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COULOMB-NUCLEAR INTERFERENCE IN PION-PROTON
PION-NUCLEUS SCATTERING IN THE (3, 3) RESONANCE
REGION USING THE GLAUBER FORMALISM

by

Bun-Woo Bertram Chang

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's Signature:

[Signature]

Houston, Texas

July, 1976
To Judy
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INTRODUCTION

In the usual analyses of charged-pion-proton and charged-pion-nucleus scattering data at high energies, electromagnetic contributions are accounted for by a simple subtraction of the Coulomb scattering cross-section. \(^{(1)}\) But as the energy is lowered, the importance of Coulomb contribution increases. Such effects are most pronounced at small scattering angles, because the Coulomb cross-section is strongly forward peaked.

The total scattering amplitude is not merely the sum of the pure Coulomb amplitude \(f_c\) and the pure nuclear amplitude \(f_N\), but rather

\[
d_{\text{int}} = f_c + f_{\text{CN}},
\]

where \(f_{\text{CN}}\) contains still Coulomb contributions in addition to the nucleus ones. The observed scattering cross-section is

\[
\frac{d\sigma}{d\Omega} = |f_c|^2 + |f_{\text{CN}}|^2 + 2 \Re (f_c \cdot f_{\text{CN}}).
\]

Since the Coulomb scattering amplitude is known, the residual amplitude \(f_{\text{CN}}\) (which is sometimes called ) can in principle be determined from the observed data and a knowledge of the experimental conditions. The difficulty is to obtain the pure nuclear amplitude \(f_N\) from the residual amplitude. In terms of partial wave phase shifts, we note that the total phase shift \(\delta_\lambda\) is not just the sum of the pure Coulomb and the pure nuclear phase shifts, \(\delta^c_\lambda\) and \(\delta^N_\lambda\) respectively, but instead there is an extra term \(\Delta_\lambda\) due to interference effects, i.e.
\[
\delta_e = \delta_e^c + \delta_e^N + \Delta_e.
\]  

(3)

The pure nuclear amplitude can be obtained from the residual amplitude in the limit

\[
\hat{f}^N = \lim_{\delta_e^c \to 0} \hat{f}^{CN}.
\]  

(4)

where \(\delta_e^c\) goes to zero when the Coulomb parameter \(n\) goes to zero.

Besides the long range effects of the Coulomb force, another difficulty in solving the Coulomb-nuclear interference problem is the uncertainty of the Coulomb potential within the nuclear interaction region. Therefore, model calculations are necessary to account for Coulomb interaction in this region.\(^{(3-5)}\) On the other hand, this uncertainty of Coulomb potential in the nuclear region means that a complicated nuclear potential will not necessarily give more precise results than a simple one. Therefore, in this thesis, the nuclear potentials chosen for pion-proton scattering are the simple square well potential and the Yukawa potential.

Although the problem of Coulomb-nuclear interference in charged-particle scattering is a difficult one, it can offer some important application. From the optical theorem and total scattering cross-section data, we can deduce the sign and magnitude of the imaginary part of the scattering amplitude for small scattering angles. From Coulomb-nuclear interference, the sign and magnitude of the real part of the nuclear scattering amplitude can be obtained.
The real part of the nuclear amplitude has important physical significance such as (1) offering a check on the forward dispersion relations which relate the real part of the scattering amplitude to the imaginary part, (2) revealing the possible behavior of the total cross-section in the asymptotic region, and (3) serving as a test for various theoretical models. Other applications that Coulomb-nuclear interference studies offer are: the determination of mass splitting of the $(3, 3)$ resonance particles as basis for testing the SU(3) mass relations, and possible test for charge independence.

In this thesis, we are interested in Coulomb-nuclear interference in the energy region of the $(3, 3)$ resonance. The Glauber formalism has been surprisingly successful in that resonance region, in spite of its being a high energy approximation (see section 2, Chapter I). And since it is also supposed to give good results at small scattering angles where Coulomb contribution is significant, we have chosen to use the Glauber formalism in our investigations of Coulomb-nuclear interference.

The processes we are interested in are charged-pion-proton scattering and charged-pion-nucleus scattering; for the latter, we are concerned only with light nuclei with $4 \leq A \leq 16$. Being a multiple scattering theory, the Glauber formalism is particularly adapted to our calculations.

The charged-pion-proton scattering processes involved are the single channel process $\pi^+ p \rightarrow \pi^+ p$ and the couple-channel pro-
cesses $\pi^- + p \rightarrow \pi^- + p$ and $\pi^- + p \rightarrow \pi^0 + n$. (Respectively, elastic and charge-exchange scattering). As was mentioned before, it is reasonable to choose simple nuclear potentials and we have used attractive square well and Yukawa potentials in our calculations. We superimpose on each of these potential the screened Coulomb potential to account for the electromagnetic interaction between charges.

For charged-pion-nucleus scattering, we have employed the independent particle model for the ground state of the nucleus and also use a density corresponding to the harmonic oscillator potential\(^{(25)}\) for the independent particle densities. For nuclei with $4 \leq A \leq 16$, there are 4 s-shell and (A-4) p-shell nucleons. The nuclear phase-shift function is expressed in terms of individual pion-nucleon scattering amplitudes through a Bessel inversion of the individual particle profile function. Assuming that the charge density for an individual proton is the same as its nuclear densities, the Coulomb potential between the pion, whose charge density is taken to be Gaussian, and the nucleus is calculated. In our equations, we have fully considered all Coulomb contributions including the effects of the charge form factors of the pion and the nucleus, the latter through the form factors of the individual nucleons.

Chapter I gives a review of what has been done previously in Coulomb nuclear interference. Three approaches using potential phase-shift analysis, Bethe's relative phase formulation and dispersion theoretical calculations are discussed. A review of the Glauber formalism is then
followed by a description of Franco's calculations within the Glauber framework of proton-deuteron scattering with the inclusion of Coulomb potentials.

Chapter II discussed how scattering cross-section data for charged particles are obtained and treated in the presence of multiple Coulomb scattering and Coulomb-nuclear interference. The analysis of total scattering data depends on a fraction \( D(\Omega) \) which is a ratio of the number of beam particles that hit the counter to that of the incident beam. An expression for \( D(\Omega) \) is obtained in Appendix A.

In Chapter III, we calculate the scattering amplitude and cross-sections for the Coulomb, square well and Yukawa potentials, individually at first. Then expressions are obtained for the scattering amplitude and cross-sections for each of the nuclear potentials when the Coulomb potential is included. Non-spin-flip and spin-flip amplitudes are discussed at small scattering angle. Isospin considerations are investigated and expressions obtained for scattering amplitudes for elastic \( \pi^- + p \rightarrow \pi^+ + p \) scattering and the charge-exchange process \( \pi^- + p \rightarrow \pi^o + n \).

In Chapter IV, we obtain expressions for pion-nucleus scattering amplitudes with Coulomb effects included. The Glauber multiple scattering formalism is discussed and used in the calculation. Individual nucleon particle nuclear densities in the nucleus are chosen to correspond to harmonic oscillator potentials. We take into account all charge distributions for the pion and proton in the nucleus. The charge form factor for the pion is Gaussian, while that of protons is assumed
equal to their nuclear distribution. Some of the expressions involved in this chapter are derive separately in Appendices C and D.

Some of the expressions obtained are employed in numerical calculations in Chapter V. This is done by first determining the parameters involved. A discussion of why the factor \( \frac{l}{\hbar \nu} \) in Glauber equations has to be changed to \( \frac{m}{\hbar^2 \lambda} \) follows. We then discuss the convergence of the infinite series involved in some of the results obtained in Chapter IV. Then additional equations which are required for numerical calculation on the computer are given. Finally, we presented numerical results with graphs and discussions.
References


Chapter I

REVIEW

1) **Coulomb-Nuclear Interference**

In attempting to properly account for Coulomb effects in charged-pion-proton and charged-pion-nucleus scattering, many approaches have been taken. Some of the more popular approaches are (1) potential model calculations \(^{1-13, 15, 16}\) in which phase-shift analysis is applied, (2) Bethe-phase formulations \(^{18-29, 31-34}\) where the Coulomb-nuclear phase difference is calculated, and (3) dispersion theoretical calculations \(^{14, 17, 35-37}\).

In this thesis, yet another approach is used. We calculate the effects of Coulomb-nuclear interference in pion-proton and pion-nucleus scattering using the Glauber formalism. The Glauber formalism is chosen because it is supposed to give good results at small scattering angles where Coulomb contributions are significant. Also Glauber calculations have been found to be very successful in the region we are interested in, i.e. the \((3, 3)\) resonance region. Here, we review what has been done in Coulomb-nuclear interference calculations using other methods such as listed above, and leave the discussion on the Glauber formalism approach till after its derivation is reviewed in section 2. We begin with the potential model phase-shift calculation.\(^{(1)}\)

Van Hove was the first to show how Coulomb effects separated from nuclear effects in \(\pi^+p\) scattering at non-relativistic
energies. We note that $\pi^+p$ is a single-channel scattering process whereas $\pi^-p$ is a coupled-channel scattering process when the radiative channel ($\gamma N$) is ignored. The two coupled channels are ($\pi^-p$) and ($\pi^0N$) which corresponds to elastic scattering and charge-exchange scattering respectively. The procedure taken by Van Hove is as follows: first the pure Coulomb scattering amplitude is removed from the observed scattering amplitude, whereby the very strong forward peak of the Coulomb interaction is eliminated. Then, from the resulting scattering amplitude the phase-shifts $\gamma$ are deduced using the s- and p-wave radial Schroedinger equations. These phase-shifts obviously still contain Coulomb effects. To obtain 'strictly' nuclear phase-shifts, the outer-Coulomb correction is applied. In the outer-Coulomb correction process, it is assumed that within the nuclear interaction radius $r_N$, there is only nuclear interactions, and that outside $r_N$ there is only Coulomb interaction. Then, the logarithmic derivative of the interior wave function determined by the nuclear potential is matched at $r_N$ with the logarithmic derivative of the exterior wave function determined by the Coulomb potential, which is chosen to correspond to point charges. By doing so $\gamma^N$ is obtained corresponding to the nuclear potential only. The outer-Coulomb correction is therefore equal to $(\gamma^N - \gamma)$. It was demonstrated that Van Hove's result depended appreciably on the nuclear interaction radius $r_N$.\(^{(2)}\)

From arguments using the Klein-Gordon equation, Van Hove also
attempted to generalize his result to relativistic energies by modifying the Coulomb amplitude and suggested that the Coulomb parameter be divided by \( (1 - \beta^2)^{\frac{1}{2}} \), where \( \beta c \) is the relative velocity of pion and proton. Solmitz\(^{(3)}\) pointed out that such a procedure has problems for large scattering angles, and presented another approach to generalize the Coulomb amplitude into relativistic region. He used an electromagnetic interaction Hamiltonian which included the one-photon exchange contribution to charged-pion-proton scattering and also the anomalous magnetic moment of the proton to obtain the Coulomb scattering amplitudes to first order of the fine structure constant \( \frac{e^2}{\hbar c} \). The spin-flip Coulomb amplitude turns out to be non-zero in this case.

The first order relativistic correction due to Solmitz was used by Foote et al.\(^{(4)}\) in a procedure originated by Stapp et al.\(^{(5)}\) to calculate non-spin-flip and spin-flip pion-proton elastic scattering amplitudes, but they ignored Coulomb-nuclear interference effects. Their method was later employed by Roper et al.\(^{(6)}\) with modifications to account for electromagnetic effects in pion-proton scattering. One major modification is in the relativistic Coulomb amplitudes which are the full one-photon exchange amplitudes with the inclusion of the Pauli term contribution for the anomalous magnetic moment of the proton. No additional approximations were employed. Although Roper et al. also neglected Coulomb-nuclear interference, their method has been used by most subsequent phase-shift analyses on pion-proton scattering data to account for electromagnetic effect. This was sufficient until the
high accuracy in experimental data obtained by Bugg et al.\(^{(7)}\)

In trying to remove the dependence and Woolcock\(^{(8)}\) suggested the necessity for the inner Coulomb correction, which comes about because inside \(\gamma_{p}\), there is still electromagnetic interaction. In fact, due to the finite spatial distribution of charges, the Coulomb potential increases to a finite value at \(r=0\) from its value at \(\gamma_{p}\). The inner Coulomb corrected was estimated by Hamilton and Woolcock using a weak-scattering approximation. They found that when the inner and outer Coulomb corrections are combined, the result is much less dependent on \(\gamma_{p}\).

Schnitzer\(^{(9)}\) presented an extension of Hamilton and Woolcock's treatment of inner Coulomb corrections but did not treat coupled-channel effects. The Coulomb potential be used was that between a point charge and a uniformly charged sphere. The nuclear potentials used were a hard-sphere potential for the s-wave and a square well potential for the p-wave. An equivalent formulation of Coulomb corrections that also applied to single-channel processes only was obtained by Auvil.\(^{(10)}\)

Later, Auvil generalized his results to coupled-channel processes.\(^{(11)}\) We should note here that in the coupled channels (\(\pi^{-}\rho\)) and (\(\pi^{-}\pi\))

\(\pi^{-}\rho\)) and (\(\pi^{-}\pi\)), there are mass differences between \(\pi^{-}\) and \(\rho\), and between \(\pi^{-}\) and \(\pi^{0}\) respectively. While Auvil did not explicitly treat these mass differences, he argued that their effects will be negligible. Oades and Rasche,\(^{(12)}\) on the other hand, did not consider the mass differences in presenting their treatment for the correction of Coulomb effects in coupled-channel
processes. But in a subsequent paper, they formulated the first complete treatment of the effect of mass differences, where they used the reduced masses of the coupled \((\pi^-p)\) and \((\pi^0n)\) channels. 

Bugg et al. \(^{(7)}\) found from their pion-proton experimental cross-section data before electromagnetic corrections that the positions and width of the \(N^{*++}\) and \(N^{*0}\) resonances are all different. Their results are

\[
M(N^{*0}) - M(N^{*++}) = 1.4 \pm 0.4 \text{ Mev.} \tag{1}
\]

and

\[
\Gamma(N^{*0}) - \Gamma(N^{*++}) = 10.3 \pm 1.3 \text{ Mev.} \tag{2}
\]

They tried to eliminate these differences by applying electromagnetic correction and were not successful. Another attempt by Tromborg and Hamilton \(^{(14)}\) using the s-matrix method gave results similar to those of Bugg. It was Zimmermann \(^{(15)}\) who recently determined that a charge-independent phase-shift analysis is only possible by including the mass differences of the two channels in making Coulomb corrections. Therefore, the mass differences are what Bugg et al., and Tromborg and Hamilton should have considered in their correction, but did not.

Zimmermann \(^{(16)}\) also suggested that since Coulomb corrections depend principally on the little known electromagnetic potential in the nuclear region, it may be sufficient to use simple nuclear potentials (such as the one he used which is close to a square well) in calculating Coulomb corrections, or that otherwise a full dispersion theoretical calculation
such as the one by Hamilton, Øverbø and Tromborg\(^{(17)}\) should be used.

Let us now consider another popular approach to analyse charged-pion-proton and charged-pion-nucleus data. This approach involves the assumption that the total scattering amplitude can be written as

\[
f_{\text{tot}} = f_c e^{i\phi} + f_N,
\]

where \(f_c\) and \(f_N\) are pure Coulomb and pure nuclear scattering amplitudes.\(^{(18-27)}\) Bethe\(^{(28)}\) was the first to formulate the problem in terms of the relative phase \(\phi\) (i.e., the phase difference between Coulomb and nuclear amplitudes) and obtained an equation for it. Using a specific potential for the proton-nucleus interaction in a semi-classical calculation with the W.K.B. approximation, he obtained a formula for \(\phi\) at small c.m. scattering angles \(\theta\)

\[
\phi \approx 2n \ln \left( \frac{1.06}{k a \theta} \right),
\]

where \(k\) is the c.m. momentum and the nuclear interaction range. There have been quite a few efforts to derive other expressions for the relative phase since Bethe's work. The following reviews some of them.

A quite different equation obtained by Solov'ev\(^{(29)}\) is

\[
\phi \approx 2n \ln \left( \frac{2}{\theta} \right).
\]

He used relativistic quantum electrodynamics and techniques developed by Yennie, Frautschi and Saura\(^{(30)}\) to deal with the problem of infrared divergences in some of the Feynmann diagrams (with the exchange of photons between the charged particles) in his calculation. He did not consider any form factor for the charged particles. This was discussed
by Locher\textsuperscript{(31)} who included electromagnetic form factors at photon
vertices and obtained
\[ \phi \approx 2 \pi \left( \ln \frac{2}{\delta} - \ln \left( \frac{\alpha}{3} R^3 \right) - \frac{1}{2} C \right), \]
where \( R \) is the r.m.s. radius of the electromagnetic proton form factor,
which is taken to be exponential (the pion structure has been ignored) and
\( C \) the Euler's constant. This equation lends agreement to Bethe's result.
In fact, it is exactly equal to Bethe's calculation before the impact
parameter is averaged over to give eq. (4) when \( a = \sqrt{\frac{2}{3}} R \). Locher
used other simple electromagnetic proton form factors in his study with
similar results. He also estimated that the difference in Bethe's and
Solov'ev's equations would give a 10\% uncertainty in the ratio \( \frac{\text{Re} f_N}{\text{Im} f_N} \).

The relative phase formula by Bethe was generalized by Rix and
Thaler.\textsuperscript{(32)} They assumed the additivity of phase-shifts, i.e.
\[ \delta_{\text{tot}} = \delta_{\text{em}} + \delta_{\text{rud}}. \]
Their result involves an integral over the observed
(pure Coulomb subtracted) scattering amplitude.

Next, Bethe's formula was analyzed by Islam\textsuperscript{(33)} who employed
a relativistic impact parameter approach and a specific Coulomb potential
that is similar to the screened point-Coulomb potential but with finite
charge distribution effects included. The result is a formula which
differs only slightly from that of Bethe's.

A quite extensive examination of the Coulomb interference problem
and Bethe's formulation was performed by West and Yennie\textsuperscript{(33)} By using
quantum electrodynamics, they obtained an expression for \( \phi \)
\[
\phi = -2 n \ln \sin \frac{\theta}{2} - n \int_{-\kappa}^{\kappa} \frac{dt'}{|t'-t|} \left[ 1 - \frac{f_N(t')}{f_N(t)} \right],
\]

(7)

which is quite often used in analyses of pion-nucleus scattering data.\(^{(23-27)}\)

In addition, they also derived a similar expression to Bethe's when the nuclear scattering amplitude is parameterized to be

\[
f_N(t) = e^{\exp (A + B^2 t)}
\]

(8)

and the pion and target form factors

\[
F_\pi = e^{\exp (\nu_\pi^2 t)} \quad \text{and} \quad F_{\text{target}} = e^{\exp (\nu_f^2 t)}
\]

(9)

are used, except now \(A\) in Bethe's formula has to be interpreted as

\[
a = (B^2 + \nu_\pi^2 + \nu_f^2)^{\frac{1}{2}}.
\]

Recently, by doing eikonal calculation on some Feymann diagrams, the relative phase as given by West and Yennie was again obtained by Hayot and Itzykson\(^{(34)}\) who also give a formula for large momentum transfer corrections.

S-matrix method (or dispersion theoretical calculation) has recently begun to be applied in electromagnetic correction calculations. Sauter\(^{(35)}\) used the s-matrix method of Dashen and Frutschi\(^{(36)}\) to calculate Coulomb corrections to pion-proton phase-shifts and obtained

---

* A small photon mass is used to put electromagnetic interaction into dispersion theoretical calculations, and is allowed to go to zero at the end of the calculations.
good agreement with the work of Van Hove and Schnitzer. He later extended his calculation to double-channel processes. Further work within the dispersion theory framework can be found in Hamilton, Øverbø and Tromborg, and Tromborg and Hamilton.

There have been a few formulations of Coulomb-nuclear interference using the Glauber approximation. These will be presented after a brief introduction to Glauber theory is given in the next section.

b) **The Glauber Formalism**

In potential scattering, the scattering amplitude is given by the equation

\[ f(k, k') = \frac{-m}{2\pi\hbar} \int e^{-i\frac{k-k'}{\hbar} \cdot r} \sqrt{\phi(r)} \frac{\gamma_k}{\phi_k} dr, \tag{10} \]

where \( \frac{\gamma_k}{\phi_k} \) is the wave function of the particle of mass \( m \) being scattered by a potential \( \sqrt{\phi} \), whose initial and final wave vectors are \( k \) and \( k' \) respectively. In order to calculate the wave function \( \frac{\gamma_k}{\phi_k} \), we proceed to solve the Schroedinger equation

\[ \sqrt{\phi_k} \partial_x^2 \frac{\gamma_k}{\phi_k} + k^2 \frac{\gamma_k}{\phi_k} = \frac{2m}{\hbar^2} \sqrt{\phi} \frac{\gamma_k}{\phi_k}. \tag{11} \]

For high energy scattering such that the energy of the incident particle is much bigger than the potential and that the wavelength of the incident particle is much smaller that the width of the potential, i.e.

\[ E >> \sqrt{\phi} \quad \text{and} \quad |\frac{k}{\alpha}| >> 1, \tag{12} \]

it is a good approximation to write the particle wave function as a plane wave in the incident direction \( z \), with a modulating factor \( \phi(z) \) such that
\[ \psi_\lambda(x) = \exp(ikz) \phi(x), \]  

where \( \phi(x) \) varies very slowly over a particle wavelength. Putting this wave function into the Schroedinger equation (eq. (11)) gives

\[ e^{ik \left( \frac{\partial}{\partial x} + \frac{5i}{2y} \right)} \phi(x,y,\bar{z}) + 2ik e^{ik \frac{\partial}{\partial \bar{z}}} \phi(x,y,\bar{z}) + e^{ik \frac{\partial^2}{\partial \bar{z}^2}} \phi(x,y,\bar{z}) = \frac{im}{\hbar} e^{ik \phi(x,y,\bar{z})}. \]  

(14)

Neglecting the second derivatives, we obtain an equation for \( \phi(x,y,\bar{z}) \) such that

\[ \frac{\partial \phi(x,y,\bar{z})}{\partial \bar{z}} = -\frac{i m}{\hbar k} V(x,y,\bar{z}) \phi(x,y,\bar{z}). \]  

(15)

Since at \( \bar{z} = -\infty \), the wave function \( \psi_\lambda(x) \) is a plane wave, we have

\[ \phi(x,-\infty) = 1. \]  

(16)

With this boundary condition, the solution of eq. (15) is

\[ \phi(x,y,\bar{z}) = \exp\left\{ -\frac{i m}{\hbar k} \int_{-\infty}^{\bar{z}} V(x,y,z') dz' \right\}. \]  

(17)

Therefore,

\[ \psi_\lambda(x) = \exp\left\{ ik \bar{z} - \frac{i m}{\hbar k} \int_{-\infty}^{\bar{z}} V(x,y,z') dz' \right\}. \]  

(18)

When \( \psi_\lambda(x) \) is put into eq. (1), the scattering amplitude becomes

\[ f(k,k',\ell) = -\frac{m}{2\pi \hbar^2} \int d^2\theta \ exp\left\{ i \frac{k}{\hbar} \cdot \frac{k'}{2} \right\} \int_{-\infty}^{\infty} V(k+z,\ell) \exp\left\{ -\frac{i m}{\hbar k} \int_{-\infty}^{z} V(e+z,\ell) dz \right\} dz. \]  

(19)

where \( \ell \) is a vector in the impact parameter plane which is perpendicular to \( \ell' \), and the approximation that for small scattering angles ( \( \ell' - \ell \) ) is perpendicular to \( \ell' \) (due to \( |\ell'| = |\ell| \) from energy conservation) has been used. Then, by noting that
\[
\frac{d}{d\gamma} \exp \left\{ -\frac{i m}{\hbar^2} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz' \right\} = \left[ \frac{d}{d\gamma} \left( -\frac{i m}{\hbar^2} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz' \right) \right].
\]

\[
\exp \left\{ \frac{i m}{\hbar^2} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz' \right\} = -\frac{i m}{\hbar^2} \nabla (k + \frac{3}{2} \hat{z}) \exp \left\{ -\frac{i m}{\hbar^2} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz' \right\}
\]

(20)

The scattering amplitude \( f(k', k) \) finally becomes

\[
\begin{align*}
 f(k', k) &= \frac{\hbar}{2\pi} \int d^2 b \; e^{i(b \cdot k')/\hbar} \left[ \exp \left\{ -\frac{i m}{\hbar^2} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz' \right\} \right] \\
 &= \frac{\hbar}{2\pi} \int d^2 b \; e^{i(b \cdot k')/\hbar} \left[ 1 - \exp \left( -\frac{i m}{\hbar^2} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz' \right) \right]
\end{align*}
\]

(21)

where

\[
\rho = \frac{\hbar}{\hbar} = m \nu
\]

(22)

have been used. (See section 1 of Chapter 5 for discussion)

Defining the momentum transfer

\[
\vec{q} = \vec{k}' - \vec{k}
\]

(23)

the phase-shift function

\[
\chi(k) = \frac{-i}{\hbar \nu} \int_{-\infty}^{\infty} \nabla (k + \frac{3}{2} \hat{z}) \, dz
\]

(24)

and the profile function

\[
\Gamma(k) = 1 - \exp \left( i \chi(k) \right)
\]

(25)

the scattering amplitude can be written as

\[
\begin{align*}
 f(q) &= \frac{i \hbar}{2\pi} \int d^2 b \; e^{i \vec{q} \cdot \vec{b}} \Gamma(k).
\end{align*}
\]

(26)

Remembering that \( \vec{q} \) is in the plane perpendicular to \( \vec{k} \), eq. (26) can be inverted to give...
\[ T'(k) = \frac{i}{2\pi \lambda^2} \int d^3\tau \ e^{-\frac{i}{\lambda^2} \cdot \kappa} f(\tau) . \quad (27) \]

It is seen from the above that the expression obtained for the scattering amplitude is a good approximation for high energy, small angle scattering. The above is only one of the many ways to derive eq. (26). Other derivations may be obtained by the use of stationary-phase argument, the Green's function, the refractive index technique etc. \(^{(39-41)}\)

The way to apply the Glauber approximation to hadron-nucleus scattering or multiple potential scattering is through the additivity of potentials, i.e.

\[ \sqrt{\chi} = \sum_{\ell=1}^{A} V_{\chi} \left( \gamma_\ell \right) \quad , \quad (28) \]

from which follows the additivity of the phase-shift function

\[ \chi(k) = \sum_{\ell=1}^{A} \chi_\ell (k) . \quad (29) \]

The total profile function \( T(k) \) can therefore be expressed in terms of the individual profile functions \( T_\ell(k) \) as

\[ T'(k) = \sum_{n=1}^{A} T_\ell^{(n)}(k) \quad (30) \]

with

\[ T_\ell^{(n)}(k) = (-1)^{n+1} \sum_{\ell_1, \ldots, \ell_n} \prod_{\ell=1}^{n} T_{\ell_1}(k_{\ell_1}) \cdots T_{\ell_n}(k_{\ell_n}) . \quad (31) \]

And the total scattering amplitude \( F(\tau) \) is in the form

\[ F(\tau) = \sum_{k=1}^{A} f^{(n)}(\tau) \quad (32) \]

with
\[ f^{(n)}(q) = \frac{i \hbar}{2 \pi} \int \exp(i \mathbf{q} \cdot \mathbf{k}) \mathbf{T}^{(n)}(k) \, d^2 k. \]  

(33)

Eqs. (32) and (33) can be interpreted as follows. The scattering amplitude can be split into A amplitudes \[ f^{(n)}(q), \quad n = 1, \ldots, A \]
where A is the number of nucleons in the nucleus or the number of potentials that make up the total potential. The amplitude \[ f^{(n)}(q) \]
the type of multiple scattering interaction that involves \( n \) nucleons or potentials, one after the other with none of the nucleons or potentials interacting more than one time.

3) **Coulomb-Nuclear Interference in the Glauber Framework**

Franco\(^{(42-45)}\) has been the main source of Coulomb-nuclear interference studies using the Glauber formalism. In his first paper\(^{(42)}\) on the subject, he included electromagnetic interactions in his calculation of proton-denteron scattering. Three phase-shift functions \( \chi_n, \chi_{ps} \), and \( \chi_c \) were used for the p-n, p-p nuclear and the p-p Coulomb interactions respectively. The total p-p phase-shift function is equal to \( \chi_c + \chi_{ps} \).

Arguing that the Coulomb phase-shift function varies only slowly in the nuclear interaction range and that outside that range the profile functions \( T_n \) and \( T_{ps} \) are negligibly small, he approximated the Coulomb phase-shift \( \chi_c(k) \) functions in the terms
\[ \exp(i \chi_c(k)) T_{ps}, \quad \exp(i \chi_c(k)) T_n \quad \text{and} \quad \exp(i \chi_c(k)) T_{ps} T_n \]
which appear in the total profile function by its appropriate average values \( \chi_1, \chi_2 \) and \( \chi_3 \) such that
\[ \chi_1 = \frac{\int \kappa_c \tau_{ps} \, d^2 b}{\int \tau_{ps} \, d^2 b}, \quad \chi_2 = \frac{\int \kappa_c \tau_n \, d^2 b}{\int \tau_n \, d^2 b}, \quad \text{and} \quad \chi_3 = \frac{\int \kappa_c \tau_{ps} \tau_n \, d^2 b}{\int \tau_{ps} \tau_n \, d^2 b} \]

His final results for the elastic and total differential cross-sections were then easily obtained.

In his next paper\(^{(43)}\) he considered the same proton-deuteron problem, but this time, he calculated the Coulomb profile function by assuming point charges. For the other two nuclear profile functions \(\tau_n\) and \(\tau_{ps}\), he expressed them respectively in terms of p-n and p-p scattering amplitudes which were taken to be Gaussian. Finally the deuteron wave function was taken to be Gaussian to obtain some quite complicated results for the differential scattering amplitudes.

