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HYDRODYNAMIC STABILITY OF VISCOELASTIC FLUIDS IN EXTRUSION

by

L. M. Skorge

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE Degree OF

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CHAPTER 1

INTRODUCTION

1.1. Some General Remarks

One of the most interesting and challenging problems in modern rheology is the prediction of hydrodynamical stability of polymers in extrusion. The problem, which is an industrial one experienced primarily within the fiber spinning industry, has become increasingly important during recent years due to the increasing popularity of synthetic commodities within the modern society.

This problem, in industrial parlance known as the "melt fracture problem," presents itself through the occurrence of irregularities on the free surface of the extrudate from a die. Where the shape and uniformity of the product is a significant factor, this break-down is wholly unacceptable.

Typical instabilities which have been observed industrially are:

i) Minor irregularities on the surface, these irregularities having the character of matteness or sharkskin.
ii) Helical or spiralling ripples.

iii) Varicose deformations such as "bambooing" of the extrudate associated with noticeable fluctuations in flow rate.

iv) Severe local necking of the threadline followed by ultimate rupture.

The irregularities covered by i), ii) and iii) are normally of global form since the entire free surface is affected, and may imply the existence of a resonance phenomenon much like a standing wave. This is known as the "melt fracture phenomenon." The instability described under iv), on the other hand, seems to be of local nature. Although the relationships between surface irregularities, operating conditions and material properties are not fully understood yet, experimental results seem to confirm that an increase in throughput results in larger deviations from a smooth, free surface.

The extrusion process may, from the rheological standpoint, be divided, rather heuristically, into three regimes (see Figure 1).

i) The feed region. This region comprises physically a hopper at the rear end through which solid polymers are fed to a feed barrel. This part of the barrel is usually quite deep. The solid polymer is driven forward by some conveying
Figure 1
device such as a rotating screw into a compression section. The depth of the barrel is gradually reduced, heat is supplied through the barrel walls, and a transition to melt takes place. In the next section, the metering section, the polymer is all melted. The pressure in this section increases gradually due to the restriction of the die at the end of the barrel. Heat is generated due to internal shearing. The rheologically interesting part of this regime is the metering section.

ii) The die-swell regime. This regime comprises the die and its near proximity. The complex phenomenon of die-swell is encountered here.

iii) Draw-down regime. The diameter of the extrudate is drastically reduced in this regime due to stretching of the molten polymer. Cooling may be provided by simple air cooling or by running the extrudate through a liquid bath. The polymer solidifies at some point downstream from the die, beyond which only minute changes in threadline diameter take place. The process terminates in a wind-up device which provides the tensile force. The wind-up device may be operated either at constant velocity or constant force. Beyond the point of solidification the extrudate behaves
essentially as an undeformed solid body of little interest to the stability of the system.

The extrusion process is comprised of a sequence of complicated flows. Sufficiently far upstream the die, the flow in the metering section is primarily pure shear flow. As the melt approaches the die, an elongational component in the flow field becomes increasingly important. Secondary flows may be experienced in the die entry region. The existence of such secondary flows depends on the rheological properties of the melt as well as on the geometry of the die inlet. The flow field in the die-swell regime is very complex. It is a combination of shear and elongational flows complicated by a sequential exposure to fixed and free boundaries. No success has been achieved in analyzing this regime. The maximum extrudate diameter in the die-swell region may be several times the die diameter. The die-swell phenomenon has sometimes been interpreted as a manifestation of the elastic properties of the melt. The melt "remembers" the larger diameter upstream from the die, and attempts to restore this configuration once it is subjected to a free boundary. But many inelastic fluids such as the power-law fluid exhibit some die-swell.

The kinematics in the draw-down regime is thought to be a close approximation to purely elongational flow.

Most polymer melts belong to the important class of fluids known as viscoelastic fluids. Such fluids respond
in a strain field as a combination of elastic and viscous bodies; the stress developed depends both on the strain and the rate of strain.

In discussing categories of response it is important to note that in a given situation, response may not depend only on the material but also on the time scale of the process. Water, for instance, behaves like Newtonian fluid in ordinary situations, but, if subjected to ultra high frequency excitations, it propagates waves like an elastic solid. The reason for this apparent change in behavior lies in the fact that response is ultimately molecular in nature, and includes both stretching of intramolecular bonds and motion of molecules past one another. In general, bonds can be stretched very quickly by an imposed stress since little motion is involved. Considerable more time is involved in causing molecules to flow. In a stress field with very short time scale, the stress may cease before molecules have had time to move appreciably, and only mechanisms giving rise to elastic response may have time to be exited.

1.2. Review of Previous Work

The draw resonance phenomenon is relatively well understood from an empirical and operating standpoint. The precise theoretical basis of the origin of this instability is in the process of development. The first results were established independently around 1966 by Kase and coworkers
(40) and Matovich and Pearson (49,50). The former work was originally published in Japanese and was unrecognized until recently. Most of its important content is repeated in a later paper (41). Matovich and Pearson showed by an order of magnitude analysis that provided \( \frac{3m}{\partial z} \ll 1 \), that is, the extrudate diameter, \( m \), changes only slightly in the axial direction, \( z \), it is sufficient to consider only one momentum equation and the continuity equation. Steady state solutions for the cases of Newtonian and power-law fluids were obtained.

In a subsequent paper Pearson and Matovich (65) consider infinitesimal disturbances of a steady, non-uniform extensional flow of a Newtonian fluid in isothermal motion. The authors show that only one parameter, the draw-ratio, remains in the mathematical model. In the absence of inertial, gravitational and surface tension effects, all of which may usually be neglected for melt spinning, the threadline was predicted to become unstable above a critical draw-ratio of 20.21. Infinitesimal disturbances of a steady, uniform extensional flow of an isothermal Newtonian fluid were also considered. No results were reported since the authors found that this approach was "quantitatively incorrect in its predictions."

Gravity, inertial and surface tension effects were later considered (80) for an isothermal, Newtonian fluid. It was found that inertia had the strongest effects on the
stability of the spinning process. It always enhances the stability. In fact, for a Reynolds number \( \text{Re} = \xi \vec{V}L/3\eta_0 \), where \( \vec{V} \) is the velocity in the axial direction, \( L \) is the length of the draw-down section and \( \eta_0 \) is the viscosity at ambient temperature) of approximately 0.16 the spinning process was for all practical purposes found stable. Gravitational effects promote stability while surface tension effects hinder stability. The same paper treats also the non-isothermal case. It was assumed that the only temperature dependent quantity in the equations of motion was the viscosity, which was taken to be an exponential function of temperature. It was found that cooling effects promote stability of the spinning process for operational conditions typical for commercial extruders. Another paper (81) dealing in more detail with the stability of non-isothermal fluids was also published.

The analysis was subsequently extended to non-isothermal, power-law fluids (66,82). The approach was philosophically identical to that for non-isothermal Newtonian fluids in the sense of identical major assumptions. A new basic parameter appears, the power law constant \( q \),

\[
\eta_T = \eta_p \left| \frac{\partial \vec{V}}{\partial z} \right|^{q-1} \quad 0 \leq q \leq \infty
\]

where \( q \) is the power law constant, \( \eta_p \) some material constant and \( \eta_T \) the Trouton viscosity. The authors found that the stability is enhanced as the numerical value of the power
law constant increases. For $q > 1.5$ sufficiently large draw-ratios were obtained that for all practical purposes the spinning process is stable to infinitesimal disturbances.

The paper by Pearson and Shah (68) deals in detail with the power law fluid, assuming only viscous forces are important. A largely analytical approach was used. The results obtained serve as a check on and are complimentary to earlier results found by solving the differential equations numerically. It was found that "shear thinning" fluids ($q < 1$) are unstable at smaller draw-ratios than "shear thickening" fluids ($q > 1$). Cohesionless fluids ($q = 0$) cannot be spun at all.

Few attempts have been made in analyzing the spinning stability of viscoelastic fluids. A linear analysis was carried out by Fisher and Denn (32) for a shear thinning generalization of a Maxwell fluid, following an earlier, approximate analysis by Zeichner (99). The relaxation time was defined in terms of the viscosity and a shear modulus where the shear modulus was assumed constant. The critical draw-ratio depends on $q$, the power-law constant, and the viscoelastic parameter $b$

$$b = \frac{3}{2} \frac{(q-1)}{G} \eta_p \left( \frac{v_o}{L} \right)^q$$

1.2

where $v_o$ is the velocity at maximum die-swell, $L$ is the length of the melted part of the draw-down section and $\eta_p$
and $q$ have the same meaning as in equation 1.1. Pearson and Shah's results are properly recovered for $b = 0$ and Lee and Rubin's results (43) for a second order fluid recovered for $q = 1$ and $b$ small. The results indicate a stabilizing effect of elasticity and, since the stability envelope is double-valued, a second stability regime at high draw-ratios exists.

Two non-linear analyses of the draw-down section have been performed. Ishihara and Kase solved the non-linear equations for Newtonian fluids (38) and power-law fluids (39). Fisher and Denn applied finite, non-linear methods for Newtonian (31) as well as Maxwell fluids (32). The results show that the systems are stable to finite perturbations below the critical draw-ratios and that a limit cycle exists for higher draw-ratios. The frequencies and amplitudes agree reasonably well with their experimental results. The stable region at high draw-ratios was found stable also to finite amplitude disturbances. The diameter oscillations near this upper limit agree quite well with the experimental results of Ishihara and Kase (39).

A large body of experimental results concerning the stability of fiber spinning has been compiled. An excellent review is delivered by Petrie and Denn (70). The results are rather inconclusive due to numerous, apparent conflicts. Only a few are in compliance with the theoretical predictions. Extremely interesting results have obtained recently by Donnelly and Weinberger (30). These researchers applied
an apparently Newtonian silicone oil and found a critical draw-ratio of 22.2 when referred to the maximum die-swell diameter. Furthermore, the stability seemed to be independent of length and flow rates. The results were obtained under approximately isothermal conditions. Identical results were found by Cruz-Saenz et al. (26) and Ishihara and Kase (39) under similar spinning conditions. There results are in very close agreement with those predicted by Kase (40) and Matovich (49). Experimental results (26,39,99) for other polymers spun isothermally, some of which "extension thickening" and some "extension thinning" reveal critical draw-ratios less than twenty, in some cases an order of magnitude less. Such results are apparently in conflict with the theoretical predictions.

Isothermal spinning experiments on two polyethylene melts (39) showed that the critical draw-ratio depends on the length of the melted part of the draw-down section. Draw-ratios of around 50 were found for lengths in the order of 5·10⁻³ meter.

The effect of heat transfer is not clear mainly due to difficulties in obtaining reliable experimental results. It seems to be the opinion of some researchers that if the cooling is sufficient to allow solidification prior to take up, then draw resonance is not observed at all. This seems to be the case for Z Jabicki and Kedzierska (100,101) who reported stable spinning at extremely high draw-ratios.
Christensen (15), Bergonzoni (9), Vassilatos (96) and Ishihara (39) all reported enhanced stability the more effective the cooling is. The contrary was found by Han et al. (36).

Melt fracture is an instability which may occur beyond a critical throughput in a capillary or die. The phenomenon is not well understood nor is its theory well developed. It is significant that the Reynold's number does not seem to be a relevant parameter. Melt fracture has been observed for Reynold's numbers as low as $10^{-15}$ (93). Early studies on polystyrene (84,85) seem to indicate that such instabilities occur at a constant value of the wall shear stress of around $10^5 \text{Nm}^{-2}$ which was essentially independent of temperature and inversely proportional to the molecular weight. These observations are typical for many subsequent studies on a variety of polymers. A regular wavy or spiral product is typical for a number of polymers, and the term "melt fracture" must not be interpreted to mean that the extrudate invariably presents a grossly distorted or fractured appearance.

It is commonly accepted that flow instabilities occur at critical values of the recoverable shear, nearly always reported in the range one to ten. The mechanisms of melt fracture which have been proposed involve one or more of five features; inertia, thermal effects, die entry effects, rheological effects and the wall slip.
Tordella (93) demonstrated convincingly that inertia effects are of minor importance. Thermal effects are often discounted on the basis of calculations made by Lupton (47) and results obtained by Clegg (17). The geometry of the die is clearly relevant as suggested by a number of authors (6, 7, 35, 56). The possibility that melt fracture is initiated by slip has often been dismissed. The slip hypothesis was convincingly argued by Benbow and Lamb (7) but was later seriously contested by den Otter (28, 29). Rheological effects are commonly accepted as important.

A large number of theoretical papers relevant to the melt fracture problem have been published. Most authors study the flow between infinite, plane, parallel plates which approximates a long slit die. "Squire's Theorem" can be proved for a Newtonian fluid in such geometry. This theorem states that the most unstable, infinitesimal disturbance is a two-dimensional perturbation in a plane normal to the walls and in the direction of the flow. It has been shown that this theorem is not valid for viscoelastic fluids in the above mentioned geometry.

The stability of second order fluids in the above geometry assuming the validity of Squire's Theorem was studied by a number of authors (16, 73, 58, 59, 74). The results must be considered with caution since the second order fluid model gives rise to constitutive instabilities which have no physical relevance whatsoever. This was first shown
convincingly by Coleman et al. (22), and later discussed in the context of stability by Craik (25), McIntire (51) and Porteous and Denn (74). The second order model is valid only for flows which vary slowly compared to the relaxation time of the fluid.

The stability of Oldroyd A and B and Maxwell fluids in long slits and again assuming Squire's Theorem valid, was also investigated (42, 74, 87, 90). The fluid elasticity was most often characterized by the Weissenberg number

\[ \text{We} = \lambda U/L \]

were \( \lambda \) is the fluid relaxation time, \( U \) the centerline velocity and \( L \) the half width of the slit (or half the width between the plates). The Weissenberg number is correlated to the recoverable shear by a proportionality constant in the slit geometry. The general result is that the critical Reynolds number decreases with increasing Weissenberg numbers. All calculations are in reasonable good agreement up to \( \text{We} = 1 \).

