Z}_p . This is a partial converse of Theorem 2.

6. Complex K-Theory has associated to it a set of ring homomorphisms ψ^n , $n \in \mathbb{Z}$, which are cohomology operations, called Adams operations. In the case of our complex B , ψ^n acts on $K^*(B)$ in the same manner as the self-map of degree n^2 . Thus ψ^n of our chosen generator x is a polynomial in x of degree n . By methods similar to the proof of Theorem 1, it can be shown that $\psi^n x$ is a Tchebysheff polynomial of the First Kind. Thus, for the fixed generator x ,

$$\mathbb{Z}[x] \hookrightarrow \mathbb{Z}[[x]] = K^*(B)$$

has an inner product

$$\langle z, w \rangle = \int z(x)w(x)d\phi$$

where $\phi(x)$ is the Tchebysheff weight function. The Adams operations are isometries of this inner product, i.e.,

$$\langle z, w \rangle = \langle \psi_z^n, \psi_w^n \rangle, \quad z, w \in \mathbb{Z}[x], \quad n \in \mathbb{Z}^+.$$

This suggests the existence of analytic tools for studying completions and localizations of B and related spaces. It is not clear whether a similar polynomial ring exists for mixes of B .

IV. Proof of Theorem 1

The classifying space of S^1 with the usual (complex) multiplication is $\mathbb{C}P^\infty$, infinite dimensional complex projective space. We will study the complex K-theory of the classification $\mathbb{C}P^\infty \xrightarrow{i} B$ of the canonical torus $S^1 \rightarrow S^3$.

Associated to complex K-theory is the Chern class, a natural map from $K^*(x)$ to $H^*(x, \mathbb{Z})$ which has the property that $c(x+y) = c(x)c(y)$ where multiplication is the cup product. The Chern character is a natural ring homomorphism of $K^*(x)$ into $H^{**}(x, \mathbb{Q})$, denoted ch .

Proposition: (Berstein [3]) There exists $\eta \in K^*(\mathbb{C}P^\infty)$ and $\xi \in K^*(B)$ such that the Chern class $c(\eta) = 1+t$ where $t \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ is a generator, $c(\xi) = 1-y$ where $y \in H^4(B, \mathbb{Z})$ is a generator, and $i^*(y) = t^2$.

By the naturality of the Chern class,

$$c(i^!(\xi)) = i^*(c(\xi)) = 1 - t^2 = (1-t)(1+t) .$$

By definition of the Chern character,

$$ch(i^!(\xi)) = e^t + e^{-t} = 2 + t^2 + \frac{2}{4!}t^4 + \dots ,$$

therefore

$$ch(\xi) = 2 + y + \frac{2}{4!}y^2 + \dots + \frac{2}{(2n)!}y^n + \dots .$$

The generator $K^*(B)$ in which we will be interested is $x = \xi - 2$, so

$$\text{ch}(x) = y + \frac{2}{4!} y^2 + \dots + \frac{2}{(2n)!} y^n + \dots .$$

Since B is without torsion, ch is 1-1 by Atiyah Integrality [5]. The set $\{\text{ch}(x^m)\} = \{(\text{ch}(x))^m\} = \{y^m + \dots\}$ is an additive basis for the image of ch in $H^{**}(B, Q)$, so the set $\{x^m\}$ is an additive basis for $K^*(B)$.

Let $f: B \rightarrow B$ have degree $a \in \mathbb{Z}$. Then

$$\text{ch}(f'(x)) = ay + \frac{2}{4!} a^2 y^2 + \dots + \frac{2}{(2n)!} a^n y^n + \dots .$$

If we can express this formula in terms of the basis $\{\text{ch}(x^m)\}$, the same coefficients will express $f'(x)$ as a power series in x .

Under a change of variables $y = t^2$, the problem is to express

$$w = at^2 + \frac{2}{4!} a^2 t^4 + \dots + \frac{2}{(2n)!} a^n t^{2n} + \dots$$

as a power series in

$$z = t^2 + \frac{2}{4!} t^4 + \dots .$$

These series lead to the differential equations

$$w'' = a(w+2)$$

$$(z')^2 = z(z+4)$$

$$z'' = z+2 .$$

Substituting z for t in the first equation and using the relation

$$\frac{d^2 w}{dt^2} = \frac{d^2 w}{dz^2} \left(\frac{dz}{dt}\right)^2 + \frac{dw}{dz} \frac{d^2 z}{dt^2}$$

we get

$$w''z(z+4) + w'(z+2) - aw = 2a \quad (1)$$

where w is now a function of z .