Again using the Glauber model, expressing the total phase-shift function as a sum of Coulomb and strong phase-shift functions, Franco\(^{(44)}\) next obtained an expression for the Bethe relative phase by assuming a screened point-charged Coulomb potential. When the screening radius is extended to infinity, the formula of West and Yennie\(^{(33)}\) is obtained. Scattering amplitudes were also calculated with extended charge distribution effects included. The resulting cross-section compares very favorably with those from Bethe's and West and Yennie's. Specific equations were obtained by Franco and Varma\(^{(45)}\) for proton-deuteron scattering when the charge form factors for the incident and bound protons are assumed to be Gaussian. Comparison with calculations using much simpler equations obtained in Ref. 42 shows that the simpler equations are a good approximation.
References


(37) E. Sauter, Nuovo Cim. 6A, 335 (1971).


(40) L. I. Schiff, Phys. Rev. 103, 443 (1956).


Chapter II

EXPERIMENTAL ASPECTS IN THE MEASUREMENT OF CROSS-SECTION

In order for a theorist to study the effects of Coulomb-nuclear interference when the beam and target contain charged particle, it is important for him to understand how the experimentalists treat their data to obtain results for both the total and the differential cross-section. The deflection of particles due to multiple Coulomb scattering on traversing the target and Coulomb-nuclear interference effects have to be taken into account before scattering cross-sections due only to the nuclear force can be obtained.

Up to the present time, there have been few experiments that attempt to take all effects of the Coulomb forces into consideration. People usually argue that the energy range or angular range of their data is such that the effects of Coulomb-strong interference and multiple Coulomb scattering are not important. In this regard, it is unfortunate that, for pion-nucleon scattering, in the region close to the (3,3) resonance, we do not have experimental data at small angles where the effect of Coulomb-nuclear interference would be important. On the other hand, the errors in some scattering experiments are so large that consideration of Coulomb correction is not worthwhile.

While Coulomb effects in differential scattering data are relatively simpler to correct once particular equations have been selected for them, it is more interesting to study how they are handled in total scattering
cross-section experiments. Attenuation measurement of the incident beam transmitted through the target are made to determine the total cross-section. The total scattering cross-section $\sigma_T$ may be defined by the equation

$$d \mathcal{I}(\mathcal{J}) = -\mathcal{I}(\mathcal{J}) \alpha_T \mathcal{N} d\mathcal{J},$$

(1)

where $\mathcal{I}(\mathcal{J})$ is the unscattered intensity of the beam after traversing a length $z$ of the target, $d \mathcal{I}(\mathcal{J})$ is the change in intensity due to scattering of the beam particle by the target after traversing an additional target thickness of $dz$, and $\mathcal{N}$ is the number density of the target (usually the number of nuclei per cm$^{-3}$). Eq. (1) can easily be integrated to give the attenuation equation

$$I_{out}(L) = I_{in} \exp[-\mathcal{N} \sigma_T L],$$

(2)

where $I_{in}$ is the incident intensity, $I_{out}$ is the unscattered transmitted intensity, and $L$ is the total target thickness traversed. The scattered intensity $I_{scatt}$ is

$$I_{scatt} = I_{in} - I_{out} = I_{in} \left( 1 - \exp(-\mathcal{N} \sigma_T L) \right).$$

(3)

Therefore measurement of the attenuation, $\frac{I_{out}}{I_{in}}$ will give the value for the total scattering cross-section. Ideally, if we had a point-size counter and if we could ignore any Coulomb multiple scattering, then we could direct the beam such that its unscattered portion would converge at the counter, which would then record only those particles in the beam that have not undergone any reaction with the target. $I_{in}$ and $I_{o}$ can be obtained by using an empty and a filled target respectively. The total cross-section is then given by
\[ \Omega_T = \frac{1}{NL} \ln \left( \frac{I_{\text{in}}}{I_{\text{out}}} \right) \]  \hspace{1cm} (4)

Unfortunately, this ideal situation is not attainable because of the finite size of the counter and Coulomb multiple scattering. As a result of the finite size of the counter, it subtends a finite solid angle \( \mathcal{N} \) at the center of the target and all beam particles that are scattered within the solid angle \( \mathcal{N} \) will be recorded. This effect causes an increase in the ratio of attenuation \( \frac{I_{\text{out}}}{I_{\text{in}}} \) and, consequently, a decrease in the measured total cross-section as a function of the solid angle \( \mathcal{N} \) given by

\[ \Omega_T(\mathcal{N}) = \frac{1}{NL} \ln \left( \frac{I_{\text{in}}}{I_{\text{out}}(\mathcal{N})} \right) \]  \hspace{1cm} (5)

(see Fig. (1)).

The effect of Coulomb multiple scattering is such that beam particles which do not react strongly with the target may be deflected by Coulomb scattering and not reach the counter. Therefore, this will cause a decrease in the attenuation ratio and an increase in the total cross-section \( \Omega_T(\mathcal{N}) \). This effect is more prominent when \( \mathcal{N} \) is small because Coulomb scattering is sharply peaked in the forward direction. Once the solid angle \( \mathcal{N} \) is larger than the forward cone of Coulomb scattering, most of the Coulomb scattered particle reach the counter and Coulomb multiple scattering effects are diminished.

The above effects are illustrated in Figs. (1) and (2). Fig. (1) shows how the attenuation \( \frac{I_{\text{out}}(\mathcal{N})}{I_{\text{in}}} \) varies with the solid angle, and Fig. (2) shows how the total cross-section \( \Omega_T(\mathcal{N}) \), obtained from the attenuation, changes with the solid angle.
Solid Angle ($\mathcal{N}$) (msr)

Fig. (1-1) Attenuation as a function of solid angle subtended to the target.

Each of Fig. s(1) and (2) is divided into three regions. In region 1, the attenuation ($\frac{I_{\text{out}}}{I_{\text{in}}}$) decreases sharply to zero and the measured total cross-section ($\sigma(\mathcal{N})$) increases sharply as the solid angle goes to zero. This is due to Coulomb scattering as mentioned earlier. Coulomb effects are less pronounced in region 2, but its interference with nuclear scattering may be important here. In region 3 Coulomb effects are negligible.

From the above discussion it is seen that eq. (2) has to be modified to give the correct dependence on the solid angle ($\mathcal{N}$) subtended by the counter. Suppose that due to Coulomb multiple scattering, only a fraction $D(\mathcal{N})$ of beam particles that are unaffected by nuclear force reaches the counter. Therefore, the measured transmitted intensity is

$$I_{\text{in}} D(\mathcal{N}) \exp(-N_{\text{eff}} L)$$

instead. In Appendix A we consider the case of a finite-sized circularly symmetric beam which, in the absence of
Coulomb scattering, would arrive at the counter with an area density

\[ \mathcal{S}(r) \text{ such that } \mathcal{S}(r) = 0 \text{ for } r > R_b \]

where \( R_b \) is the beam radius. Then*

\[ D(\Omega) = \frac{\int_{R_b}^{\infty} F(r, r) \mathcal{S}(r) r \, dr}{\int_{0}^{R_b} \mathcal{S}(r) r \, dr} \]  \[ (6) \]

where \( F(r, \cdot, r) \) is the probability that the beam particle hits the counter when in the absence of Coulomb scattering it would have done so at the distance \( r \) from the center of the counter. The other argument \( r_o \) of \( F(r_o, r) \) depends on the energy of the beam particle, the radiation length of the target (\( X_o \)), the target thickness (\( L \)), and the distance of the counter to the target (\( e \)). Explicit expressions for \( F(r_o, r) \) and \( r_o \) are given in Appendix A. The dependence of \( D(\Omega) \) on the solid angle comes implicitly from its dependence on \( F(r_o, r) \). Eq. (6) takes into account not only multiple Coulomb scattering, but also the finite beam size and the beam divergence.

One should also correct eq. (2) for particles that have undergone small angle nuclear scattering because of finite beam size. Integrating the differential cross-section \( \frac{d\sigma}{d\Omega} \) over the solid angle \( \Omega \) gives us the cross-section for such beam particles, i.e., \( \int_{\Omega} d\sigma \frac{d\sigma}{d\Omega} d\Omega \). Unfortunately, in this case, we cannot use eq. (3) to determine the intensity of the particles scattered into the solid angle \( \Omega \) after traversing the target. The reason is that the scattered intensity obtained from eq. (3) in-

*An explicit expression for \( D(\Omega) \) is also obtained in Appendix A for the case when \( \mathcal{S}(r) \) is a Gaussian distribution.
cludes all particles that are first scattered into solid angle $\mathcal{N}$, but some of these particles can be subsequently scattered away from the solid angle $\mathcal{N}$ and therefore not reach the counter. This situation can be handled by considering multiple nuclear scattering. Appendix A, where multiple Coulomb scattering is considered, also indicates how multiple scattering effect can be calculated for nuclear scattering.

A much simpler method for correcting for counter size in the case of nuclear scattering results from the observation that most of the particles scattered by nuclear forces are singly scattered because of the small target thickness and the small size of cross-section. Therefore by calculating the intensity of single nuclear scattering to solid angle $\mathcal{N}$, most of the overcounting will be accounted for. Let $I^{(1)}(z)$ be the intensity of the beam particles that have been scattered once into the solid angle $\mathcal{N}$ after traversing a length $z$ of the target. On traversing an additional target length of $dz$, some of the particles that have not undergone any scattering at all ($I(z)$), will be scattered into $\mathcal{N}$, this will add to $I^{(1)}(z)$ by the amount $I(z)N\left[\int_0^z \frac{d\omega}{d\mathcal{N}} N \, dz\right]$. On the other hand, some of the particles in $I^{(1)}(z)$ will be scattered again, this will then be subtracted from $I^{(1)}(z)$ by the amount $I^{(1)}(z)N \sigma_2 \, dz$.

Therefore, the change in $I^{(1)}(z)$ after traversing an additional target length $dz$ is given by the differential equation,

$$dI^{(1)}(z) = I(z)N\left[\int_0^z \frac{d\omega}{d\mathcal{N}} N \, dz\right] - I^{(1)}(z)N \sigma_2 \, dz.$$  

(7)

By using eq. (2), eq. (7) can easily be solved to give
\[ I''(L) = I_{in} N L \left[ \int_0^\pi \frac{d\phi}{d\Omega} \, d\Omega \right] \exp(-N\sigma T L) \]  

(8)

Combining the effects discussed above, eq. (2) is modified to be

\[ \frac{I_{out}(\Omega)}{I_{in}} = \exp(-N\sigma T L) \left\{ D(\Omega) + N L \left[ \int_0^\pi \frac{d\phi}{d\Omega} \, d\Omega \right] \right\} \]  

(9)

with \( D(\Omega) \) given by eq. (6).

An equation similar to (9) was given by Amaldi et al.\(^{(1)}\), except that their first term was

\[ \exp(-N\sigma T L) 2\pi \int_0^{R_c} f(\eta, \xi) \, r \, dr \]  

where \( R_c \) is the radius of the counter, \( f(\eta, \xi) \) is the distribution of the beam particles at the counter after undergoing multiple Coulomb scattering traversing the target. The solid angle subtended by the counter is

\[ \Omega = \pi \frac{R_c^2}{\xi^2} \]  

(2, 3)

Bugg et al. have also given equations for both single and multiple scattering correction. Unfortunately, there was not sufficient information to show how the equations were obtained.

The typical behavior of the measured attenuation \( \frac{I_{out}(\Omega)}{I_{in}} \) in eq. (9) is shown in Fig. (1). In region 1, \( D(\Omega) \) changes sharply and is dominant in its effect on the variation in \( \frac{I_{out}(\Omega)}{I_{in}} \). In region 2, the second term becomes as important as the first, and finally, \( D(\Omega) \) approaches 1 \( (D(\Omega) = 1) \) in region 3 leaving all the correction to the second term in eq. (9).

The above discussion suggest a procedure for determining the total
scattering cross-section $\sigma_T$. The attenuation $\frac{I_{out}(\Omega)}{I_{in}}$ will be measured at various solid angle except for solid angles in region 1, where the multiple Coulomb scattering and finite beam size effects dominate. This is then corrected for effects of multiple Coulomb scattering, finite beam size and finite counter size by the use of eq. (9). The corrected attenuation $\left(\frac{I_{out}(\Omega)}{I_{in}}\right)_{corr.}$ due to nuclear scattering is then used to calculate the total cross-section $\sigma_T(\Omega)$ as a function of angle by using the equation

$$\sigma_T(\Omega) = -\frac{1}{N L} \ln \left(\frac{I_{out}(\Omega)}{I_{in}}\right)_{corr.}.$$  \hspace{1cm} (11)

Extrapolating $\sigma_T(\Omega)$ to zero solid angles gives the total cross-section $\sigma_T$ due to nuclear scattering. Extrapolation is usually done by fitting the values of $\sigma_T(\Omega)$ obtained from eq. (11) to a quadratic equation

$$\sigma_T^{fit}(\Omega) = C_0 + C_1 \Omega + C_2 \Omega^2.$$  \hspace{1cm} (12)

then total cross-section $\sigma_T$ is given by

$$\sigma_T = \sigma_T(\Omega = 0) = C_0.$$  \hspace{1cm} (13)

The reason for the exclusion of region 1 in measuring the attenuation is that the dominant multiple Coulomb and beam size effects to be corrected can be much bigger than the corrected result. Large errors can be introduced if data in this region is used.

Another thing to notice is that $\frac{d\sigma}{d\Omega}$ in eq. (9) has to be determined before that part of the correction can be made. Usually it is assumed
constant or assumed to come from some theoretical model. But it is also possible to contain nuclear-Coulomb interference effect especially when the solid angle is in the region 2 in Fig. 2.

There are, of course, other effects such as beam contamination and beam normalization, which requires correction in the course of data treatment. These have been ignored in our discussion because they have no bearing on this thesis.
Reference

(1) U. Almoldi, Jr. et al., Nuovo Cimento 34, 5873 (1964)


Chapter III

COULOMB-NUCLEAR INTERFERENCE IN PION-NUCLEON POTENTIAL SCATTERING

Interference effects will be discussed separately for the cases of pion-nucleon scattering and pion-nucleus scattering. People sometimes make the distinction that the interference effect for the first case is called Coulomb-strong interference and that for the latter case the Coulomb-nuclear interference. We start out with \( \pi-N \) scattering.

1) Coulomb Potential

Before we consider Coulomb-strong interference using the Glauber Formalism, Coulomb and nuclear scattering are considered individually. For Coulomb scattering, we first look at the point-charge case,

\[
V_C(r) = \frac{z_1 z_2 e^4}{r},
\]

(1)

where \( z_1 e \) and \( z_2 e \) are the charge of the projectile and target respectively.

In the extended charge case, (1) will give the long range effect of \( V_C(r) \) and present problems because when we calculate the phase shift-function using the integral*

\[
\chi_{r}(b) = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} V(k + 3 \hat{z}) \, dz,
\]

(2)

---

* \( \frac{1}{\hbar v} \) factor in this and later equations should be changed to \( \frac{m}{\hbar L} \) in this and the next chapter. Please see Chapter V for discussion.
it diverges at both limits. To remedy this situation, we note that in any nuclear scattering experiment, the actual Coulomb potential decreases faster than in eq. (1) because of electronic shielding; therefore, a multiplicative cut-off function can be used to eliminate the divergence. This cut-off function is such that it decreases from one at short range to zero at atomic distances. The simplest cut off is a step function that it is one for \( r \) is smaller than some atomic distance \( A \) and zero otherwise. In this case, \( V_c(r) \) in eq. (1) becomes

\[
V_c(r) = \begin{cases} \frac{z_1 z_2 e^2}{r} & r \leq A \\ 0 & r > A \end{cases}
\]  

(3)

Putting eq. (3) into eq. (2) gives the phase-shift function

\[
\chi_c(b) = -\frac{z_1 z_2 e^2}{\hbar v} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{b^2 + z^2}} \quad b \leq A
\]

\[
= 0 \quad b > A
\]

(4)

or

\[
\chi_c(b) = -2 \pi \ln \left( \frac{A + \sqrt{A^2 - b^2}}{b} \right) \quad b \leq A
\]

\[
= 0 \quad b > A
\]

(5)

where

\[
h = \frac{z_1 z_2 e^2}{\hbar v}
\]  

(6)

The scattering amplitude, \( f_c(\theta) \), is given by

\[
f_c(\theta) = \frac{k}{i} \int_0^\infty \mathcal{J}(gb) \left\{ e^{i \chi_c(b)} - 1 \right\} \, db
\]  

(7)

where

\[
gb = 2 k \sin \frac{\theta}{2}
\]  

(8)

The Bessel function \( J_0(2kb \sin \frac{\theta}{2}) \) oscillates with rapidly decreasing
amplitude with the first zero when the argument has a value around 2.4; therefore, its contribution to the integral in eq. (1) will come almost entirely from the region where

$$\kappa b \theta \sim 1$$  \hspace{1cm} (9)

(remembering that $\theta$ is small), or

$$b \sim \frac{1}{\kappa \theta}.$$  \hspace{1cm} (10)

Note that, for the energies we are interested in, $\kappa$ has a value of order one fermi, whereas $A$ is of the order $10^5$ fermi; therefore the ratio

$$\frac{b}{A} \sim \frac{1}{\kappa a \theta} \sim 10^{-5} \theta$$  \hspace{1cm} (11)

is very small as long as extremely small angles of order $10^{-15}$ rad are not considered. Since we have $\frac{b}{A} \ll 1$, we can simplify $\chi_c(b)$ in eq (5) by expanding it in powers of $\frac{b}{A}$. We invert the argument inside the log because we want the numerator to be smaller than the denominator, and, after expansion in powers of $\frac{b}{A}$ and comparison with the series expansion of $\exp\left(\frac{b^2}{4A^2}\right)$, the argument becomes

$$\frac{b}{A + \sqrt{A^2 - b^2}} = \frac{b}{2A} \left[ \exp\left(\frac{b^2}{4A^2}\right) + O\left(\frac{b^4}{A^2}\right) \right].$$  \hspace{1cm} (12)

As a result of (12), $\chi_c(b)$ in (5) becomes

$$\chi_c(b) = 2n \ln\left(\frac{b}{2A}\right) + n\left(\frac{b^2}{2A^2}\right) + O\left(\frac{b^4}{A^4}\right).$$  \hspace{1cm} (13)

By keeping only the $\left(\frac{b}{A}\right)$ term, eq. (6) is approximated by
\[
\frac{f_c(\theta)}{\theta} = \frac{k}{i} \int_0^\infty J_0(qb) \exp(izn \ln (\frac{n}{2A})) b \, db 
\]

or

\[
f_c(\theta) = \frac{k}{i} \exp(-izn \ln 2A) \int_0^\infty J_0(qb) b^{1+2i\ln b} \, db \quad (15)
\]

After evaluating the integral, we have \(^{(1)}\)

\[
f_c(\theta) = -\frac{2k}{\theta} \exp \left\{ -i \left[ 2n \ln(2kA \sin \frac{\theta}{2}) - 2 \arg \Gamma(1+i\ln b) \right] \right\}
\]

which gives agreement with the Rutherford scattering formula when the differential cross-section is calculated, i.e. \(^{(2)}\)

\[
\frac{d\sigma}{d\Omega} = \left| f_c(\theta) \right|^2 = \frac{n^2}{4k^2 \sin^2 \frac{\theta}{2}} \quad (17)
\]

It is evident from eq. (17) that the Coulomb differential cross-section is strongly forward-peaked.

2) **Square Well Potential**

In order to obtain analytic equations for Coulomb-strong interference effects, simple potentials (the square well potential and the Yukawa potential) are used to represent the pion nucleon potential. \(^*\) We first discuss the square well potential without coupling to the Coulomb potential, i.e.

\*Please see section 6 for discussions on spin and isospin.
\[ V_s = -V_0 \quad r \leq a \]
\[ = 0 \quad r > a, \]  
(18)

where \( a \) is of nuclear dimension and where \( V_0 \) is real as long as we confine our attention to elastic scattering.

From eq. (2), we get \( \chi_s(b) \) to be

\[ \chi_s(b) = \frac{2V_0}{\hbar V} \sqrt{a^2 - b^2} \quad b \leq a \]
\[ = 0 \quad b > a. \]  
(19)

Unlike the result for Coulomb scattering, the total scattering cross-section \( \sigma_{tot} \) for a square well potential is finite. Since it is easier to calculate than the differential scattering cross-section, we evaluate it first. From the optical theorem,

\[ \sigma_{tot} = \frac{4\pi}{\hbar^2} \text{Im} \int f'(\theta=0) \]
(20)

and from eq. (7), we obtain

\[ \sigma_{tot} = 4\pi \int_0^\infty \text{Re} \left\{ b \left[ 1 - \exp \left( \frac{i V_0}{\hbar V} \sqrt{a^2 - b^2} \right) \right] \right\} db, \]
(21)

where we have used \( J_0(\theta) = 1 \). Putting eq. (18) into eq. (21), we have

\[ \sigma_{tot} = 4\pi \int_0^a \text{Re} \left\{ b \left[ 1 - \exp \left( \frac{i V_0}{\hbar V} \sqrt{a^2 - b^2} \right) \right] \right\} db \]
\[ = 2\pi a^2 - 4\pi \int_0^a \text{Re} \left[ \cos \left( \frac{2V_0}{\hbar V} \sqrt{a^2 - b^2} \right) \right] b \, db. \]
(22)

By changing the variable of integration to \( X = \sqrt{a^2 - b^2} \)
\[ \theta_{\text{tot}}^{\text{tot}} = 2\pi a^2 - 4\pi \int x \cos \left( \frac{\sqrt{V_0}}{\hbar} x \right) \, dx. \]  

(23)

From the integral

\[ \int (a + bx) \cos (kx) \, dx = \frac{a + bx}{k} \sin (kx) + \frac{b}{k^2} \cos (kx), \]

(24)

we obtain

\[ \theta_{\text{tot}}^{\text{tot}} = 2\pi a^2 - 4\pi a^2 \left\{ \frac{\cos (2\lambda)}{\lambda^2} + \frac{2\sin (2\lambda)}{\lambda} - \frac{1}{\lambda^2} \right\}. \]

(25)

with

\[ \lambda = \frac{\sqrt{V_0} a}{\hbar}. \]

(26)

The scattering amplitude \( f_s(\theta) \) for a square-well potential can be obtained from eq. (7), with the phase-shift function given by (19):

\[ f_s(\theta) = \frac{k}{i} \int_0^a \mathcal{J}_0(b) \left\{ \exp \left( i \frac{\sqrt{V_0}}{\hbar} \sqrt{a^2 - b^2} \right) - 1 \right\} b \, db. \]

(27)

This can be written as a sum of three integrals:

\[ f_s(\theta) = k \int_0^a \mathcal{J}_0(b) \sin \left( \frac{\sqrt{V_0}}{\hbar} \sqrt{a^2 - b^2} \right) b \, db + ik \int_0^a \mathcal{J}_1(b) b \, db - ik \int_0^a \mathcal{J}_0(b) \cos \left( \frac{\sqrt{V_0}}{\hbar} \sqrt{a^2 - b^2} \right) b \, db. \]

(28)

Now we consider the integral

\[ \int_0^a x^{\nu+1} \sin \left( b \sqrt{a^2 - x^2} \right) \mathcal{J}_\nu(x) \, dx = \left[ \frac{\pi}{2} a^{\nu+\frac{3}{2}} b^{1+\frac{3}{2}} \mathcal{J}_{\nu+\frac{3}{2}} (a\sqrt{1+b^2}) \right]_{\Re \nu > -1}. \]

(29)

The first integral in eq. (28) matches the above when \( \nu = 0 \), and so after change of variable of integration,
\[ \int_0^b J_0(bq) \sin \left( \frac{2V_0}{\hbar \nu} \sqrt{a^2 - bq^2} \right) b \, dq = \frac{k}{\hbar} \int_0^\frac{2V_0}{\hbar \nu} J_0(x) \sin \left( \frac{2V_0}{\hbar \nu} \sqrt{a^2 - x^2} \right) x \, dx \]

\[ = \frac{k}{\hbar^2} \int_0^\frac{2V_0}{\hbar \nu} \left( \frac{2V_0}{\hbar \nu} \right)^{3/2} \frac{1}{(\hbar \nu)^{3/2}} \left[ 1 + \left( \frac{2V_0}{\hbar \nu} \right)^{-1} \right] \frac{2}{\sqrt{1 + \left( \frac{2V_0}{\hbar \nu} \right)^2}} J_\frac{3}{2} \left( \frac{2V_0}{\hbar \nu} \sqrt{1 + \left( \frac{2V_0}{\hbar \nu} \right)^2} \right) \]

From the integral

\[ \int_0^1 x^\nu \nu J_\nu(ax) \, dx = a^{-1} J_{\nu+1}(a) \qquad [\Re \, \nu > -1] \]

the second integral in eq. (28) can be calculated to be,

\[ i \, \frac{k}{\hbar} \int_0^a J_0(bq) \, dq \, db = i \, \frac{k}{\hbar} \int_0^1 J_0(xa) \, x \, dx \]

\[ = i \, \frac{k}{\hbar} a J_1(aq) . \]

In order to evaluate the third integral in eq. (28), we make use of the
series expansion of the cosine function and the integral

\[ \int_0^1 x^{\nu+1} (1-x^2)^\mu J_\nu(bx) \, dx = 2^\mu \Gamma(\mu+1) b^{-\nu-1} \int_0^b J_{\nu+\mu+1}(b) \]

\[ \qquad [b > 0, \Re \, \nu > -1, \Re \, \mu > -1] . \]

To put the third integral into the above form, we again change the
variable of integration and let \( \nu = 0 \), obtaining

\[ -i \, \frac{k}{\hbar} \int_0^a J_0(bq) \cos \left( \frac{2V_0}{\hbar \nu} \sqrt{a^2 - bq^2} \right) b \, dq \, db = -i \, \frac{k}{\hbar} a^2 \int_0^1 J_0(xa) \cos \left( \frac{2V_0}{\hbar \nu} \sqrt{1 - x^2} \right) x \, dx \]

\[ = -i \, \frac{k}{\hbar} a^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2V_0)^{2m}}{(2m)!} \int_0^1 J_m(aq) (1-x^2)^m \, dx \]

\[ = -i \, \frac{k}{\hbar} a^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2V_0)^{2m}}{(2m)!} 2^m \Gamma(m+1) (aq)^{m+1} \int_0^1 J_{m+1}(aq) . \]
By combining eqs. (30), (32) and (34), we find the scattering amplitude for a square well potential to be

$$f_s(q) = \frac{k}{\beta} \sqrt{\frac{2}{\pi}} \left( \frac{E}{\beta^2} \right)^{3/2} \left[ 1 + \left( \frac{2E}{\beta^2} \right)^{1/2} \right] \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \left( \frac{q}{\beta} \right)^3 \left[ 1 + \left( \frac{2E}{\beta^2} \right)^{1/2} \right] \int_{-\pi}^{\pi} \frac{d\phi}{\phi} \left( \frac{q}{\beta} \right)^3 \left[ 1 + \left( \frac{2E}{\beta^2} \right)^{1/2} \right]$$

$$+ k \alpha \left\{ \frac{J_1(q\phi)}{q} - a \sum_{m=0}^{\infty} \frac{(-1)^m (2V)a^{3m}}{(2m)!} 2^m \Gamma(m+1)(a\phi)^{-(m+1)} J_{m+1}(a\phi) \right\}.$$  \[ (35) \]

Discussions concerning the convergence of the series in the above equation and in later equations will be postponed until after appropriate values are obtained for the variables involved in the series. It will be found that this series converges too slowly for the relevant values of the variable and so makes computation difficult. Another equation for the differential scattering amplitude is therefore obtained in what follows.