McIntire (52) and Bonnett and McIntire (13) studied low Reynolds number instabilities in the presence of temperature gradients using Bird-Carreau's integral fluid model. Both convective and purely rheological instabilities were found. These instabilities turn out to be different in nature. The convective instability is characterized by a relatively long wavelength, the rheological instability by a very short wavelength.
Rothenberger et al. (77) investigated circular and plane Poiseuille flow of a Maxwell fluid in the limit of zero Reynold's number. Their results seem to agree well with experiments.

A few papers on non-linear stability analysis of plane Poiseuille flow have been published (24, 53, 75). The qualitative result is that finite amplitude disturbances result in lower critical Reynold's numbers than the infinitesimally small perturbations.
CHAPTER 2

SOME ELEMENTARY KINEMATICS

The flow or deformation of a continuous material through space can most naturally be described mathematically in terms of two physically distinct manifolds, i.e. totalities of coordinate systems:

i) The body manifold, consisting of the material under consideration, in which points are particles.

ii) The space manifold, through which the body moves and in which points are places.

In a body coordinate system a given particle always has the same coordinates. Such a coordinate system is often called a convected coordinate system since it moves along with any given particle. Convected or body manifolds are particularly useful when one wishes to trace properties or conditions pertaining to any given particle of a continuum as it moves through space.

Both manifolds are important. To ensure material objectively, constitutive relationships are most often formulated in convected coordinates, while, of course, the
ultimate mathematical description must refer to a space coordinate system to be meaningful and for measurements to be taken.

One shall insist on a one-to-one correspondence or isomorphism between these two manifolds. Lodge (45) has discussed this problem in great length. Given a field, \( Q^{ij}(x) \), say, defined over the space manifold in any permissible space coordinate system \( x \), then, the corresponding field over the body manifold, \( \phi^{ij}(\xi) \), in any permissible body coordinate system, \( \xi \), may be found as follows;

\[
Q^{ij}(x) \rightarrow \phi^{ij}(\xi) = \frac{\partial x^i}{\partial \xi^m} \frac{\partial x^j}{\partial \xi^n} Q^{mn}(x)
\] 2.1

The summation convention is here and hereafter invoked. The space field \( Q^{ij}(x) \) induces at time \( t \) the field \( \phi^{ij}(\xi) \) in the body. The last part of equation 2.1 is a direct consequence of Oldroyd's transformation rules (61).

For the inverse mapping,

\[
\phi^{ij}(\xi) \rightarrow Q^{ij}(x) = \frac{\partial x^i}{\partial \xi^m} \frac{\partial x^j}{\partial \xi^n} \phi^{mn}(\xi)
\] 2.2

Analogous expressions can be found for covariant and contravariant fields of any order, a second order contravariant field was here arbitrarily chosen.

The existence and uniqueness of the partial derivatives in equations 2.1 and 2.2 define what was meant above
with permissible coordinate systems. An important property of the isomorphism is that it reproduces invariant relations of all types, whether they involve addition, subtraction, contraction, covariant differentiation or the formation of outer products of tensor fields.

The separation $dS$ of two material points at time $t'$, some previous time, relative to the separation $ds$ of the same points at time $t$, the present time, is

$$
ds^2 - dS^2 = [\hat{g}^{ij} (\xi, t) - \hat{g}^{ij} (\xi, t')] d\xi^i d\xi^j$$

$$= [g^{ij} (\xi, t) - \hat{g}^{ij} (\xi, t')] d\xi^i d\xi^j \quad 2.3$$

where $\hat{g}(\xi)$ is the convected body metric tensor. Note that it is time dependent. The quantities in brackets represent the covariant and contravariant strains, respectively, suffered by the material in motion. The strains may be transformed to the space manifold. Equations 2.1, 2.2 and 2.3 yield,

$$\hat{g}_{ij} (\xi, t) - \hat{g}_{ij} (\xi, t') = g_{ij} (x) - \frac{\partial x^m}{\partial \xi^i} \cdot \frac{\partial x^n}{\partial \xi^j} g_{mn} (x)$$

$$= g_{ij} (x) - C_{ij} (x, t') \quad 2.4$$

$$\hat{g}^{ij} (\xi, t) - \hat{g}^{ij} (\xi, t') = g^{ij} (x) - \frac{\partial x^i}{\partial \xi^m} \cdot \frac{\partial x^j}{\partial \xi^n} g^{mn} (x)$$

$$= g^{ij} (x) - B^{ij} (x, t') \quad 2.5$$
where \( g(\mathbf{x}) \) is the space metric tensor. It is, of course, time independent. The tensors \( \mathcal{G}(\mathbf{x}, t') \) and \( \mathcal{B}(\mathbf{x}, t') \) are the well known Cauchy and Finger's relative strain tensors. It follows from equations 2.4 and 2.5 that

\[
C_{ij}(\mathbf{x}, t) = g_{ij}(\mathbf{x}), \quad B^{ij}(\mathbf{x}, t) = g^{ij}(\mathbf{x})
\]

Researchers have pointed out an interesting property of the \( \mathcal{G} \) isomorphism, of particular relevance to the strain tensors. They show that the operator \( \frac{\partial}{\partial t'}\) does not commute with raising and lowering of indices. For fluids the stress correlates to the rate of strain, which implies that the result will depend on whether the covariant or the contravariant strain formalism is applied in the problem formulation. Only in rectilinear, cartesian coordinates are the two formalisms identical.

Consider a material point of a continuum. This point occupies the position \( \mathbf{x} \) at time \( t \), the present time. Suppose the same particle occupied the position \( \xi \) in the body manifold at some previous time, \( t' \). Then

\[
\xi = X_t(\mathbf{x}, t')
\]

where \( X_t(\mathbf{x}, t') \) is the displacement function relative to the configuration at time \( t \). If the velocity field, \( \mathbf{v} \), is given as a function of \( \xi \), \( t' \), then

\[
\mathbf{v} = v(\xi, t')
\]
and the position of each particle at time $t'$,

$$\xi(t') = X_t(x, t')$$  \hspace{1cm} 2.9

may be found if its position at time $t$ is known. It is simply the solution of the following differential equation,

$$\frac{d\xi(t')}{dt'} = v(\xi(t'), t)$$  \hspace{1cm} 2.10

The obvious initial value for equation 2.10 is

$$\xi(t') = x, \quad t' = t.$$  \hspace{1cm} 2.11
CHAPTER 3

CONSTITUTIVE MODELS

A meaningful mathematical description of the process may only be accomplished if a realistic constitutive model is applied. Some of the most common properties exhibited by polymer systems are:

i) Shear dependent viscosity.

ii) Finite normal stress differences in steady shear flow.

iii) Stress relaxation and recoil. This distinguishes viscoelastic from viscoinelastic fluids.

iv) Phase lag in oscillatory motion.

Properties i) and ii) are of most interest in this work.

The Newtonian model is a poor approximation for polymers, since it cannot predict any of the above mentioned properties. The first generalization of this model was proposed by Stokes more than 100 years ago. He postulated,

\[ T_{ij} + p\bar{e}_{ij} = f(e_{ij}) \]

where \( \bar{e} \) is the rate of strain tensor, \( f() \) is some tensor valued function and \( p \) the isotropic part of the stress.
Reiner and Rivlin showed in 1945 that equation 3.1 can be written in a much more compact form,

\[ T_{ij} + p g_{ij} = a_1 e_{ij} + a_2 e_{ik} g^{km}(x) e_{mj} \]  \[ 3.2 \]

where \( a_1 \) and \( a_2 \) are scalar functions of the two principal invariants of \( \varepsilon \). This model predicts finite and equal normal stress differences in simple shear flow, but does yield Newtonian viscosity.

Rivlin and Ericksen (76) derived around 1955 a more general model:

\[ T_{ij} + p g_{ij} = f(A^1_{ij}, \ldots, A^n_{ij}) \]  \[ 3.3 \]

where

\[ A^1_{ij} = 2 e_{ij} \]

\[ A^{n+1}_{ij} = \frac{\partial A^n_{ij}}{\partial t} + e_{ik} g^{km}(x) A^n_{kj} + A^n_{ik} g^{km}(x) e_{kj}, \quad n > 1 \]

The operator \( \frac{\partial}{\partial t} \), the Jauman derivative, is the time derivative as seen by an observer who is translating and rotating with the fluid. The tensors \( A^n \) are commonly referred to as the Rivlin-Ericksen tensors. This model is of little or no use for the general fluid due to its complexity. For viscometric flows, however, \( A^n = 0 \) for \( n > 2 \). The model then predicts finite normal stress differences for pure shear flow.
None of the above mentioned constitutive relations can account for stress relaxation, however.

Another class of constitutive models, developed primarily by physical chemists, comprises the "differential models." These models are based on the assumption that dilute suspensions of polymers behave much like dilute suspensions of elastic spheroids. One of the simplest is:

\[(1 + \lambda \frac{\partial}{\partial t})(T_{ij} + p\varepsilon_{ij}) = 2\alpha_1(1 + \alpha_2 \frac{\partial}{\partial t})\varepsilon_{ij}\]  

where \(\lambda, \alpha_1\) and \(\alpha_2\) are material constants.

Oldroyd (61) proposed a more complicated model of this class, introducing eight material constants. A three constant form of this model has been studied extensively by Bird and coworkers (97,98).

This class of models predicts stress relaxation, and with a large number of material constants, can be adjusted to fit almost any set of experimental data.

The next class of constitutive models, the integral models, is probably the most promising both for mathematical tractability and range of validity.

A few models have been developed assuming the fluid is basically elastic rather than viscous, notably the Bernstein, Kearsley and Zapas model (8) for isothermal, incompressible flows:
\[ T_{ij} = p g_{ij} + 2 \int_{-\infty}^{t} \left\{ \frac{a U}{a t'} g_{im} g_{jn} B^{mn}(t') \right\} dt' \]

where \( U = U(I, II, t-t') \), a scalar function, and I and II are the first and second invariants, respectively, of the Cauchy-Green strain tensor. The motivation for equation 3.5 can be seen by comparing it with purely elastic theory:

\[ T_{ij} + p g_{ij} = 2 \frac{a W}{a t} g_{im} g_{jn} B^{mn} - 2 \frac{a W}{a II} C_{ij} \]

where \( W \), a scalar function, is the strain energy function. The connection between \( W \) and \( U \) can be expressed by:

\[ \frac{\partial U(I, II, t-t')}{\partial t} = \frac{\partial^2 W(I, II, t-t')}{\partial t^2} \]

\[ \frac{\partial U(I, II, t-t')}{\partial II} = \frac{\partial^2 W(I, II, t-t')}{\partial II^2} \]

Note that the strain energy function is independent of time for an elastic body while it does depend on time for a fluid. This model is non-linear and predicts, consequently, non-Newtonian viscosity. The strain energy function is usually determined by correlation of experimental data.

Coleman and Noll (18,20) have recently formulated a general theory of importance in the search for a unified approach to the development of constitutive relations. This
is the simple fluid theory. An incompressible, simple fluid satisfies two postulates:

i) The present stress at a given at a time material point is determined up to an arbitrary constant by the history of the relative deformation gradient in the immediate neighborhood of that point.

ii) There is no directional preferences, i.e. the fluid is isotropic in the rest state.

The resulting constitutive relationship is:

\[ T_{ij} + p g_{ij} = \lim_{t' \to -\infty} J_{ij}(t') \]

\[ J_{ij}(t') = C_{ij}(t') - g_{ij}(x) \]  \( 3.8 \)

Green and Rivlin (34) and Coleman and Markovitz (21) have applied well-known results from functional analysis and shown that the tensor valued functional \( \lim_{t' \to -\infty} () \) can be expressed as an infinite integral series:

\[ T_{ij} + p g_{ij} = \int_0^\infty m(x) J_{ij}(s) ds \]

\[ + \int_0^\infty \int_0^\infty (a(s_1, s_2) J_{ik}(s_1) J_{jm}(s_2)) ds_1 ds_2 \]

\[ + b(s_1, s_2) g^{mn}(x) J_{mn}(s_1) J_{ij}(s_2)) ds_1 ds_2 \]

\[ + \ldots + (| | J_{mn}(s) | |)^n \]  \( 3.9 \)
where

\[ s = t - t', \quad ||J_{mn}(s)|| \]

\[ = (\int_{t}^{\infty} h^2 J_{mk}(s) g^m(x) g^n(x) J_{ml}(s) ds)^{1/2} \]

The quantity \( ||J|| \) represents a norm in function space. The quantities \( m(s), a(s) \) and \( b(s) \) are material functions and \( h \) an influence function. When \( ||J_{nm}(s)|| \) is so small that terms of order \( ||J_{nm}(s)||^2 \) can be neglected, one arrives at the equation for linear viscoelasticity:

\[ T_{ij} + p g_{ij} = \int_{0}^{\infty} m(s) J_{ij}(s) ds \quad 3.10 \]

This equation does not predict non-Newtonian viscosity but does exhibit stress relaxation.

If equation 3.10 shall represent a valid approximation for equation 3.9, it is necessary that the considered flow history possess small norms in function space. One notes that steady Poiseuille and Couette flows do not have this property, while oscillatory flows of small amplitudes do.