Lemma 1: The homogeneous equation

$$w''z(z+4) + w'(z+2) - aw = 0 \quad (2)$$

has a general solution given by linear combinations of

$$1 + \frac{a}{2}z + \frac{a(a-1)}{24}z^2 + \dots + \frac{z^n}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) + \dots \quad (3)$$

and

$$\begin{aligned} & \frac{1}{z^2} \left[1 + \frac{4a-1}{24}z + \frac{(4a-1)(4a-9)}{1920}z^2 + \dots \right. \\ & \left. + \frac{1}{4^n (2n+1)!} \prod_{i=0}^{n-1} (4a - (2i+1)^2) z^n + \dots \right] . \quad (4) \end{aligned}$$

A general solution of (1) is then given by the sum of a particular solution of (1) and a general solution of (2) [9]. By inspection, a particular solution of (1) is $w = -2$. Since we are only interested in power series solutions, we want an expression of the form $c \times (3) - 2$ where c is a real constant.

By inspection of the original problem, we have initial conditions $w(0) = 0$ and $w'(0) = a$. Therefore the solution we want is

$$w(z) = az + \frac{2a(a-1)}{4!}z^2 + \dots + \frac{2}{(2n)!} \prod_{i=0}^{n-1} (a-i^2)z^n + \dots$$

by setting $c = 2$. Thus

$$f'(x) = ax + \frac{2}{4!}x^2 a(a-1) + \dots + \frac{2x^n}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) + \dots$$

Q.E.D.

Proof of Lemma 1: Zero is a regular singular point of (2) so that by the Method of Frobenius [9] we seek solutions of the form

$$w = z^\alpha [a_0 + a_1 z + \dots + a_n z^n + \dots] \text{ where } a_0 \neq 0 .$$

Differentiating

$$w' = \alpha a_0 z^{\alpha-1} + (\alpha+1)a_1 z^\alpha + (\alpha+2)a_2 z^{\alpha+1} + \dots ,$$

$$w'' = \alpha(\alpha-1)a_0 z^{\alpha-2} + \alpha(\alpha+1)a_1 z^{\alpha-1} + (\alpha+1)(\alpha+2)a_2 z^\alpha + \dots .$$

So

$$w'(z+2) = 2\alpha a_0 z^{\alpha-1} + (\alpha a_0 + 2(\alpha+1)a_1)z^\alpha + \dots$$

$$w''z(z+4) = 4\alpha(\alpha-1)a_0 z^{\alpha-1} + (4\alpha(\alpha+1)a_1 + \alpha(\alpha-1)a_0)z^\alpha + \dots .$$

Substituting in (1), we get

$$0 = 2a_0(\alpha + 2\alpha(\alpha-1))z^{\alpha-1} + \dots .$$

So the indicial equation is $2\alpha^2 - \alpha = 0$ which has solutions $\alpha = 0$, $\alpha = \frac{1}{2}$.

Let $\alpha = 0$. Then let

$$w = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

Differentiating

$$w' = a_1 + 2a_2 z + \dots + a_n z^{n-1} + \dots$$

$$w'' = 2a_2 + 6a_3 z + \dots + n(n-1)a_n z^{n-2} + \dots$$

$$z(z+4)w'' = 8a_2 z + (24a_3 + 2a_2)z^2 + \dots + [n(n-1)a_n + 4(n+1)a_{n+1}]z^n + \dots$$

$$(z+2)w' = 2a_1 + (a_1 + 4a_2)z + \dots + [na_n + 2(n+1)a_{n+1}]z^n + \dots$$

$$-aw = -aa_0 - aa_1 z - aa_2 z^2 + \dots - aa_n z^n + \dots$$

Substituting in (1) we get

$$0 = (2a_1 - a \cdot a_0) + [8a_2 + (a_1 + 4a_2) - aa_1]z + \dots$$

$$+ [n(n-1)a_n + 4n(n+1)a_{n+1} + na_n + 2(n+1)a_{n+1} - aa_n]z^n + \dots$$

Setting each coefficient to zero

$$a_1 = a_0 \cdot \frac{a}{2}, \quad a_2 = \frac{1}{12} a_1 (a-1) = \frac{1}{24} a(a-1)a_0$$

$$a_{n+1} = \frac{1}{4n(n+1) + 2(n+1)} [a_n (a - (n(n-1) + n))] = \frac{1}{(2n+2)(2n+1)} [a_n (a - n^2)]$$

By induction on n

$$a_n = \left(\frac{1}{(2n)!} \prod_{i=0}^{n-1} (a - i^2) \right) \cdot a_0$$

Thus we have a set of solutions

$$a_0 \left(1 + \frac{a}{2}z + \frac{a(a-1)}{24}z^2 + \dots + \frac{z^n}{(2n)!} \prod_{i=0}^{n-1} (a - i^2) + \dots \right)$$