In eq. (34), a series expansion of the cosine function was used. An alternative is to make use of the series expansion of the Bessel function, \( J_0(bq) \), which can be obtained from

$$J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{m=0}^{\infty} \frac{(-\frac{1}{2}z^2)^m}{m! \Gamma(\nu+m+1)}.$$  \[ (36) \]

By putting \( \nu = 0 \) and noting that \( \Gamma(m+1) = M! \), we have

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{4^m (m!)^2}.$$  \[ (37) \]

It will be shown later that this series is rapidly convergent for the values of \( z \) we are interested in. (See p. 98). The third integral in eq. (28) therefore becomes
\[-i \mathcal{K} \int_{0}^{\infty} J_{\nu}(b \gamma) \cos \left( \frac{2V_0}{\hbar \nu} \sqrt{a^2 - b^2} \right) b \, db \]

\[= -i \mathcal{K} \sum_{m=0}^{\infty} \frac{(-1)^m b^{2m}}{4^m (m!)^2} \int_{0}^{a} b^{2m+1} e^{-ib \gamma} \cos \left( \frac{2V_0}{\hbar \nu} \sqrt{a^2 - b^2} \right) dB \]

By considering the integral *

\[\int_{x}^{\infty} (u^2 - x^2)^{-\nu} \cos(cx) \, dx \]

\[= \frac{\Gamma(\nu + \frac{1}{2})}{2^{\nu} \Gamma(\nu)} \frac{u^{\nu - \frac{1}{2}}}{\Gamma(\nu + \frac{1}{2})} \mathcal{H}_{\nu + \frac{1}{2}}(cu) \]

\[(c > 0, u > 0, \text{Re} \nu > -\frac{1}{2}) \]

where \( \Gamma(\nu + \frac{1}{2}) \) is a gamma function and \( \mathcal{H}_{\nu + \frac{1}{2}}(c) \) is a struve function of order \( \nu + 1 \), we note that (38) can be integrated by using the new integration variable \( x = \frac{a^2 - b^2}{c} \), and letting \( \nu = m + \frac{1}{2} \). Then

\[-i \mathcal{K} \int_{0}^{\infty} J_{\nu}(b \gamma) \cos \left( \frac{2V_0}{\hbar \nu} \sqrt{a^2 - b^2} \right) b \, db \]

\[= -i \mathcal{K} \sum_{m=0}^{\infty} \frac{(-1)^m b^{2m}}{4^m (m!)^2} \int_{0}^{a} (a^2 - x^2)^m \cos \left( \frac{2V_0}{\hbar \nu} x \right) x \, dx \]

\[= -i \mathcal{K} \sum_{m=0}^{\infty} \frac{(-1)^m b^{2m}}{4^m (m!)^2} \left( \frac{a^{2m+1}}{2^{m+1}} - \frac{\nu \pi}{2} \frac{a}{V_0} \left( \frac{a}{V_0} \right)^{\nu + \frac{1}{2}} \Gamma(m+\frac{3}{2}) \mathcal{H}_{m+\frac{3}{2}} \left( \frac{2V_0 a}{\hbar \nu} \right) \right) \]

\[= -i \mathcal{K} \frac{a}{b} \sum_{m=0}^{\infty} \frac{(-1)^m b^{2m}}{4^m (m!)^2} \left( \frac{a}{V_0} \right)^{\nu + \frac{1}{2}} \mathcal{H}_{m+\frac{3}{2}} \left( \frac{2V_0 a}{\hbar \nu} \right) \]

because, according to eq. (36) for \( \nu = 1 \), we have

---

*The integral in Gradshteyn and Ryzhik \(^8\) has an error in it which is corrected here. When traced, it is found that the error is present in the source of the integral Erdelyi, et al. \(^9\). The location and correction of this error is given on p. 90. (See also Appendix B).*
\[
J_z^\gamma = \frac{3\pi}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \frac{3^m}{m!}}{4^m (m+1)!}
\]  

(41)

By combining eqs. (30), (32), (40), another equation (cf. eq. (35)) is obtained for the scattering amplitude \(f_s(\theta)\), i.e.

\[
f_s(\theta) = \frac{k}{\sqrt{2}} \left[ \left( \frac{2V_0}{\hbar \omega_0} \right)^{1/2} \left[ 1 + \left( \frac{2V_0}{\hbar \omega_0} \right)^{1/2} \right] \right] \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \frac{3^m}{m!}}{4^m (m+1)!} \left[ \left( \frac{a_0}{\hbar \omega_0} \right)^{3/2} \right] \right]
\]

(42)

\[
+ i \frac{k}{\sqrt{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \frac{3^m}{m!}}{4^m (m+1)!} \left( \frac{2V_0}{\hbar \omega_0} \right)^{m+1/2} \left( \frac{2V_0}{\hbar \omega_0} \right)
\]

It is interesting to note that if we use the above result in eq. (20), to calculate \(O_5^{\text{tot}}\), we obtain

\[
O_5^{\text{tot}} = 2 \pi^{3/2} a^2 \left( \frac{\hbar \omega_0}{a_0} \right)^{1/2} \left[ \frac{2V_0 a}{\hbar \omega_0} \right]
\]

(43)

By noting that

\[
H_{\gamma}(\gamma) = \left( \frac{3}{2\pi} \right)^{1/2} \left( 1 + \frac{2}{3} \right) - \left( \frac{2}{\pi \gamma} \right)^{1/2} \left( \sin \gamma + \frac{\cos \gamma}{\gamma} \right)
\]

and calling eq. (26), \(O_5^{\text{tot}}\) in eq. (43) can be simplified to give the same result that was obtained in eq. (25).

3) Yukawa Potential

When potential \(V_y\) is used as the pion-nucleon potential, i.e.

\[
V_y = \gamma \frac{e^{-\rho r}}{r}
\]

(45)

we obtain the phase shift function \(\chi_y(\gamma)\) from eq. (2):
\[ \chi_y(b) = -\frac{2\gamma}{\hbar v} \int_0^\infty \frac{\exp\left(-\frac{\mu^2 b^2 + \beta^2}{b^2 - \delta^2}\right)}{(b^2 - \delta^2)^{1/2}} \, db \]  

(46)

Changing the variable of integration to \( x = \sqrt{b^2 + \delta^2} \), we have

\[ \chi_y(b) = -\frac{2\gamma}{\hbar v} \int_b^\infty \frac{\exp\left(-\mu x\right)}{(x^2 - b^2)^{1/2}} \, dx \]

(47)

Using the integral\(^{11}\)

\[ \int_b^\infty (x^2 - u^2)^{-\nu/2} \exp(-\mu x) \, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2 \mu}{\nu} \right)^{-\nu/2} \Gamma(\nu) K_{\nu/2}(u \mu) \]

(48)

( here \( K_{\nu/2}(u \mu) \) is a modified Bessel function of order \( \nu - \frac{1}{2} \) ), setting \( \nu = \frac{1}{2} \) , and noting that

\[ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]

(49)

we get

\[ \chi_y(b) = -\frac{2\gamma}{\hbar v} K_{\nu/2}(b \mu) \]

(50)

The scattering amplitude and the total scattering cross-section for the Yukawa potential are therefore given by (cf. eq. (7) and eq. (21))

\[ f_y(\theta) = -\frac{k}{\lambda} \int_0^\infty J_0(\theta b) \left\{ \exp\left[-i \frac{2\gamma}{\hbar v} \frac{K_{\nu/2}(b \mu)}{b}\right] - 1 \right\} \, db \]

(51)

\[ \sigma_{y,\text{tot}} = 4\pi \int_0^\infty \Re \left\{ b \left[ 1 - \exp\left(-i \frac{2\gamma}{\hbar v} K_{\nu/2}(b \mu)\right) \right] \right\} \, db \]

(52)

Numerical calculation is required in the above.
A point worth noting here is that in the case of Coulomb scattering, if we use an exponential cut-off such as

$$V_c = \frac{2z_2 e}{r} e^{-r/A}$$  \hspace{1cm} (53)

instead of a sharp cut-off as in eq. (3), then the phase-shift function is similar to eq. (50):

$$\chi_c(b) = -\frac{2z_2 e}{\hbar v} K_0\left(\frac{b}{A}\right)$$  \hspace{1cm} (54)

According to the equations

$$K_0\left(\frac{b}{A}\right) = -\left\{ \ln\left(\frac{b}{A}\right) + C \right\} I_0\left(\frac{b}{A}\right) + \left(\frac{1}{2}\right) \frac{(b/A)^2}{(1!)} + \left(1 + \frac{1}{2}\right) \frac{(b/A)^3}{(2!)} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{(b/A)^4}{(3!)} + \ldots$$  \hspace{1cm} (55)

and

$$I_0\left(\frac{b}{A}\right) = 1 + \frac{2b^2}{(1!)} + \frac{(4b^2)^2}{(2!)} + \frac{(4b^2)^3}{(3!)} + \ldots$$  \hspace{1cm} (56)

where $I_0\left(\frac{b}{A}\right)$ is another modified Bessel function and $C$ is the Euler's constant ($C = .57721566491771$), the expansion of eq. (54) in power of $\frac{b}{A}$, with only the first order term in $\frac{b}{A}$ being retained, gives

$$\chi_c(b) = \frac{2z_2 e}{\hbar v} \left\{ \ln\left(\frac{b}{2A}\right) + C \right\}$$  \hspace{1cm} (57)

When this approximation for $\chi_c(b)$ is used to calculate the scattering amplitude $f_c(\theta)$, an extra phase factor of $e^{i2nc}$ to eq. (14) is found. So a change in the cut-off factor from a step-function to an exponential function only results in an additional unobservable phase factor.

4) **Square-well Potential Plus Coulomb Potential**

As we have mentioned before, in nuclear scattering the beam
particles are also affected by the Coulomb field. The interference effect between Coulomb and nuclear forces is most important for small angle scatterings, i.e., in regions where Coulomb and nuclear effects have similar magnitude. To calculate the scattering amplitudes and cross-sections for such situation using the Glauber formalism requires the superposition of phase-shift functions for the nuclear and Coulomb potentials, i.e.,

$$\chi(b) = \chi_{\text{Nuclear}}(b) + \chi_{\text{Coulomb}}(b). \quad (58)$$

This procedure takes proper account of each of the two kinds of scattering and their interference.\(^{(1)}\) From eq. (2), addition of phase-shift function is seem to be equivalent to the assumption that the total scattering potential is equal to the sum of Coulomb and nuclear potentials.

(a) **Total Cross-section**

With the strong potential represented by the square well potential, the total scattering cross-section \(\sigma^{\text{tot}}\) obtained from eq. (21) is

$$\sigma^{\text{tot}} = 4\pi \int_0^\infty \left(1 - \text{Re} \ e^{-i\chi(b)} \right) b \ dB \quad (59)$$

$$= 4\pi \int_0^\infty \left(1 - \text{Re} \ e^{-i\chi(b)} \right) b \ dB + 4\pi \text{Re} \int_0^\infty e^{-i\chi(w)} (1 - e^{-i\chi(w)}) b \ dB.$$  

Therefore, we obtain for \(\sigma^{\text{tot}'}\), the total cross-section minus the effects of pure Coulomb scattering

$$\sigma^{\text{tot}'} = \sigma^{\text{tot}} - \sigma_e = 4\pi \text{Re} \int_0^\infty e^{-i\chi(b)} (1 - e^{-i\chi(b)}) b \ dB. \quad (60)$$
Since
\[
\left\{ 1 - \exp \left[ -i \chi_c(b) \right] \right\} = 0 \quad b > a
\]  \hspace{1cm} (61)

where \( a \) is the width of the square well, we can write
\[
\alpha_{\text{tot}}' = 4 \pi \Re \int_0^a e^{i \chi_c(b)} \left[ 1 - e^{i \chi_c(b)} \right] b \, db
\]  \hspace{1cm} (62)

It is obvious that
\[
\frac{b}{A} \leq 10^{-5} \quad b < a
\]  \hspace{1cm} (63)

because \( a \) is of nuclear dimensions whereas \( A \), the cut-off distance for the Coulomb potential is of atomic dimensions (eq. (3)). Therefore, it is a very good approximation to keep only the first term in eq. (13) so that the phase-shift function for the Coulomb potential is
\[
\chi_c(b) = 2 \pi \ln \left( \frac{b}{2A} \right) \quad b < a
\]  \hspace{1cm} (64)

Therefore,
\[
e^{i \chi_c(b)} = \left( \frac{b}{2A} \right)^{2i\pi n} \quad b < a
\]  \hspace{1cm} (65)

By putting eqs (19) and (65) into eq. (62) we obtain
\[
\alpha_{\text{tot}}' = 4 \pi \Re \left\{ \left( \frac{1}{2A} \right)^{2i\pi n} \left( I_1 - I_2 \right) \right\}
\]  \hspace{1cm} (66)

with
\[
I_1 = \int_0^a b^{1+2i\pi n} \, db
\]  \hspace{1cm} (67)

and
\[
I_2 = \int_0^a b^{1+2i\pi n} \exp \left( i \frac{2V_0}{\hbar^2} \sqrt{a^2 - b^2} \right) \, db
\]  \hspace{1cm} (68)
\( I_1 \) can easily be integrated to give
\[
I_1 = \frac{d^{2(1+i\nu)}}{2(1+i\nu)} \tag{69}
\]

To evaluate \( I_2 \), we make a change of integration variable to \( X = \sqrt{a^2 + b^2} \) and use the integral \(^{(14)}\)
\[
\int_0^\infty x (u^2 - x^2)^{\nu-1} e^{\mu x} \, dx = \frac{\nu}{2\nu} + \frac{i\pi}{2} \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \Gamma(\nu) \left[ \int \gamma_{\nu+\frac{1}{2}}(\mu x) + L_{\nu+\frac{1}{2}}(\mu x) \right] \tag{70}
\]
where \( \int \gamma_{\nu+\frac{1}{2}}(\mu x) \) and \( L_{\nu+\frac{1}{2}}(\mu x) \) are a modified Bessel function and a modified Struve function, respectively. With \( \nu = 1 + i\eta \), the result is
\[
I_2 = \int_0^\infty (a^2 - X)^{i\eta} X \exp\left(i \frac{2\sqrt{a}}{\eta \nu} X\right) \, dX
\]
\[
= \frac{d^{2(1+i\nu)}}{2(1+i\nu)} + \frac{i\pi}{2} \left(\frac{\nu}{\eta \nu}\right)^{\frac{1}{2}} \Gamma(1+i\nu) \left[ \int \frac{d^{2+i\eta}}{2+i\eta} \left(\frac{2\sqrt{a}}{\eta \nu}\right) \right] \tag{71}
\]

Using the relationships \(^{(15)}\)
\[
\int_\nu (z) = \exp\left(-\frac{i}{2} \pi \nu i\right) J_\nu \left(e^{i\frac{\pi}{2} \frac{z}{\nu}}\right) \quad [-\pi < \arg z \leq \pi] \tag{72}
\]
and
\[
L_\nu (z) = -i \exp\left(-\frac{i}{2} \pi \nu i\right) H_\nu \left(e^{i\frac{\pi}{2} \frac{z}{\nu}}\right) \tag{73}
\]
we obtain

\[ I_{3+in}(2i\lambda) = \left[ I_{3-in}(-2i\lambda) \right]^* = \left[ \exp\left( -\frac{3}{4} \pi i \right) \exp(-\frac{\pi i n}{2}) J_{\frac{3}{2} - in}(2\lambda) \right]^* \]

and

\[ L_{3+in}(2i\lambda) = \left[ L_{3-in}(-2i\lambda) \right]^* = \left[ \exp\left( \frac{3}{4} \pi i \right) \exp(-\frac{\pi i n}{2}) H_{\frac{3}{2} + in}(2\lambda) \right]^* \]

where \( \lambda \) is given in eq. (26). By noting that the factor \((-\lambda)^{-\frac{1}{2} - in}\)

in eq. (69) is

\[ (-\lambda)^{-\frac{1}{2} - in} = \left( e^{i\frac{\pi}{2}} \right)^{-\frac{1}{2} - in} = e^{-\frac{\pi i}{2}} e^{i\frac{\pi i n}{2}} \]

and by also substituting eqs. (74), (75) into eq. (71) gives

\[ I_2 = \frac{\alpha^{2(n+in)}}{2(n+in)} - \frac{4\pi}{2} \left( \frac{V_0}{\hbar^2} \right)^{\frac{1}{2} - in} \alpha^{\frac{3}{2} + in} \Gamma(1+in) \left\{ H_{\frac{3}{2} + in}(2\lambda) - \frac{1}{2} \int_{\lambda}^{\infty} (a^2 - X^2)^{\frac{3}{2} + in} \sin(2\lambda X) X dX \right\} \]

As a confirmation of eq. (77), we calculate \( I_2 \) using other integral formulae. \( I_2 \) in eq. (71) is written as

\[ I_2 = \int_0^\alpha (a^2 - X^2)^{\frac{3}{2} + in} \cos(2\lambda X) dX + i \int_0^\alpha (a^2 - X^2)^{\frac{3}{2} + in} \sin(2\lambda X) X dX \]

Then, we use the integrals as given in eq. (39) and (16)

\[ \int_0^\alpha (u^2 - x^2)^{\frac{3}{2} + in} \sin(ax) dx \]

\[ = \frac{\sqrt{\pi}}{2} \left( \frac{2u}{a} \right)^\nu \Gamma(\nu + \frac{1}{2}) \int_{a+1}(au) \left[ a > 0, \ u > 0, \ Re \nu > -\frac{1}{2} \right] \]

An error has been found in the formula given by Gradshteyn and Ryzhik (see footnote on p. 43) on comparison with the result in eq. (77).
This formula is corrected by a derivation from a more general formula. This is done in Appendix B. The corrected result is given as eq. (39). It is easily seen that the above when used in combination with eq. (79) in eq. (78) gives exact agreement with eq. (77).

The first term in eq. (77) exactly cancels the contributions from $I_1$ in eq. (69), when they are put into eq. (66). The result for the total cross-section with pure Coulomb effects subtracted is

$$
\sigma^{\text{tot}} = \frac{\pi}{2} a^2 \text{Re} \left\{ \left( \frac{a}{2A} \right)^{2n} \Gamma(1+i\eta) \left[ \frac{H_{\frac{3}{2}+i\eta}(2\alpha) - i \frac{J_{\frac{3}{2}+i\eta}(2\alpha)}{2} }{2} \right] \right\}.
$$

(80)

When $\eta$ is equated to zero, the above result correctly reduces to eq. (25) (cf eq. s(43) and (44)) for the case of square well potential only. We note here that $\eta$ can be positive or negative depending on whether the incident particle has positive or negative charge.

b) Differential Cross-section

To calculate the scattering amplitude for the strong potential plus a screened Coulomb potential, we use the formula

$$
\mathcal{f}(\varphi) = \frac{k}{i} \int_0^\infty J_0(b \eta) \left\{ e^{i(x_0 \varphi + x_c \eta)} - 1 \right\} b \, db.
$$

(81)

Following what was done in the case of total cross-section, the pure Coulomb scattering amplitude is separated so that

$$
\mathcal{f}_{c}(\varphi) = \frac{k}{i} \int_0^\infty J_0(b \eta) \left( e^{i x_0 \varphi} - 1 \right) b \, db + \frac{k}{i} \int_0^\infty J_0(b \eta) e^{i x_c \eta} \left( e^{i x_0 \varphi} - 1 \right) b \, db.
$$

(82)

The scattering amplitude minus the pure Coulomb amplitude $\mathcal{f}_{c}(\varphi)$, is thus

$$
\mathcal{f}'(\varphi) = \mathcal{f}(\varphi) - \mathcal{f}_{c}(\varphi) = \frac{k}{i} \int_0^\infty J_0(b \eta) e^{i x_c \eta} \left( e^{i x_0 \varphi} - 1 \right) b \, db.
$$

(83)
Using eq. (19) and the same argument as on p. 48, we can approximate \( e^{iX(b)} \) by eq. (65), and therefore
\[
\hat{f}(\theta) = \frac{k}{i} \left( \frac{1}{2\hbar} \right)^{2i\theta} \int_{0}^{a} J_0(b \delta) b^{1+2i\theta} \left( \exp\left(-\frac{2V_0}{\hbar v} \sqrt{a^2 - b^2}\right) - 1\right) db. \tag{84}
\]
This can be separated into three integrals
\[
\hat{f}(\theta) = \frac{k}{i} \left( \frac{1}{2\hbar} \right)^{2i\theta} \left\{ N_1 + N_2 + N_3 \right\}, \tag{85}
\]
where
\[
N_1 = -\int_{0}^{a} J_0(b \delta) b^{1+2i\theta} db, \tag{86}
\]
\[
N_2 = -i\int_{0}^{a} J_0(b \delta) b^{1+2i\theta} \sin\left(\frac{2V_0}{\hbar v} \sqrt{a^2 - b^2}\right) db, \tag{87}
\]
and
\[
N_3 = \int_{0}^{a} J_0(b \delta) b^{1+2i\theta} \cos\left(\frac{2V_0}{\hbar v} \sqrt{a^2 - b^2}\right) db. \tag{88}
\]
Although the above integrals are quite similar to those in eq. (28) except for a factor of \( b^{2i\theta} \) in the integrand, the difference is enough so that eqs. (29), (31) and (33) cannot be used. To calculate \( N_1 \), we expand \( J_0(b \delta) \) in series according to eq. (37):
\[
N_1 = -\frac{2^{-2-2i\theta}}{b^2} \int_{0}^{a/b} J_0(b \delta) \left( b \delta \right)^{1+2i\theta} db \delta
\]
\[
= -\frac{2^{-2-2i\theta}}{b^2} \sum_{m=0}^{\infty} \frac{(-1)^m (a \delta)^{2m+1+2i\theta}}{(2m+1)!} \int_{0}^{a/b} x^{2m+1+2i\theta} dx \tag{89}
\]
\[
= -\frac{2^{-2-2i\theta}}{b^2} \sum_{m=0}^{\infty} \frac{(-1)^m (a \delta)^{2m+2+2i\theta}}{(2m+1)! (m+1+2i\theta)^3}.
\]
Using the series expansion for the Bessel function, $J_0(z)$ in

\[ N_2 = i \sum_{m=0}^{\infty} \left( \frac{\theta}{4m} \right) \int_0^\alpha b^{2m+2i\nu} \sin \left( \frac{2V_0}{\hbar \nu} a^2 - b^2 \right) db, \tag{90} \]

and, substituting $X = \sqrt{a^2 - b^2}$, we have

\[ N_2 = i \sum_{m=0}^{\infty} \left( \frac{\theta}{4m} \right) \int_0^\alpha X^{m+2i\nu} \sin \left( \frac{2V_0}{\hbar \nu} X \right) dX \tag{91} \]

By noting the integral (16)

\[ \int_0^\pi x (u^2 - x^2)^{\nu-\frac{1}{2}} \sin (cu) dx = \frac{\sqrt{\pi}}{2} u \left( \frac{2u}{c} \right)^{\nu} \Gamma(\nu + \frac{1}{2}) J_{\nu+1}(cu) \tag{92} \]

we obtain

\[ N_2 = i 2^{-\frac{1}{2}+i\nu} \pi \left( \frac{\theta}{2V_0} \right)^{\nu} \sum_{m=0}^{\infty} \left( \frac{\theta}{2m} \right)^{2m} a^m \left( \frac{\theta}{2V_0} \right)^m \Gamma(m+1+i\nu) J_{m+\frac{1}{2}+i\nu}(2V_0a) \tag{93} \]

with $\nu = m + \frac{1}{2} + i\nu$. For the case $n = 0$, the above will correctly reduce back to eq. (30) with appropriate factors by means of the following multiplication theorem for Bessel functions (17):

\[ J_\nu (a \beta) = \beta^\nu \sum_{m=0}^{\infty} \left( \frac{-1}{m!} (a^2 - 1)^m \right) \frac{\beta^m}{m!} J_{\nu+m}(\beta) \tag{94} \]

Unfortunately, when $n$ is non-zero the in term in the argument of the Gamma function $\Gamma(m+1+i\nu)$ prevents a similar reduction for eq. (93).

In a manner similar to $N_2$ (eq. (88)), $N_3$ is found to be

\[ N_3 = \sum_{m=0}^{\infty} \left( \frac{\theta}{4m} \right)^{2m} \int_0^\alpha X^{m+2i\nu} \cos \left( \frac{2V_0}{\hbar \nu} X \right) dX \tag{95} \]
Applying eq. (39), with \[ \nu = m + \frac{1}{2} + i\eta \], we have

\[ N_3 = \sum_{m=0}^{\infty} \frac{(-1)^m \frac{2m}{\eta} a^{2m+2+2i\eta}}{\nu (2m!)^2} \sum_{m=0}^{\infty} \left( \frac{-i}{\eta} \right)^{2m} a^{m} \left( \frac{\eta}{2\nu} \right)^{m} \Gamma(m+1+i\eta) \Gamma(m+\frac{3}{2}+i\eta) \left( \frac{2V\alpha}{\hbar \nu} \right). \tag{96} \]

We note that after rearrangement the first term in eq. (96) is equal to \( N_1 \) in eq. (89) with a opposite sign. Therefore, they cancel each other out when eqs (89), (93) and (96), are substituted into eq. (90). The scattering amplitude for a square-well potential plus a Coulomb potential, with the pure Coulomb amplitude subtracted out, is then found to be

\[ f'(\theta) = i \kappa \left( \frac{1}{2A} \right)^{2i\eta} \frac{a^{\nu}}{\nu} \sum_{m=0}^{\infty} \left( \frac{-i}{\eta} \right)^{2m} a^{m} \left( \frac{\eta}{2\nu} \right)^{m} \Gamma(m+1+i\eta) \Gamma(m+\frac{3}{2}+i\eta) \left( \frac{2V\alpha}{\hbar \nu} \right) \left[ \eta \right] (m+\frac{3}{2}+i\eta) \left( \frac{\eta}{2\nu} \right)^{m} \right]. \tag{97} \]

We can remove the phase factor \[ \left( \frac{1}{2A} \right)^{2i\eta} = \exp (-2i\eta \ln 2A) \] by defining a new amplitude \( \tilde{f}'(\theta) \) such that

\[ \tilde{f}'(\theta) = f'(\theta) \exp (2i\eta \ln 2A). \tag{98} \]

With \( \tilde{f}(\theta) \) and \( \tilde{f}_{c}(\theta) \) similarly defined corresponding from \( f(\theta) \) and \( f_{c}(\theta) \), we obtain, from eq. (16)

\[ \tilde{f}_{c}(\theta) = -\frac{2\kappa n}{\hbar^2} \exp \left\{ -i \left[ 2n \ln (\kappa \sin \frac{\theta}{2}) - 2 \arg \Gamma(1+i\eta) \right] \right\}. \tag{99} \]
and, from eq. (81)
\[ \tilde{f}(\phi) = \tilde{f}(\phi) - \tilde{f}_c(\phi). \]  
(100)

Since both \( \tilde{f}(\phi) \) and \( f(\phi) \) give the same differential scattering cross-section from the relation
\[ \frac{d\sigma}{d\Omega} = |\tilde{f}(\phi)|^2 = |\tilde{f}(\phi)|^2, \]
(101)
the newly defined amplitudes \( \tilde{f}(\phi), \tilde{f}_c(\phi) \) and \( \tilde{f}(\phi) \) can be used consistently with the indicated phase factor removed. Therefore, the new scattering amplitude \( \tilde{f}(\phi) \) for square-well potential plus the Coulomb potential, with the pure Coulomb amplitude \( \tilde{f}_c(\phi) \) subtracted out is
\[ \tilde{f}(\phi) = i \Phi 2^{-\frac{1}{2}} \gamma^n \left( \frac{a \hbar \nu}{2 V_0} \right)^{\frac{1}{2} + i n} \sum_{m=0}^{\infty} \frac{(-1)^m a^m}{2^m (m!)^2} \times \left( \frac{\hbar \nu}{2 V_0} \right)^m \left\{ \frac{2 V_0 a}{\hbar \nu} \right\}^{m+\frac{1}{2} + i n} \int \frac{2 V_0 a}{\hbar \nu} \right\} - i \int \frac{2 V_0 a}{\hbar \nu} \right\}. \]
(102)

5) **Yukawa Potential Plus Coulomb Potential**

When a Yukawa potential is used for the pion-nucleon potential, and the Coulomb potential, is included, \( O_{\text{tot}}' \), the total cross-section with pure Coulomb subtracted, is (cf. eq. (60))
\[ O_{\text{tot}}' = 4 \pi \int e^{i \chi_{\phi}^{(b)}} \left[ 1 - e^{i \chi_{\phi}^{(b)}} \right] b dB. \]
(103)
From eq. (50), $\chi_y(b)$ is proportional to the modified Bessel function $K_\nu(\mu b)$, where $\left(\frac{1}{\mu^2}\right)$ is of nuclear dimensions. From the asymptotic expansion

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \left\{ 1 + \frac{\mu - 1}{8z} + \frac{(\mu - 1)(\mu - 3)}{24 \cdot 8z^2} + \cdots \right\} \quad (104)$$

with

$$\mu = 4\frac{\nu^2}{\pi} \quad \left( \log \frac{3}{2} < \frac{3}{2} \pi \right)$$

we see that $K_\nu(\mu b)$ decreases so fast that at $b \gg \frac{1}{\mu}$, the integrand in eq. (103) is practically zero. Therefore, it is a good approximation as in the case of square well potential plus Coulomb potential to use eq. (64) for the phase-shift function $\chi_\nu(b)$ instead of eq. (13).

Then

$$Z_{tot} = 4\pi \Re \left\{ \left(\frac{1}{2A}\right)^{2i\nu} \int_0^\infty b^{1+2i\nu} \left( 1 - \exp[-i\frac{2\pi}{\nu} K_\nu(\mu b)] \right) db \right\} \quad (105)$$

Similarly, the scattering amplitude with pure Coulomb amplitude subtracted is

$$f'(\theta) = iK \left(\frac{1}{2A}\right)^{2i\nu} \int_0^\infty J_\nu(\rho b) b^{1+2i\nu} \left\{ 1 - \exp[-i\frac{2\pi}{\nu} K_\nu(\mu b)] \right\} db \quad (106)$$

After removing the phase factor $\left(\frac{1}{2A}\right)^{2i\nu}$, following the same reasoning in the last section, we obtain

$$\overline{f}'(\theta) = iK \int_0^\infty J_\nu(\rho b) b^{1+2i\nu} \left\{ 1 - \exp[-i\frac{2\pi}{\nu} K_\nu(\mu b)] \right\} db \quad (107)$$

Eq. (105) and (106) (or (107)) differs from eqs. (51) and (52) by a factor of $b^{2i\nu}$ in the integrand and numerical calculation is once more required. In calculating the scattering cross-section,
we have to be careful to use the full scattering amplitude (Coulomb part not subtracted) in eq. (101) and that is what we imply to do even when we may simply refer to getting the differential scattering cross-section from the amplitude with pure Coulomb part subtracted.

One may be distributed to note that the differential cross-section obtained from eq. (106) or eq. (107) is not dependent on the Coulomb cut-off, $A$, whereas the total scattering cross-section en eq. (105) is cut-off dependent. This should not be the case, because when we integrate the differential cross-section over all solid angles, the result is the total scattering cross-section (remembering that there is only elastic scattering when the potential is real). In fact, the following Lemma concerning the argument and modulus of the differential scattering amplitude can be obtained from the optical theorem in the case of a real potential:

**Lemma 1:** The modulus $|f(\omega)|$ of the differential scattering amplitude $f(\omega) = |f(\omega)| e^{i\phi}$ is a function of all the parameters that appear in the argument $\phi$.

The optical theorem for a real potential is

$$\int_{\text{all angle}} |f(\omega)|^2 d\Omega = \frac{4\pi}{k} \text{Im} f(\theta = 0) \tag{108}$$

Now it is simple to see that the above Lemma is true because if $f(\omega)$ contains a parameters that is not in $|f(\omega)|$, then we will find the absurd situation when $f(\omega)$ is put into eq. (108), that the R.H.S. is
dependent on that parameter while the L.H.S. is not.