Many non-linear models have resulted in recent years from generalizations of equation 3.10. The modifications have usually been motivated by results from the network theories. One of the most promising constitutive models has
been proposed by Bird and Carreau (11). To ensure material objectivity, the model has been formulated in a convected frame of reference:

\[
\hat{T}^{ij} = -\int_{\infty}^{t} m(t-t', II(t')) \left( (1+1/2E)(\hat{g}^{ij}(t') - \hat{g}^{ij}(t)) + 1/2E \cdot \hat{g}^{im}(t) \hat{g}^{jn}(t) (\hat{g}_{mn}(t') - \hat{g}_{mn}(t)) \right) dt'
\]

where

\[
m(t-t', II(t')) = \sum_{p=1}^{\infty} \frac{n_p}{\lambda_{2p}^{1/2}} \exp(-\frac{(t-t')/\lambda_{2p}}{1+1/2 \cdot \lambda_{1p}^{1/2} II(t')})
\]

is a memory or influence function, and

\[
II(t') = \hat{g}_{kl}(t') \hat{g}_{mn}(t') \frac{\partial \hat{g}^{km}(t')}{\partial t'} \cdot \frac{\partial \hat{g}^{ln}(t')}{\partial t'}
\]

is the general form for the second principle invariant of the rate of deformation tensor.

Two sets of time constants have been employed as suggested by the network theories. The set \( \lambda_{1p} \) accounts for the rate of formation of network junctions and depends on the motion through the inclusion on II(t'). The model thus predicts non-Newtonian viscosity. The set \( \lambda_{2p} \) is associated with the loss of network junctions. The quantity E carries physical significance in steady shear flow, in which

\[
1/2E = (T_{22}-T_{33})/(T_{11}-T_{22}).
\]

Since E is a scalar, it can at the most be a function of II(t'). The available experimental
data thus far do not suggest a reasonable relationship
$E = E(II)$, and it is usually considered constant.

Both the covariant and the contravariant strain
measures have been included in the model. Continuum me-
chanics arguments alone provide no reason for preferring
one over the other. They should, ideally, lead to identi-
cal results in terms of physical components in a stationary
frame of reference. Since not all operators commute with
index raising and lowering when transformed from the con-
vected to a stationary frame of reference, different physi-
cal results will be experienced depending on whether the
covariant or the contravariant representation has been em-
ployed. Neither the contravariant nor the covariant formal-
isms alone provide a good fit to existing experimental
results. Network theories seem to indicate a slight
preference for the contravariant representation. Equation
3.11 is written in a way which reflects the same preference.

The authors have suggested the following empiricisms
for the material constants $\eta_p, \lambda_{1p}$ and $\lambda_{2p}$:

$$\eta_p = \eta_0 \lambda_{1p} / \sum_{p=1}^{\infty} \lambda_{1p}, \quad \lambda_{1p} = \lambda_1 ((1+n_1)/(p+n_1))^{a_1}$$

$$\lambda_{2p} = \lambda_2 ((1+n_2)/(p+n_2))^{a_2} \quad 3.14$$

where $\eta_0$ is a zero shear rate viscosity and $n_1, n_2, a_1$ and
$a_2$ are constants. For the particular choice
\[ n_1 = n_2 = 0, \quad \lambda_1 = \lambda_2 = \lambda, \quad \alpha_1 = \alpha_2 = 2 \quad 3.15 \]

one gets

\[ \eta_p = \eta_0^\lambda / \sum_{p=1}^\infty \lambda_p, \quad \lambda_p = \lambda / p^2 \quad 3.16 \]

which are the results for the linear viscoelastic fluids derived by Rouse (78) from "molecular" theories.

The Bird-Carreau constitutive model has been moderately successful in matching analysis to experimental results obtained with a wide variety of materials under a variety of flow conditions. Their model is used in this work.

Before the model can be used for physical flow situations, it must be transformed to a stationary frame of reference. When using Oldroyd's transformation rules on equation 3.11, one obtains

\[
T^{ij} + \rho g^{ij}(\mathbf{x}) = \int_0^\infty m(s, II(t-s))((1+1/2E)(B^{ij}(t-s) - g^{ij}(\mathbf{x})) + 1/2Eg^{im}(\mathbf{x})g^{jn}(\mathbf{x})(C_{mn}(t-s) - g_{ij}(\mathbf{x})))ds \quad 3.17
\]

where
\[ C_{ij} = \frac{\partial \xi^m}{\partial x^i} \frac{\partial \xi^n}{\partial x^j} g_{mn}(\xi), \quad B^{ij} = \frac{\partial x^i}{\partial \xi^m} \frac{\partial x^j}{\partial \xi^n} g^{mn}(\xi) \]

A very complicated model has been derived by Pao (62,63). It cannot be described in detail in reasonable space. The qualitative idea of his theory is:

\[ \text{Stress} = \int \text{(memory function)} \frac{d(\text{strain})}{dt'} dt' \]

The integral is viewed in a coordinate system which rotates with the particle. The time derivative is much like the Jauman derivative, except that the fluid rotation is tracked for a finite time rather than differentially. It is this property which permits the prediction of non-Newtonian viscosity. The mathematical complexity associated with this model makes it unsuitable for calculations of flows other than steady shear flows.

An important class of flows useful for laboratory determination of material properties are called the viscometric flows. Any simple fluid in a viscometric flow can be completely characterized by the three viscometric functions, the two normal stress differences and the shear stress function. Pipkin and Owen (72) have shown, however, that thirteen functionals are needed to describe even an infinitesimal departure from a general, steady viscometric flow.

The situation is, in general, much more complex for non-viscometric flows. Simple fluid theory offers little
help since for an arbitrary flow there is no way by which to reduce the stress functional to functions.

One case in which simple fluid theory does provide a simple result is steady extensional or elongational flow.

Coleman and Noll (19) have shown that only two material functions are necessary to describe this flow. However, there is no a priori relationship between these functions and the viscometric functions. One must, consequently, be able to do the particular flow to determine the extensional material functions. Applications of various integral models to this class of flows have uncovered large discrepancies in the predictions (45,88,89). A peculiarity associated with most integral models is the prediction of infinite stresses for finite extension rates.

In this work, one has assumed the Bird-Carreau integral model to be valid both for the steady flows and infinitesimal departures away from these steady flows.
CHAPTER 4

THE PHYSICAL PROBLEM

A simplified schematic of the extrusion process is shown in Figure 1. Cylindrical, polar geometry is implied. The sections of interest in this work are the metering and the draw-down sections. The die-swell section is extremely complex, and no attempt was made to analyze this region. It does not imply, of course, that the die-swell section is rheologically unimportant. Experimental results for specific polymer systems have, in fact, led researchers to believe that sometimes melt instabilities originate in this section.

The entire extrusion process performs as a unit, but since the intermediate section, the die swell section, is ignored, one is forced to treat the metering and the draw-down sections as independent, uncoupled processes. This restriction is not considered severe since the coupling between sections only would take place through the boundary values, such that the terminal values of the dependent variables in one section serve as starting values for the next section.

The fluid is assumed incompressible. Although polymer
melts are slightly compressible, detectably so when exposed
to pressures of 10000 lb/in² or higher, which are common in
injection molding for instance, the mechanics of melt flow
are quite adequately represented by assuming the fluid to
be incompressible, i.e. the density may be assumed constant
everywhere.

The entire extrusion process is considered iso-
thermal. This assumption is made solely for the purpose of re-
ducing the complexity of the mathematical problem. Although
many laboratory extruders are operating under approximately
isothermal condition, most commercial units are not. Heat
is usually supplied to the metering section through the wall
of the barrel and for highly viscous polymers, the viscous
or internal heating may be important also. Since the mixing
in this section is poor, substantial temperature gradients
may result. The extrudate is normally exposed to cooling
immediately after leaving the die. The possibility for sig-
nificant temperature gradients thus exist for the draw-down
section as well. The viscous heating effects are considered
of little importance for this section. Some extruders
employ quench cooling. In this mode of operation the extru-
date is cooled very rapidly by being run through a liquid
bath at some distance downstream the die. The solidifica-
tion zone is well defined and situated close to the bath
entrance. The draw-down section terminates in such a case
in the proximity of the bath entrance, and most of this
section experiences only slight cooling. In such a mode of operation, the draw-down section remains approximately isothermal.

The fluid is assumed homogeneous. This means that all the fluid passing through a limited region of the flow in any short time interval is in the same uniform state as it does so. The region may have to be small, but must remain finite. What one is not permitting is the nature of the fluid to vary much over lengths short compared to the processing length. Exceptions to this occur for poorly mixed polymer blends, but these will not be considered here. This assumption is considered good with the possible exception of the region close to the solidification zone where crystallization effects may become important.

Gravitational effects are assumed negligible. This is certainly true for the metering section of a horizontally mounted extruder. If the flight distance, i.e. the distance between the die and the wind-up device, is large, then gravitational effects may play an important role. This is indicated by sagging of the threadline when extruded in the horizontal plane.

Surface tension effects are assumed negligible. Such effects may be important only for the draw-down sections where the extrudate maintains a free surface. These effects do not depend only on the surface tension itself, but also on the principle curvatures of the boundary. Thus, such
effects become increasingly important the thinner the threadline is. Usually, the boundary is between melt and air or other gas. Earlier results indicate (80) that surface effects are of negligible magnitude for most commercial extruders of the type considered in this work. In film casting processes, on the other hand, these effects may be highly relevant.

A large body of experimental work* has been performed in order to uncover the particle flow patterns within typical extruders. Most of the work pertains to the metering and the die-swell sections. For stable flows basically two distinctly different types of flow patterns were found. These are depicted in Figures 2a and 2b.

![Fig. 2a](image1)

![Fig. 2b](image2)

The pattern shown in Figure 2a was found typical for Newtonian fluids and some polymer systems such as high density polyethylene. This pattern reveals organized,

*See the following literature references: 1,2,3,4,5, 10,12,48,56,57,79,91,92,94.
primarily channel flow.

Most polymer systems, low density polyethylene and polypropylene for instance, exhibited flow patterns of the type shown in Figure 2b. One notes the existence of secondary flows. The secondary flow regime is isolated in the sense that it is purely rotational. There is, usually, minute material exchange between the primary and the secondary regimes. It is believed that the secondary flows originate through incompatibilities in the equations of motion. Some polymers, nylon for example, are thermodynamically unstable at the elevated temperatures usually experienced in the metering section, and it is important to move the fluid through this section rapidly. Since the fluid in the secondary flow regime remains at high temperatures for very long periods of time, it may degenerate and thus represents a source of product contamination. The fiber spinning industry has often gone to great lengths in order to reduce or eliminate such secondary flows.

The mathematical description of the secondary flow regime is very difficult and normally not needed. One simply assumes the secondary flow regime to be truely stagnant. But even so, the problem of determining the position of the interface between the two regimes where the boundary conditions for the primary flow apply, and furthermore, what these boundary conditions are, still remains. These problems will be discussed later in this report.
The free surface in the draw-down section can only exist if specified force relationships are satisfied on the surface. In the case of negligible surface tension effects, the surface may remain free and stationary only if the normal stresses as well as the tangential stresses each vanish locally on the boundary.
CHAPTER 5

THE MATHEMATICAL MODEL

The mathematical models for the metering and the draw-down sections will be developed in this chapter. The models are based on the physical assumptions of the previous chapter and will employ the theory of linear hydrodynamical stability in an effort to predict instability conditions for the two flow sections in question.

5.1. The Linear Theory

The linear, hydrodynamical stability theory is used to predict the onset of instabilities via the indefinite growth of infinitesimal disturbances in laminar flow. The conditions offered by the theory are thus sufficient conditions for instability or, equivalently, necessary conditions for stability. The theory is unable to predict such important phenomena as limit cycles, since these can not exist in linear systems. Nevertheless, the theory has been applied for a wide variety of problems in fluid mechanics and rheology. A number of these applications have been given in texts by Chandrasekhar (14) and Lin (44).
The theory presupposes the existence of a sufficiently smooth steady state solution of the field equations. Under the assumptions of Chapter 4, this implies:

\[ \nabla \cdot \overline{\mathbf{V}}(\mathbf{x}) = 0 \quad 5.1 \]
\[ \rho \overline{\mathbf{V}}(\mathbf{x}) \cdot \nabla \overline{\mathbf{V}}(\mathbf{x}) = -\nabla \overline{P}(\mathbf{x}) + \nabla \cdot \overline{T}(\mathbf{x}) \quad 5.2 \]
\[ \overline{\mathbf{T}}(\mathbf{x}) + \overline{P}(\mathbf{x}) \mathbb{I} = \mathcal{T} \left[ \overline{\mathbf{G}}(\mathbf{x}, t' - t) \right] \quad 5.3 \]

Symbolic vector and tensor notation has been applied for simplicity. Superbar denotes steady state and \( \nabla = \overline{\nabla} \). The theory then considers time dependent departures in the variables from the steady state values which are sufficiently small that the superposition principle may be applied:

\[ \mathbf{v}(t, \mathbf{x}) = \overline{\mathbf{V}}(\mathbf{x}) + \mathbf{v}'(t, \mathbf{x}) \quad 5.4 \]
\[ p(t, \mathbf{x}) = \overline{P}(\mathbf{x}) + p'(t, \mathbf{x}) \quad 5.5 \]
\[ \zeta(t, \mathbf{x}) = \overline{\zeta}(\mathbf{x}) + \zeta'(t, \mathbf{x}) \quad 5.6 \]

The field equations are now:

\[ \nabla \cdot \overline{\mathbf{v}}(t, \mathbf{x}) = 0 \quad 5.7 \]
\[ \rho \left[ \frac{\partial \overline{\mathbf{v}}(t, \mathbf{x})}{\partial t} + \overline{\mathbf{v}}(t, \mathbf{x}) \cdot \nabla \overline{\mathbf{v}}(t, \mathbf{x}) \right] = \]
\[ - p(\mathbf{x}, t) + \overline{T}(t, \mathbf{x}) \quad 5.8 \]
\[ T(t, x) + p(t, x) = \mathcal{T}_{\infty}^{t}(\overline{c}(x, t-t')) \]

\[ + \mathcal{T}_{\infty}^{t}(\overline{c}'(x, t-t'); \overline{c}(x, t-t')) \quad 5.9 \]