Let $\alpha = \frac{1}{2}$. Let

$$w = a_0 z^{\frac{1}{2}} + a_1 z^{\frac{2}{3}} + \dots$$

A similar recursive calculation gets a set of solutions

$$w = a_0 z^{\frac{1}{2}} \left(1 + \frac{4a-1}{24} z + \frac{(4a-1)(4a-9)}{1920} z^2 + \dots \right) + \frac{1}{4^n (2n+1)!} \prod_{i=0}^{n-1} (4a - (2i+1)^2) z^n + \dots \quad \text{Q.E.D.}$$

|

V. Proof of Theorem 2

As in the global case, our method will be to decide when the coefficients of

$$f'(x) = \sum_{n=1}^{\infty} \frac{2}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) x^n$$

cannot all be integral. Since the ring in which the power series lies is $\mathbb{Z}_{(p)}[[x]]$, "integral" means lying in $\mathbb{Z}_{(p)}$; i.e., a rational number with denominator prime to p .

p odd:

Lemma: $a \in \hat{\mathbb{Z}}_p$, the ring of p -adic integers, is a square if and only if $a = p^{2n}u$ where u is a unit whose reduction mod p is a square class in $\mathbb{Z}/p\mathbb{Z}$.

Proof: An immediate consequence of Hensel's lemma [2].

Lemma: Let $d_p(n)$ be the largest integer such that $p^{d_p(n)} \mid n!$, and let $n = \sum a_i p^i$, $0 \leq a_i \leq p-1$. Then $d_p(n) = \frac{1}{p-1}(n - \sum a_i)$.

Proof: A straightforward calculation.

Assume a is not a p -adic square. Then $a = p^m b$, b a unit, must have either m odd or b not congruent to a square class in $\mathbb{Z}/p\mathbb{Z}$.

It is sufficient for our purpose to establish that for

some n , $(2n)!$ is divisible by a larger power of p than $\prod_{i=0}^{n-1} (a-i^2)$ is; for then the corresponding coefficient of x^n would fail to be a p -adic integer.

First notice that the number of powers of p dividing $a - i^2$ can only exceed the number dividing i^2 when

- (1) a and i^2 have the same p -divisibility and
- (2) if $i^2 = c^2 p^{2k}$, c a unit in $\hat{\mathbb{Z}}(p)$, then $c^2 \equiv b \pmod{p}$.

But these are precisely the conditions we have assumed a does not meet. Adding together the powers of p dividing i^2 (at least as numerous as those dividing $a - i^2$) for each value of i , we get that the power of p dividing $\prod_{i=0}^{n-1} (a-i^2)$ is at most $m + 2d_p(n-1)$.

Let $n = p^r$, $r \in \mathbb{Z}^+$.

$$d_p(2p^r) = 2(p^{r-1} + p^{r-2} + \dots + 1) \quad \text{and}$$

$$d_p(p^r - 1) = (p^{r-1} + p^{r-2} + \dots + 1 - r)$$

by our formula.

Thus, $d_p(2p^r) > m + 2d_p(p^r - 1)$ whenever $r > \frac{m}{2}$.

Therefore, for r sufficiently large, $n = 2^r$ corresponds to a non-integral coefficient of the power series, and a is not the degree of any map

$$B_p \rightarrow B_p .$$

$p = 2$:

Lemma: $a \in \hat{\mathbb{Z}}_p$ is a square if and only if $a = 2^{2m}b$ where $b \in \hat{\mathbb{Z}}_p$ is a unit and $b \equiv 1 \pmod{8}$.

Proof: Consequence of Hensel's lemma [2].

As in the odd case, we will determine when the coefficients of

$$f'(x) = \sum_{n=1}^{\infty} \left[\frac{2}{(2n)!} \prod_{i=0}^{n-1} (a-i^2) \right] x^n$$

cannot all be two-integral. In that case, $\prod_{i=0}^{n-1} (a-i^2)$ would be divisible by fewer powers of two than $\frac{(2n)!}{2}$.

Case 1: $a = 2^{2r+1}b$, b odd.

Let $n = 2^{2r+1}$, then $\frac{(2n)!}{2}$ is divisible by $2^{2r+2} - 2$ factors of 2.

Since a and i^2 never have the same 2-divisibility, then the divisibility of i^2 must be at least that of $a - i^2$ for $i \neq 0$. Thus $\prod_{i=0}^{2^{2r+1}-1} (a-i^2)$ has at most $(2r+1) + 2(2^{2r+1} - 1 - (2r+1))$ factors of 2.