Therefore, both the differential scattering amplitudes obtained for both the case of a square-well potential plus a Coulomb potential (eq. (97) and the case of a Yukawa potential plus Coulomb potential (eq. (106)) violate the above Lemma. These violations come from the approximation used for the Coulomb phase-shift function \( \chi_c \) in eq. (64), which leads to eq. (65), i.e. 
\[
 e^{i \chi_{\text{C}}(b)} = \left( \frac{b}{2A} \right)^{2\text{in}} .
\]
However, we have argued that the approximation was an excellent one. This is substantiated when we look at the phase factor \( \left( \frac{1}{2A} \right)^{2\text{in}} \), which is retained in the equations for the total cross-sections (eqs. (80) and (105)). \( \left( \frac{1}{2A} \right)^{2\text{in}} \) can be estimated to be

\[
 \left( \frac{1}{2A} \right)^{2\text{in}} \approx 0.983 \mp 0.184 i \quad \left( \begin{array}{c}
 n = \pm 8 \times 10^{-3} \\
 A = 0.529 \times 10^5 \text{fm}
\end{array} \right),
\]

(109)

where \( A \) is assumed to be the Bohr radius \( (0.529 \times 10^5 \text{ fm.}) \) and \( n \) is obtained from Chapter 5. We next note that \( \left( \frac{1}{2A} \right)^{2\text{in}} \) varies very little over a wide range values of \( A \). For example it only increases by about 2% when \( A \) is increased from one Bohr radius to 10 times Bohr radius. Therefore, the violation of Lemma 1 by the scattering amplitudes under discussion is not serious.

6) **Spin and Isospin**

For pion-nucleon scattering, the scattering amplitude \( F(\theta) \) can be given in terms of two amplitudes \( f(\theta) \) and \( g(\theta) \) such that

\[
 F(\theta) = f(\theta) + i g(\theta) \mathbf{k}_i \times \mathbf{k}_f ,
\]

(110)
where \( k_i \) and \( k_f \) are unit vectors in the direction of the initial and final momenta respectively of the pion, \( k_i \times k_f \) is normal to the plane of scattering, and \( \mathbf{Q} \) is the vector form of the Pauli matrices. Therefore, \( F(\theta) \) is a 2 x 2 matrix in spin space. The amplitude \( f(\theta) \) is called the non-spin flip amplitude because it does not change the spin state of the initial pion-nucleon state and \( g(\theta) \) is called the spin flip amplitude because it changes the initial pion-nucleon state to a final state with opposite spin.

To calculate the differential cross-section for an unpolarized target, we need to sum over all the possible incoming \( S_i \) and outgoing spin states \( S_f \) and divide by the total number of possible states, i.e.

\[
\frac{d\sigma(\theta)}{d\Omega} = \frac{1}{2} \sum \left| \langle S_f | f(\theta) | S_i \rangle \right|^2 .
\]  
(111)

By using eqs. (109), and noting that

\[
\hat{k}_i \times \hat{k}_f = \sin \theta \, \hat{n} ,
\]  
(112)

where \( \hat{n} \) is a unit vector normal to the scattering plane, and \( \Theta \) is the scattering angle, and by also using the equation (20)

\[
(\mathbf{Q} \cdot \mathbf{A}) (\mathbf{Q} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i \mathbf{Q} \cdot (\mathbf{A} \times \mathbf{B})
\]  
(113)

the differential cross-section can be written in the form

\[
\frac{d\sigma(\theta)}{d\Omega} = \frac{1}{2} \text{Tr} \left\{ (f(\theta))^2 - i f(\theta) g(\theta) \mathbf{Q} \cdot \hat{n} (f(\theta) + i g(\theta) \sin \theta \, \mathbf{Q} \cdot \hat{n}) \right\}
\]  
\[
= \frac{1}{2} \text{Tr} \left\{ (f(\theta))^2 I + i \left[ f(\theta) g(\theta) - f(\theta) g(\theta) \sin \Theta \, \mathbf{Q} \cdot \hat{n} + \sin^2 \Theta \, g(\theta) \right] (\mathbf{Q} \cdot \hat{n}) I \right\}
\]  
(114)
where $I$ is the $2 \times 2$ identity matrix. Since the trace of the identity element $I$ is equal to 2 and those of the Pauli matrices are equal to zero, i.e.,

$$\text{Tr} \, I = 2, \quad \text{Tr} \, \sigma_i = \text{Tr} \, \sigma_2 = \text{Tr} \, \sigma_3 = 0$$

the differential cross-section becomes

$$\frac{d \sigma(\theta)}{d \Omega} = \left| f(\theta) \right|^2 + \sin^2 \theta \left| g(\theta) \right|^2.$$  

Since we are only concerned with small angle scattering, we shall ignore the contribution from the spin-flip amplitudes.

We next consider the effect of isospin. The \( \pi \)-\( N \) scattering processes which also involve Coulomb scattering are \( \pi^+p \) and \( \pi^-p \) scattering. Since \( \pi^+p \) is a pure \( T \) (isospin) = $\frac{3}{2}$ state it involves only the single channel \( \pi^+p \rightarrow \pi^+p \), whereas for \( \pi^-p \) there are two possible final channels due to isospin:

\[
\pi^-p \rightarrow \pi^+n \quad \text{and} \quad \pi^-p \rightarrow n^+ + n .
\]

We first remember that the scattering states \( |\pi^-p\rangle \) and \( |n^0n\rangle \) can be given in terms of total isospin states such that

\[
|\pi^-p\rangle = \frac{1}{\sqrt{3}} | \frac{3}{2}, -\frac{1}{2} \rangle - \frac{1}{\sqrt{3}} | \frac{1}{2}, \frac{1}{2} \rangle ,
\]

\[
|n^0n\rangle = \frac{1}{\sqrt{3}} | \frac{3}{2}, -\frac{1}{2} \rangle + \frac{1}{\sqrt{3}} | \frac{1}{2}, \frac{1}{2} \rangle .
\]

From the above equations, the transformation matrix $U$ from total isospin space to \( \pi \)-\( N \) particle space is

\[
U \approx \frac{1}{\sqrt{3}} \left( \begin{array}{cc} -\frac{\sqrt{2}}{2} & 1 \\ \frac{1}{2} & \frac{\sqrt{2}}{2} \end{array} \right) ;
\]

i.e.
\[
\left(\begin{array}{c}
|\pi^- p\rangle \\
|\pi^0 n\rangle
\end{array}\right) = \bigcup \left(\begin{array}{c}
|\frac{1}{2}, -\frac{1}{2}\rangle \\
|\frac{3}{2}, -\frac{1}{2}\rangle
\end{array}\right)
\]

(119)

Since
\[
\frac{\hat{L}^2}{\hbar^2} = 1,
\]

(120)
eq (119) can be inverted to give
\[
\left(\begin{array}{c}
|\frac{1}{2}, -\frac{1}{2}\rangle \\
|\frac{3}{2}, -\frac{1}{2}\rangle
\end{array}\right) = \bigcup \left(\begin{array}{c}
|\pi^- p\rangle \\
|\pi^0 n\rangle
\end{array}\right).
\]

(121)

In total isospin space, if we let the nuclear potential be \(V_1\) and \(V_3\) for the total isospin states with \(T = \frac{1}{2}\) and \(T = \frac{3}{2}\) respectively, the Schrödinger equation can be written in the form
\[
(\nabla^2 + \hat{k}^2) \left(\begin{array}{c}
|\frac{1}{2}, -\frac{1}{2}\rangle \\
|\frac{3}{2}, -\frac{1}{2}\rangle
\end{array}\right) = \frac{2m}{\hbar^2} \left(\begin{array}{cc}
V_1 & 0 \\
0 & V_3
\end{array}\right) \left(\begin{array}{c}
|\frac{1}{2}, -\frac{1}{2}\rangle \\
|\frac{3}{2}, -\frac{1}{2}\rangle
\end{array}\right).
\]

(122)

Using eqs. (118)-(121), the above equation can be rewritten for the \(\pi-N\) particle space as
\[
(\nabla^2 + \hat{k}^2) \left(\begin{array}{c}
|\pi^- p\rangle \\
|\pi^0 n\rangle
\end{array}\right) = \frac{2m}{3\hbar^2} \left(\begin{array}{cc}
\tilde{m} & 1 \\
1 & \tilde{m}
\end{array}\right) \left(\begin{array}{cc}
V_1 & 0 \\
0 & V_3
\end{array}\right) \left(\begin{array}{c}
|\pi^- p\rangle \\
|\pi^0 n\rangle
\end{array}\right)
\]
\[
= \frac{2m}{3\hbar^2} \left(\begin{array}{cc}
2V_1 + V_3 & \sqrt{2} (V_3 - V_1) \\
\sqrt{2} (V_3 - V_1) & V_1 + 2V_3
\end{array}\right) \left(\begin{array}{c}
|\pi^- p\rangle \\
|\pi^0 n\rangle
\end{array}\right).
\]

(123)

Now we are ready to incorporate the Coulomb potential, which in particle space is \(22\)
\[ \nabla_c \approx \left( \begin{array}{cc} V_c & 0 \\ 0 & 0 \end{array} \right). \]  

(124)

Therefore, the total combined potential is

\[ \nabla_{\approx}^{\text{tot}} = \frac{1}{3} \begin{pmatrix} 2V_1 + V_3 + 3V_c & \sqrt{2} (V_3 - V_1) \\ \sqrt{2} (V_3 - V_1) & V_1 + 2V_3 \end{pmatrix}. \]  

(125)

It is well known that around the (3, 3) resonance region the contribution from \( V_1 \) is small in comparison with that from \( V_3 \). We, therefore, set

\[ V_1 = 0, \]  

(126)

and \( \nabla_{\approx}^{\text{tot}} \) becomes

\[ \nabla_{\approx}^{\text{tot}} = \frac{1}{3} \begin{pmatrix} V_3 + 3V_c & \sqrt{2} V_3 \\ \sqrt{2} V_3 & 2V_3 \end{pmatrix}. \]  

(127)

This can be applied to calculate the 2 x 2 scattering amplitude matrix \( \vec{F}(\theta) \) using the Glauber formalism such that

\[ \vec{F}(\theta) = \frac{R}{i} \int_0^\infty J_0(qb) \left\{ \exp\left[ i \chi_q(b) \right] - 1 \right\} \, db \, db, \]  

(128)

where \( \chi_q(b) \) is given as

\[ \chi_q(b) = -\frac{i}{\pi \nu} \int_0^\infty \frac{\gamma(b + \frac{3\bar{\gamma}}{2}) \, db}{\bar{\gamma}}. \]  

(129)

From eq. (127), \( \chi_q(b) \) is obtained as

\[ \chi_q(b) = \frac{1}{3} \begin{pmatrix} \chi_3(b) + 3\chi_7(b) & \sqrt{2} \chi_3(b) \\ \sqrt{2} \chi_3(b) & 2 \chi_3(b) \end{pmatrix}. \]  

(130)
where $\chi_3(b)$ and $\chi_c(b)$ are one-dimensional phase-shift functions obtained from $V_3$ and $V_c$ respectively. We can use the square-well or the Yukawa phase-shift function for $X_3(b)$.

Using the $2 \times 2$ identity matrix $I$, and the Pauli matrices,

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(131)

$\chi(b)$ can be written as

\[
\chi(b) = \frac{1}{2} \left( \chi_3(b) + \chi_c(b) \right) I + \frac{\mu}{3} \chi_3(b) \sigma_i + \left( \frac{1}{6} \chi_3(b) + \chi_c(b) \right) \sigma_3
\]

or

\[
\chi(b) = \frac{1}{2} \left( \chi_3(b) + \chi_c(b) \right) I + \lambda \hat{n} \cdot \vec{a},
\]

(132)

(133)

where

\[
\hat{n} = \frac{\frac{\mu}{3} \chi_3(b) \hat{e}_i + \frac{1}{6} (3 \chi_c(b) - \chi_3(b)) \hat{e}_3}{\lambda}
\]

(134)

and

\[
\mathcal{L}_z = \left( \frac{\mu}{3} \chi_3(b) \right)^2 + \left[ \frac{1}{6} (3 \chi_c(b) - \chi_3(b)) \right]^2
\]

\[
= \frac{1}{12} \left( 3 \chi_3^2(b) - 2 \chi_3(b) \chi_c(b) + 3 \chi_c^2(b) \right).
\]

(135)

The matrix function $\exp(i \chi)$ can then be calculated from eq. (133) as

\[
\exp[i \chi] = \exp\left[i \frac{1}{2} \left( \chi_3 + \chi_c \right) \right] \exp\left[i \lambda \hat{n} \cdot \vec{a} \right].
\]

(136)

Using the equation

\[
\exp(i \lambda \hat{n} \cdot \vec{a}) = \cos \lambda + i \hat{n} \cdot \vec{a} \sin \lambda
\]

(137)

we have

\[
\exp[i \chi] = \exp\left[i \frac{1}{2} (\chi_3 + \chi_c) \right] (\cos \lambda + i \hat{n} \cdot \vec{a} \sin \lambda).
\]

(138)
or explicitly

\[
\exp \left[ i \frac{\chi_b}{\lambda} \right] = \exp \left[ i \left( \frac{\chi_g}{\lambda} + \chi_e/\lambda \right) \right] \begin{pmatrix}
\cos \theta + i \frac{3 \chi_g \chi_e}{6 \lambda} \sin \theta & i \frac{\chi_g}{\lambda} \sin \theta \\
\frac{i \chi_g}{\lambda} \sin \theta & \cos \theta - i \frac{3 \chi_g \chi_e}{6 \lambda} \sin \theta
\end{pmatrix}
\]  \tag{139}

The above can be used to calculate the 2 x 2 scattering amplitude matrix \( \mathbf{F}(\theta) \) from which scattering matrices for the two \( n^-p \rightarrow n^-p \) scattering channels can be evaluated. For \( n^-+p \rightarrow n^-+p \), we have

\[
\langle n^-p | \mathbf{F}(\theta) | n^-p \rangle = \frac{k}{\lambda} \int \mathcal{J}(g \delta) \left\{ e^{i \frac{\chi_g + \chi_e}{\lambda}} \left[ \cos \theta + i \frac{3 \chi_g \chi_e}{6 \lambda} \sin \theta \right] - 1 \right\} b \, d \, b \tag{140}
\]

And, for \( n^-+p \rightarrow n^0+n \), we have

\[
\langle n^-n | \mathbf{F}(\theta) | n^-p \rangle = \frac{\sqrt{3} k}{\lambda} \int \mathcal{J}(g \delta) \left\{ e^{i \frac{\chi_g + \chi_e}{\lambda}} \frac{\chi_e(\delta)}{\lambda} - 1 \right\} b \, d \, b \tag{141}
\]

Using the optical theorem, the corresponding equations for the total scattering cross-section are

\[
\sigma^{tot} (n^-p \rightarrow n^-p) = 4 \pi \int_0^\infty \left\{ 1 - e^{i \frac{\chi_g + \chi_e}{\lambda}} \left[ \cos \theta + i \frac{3 \chi_g \chi_e}{6 \lambda} \sin \theta \right] \right\} b \, d \, b \tag{142}
\]

and

\[
\sigma^{tot} (n^-p \rightarrow n^0n) = \frac{4 \pi \sqrt{3}}{3} \int_0^\infty \left\{ 1 - e^{i \frac{\chi_g + \chi_e}{\lambda}} \frac{\chi_e(\delta)}{\lambda} \right\} b \, d \, b \tag{143}
\]


(4) Gradshteyn and Ryzhik, P. 761, eq. 6.738-1.


(6) Gradshteyn and Ryzhik, p. 688, eq. 6.567-1.


(8) Gradshteyn and Ryzhik, p. 427, eq. 3.771-12.


(10) Abramowitz and Stegun, p. 497, eq. 12.1.17.


(12) Abramowitz and Stegun, p. 255, eq. 6.1.8.

(13) Abramowitz and Stegun, p. 375, eq. 9.6.12 and eq. 9.6.12.

(14) Gradshteyn and Ryzhik, p. 323, eq. 3.389-3.


(18) Abramowitz and Stegun, p. 378, eq. 9.7.2.


Chapter IV

COULOMB-NUCLEAR INTERFERENCE IN PION-NUCLEUS SCATTERING

Besides pion-nucleon scattering, we are interested in pion-nucleus scattering, e.g. \( \pi-Z^2 \) scattering. We develop here methods by which Coulomb interactions can be incorporated into the Glauber multiple scattering theory and by which effects of Coulomb-nuclear interference can thus be obtained. Incidentally the formalae obtained can also be used in other charged-hadron-nucleus scattering.

1) Glauber's Multiple Diffraction Theory

We treat pion-nucleus scattering as a pion being scattered by the individual nucleons which constitute the nucleus. A pion can be scattered singly by one such nucleon or scattered in succession by many nucleons. We can therefore consider the interaction between the pion and nucleus as a sum of interactions between the pion and the individual nucleons of the nucleus. By employing the Glauber approximation and the properties that the phase-shift functions are additive, i.e. by setting the phase-shift function for pion-nucleus scattering equal to the sum of the phase-shifts for interaction between the pion and individual nucleons, the multiple scattering theory is greatly simplified. The superposition of phase-shift functions (or potentials) in this case is different from the previous case, where phase-shift functions for strong and Coulomb interactions were superimposed. The difference arising from the fact that the nucleons are randomly distributed in the nucleus, so the phase-shift functions for the
pion-nucleon interaction is dependent on the positions of the nucleons. An average over the various positions of the nucleons is thus required. The scattering amplitude for pion-nucleus collision without Coulomb effects, which are to be accounted for later, in the Glauber multiple scattering theory is

$$\Gamma_{N_i}(q) = \frac{i\hbar}{2\pi} \int e^{i\frac{\hbar}{\tau} \frac{\mathbf{p}^2}{2m}} \psi^*(\{\mathbf{r}_i\}) \Gamma(\mathbf{b}; \mathbf{z}_i, \ldots, \mathbf{z}_A) \psi(\{\mathbf{r}_j\}) d^3r_1 \ldots d^3r_A d^3b$$

(1)

where $\psi^*(\{\mathbf{r}_i\})$ and $\psi(\{\mathbf{r}_j\})$ are wave-functions of the nucleus before and after the scattering of the pion respectively and $\Gamma(\mathbf{b}; \mathbf{z}_i, \ldots, \mathbf{z}_A)$ is the profile function such that

$$\Gamma(\mathbf{b}; \mathbf{z}_i, \ldots, \mathbf{z}_A) = 1 - \exp\left\{ i \chi_N(\mathbf{b}; \mathbf{z}_i, \ldots, \mathbf{z}_A) \right\}. \quad (2)$$

The vectors $\mathbf{z}_i$, $i = 1, \ldots, A$ define the positions of the nucleons in the nucleus, and $\mathbf{z}_i$ is the projections of $\mathbf{z}_i$ on to the plane of the impact parameter $\mathbf{b}$, which plane is perpendicular to the incident momentum $\mathbf{b}$. It is assumed that the nucleons are frozen during their interaction with the incident pion. And the integrals over the various $\chi_N$'s with the nuclear state wave functions represent the average effect taken over all the possible positions of the nucleons. We note here also that the effect of c.m. motion has been neglected. This effect comes in when we consider fully the momentum and energy conservation, and results in a delta function $\delta\left( \frac{\mathbf{r}_1 + \ldots + \mathbf{r}_A}{A} \right)$ in eq. (1).\(^{(1)}\)

The delta function can be removed by the use of a Gartenhaus and Schwartz transformation.\(^{(2)}\) In the case of the harmonic-oscillator wave functions
which we shall use, the result is simply a correction factor by which eq. (1) is to be multiplied. The correction factor is equal to \( \exp(q^2 \alpha^2 / 2a) \), where \( \alpha \) is the radial parameter of the harmonic-oscillator potential. \(^{(1,3)}\) The effect of c.m. motion can therefore be accounted for and is estimated to contribute less than \( \frac{1}{\sqrt{3}} \) to the differential scattering, so eq. (1) becomes

\[
\Gamma_N^{\text{el}}(q) = \frac{i k}{2\pi} \int e^{i q \cdot b} \left| \psi(s_1, \ldots, s_N) \right|^2 \Gamma(b; s_1, \ldots, s_N) d^4 s_1 \cdots d^4 s_N d^2 b ,
\]

where \( \psi(s_1, \ldots, s_N) \) is the ground-state.

We assume that only two-body forces are important and write

\[
\chi_N(b; s_1, \ldots, s_N) = \sum_{j=1}^{A} \chi_j(b - s_j),
\]

with eq. (4), we are able to get from eq. (2)

\[
\Gamma(b; s_1, \ldots, s_N) = 1 - \prod_{j=1}^{A} \left[ 1 - \Gamma_j(b - s_j) \right],
\]

where \( \Gamma_j(b - s_j) \) is the profile function of the \( j \)th nucleon such that the hadron-nucleon scattering amplitude is

\[
f(q) = \frac{i k}{2\pi} \int \exp(i q \cdot b) \Gamma(b) d^2 b .
\]

Eq. (6) can be inverted to give \( \Gamma(b) \) as

\[
\Gamma(b) = \frac{1}{2\pi i k} \int \exp(-i q \cdot b) f(q) d^2 q ,
\]
For elastic scattering, the momentum transfer $\mathbf{q}$ actually varies over a sphere with its vector origin at a point on the sphere. But, the above inverse transform integrates $\mathbf{q}$ over a plane perpendicular to the direction of the incident pion. Therefore, this is accurate only when the contribution to the integral comes mainly from small $\mathbf{q}$ values, i.e., when the scattering is predominantly in the forward direction.

Therefore, from equations (5) & (7), we are able to express the nuclear profile function for the nucleus in terms of the scattering amplitudes of the individual nucleons. The usual hadron-nucleon scattering amplitude used is a Gaussian such that

$$f(b) = \frac{i \kappa}{4\pi} (1 - i\lambda) \exp\left(-\frac{1}{2} \beta \mathbf{q}^2\right).$$ \hfill (8)

Since $\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_A)$ is normalized to

$$\int |\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_A)|^2 d^3r_1 \ldots d^3r_A = 1,$$ \hfill (9)

we write eq. (3) with the help of eq. (5) as

$$\overline{F}_N(q) = \frac{i \kappa}{2\pi} \int e^{i\mathbf{q} \cdot \mathbf{b}} \left\{1 - \int |\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_A)|^2 \prod_{i=1}^A \left[1 - (b-b_i)d_i^3 \ldots d_A^3\right] d^3b \right\} d^3b$$

$$= \frac{i \kappa}{2\pi} \int e^{i\mathbf{q} \cdot \mathbf{b}} \left\{1 - e^{i\mathbf{b} \cdot \mathbf{b}}\right\} d^3b.$$ \hfill (10)
Thus we obtain
\[ \exp \{ i \chi_N(b) \} = \int |\psi(x_1, \ldots, x_A)|^2 \prod_{i=1}^A [1 - \Gamma_i(k_x - k_{\text{lab}})] \, \text{d}x_1 \ldots \text{d}x_A. \] (11)

But when Coulomb effects are not negligible, such as at small angles where Coulomb and nuclear effects have comparable magnitudes, the elastic scattering amplitude is
\[ \bar{F}(q) = i \hbar \int_0^\infty J_0(qb) \left\{ 1 - e^{i \chi(b)} \right\} b \, \text{d}b, \] (12)

with
\[ \chi(b) = \chi_C(b) + \chi_N(b). \] (13)

where \( \chi_N(b) \) is given by eq. (11) and \( \chi_C(b) \) is the Coulomb phase function. We note that azimuthal symmetry is assumed in eq. (12).

So we see from the above that the problem of solving for the scattering amplitude and therefore the scattering cross-section using the multiple-diffraction theory comes down to calculating the nuclear phase function (eq. (11) and the Coulomb phase function, \( \chi_C(b) \)) and the final result is obtained from eq. (12). (Note that from eq. (11) and (12) we see that we do not actually need to evaluate \( \chi_N(b) \) explicitly).
2) Calculation Using the Harmonic Oscillator Potential Model for the Nucleus.

For light nuclei with $4 \leq A \leq 16$, we employ the independent particle model to account for the ground state of the nucleus. In this model

$$|\psi(x_1, \ldots, x_A)|^2 = \prod_{i=1}^{A} \rho_i(x_i), \quad (14)$$

where $\rho_i(x_i)$ is a single particle density with normalization such that

$$\int \rho_i(x_i) d^3r_i = 1 \quad (15)$$

In doing so, the position correlations of the nucleons in the nucleus are neglected. The single particle densities are chosen to correspond to the harmonic oscillator potential. For the s-shell and p-shell nucleons, we have

$$\rho_s(r) = \frac{1}{n^{3/2} a_0^3} \exp\left(-\frac{r^2}{a_0^2}\right) \quad (16)$$

and

$$\rho_p(r) = \frac{2}{3 n^{3/2} a_0^5} r^2 \exp\left(-\frac{r^2}{a_0^2}\right) \quad (17)$$

respectively, where $a_0$ is the radial parameter. Since there are 4 s-shell nucleons and $(A-4)$ p-shell nucleons in a nucleus with $4 \leq A \leq 16$, the total nuclear density which is equal to the sum of the individual single particle densities, is given by

$$\rho_N(r) = \frac{2}{3 n^{3/2} a_0^3} \left(6 + (A-4) \frac{r^2}{a_0^2}\right) \exp\left(-\frac{r^2}{a_0^2}\right). \quad (18)$$

Although calculations were done by Glauber and Matthiae\(^{(5)}\) using harmonic oscillator particle densities for the nucleons, they did not employ any charge distribution and only suggested that, assuming a
point-charged incident pion, the coulomb phase-shift function can be calculated by the following equations:

$$\chi_c(b) = \chi_c'(b) + \chi_{scr}(b),$$  \hspace{1cm} (19)

where

$$\chi_{scr}(b) = -\frac{2\pi e^2}{\hbar v} \ln (2R_{scr})$$  \hspace{1cm} (20)

is the contribution to $\chi_c(b)$ from the screening radius $R_{scr}$ and

$$\chi_c'(b) = \frac{2\pi e^2}{\hbar v} \left[ \ln b \int_0^b T_c(b') 2\pi b' db' + \int_b^\infty T_c(b') \ln b' 2\pi b' db' \right],$$  \hspace{1cm} (21)

with the thickness function defined as

$$T_c(b) = \int_0^b \mathcal{P}(\gamma b + \frac{3}{2} b) d\gamma .$$  \hspace{1cm} (22)

We note that $\chi_c(b)$ have been expanded in powers of $\frac{b}{R_{scr}}$ and only the lowest order terms has been retained in density to the above equations. Actually, as we shall show later, such an approximation is not necessary for the choice of nuclear charge distribution we are going to make. In fact our result will be exact. Furthermore, the form factor of the pion will also be considered instead of assuming to a point-charge.

The choice of the form of the nuclear density is quite natural in view of the harmonic oscillator particle densities used for the nucleon distributions in the nucleon. We simply assume that nuclear charge density is the sum of individual proton charge densities, and that the proton charge density is equal to the proton nuclear density. With $2 \leq Z \leq 8$, there are 2 s-shell and $(Z-2)$ p-shell protons. Therefore
the total nuclear charge density is

\[ S_\text{c}(r) = 2 S_\text{s}(r) + (z - 2) S_\text{p}(r) \]

\[ = \frac{2}{3} \frac{Z}{a_0^2} \left( 3 + (z - 2) \frac{r^2}{a_0^2} \right) \exp \left( - \frac{r^2}{a_0^2} \right) \]  

(23)

At the same time, we also allow a charge structure for the incoming pion to be a gaussian

\[ S_\pi(r) = \frac{1}{\pi b_0^2} \exp \left( - \frac{r^2}{b_0^2} \right) \]  

(24)

where \( b_0 \) is the radial parameter for pion.

a) **Nuclear Phase-shift Function**

The nuclear phase-shift function can now be evaluated by putting

eq. (14) into eq. (11) which then becomes

\[ \exp (i \chi_\text{N}(b)) = \int \left[ \prod_{\alpha} \Phi_\text{s}(r_\alpha) \right] \left[ \prod_{\alpha} \Phi_\text{p}(r_\alpha) \right] \prod_{\alpha} \left[ 1 - \Gamma_\alpha(b - \xi) \right] d^3r_1 \ldots d^3r_4. \]  

(25)

Separating the above into independent integrals we have

\[ \exp (i \chi_\text{N}(b)) = \prod_{\alpha} \int \Phi_\text{s}(r_\alpha) \left[ 1 - \Gamma_\alpha(b - \xi) \right] d^3r_\alpha \prod_{\alpha} \int \Phi_\text{p}(r_\alpha) \left[ 1 - \Gamma_\alpha(b - \xi) \right] d^3r_\alpha. \]  

(26)

Collecting identical integrals together, we have

\[ \exp (i \chi_\text{N}(b)) = \left\{ \int \Phi_\text{s}(r) \left[ 1 - \Gamma(b - \xi) \right] d^3r \right\} \left\{ \int \Phi_\text{p}(r) \left[ 1 - \Gamma(b - \xi) \right] d^3r \right\}. \]  

(27)

Remembering eq. (15) we obtain

\[ \exp (i \chi_\text{N}(b)) = \left[ 1 - M_\text{s}(b) \right] \left[ 1 - M_\text{p}(b) \right]. \]  

(28)

with

\[ M_\text{s}(b) = \int \Phi_\text{s}(r) \Gamma(b - \xi) d^3r \]  

(29)

and

\[ M_\text{p}(b) = \int \Phi_\text{p}(r) \Gamma(b - \xi) d^3r. \]  

(30)

To calculate \( M_\text{s}(b) \) and \( M_\text{p}(b) \), we use the inverse transform for \( \Gamma(b) \) eq. (7). Take \( M_\text{s}(b) \) for example

\[ M_\text{s}(b) = \frac{1}{2\pi i k} \int \Phi_\text{s}(r) \exp \left[ -i \frac{q_0}{k} (b - \xi) \right] f(q) d^3q \]  

(31)
Note that the nucleon form factor is defined as

\[ S_s(q) = \int \exp(i \frac{q \cdot x}{2}) f_s(x) \, d^3r \]  

(32)

\( M_s(b) \) becomes

\[ M_s(b) = \frac{i}{2 \pi i k} \int \exp(-i \frac{q \cdot k}{2}) S_s(q) f_s(q) \, d^2q, \]  

(33)

where we have used the relationship \( \frac{q \cdot x}{2} = \frac{q \cdot s}{2} \) because \( q \) is taken to be in the plane perpendicular to the incident pion. Similarly

\[ M_p(b) = \frac{i}{2 \pi i k} \int \exp(-i \frac{q \cdot k}{2}) S_p(q) f_p(q) \, d^2q, \]  

(34)

with \( S_p(q) \) defined as \( S_s(q) \) in eq. (32).