By substituting equations 5.1 through 5.3 into 5.7 through 5.9, one obtains the differential equations for the perturbations:

\[ \nabla \cdot \nabla'(t, x) = 0 \quad 5.10 \]

\[ \rho \left[ \frac{3\nabla'(t, x)}{3t} + \nabla(x) \cdot \nabla'(t, x) + \nabla'(t, x) \cdot \nabla'(x) \right] = \]

\[ - \nabla p'(t, x) + \nabla \cdot \mathcal{T}'(t, x) \quad 5.11 \]

\[ \mathcal{T}'(t, x) + p'(t, x) = \mathcal{T}_{\infty}^{t}(\overline{c}'(t', x)) \quad 5.12 \]

provided that higher order terms are negligible. A necessary condition for the latter is that the disturbed velocity field must be of infinitesimal magnitude. It will be shown later in this thesis that

\[ \overline{c}'(t', x) = f(\nabla \overline{v}(x), \nabla'(t', x)) \quad 5.13 \]

where \( f() \) is a tensor valued function of its argument. The curl of equation 5.11 is taken in order to eliminate the pressure variable \( (\nabla \times \nabla p = 0) \). A solution of the form
\[ \mathbf{v}'(t, \mathbf{x}) = Q(t) \mathbf{W}(\mathbf{x}) \] is assumed and the following is obtained:

\[ \frac{dQ(t)}{dt} \nabla \times \mathbf{W}(\mathbf{x}) = Q(t) (\text{Spatial operators}) \mathbf{W}(\mathbf{x}) \]

This equation requires

\[ \frac{dQ(t)}{dt} \cdot \frac{1}{Q(t)} = \text{constant} = \sigma \]

\[ \Rightarrow Q(t) = \exp(\sigma t) \]

The perturbation velocity field may then always be expressed as

\[ \mathbf{v}'(t, \mathbf{x}) = \exp(\sigma t) \mathbf{W}(\mathbf{x}) \]

no matter what the stress functional, \( \mathcal{R}(\cdot) \), is. Since uniqueness can not be proved in general, other solution forms may exist. Both \( \sigma \) and \( \mathbf{W}(\mathbf{x}) \) must, in general, be considered complex. The real part of \( \sigma \) indicates whether or not the disturbances grow or decay in time. The imaginary part of \( \sigma \) reveals a perturbation of either stationary or oscillatory nature. One encounters three possibilities for the real part of \( \sigma \):

i) Real \( \sigma < 0 \). The disturbances decay in time and no instability develops.

ii) Real \( \sigma = 0 \). The disturbance is either purely oscillatory in time \( (\text{Im}(\sigma) \neq 0) \), or independent
of time altogether \((\text{Im}(\sigma)=0)\). This case is known as the marginal or neutral state.

iii) Real \((\sigma) > 0\). The disturbance grows in time and instabilities develop.

In connection with the above cases the "Principle of the Exchange of Stabilities" (14) have been defined. The principle is said to hold if \(\sigma\) as a matter of necessity must remain purely real. Such a situation is, of course, characterized by stationary marginal states.

Solving the system of linear, partial differential equations is quite an ambitious task, and it has become customary to assume a periodic or modal dependence in all but one of the independent variables:

\[
W(x) = u(x_N) \exp \left( i \sum_{p=1}^{N-1} \nu_p x_p \right)
\]

where

\[
x = [x_1, x_2, \ldots, x_N]^T, \quad N \leq 3
\]

and \(\nu_p\) is a real number known as the wave-number. The validity of the modal assumption must be judged on its ability to satisfy the boundary conditions to be imposed on the velocity field \(v'(t,x)\). If the physical system can be considered unbounded in the \(x_1, \ldots, x_{N-1}\) coordinate directions, i.e. \(-\infty \leq x_1, \ldots, x_{N-1} \leq \infty\), then it seems plausible to expect that no influence from these boundaries
may be felt and the boundary conditions one wishes to satisfy are those for specified values of the remaining independent variable, \( x_N \). The boundary conditions in stability problems are normally of a homogeneous nature. They may then be expressed as:

\[
L(u(x_N)) = 0, \text{ for boundary values of } x_N,
\]

where \( L() \) is some homogeneous, ordinary differential operator. From equation 5.17 it is clear, then, that all wave-numbers can exist. The mode which yields the lowest value of some particular system parameter (Reynold's number for example) is properly judged to be the critical mode. An example of a system as described above is flow between infinite, plane, parallel plates (52,87).

When the physical system is of finite extension in more than one coordinate direction, the wave-numbers are usually restricted since a larger set of boundary conditions must be satisfied. The modal assumption may still be valid. Most functions can be build up by a superposition of periodic modes. Exceptions include discontinuous functions. However, such functions introduce other complications, and are not of interest in linear stability analysis. The determination of permissible wave-number configurations is often a very difficult undertaking and may not be unique. It has become customary also for such systems to consider all wave-numbers possible if they for
other reasons can exist. The result is too conservative stability conditions.

5.2. Steady State Solutions, Compatibility

When dealing with complex flows, the problem of compatibility with respect to the equations of motion becomes one of paramount importance. It represents, for a given flow situation, a restriction on the constitutive relation in the sense that only those constitutive models resulting in a compatible system of equations are permissible. Flows satisfying the equations of motion no matter what the constitutive relationship is, are known in Pipkin's terminology as "controllable flows." Only one controllable flow is cited in the literature, steady shearing motion itself.

When integral constitutive relations are employed, one is most often unable to solve the steady state equations explicitly for the velocity and pressure fields. One is forced to postulate the steady state velocity field, for instance, and then establish restrictions necessary for compatibility.

The contravariant vector and tensor formalism has been applied in this work in view of equation 3.17. Super-bracketed quantities indicate physical components, superbar denotes steady state.
5.2.1. **Draw-down section**

Uniform elongational flow is postulated:

\[
\bar{v}^{<z>} = \cdot_\gamma z + v_0
\]

\[
\bar{v}^{<\theta>} = 0
\]

Experimental results indicate the existence of such a field. The remaining velocity component, \(\bar{v}^{<r>}\), may be found by means of the equation of continuity:

\[
\bar{v}^{<r>} = -\cdot_\gamma r/2
\]

In order to evaluate the stresses, the steady state particle pathline vector must be found. Using equations 2.10, 2.11, 5.18 and 5.19 one gets

\[
\frac{d\xi^r}{ds} = \cdot_\gamma r/2, \quad \xi^r = r \quad \text{for } s = 0
\]

\[
\frac{d\xi^\theta}{ds} = 0, \quad \xi^\theta = \theta \quad \text{for } s = 0
\]

\[
\frac{d\xi^z}{ds} = -(\cdot_\gamma \xi^z + v_0), \quad \xi^z = z \quad \text{for } s = 0
\]

The system 5.20 renders the following solution:
\[ \bar{\xi}^T = \text{r} \exp(\dot{\gamma}s/2) \]
\[ \bar{\xi}^0 = 0 \]
\[ \bar{\xi}^Z = (z+v_o/\dot{\gamma})\exp(-\dot{\gamma}s) - v_o/\dot{\gamma} \]

5.21

The steady state strains and corresponding stresses for the Bird-Carreau fluid are now easily evaluated. The non-zero elements are:

\[ \bar{C}_{rr} = \exp(\dot{\gamma}s) - 1, \quad \bar{C}_{zz} = \exp(-2\dot{\gamma}s) - 1 \]
\[ \bar{B}_{rr} = \exp(-\gamma s) - 1, \quad \bar{B}_{zz} = \exp(2\gamma s) - 1 \]

\[ \bar{T}^{<rr>} = K_\gamma \frac{(1+E/2)\lambda_{2p}\dot{\gamma} - 1}{1 - (\lambda_{2p}\dot{\gamma})^2} \]
\[ \bar{T}^{<zz>} = 2\gamma K \frac{1+2(1+E)\lambda_{2p}\dot{\gamma}}{1-2(\lambda_{2p}\dot{\gamma})^2} \]

\[ K = \sum_{p=1}^{\infty} \frac{\eta_p}{1+2\dot{\gamma}\lambda_{1p}} , \quad \lambda_{2p}\dot{\gamma} < 1/2 \]

5.22

The well known result for the Trouton viscosity is immediately recovered:

\[ \eta_T = \lim((\bar{T}^{<zz>} - \bar{T}^{<rr>})/\dot{\gamma}) \rightarrow 3 \sum_{p=1}^{\infty} \eta_p = 3\eta \]
Recalling the physical assumptions of Chapter 4, the equations of motion reduce to:

\[ \rho \left( \frac{\dot{y}}{2} \right)^2 r = - \frac{\partial P}{\partial r} + \overline{T^{rr}} / r \]  
\[ 5.23 \]

\[ \rho \dot{y} (z + v_o) = \frac{\partial P}{\partial z} \]  
\[ 5.24 \]

Equation 5.23 is singular for \( T^{rr} \) \( \neq 0 \). By L'Hospital's rule it is easily seen that the singularity is removed if \( T^{rr} = 0 \), which requires

\[ (1+E)\dot{y} \sum_{p=1}^{\infty} \lambda_{2p} = 1 \]  
\[ 5.25 \]

Equation 5.25 expresses, however, a physically unrealistic condition. The only remaining possibility is to disregard equation 5.23 altogether. This is permissible provided \( \overline{V^{rr}} \) is small compared to \( \overline{V^{zz}} \). By examining equations 5.18 and 5.19, it is seen that this condition is satisfied if \( \dot{y} \) is very small. It should be noted that an identical assumption was made, although for other reasons, by Pearson and his coworkers and others.

Equation 5.24 is then integrated:

\[ \overline{P}(z) = \rho \dot{y} \left( \frac{\dot{y} z^2}{2 + v_o} \right) + \text{Constant} \]  
\[ 5.26 \]

The physical boundary condition to be satisfied is:
\( \bar{p}(z) = p_0 \) on the free surface, \( r = \bar{r} \), say

But equation 5.26 can never satisfy this boundary condition, thus, the pressure must remain everywhere constant. This result is in compliance with physical expectations. Equations 5.23 and 5.24 turn into identities for very small \( \dot{\gamma} \). Necessary conditions for steady state compatibility are then, in summary:

i) Slow flow approximation applies \( \Rightarrow \dot{\gamma} \) small.

ii) \( \bar{T}^{<rr>} \Rightarrow \dot{\gamma} \) small.

iii) The isotropic pressure is everywhere constant.

The compatible, non-zero, steady state values are:

\[
\bar{v}^{<z>} = \dot{\gamma}z + v_o, \quad \bar{\xi}^r = r, \quad \bar{\xi}^\theta = \theta,
\]

\[
\bar{\xi}^z = (z + v_o/\dot{\gamma}) \exp(-\dot{\gamma} s) - v_o/\dot{\gamma}
\]

\[
\bar{C}_{zz} = \exp(-2\dot{\gamma} s) - 1, \quad \bar{E}^{zz} = \exp(2\dot{\gamma} s) - 1,
\]

\[
\bar{T}^{<zz>} = 2K \cdot (1+2(1+\nu)\lambda_2 p \dot{\gamma}/(1-2\lambda_2 p \dot{\gamma})^2) \quad 5.27
\]

One notes that the Trouton viscosity for the compatible system is:

\[
\eta_T = 2 \sum_{p=1}^{\infty} \eta_p = 2\eta
\]
The differential field equations for the perturbated variables are

$$\rho \left[ \frac{\partial \mathbf{v}'(t,x)}{\partial t} + \mathbf{v}(x) \cdot \nabla \mathbf{v}'(t,x) \right] =$$

$$- \nabla p'(t,x) + \nabla \cdot \mathbf{T}'(t,x)$$

$$\nabla \cdot \mathbf{v}'(t,x) = 0 \quad 5.28$$

Pearson and coworkers (49,66,80) have pointed out that slow flow approximation may not be valid for all operating conditions of interest to the fiber spinning industry. It is conceivable that the linear velocities may become sufficiently large, i.e. the extruder diameter small enough, that the inertial forces predominate the stress forces. Such a case would then serve as a complimentary to the situation in which the stress forces dominate over the inertial forces. The feasibility of this idea rests on the existence of a non-trivial, compatible solution of the steady state equations, a problem which will now be investigated.

Postulate: \( \bar{p} \) is constant, \( \bar{v}^{<\theta>} = 0 \), \( \frac{\partial \bar{v}}{\partial \theta} = 0 \). The equations to solve are:

$$\bar{v}(x) \cdot \nabla \bar{v}(x) = 0, \quad v \cdot \bar{v}(x) = 0$$

In component form:
\[ \frac{1}{r} \frac{\partial}{\partial r} (r \vec{v} \cdot \vec{r}) + \frac{\partial \vec{v}}{\partial z} = 0 \]  

5.29

\[ \vec{v} \cdot \frac{\partial \vec{v}}{\partial r} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial z} = 0 \]  

5.30

\[ \vec{v} \cdot \frac{\partial \vec{v}}{\partial r} + \vec{v} \cdot \frac{\partial \vec{v}}{\partial z} \]  

5.31

One notes that the above system of equations is over-determined, since \( \vec{v} \cdot \vec{r} = 0 \), the hope then is that for acceptable boundary conditions two equations may become linearly dependent or one identically satisfied. A stream function \( \psi = \psi(r, z) \) satisfying the continuity equation identically is defined:

\[ \vec{v} \cdot \frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial}{\partial z} \left( \frac{a(r \psi)}{\partial z} \right) , \quad \vec{v} \cdot \frac{\partial \psi}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{a(r \psi)}{\partial r} \right) \]  

5.32

The resulting equations are then:

\[ \frac{\partial \psi}{\partial z} \cdot \frac{\partial^2 \psi}{\partial r \partial z} - \left( \frac{\partial \psi}{\partial r} \cdot \frac{\partial \psi}{\partial r} \right) \frac{\partial^2 \psi}{\partial z^2} = 0 \]  

5.33

\[ \frac{\partial \psi}{\partial z} \cdot \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right) - \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right) \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right) = 0 \]  

5.34

Equations 5.33 and 5.34 remain consistent only if

\[ \frac{\partial}{\partial r} \cdot \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} - \frac{\partial \psi}{\partial z} \right) = 0 \]  

5.35
\[
\frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} - \frac{\partial \psi}{\partial z} \right) = 0
\]

These equations imply that

\[
\frac{\partial \psi}{\partial r} + \frac{\psi}{r} - \frac{\partial \psi}{\partial z} = \text{constant} = C
\]

Employing the method of characteristics (23,33,69), the general solution of last equation is found. It is:

\[
\psi(r,z) = f(\eta)/r + C \frac{r}{z}, \quad \eta = r + z
\]

where \(f(\eta)\) is any differentiable function in \(\eta\). One notes the existence of singularities in equation 5.38 for \(r = 0\), \(z \neq 0\). In order to proceed, this anomaly must be accepted.