But $2^{2r+2} - 2 > 2^{2r+2} - 2 - (2r+1)$ for $r \geq 0$.

Thus, the r^{th} coefficient is not 2-integral, and a is not the degree of a map.

Case 2: $a = 2^{2m}b$, $b \equiv 3, 5$ or $7 \pmod{8}$.

Let $n = 2^r$, $r > m + 1$. $\frac{(2n)!}{2}$ has $2^{r+1} - 2$ factors of 2. In order to find a bound for factors of 2 in

$\prod_{i=0}^{n-1} (a-i^2)$, we will sum over the contributions of terms corresponding to various values of i .

(i) $i = 0$: a has $2m$ factors of 2 .

(2) i even, but i^2 is divisible by fewer powers of 2 than is a : $i = 2^j c$, where $2j < 2m$

$i = 2 \times \{1, 3, 5, \dots, 2^{r-1} - 1\}$ gives 2^{r-2} powers of 2 .

$i = 4 \times \{1, 3, 5, \dots, 2^{r-2} - 1\}$ contributes $2 \cdot 2^{r-3}$ powers of 2 .

$i = 2^j \times \{1, 3, \dots, 2^{r-j} - 1\}$ contributes $j \cdot 2^{r-j-1}$ powers of 2 .

$i = 2^{m-1} \times \{1, \dots, 2^{r-m+1} - 1\}$ contributes $(m-1)2^{r-m}$ powers of 2 .

The sum of this list is $\sum_{j=1}^{m-1} j \cdot 2^{r-j-1}$ which can be rewritten as

$$\begin{aligned}
 & 2^{r-2} \\
 & + 2^{r-3} + 2^{r-3} \\
 & + 2^{r-4} + 2^{r-4} + 2^{r-4} \\
 & + \dots \\
 & + \underline{2^{r-m} + 2^{r-m} + \dots + 2^{r-m}} \\
 & = (2^{r-1} - 2^{r-m}) + (2^{r-2} - 2^{r-m}) + \dots + (2^{r-m+1} - 2^{r-m}) \\
 & = 2^r - 2^{r-m+1} + (m-1)2^{r-m}.
 \end{aligned}$$

These are the 2-divisibilities of i , so the terms $(a-i^2)$ in this case contribute at most

$2[2^r - 2^{r-m+1} - (m-1)2^{r-m}]$ powers of 2 .

(3) $i = 2^m c$, c odd: In this case
 $a - i^2 = 2^{2m}((1+2d)+(1+8e))$ or $2^{2m}((1+4d)-(1+8e))$ so
each is divisible by at most $2m + 2$ powers of 2 . Our
cases are $i = 2^m \times \{1, 3, 5, \dots, 2^{r-m} - 1\}$ which are 2^{r-m-1}
in number. Thus these values of i contribute at most

$(2m+2)2^{r-m-1}$ powers of 2 .

(4) $i = 2^j c$, $j > m$: In this case the divisibility
of $a - i^2$ is determined by that of a , which is $2m$.

Counting cases

$i = 2^{m+1} \times \{1, 3, 5, \dots, 2^{r-m-1} - 1\}$ gives
 2^{r-m-2} cases;

$i = 2^{m+2} \times \{1, 3, \dots, 2^{r-m-2} - 1\}$ gives 2^{r-m-3}
cases;

$i = 2^{r-1} \times \{1\}$ gives 1 case;

so these values of i contribute $2m(2^{r-m-1} - 1)$ factors
of 2 .

Summing over values of i , $\prod_{i=0}^{n-1} (a-i^2)$ has at most

$$\begin{aligned} & 2m+2(2^r - 2^{r-m-1} - (m-1)2^{r-m}) + (2m+2)2^{r-m-1} + 2m(2^{r-m-1} - 1) \\ &= 2^{r+1} - 2^{r-m+2} + 2^{r-m+1} + 2^{r-m} \\ &= 2^{r+1} - 2^{r-m} \text{ powers of 2 .} \end{aligned}$$

Now $2^{r+1} - 2 > 2^{r+1} - 2^{r-m}$ whenever $r > m + 1$ so the
 n^{th} coefficient in our power series is not 2-integral, and

a is not the degree of a map.

Case 3: a odd but not congruent to $1 \pmod{8}$.

$\frac{2}{8} (a(a-1)(a-4)(a-9))$ is not a 2 -adic integer
which completes the case $p = 2$, and therefore the proof
of Theorem 2. Q.E.D.

VI. Bibliography

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