Using a gaussian form for the pion-nucleon differential scattering amplitude, as given in eq. (8), \( M_s(b) \) and \( M_p(b) \) can be evaluated to be (4)

(see Appendix C)

\[ M_s(b) = \frac{(1 - i \delta) \sigma}{2 \pi (\sigma^2 + 2 \beta^2)} \exp\left(-\frac{b^2}{\sigma^2 + 2 \beta^2}\right), \]  

(35)

\[ M_p(b) = \frac{(1 - i \delta)}{2 \pi (\sigma^2 + 2 \beta^2)} \left\{ 1 - \frac{2 \sigma^2}{3(\sigma^2 + 2 \beta^2)} \left(1 - \frac{b^2}{(\sigma^2 + 2 \beta^2)}\right) \right\} \exp\left(-\frac{b^2}{\sigma^2 + 2 \beta^2}\right). \]  

(36)

b) **Coulomb Phase-shift Function.**

The coulomb phase-shift function is calculated from the coulomb interaction potential \( V_c (\mathbf{r}) \) between pion and the nucleus, using the usual equation in the Glauber formalism,

\[ \chi_c(b) = -\frac{1}{\hbar \nu} \int_{-\infty}^{\infty} V_c (k + z \hat{z}) \, dz. \]  

(37)
with \( R = \ell + z \hat{\jmath} \). The interaction potential between two charge distributions is equal to

\[
V_c(R) = \int \frac{\mathcal{P}_p(x'') \mathcal{P}_c(x')}{|x' - x|} d^3r'' d^3r',
\]

(38)

where \( \mathcal{P}_p(x'') \) and \( \mathcal{P}_c(x') \) are the charge distributions of the incident pion and the target nucleus respectively. The meaning of this equation and its notation used there is demonstrated in Fig. (1), where it is shown that an infinitesimal portion of the incident pion particle charge density at position \( y'' \) from its own center and \( x \) from the target center, is interacting with a portion of the target nuclear charge density at position \( y' \) from the target center. \( R \) is the position of the incident pion center with respect to the target nuclear center. \( V(R) \) is

\[ \text{Fig. 4-1 Two charge spheres (incident pion and target nucleus) in interaction. Coulomb potential is given in eq. (38)} \]
obtained in eq. (38) from summing up all portions of both charges by integrating over $r'$ and $r''$. In performing this integration, we have arbitrarily taken the target particle center to be the center of the coordinate system. The final result is independent of this choice.

As was previously mentioned, we assume that the nuclear charge density is equal to the sum of charge distribution of the individual protons, and that the proton charge distribution is equal to the proton nuclear distribution. In that case, eq. (38) can be written using eq.(24) as

$$V_c(r) = 2V_{cs}(r) + (z^2 - 2)V_{cp}(r),$$  

(39)

where

$$V_{cs}(r) = e^2 \int \frac{\rho_s(r') \rho_s(r'')}{|r' - r''|} \, d^3r'' \, d^3r'$$  

(40)

and

$$V_{cp}(r) = e^2 \int \frac{\rho_p(r') \rho_p(r'')}{|r' - r''|} \, d^3r'' \, d^3r'$$  

(41)

with $\rho_s(r')$ and $\rho_p(r')$ given in eqs(16) and(17) as s- and p-shell distributions corresponding to harmonic oscillator potential and $\rho_p(r)$ given in eq. (24) corresponding to a gaussian distribution. We note also that the incident pion is assumed to be $\pi^+$, but the equations can be used for $\pi^-$ by changing $e^+$ into $-e^+$. The rule applies to the equations which follows in this chapter.

The corresponding phase-shift functions can be written as
\[ \chi_c(b) = 2 \chi_{cs}(b) + (z-2) \chi_{cp}(b) \]  
\[ \chi_{cs}(b) = -i \frac{1}{\hbar v} \int_{-\infty}^{\infty} V_{cs}(k + \frac{\hbar \delta}{m}) d\delta \]  
\[ \chi_{cp}(b) = -i \frac{1}{\hbar v} \int_{-\infty}^{\infty} V_{cp}(k + \frac{\hbar \delta}{m}) d\delta \]

Now we can proceed to calculate \( \chi_c(b) \) by calculating \( \chi_{cs}(b) \) and \( \chi_{cp}(b) \). For the s-wave protons, we note that Franco has calculated the phase-shift function for the case when both the incident particle and the target have gaussian target distribution, \(^7\) and that the s-wave proton particle density corresponding to the harmonic oscillator potential is actually equivalent to the gaussian. Therefore, we can use Franco's result for \( \chi_{cs}(b) \), i.e.

\[ \chi_{cs}(b) = \chi^p_{\text{e}}(b) + \eta \frac{\alpha_s^2}{\alpha_s^2 + \frac{b}{\hbar v}} \]  

where \( \chi^p_{\text{e}}(b) \) is the point-charge phase-shift function, \( \eta = \frac{e^2}{\hbar v} \), and the function \( E_1(x) \) can be defined from the exponential integral function \( E_\zeta(x) \), such that

\[ E_1(x) = -E_\zeta(x) \]

In Appendix D, my calculation to obtain eq. (45) for \( \chi_{cs}(b) \) is presented, and it is also obtained that the potential between two gaussian charge densities each with unit charge is

\[ V_{cs}(R) = \frac{e^2}{R} \Phi\left( \frac{R}{\sqrt{\alpha_s^2 + \frac{b}{\hbar^2}}} \right) \]  

(\text{two gaussian charge densities})

where \( \Phi(x) \) is probability integral.

As for the p-shell protons, we again calculate the phase-shift
function by first calculating the potential $V_p(\mathbf{r})$ using eqs (41), (17) and (24):

$$V_p(\mathbf{r}) = \frac{-2e^2}{3\pi\alpha_0^2 b_0^3} \int \frac{r' e^{-\frac{r'^2}{2\sigma}} e^{-\frac{r'^2}{2\tau}}}{|r' - r|^3} \, d^3r' \, d^3r' \quad \text{with} \quad r'' = r - \mathbf{R}. \quad (48)$$

Changing the integration variable from $r''$ to $\gamma = r'' + \mathbf{R}$, we obtain

$$V_p(\mathbf{r}) = M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{(r'^2 + r''^2 - 2rr'' \cos\theta)^2} \, d^3r' \, d^3r, \quad (49)$$

where

$$M = \frac{2e^2}{3\pi\alpha_0^2 b_0^3}. \quad (50)$$

Integrating the angular part of $\gamma'$, we get

$$V_p(\mathbf{r}) = 2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \left( \frac{|r |}{r'^2 + r''^2 - 2rr'' \cos\theta} \right)^{\frac{1}{2}} \, d^3r' \, d^3r'$$

$$= 2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \, d^3r' \, d^3r - 2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \, |r - r'| \, d^3r' \, d^3r$$

$$= V_1 + V_2 + V_3 + V_4, \quad (51)$$

where

$$V_1 = 2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \, d^3r' \, d^3r; \quad V_2 = 2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \, (r'' - r) \, d^3r' \, d^3r;$$

$$V_3 = -2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \, (r - r'') \, d^3r' \, d^3r; \quad V_4 = 2\pi M \int \frac{r'^2 e^{-\frac{r'^2}{2\sigma}} e^{-\frac{\alpha \cdot B'^2}{k^2}}}{r} \, (r' - r) \, d^3r' \, d^3r. \quad (52)$$

Therefore, recombining terms

$$V_p(\mathbf{r}) = 4\pi M \int I_1(r) \frac{-\alpha \cdot B'^2}{k^2} \, d^3r + 4\pi M \int I_2(\mathbf{r}) \, e^{-\frac{\alpha \cdot B'^2}{k^2}} \, d^3r, \quad (53)$$
with
\[ I_1(r) = \int_0^r r'^4 e^{-r'/a^2} \, dr' \] (54)
and
\[ I_2(r) = \int_r^\infty r'^3 e^{-r'/a^2} \, dr'. \] (55)

Changing the variable of integration to \( u = r'^2 \),
\[ I_1(r) = \frac{1}{2} \int_0^{r^2} u^{3/2} \exp\left(-\frac{u}{a^2}\right) \, du \] (56)
and
\[ I_2(r) = \frac{1}{2} \int_{r^2}^{\infty} u \exp\left(-\frac{u}{a^2}\right) \, du. \] (57)

Using the integral
\[ \int_0^x x^{\nu-1} \exp\left(-\mu x\right) \, dx = \mu^{-\nu} \gamma(\nu, \mu x) \] \([\text{Re} \nu > 0]\) (58)

with \( \nu = \frac{5}{2} \), \( I_1(r) \) becomes
\[ I_1(r) = \frac{1}{2} a^{5/2} \gamma\left(\frac{5}{2}, \frac{r^2}{a^2}\right), \] (59)

where \( \gamma(\nu, \mu) \) is an incomplete gamma function. With the recursion relation
\[ \gamma(\nu+1, x) = \nu \gamma(\nu, x) - x^{\nu} e^{-x} \] (60)
and the equality\(^{(10)}\)
\[ \gamma\left(\frac{1}{2}, x^2\right) = \pi^{1/2} \Phi(x) \] (61)
we obtain
\[ \gamma\left(\frac{5}{2}, \frac{r^2}{a^2}\right) = \frac{3}{2} \gamma\left(\frac{3}{2}, \frac{r^2}{a^2}\right) - \left(\frac{r}{a}\right)^3 e^{-\frac{r^2}{a^2}}. \] (62)
where

\[ \Gamma(n, x) = \int_{x}^{\infty} t^{n-1} e^{-t} dt \quad [n > 0] \] (65)

and

\[ n = 1 \quad I_2(r) \] becomes

\[ I_2(r) = \frac{a_o^4}{2} \left[ 1 + \frac{r^2}{a_o^2} \right] e^{-\frac{r^2}{a_o^2}} . \] (66)

Putting \( I_1(r) \) and \( I_2(r) \) as given in eqs. (62) and (66) back into eq. (53) gives after cancellations

\[ \mathcal{V}_{\text{cp}}(R) = \frac{3}{2} \pi \frac{3}{2} M a_o^5 I_3(R) - M \pi a_o^4 I_4(R) \] (67)

with

\[ I_3(R) = \frac{1}{r} \Phi \left( \frac{r}{a_o} \right) \exp \left( - \frac{(r - b_o)^2}{b_o^2} \right) r^3 \] (68)
and
\[ I_4(R) = \int \exp \left[ -\frac{r^2}{\alpha^2} - \frac{(x^2 - y^2)}{b^2} \right] d^3r. \] (69)

To calculate \( I_3(R) \), we first integrate over the angular part of \( \chi \) and obtain
\[ I_3(R) = \int \frac{1}{r} \Phi \left( \frac{r}{\alpha_o} \right) \exp \left[ -\frac{(r^2 + R^2 - 2rR\cos\theta)}{b^2} \right] d\theta \]
\[ = \frac{\pi R b^2}{R} \exp \left( \frac{R^2}{b^2} \right) \Phi \left( \frac{R}{\alpha_o} \right) \left\{ \exp \left[ -\frac{(r^2 + R^2)}{b^2} \right] - \exp \left[ -\frac{(r^2 + 2rR)}{b^2} \right] \right\} dr. \] (70)

The integral in the above equation involves the probability integral
\[ \Phi \left( \frac{r}{\alpha_o} \right) \], and cannot be simply integrated. Even the usual integral representations of \( \Phi \left( \frac{r}{\alpha_o} \right) \) (13) do not lead to the evaluation of \( I_3(R) \).

However, the following equation has proven to be useful (14)
\[ \int_0^\infty \exp \left( -\beta x^2 \right) \sinh(\lambda x) dx = \frac{1}{2} \left( \frac{\pi}{\rho} \right)^{\frac{1}{2}} \exp \left( \frac{\lambda^2}{4\rho} \right) \Phi \left( \frac{\lambda}{2\rho} \right) \] \( [Re \beta > 0] \). (71)

Using the above eqn, letting \( \beta = \frac{1}{\alpha^2} \) and \( \lambda = \frac{2r}{\alpha^2} \), \( I_3(R) \) becomes
\[ I_3(R) = \frac{b^2}{\alpha_o R} \int_0^\infty \int_0^\infty e^{-\frac{x^2}{b^2}} \left[ e^{\frac{(r^2 + 2rR)}{b^2}} - e^{\frac{(r^2 + 2rR)}{b^2}} \right] \left[ e^{-\frac{(r^2 + 2rR)}{b^2}} - e^{-\frac{(r^2 + 2rR)}{b^2}} \right] dx \] (72)

Expanding the integrand,
\[ I_3(R) = \frac{b^2 \pi^2}{\alpha_o R} e^{-\frac{R^2}{b^2}} \int_0^\infty \int_0^\infty e^{-\frac{x^2}{b^2}} \left\{ \exp \left( -\frac{r^2}{\alpha^2} - \frac{R^2}{b^2} + \frac{2rR}{\alpha_o^2} + \frac{2rR}{b^2} \right) \right\} \]
\[ - \exp \left( -\frac{r^2}{\alpha^2} - \frac{r^2}{b^2} - \frac{2rR}{\alpha_o^2} - \frac{2rR}{b^2} \right) - \exp \left( -\frac{r^2}{\alpha^2} - \frac{r^2}{b^2} + \frac{2rR}{\alpha_o^2} - \frac{2rR}{b^2} \right) \]
\[ - \exp \left( -\frac{r^2}{\alpha^2} - \frac{r^2}{b^2} - \frac{2rR}{\alpha_o^2} - \frac{2rR}{b^2} \right) \] (73) \[ d\tau \ dx. \]
Using the integral\(^{(15)}\)

\[
\int_0^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right)dx = (\pi \beta)^{\frac{1}{4}} \exp(\beta \gamma)\left[1 - \Phi\left(\sqrt[4]{\beta}\right)\right] \\
\text{[Re} \beta > 0]\]

(74)

the integrals over \(r\) in eq. (71) can be performed,

\[
\int_0^\infty \exp\left(-\frac{a^2 + b^2}{a^2 b_0^2} r^2 + \frac{2b^2 x - 2a^2 R}{a^2 b_0^2} r\right)dr = \frac{\pi a b_0}{2(a^2 + b_0^2)^{\frac{1}{2}}} \exp\left[\frac{(2b^2 x - 2a^2 R)^3}{4(a^2 + b_0^2)^2 a b_0^2}\right] [1 - \Phi\left(\frac{2b^2 x - 2a^2 R}{4(a^2 + b_0^2)^{\frac{1}{2}} a b_0}\right)]
\]

(75)

Due to the property of the probability integral function, i.e. \(^*\)

\[
\Phi(-z) = -\Phi(z) \\
(76)
\]

\(^*\) This comes from the definition of the probability integral\(^{(16)}\)

\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,
\]

where a change of variable of integration from \(t' = -t\) would lead the property given in eq. (76)
we find cancellations in all terms containing $\Phi_3$ when eq. (75) is put into eq. (73) $I_3(R)$ becomes

$$I_3(R) = \frac{\pi b_o^3}{(a_o^2 + b_o^2)^2 R} \exp\left(-\frac{R^2}{b_o^2} + \frac{a_o^3 R^2}{(a_o^2 + b_o^2) b_o^2}\right) \cdot$$

$$\int_0^\infty dx \left\{ \exp\left[-\frac{x^2}{a_o^2} + \frac{1}{2} \frac{x^2}{a_o^4 + b_o^4} + \frac{2 R x}{(a_o^2 + b_o^2) a_o^2}\right] - \exp\left[-\frac{x^2}{a_o^2} + \frac{1}{2} \frac{x^2}{a_o^4 + b_o^4} - \frac{2 R x}{(a_o^2 + b_o^2) a_o^2}\right] \right\}$$

$$= \frac{\pi b_o^3}{(a_o^2 + b_o^2)^2 R} \int_0^\infty dx \left\{ \exp\left[-\frac{x^2}{a_o^2} + \frac{2 R x}{(a_o^2 + b_o^2) a_o^2}\right] - \exp\left[-\frac{x^2}{a_o^2} - \frac{2 R x}{(a_o^2 + b_o^2) a_o^2}\right] \right\}$$

(77)

upon simplification. Again using eq. (74),

$$I_3(R) = \frac{2 \pi b_o^3}{(a_o^2 + b_o^2)^2 R} \left[ (a_o^2 + b_o^2) \exp\left[\frac{(a_o^2 + b_o^2) x^2}{2 (a_o^4 + b_o^4)}\right] - \exp\left[\frac{x^2}{a_o^2} + \frac{2 R x}{(a_o^2 + b_o^2) a_o^2}\right] \right]$$

(78)

which simplifies to

$$I_3(R) = \frac{\pi b_o^3}{R} \Phi_3\left(\frac{R}{a_o^2 + b_o^2}\right).$$

(79)

For $I_4(R)$, integration over angles similar to $I_3(R)$ (see eq. (70)) leads to

$$I_4(R) = \frac{\pi b_o^2}{R} \exp\left(-\frac{R^2}{b_o^2}\right) \int_0^\infty r \exp\left(-\frac{r^2}{a_o^2}\right) \left\{ \exp\left[\frac{(a_o^2 + b_o^2) r^2}{2 (a_o^4 + b_o^4)}\right] - \exp\left[\frac{(a_o^2 + b_o^2) r^2}{2 (a_o^4 + b_o^4)}\right] \right\} dr,$$

(80)

which can be rewritten as

$$I_4(R) = \frac{2 \pi b_o^2}{R} \exp\left(-\frac{R^2}{b_o^2}\right) \int_0^\infty r \exp\left[-\frac{(a_o^2 + b_o^2) r^2}{a_o^2 b_o^2}\right] \sinh\left(\frac{2 r R}{b_o^2}\right) dr.$$

(81)

Using the integral(17)
\[ \int_{-\infty}^{\infty} x \exp(-\beta x^2) \text{sh}(\gamma x) \, dx = \frac{\delta}{4\beta} \left( \frac{\pi^{\frac{1}{2}}}{\beta} \right) e^{\frac{\delta^2}{4\beta}} \]

we obtain \[ I_+ (R) = \frac{2\pi b^2}{R} \exp\left(-\frac{R^2}{b^2}\right) \frac{\gamma^2 b^2}{4(a^2 + b^2)^{\frac{3}{2}}} \exp\left\{ -\frac{R^2}{4(a^2 + b^2)^{\frac{3}{2}}} \right\} \cdot \frac{a^3 b^3}{4(a^2 + b^2)^{\frac{3}{2}}} \] \[ \exp\left\{ -\frac{R^2}{4(a^2 + b^2)^{\frac{3}{2}}} \right\} \] (83)

After simplification, the result for \( I_+ (R) \) is

\[ I_+ (R) = \frac{\pi^{\frac{3}{2}} a^3 b^3}{(a^2 + b^2)^{\frac{3}{2}}} \exp\left\{ -\frac{R^2}{(a^2 + b^2)^{\frac{3}{2}}} \right\} \] (84)

Putting \( I_3 (R) \) and \( I_+ (R) \) (eqs. (79) and (84) back into eq. (67), the potential \( V_{cp} (R) \) is evaluated to be

\[ V_{cp} (R) = \frac{e^2}{R} \Phi\left( \frac{R}{(a^2 + b^2)^{\frac{3}{2}}} \right) - \frac{2a^2 e^2}{3 \pi (a^2 + b^2)^{\frac{3}{2}}} \exp\left\{ -\frac{R^2}{(a^2 + b^2)^{\frac{3}{2}}} \right\} \] (85)

When we calculate the phase-shift function \( \chi_{cp} (b) \) from \( V_{cp} (R) \) using eq. (44), we note that the first term in eq. (85) is equal to \( V_{cs} (R) \) and, therefore, would lead to \( \chi_{cs} (b) \) as given in eq. (45) (see also Appendix D). Whereas the second term in eq. (85) would contribute to the phase-shift function \( \chi_{cp} (b) \) a term equal to

\[ \chi'_{cp} (b) = \frac{2a^2 e^2}{3 \pi (a^2 + b^2)^{\frac{3}{2}}} \exp\left[ -\frac{b^2}{a^2 + b^2} \right] \int_{-\infty}^{\infty} \exp\left[ -\frac{\beta^2}{a^2 + b^2} \right] \, d\beta . \] (86)

Using the integral (18)

\[ \int_{0}^{\infty} \exp (-q^2 x^2) \, dx = \frac{\pi^{\frac{1}{2}}}{q} \quad [q > 0], \] (87)

\( \chi'_{cp} (b) \) becomes

\[ \chi'_{cp} (b) = \frac{2a^2 e^2}{3 \pi (a^2 + b^2)^{\frac{3}{2}}} \exp\left[ -\frac{b^2}{a^2 + b^2} \right] \] (88)
Therefore, for a \( \nu \)-shell proton in the harmonic oscillator potential interacting with an incident pion with gaussian charge distribution the phase-shift function is

\[
\chi_{\nu p}(b) = \chi_{\nu s}(b) + \frac{e^2}{\hbar v} E_1 \left( \frac{b^2}{a_s^2 + b^2} \right) + \frac{2a_s^2 e^2}{3 \hbar v (a_s^2 + b^2)} \exp \left[ -\frac{b^2}{a_s^2 + b^2} \right]. \tag{89}
\]

Putting \( \chi_{\nu s}(b) \) and \( \chi_{\nu p}(b) \) (eqs. (45) and (89)) into eq. (42), we obtain the result for the coulomb phase-shift function for a light nuclei with \( z \) protons \( (2 \leq z \leq 8) \)

\[
\chi_{\nu}(b) = \overline{\chi}_{\nu s}(b) + \frac{2e^2}{\hbar v} E_1 \left( \frac{b^2}{a_s^2 + b^2} \right) + \frac{2}{3} (z-2) \frac{e^2}{\hbar v} \frac{a_s^2}{a_s^2 + b^2} \exp \left[ -\frac{b^2}{a_s^2 + b^2} \right], \tag{90}
\]

where we call the phase-shift function due to all nuclear charges concentrated at a point to be \( \overline{\chi}_{\nu s}(b) \) such that

\[
\overline{\chi}_{\nu s}(b) = z \chi_{\nu s}(b). \tag{91}
\]

Eq. (89) applies to \( \Sigma^+ \). As was previously noted a change of \( e^2 \) to \( -e^2 \) would give an equation which applies to \( \Sigma^- \). The result of such a change is an overall minus sign, i.e.

\[
\chi_{\nu}^{\Sigma^+}(b) = - \chi_{\nu}^{\Sigma^-}(b). \tag{92}
\]

c) **Pion-nucleus scattering with coulomb effects.**

Now we are ready to gather together the results obtained in the previous sections to arrive at an equation for the differential amplitude \( F(q) \) for the scattering of pion by a nucleus with \( 4 \leq A \leq 16 \) and \( 2 \leq z \leq 8 \). Using \( \chi_{\nu}(b) \) (eq. 90), we can write \( F(q) \) from eqs. (12) and (13) as
\[ F(q) = i \frac{k}{b} \int_0^\infty J_b(qb) \left\{ 1 - \exp \left[ i \left( \chi_N(b) + \chi_c^t(b) + \chi_c(b) \right) \right] \right\} b \, db, \]

where

\[ \chi' = \pm \left\{ \frac{2 e^2}{\hbar} E_i \left( \frac{b^2}{a_s^2 + b_0^2} \right) + \frac{2}{3} (z - 2) \frac{a_s^3 e^5}{\hbar r (a_s^2 + b_0^2)} \exp \left[ -\frac{b^2}{(a_s^2 + b_0^2)} \right] \right\} \]

(plus sign for \( \pi^+ \) and minus sign for \( \pi^- \)). \( F(q) \) can be written as

\[ F(q) = F^t_c(q) + i \frac{k}{b} \int_0^\infty J_b(qb) e^{i \chi_c^t(b)} \left\{ 1 - \exp \left[ i \left( \chi_N(b) + \chi_c^t(b) \right) \right] \right\} b \, db \]

with

\[ F^t_c(q) = i \frac{k}{b} \int_0^\infty J_b(qb) \left\{ 1 - \exp \left[ i \chi_c^t(b) \right] \right\} b \, db. \]

As was obtained in the previous chapter with the assumption of a step function cut-off,

\[ F^t_c(q) = -\frac{2 \ln n}{q^2} \exp \left\{ -i \left[ 2n \ln \left( 2kA \sin \frac{\theta}{2} \right) - 2 \arg (i + in) \right] \right\} \]

with \( n = \left\{ \begin{array}{ll} + \frac{2 e^3}{\hbar} & \text{for } \left( \frac{\pi^+}{\pi^-} \right) \end{array} \right. \) and \( A \) the cut-off distance. Since \( \exp \left[ i \left( \chi_N(b) + \chi_c(b) \right) \right] \approx 1 \) for large \( b \) values, it is a good approximation, according to the discussion in the last chapter to write \( e^{i \chi_c^t(b)} \) as

\[ \exp \left[ i \chi_c^t(b) \right] = \left( \frac{b}{2A} \right)^{2in}. \]
Therefore, both terms in eq. (95) contains a phase factor \( \exp(-2i\ln \ln 2A) \).

Since this phase factor is unobservable, we again use a new scattering amplitude \( \mathcal{F}(q) \) such that

\[
\bar{\mathcal{F}}(q) = \mathcal{F}(q) \exp(2i\ln \ln 2A)
\]

(99)

then we have,

\[
\bar{\mathcal{F}}(q) = \bar{\mathcal{F}}_c^{pt}(q) + i k \int db \mathcal{S}(q) b^{1+2i\ln \left\{ 1 - [1 - M_s(b)][1 - M_p(b)]^{A-4} \right\} \exp(i \chi_c(b))}
\]

(100)

where eq. (28) has also been used,

\[
\bar{\mathcal{F}}_c^{pt}(q) = -\frac{2k_n}{g^2} \exp\left\{ -i \left[ 2n \ln \left( \ell \sin \frac{q}{2} \right) - 2\text{arctg} \sqrt{1 + i \eta} \right] \right\}
\]

(101)

\[M_s(b) = \frac{(1 - i\lambda)}{2\pi (a_0^2 + 2\beta^2)} \exp\left( \frac{-y^2}{a_0^2 + 2\beta^2} \right)
\]

(102)

\[M_p(b) = \frac{(1 - i\lambda)}{2\pi (a_0^2 + 2\beta^2)} \left\{ 1 - \frac{2a_0^2}{3(a_0^2 + 2\beta^2)} \left[ 1 - \frac{b^2}{(a_0^2 + 2\beta^2)} \right] \right\} \exp\left( \frac{-b^2}{a_0^2 + 2\beta^2} \right)
\]

(103)

and \( \chi_c'(b) \) is given in eq. (90).

\( \mathcal{F}(q) \) can be expanded into many single integrals, but because of the presence of the exponential integral function in \( \chi_c'(b) \), analytical evaluation is very difficult, so \( \mathcal{F}(q) \) is best numerically calculated as one single integral. Needless to say, the differential and total cross-sections can be obtained from \( \mathcal{F}(q) \) as usual.
References


(11) Gradshteyn and Ryzhik, p. 317, eq. 3.381-3.


(13) Gradshteyn and Ryzhik, pages 930-931.

(14) Gradshteyn and Ryzhik, p. 357, eq. 3.546-1.


(16) Gradshteyn and Ryzhik, p. 930, eq. 8.250-1.

(17) Gradshteyn and Ryzhik, p. 365, eq. 3.562-3.

Chapter V

NUMERICAL CALCULATIONS

In this chapter, we present some numerical results using some of the equations obtained so far. Our numerical calculations is severely limited by the amount of computer funding available. Nevertheless, we have obtained some interesting results. We have also found out that it is necessary to change the factor \( \frac{1}{\pi \nu} \) in eq. (3-02) to \( \frac{m}{\pi \nu} \). Corresponding changes must also be made in subsequent equations where the phase-function is used. But these changes are simple and easy with no other effects on the equations obtained.

1) Factor in Phase-shift Function Changed from \( \frac{1}{\pi \nu} \) to \( \frac{m}{\pi \nu} \)

For the square-well potential, there are two parameters we need to determine before numerical results can be obtained. They are the depth \( (V_o) \) and the width \( (a) \) of the potential. We first obtain a graph of the total cross-section for the square-well potential, as a function of \( \frac{1}{\lambda} \left( = \frac{\pi \nu}{V_o a} \right) \) according to eq. (3-25) (Fig. 5-1). A positive value of \( \lambda \) is chosen, because for an attractive \( V_p \) potential \( V = -V_o \) is negative. Nevertheless, the graph is also valid if we change the sign of \( \lambda \) because \( \frac{\sigma^{tot}}{\pi a^2} \) is an even function of \( \lambda \). We note from the graph that \( \frac{\sigma^{tot}}{\pi a^2} \) has a broad peak at \( \frac{1}{\lambda} = 0.49 \). Fitting this
peak to the $\pi^+p$ (3, 3) resonance ($pN^{**+}$) peak at about 190 Mev\(^{(1)}\)* will determine $V_o$ and $a$. With $\lambda = \frac{Ve}{h\nu}$, we need to calculate the velocity of a pion which has kinetic energy $T = 190$ Mev. in the laboratory frame. ** The following equations are used:

\[ p^2c^2 = T^2 + 2mc^2T \]  

\[ \frac{\nu}{c} = \frac{pc}{(mc^2 + p^2c^4)^{1/2}} \]  

where $m$ and $p$ are the mass and momentum of the incident pion respectively. For the mass of pion, we use the value $mc^2 = 139.57$ Mev\(^{(2)}\)

\[ \frac{\nu}{c} \]  

is calculated to be 0.906 for a pion with kinetic energy 190 Mev.