For the stream-function of equation 5.38, the equations 5.33 and 5.34 become identical. The problem is now to find particular functions \(f(\eta)\) which satisfy these identical equations. The equation to be solved is, using equations 5.33 and 5.38:

\[
Cr^2 \frac{d^2 f(\eta)}{d\eta^2} + \left( \frac{df(\eta)}{d\eta} \right)^2 = 0
\]

or, equivalently, since \(\frac{df}{d\eta} = \frac{\partial f}{\partial r} \frac{dr}{d\eta} + \frac{\partial f}{\partial z} \frac{dz}{d\eta} = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial z}\),

\[
Cr^2 \left( \frac{\partial g}{\partial z} + \frac{\partial g}{\partial r} \right) + g^2 = 0
\]

\[g = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial z}\]
The method of characteristics is again applied to find the general solution. After some lengthy and rather complicated calculus, only functions of the following form were found to satisfy equation 5.40:

\[
f(r,z) = \phi_1(r-z) + \frac{r}{\phi_2(r-z)} + \frac{\log\{r-1/\phi_2(r-z)\}}{\phi_2^2(r-z)}
\]

where \(\phi_1(\mu)\) and \(\phi_2(\mu)\) are arbitrary, twice differentiable functions in \(\mu = r - z\). Consultations of equations 5.38 and 5.41 reveal that no non-trivial function, \(f(\eta)\), exists such that equation 5.40 is satisfied, provided \(z \neq 0\). It is easily shown that for \(f(\eta) = 0\) one obtains

\[
\bar{v}^{<z>} = \text{Constant}, \quad \bar{v}^{<r>} = 0
\]

in which case the inertial forces vanish. But this contradicts the original assumption.

**Assertion 1.** It is not possible for an axisymmetric, incompressible and cylindrical flow field subjected to negligible gravity forces and constant isotropic pressure to achieve a steady state in which inertial forces may predominate the stress forces provided \(\bar{v}^{<\theta>} = 0\).

**Assertion 2.** It is not possible by perturbing infinitesimally a steady, axisymmetric, incompressible
and cylindrical flow field subjected to constant isotropic pressure to achieve a state in which the inertial forces may predominate the stress forces provided the gravity forces are always of negligible magnitudes and $\bar{v}^{<0>} = 0$.

5.2.2. **Metering section**

The steady flow in the metering section is considered to be shear Poiseuille flow upon which an elongational component, small if necessary, is superimposed.

Postulate:

$$\bar{v}^{<0>} = 0$$

$$\bar{v}^{<z>} = \dot{\gamma}_z + \nu_0 (1 - (1/r_0))^2, \dot{\gamma}, \nu_0, r_0$$

are constants 5.43

Equations 5.43 and the continuity equation give the remaining velocity component,

$$\bar{v}^{<r>} = -\dot{\gamma}_r/2$$ 5.44

The differential equations for the particle pathline vector are found from 2.10, 2.11, 5.43 and 5.44,

$$\frac{d\xi^r}{ds} = \dot{\gamma}_r/2, \quad \xi^r = r \text{ for } s = 0$$
\[ \frac{d\xi^\theta}{ds} = 0, \quad \xi^\theta = \theta \text{ for } s = 0 \]

\[ \frac{d\xi^z}{ds} = -\dot{\gamma} \xi^z - v_0 (1 - (r/r_0)^2), \quad \xi^z = z \text{ for } s = 0 \]  \hspace{1cm} \text{5.45} \]

The solutions of equations 5.45 are

\[ \xi^r = r \exp(\dot{\gamma}s/2), \quad \xi^\theta = \theta \]

\[ \xi^z = z \exp(-\dot{\gamma}s) + v_0 (\exp(-\dot{\gamma}s) - 1)/\dot{\gamma} \]

\[ + v_0 (r/r_0)^2 (\exp(\dot{\gamma}s) - \exp(-\dot{\gamma}s))/2\dot{\gamma} \]  \hspace{1cm} \text{5.46} \]

Using equation 3.17, the steady state strains may be evaluated, the following non-zero elements are found:

\[ \overline{C}_{rr} = e^{\dot{\gamma}s} - 1 + \left( \frac{v_0 r}{\dot{\gamma} r_0} \right)^2 \left( \exp(\dot{\gamma}s) - \exp(-\dot{\gamma}s) \right)^2 \]

\[ \overline{C}_{rz} = \overline{C}_{zr} = \left( \frac{v_0 r}{\dot{\gamma} r_0} \right) (1 - \exp(-2\dot{\gamma}s)) \]

\[ \overline{C}_{zz} = \exp(-2\dot{\gamma}s) - 1 \]

\[ \overline{B}^{rr} = \exp(-\dot{\gamma}s) - 1 \]

\[ \overline{B}^{rz} = \overline{B}^{zr} = \left( \frac{v_0 r}{\dot{\gamma} r_0} \right) \exp(-\frac{\dot{\gamma}}{2} s) \{1 - \exp(2\dot{\gamma}s)\} \]
\[ \overline{\mathbb{R}}^{ZZ} = \exp(2\dot{\gamma}S) - 1 + \left( \frac{v_0}{\hat{\gamma}r_o} \right)^2 (1 - \exp(2\dot{\gamma}S))^2 \]  

5.47

In order to determine the conditions for compatibility with respect to the equations of motion, it is sufficient to evaluate only the \( \overline{T}^{<\mathbf{r} \mathbf{r}>} \) and \( \overline{T}^{<\mathbf{r} \mathbf{z}>} \) components of the stress field,

\[ \overline{T}^{<\mathbf{r} \mathbf{r}>} = K \left( \frac{E}{2} \left[ \frac{\dot{\gamma}}{1 - 2\lambda_2 p \dot{\gamma}} + \left( \frac{v_0}{r_o} \right)^2 \left( \frac{4}{1 - (2\lambda_2 p \dot{\gamma})^2} \right) \right] - \frac{2}{\lambda_2 p \dot{\gamma}} \right) - \frac{\dot{\gamma}(1 + E/2)}{1 + \lambda_2 p \dot{\gamma}} \]  

5.48

\[ \overline{T}^{<\mathbf{r} \mathbf{z}>} = K \left( \frac{v_0}{r_o} \right) \left\{ \frac{3(1 + E/2)}{(1 + \lambda_2 p \dot{\gamma}/2)(1 + 5\lambda_2 p \dot{\gamma}/2)} \right\} + \frac{E}{2} \cdot \frac{1}{1 + 2\lambda_2 p \dot{\gamma}} \]  

5.49

where

\[ K = \sum_{p=1}^{\infty} \eta_{p}/\{1 + 2\lambda_1 p \dot{\gamma} \}^2 (5\dot{\gamma}^2/4 + 2[ \frac{v_0}{r_o} ]^2) \]  

5.50

and provided that \( 1 - 2\lambda_2 p \dot{\gamma} > 0 \), \( \lambda_2 p \dot{\gamma} \neq 0 \).

The singularity in stress for \( \lambda_2 p \dot{\gamma} = 0 \) is of no concern at the moment since the compatibility of the above system has not yet been established.
The equation of motion in the radial direction is, recalling the physical assumptions of Chapter 4:

\[ \rho \left( \frac{\gamma}{2} \right)^2 r = -\frac{\partial P}{\partial r} + \frac{\tau^{rr}}{r} + \frac{3}{r} \frac{\partial \tau^{rr}}{\partial r} \]  \hspace{1cm} 5.51

Since \( \tau^{rr} \neq 0 \) at \( r = 0 \), this point represents a singularity. One shall insist on boundedness at \( r = 0 \), and the singularity must be removed. It is clear that this singularity exists on account of the finite value of the radial velocity component, \( \vec{v}^{<r>} \). But this velocity component vanishes only if the extension rate vanishes, which is in conflict with the original postulate, equation 5.43. One is, consequently, forced to consider sufficiently small extension rates that the radial velocity component may be disregarded in comparison with \( \vec{v}^{<z>} \). The steady state velocity field is then:

\[ \vec{v}^{<r>} = \vec{v}^{<\theta>} = 0 \]

\[ \vec{v}^{<z>} = \gamma z + v_o \left( 1 - \left( \frac{r}{r_o} \right)^2 \right) \]

\[ v_o \text{ is usually small for melts} \]  \hspace{1cm} 5.52

The particle pathline vector is now described by the following differential equations:
\[
\frac{d\xi_r}{ds} = 0, \quad \xi_r = r \quad \text{for } s = 0
\]

\[
\frac{d\xi_\theta}{ds} = 0, \quad \xi_\theta = \theta \quad \text{for } s = 0
\]

\[
\frac{d\xi_z}{ds} = -\gamma \xi_z - v_o (1-(r/r_o)^2), \quad \xi_z = z
\]

for \( s = 0 \) \hspace{1cm} 5.53

The solutions of equations 5.53 with the stated initial values are:

\[
\xi_r = r
\]

\[
\xi_\theta = \theta
\]

\[
\xi_z = z \exp(-\gamma s) - v_o (1-(r/r_o)^2)(1-\exp(-\gamma s))/\gamma
\] \hspace{1cm} 5.54

The following non-zero elements of strains and associated stresses are found:

\[
\bar{\sigma}_{rr} = 2v_o (1-\exp(-\gamma s))/r_o^2
\]

\[
\bar{\sigma}_{rz} = \bar{\sigma}_{zr} = 2v_o r \exp(-\gamma s)(1-\exp(-\gamma s))/r_o^2
\]

\[
\bar{\sigma}_{zz} = \exp(-2\gamma s) - 1
\]
\[ \bar{E}^{rz} = \bar{E}^{zr} = (2v_0r/r_o^2)(1-\exp(\dot{\gamma}s))/\dot{\gamma} \]

\[ \bar{E}^{zz} = \exp(2\dot{\gamma}s) - 1 + ((2v_0r/r_o^2)(1-\exp(\dot{\gamma}s))^2)/\dot{\gamma} \]

\[ T^{rr} = KE(\frac{2v_0r}{r_o^2})^2 \frac{1}{(1+\lambda_{2p}\dot{\gamma})(1+2\lambda_{2p}\dot{\gamma})} \]

\[ T^{rz} = T^{zr} = KE(\frac{1}{r_o^2})(\frac{1}{1+\lambda_{2p}\dot{\gamma}})(\frac{1}{1+2\lambda_{2p}\dot{\gamma}}) \]

\[ - \left( \frac{2+E}{1-2\lambda_{2p}\dot{\gamma}} \right) \]

\[ T^{zz} = KE(\frac{2v_0r}{r_o^2})^2 \frac{\lambda_{2p}(2+E)}{(1-2\lambda_{2p}\dot{\gamma})(1-\lambda_{2p}\dot{\gamma})} \]

\[ + \dot{\gamma} \left( \frac{2+\lambda_{2p}(2+3E)}{(1-2\lambda_{2p}\dot{\gamma})(1+\lambda_{2p}\dot{\gamma})} \right) \]

\[ K = \sum_{p=1}^{\infty} \eta_p/(1+2\lambda_{2p}\dot{\gamma})^2 + 2(\frac{v_0r}{r_o^2})^2 \]

5.55

The equations of motion are now:

\[ -\rho \left( \frac{\dot{\gamma}}{2} \right)^2 r = -\frac{\partial \tilde{p}}{\partial r} + 12 \frac{v_0a_1}{r_o^2} r \]

\[ \rho \dot{\gamma} v_0 \left( \frac{r}{r_o} \right)^2 + \rho \dot{\gamma}^2 z = -\frac{\partial \tilde{p}}{\partial z} - 2(a_2-a_1) \]
\[ \alpha_1 = KE \frac{v_o}{r_o^2} \frac{\lambda_2 p}{(1+\lambda_2 \dot{\gamma})(1+2\lambda_2 \dot{\gamma})} \]

\[ \alpha_2 = K \frac{v_o}{r_o^2} \frac{2+E}{1-2\lambda_2 \dot{\gamma}} \]

provided \( 1-2\lambda_2 \dot{\gamma} > 0 \).