Since $\frac{\hbar}{c} = 197.32858$ Mev. -fm. and

\[ \frac{\hbar\nu}{V_o a} = \frac{\hbar c\left(\frac{\nu}{c}\right)}{V_o a} = \frac{(197.32858 \text{ Mev.-fm})(0.906)}{V_o a (\text{MeV.-fm})} = 0.49 \]  

*The position of the $\pi^+p$ (3, 3) resonance peak can also be calculated from the value of the mass of the resonance particle, i.e. $M_A c^2 = 122$ Mev\(^{(2)}\). To do so we use the equation for the invariant $s = (p_1 + p_2)^2$, where $p_1$ and $p_2$ are the four-momenta of the incident and target particle respectively. In the $C.M.$ frame, $s = M_A c^2$ at the resonance position. In the laboratory frame where $p_1 = 0$, $E_2 = Mc^2$ and $E_1 = Mc^2 + T (T = \text{Kinetic energy})$, $s$ is equal to $(M + m)^2c^4 + 2Mc^2T$.

Equating $s$ in the two frames, we obtain the equation for the laboratory kinetic energy $T$ of the incident particle at the resonance

\[ T = \frac{M_A^2c^4 - (M + m)^2c^4}{2Mc^2} \]  

With the mass of pion $mc^2 = 139.57$ Mev. and mass of proton $Mc^2 = 938.3$ Mev. $T$ is calculated to be 189.72 Mev.

** For small angle scattering, the Glauber scattering amplitude in the laboratory has the same form as that in the center-of-mass system\(^{(3)}\).
the result \( V_o a = 364.48 \text{ Mev.-fm} \) is obtained. Instead of going on to calculate what \( V_o \) and \( a \) are individually, we note that when we try to calculate the width of the peak in Fig. (5-1), the half-peak values \( \left( \frac{3.173}{2} \right) \) are found to occur at \( \frac{l}{\lambda} = 0.28 \) and 1.01. The latter value corresponds to the absurd result of \( \frac{V}{c} = 1.87 \). In fact we note from eq. (3) with the restriction \( \frac{V}{c} < 1, \frac{l}{\lambda} \) can never be bigger than 0.54, which mean that when total cross-section is plotted as a function of the incident pion energy, the peak in Fig. (5-1) would note be a peak, but the cross-section would increase to a maximum and then decreases slightly and levels off with increasing pion energies.

The problem comes obviously from the use of the velocity \( v \) in the equation to calculate the phase-shift function (eq. 3-02). We therefore studied very carefully the various derivations of this equation and found that a more precise form for the phase-shift function equation is

\[
\chi(b) = -\frac{m}{\hbar^2 \lambda} \int_0^\infty \sqrt{k + \frac{3\lambda}{2}} \, d\lambda
\] (5)

The difference between eq. (5) and eq. (3-02) is in the factor \( \beta \) multiplying the integral. Actually, \( \beta \) in eq. (3-02) can be obtained from that in eq. (5) only after assuming that the velocity \( v \) of the incident particle is non-relativistic, i.e.

\[
\beta = \frac{m}{\hbar^2 \lambda} = \frac{m}{\hbar p} = \frac{m}{\hbar (mv)} = \frac{l}{\hbar v}
\] (6)

It is therefore better to use \( \beta = \frac{m}{\hbar \lambda} \) than \( \beta = \frac{l}{\hbar v} \) because of the assumption involved in the latter. In addition, when \( \beta = \frac{m}{\hbar \lambda} \) is used, we can go from one side of the cross-section peak in Fig (5-1)
to the other without meeting the absurd situation of \( \frac{v}{c} > 1 \). Therefore, we see that the difficulty encountered with using \( \beta = -\frac{l}{\hbar v} \) is a result of non-relativistic approximation, and could not have come up if the correct form had been used. All previous equations that have a factor \( \frac{l}{\hbar v} \) due to the phase-shift functions should have it changed to \( \frac{m}{\hbar^2 k} \), e.g. the coulomb factor \( \frac{e^2}{\hbar \nu} \) would be written as \( \frac{e^2 m}{\hbar^2 k} \).

2) **Determination of Parameters**

Due to this change in \( \beta \), \( \frac{l}{\mathcal{L}} \) becomes \( \frac{\hbar^2 k}{m V_0 a} \) and we need to re-evaluate \( V_0 a \). From eq. (1), the momentum \( p \) of an incident particle can be calculated from its kinetic energy \( T \), and \( k \) can be obtained using the relationship \( \frac{k}{\hbar} = \frac{p}{\hbar} \). For a 190 Mev. pion \( k \) is calculated to be 1.51 fm-1, therefore

\[
\frac{l}{\mathcal{L}} = \frac{\hbar^2 k}{m V_0 a} = \frac{(h c)^2 k}{(m c) V_0 a} = \frac{(197.32858 \text{ Mev.fm})^2 (1.51 \text{ fm})}{(139.57 \text{ Mev}) V_0 a (\text{Mev.fm})} = 0.49 \tag{7}
\]

The result is \( V_0 a = 859.75 \text{ Mev. -fm} \).

The \( \pi^+ p \) scattering cross-section at the (3, 3) resonance position has a peak value of \( \sigma_{\pi^+ p} = 200 \text{ mb} = 20 \text{ fm}^2 \). Fitting that to the peak value in Fig. (5-1) \( \frac{\sigma_{\pi^+ p}}{\mathcal{L}} = 3.173 \), \( a \) is calculated to be \( a = 1.416 \text{ fm} \). Subsequently \( V_0 \) can also be calculated to be \( V_0 = 606.97 \text{ Mev} \).

Summing up, the parameters for the square-well potential are

\[
V_0 = 606.97 \text{ Mev.} \\
a = 1.416 \text{ fm} \tag{8}
\]

The half peak values \( \frac{l}{\mathcal{L}} = 0.28 \) and \( 1.01 \) are found to correspond to kinetic energies of \( 81 \text{ Mev} \) and \( 490 \text{ Mev} \) respectively (and to laboratory
momenta of 170 Mev/c and 615 Mev/c, respectively). Therefore the width of the peak is 409 Mev. This is relatively large compared with the actual value of 125 Mev. The width of the square well peak would decrease if the value of either \( V_0 \) or \( a \) is lowered, but at the same time the peak position would also be shifted to a lower energy. Lowering the value of \( a \) would also mean decreasing the total scattering amplitude.

For the Yukawa potential, the parameters to be determined are \( \mu \) and \( \gamma \). We know from Yukawa’s hypothesis that the nuclear force arises from the exchange of a particle and that the range \( \mu^{-1} \) of the resulting interaction potential is related to the mass \( M \) of the exchanged particle by the equation

\[
\mu^{-1} = \frac{4\pi}{M C^2}
\]

To find what \( M \) is, we remember the NN* reciprocal bootstrap which refers to the interesting discovery that the nuclear force that binds a pion and a nucleon together to form a neutron comes mainly from the exchange of \( N^* \) (or now commonly called \( \Delta(1232) \)) whereas the nuclear force that binds a pion and a nucleon together to form a \( N^* \) comes mainly from a nucleon. Therefore, for pion-nucleon scattering at around the \((3,3)\) resonance \((N^*)\) region, the particle exchanged is a nucleon. (see Fig. (5-2a)). In fact, for \( \pi N \) scattering, the exchanged nucleon is a neutron, so we have \( M C^2 = 939.5731^{(2)} \). Therefore, \( \mu \) is obtained from eq. (9) to be
\[ \mu = \frac{MC^2}{\hbar c} = \frac{939.5731 \text{ MeV}}{197.32858 \text{ fm}} = 4.76 \text{ fm}^{-1} \]  

(10)

In order to determine \( \Upsilon \), we first obtain a graph of the total scattering cross-section due to the Yukawa potential to see if we can get any help from it. We note from eq. (3-52) that the integral can be written, after a change of variable of integration from \( b \) to \( B = \mu b \) as

\[ O_{\gamma}^{\text{tot}} = \frac{4\pi}{\mu^2} \int_0^\infty \Re \left\{ B \left\{ 1 - \exp\left[-i \frac{2\pi m}{\hbar \kappa} K_0(B) \right] \right\} \right\} dB \]

\[ = \frac{4\pi}{\mu^2} \int_0^\infty B \left\{ 1 - \cos \left[ \frac{\pi m}{\hbar \kappa} K_0(B) \right] \right\} dB \]

(11)

where

\[ \Upsilon = \frac{2m}{\hbar^2 \kappa} = \frac{2mc^2 \Upsilon}{(\hbar c)^2 \kappa} = \frac{2(139.57 \text{ MeV})(\Upsilon \text{ MeV.fm})}{(197.32858 \text{ MeV.fm})^2 \kappa \text{ fm}^{-1}} \]

(12)

\( \mu \) is therefore seem to have only a scaling effect on \( O_{\gamma}^{\text{tot}} \). Fig. (5-3) shows the total scattering amplitude \( O_{\gamma}^{\text{tot}} \) as a function of \( \Upsilon \) from zero to 200. With \( k \) approximately equal to 1.5 fm\(^{-1}\), this range of \( \Upsilon \) correspond to a range of 0 Mev.fm to 41848 Mev.fm for \( \Upsilon \). We see that \( O_{\gamma}^{\text{tot}} \) increases monotonically throughout the above range, therefore there is not going to be a peak as in the case of a square well potential. In fact, with fixed \( \Upsilon \), increasing pion energy would mean increasing \( k \) and therefore decreasing \( \Upsilon \). That means the scattering amplitude is going to decrease monotonically with increasing...
Energy. That there is no resonance should not be surprising if we remember that the Regge trajectories of simple Yukawa potentials usually turn over so quickly that they rarely lead to resonances. \(^{(7)}\)

One method to determine \(\gamma\) is to match the value of the scattering amplitude at a particular energy to that obtained experimentally. With lack of a peak the choice of a particular energy for such a matching procedure can be quite arbitrary. In addition the fact that the total cross-section is scaled by the square of the interaction range \(\kappa^{-1}\) which is much smaller than the interaction range \(a\) of the square well potential \((\kappa^{-1} = 0.21 \text{ fm. and } a = 1.416 \text{ fm.})\) indicates that matching may not be a good method to determine \(\gamma\). The best way to determine \(\gamma\) is actually from looking at the nucleon-nucleon interaction. Fig. (5-2b) illustrates the nucleon-nucleon interaction as coming from one-pion exchange. The effect of one-pion-exchange is found to be equal to the Born approximation to the Yukawa potential

\[
\gamma(r) = \frac{-g^2 \exp(-\kappa r)}{r},
\]

where \(\kappa\) is related to the pion mass by eq. (9) and \(g\) is the pion-nucleon coupling constant. \(^{(8)}\) As is shown in Fig. (5-26) the coupling constants for the nucleon-nucleon scattering should be the same as that of pion-nucleon scattering because the particles involved at the vertices are the same. We, therefore, conclude that

\[
\gamma = -g^2.
\]

(14)
But $g^2$ was obtained such that (8, 9)\[\begin{align*}
\frac{g^2}{\hbar c} &= 14.8 \\
\text{(15)}
\end{align*}\]
Usually $g^2$ is given as 14.5 in the literature, that is because the convention $\hbar = c = 1$ is commonly used implicitly. From the above two equations, $\gamma$ is obtained to be
\[\begin{align*}
\gamma &= -(14.8) \frac{\hbar}{c} = -(14.8) (197.32858 \text{ Mev.fm.}) \\
&= -2920.46 \text{ Mev.fm.} \\
\text{(16)}
\end{align*}\]
Summing up, the parameters for the Yukawa potential are
\[\begin{align*}
\mu &= 4.76 \text{ fm}^{-1} \\
\gamma &= -2920.46 \text{ Mev.fm.} \\
\text{(17)}
\end{align*}\]
This together with the square well potential is plotted in Fig. (5-4).

* There are different definitions of pion-nucleon coupling constants due to rationalization, and the use of different coupling such as the pseudoscalar coupling and pseudovector coupling. To eliminate any possible confusion, the coupling constants $g, G, f$ and $F$ are explained. $g$ given in eq. (13) is unrationaized, analogous to the unrationaized form of the coulomb equation for electric charges, and since $g$ also comes as the coupling constant for pseudoscalar coupling interaction Lagrangian $i g \bar{\psi} \sigma_{\mu \nu} \gamma^\nu \phi$, $g$ is called the unrationaized pseudoscalar coupling constant. The rationaized pseudoscalar coupling constant is defined as $G$ such that
\[\begin{align*}
g^2 &= \frac{G^2}{4\pi} \\
\text{analogous to the rationaized coulomb equation. The unrationaized pseudovector coupling constant comes in from the pseudovector coupling interaction Lagrangian } f \bar{\psi} \gamma^\mu \gamma_5 \psi \left( \frac{1}{2} \gamma^\mu \gamma_5 \right). \text{ In first order calculation pseudovector and pseudoscalar Lagrangian gives equivalent results with}
\end{align*}\]
\[\begin{align*}
f^2 &= \left( \frac{m}{2M} \right)^2 g^2 \\
\text{(19)}
\end{align*}\]
Similar to $G$ in eq. (18), a rationaized pseudovector coupling constant $F$ can also defined from $f$. With $\hbar = c = 1$, $F^2$ is usually quoted to have the value $0.982$ which lead to the result for $g^2$ as given in eq. (15).
Lastly, the coulomb parameter $\eta$ is easy to calculate. We remember from the first section of this chapter that instead of $\eta = \frac{e^2}{\hbar c}$ we have $\eta = \frac{e^2 m}{\hbar c^2 \hbar}$. With the fine structure constant $\frac{e^2}{\hbar c} = 0.0072973506 \left( \approx \frac{1}{137} \right)$, (20)

$$\eta = \frac{e^2 m c^2}{\hbar c (\hbar c) \hbar} = \frac{(0.0072973506) (139.57 \text{ MeV})}{(197.3285 \text{ MeV}) k_{\text{fm}}} = \frac{(0.005161397)}{k \text{ fm}}$$

Around the $(3,3)$ resonance region, $k$ is approximately equal to 1.5 fm; therefore $\eta$ is approximately $3.4 \times 10^{-3}$ and is dimensionless.

3) **Convergence of Series Obtained**

In some of the results obtained in chapter 3, there are infinite series. Up to now, we have not discussed about their convergence. We have emphasized earlier that the convergence of these series does not depend on the actual values of the parameters involved. Nevertheless, their values indicate how fast these series converge.

Therefore, we postpone this discussion till now.

The first infinite series in our results appears in eq. (3-35).

It is

$$S_t = \sum_{m=1}^{\infty} \frac{(-1)^m (2m)^2}{(2m)!} \frac{2^m m!}{(2m)!} (a_q)^{-(m+1)} \mathcal{J}_{m+1}(a_q),$$

The $m = \ell$ and $m = \ell + 1$ terms are

$$u_{\ell} = \frac{(-1)^\ell (2\ell)^2 2^\ell \ell!}{(2\ell)!} (a_q)^{-(\ell+1)} \mathcal{J}_{\ell+1}(a_q)$$

and

$$u_{\ell+1} = \frac{(-1)^{\ell+1} (2\ell)^2 2^{\ell+1} (\ell+1)!}{(2\ell+2)!} (a_q)^{-(\ell+2)} \mathcal{J}_{\ell+2}(a_q).$$
Applying the ratio test to the series, we have

$$\lim_{l \to \infty} \left| \frac{V_{l+1}}{V_l} \right| = \lim_{l \to \infty} \frac{(2l)^3}{(2l+1)(a^2)} \left| \frac{J_{l+2}(a\eta)}{J_{l+1}(a\eta)} \right| .$$  \hspace{1cm} (24)

Noting that the asymptotic expansion for large orders of Bessel function

$$J_{\nu}(z) \approx \frac{1}{\sqrt{2\pi \nu}} \left( \frac{e^{\nu} \eta}{2\nu} \right)^{\nu} \hspace{1cm} (25)$$

the ratio

$$\lim_{l \to \infty} \left| \frac{J_{l+2}(a\eta)}{J_{l+1}(a\eta)} \right| = \left( \frac{e}{\nu} \right) \left( \frac{l+1}{l+2} \right) \left( \frac{1}{l+2} \right) \frac{a^2}{\eta} \hspace{1cm} (26)$$

Putting this back into eq. (24)

$$\lim_{l \to \infty} \left| \frac{V_{l+1}}{V_l} \right| = \lim_{l \to \infty} \frac{e (2l)^3}{2(l+2)(2l+1)(l+2)} \left( \frac{l+1}{l+2} \right) \frac{a^2}{\eta} = 0 \hspace{1cm} (27)$$

Therefore the infinite series in eq. (3-35) converges. Remembering that $\lambda$ is approximately equal to 2 from eq. (7), we see that this series converges very slowly, the ratio in eq. (27) decreases to less than .1 only for $l \geq 6$.

The next infinite series appears in eq. (3-42). It is

$$S_2 = \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m}}{4^m m!} \left( \frac{a^2}{\lambda} \right)^{m+\frac{1}{2}} H_{m+\frac{3}{2}}(2\lambda) \hspace{1cm} (28)$$

The $m = l$, and $l+1$ terms are

$$V_l = \frac{(-1)^l q^{2l}}{4^l \lambda^l} \left( \frac{a^2}{\lambda} \right)^{l+\frac{1}{2}} H_{l+\frac{3}{2}}(2\lambda) \hspace{1cm} (29)$$

and

$$V_{l+1} = \frac{(-1)^{l+1} q^{2l+2}}{4^{l+1} (l+1)!} \left( \frac{a^2}{\lambda} \right)^{l+\frac{3}{2}} H_{l+\frac{5}{2}}(2\lambda) \hspace{1cm} (30)$$
Again, applying the ration test, we have

\[
\lim_{L \to \infty} \left| \frac{v_{e+1}}{v_e} \right| = \lim_{L \to \infty} \frac{\frac{9^2}{4(L+1)}}{H_{2+\frac{3}{2}(L)}(2L)} = 0.
\]  

That the ratio of struve functions is less than one for large \( L \) can be deduced from the equation

\[
H_{\nu}(Z) = (\frac{3}{2})^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{3}{2})^{2k}}{\Gamma(\frac{3}{2} + k) \Gamma(\nu + \frac{3}{2} + k)}
\]

with \( \nu = L + \frac{3}{2} \),

\[
H_{2+\frac{3}{2}(L)}(Z) = (\frac{3}{2})^{\nu+\frac{3}{2}} \left\{ \frac{1}{(\frac{1}{2} Z)!(L+2)!} - \frac{(3/2)^{\nu}}{L^{\frac{3}{2}}(\frac{1}{2} Z)!(L+3)!} \right. 
\]

\[
+ \frac{(3/2)^{\nu}}{L^{\frac{3}{2}}(\frac{1}{2} Z)!(L+4)!} - \ldots \left\}.
\]

where

\[
\Gamma(\nu + \frac{3}{2}) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2\nu-1)}{2^{\nu}} \Gamma(\frac{3}{2}) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2\nu-1)}{2^{\nu}} \sqrt{\pi}
\]

has been used. We see that for \( L \) sufficiently large, only the first term is important, or

\[
\lim_{L \to \infty} H_{2+\frac{3}{2}(L)}(Z) = (\frac{3}{2})^{\nu+\frac{3}{2}} \frac{L+\frac{3}{2}}{(\frac{1}{2} Z)!(L+2)!}.
\]

The ration of Struve functions becomes

\[
\lim_{L \to \infty} \left| \frac{H_{2+\frac{3}{2}(L)}(2L)}{H_{2+\frac{3}{2}(L)}(2L)} \right| = \lim_{L \to \infty} \frac{L}{L+3} = 0.
\]

Putting this back into eq. (31),

\[
\lim_{L \to \infty} \left| \frac{v_{e+1}}{v_e} \right| = \lim_{L \to \infty} \frac{(\frac{9a}{2})^2}{4(L+1)(L+3)} = 0.
\]
Therefore the infinite series in eq. (3-42) converges. In addition, it also converges quite rapidly, because of the \((qa)^2\) term in eq. (37), which is less than 0.2 for scattering angle \(\theta\) of below 12 degrees
\[
q \leq 0.31 \text{ f.m}^{-1}
\]

The infinite series in eq. (3-102) can be split up into two. They are
\[
S_3 = \sum_{m=0}^{\infty} \frac{(-1)^m q^m a^{2m}}{4^m (m!)^2} \frac{\Gamma'(m+1+in)}{\Gamma(m+\frac{3}{2}+in)} \left(2\lambda\right)_{m+\frac{3}{2}+in}
\]
and
\[
S_4 = \sum_{m=0}^{\infty} \frac{(-1)^m q^m a^{2m}}{4^m (m!)^2} \frac{\Gamma'(m+1+in)}{\Gamma(m+\frac{3}{2}+in)} \left(2\lambda\right)_{m+\frac{3}{2}+in}
\]

Apply the ratio test,
\[
\lim_{\lambda \to \infty} \left| \frac{S_{n+1}}{S_n} \right| = \lim_{\lambda \to \infty} \frac{g^2(\frac{a^2}{\lambda^2})(\lambda+1+in)}{4(\lambda+1)^2} \left| \frac{J_{\lambda+\frac{3}{2}+in}(2\lambda)}{J_{\lambda+\frac{3}{2}+in}(2\lambda)} \right|
\]
where eq. (25) has been used, and it is also noted that for \(n\) estimated to be approximately 0.03, it is too small to be significant when compared with asymptotic \(\lambda\) values. \(S_3\) is therefore shown to be convergent.

Similarly, \(S_4\) can be shown to be convergent when eq. (35) is used

4) **Additional Equations Used for Numerical Calculation.**

We have used some special functions in the result obtained in the last two chapters. These special functions have to be treated before numerical calculation can be done using the computer. We begin first
with the gamma function \( T'(1 + in) \) which appears in the Coulomb scattering amplitude and other equations where Coulomb effect is present.

Using the relationship (14)

\[
T'(z+1) = z \ T'(z)
\]

we write

\[
T'(1 + in) = in \ T'(in)
\]

To calculate \( T'(in) \), we use the series expansion for \( \frac{1}{T'(z)} \)

\[
\frac{1}{T'(z)} = \sum_{k=1}^{\infty} C_k z^k
\]

with \( C_1 = 1 \) and \( C_2 = C_*( \text{the Euler's constant}) = 0.5772156649 \) and

the other coefficients given in reference 15. Their magnitude are all diminishingly small. Letting \( z = in \),

\[
\frac{1}{T'(in)} = \left( -Cn^2 + C_4n^4 - \ldots \right) + i \left( n - C_3n^3 + \ldots \right)
\]

Remembering that \( n \approx 3 \times 10^{-3} \) around the resonance region, we can ignore the \( C_4n^4 \) term as compared with \(-Cn^2\) and the \(-C_3n^3\) term as compared with \( n \). We then have

\[
\frac{1}{T'(in)} = -Cn^2 + in
\]

or \( T'(in) \) becomes

\[
T'(in) = \frac{\left( \frac{1}{T'(in)} \right)^k}{\left| \frac{1}{T'(in)} \right|} = \frac{-n(Cn+i)}{n^2(Cn^2+1)} = \frac{(Cn+i)}{n}
\]

Putting this back into eq. (42)

\[
T'(1 + in) = 1 - i \ Cn \quad n \ll 1
\]

An estimate of the error involved in the above equation was made.

The real part and the imaginary part has error factor of \( 10^{-5} \) and \( 10^{-4} \) respectively.
For the Coulomb amplitude, the argument of $\mathcal{F}(1+i\eta)$ is required. This is easily obtained, when we remember that for $z = x + iy$,
\[
\arg(z) = \arctan \left( \frac{y}{x} \right).
\]
Therefore, we have
\[
\arg \mathcal{F}(1+i\eta) = \arctan \left( -\eta \right) \quad \eta \ll 1
\] (48)

To calculate $J_{\frac{1}{2}}(a \sqrt{1 + (\frac{2\pi}{a})^2})$ amplitude in eq. (3-42), we use the relationships (16)
\[
j_n(z) = \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z)
\] (49)
and
\[
j_1(z) = \frac{\sin \frac{z}{2}}{\frac{z}{2}} - \frac{\cos \frac{z}{2}}{\frac{z}{2}}
\] (50)

to obtain
\[
J_{\frac{1}{2}}(z) = \left( \frac{2z}{\pi} \right)^{\frac{1}{2}} \left( \frac{\sin \frac{z}{2}}{\frac{z}{2}} - \frac{\cos \frac{z}{2}}{\frac{z}{2}} \right).
\] (51)

The second part of eq. (3-42) involves an infinite series of Struve functions $H_{m+\frac{1}{2}}(2\lambda)$. From the values of the parameters involved it is estimated that the first four terms $m = 0, 1, 2, 3$ in the series are sufficient for its calculation. The Struve functions involved are $H_{\frac{1}{2}}(2\lambda)$, $H_{\frac{3}{2}}(2\lambda)$, $H_{1}(2\lambda)$ and $H_{\frac{3}{2}}(2\lambda)$. From Abramowitz and Stigun (17) we obtain
\[
H_{\frac{1}{2}}(z) = \left( \frac{2}{\pi \frac{z}{2}} \right)^{\frac{1}{2}} (1 - \cos \frac{z}{2})
\] (52)
and
\[
H_{\frac{3}{2}}(z) = \left( \frac{2}{2\pi \frac{z}{2}} \right)^{\frac{1}{2}} \left( 1 + \frac{2}{\pi \frac{z}{2}} \right) - \frac{1}{\frac{2}{\pi \frac{z}{2}}} \left( \frac{\cos \frac{z}{2} + \sin \frac{z}{2}}{\frac{z}{2}} \right).
\] (53)

Furthermore, $H_{\frac{5}{2}}(z)$, $H_{\frac{7}{2}}(z)$ and $H_{2}(z)$ can be obtained from the above two equations by using the recurrence relation (18)
\[
H_{\nu+1}(z) + \frac{\nu}{z} H_\nu(z) = \frac{2}{\pi \frac{z}{2}} H_{\nu-1}(z) + \frac{\lambda}{\pi \frac{z}{2}} H_{\nu-2}(z)
\] (54)
to be
\[
H_{\frac{5}{2}}(z) = \frac{3}{\pi \frac{z}{2}} H_{\frac{3}{2}}(z) - H_{\frac{1}{2}}(z) + \frac{\lambda}{\pi \frac{z}{2}} H_{\frac{3}{2}}(z)
\] (55)
\[ H_{\frac{1}{2}}(3) = \frac{5}{2} H_{\frac{1}{2}}(3) - H_{\frac{3}{2}}(3) + \frac{(\frac{5}{2})^5}{6 \pi^5} \]  \hspace{1cm} (56)

and

\[ H_{\frac{3}{2}}(3) = \frac{7}{3} H_{\frac{1}{2}}(3) - H_{\frac{3}{2}}(3) + \frac{(\frac{7}{3})^5}{24 \pi^5}. \]  \hspace{1cm} (57)

When the Coulomb interaction is included in addition to the square well potential, the result obtained for the total scattering cross-section with pure Coulomb cross-section subtracted is eq. (3-80). This contains complex phase factors such as \( e^{-i\theta} \) which is calculated as \( \exp(-i\theta) \) using the computer. In addition, the eq. (3-80) also contains the Bessel function \( J_{\frac{3}{2}+i\theta}(2d) \) and the Struve function \( H_{\frac{3}{2}+i\theta}(2d) \). Although they can be calculated using their respective power series expansions (eq. (3-36) and eq. (32)), their complex orders introduce much complication in their calculation involving the calculation procedure and save computer time, we expand the Bessel function

\[ J_{\frac{3}{2}+i\theta}(2d) \] is expanded in a Taylor series expansion as

\[ J_{\frac{3}{2}+i\theta}(3) = J_{\frac{3}{2}}(3) + i\theta \frac{\partial}{\partial \theta} J_{\frac{3}{2}}(3) \bigg|_{\theta=\frac{1}{2}} + \frac{i\theta^2}{2!} \frac{\partial^2}{\partial \theta^2} J_{\frac{3}{2}}(3) \bigg|_{\theta=\frac{1}{2}} + \frac{i\theta^3}{3!} \frac{\partial^3}{\partial \theta^3} J_{\frac{3}{2}}(3) \bigg|_{\theta=\frac{1}{2}} + \ldots \]  \hspace{1cm} (58)

\[ \begin{align*}
= \left\{ J_{\frac{3}{2}}(3) - \frac{\pi^2}{2} \frac{\partial^2}{\partial \theta^2} J_{\frac{3}{2}}(3) \bigg|_{\theta=\frac{1}{2}} + \cdots \right\} + i\theta \left\{ \frac{\partial}{\partial \theta} J_{\frac{3}{2}}(3) \bigg|_{\theta=\frac{1}{2}} - \frac{\pi^2}{6} \frac{\partial^2}{\partial \theta^2} J_{\frac{3}{2}}(3) \bigg|_{\theta=\frac{1}{2}} + \ldots \right\}.
\end{align*} \]

We note from the following equation\(^{(19)}\) that the Bessel function \( J_{\nu}(3) \) has well-behaved derivatives with respect to the order \( \nu \)

\[ \frac{\partial}{\partial \nu} J_{\nu}(3) = J_{\nu}(3) \ln \left( \frac{3}{2} \right) - \left( \frac{3}{2} \right)^\nu \sum_{\ell=0}^\infty \frac{(-\nu)^\ell \psi(\nu+\ell+1) \left( \frac{3}{2} \right)^{2\ell}}{\ell!}, \]  \hspace{1cm} (59)
where $\psi(\nu+\ell+1)$ is the digamma function (or called the psi function) such that

$$\psi(3) = \frac{d}{d \nu} \left[ \ln \Gamma(3) \right] = \frac{T'(3)}{T(3)} \quad (3 = 2\xi \approx 4 \text{ \ in our case}).$$

We also note that $n^2 \approx 10^{-5}$.