This system is compatible. The left hand sides of the equations of motion, which represents the inertial forces, must necessarily remain comparatively very small since \( \dot{\gamma} \) is small, and may safely be ignored. Thus,

\[ \frac{\partial \overline{P}}{\partial r} = 12 \frac{v_o \alpha_1}{r_o^2} r \]

\[ \frac{\partial \overline{P}}{\partial z} = -(2\alpha_2 - \alpha_1) \]

Since the pressure is an analytic function:

\[ d\overline{P} = \frac{\partial \overline{P}}{\partial r} dr + \frac{\partial \overline{P}}{\partial z} dz = 12 \frac{v_o \alpha_1}{r_o^2} r dr - 2(\alpha_2 - \alpha_1) dz \]

Integration of equation 5.58 yields:

\[ \overline{p} = 6v_o \alpha_1 (r/r_0)^2 - 2(\alpha_2 - \alpha_1)z + \text{Constant} \]

The constant is determined by specifying the pressure at some point in the system:
\[ \bar{p} = p_0 \text{ at } z = 0, \ r = 0, \text{ say,} \]

which when substituted into equation 5.59 yields:

\[ \bar{p} - p_0 = 6v_0 a_1 (r/r_0)^2 - 2(a_2 - a_1)z \tag{5.60} \]

The quantities \( a_1 \) and \( a_2 - a_1 \) are positive definite. The pressure is, consequently, parabolic in the radial direction and linear in the longitudinal direction, an altogether believable result. Then, in summary:

i) The system is compatible only if \( \dot{\bar{v}}^{\text{tr}} \) is very small \( \Rightarrow \dot{\gamma} \) must be very small.

ii) When \( \dot{\gamma} \) is very small, then the inertial forces may safely be ignored.

The differential equations for the perturbed velocity field are now easily found. They are, utilizing symbolic vector and tensor notation:

\[ \rho \frac{\partial \bar{v}'(t,x)}{\partial t} + \rho \bar{v}(x) \cdot \nabla \bar{v}'((t,x)) = \\
- \bar{v} p'(t,x) + \nabla \cdot T'(t,x) \]

\[ \nabla \cdot \bar{v}'(t,x) = 0 \tag{5.61} \]

The associated steady state solutions are given by equations 5.52 through 5.55.
5.3. **The Particle Pathline Vector for the Perturbed System**

The disturbance in stresses due to a perturbation in the velocity field can only be evaluated if the particle pathline vector is known. The problem is clearly implicit since the particle path line vector may only be determined when the velocity field is completely known. The general problem is therefore of immense mathematical complexity. Restating equations 2.9, 2.10 and 2.11:

\[ \xi = X_t(x,s) \]  
\[ \frac{dX_t(x,s)}{ds} = -v(X_t(x,s), t-s) \]  
\[ \xi = x, \quad s = 0 \]

where \( X_t(x,s) \) is regarded as a displacement function relative to the configuration at time \( t \), and \( \xi \) is the position vector or pathline vector for any fluid particle in the past. Since the steady state solution of the field equations are assumed known and well behaved, the following equations render well behaved solutions:

\[ \bar{\xi} = \bar{X}(x,s) \]
\[ \frac{d\bar{X}_t(x,s)}{ds} = -\bar{v} (\bar{X}(x,s)) \]
Linear stability theory considers infinitesimal departures from the steady state velocity field:

\[ \mathbf{v}(t, \mathbf{x}) = \overline{\mathbf{v}}(\mathbf{x}) + \mathbf{v}'(t, \mathbf{x}) \]  

Equations 5.63 and 5.64 give

\[ \frac{dX_t(x, s)}{ds} = -\overline{\mathbf{v}}(X_t(x, s)) - \mathbf{v}'(X_t(x, s), t-s) \]  

From equation 5.65 it is immediately obvious that the assumption of infinitesimal perturbations alone does not change the implicit nature of the problem. In order to make the problem tractable mathematically, the implicitness must be eliminated through additional assumptions. One hopes to express the departures of the displacement function from its steady state values in terms of the steady state properties of the system and the perturbed velocity vector.

For the mathematical problem to remain linear the following assumption must be made.

\[ X_t(x, s) = \overline{X}(x, s) + X_t'(x, s) + ... \]  

This is a kinematic assumption whose validity must be assessed by considering the dynamics of the system. The assumption states that only very short histories of the perturbed velocity vector is important. One can certainly find flow histories and fluids for which this assumption
is not valid. It is at this point merely required that the fluid memory fades sufficiently fast that the assumption is reasonable.

Equations 5.65 and 5.66 now give:

$$\frac{d}{ds}(\mathbf{X}(x,s)) + \frac{d}{ds}(\mathbf{X}'(x,s)) = -\mathbf{V}(\mathbf{X}(x,s)) + \mathbf{X}'_t(x,s)$$

$$- \mathbf{v}'(\mathbf{X}(x,s)) + \mathbf{X}'_t(x,s), t-s)$$

5.67

It follows from equation 5.66 that

$$\mathbf{X}'_t(x,s) = 0 \quad \text{for} \quad s = 0 \quad 5.68$$

Since $\mathbf{X}'_t(x,s)$ is very small, it is permissible to perform a Taylor series expansion of $\mathbf{X}_t(x,s)$ around $\mathbf{X}(x,s)$.

$$\mathbf{v} = [v^r, v^\theta, v^z]^T, \quad \mathbf{X}_t = [X_t^r, X_t^\theta, X_t^z]^T$$

$$\mathbf{V}(\mathbf{X}(x,s)) + \mathbf{X}'_t(x,s)) = \mathbf{V}(\mathbf{X}(x,s)) +$$

$$\begin{bmatrix}
\left( \frac{\partial \mathbf{V}^r(\mathbf{X}(x,s))}{\partial X_t^r(x,s)} \right)_{s=0} X_t^r(x,s), \ldots, \left( \frac{\partial \mathbf{V}^r(\mathbf{X}(x,s))}{\partial X_t^z(x,s)} \right)_{s=0} X_t^z(x,s) \\
\vdots \\
\left( \frac{\partial \mathbf{V}^z(\mathbf{X}(x,s))}{\partial X_t^r(x,s)} \right)_{s=0} X_t^r(x,s), \ldots, \left( \frac{\partial \mathbf{V}^z(\mathbf{X}(x,s))}{\partial X_t^z(x,s)} \right)_{s=0} X_t^z(x,s)
\end{bmatrix}$$
\[
\begin{bmatrix}
\frac{a_v^T(\mathbf{x}(x,s))}{a_r} & \ldots & \frac{a_v^T(\mathbf{x}(x,s))}{a_z} \\
\vdots & \ddots & \vdots \\
\frac{a_v^2(\mathbf{x}(x,s))}{a_r} & \ldots & \frac{a_v^2(\mathbf{x}(x,s))}{a_z}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_t^r(x,s) \\
\mathbf{x}_t^\theta(x,s) \\
\mathbf{x}_t^z(x,s)
\end{bmatrix}
\]

In symbolic form,

\[
\overline{V}(\mathbf{x}(x,s) + \mathbf{x}_t'(x,s)) = \overline{V}(\mathbf{x}(x,s))
\]

\[+ [\overline{V}(\mathbf{x}(x,s))]^T \cdot \mathbf{x}_t'(x,s) \quad 5.69\]

Furthermore,

\[
\overline{v}'(\mathbf{x}(x,s) + \mathbf{x}_t'(x,s), t-s) = \overline{v}'(\mathbf{x}(x,s)) + \text{ higher order terms} \quad 5.70
\]

Substituting equations 5.69 and 5.70 into 5.67,

\[
\frac{d\mathbf{x}(x,s)}{ds} + \frac{d\mathbf{x}_t'(x,s)}{ds} = -\overline{V}(\mathbf{x}(x,s))
\]

\[-\overline{v}'(\mathbf{x}(x,s), t-s) - [\overline{V}(\mathbf{x}(x,s))]^T \cdot \mathbf{x}_t'(x,s) \quad 5.71\]

and since \(\frac{d\overline{x}(x,s)}{ds} = -\overline{V}(\overline{x}(x,s))\),
\[
\frac{dX_t'(x,s)}{ds} + [\nabla W(X(x,s))]^T \cdot X_t'(x,s) \\
+ \nu'(X(x,s), t-s) = 0 \\
\]

5.72

\[X_t'(x,s) = 0 \quad \text{for} \quad s = 0\]

One has now obtained a relationship between the perturbed particle pathline vector, the steady state velocity vector and the perturbed velocity vector.

5.3.1. **Draw-down section**

The differential equations and associated initial values for the perturbed pathline vector are found from equations 5.18 and 5.72. Assuming \(\nu'(t,x) = h(t)W(x),\)

\[
\frac{d\xi_t'^R}{dt} = h(t')W^R(x), \quad \xi_t'^R = 0 \quad \text{for} \quad t' = t
\]

\[
\frac{d\xi_t'^\theta}{dt} = h(t')W^\theta(x), \quad \xi_t'^\theta = 0 \quad \text{for} \quad t' = t
\]

\[
\frac{d\xi_t'^Z}{dt} = h(t')W^Z(x) + \gamma \xi_t'^Z, \quad \xi_t'^Z = 0 \quad \text{for} \quad t' = t
\]

5.73

The solutions are:

\[
\xi_t'^R = W^R(x) \int_t^{t'} h(q) dq
\]
\[ \xi' \theta = W^\theta(x) \int_t^{t'} h(q) dq \]

\[ \xi' \gamma = W^\gamma(x) e^{\gamma t} \int_t^{t'} h(q) e^{-\gamma q} dq \]

When \( h(t) = \exp(\sigma t) \) equation 5.16):

\[ \xi' \Gamma = \frac{W^\Gamma(x)}{\sigma} e^{\sigma t}(e^{-\sigma s} - 1) \]

\[ \xi' \theta = \frac{W^\theta(x)}{\sigma} e^{\sigma t}(e^{-\sigma s} - 1) \]

\[ \xi' \gamma = \frac{W^\gamma(x)}{\sigma - \gamma} e^{\sigma t}(e^{-\sigma s} - e^{-\gamma s}) \]

\[ s = t - t' \]

5.3.2. **Metering section**

For this section the differential equations and the associated initial values are:

\[ \frac{d\xi' \Gamma}{dt'} = h(t')W^\Gamma(x), \quad \xi' \Gamma = 0 \quad \text{for} \ t' = t \]

\[ \frac{d\xi' \theta}{dt'} = h(t')W^\theta(x), \quad \xi' \theta = 0 \quad \text{for} \ t' = t \]
\[ \frac{d\xi'\zeta}{dt'} = h(t')W^z(x) + \dot{\gamma}\xi'\zeta - 2\nu_0 (r/r_0^2) \xi'\gamma, \]

\[ \xi'\zeta = 0 \quad \text{for} \quad t' = t \quad 5.76 \]

The solutions of equations 5.75 are:

\[ \xi'\gamma = W^\gamma(x) \int_t^{t'} \frac{h(p)}{dp} dp \]

\[ \xi'\theta = W^\theta(x) \int_t^{t'} \frac{h(p)}{dp} dp \]

\[ \xi'z = e^{\dot{\gamma}t'} (W^z(x)) \int_t^{t'} e^{-\dot{\gamma}p} h(p) dp \]

\[ - 2\nu_0 (r/r_0^2) \int_t^{t'} e^{-\dot{\gamma}q} \int_t^q \frac{h(p) dp dq}{d} 5.77 \]

When \( h(t) = \exp(\sigma t) \) (equation 5.16),

\[ \xi'\gamma = \frac{W^\gamma(x)}{\sigma} e^{\sigma t} (e^{-\sigma s} - 1) \]

\[ \xi'\theta = \frac{W^\theta(x)}{\sigma} e^{\sigma t} (e^{-\sigma s} - 1) \]

\[ \xi'z = \frac{e^{\sigma t}}{\sigma - \dot{\gamma}} \{ W^z(x) (e^{-\sigma s} - e^{-\dot{\gamma}s}) \]

\[ + 2 \frac{\nu_0 r}{r_0^2} W^\gamma(x) \left( \frac{1-e^{-\sigma s}}{\sigma} - \frac{1-e^{-\dot{\gamma}s}}{\dot{\gamma}} \right) \} 5.78 \]
5.4. The Perturbed Strains and Stresses

The Bird-Carreau constitutive model (equation 3.17) which is employed in this work, applies both the covariant and the contravariant strain measures for reasons discussed in Chapter 3.

\[
(C_{ij})_t = \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} g_{mn}(\xi) - g_{ij}(x) \tag{5.79}
\]

\[
B_{ij}^t = \frac{\partial x^i}{\partial \xi^m} \cdot \frac{\partial x^j}{\partial \xi^n} g_{mn}(\xi) - g_{ij}(x) \tag{5.80}
\]

The tensors \(g(x)\) and \(g(\xi)\) signify the space and body or imbedded metric tensors, respectively, \(\xi = \xi(x,s), C_t = C_{\xi t}(x,s)\) and \(B_t = B_{\xi t}(x,s)\) is implied. The subscript \(t\) indicates the quantity to be measured relative to the configuration at time \(t\). Thus,

\[
\lim_{s \to 0} (C_{\xi t}) = \lim_{s \to 0} (B_{\xi t}) \to 0.
\]

The matrices of the various metric tensors when cylindrical polar coordinates are used are:

\[
g_{ij}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
g_{ij}(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\xi^r)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij}(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\xi^r)^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The body metric tensor is, in general, a function of the perturbed velocity vector. It departs only infinitesimally from its steady state values. It simplifies the mathematics largely if one assumes that this tensor retains its steady state values at all times. It is believed that only minute errors would result from this assumption. Since \( \xi^r(x,s) = r \) for both flow sections considered (equations 5.27 and 5.54),

\[
g_{ij}(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij}(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

5.82

The non-linear strains are now linearized with respect to pathline perturbations from the steady state flow field:

\[
(C_{ij})_t = \left\{ \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} g_{mn}(\xi) - g_{ij}(x) \right\}
\]
\[ + \left\{ \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} + \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} \right\} g_{mn}(\xi) \]

+ higher order terms

\[ = (C^t_{ij}) + (C^t_{ij}) \]

5.83

The steady state covariant strain tensor is obviously

\[ (C^t_{ij}) = \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} g_{mn}(\xi) - g_{ij}(x) \]

5.84

This tensor was calculated earlier in the report, equations 5.27 and 5.55. The perturbed covariant strain tensor is

\[ (C^t_{ij}) = \left( \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} + \frac{\partial \xi^m}{\partial x^i} \cdot \frac{\partial \xi^n}{\partial x^j} \right) g_{mn}(\xi) \]

5.85

The matrices for the perturbed, covariant strain tensors pertaining to the two flow sections are readily found from equations 5.85, 5.75 and 5.78. Evaluation of the perturbed contravariant strain tensors is much more complicated. These tensors can not be found directly. They involve terms of the form \( \partial x^i / \partial \xi^j \). Consulting equations 5.75 and 5.78 it is seen that terms of the form \( f(s) \partial x^i / \partial W^j(x) \) would emerge. Such forms do not express meaningful results. Assuming the matrices for the covariant strain tensors are invertible, then
\[ \mathbf{C} \cdot \mathbf{B} = \mathbb{I} \]

\[ \mathbf{B} = \mathbf{C}^{-1} = (F_t^T \cdot F_t)^{-1} = F_t^{-1} \cdot (F_t^T)^{-1} \quad 5.86 \]

where \( F_t \) is the deformation gradient tensor whose elements are

\[ (F_t)^i_j = \frac{\partial x^i}{\partial \xi^j} \quad 5.87 \]

The subscript \( t \) again indicates that the deformation is measured relative to its configuration at time \( t \).

The contravariant strain tensors are then found by inverting the deformation gradient matrices. From equations 5.80 and 5.86:

\[ B^{ij}_t = (F^m_i)_t^{-1} \cdot (F^n_j)_t^{-1} g^{mn}(\xi) - g^{ij}(x) \quad 5.88 \]

The inversion procedure is rather lengthy and somewhat intricate. The reader is referred to Appendix 1 for details. It is shown in that appendix that the contravariant tensors may be decomposed into a steady state and a perturbed part:

\[ B^{ij}_t = B^{ij}_t + B'^{ij}_t \quad 5.89 \]

where
is the steady state part of the contravariant strain tensor calculated earlier in this report, equations 5.27 and 5.55, and $B_{t}^{ij}$ is the perturbed part.

The inversion was possible only for

$$\text{Real}(\sigma) \geq 0$$  \hspace{1cm} (5.91)

Equation 5.91 implies that the analysis is valid only for unstable and marginally stable systems. This is not a severe restriction, however, since the linear theory can only furnish sufficient conditions for instability. Furthermore, one is primarily interested in marginal or neutral stability in this work.

Once the perturbed strains are known, the associated perturbed stresses may be evaluated. From equations 3.17, 5.85 and 5.89:

$$T'_{ij} = \int_{0}^{\infty} m(s, \Pi(t-s))((1+1/2E)B_{t}^{ij}$$

$$+ 1/2Eg_{im}(\xi)g_{jn}(\xi)c_{mn}' \text{d}s$$  \hspace{1cm} (5.92)

The memory or influence function $m()$ is defined in equation 3.12. The calculations are very long and tedious but principally straightforward. No need is seen to report
them in detail.

It was noticed in the evaluation of the steady state stresses that the stress integrals did not converge unconditionally for bounded values of the parameters. Unbounded stresses were found for

\[ 1 - 2\dot{\lambda}_p \dot{\gamma} < 0 \]

This restriction has been discussed in detail by others (27). Similar results were experienced for the stress perturbations. Unbounded stresses occurred if

\[ 1 - 3\dot{\lambda}_p \dot{\gamma} < 0. \]

Such conditions arise from the combination of an elongational flow field and an exponentially fading memory constitutive model.

The stress-unboundedness has been of some concern to researchers, and the very existence of purely elongational flow fields questioned. Experimental results (83, 26) seem to confirm that extensional flows are indeed realizable. Extremely interesting experimental results were reported by Stevenson and Meissner (54, 55, 86). These researchers found that for some polymers in uniform extensional flow the stresses grew in time much like S-shaped curves without achieving any steady state values, provided the extension rates were sufficiently large. There is in
the author's opinion sufficient experimental evidence at this time to support the existence of the "unbounded stress phenomenon."

5.4.1. **Draw-down section**

\[
C'_{rr} = 2\alpha_1 \frac{\partial W^r}{\partial r}, \quad C'_{r\theta} = C'_{\theta r} = \alpha_1 \left( \frac{\partial W^r}{\partial \theta} + r^2 \frac{\partial W^\theta}{\partial r} \right)
\]

\[
C'_{rz} = C'_{zr} = \alpha_1 \frac{\partial W^r}{\partial z} + \alpha_2 e^{-\gamma_s} \frac{\partial W^z}{\partial r}
\]

\[
C'_{\theta\theta} = 2\alpha_1 r^2 \frac{\partial W^\theta}{\partial \theta}
\]

\[
C'_{\theta z} = C'_{z\theta} = \alpha_1 r^2 \frac{\partial W^\theta}{\partial z} + \alpha_2 e^{-\gamma_s} \frac{\partial W^z}{\partial \theta}
\]

\[
C'_{zz} = 2\alpha_1 e^{-\gamma_s} \frac{\partial W^z}{\partial z}
\]

\[
B'_{rr} = -2\alpha_1 \frac{\partial W^r}{\partial r}, \quad B'_{r\theta} = B'_{\theta r} = -\alpha_1 \left( \frac{\partial W^\theta}{\partial r} + \frac{1}{r^2} \frac{\partial W^r}{\partial \theta} \right)
\]

\[
B'_{rz} = B'_{zr} = -\alpha_1 \frac{\partial W^r}{\partial z} - \alpha_2 e^{-\gamma_s} \frac{\partial W^z}{\partial r}
\]

\[
B'_{\theta\theta} = -2 \frac{\alpha_1}{r^2} \frac{\partial W^\theta}{\partial \theta}
\]

\[
B'_{\theta z} = B'_{z\theta} = -\gamma_s \left( \frac{\alpha_2}{r^2} \frac{\partial W^z}{\partial \theta} + \alpha_1 \gamma_s \frac{\partial W^\theta}{\partial z} \right)
\]
\[ B'rzz = -2\alpha_2 e^{3i\gamma_s} \frac{\partial W_z}{\partial z} \]

\[ T'rr = 2e^{\sigma t} \sum_{p=1}^{\infty} k_p^1 N_p^1 \frac{\partial W_r}{\partial r} \]

\[ T'r\theta = T'\theta r = e^{\sigma t} \sum_{p=1}^{\infty} k_p^1 N_p^1 \left( \frac{\partial W_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial W_r}{\partial \theta} \right) \]

\[ T'r z = T'z r = e^{\sigma t} \sum_{p=1}^{\infty} k_p^1 \left\{ (1+E/2) \left( N_p^4 \frac{\partial W_r}{\partial z} + N_p^2 \frac{\partial W_z}{\partial r} \right) + N_p^3 \frac{\partial W_z}{\partial r} \right\} - \frac{E}{2} \left( N_p^1 \frac{\partial W_\theta}{\partial z} + N_p^2 \frac{\partial W_z}{\partial r} \right) \]

\[ T'\theta \theta = 2e^{\sigma t} \sum_{p=1}^{\infty} k_p^1 \frac{1}{r^2} \frac{\partial W_\theta}{\partial \theta} \]

\[ T'\theta z = T'z \theta = e^{\sigma t} \sum_{p=1}^{\infty} k_p^1 \left\{ (1+E/2) \left( N_p^4 \frac{\partial W_\theta}{\partial z} + \frac{N_p^3}{r^2} \frac{\partial W_z}{\partial \theta} \right) \right\} - \frac{E}{2} \left( N_p^1 \frac{\partial W_\theta}{\partial z} + \frac{N_p^2}{r^2} \frac{\partial W_z}{\partial \theta} \right) \]

\[ T'zz = 2e^{\sigma t} \sum_{p=1}^{\infty} k_p^1 \left\{ (1+E/2) N_p^5 - \frac{E}{2} N_p^2 \right\} \frac{\partial W_z}{\partial z} \]

The constants are defined in the parameter list at the end of this report. The condition for convergence of the stress integrals is
\[ 0 \leq \frac{\lambda_2}{p} \leq \frac{1}{3} \]

### 5.4.2. Metering section

\[ C_{rr}' = 2\left\{ (\beta_1 + 2\beta_2 \beta_3 r^2) \frac{\partial W^r}{\partial r} + \beta_2 \beta_3 r W^r \right\} \]

\[ C_{r\theta}' = C_{\theta r}' = (\beta_1 + 2\beta_2 \beta_3 r^2) \frac{\partial W^r}{\partial \theta} + \beta_1 r^2 \frac{\partial W^\theta}{\partial r} \]

\[ + 2\beta_2 \beta_3 r \frac{\partial W^z}{\partial \theta} \]

\[ C_{r z}' = C_{z r}' = (\beta_1 + 2\beta_2 \beta_3 r^2) \frac{\partial W^r}{\partial z} + 2\beta_2 \beta_3 r \frac{\partial W^z}{\partial z} \]

\[ + \beta_1 \beta_2 \frac{\partial W^z}{\partial r} + \beta_1 \beta_3 \frac{\partial (r W^r)}{\partial r} \]

\[ C_{\theta \theta}' = 2\beta_1 r^2 \frac{\partial W^\theta}{\partial \theta} \]

\[ C_{\theta z}' = C_{z \theta}' = \beta_1 r^2 \frac{\partial W^\theta}{\partial z} + \beta_1 \beta_2 \frac{\partial W^z}{\partial \theta} + \beta_1 \beta_3 \frac{\partial W^r}{\partial \theta} \]

\[ C_{z z}' = 2\left\{ \beta_1 \beta_2 \frac{\partial W^z}{\partial z} + \beta_1 \beta_3 r \frac{\partial W^r}{\partial z} \right\} \]

\[ B^{rr} = -2\beta_1 \left( \frac{\partial W^r}{\partial r} + \beta_1 \frac{\partial W^r}{\partial z} \right) \]

\[ B^{r \theta} = B^{\theta r} = -\beta_1 \left\{ \beta_1 \frac{\partial W^\theta}{\partial z} + \frac{\partial W^\theta}{\partial r} + \frac{1}{r^2} \frac{\partial W^r}{\partial \theta} \right\} \]
\[ B'_{r z} = B'_{z r} = \{- \frac{\beta_2}{\beta_1} \frac{\partial W^z}{\partial r} + \frac{\beta_3}{\beta_1} W^r + \beta_{11} \frac{\partial W^\theta}{\partial \theta} \]

\[ + \frac{\beta_1 \beta_2}{\beta_1} \frac{1}{r^2} \frac{\partial W^z}{\partial \theta} + \beta_{12} \frac{\partial W^r}{\partial r} + \beta_7 \frac{\partial W^r}{\partial z} \}\]

\[ B'_{\theta \theta} = -2 \frac{\beta_1}{r^2} \frac{\partial W^\theta}{\partial \theta} \]

\[ B'_{\theta z} = B'_{z \theta} = \{- \beta_8 \frac{\partial W^\theta}{\partial z} + \frac{\beta_6}{r^2} \frac{\partial W^r}{\partial \theta} \]

\[ + \frac{\beta_1 \beta_2}{\beta_1} \frac{1}{r^2} \frac{\partial W^z}{\partial \theta} + \beta_1 \beta_10 \frac{\partial W^\theta}{\partial r} \}\]

\[ B'_{z z} = \{- \frac{\beta_2 \beta_10}{\beta_1} \frac{\partial W^z}{\partial r} + \beta_6 \beta_10 \frac{\partial W^r}{\partial r} + \frac{\beta_3 \beta_10}{\beta_1} r W^r \]

\[ + \beta_{11} \beta_10 \frac{\partial W^\theta}{\partial \theta} + \beta_{13} \frac{\partial W^z}{\partial z} + \beta_{14} \frac{\partial W^r}{\partial z} \}\]

Due to the vast mathematical complexity, the stresses for this section were only evaluated for the case of rotational symmetry. This assumption was later made for the drawdown section as well, and will be discussed later.
\[ T'_{rr} = e^{\sigma t} \sum_{p=1}^{\infty} K_p^2 \left\{ 2\beta_{15} - \beta_{33} \left( \frac{r}{r_0} \right)^2 \right\} \frac{\partial W^r}{\partial r} \]

\[ + 2\beta_{34} \left( \frac{r}{r_0} \right) \frac{\partial W^r}{\partial z} - \beta_{33} \frac{r}{r_0} W^r \}

\[ T'_{r\theta} = T'_{\theta r} = e^{\sigma t} \sum_{p=1}^{\infty} K_p^2 \left\{ \beta_{34} \left( \frac{r}{r_0} \right) \frac{\partial W^\theta}{\partial z} + \beta_{15} \frac{\partial W^\theta}{\partial r} \right\} \]

\[ T'_{rz} = T'_{zr} = e^{\sigma t} \sum_{p=1}^{\infty} K_p^2 \left\{ \beta_{35} \frac{\partial W^z}{\partial r} \right\} \]

\[ + \frac{\beta_{36}}{r_0} W^r + \beta_{37} \left( \frac{r}{r_0} \right) \frac{\partial W^z}{\partial z} + \beta_{38} \left( \frac{r}{r_0} \right) \frac{\partial W^r}{\partial r} \]

\[ + \left( \beta_{39} + \beta_{40} \left( \frac{r}{r_0} \right)^2 \right) \frac{\partial W^r}{\partial z} \}

\[ T'_{\theta z} = T'_{z\theta} = e^{\sigma t} \sum_{p=1}^{\infty} K_p^2 \left\{ \left[ \beta_{39} + \beta_{34} \left( \frac{r}{r_0} \right) \right] \frac{\partial W^\theta}{\partial z} \right\} \]

\[ + \beta_{20} \left( \frac{r}{r_0} \right) \frac{\partial W^\theta}{\partial r} \}

\[ T'_{zz} = e^{\sigma t} \sum_{p=1}^{\infty} K_p^2 \beta_{41} \left( \frac{r}{r_0} \right) \frac{\partial W^z}{\partial r} + \beta_{42} \left( \frac{r}{r_0} \right)^2 \frac{\partial W^r}{\partial r} \]

\[ + \beta_{43} \frac{r}{r_0} W^r + \left[ \beta_{44} + \beta_{45} \left( \frac{r}{r_0} \right)^2 \right] \frac{\partial W^z}{\partial z} \]
\[
\begin{align*}
+ \left[ \beta_{46} \left( \frac{r}{r_0} \right)^3 + \beta_{47} \left( \frac{r}{r_0} \right) \right] \frac{\partial W}{\partial z}
\end{align*}
\]