Therefore $\tilde{J}_{\frac{3}{2} + i\nu}(3)$ can be approximated as

$$\tilde{J}_{\frac{3}{2} + i\nu}(3) = J_{\frac{3}{2}}(3) + \frac{i \nu}{\nu} J_{\nu}(3) \bigg|_{\nu = \frac{3}{2}} \quad (\nu \geq 1)$$

Putting in eq. (59),

$$\tilde{J}_{\frac{3}{2} + i\nu}(3) = J_{\frac{3}{2}}(3) + \frac{i \nu}{\nu} \left[ J_{\frac{3}{2}}(3) \ln(\frac{3}{2}) - \left( \frac{3}{2} \right)^{\frac{3}{2}} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \psi(\ell+\frac{3}{2}) \left( \frac{3}{2} \right)^{2\ell}}{\Gamma(\ell+3) \ell!} \right]$$

where $T(\ell+\frac{5}{2})$ and $\psi(\ell+\frac{3}{2})$ can be calculated using eq. (34) and (20)

$$\psi(n + \frac{1}{2}) = -C - 2 \ln 2 + 2 \left( \frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right)$$

respectively.

Using similar arguments, $\tilde{H}_{\frac{3}{2} + i\nu}(3)$ can be written as

$$\tilde{H}_{\frac{3}{2} + i\nu}(3) = H_{\frac{3}{2}}(3) + \frac{i \nu}{\nu} H_{\nu}(3) \bigg|_{\nu = \frac{3}{2}} \quad (\nu \geq 1)$$

The derivative of the Struve function with respect to order is not given in Abramowitz and Stegun or any other handbook we can find, but it can be derived simply from the power series expansion (eq. (32)) as follows

$$\frac{\partial}{\partial \nu} H_{\nu}(3) = \left( \frac{3}{2} \right)^{\nu+1} \ln(\frac{3}{2}) \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \psi(\ell+\frac{3}{2}) \left( \frac{3}{2} \right)^{2\ell}}{\Gamma(\ell+3+\nu) \ell!} + \left( \frac{3}{2} \right)^{\nu+1} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \psi(\ell+\nu+\frac{3}{2}) \left( \frac{3}{2} \right)^{2\ell}}{\Gamma(\ell+3+\nu) \ell!}$$

where eq. (60) has been used. Therefore, we have

$$\frac{\partial}{\partial \nu} H_{\nu}(3) = H_{\nu}(3) \ln(\frac{3}{2}) - \left( \frac{3}{2} \right)^{\nu+1} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \psi(\ell+\nu+\frac{3}{2}) \left( \frac{3}{2} \right)^{2\ell}}{\Gamma(\ell+3+\nu) \ell!}$$

Putting this in eq. (64),

$$\tilde{H}_{\frac{3}{2} + i\nu}(3) = H_{\frac{3}{2}}(3) + \frac{i \nu}{\nu} \left[ H_{\frac{3}{2}}(3) \ln(\frac{3}{2}) - \left( \frac{3}{2} \right)^{\nu+1} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} \psi(\ell+\frac{3}{2}) \left( \frac{3}{2} \right)^{2\ell}}{\Gamma(\ell+3+\nu) \ell!} \right]$$
where \( \psi(\ell+3) = (\ell+2)! \) and \( \psi(\ell+3) \) is calculated using the equation (21)

\[
\psi(n) = -C + \sum_{k=1}^{n-1} \frac{k}{(n-k)^{n}} \quad (n \geq 2).
\]

In the case of the differential scattering amplitude (eq. 3-102) for the coulomb plus square well potentials, it is again estimated to be necessary to keep only \( m=0, 1, 2, \) and 3 terms. The Bessel function \( J_{m+\frac{3}{2}+\text{in}}(3) \) and the Struve function \( H_{m+\frac{3}{2}+\text{in}}(3) \) involved in these terms are treated similarly to \( J_{\frac{3}{2}+\text{in}}(3) \) and \( H_{\frac{3}{2}+\text{in}}(3) \) in the above. The result are

\[
J_{m+\frac{3}{2}+\text{in}}(3) = J_{m+\frac{3}{2}}(3) + \text{in} \left\{ J_{m+\frac{3}{2}}(3) \ln \left( \frac{3}{2} \right) - \sum_{l=0}^{\infty} \frac{(-1)^l \psi(m+l+\frac{3}{2})}{\Gamma(m+l+\frac{3}{2})} \left( \frac{3}{2} \right)^l \right\}
\]

and

\[
H_{m+\frac{3}{2}+\text{in}}(3) = H_{m+\frac{3}{2}}(3) + \text{in} \left\{ H_{m+\frac{3}{2}}(3) \ln \left( \frac{3}{2} \right) - \sum_{l=0}^{\infty} \frac{(-1)^l \psi(m+l+\frac{3}{2})}{\Gamma(m+l+\frac{3}{2})} \left( \frac{3}{2} \right)^l \right\},
\]

where the gamma functions, digamma functions, the Struve functions \( H_{m+\frac{3}{2}}(3) \) with \( m = 0, 1, 2, 3 \) and the Bessel function \( J_{\frac{3}{2}}(3) \) have been given previously. For \( J_{\frac{3}{2}}(3) \), we note that

\[
J_{\frac{3}{2}}(3) = \left( \frac{3}{2} \right) \sin \frac{3}{2} - \frac{3}{2} \cos \frac{3}{2}
\]

therefore, from eq. (49)

\[
J_{\frac{3}{2}}(3) = \left( \frac{3}{2} \right)^\frac{1}{2} \left\{ \left( \frac{3}{2} \right) \sin \frac{3}{2} - \frac{3}{2} \cos \frac{3}{2} \right\}
\]

\( J_{\frac{3}{2}}(3) \) and \( J_{\frac{3}{2}}(3) \) can then be obtained by using the recurrence relation (23)

\[
J_{\nu-1}(3) + J_{\nu+1}(3) = \frac{2\nu}{3} J_{\nu}(3)
\]

as

\[
J_{\frac{3}{2}}(3) = \frac{3}{2} J_{\frac{3}{2}}(3) - J_{\frac{1}{2}}(3).
\]
and
\[ J_2(z) = \frac{7}{8} J_2(z) - J_2(z). \] (75)

In the case of Yukawa potential, the special functions involved are \( J_0(qb) \) and \( K_0 qb \) (see eq. (107) for example). To calculate this function on the computer, we use the subroutines BESJ for \( J_0(qb) \) and BESKO for \( K_0 qb \). BESJ comes from the Fortran and PL/I Scientific Subroutine Package (SSP) and BESKO from the International Mathematical & Statistical Libraries, Inc. Houston, Texas. (IMSL). Both SSP and IMSL are available at ICSA, Rice University. Although other methods of calculating these functions are available from sources such as Abramowitz and Stegun, (24) and Watson, (25) the subroutines we have chosen are more useful to our purpose.

5) Results and Discussions

In this section, we present some results from numerical calculation. In all cases, we have done calculation with \( n = \frac{e^2}{\hbar^2} \) and \( n = -\frac{me^2}{\hbar^2} \). The former would correspond to \( \pi^+ \rho \) scattering, but the latter would not correspond to \( \pi^- \rho \) scattering (see section 6, Chapter 3). Nevertheless, it is interesting to observe what happens when we have an attractive coulomb force instead of a repulsive coulomb force. From now on, we refer to the curves with \( n = \frac{me^2}{\hbar^2} \) as the \( \pi^+ \rho \) curves (or \( n(+) \) curves) and those with \( n = -\frac{me^2}{\hbar^2} \) as the \( n(-) \) curves.

Plotted in Fig. (5-5) is the total scattering cross-sections with
the square well potential as the nuclear potential. We see that the peak of the $\pi^+p$ curve have been shifted to lower energy at 183.5 Mev. On the other hand the n(-) curve has its peak shifted to higher energy at 196.5 Mev. When we calculate the resonance mass $M(n(+))$ corresponding to the $\pi^+p$ peak, we get $M(n(+))c^2 = 1228.56$ Mev. using eq. (5-4). Similarly $M(n(-))c^2 = 1238.46$ Mev. Therefore, we have $M(n(-))c^2 - M(n(+))c^2 = 9.9$ Mev. Although we actually should not treat $M(n(-))$ as $M(N^{*0})$, but if we do so as a first approximation we note that this gives the correct sign when compared with the mass splitting equation (eq. 1-1).

From Fig. (5-5), we further notice that the width for the $\pi^+p$ curve is different from that for the n(-) curve. The difference $\Gamma(n(-)) - \Gamma(n(+))$ is calculated to be about 19 Mev., which compares favorably well with eq. (1-2), considering that it is a very rough approximation to treat $\Gamma(N^{*0})$ as $\Gamma(N^{*0})$. Fig. (5-5), also shows that the total cross-section of $\pi^+p$ is smaller than that of the pure nuclear scattering. This is because the repulsive Coulomb potential forces the incident pion away from the proton, therefore less nuclear potential is encountered leading to a smaller total cross-section. By the same reasoning the increase in n(-)

cross-section can be understood.

In Figs. (5-6), (5-7), and (5-8), the differential cross-sections are given at pion energies of 140 Mev., 190 Mev. and 240 Mev. respectively. It can be seen that the interference effect is most signi-
significant at small angles and also at lower pion energies.

Finally in Fig. (5-9), total cross-sections for Yukawa potential is shown. As is expected from Fig. (5-3), a peak is not obtained. Nevertheless, the cross-section for $n^p$ is smaller than that of the pure nuclear scattering as have been reasoned before.

The effect of change in the parameters $V_0, a, \gamma$ and $\mu$ has been studied. We have found that for square well potential, when the product $(V_0 a)$ is decreased, the peak of the total cross-section will be shifted to a lower energy, the width will also be decreased. As for the Yukawa potential, we see that, from the definition of $\bar{r}$ in eq. (12) and from Fig. (5-3), decreasing $\gamma$ would have the same effect as decreasing $k$, and would therefore lead to smaller total cross-section. Change of $\mu$ only affect the magnitude of the cross-sections. We also observe from the equations for the total cross-sections and scattering amplitudes for the square well potential and Yukawa potential that their resulting cross-sections would not change if we simultaneously change the signs of both $V_0$ (or $\gamma$) and $n$.

By using fig. (5-5) and a short calculation, we find that if we wish to shift the $n^p$ peak to 190 Mev., we only have to change $V_0$ and a to 598.59 Mev. and 1.413 fm. respectively. The whole graph in fig. (5-5) will shift by 6.5 Mev. with essentially no other changes.
\[
\frac{\sigma_{\text{tot}}}{\pi a^2} = 2 \left( \frac{\cos(\pi \xi)}{\xi^2} + \frac{2 \sin(\pi \xi)}{\xi} - \frac{1}{\xi^2} \right)
\]

Fig. (5-1) Total scattering cross-section for the square well potential.
(a) \( \pi \)-N scattering: Formation of \( N^* \) in the \( \pi \)-N s-channel (right) due to contribution from N-exchange in the u-channel (left).

(b) N-N Scattering via one-pion exchange (left) and \( \pi \)-N scattering via one nucleon exchange (right). Notice that the coupling constants \( g_{NN\pi} \) are all the same because of the same particle \( NN\pi \) involved at the vertices.
Fig. (5-3) Total scattering cross-section for the Yukawa potential as a function of 
\[ \gamma = \frac{2 \gamma m}{\hbar^2 k} \].

\[
\sigma_y^{\text{tot}} = \frac{4 \pi}{\mu^2} \int B \left\{ 1 - \cos \left[ \gamma K_0(B) \right] \right\} dB
\]
Fig. (5-4) Square well and Yukawa potentials as functions of nuclear distance (fm)

\[ V_s(r) = \begin{cases} 
-606.97 \text{ MeV} & \text{if } r \leq 1.446 \text{ fm} \\
0 & \text{if } r > 1.446 \text{ fm} 
\end{cases} \]

\[ V_y(r) = -2920.46 \frac{\exp(-4.76 r)}{r} \text{ MeV} \]
Fig. (5-5) Total scattering cross-sections for the square well potential, with attractive Coulomb interaction, with repulsive Coulomb interaction, and with no Coulomb interaction.
Fig. (5-6) Differential scattering cross-sections for square well potential at pion energy of 140 Mev.
Fig. (5-7) Differential scattering cross-sections for square well potential at pion energy of 190 Mev.
Fig. (5-8) Differential scattering cross-sections for square well potential at pion energy of 240 Mev.

1----- Pure nuclear
2----- $\pi^+p$
3----- Nuclear with attractive Coulomb
Fig. (5-9) Total scattering cross-sections for the Yukawa potential, with attractive Coulomb interaction, with repulsive Coulomb interaction, and with no Coulomb interaction.
References


(02) Particle Data Group, Rev. Mod. Phys. 48, S1 (1976).


(12) Abramowitz and Stegun, p. 496, eq. 12.1.3.

(13) Abramowitz and Stegun, p. 255, eq. 6.1.8. and 6.1.12.

(14) Abramowitz and Stegun, p. 256, eq. 6.1.15.

(15) Abramowitz and Stegun, p. 256, eq. 6.1.34.

(16) Abramowitz and Stegun, p. 437, eq. 10.1.1 and p. 438, eq. 4.1.11.

(17) Abramowitz and Stegun, p. 497, eqs. 12.1.16 and 12.1.17.

(18) Abramowitz and Stegun, p. 497, eq. 12.1.9.

(19) Abramowitz and Stegun, p. 362, eq. 9.1.64.

(20) Abramowitz and Stegun, p. 358, eq. 6.3.4.

(21) Abramowitz and Stegun, p. 358, eq. 6.3.2.

(22) Abramowitz and Stegun, p. 439, eq. 10.1.11.

(23) Abramowitz and Stegun, p. 361, eq. 9.1.27.

(24) Abramowitz and Stegun, Ch. 9, eqs. 9.4.1, 9.4.3, 9.8.1, 9.8.5 and 9.8.6.

Appendix A

FRACTION $D(a)$ OF BEAM PARTICLES HITTING COUNTER

As is explained in Chapter 2, corrections have to be made on experimental data to take into account of multiple coulomb scattering and the finite size of the beam and counter before cross-sections due only to nuclear scattering can be obtained. Here, the derivation of eq. (6) for $D(N)$, the fraction of incident beam particles that hit the counter, of $D(N)$ (eq. (6)) will be presented. The system to be considered is given in Fig. 1. A finite sized particle beam travelling in the z-direction impinges upon a target at the origin of the cartesian coordinates. The beam is dispersed due to multiple Coulomb scattering by the target whose thickness is $L$. After traversing the target, a beam particle has a certain probability is dependent on the lateral distance $(\gamma_b)$ of the particle from the axis of the beam before hitting the target, the dimensions of the set-up, and the differential cross-section involved.

Fig. A1. Diagram of the system under discussion (x-z) plane is shown with y-axis pointing out of paper.

* This derivation comes mainly from sternheimer$^1$ and Rossi.$^2$
From the definition of differential scattering amplitude, the scattering probability \( \hat{f}(\vartheta) d\Omega d\varphi \), that a particle, after travelling a target length undergoes a collision which deflects the trajectory of the particle into the solid angle \( d\Omega \) at an angle \( \vartheta \) to its original motion, is given by

\[
\hat{f}(\vartheta) d\Omega d\varphi = N \frac{d\sigma(\vartheta)}{d\Omega} d\Omega d\varphi ,
\]

where \( N \) is the number density of target particles per cm\(^3\).

We consider only the projection of particle motion into the \((x, z)\) plane. The projected motion onto the \((y, z)\) plane will be described by the same functions except \( x \) is substituted by \( y \), because of cylindrical symmetry. Calling the projected angle of \( \vartheta \) onto the \((x, z)\) and \((y, z)\) planes respectively by \( \vartheta_x \) and \( \vartheta_y \), we note that for small angles, \( \vartheta \) is related to \( \vartheta_x \) and \( \vartheta_y \) by

\[
\vartheta^2 = \vartheta_x^2 + \vartheta_y^2 .
\]

Integrating \( \hat{f}(\vartheta) \) over \( \vartheta_y \), we obtain the probability per unit target thickness \( \hat{f}(\vartheta) d\vartheta d\varphi \) that a particle will deflect into \( d\vartheta_x \) at the projected angle \( \vartheta_x \) to its original motion. That is

\[
\int \hat{f}(\vartheta) d\vartheta d\varphi = N \int \frac{d\sigma(\vartheta_x, \vartheta_y)}{d\Omega} d\vartheta_x d\vartheta_y d\varphi .
\]

Now we find an expression for the number of beam particles, \( M(z, x, \vartheta_x) dx d\vartheta_x \), that have a displacement in \( dx \) at \( x \) and travel at an angle \( \vartheta_x \) in \( d\vartheta_x \) with the z-axis after a target thickness of \( z \) is traversed. In order to do so, we compute the charge in \( M(z, x, \vartheta_x) \) as additional thickness of target, \( dz \), is traversed. This change in
$M(z, x, \theta_x) \, dx \, d\theta_x$ can come in from two causes; scattering by the target and particle drift. Each of these two causes can lead some particles to be added to $M(z, x, \theta_x) \, dx \, d\theta_x$ and some others to be removed from $M(z, x, \theta_x) \, dx \, d\theta_x$.

In the case of scattering, some particles of $M(z, x, \theta_x) \, dx \, d\theta_x$ will be removed when they undergo a collision and leave the interval $d\theta_x$. The number of these particles is

$$[M(z, x, \theta_x) \, dx \, d\theta_x] \, dz = \int F(\theta_x') \, d\theta_x'. \quad (A4)$$

Some of the particles that were originally not in $M(z, x, \theta_x) \, dx \, d\theta_x$ but had lateral displacement in $dx$ at $x$ may be scattered into $d\theta_x$ at $\theta_x$ and, therefore, add to $M(z, x, \theta_x) \, dx \, d\theta_x$ by the number

$$dz \int [M(z, x, \theta_x + \theta_x') \, dx \, d\theta_x] \, F(-\theta_x') \, d\theta_x'. \quad (A5)$$

From eq. (A3), $F(\theta_x)$ is an even function, therefore we can write (A5) as

$$dz \int [M(z, x, \theta_x + \theta_x') \, dx \, d\theta_x] \, F(\theta_x') \, d\theta_x'. \quad (A6)$$

Combining (A4) and (A6), we obtain the net change in $M(z, x, \theta_x)$ due to scattering

$$dz \, dx \, d\theta_x \int [M(z, x, \theta_x + \theta_x') - M(z, x, \theta_x)] \, F(\theta_x') \, d\theta_x'. \quad (A7)$$

The lateral displacement of particles travelling at an angle $\theta_x$ will change by $\theta_x \, dz$ as additional thickness $dz$ of target is traversed. Therefore, particles that have lateral displacement $x$ at target thickness $z$, will have a different lateral displacement. Particles that have lateral displacement $x$ at target thickness $z$ will come from those that have lateral displacement $x - \theta_x \, dz$. Therefore, effect of particle
Drift will be a net change in $M(z, x, \theta_x) \, dx \, d\theta_x$ equal to

$$M(\bar{z}, x - \theta_x d\bar{z}, \theta_x) \, dx \, d\theta_x - M(\bar{z}, x, \theta_x) \, dx \, d\theta_x.$$  \hspace{1cm} \text{(A8)}

Since $\theta_x dz$ is infinitesimal, (A8) can be written as

$$-\theta_x d\bar{z} \frac{\partial}{\partial x} M(\bar{z}, x, \theta_x) \, dx \, d\theta_x.$$  \hspace{1cm} \text{(A9)}

Combining eqs (A7) and (A9), we obtain the change in $M(x, x, \theta_x) \, dx \, d\theta_x$ as length $dz$ of target is traversed, and a particle differential equation for $M(z, x, \theta_x)$ as

$$\frac{\partial}{\partial x} M(z, x, \theta_x) = -\theta_x \frac{\partial}{\partial x} M(z, x, \theta_x) + \int d\theta' [M(z, x, \theta_x + \theta') - M(z, x, \theta_x)] \xi(\theta').$$  \hspace{1cm} \text{(A10)}

Since coulomb scattering is sharply forward peaked, $\xi(\theta)$ decreases rapidly with $\theta_x$. Assuming that that only small angle scatterings are important, we can approximate the difference $M(z, x, \theta_x + \theta_x')$

$$M(z, x, \theta_x + \theta_x') - M(z, x, \theta_x) = \theta_x' \frac{\partial}{\partial \theta_x} M(z, x, \theta_x) + \frac{\theta_x'^2}{2} \frac{\partial^2}{\partial \theta_x^2} M(z, x, \theta_x).$$  \hspace{1cm} \text{(A11)}

When eq. (A11) is substituted into (A10), it gives two integrals:

$$\int \theta_x' \xi(\theta_x') \, d\theta_x' \quad \text{and} \quad \int \theta_x'^2 \xi(\theta_x') \, d\theta_x'.$$

The first one

$$\int \theta_x' \xi(\theta_x') \, d\theta_x' = 0$$  \hspace{1cm} \text{(A12)}

because $\xi(\theta)$ is an even function of $\theta_x'$.

For the second integral, we define from (A1) the root mean square angle of scattering per unit length of target traversed as $\theta_s$ such that

$$\theta_s^2 = N \int \theta^2 \left( \frac{d^2 \sigma(\theta)}{d\theta \, dN} \right) \, d\theta \, dN.$$  \hspace{1cm} \text{(A13)}
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thickness of the target, \( M(L, x, \theta_x) \) is, therefore, the probability
density function for finding the particle with lateral displacement \( x \)
and travelling at an angle \( \theta_x \) with the \( z \)-axis, on exit from the target.
Such a particle will reach the counter with lateral displacement equal

![Diagram](image)

**Fig. (A2)** A trajectory of a beam particle is shown with
(projected) lateral displacement, \( x \), upon exit from the target, and lateral displacement \( X \)
on hitting the counter

to \( X \) as is shown in Fig. (A2). We can express \( \theta_x \) in terms of \( x \)
and \( X \) by

\[
\theta_x = \frac{X - x}{L}
\]

(A19)

remembering that \( \theta_x \) is assumed small. From (A19),

\[
d\theta_x = \frac{dX}{L}.
\]

(A20)

Since \( \theta_x \) is not dependent on \( x \).

Eq. s(A19) and (A20), when put into \( M(L, x, \theta_x) \) \( d \theta_x \),
to replace $\theta_x$, give

$$M(L, x, \theta_x) d\theta_x = M(L, x, \frac{(X-x)}{L}) d\frac{X}{L}.$$  \hspace{1cm} (A21)

Then, an integration of the R.H.S. in eq. (A21) over $x$ results in the probability density function, $N(L, l, X)$, for finding the particle with

$$N(L, l, X) = \int_{-\infty}^{\infty} [M(L, x, \frac{(X-x)}{L}) L^{-1}] dx.$$ \hspace{1cm} (A22)

With eq. (A18), $N(L, l, X)$ becomes

$$N(L, l, X) = \frac{2\sqrt{3}}{\pi \theta^2 \lambda^3} \int_{-\infty}^{\infty} \exp \left\{ -\frac{4}{\theta^2} \left[ \frac{(X-x)^2}{\lambda^2} - \frac{3(x-X-y)}{\lambda^2} + \frac{3 \lambda^2}{\lambda^2} \right] \right\} dx

= \frac{2\sqrt{3}}{\pi \theta^2 \lambda^3} \exp \left\{ \frac{4}{\theta^2} \left[ \frac{X^2}{\lambda^2} - \frac{2}{\lambda^2} - \frac{3 \lambda^2}{\lambda^2} \right] \right\} \int_{-\infty}^{\infty} \exp \left\{ -\frac{4}{\theta^2} \left( \frac{1}{\lambda^2} + \frac{3}{\lambda^2} \right) X^2 \right\} dx.$$ \hspace{1cm} (A23)

Now, we have

$$\int_{-\infty}^{\infty} \exp \left( -\frac{1}{\rho^2} x^2 + \frac{2}{\rho^2} x \right) dx = \exp \left( -\frac{\theta^2}{\rho^2} \right) \frac{\theta^{\frac{\rho}{2}}}{\rho} \quad [\rho > 0].$$ \hspace{1cm} (A24)

Therefore, using eq. (A24) in eq. (A23) and after simplification, we obtain

$$N(L, l, X) = \frac{1}{\gamma^2 \nu_0} \exp \left( -\frac{X^2}{\gamma^2 \nu_0} \right)$$ \hspace{1cm} (A25)

with

$$\nu_0 = \ell^2 \left[ \frac{3}{\theta^2} (\lambda^2 + 3 \ell^2 + 3 \ell^2) \right].$$ \hspace{1cm} (A26)

The probability density function for finding the particle with lateral displacement $Y$ along $7$-axis is obtained from eq. (A.25) by
replacing $X$ by $Y$. Combining the two functions, we obtain the probability density function, $P(r_o, s)$, for finding the particle to be $G$, a distance $s$ from $H$ where it would be if there is no coulomb scattering to be

$$P(r_o, s) = N(L, l, X) dX N(L, l, Y) dY = \frac{1}{\pi r_o^3} \exp \left( -\frac{s^2}{r_o^2} \right) dA$$  (A27)

where $X^2 + Y^2 = s^2$ is used, and $dA$ is an infinitesimal segment of area.

With Fig. (A3), we see that $s$ is give by

$$s = \left( r_c^2 + r_b^2 - 2r_c r_b \cos \theta_c \right)^{\frac{1}{2}}$$  (A28)

By integrating $P(r_o, s)$ over the area of the counter gives the

Fig. (A3). Counter with center $O_c$, and positions where the particle would hit with or without coulomb scattering $G$ and $H$ respectively.

probability, $F(r_o, r_b)$, of the beam particle hitting the counter when, otherwise in the absence of coulomb scattering, it would arrive at distance $r_b$ from the centre of the counter.

$$F(r_o, r_b) = \int_{\text{counter area}} P(r_o, s) dA$$

$$= \frac{1}{\pi r_o^3} \int_0^{2\pi} \int_0^r \exp \left[ -\frac{1}{r_o^2} \left( r_c^2 + r_b^2 - 2r_c r_b \cos \theta_c \right) \right] r_c dr_c d\theta_c$$  (A29)

Using the integral representation of $I_o(3)$,

$$I_o(3) = \frac{1}{\pi} \int_0^{\pi} \exp \left( \pm 3 \cos \theta \right) d\theta$$  (A30)
\[ F(r_o, r_b) = \frac{2}{r_o^2} \exp\left(-\frac{r_b^2}{r_o^2}\right) \int_0^{r_c} \exp\left(-\frac{r_c^2}{r_o^2}\right) I_o\left(\frac{2r_c r_b}{r_o^2}\right) r_c \, dr_c, \quad (A31) \]

Eq. (A31) was numerically integrated for some values of \( r_o \) and \( r_b \), (A1) The typical behavior of \( F(r_o, r_b) \) is that it decreases with increasing \( r_o \) or increasing \( r_b \). It is easy to understand the effect of varying \( r_b \) on \( F(r_o, r_b) \), because the bigger \( r_b \) is, the further is the particle from the beam axis and therefore easier to miss the counter after multiple coulomb scattering. From eq. (A25), we see that \( r_o \) increases with the target thickness \( L \), the r.m.s. angle of coulomb scattering \( \theta_s \), and the distance of the counter from the target \( L \). Increase of \( L \) or \( \theta_s \) will mean increase in coulomb scattering effect, while the increase of \( L \) will mean that the scattered beam will diverge over a longer distance. These effects explains why \( F(r_o, r_b) \) decreases with increasing \( r_o \).

The probability function \( F(r_o, r_b) \) can be used to calculate the fraction \( D(\mathcal{N}) \), of incident beam particles hitting the counter. Assume that the beam distribution on the counter without multiple coulomb scattering is \( \mathcal{P}(r_b) \). Then integrating \( F(r_o, r_b) \) over the beam gives \( D(\mathcal{N}) \) after appropriate normalization, \[ D(\mathcal{N}) = \frac{\int_0^{r_b} F(r_o, r_b) \mathcal{P}(r_b) r_b \, dr_b}{\int_0^{r_b} \mathcal{P}(r_b) r_b \, dr_b} \quad (A32) \]
(cylindrical symmetry is used).

In calculating the fraction $D(\Omega)$, $F(\mathbf{r}_o, \mathbf{r}_b)$ is required. Since the integral in eq. (A31) is not calculated analytically, numerical evaluation of $F(\mathbf{r}_o, \mathbf{r}_b)$ is necessary. This, however, can sometimes be avoided depending on the actual form for the function $\mathcal{P}(\mathbf{r}_b)$. If, for example, we can assume $\mathcal{P}(\mathbf{r})$ to be gaussian with the peak at the center of the beam, i.e.

$$\mathcal{P}(\mathbf{r}) = \frac{1}{2\pi \sigma^2} \exp \left(-\frac{\mathbf{r}^2}{2 \sigma^2}\right) \quad 0 < r < \infty, \quad (A33)$$

where the normalization is chosen such that

$$\int \mathcal{P}(\mathbf{r}) dA = \frac{1}{2\pi \sigma^2} \int_0^{2\pi} \int_{\mathcal{R}_b} \exp \left(-\frac{r^2}{2 \sigma^2}\right) r dr d\theta\]

$$= \frac{1}{\sigma^2} \int_0^{\infty} \exp \left(-\frac{r^2}{2 \sigma^2}\right) r dr = 1. \quad (A34)$$

The fact that $\mathcal{R}_b$ is assumed infinitely large does not worry us, because $\mathcal{P}(\mathbf{r})$ is practically zero for $r \gg \sigma$. Using the $\mathcal{P}(\mathbf{r})$ and eq. (A31), we obtain

$$D(\Omega) = 2\pi \int_0^{\infty} \frac{2}{\sigma^2} \exp \left(-\frac{\mathbf{r}_c^2}{\sigma^2}\right) \int_{\mathcal{R}_c} \exp \left(-\frac{2 \mathbf{r}_c \mathbf{r}_b}{\sigma^2}\right) r_c dr_c \frac{1}{2\pi \sigma^2} \exp \left(-\frac{\mathbf{r}_b^2}{2 \sigma^2}\right) r_b dr_b, \quad (A35)$$

where the factor $2\pi$ comes from the evaluation of the integral in the denominator of eq. (A32). Interchanging the integral, we get

$$D(\Omega) = \frac{2}{\sigma^2} \int_{\mathcal{R}_c} \exp \left(-\frac{\mathbf{r}_c^2}{\sigma^2}\right) r_c dr_c \int_0^{\infty} \exp \left[-\frac{(\mathbf{r}_c + \frac{1}{2\sigma^2}) \mathbf{r}_b^2}{\sigma^2}\right] \int_{\mathcal{R}_b} \exp \left(\frac{2 \mathbf{r}_c \mathbf{r}_b}{\sigma^2}\right) r_b dr_b. \quad (A36)$$
Noting that

\[ I_0(z) = J_0(iz) \]  \hspace{1cm} (A37)

and the integral

\[ \int_0^\infty r_b \exp(-\chi r_b^2) J_0(\beta r_b) dr_b = \frac{1}{2\chi} \exp\left(-\frac{\beta^2}{4\chi}\right) \quad [\text{Re } \chi > 0] \]  \hspace{1cm} (A38)

\( D(\Omega) \) in eq. (A35) becomes (with \( \beta \) equal to \( i \frac{2 \frac{R_c}{r_o}}{2^\chi} \)),

\[ D(\Omega) = \frac{2}{(2\alpha^2 + r_o^2)} \int_0^{R_c} \exp\left(-\frac{\frac{R_c^2}{r_o^2 + 2\alpha^2}}{2\alpha^2 + r_o^2} \right) r_c dr_c \]  \hspace{1cm} (A39)

and then

\[ D(\Omega) = 1 - \exp\left(-\frac{R_c^2}{r_o^2 + 2\alpha^2} \right) \]  \hspace{1cm} (A40)

Hence, numerical calculation of eq. (A31) is avoided when the beam distribution \( f(r) \) is assumed to be gaussian.


(A3) Rossi and Greissen, Rev. Mod. Phys. 13, 240 (1942).

(A4) B. Rosi, p. 66.


(A6) I. S. Gradshteyn and I. M. Ryzhik, p. 958. eq. 8. 431. 3.

(A7) I. S. Gradshteyn and I. M. Ryzhik, p. 952. eq.8, 406. 3.
Appendix B

DERIVATION OF EQUATION (3-39)

On P. 43, an error was found in a formula in Gradshteyn and Ryzhik\(^{(B1)}\). A corrected version is given in eq (3-39) without proof. Here we derive eq (3-39) from a more general formula. Verification comes from the used of this corrected formula in obtaining agreement in eq. (75).

We start out from the integral\(^{(B2)}\)

\[
\int_0^\infty x^{2\gamma-1} (u^2-x^2)^{\nu-1} \cos(ax) \, dx = \frac{1}{2} u^{2\gamma-2} B(\nu, \frac{1}{2}) \left[ \frac{\Gamma(\gamma; 1/2, \nu, \frac{a^2}{4})}{\Gamma(\gamma; 1/2, \nu, \frac{a^2}{4})} \right] \quad [\text{Re} \mu > 0, \text{Re} \gamma > 0].
\]

(B1)

Let \( \gamma = 1 \) and \( \nu = \nu + \frac{1}{2} \), then

\[
\int_0^\infty x (u^2-x^2)^{\nu+\frac{1}{2}} \cos(ax) \, dx = \frac{1}{2} u^{2\nu+1} B(\nu+\frac{1}{2}, 1) \left[ \frac{\Gamma(\nu; 1/2, \nu + \frac{3}{2}, -\frac{a^2}{4})}{\Gamma(\nu; 1/2, \nu + \frac{3}{2}, -\frac{a^2}{4})} \right] \quad [\text{Re} \nu > \frac{3}{2}]
\]

(B2)

where \( B(\gamma, \omega) \) is a Beta function and \( \sum_{\gamma} F_\gamma (\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; \gamma) \) is a generalized hypergeometric series. The Beta function can be defined as

\[
B(\nu + \frac{1}{2}, 1) = \frac{\Gamma(\nu + \frac{1}{2}) \Gamma(\nu)}{\Gamma(\nu + 1)}.
\]

(B3)

Therefore

\[
B(\nu + \frac{1}{2}, 1) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + \frac{3}{2})} = \frac{1}{\nu + \frac{1}{2}},
\]

(B4)

where we have used \( \Gamma(1) = 1 \) and \( \Gamma'(1) = 1 \). The generalized hypergeometric series is defined as

\[
\sum_{\gamma=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \ldots (\alpha_p)_k (\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k}{(\gamma)_k (\gamma + 1)_k \ldots (\gamma + k)_k k!} z^k,
\]

(B5)

where the notation \( (\alpha)_k \) mean
\[(a)_{k} = \alpha (\alpha + 1) \cdots (\alpha + k - 1) = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}. \quad \text{(B6)}\]

From eq. (B6), we note that
\[(1)_{k} = k!, \quad (\alpha)_{k} = \alpha (\alpha + 1) \cdots (\alpha + k - 1). \quad \text{(B7)}\]

Then, we can write the function \( _1F_2 (1; a, b; z) \) as
\[
_1F_2 (1; a, b; z) = \sum_{k=0}^{\infty} \frac{(1)_{k} z^{k}}{(a)_{k} (b)_{k} k!}
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{(3)_{k-1}}{a(a+1)_{k-1} b(b+1)_{k-1} k!}
\]

\[
= 1 + \frac{3}{ab} \sum_{k=0}^{\infty} \frac{(3)_{k-1}}{(a+1)_{k-1} (b+1)_{k-1} k!}
\]

or
\[
_1F_2 (1; a, b; z) = 1 + \frac{3}{ab} _1F_2 (1; a+1, b+1; z) \quad \text{(B8)}
\]

Note that \(a, b\) is symmetric. For \(a = \frac{1}{2}\), we have
\[
_1F_2 (1; 1, 1, \frac{3}{2}, \frac{3}{2}; -\frac{a u^2}{4}) = 1 - \frac{a^2 k^2}{(2 \mu + 3)} _1F_2 (1; \frac{3}{2}, \frac{3}{2}; \frac{3}{2}, -\frac{a u^2}{4}) \quad \text{(B9)}
\]

or
\[
_1F_2 (1; 1, 1, \frac{3}{2}, \frac{3}{2}; -\frac{a u^2}{4}) = 1 - \frac{a^2 k^2}{(2 \mu + 3)} _1F_2 (1; \mu+\frac{3}{2}, \frac{3}{2}; -\frac{a u^2}{4}) \quad \text{(B10)}
\]

from eq. (B5). Comparing eq. (B10) with the equation (B5)
\[
H_{\delta} (\omega) = \frac{2(\omega/2)^{\delta+1}}{\sqrt{\pi} \Gamma(\delta+\frac{3}{2})} _1F_2 \left(1; \frac{3}{2} + \delta, \frac{3}{2}; -\frac{\omega^2}{4}\right). \quad \text{(B11)}
\]

Putting eq. s. (B10) and (B11) into (B2), we obtain
\begin{align*}
  \int_0^\mu x (u^2 - x^2)^{\nu + \frac{1}{2}} \cos (ax) \, dx &= \frac{1}{2} \mathcal{U}^{2\nu + 1} \left( \frac{1}{\nu + \frac{1}{2}} \right) \left[ -\frac{\mathcal{U}^2 \sqrt{\pi} \Gamma \left( \nu + \frac{3}{2} \right) \mathcal{V}}{(a u)^{2\nu + 2} \mathcal{W}} \right] H_{\nu + 1} (au) \\
  \text{or} \\
  \int_0^\mu x (u^2 - x^2)^{\nu + \frac{1}{2}} \cos (ax) \, dx &= \frac{\mathcal{U}^{2\nu + 1}}{2\nu + 1} - \frac{\sqrt{\pi}}{2} \mathcal{U} \left( \frac{2\nu + 1}{a} \right) \Gamma \left( \nu + \frac{3}{2} \right) H_{\nu + 1} (au)
\end{align*}

which is eq. (3-39).

(B2) Gradshteyn and Ryzhik, p. 427, eq. 3.771-4.


(B4) Gradshteyn and Ryzhik, p. 1045, eq. 9.14-1.

(B5) Abramowitz and Stegun, p. 497, eq. 12.1.21.
Appendix C

EVALUATION OF THE FUNCTIONS $M_s(b)$ AND $M_p(b)$

The interaction of pion with each nucleon in the nucleons contributes to the function $e^{i \chi_N(b)}$ (where $\chi_N(b)$ is the nuclear phase-shift function) by a factor equal to $[1 - M_s(b)]$ or $[1 - M_p(b)]$ depending on whether the nucleon is in the s- or p-shell. We shall now proceed to evaluate $M_s(b)$ and $M_p(b)$. (See Ch. 4).

1) s-shell nucleon

The single particle density corresponding to a harmonic oscillation potential for the s-shell nucleon is

$$f_s(r) = \frac{1}{\pi^{1/2} a_0^3} e^{\frac{-r^2}{2a_0^2}}$$

where $a_0$ is the radial parameter.

The normalization is chosen such that

$$\int f_s(r) \, d^3r = 1$$

(C2)

The form factor of the single particle density is defined as

$$S_s(q) = \int \exp(i \frac{q \cdot r}{2}) f_s(r) \, d^3r$$

(C3)

Using $f_s(r)$ given in eq. (C1)

$$S_s(q) = \frac{1}{\pi^{3/2} a_0^3} \int \int \int \exp(i \frac{q \cdot r \cos \theta}{2}) \exp(-\frac{r^2}{2a_0^2}) r^2 \, d(cos \theta) \, d\phi \, dr$$

(C4)

On integrating $\phi$ and $cos \theta$ we get

$$S_s(q) = \frac{4}{\pi \pi^2 a_0^3} \int sin(q r) \exp(-\frac{r^2}{2a_0^2}) r \, dr$$

(C5)

Using the integral (C2)

$$\int_0^\infty \exp(-p^2 x^2) \sin(ax) \, dx = \frac{a \sqrt{\pi}}{4p^2} \exp(-\frac{a^2}{4p^2})$$

(C6)
we obtain

\[ S_s(q) = \exp \left( -\frac{a_2^2q^2}{4} \right) \]  

(C7)

From Ch. 4, \( M_s(b) \) is shown to be

\[ M_s(b) = \frac{1}{2\pi i k} \int \exp (-i g \cdot b) S_s(q) f(q) \, d^2q, \]  

(C8)

where \( f(q) \) is assumed to be a gaussian such that

\[ f(q) = f(0) \exp \left( -\frac{\beta^2q^2}{2} \right) \]  

(C9)

with

\[ f(0) = \frac{i k_0}{4\pi} \left( 1 - i\alpha \right). \]  

(C10)

Therefore

\[ M_s(b) = \frac{f(0)}{2\pi i k} \int_0^{2\pi} \exp (-i q b \cos \phi) \exp \left[ \left( \frac{i}{2} \beta^2 + \frac{i}{4} a_0^2 \right) q^2 \right] dq \, d\phi \]  

(C11)

using the integral (C3)

\[ \frac{1}{2\pi} \int_0^{2\pi} \exp (\pm i \lambda \cos \phi) \, d\phi = J_0(\lambda). \]  

(C12)

We get

\[ M_s(b) = \frac{f(0)}{i k} \int_0^{2\pi} J_0(q b) \exp \left[ \left( \frac{i}{2} \beta^2 + \frac{i}{4} a_0^2 \right) q^2 \right] q \, dq. \]  

(C13)

Noting the integral (C4)

\[ \int_0^{\infty} x^{\nu+1} \exp (-\alpha x^2) J_\nu(\beta x) = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp \left( -\frac{\beta^2}{4\alpha} \right) \left[ \text{Re} \alpha > 0, \text{Re} \nu > -1 \right] \]  

we obtain (with \( \nu = 0 \))

(C14)
\[ M_s(b) = \frac{\mathcal{K}(\alpha)}{i \mathcal{K}} \frac{2}{\left(a_0^2 + 2\beta\right)} \exp\left(-\frac{b^2}{a_0^2 + 2\beta}\right). \]  \hspace{1cm} (C15)

Putting in eq. (C10),

\[ M_s(b) = \frac{(1-i\alpha)0^-}{2\pi \left(a_0^2 + 2\beta\right)} \exp\left(-\frac{b^2}{a_0^2 + 2\beta}\right). \]  \hspace{1cm} (C16)

2) **p-shell Nucleon**

The single particle density function corresponding to a harmonic oscillation potential for the p-shell nucleon is

\[ \rho_p(r) = \frac{2}{3 \pi^{\frac{3}{2}} a_0^5} r^2 \exp\left(-\frac{r^2}{a_0^2}\right) \]  \hspace{1cm} (C17)

which is again normalization as in eq. (C2).

The form factor defined according to eq. (C3) is therefore equal to

\[ S_p(q) = \frac{2}{3 \pi^{\frac{3}{2}} a_0^5 q^5} \int_0^{2\pi} \int_0^{\pi} \exp(i q r \cos \theta) \exp\left(-\frac{r^2}{a_0^2}\right) r^4 d\theta d\phi dr \]  \hspace{1cm} (C18)

Integrating over \( \phi \) and \( \cos \theta \) gives

\[ S_p(q) = \frac{8}{3 \pi^{\frac{3}{2}} a_0^5 q^5} \int_0^{\pi} \sin(q r) \exp\left(-\frac{r^2}{a_0^2}\right) r^3 dr \]  \hspace{1cm} (C19)

Using the integral\(^{(C5)}\)

\[ \int_0^\infty x \exp(-p^2 x^2) \sin(ax) dx = \frac{\pi^\frac{3}{2}}{16 p^7} \left(6 a^2 - a^2 \right) \exp\left(-\frac{a^2}{4 p^2}\right) \]  \hspace{1cm} (C20)

we therefore obtain

\[ S_p(q) = \left(1 - \frac{1}{6} a_0^2 q^2\right) \exp\left(-\frac{q^2 a_0^2}{4}\right) \]  \hspace{1cm} (C21)
From Ch. 4, we have

\[ M_p(b) = \frac{1}{2\pi i k} \int \exp(-i \frac{q \cdot k}{b}) S_p(q) f(q) d^3 q \]  

(C22)

Using \( f(q) \) and \( S_p(q) \) from eq. (C10) and eq. (C21) respectively,

\[ M_p(b) = M_p^{(1)}(b) + M_p^{(2)}(b) \]  

(C23)

where

\[ M_p^{(1)}(b) = \frac{\beta(0)}{2\pi i k} \int \exp(-i \frac{q b \cos \theta}{b}) \exp \left[ -\left( \frac{1}{b^2} + \frac{1}{4} \alpha^2 \right) \frac{q^2}{b} \right] q d q d \theta \]

(C24)

and

\[ M_p^{(2)}(b) = -\frac{\beta(0)}{12\pi i k} \int \exp(-i \frac{q b \cos \theta}{b}) \exp \left[ -\left( \frac{1}{b^2} + \frac{1}{2} \alpha^2 \right) \frac{q^2}{b} \right] q^2 d q d \theta \]

(C25)

\( M_p^{(1)}(b) \) is exactly the same as \( M_0(b) \) in eq. (C11) and therefore

\[ M_p^{(1)}(b) = \frac{1}{2\pi \left( \frac{\alpha^2}{b^2} + \frac{\beta^2}{b^2} \right)} \exp \left( -\frac{b^2}{\left( \frac{\alpha^2}{b^2} + \frac{\beta^2}{b^2} \right)} \right) \]

(C26)

Integrating \( \phi_0 \) in eq. (C25),

\[ M_p^{(2)}(b) = -\frac{\beta(0) \alpha^2}{6 i k} \int_0^\infty J_0(q b) \exp \left[ -\left( \frac{1}{b^2} + \frac{1}{2} \alpha^2 \right) \frac{q^2}{b} \right] q^3 d q \]

(C27)

Starting from the integral \( \text{C6} \)

\[ \int_0^\infty \exp(-x^2) x^{2n+\mu+1} J_\mu(2x \sqrt{3}) dx = \frac{\pi^\frac{\mu}{2}}{2} \exp(-3) x^{\frac{\mu}{2}} L_n^\mu(3) \]

(C28)

the following integral is obtained for \( \mu = 0 \) and \( n = 1 \)

\[ \int_0^\infty \exp(-x^2) x^3 J_0(2x \sqrt{3}) dx = \frac{1}{2} \exp(-3) L_0^0(3), \]

(C29)

where the special case for the Laguerre polynomial \( \text{C7} \)

\[ L_1^0(x) = x + 1 + x \]

(C30)
has been employed. After appropriate change of variable in eq. (C27)
\[
M_p^{(3)}(b) = -\frac{\kappa_1}{6} \lambda \begin{pmatrix} 2 \frac{\lambda^2}{\alpha^2 + 2\beta^2} \end{pmatrix}^2 \int_0^\infty \mathcal{J}_0 \left( 2 \left( \frac{\lambda^2}{\alpha^2 + 2\beta^2} \right)^{\frac{1}{2}} \left( \frac{b^2}{\alpha^2 + 2\beta^2} \right)^{\frac{1}{2}} \right) \exp \left[ -\left( \frac{\lambda^2}{\alpha^2 + \beta^2} \right)^2 \right] \left( \frac{\alpha^2 + 2\beta^2}{\alpha^2} \right)^{\frac{3}{2}} \phi^3 \, d\phi
\]  
(C31)

which can then be integrated according to eq. (C29) to be
\[
M_p^{(2)}(b) = -\frac{(1-i\delta)\lambda}{3\pi(\alpha^2 + 2\beta^2)^2} \exp \left( -\frac{b^2}{\alpha^2 + 2\beta^2} \right) \left( 1 - \frac{b^2}{\alpha^2 + 2\beta^2} \right)
\]  
(C32)

where eq. (C10) for \( f(O) \) has been used.

Adding \( M_p^{(1)}(b) \) (eq. ((26))) and \( M_p^{(2)}(b) \) (eq. ((32))) together we obtain the result for \( M_p(b) \), which is
\[
M_p(b) = \frac{\rho}{2\pi(\alpha^2 + 2\beta^2)} \left[ 1 - \frac{2\alpha^2}{3(\alpha^2 + 2\beta^2)} \left( 1 - \frac{b^2}{\alpha^2 + 2\beta^2} \right) \right] \exp \left( -\frac{b^2}{\alpha^2 + 2\beta^2} \right)
\]  
(C33)


(C4) Gradshteyn and Ryzhik, p. 717, eq. 6. 631-4.

(C5) Gradshteyn and Ryzhik, p. 495, eq. 3-952-5.

(C6) Gradshteyn and Ryzhik, p. 718, eq. 6. 631-10.

(C7) Gradshteyn and Ryzhik, p. 1038, eq. 8. 973-2.
Appendix D

Coulomb Phase-Shift Function \( \chi_{cs}(b) \)

Coulomb phase-shift function \( \chi_{cs}(b) \) for the scattering of an incident pion with gaussian charge density by an s-shell proton in the harmonic oscillator potential is to be calculated. We have noted that the particle density function for such a proton is equivalent to the gaussian, therefore, a result for \( \chi_{cs}(b) \) by Franco\(^{(D1)}\) can be employed. Our derivation for this result is presented here. The charge density functions for the incident pion and the s-wave proton are respectively

\[
\rho_{\pi}(r) = \frac{1}{\pi \frac{3}{2} b_0^3} e^x \left(-\frac{r^2}{b_0^2}\right) \tag{D1}
\]

and

\[
\rho_s(r) = \frac{1}{\pi \frac{3}{2} a_0^3} e^x \left(-\frac{r^2}{a_0^2}\right) \tag{D2}
\]

The potential between these two charge particles is

\[
V_{cs}(R) = e^2 \int \int \frac{\rho_{\pi}(r') \rho_s(r)}{(r' - r)} d^3r' d^3r \tag{D3}
\]

where notation, \( R, r, r' \) and \( \mathcal{E} \) are given in Fig. 1 in chapter 4.

Using eqs. (D1) and (D2), \( V_{cs}(R) \) becomes

\[
V_{cs}(R) = \frac{e^2}{\pi \frac{3}{2} a_0^3 b_0^3} \int \int \frac{\exp \left(-\frac{r'^2}{b_0^2} - \frac{r^2}{a_0^2}\right)}{(r' - r)} d^3r' d^3r \tag{D4}
\]
Changing integration variables from $\xi'$ to $\xi = \xi' - \xi$ and from $\xi''$ to $\xi'' + \beta$, we have

$$V_{cs}(R) = \frac{e^2}{(r_0 a_0 b_0)^3} \int \frac{1}{x} \exp\left[-\frac{(r - \xi)^2}{b_0^2} - \frac{(\xi + r)^2}{a_0^2}\right] d^3x \ d^3r \quad (D5)$$

Regrouping terms,

$$V_{cs}(R) = \frac{e^2}{(r_0 a_0 b_0)^3} \int \frac{1}{x} \exp\left[-\frac{(r + \xi)^2}{b_0^2} - \frac{(\xi + r)^2}{a_0^2}\right] \cdot$$

$$\cdot \exp\left[-\frac{2}{a_0^2 b_0^2} \beta \cdot (b_0^2 \xi + a_0^2 R)\right] d^3x \ d^3r \quad (D6)$$

Letting the vector $(b_0^2 \xi + a_0^2 R) = \frac{R}{y}$, such that when the angular parts of $r$ is integrated, we obtain

$$V_{cs}(R) = \frac{e^2}{na_0 b_0} \exp\left(-\frac{R^2}{b_0^2}\right) \int \frac{1}{xy} \exp\left[-\frac{(a_0^2 b_0^2)^2}{b_0^2} r^2 - \frac{\xi^2}{a_0^2}\right] \cdot$$

$$\cdot \left[\exp\left(-\frac{2}{a_0^2 b_0^2} r y\right) - \exp\left(-\frac{2}{a_0^2 b_0^2} a y\right)\right] r dr d^3x \quad (D7)$$

$$= \frac{2e^2}{n^2 a_0 b_0} \exp\left(-\frac{R^2}{b_0^2}\right) \int \frac{1}{xy} \exp\left[-\frac{\xi^2}{a_0^2}\right] \int_0^\infty \exp\left[-\frac{(a_0^2 b_0^2)^2}{a_0^2 b_0^2} r^2\right] r^2 \left[\frac{2}{a_0^2 b_0^2}\right] dr \ dx \quad (D7)$$

Using the integral $(D2)$

$$\int_0^\infty \exp(-\beta x^2) \sh \gamma x \ dx = \frac{\gamma}{\beta} \left(\frac{\pi}{\beta}\right)^{1/2} \exp\left(\frac{\gamma^2}{4\beta}\right)$$

$$\quad [\Re \beta > 0] \quad (D8)$$

$$V_{cs}(R) = \frac{e^2}{n^2 (a_0^2 + b_0^2)^2} \exp\left(-\frac{R^2}{b_0^2}\right) \int \frac{1}{x} \exp\left(-\frac{x^2}{a_0^2}\right) \exp\left[-\frac{\gamma^2}{\beta}\right] d^3x \quad (D9)$$
Using \( y' = (b_0 x + a_0^2) \)

\[
V_{cs}(R) = \frac{e^2}{\pi^{3/2}(a_0^2 + b_0^2)^2} \exp\left(\frac{-x^2}{a_0^2}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{y'^2}{b_0^2}\right) \, dy' \, dx \, dy \cdot \exp\left[\frac{a_0^4 x^2 + a_0^2 R^2}{a_0^2 b_0^2 (a_0^2 + b_0^2)} + \frac{2xR}{a_0^2 + b_0^2}\right] x \, d\phi \, d\mu \, dx
\]

(D10)

where \( \mu = \cos \theta \). Integrating over the angular part of \( x \), we get

\[
V_{cs}(R) = \frac{2e^2}{\pi^{3/2}(a_0^2 + b_0^2)^2} \exp\left(-\frac{R^2}{a_0^2 + b_0^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a_0^2 + b_0^2}\right) \text{sh}\left(\frac{2xR}{a_0^2 + b_0^2}\right) \, dx
\]

(D11)

Using the integral (D3)

\[
\int_{0}^{\infty} \exp(-\beta x^2) \text{sh}(ax) \, dx = \frac{1}{2} \left(\frac{\pi}{\beta}\right)^{1/4} \exp\left(\frac{a^2}{4\beta}\right) \Phi\left(\frac{a}{2\sqrt{\beta}}\right)
\]

where \( \Phi(\beta) \) is the probability integral. We note that the above result for \( V_{cs}(R) \) reduces correctly to the simple case when the incoming pion is point charged, i.e. when \( b_0 = 0 \),

\[
V_{cs}(R) = \frac{e^2}{R} \Phi\left(\frac{R}{a_0}\right)
\]

(D13)

point-charged pion

(D14)
\[ \chi_{cs}(k) = \frac{-e^2}{\hbar \nu} \int_{-\infty}^{\infty} \mathcal{V}_{cs}(\sqrt{\delta^2 + 3^2}) \, dz \]

\[ = \frac{-e^2}{\hbar \nu} \int_{0}^{\infty} \frac{\mathcal{F}(\sqrt{\frac{\delta^2 + 3^2}{a^2 + b^2}})}{\sqrt{b^2 + z^2}} \, dz \]  \hspace{1cm} (D15)

Using the integral representation of the probability integral $\mathcal{F}(xy)$

\[ \mathcal{F}(xy) = 1 - \frac{2}{\sqrt{\pi}} \exp(-x^2 y^2) \int_{0}^{\infty} \frac{\exp(-t^2 y^2)}{\sqrt{t^2 + x^2}} \, dt \]  \hspace{1cm} \text{[Re } y^2 > 0\text{]} \hspace{1cm} (D16)

\[ \chi_{cs}(k) \]  \hspace{1cm} becomes

\[ \chi_{cs}(k) = \frac{-e^2}{\hbar \nu} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{b^2 + z^2}} - \frac{4e^2}{\hbar \nu} \int_{0}^{\infty} \exp(-\frac{t^2}{a^2 + b^2}) \frac{\exp[-(\frac{b^2 + 3^2}{a^2 + b^2}) t^2]}{\sqrt{\frac{1}{a^2 + b^2} + t^2}} \, dt \, dz \]  \hspace{1cm} (D17)

We note that the first term in the above equation is actually the phase-shift function for a point-charge potential when screening is assumed.

Therefore,

\[ \chi_{cs}(k) = \chi_{cs}^{pt}(k) - \frac{4e^2}{\hbar \nu} \int_{0}^{\infty} \exp(-\frac{t^2}{a^2 + b^2}) \int_{0}^{\infty} \exp[-(\frac{b^2 + 3^2}{a^2 + b^2}) t^2] \, dt \, dz \frac{\exp(-b^2 t^2)}{\sqrt{\frac{1}{a^2 + b^2} + t^2}} \]  \hspace{1cm} (D18)

Using the integral $^{(D5)}$

\[ \int_{0}^{\infty} \exp(-q^2 x^2) = \frac{\sqrt{\pi}}{2q} \]  \hspace{1cm} \text{[q > 0]} \hspace{1cm} (D19)
we have
\[
\chi_{cs}(b) = \chi_c^{pt}(b) - \frac{2e^2}{\hbar v} \exp\left(-\frac{b^2}{a_0^2+b_0^2}\right) \int_0^\infty \frac{\exp\left(-\beta^2 t^2\right)}{\left(\frac{1}{a_0^2+b_0^2} + t^2\right)^{3/2}} t \, dt
\]
(D20)

Changing the integration variable to \( u = b^2 \left(\frac{1}{a_0^2+b_0^2} + t^2\right) \)
\[
\chi_{cs}(b) = \chi_c^{pt}(b) - \frac{e^2}{\hbar v} \int_{\frac{b^2}{a_0^2+b_0^2}}^\infty \frac{\exp(-u)}{u} \, du
\]
(D21)

Using the integral
\[
\int_\alpha^\infty \frac{\exp(-\mu x)}{x + \beta} \, dx = -\exp(\beta \mu) E_i(-\mu a - \mu \beta)
\]
where \( \Re \mu > 0, \alpha > 0, |\arg(a+\beta)| < \pi \)
(D22)

\( \chi_{cs}(b) \) becomes
\[
\chi_{cs}(b) = \chi_c^{pt}(b) - \frac{e^2}{\hbar v} E_i\left(-\frac{b^2}{a_0^2+b_0^2}\right)
\]
(D23)

where \( E_i(x) \) is exponential integral function. Noting the relationship
\[
E_i(-x) = - E_i(x)
\]
(D24)

where \( E_i(x) \) is another exponential integral function. The final result

\[ E_i(z) \]

is defined as \( E_i(z) = \int_0^\infty \frac{e^{-t}}{t} \, dt \), whereas \( E_i(x) \) is defined by some authors as \( E_i(x) = \int_e^\infty \frac{e^{-t}}{t} \, dt \) and others as \( E_i(x) = -P \int_0^\infty \frac{e^{-t}}{t} \, dt \)

Either definition can be used in our case with no conflict, because the argument of \( E_i\left(-\frac{b^2}{a_0^2+b_0^2}\right) \) is negative, so principle value does not apply. It is interesting to note that if the latter definition is used then eq. (D24) does not hold true for all \( x \). In fact for \( x > 0 \), \( E_i(-x+iz) = -E_i(x) - 2\pi i \).
for $\chi_{cs}(b)$ is

$$\chi_{cs}(b) = \chi_c^b + \frac{e^2}{\hbar v} E_1\left(\frac{b^2}{a^2 + b^2}\right)$$

(D25)


(D3) Gradshteyn and Ryzhik, p. 357, eq. 3.546-1.

(D4) Gradshteyn and Ryzhik, p. 931, eq. 8.252-4.

(D5) Gradshteyn and Ryzhik, p. 307, eq. 3.321-3.

(D6) Gradshteyn and Ryzhik, p. 311, eq. 3.352-2.