The constants are defined in the parameter list at the end of the report. Necessary and sufficient conditions for convergence of the stress integrals are

\[0 \leq \lambda_2 \gamma \leq 1/3\]

5.5 Describing Partial Differential Equations

The field equations for the two flow sections are now derived by use of equations 5.28 and 5.61 and the perturbed stresses given in Chapter 5.4.1 and 5.4.2. It is assumed that the perturbed isotropic pressure is of the same form as the velocity components,

\[p'(t,x) = e^{t \sigma} p(x)\]

5.93

\(p(x)\) is a scalar function, complex in general. It is emphasized that equation 5.93 is an assumption. For the velocity field, on the other hand, a similar form was rigorously derived (Chapter 5.1).

\[\text{Determination of physically relevant boundary conditions for the perturbed pressure poses a problem, in particular for the metering section. This problem may be circumvented, however, since the pressure variable can be eliminated with relative ease by cross differentiation and} \]

subtraction, or, equivalently, by taking the curl of the equations of motion \((\nabla \times \nabla p' = 0)\). This technique has been widely used for the same reason within hydrodynamical stability analysis. The procedure increases the order of the differential equations, and thus the possibility of introducing spurious solutions, that is solutions completely unrelated to the physical problem do exist. Furthermore, a larger set of boundary conditions for the remaining dependent variables usually has to be provided. The additional boundary conditions may, fortunately, often be derived from the physical boundary conditions and the equation of continuity.

It was found most natural to derive the field equations in terms of the physical components rather than vector and tensor components of the dependent variables. The conversion from vector and tensor quantities to the associated physical components is easily accomplished using the well known transformations:

\[
T^{ij} = (g_{ii}(x)g_{jj}(x))^{1/2} T^{ij}
\]

\[
T^{ij} = (g^{ii}(x)g^{jj}(x))^{1/2} T_{ij}
\]

\[
v^i = g_{ii}(x)^{1/2} v^i = g^{ii}(x) v_i
\]

where the underbar denotes that the summation convention
shall not be applied, $T^{ij}$ and $v^{i}$ signify the physical components of a second order tensor field and a vector field, respectively.

The derivations are principally straightforward and will not be reported in detail.

5.5.1. **Draw-down section**

Continuity equation:

$$\frac{1}{r} \frac{\partial}{\partial r} (r W^{<r>}) + \frac{1}{r} \frac{\partial W^{<\theta>}}{\partial \theta} + \frac{\partial W^{<z>}}{\partial z} = 0$$  \hspace{1cm} 5.95

Equations of motion:

$$\rho v_0 \frac{\partial W^{<r>}}{\partial z} + \rho a W^{<r>} = \sum_{p=1}^{\infty} K^\alpha_p \{ 2N^1_p \frac{\partial^2 W^{<r>}}{\partial r^2} $$

$$+ \frac{N^1_p}{r^2} \frac{\partial W^{<r>}}{\partial \theta^2} + N^6_p \frac{\partial W^{<r>}}{\partial z^2} + \frac{N^1_r}{r} \frac{\partial W^{<\theta>}}{\partial \theta} $$

$$+ N^7_p \frac{\partial W^{<z>}}{\partial \theta \partial z} + 2 \frac{N^1_p}{r} \frac{\partial W^{<r>}}{\partial r} - 3 \frac{N^1_p}{r^2} \frac{\partial W^{<\theta>}}{\partial \theta} - \frac{\partial P}{\partial r} \}$$  \hspace{1cm} 5.96

$$\rho v_0 \frac{\partial W^{<\theta>}}{\partial z} + \rho a W^{<\theta>} = \sum_{p=1}^{\infty} K^\alpha_p \{ 2N^1_p \frac{\partial^2 W^{<\theta>}}{\partial \theta^2} $$

$$+ \frac{N^1_p}{r^2} \frac{\partial W^{<\theta>}}{\partial \theta^2} + N^6_p \frac{\partial W^{<\theta>}}{\partial z^2} + \frac{N^1_r}{r} \frac{\partial W^{<z>}}{\partial \theta} $$

$$+ N^7_p \frac{\partial W^{<z>}}{\partial \theta \partial z} + \frac{N^1_p}{r} \frac{\partial W^{<r>}}{\partial r} \}$$
\begin{align}
&+ \frac{N_p^1}{r} \frac{\partial W^{<r>}}{\partial r} + \frac{N_p^1}{r^2} \frac{\partial W^{<\theta>}}{\partial \theta} - \frac{N_p^1}{r^2} W^{<\theta>} - \frac{1}{r} \frac{\partial P}{\partial \theta} } \quad 5.97 \\
\rho v_o \frac{\partial W^{<z>}}{\partial z} + \rho \sigma W^{<z>} = \sum_{p=1}^{\infty} \frac{K_p^1}{p} \left( 2 N_p^0 \frac{\partial^2 W^{<z>}}{\partial z^2} + \frac{N_p^0}{r^2} \frac{\partial^2 W^{<z>}}{\partial \theta^2} ight) \\
&+ \frac{N_p^7}{r} \frac{\partial W^{<z>}}{\partial r} - \frac{\partial P}{\partial z} } \quad 5.98
\end{align}

It is implied that \( W^{<p>} = W^{<p>}(r, \theta, z) \) and \( P = P(r, \theta, z) \).

The constants occurring in the above equations are defined in the parameter list at the end of the report.

The following dimensionless variables are now introduced:

\[ u^{<R>} = \frac{W^{<r>}}{\bar{u}}, \quad u^{<\theta>} = \frac{W^{<\theta>}}{\bar{u}}, \quad u^{<z>} = \frac{W^{<z>}}{\bar{u}} \]

\[ \pi = \frac{P}{\rho \bar{u}^2}, \quad R = \frac{r}{r_o}, \quad Z = \frac{z}{L} \]

where \( r_o \) is the radius of the extrudate at the entrance of the draw-down section and \( L \) is the length of the same section.

\[ \bar{u} = v_L - v_o = \gamma L \]

is the increase in steady state axial velocity due to stretching. The choice of \( \bar{u} \) as the reference velocity may
seem surprising since \( \bar{u} \) must remain very small on account of compatibility (see Chapter 5.2.1). However, the easily measured steady state terminal velocity and, as will be discussed later, a well defined Deborah number are thereby introduced. Both are quantities which should naturally occur in the model. Moreover, the case \( \dot{\gamma} = 0 \) is of no interest. After eliminating the pressure from the above equations, the following set of field equations result:

\[
\frac{\partial u^{<R>}}{\partial R} + \frac{u^{<R>}}{R} + \frac{1}{R} \frac{\partial u^{<\theta>}}{\partial \theta} + \Lambda \frac{\partial u^{<z>}}{\partial Z} = 0 \tag{5.101}
\]

\[
\sum_{p=1}^{\infty} \left\{ \Lambda (2N^1_p - N^6_p) \right\} \frac{\partial u^{<R>}}{\partial R} \frac{\partial^2 u^{<R>}}{\partial Z^2} + \Lambda \frac{N^1_p}{R^2} \frac{\partial^3 u^{<R>}}{\partial \theta^2 \partial Z} + \Lambda \frac{N^6_p}{R^2} \frac{\partial^3 u^{<R>}}{\partial Z^3} 
\]

- \( \text{Re}' \Lambda \frac{v_o}{v_L - v_o} \frac{\partial u^{<R>}}{\partial Z} \frac{\partial^2 u^{<R>}}{\partial \theta^2 \partial Z} + \Lambda \frac{N^1_p}{R} \frac{\partial^3 u^{<\theta>}}{\partial \theta^2 \partial \theta Z} \)

- \( N^7_p \frac{\partial^3 u^{<z>}}{\partial R^3} - \frac{N^7_p}{R^2} \frac{\partial^3 u^{<z>}}{\partial \theta^2 \partial R} + \Lambda \frac{N^7_p - 2N^8_p}{R^2} \frac{\partial^3 u^{<z>}}{\partial \theta^2 \partial Z} \)

+ \( \frac{\Lambda}{R} (2N^1_p - N^6_p) \frac{\partial^2 u^{<R>}}{\partial R \partial Z} - 3N^1_p \Lambda \frac{\partial^2 u^{<\theta>}}{\partial \theta^2 \partial Z} \)

+ \( \text{Re}' \Lambda \frac{v_o}{v_L - v_o} \frac{\partial^2 u^{<z>}}{\partial R \partial Z} - \frac{N^7_p}{R} \frac{\partial^2 u^{<z>}}{\partial R^2} \)

- \( \Lambda^2 \text{Re}'(\Lambda \frac{\partial u^{<R>}}{\partial Z} - \frac{\partial u^{<z>}}{\partial R}) \) = 0 \tag{5.102}
\[ \sum_{p=1}^{\infty} \left( \Lambda (N_p^1 - N_p^6) \frac{3u^{<R>}}{\partial R \partial \vartheta} + \Lambda \left( 2N_p^1 - N_p^6 \right) \frac{3u^{<\theta>}}{\partial R \partial Z} \right) + \Lambda N_p^1 \frac{3u^{<\theta>}}{\partial R \partial Z} + N_p^6 \Lambda \frac{3u^{<\theta>}}{\partial Z^2} + \Lambda^2 (N_p^7 - 2N_p^8) \frac{3u^{<z>}}{\partial Z^2 \partial \vartheta} - N_p^7 \frac{3u^{<z>}}{\partial R \partial \vartheta} - \frac{N_p^7}{R^2} \frac{3u^{<z>}}{\partial \vartheta^2} \]

\[ + \left[ \frac{\Lambda}{R} (N_p^1 - N_p^6) + \frac{\nu_o}{v_L \nu_o} \cdot \Lambda^2 \text{Re}' \right] \frac{2u^{<R>}}{\partial \vartheta} \]

\[ - \frac{\nu_o}{v_L \nu_o} \cdot \Lambda^3 \text{Re}' R \frac{2u^{<\theta>}}{\partial Z^2} + \Lambda N_p^1 \frac{2u^{<\theta>}}{\partial R \vartheta} \]

\[ - \frac{N_p^7}{R} \frac{2u^{<z>}}{\partial R \vartheta} + \Gamma \Lambda^2 \text{Re}' \frac{u^{<z>}}{\partial \vartheta} \]

\[ - \Lambda \left( \frac{N_p^1}{R} + \Gamma \Lambda^2 \text{Re}' \right) \frac{u^{<\theta>}}{\partial Z} \}

= 0 \quad \text{5.103}

where

\[ \text{Re}' = \rho u L / K_p^1, \text{ a Reynolds number,} \]

\[ K_p^1 = \eta_p / (1 + 2(\lambda_p \dot{\gamma})^2), \text{ a non-Newtonian viscosity,} \]

\[ \Lambda = r_o / L, \text{ a scaling factor,} \]

\[ \Gamma = \sigma / \dot{\gamma}, \text{ a non-dimensional eigenvalue.} \]

5.5.2. Metering section

One remembers that for this section an assumption
of rotational symmetry, \( a/a\theta = 0 \), was made earlier in the report.

The non-Newtonian viscosity is

\[
K_p^2 = \eta_p / (1 + 2(\gamma^2 + 2(v_o/r_o)^2 R^2 \lambda_{lp}^2))
\]

then

\[
\frac{\partial K_p^2}{\partial R} = -8\eta_p (v_o/r_o)^2 \lambda_{lp}^2 / (1 + 2(\gamma^2 + 2(v_o/R/r_o)^2 \lambda_{lp}^2))
\]

The group \( \lambda_{lp} v_o/r_o \) is a dimensionless network formation time constant. Experimental results reveal that for the majority of extrusion systems: \( \lambda_{lp} v_o/r_o \ll 1 \). Consequently,

\[
\left| \frac{\partial K_p^2}{\partial R} \right| \ll |K_p^2|
\]

It can be shown similarly that

\[
\left| \frac{\partial^3 K_p^2}{\partial R^3} \right| \ll \left| \frac{\partial^2 K_p^2}{\partial R^2} \right| \ll \left| \frac{\partial K_p^2}{\partial R} \right| .
\]

One has felt justified in ignoring the partial derivatives of \( K_p^2 \). This assumption reduces the mathematical complexity considerably.
Continuity equation,

\[ \frac{1}{r} \frac{\partial}{\partial r} (rW^{r}) + \frac{\partial W^{z}}{\partial z} = 0 \]  

Equations of motion,

\[ \rho v_{0} \left[ 1 - \left( \frac{r}{r_{0}} \right)^{2} \right] \frac{\partial W^{r}}{\partial z} + \rho \sigma W^{r} = \sum_{p=1}^{\infty} K_{p}^{2} \left[ 2 \beta_{15} \right. \]

\[ - \beta_{33} \left( \frac{r}{r_{0}} \right)^{2} \frac{\partial^{2} W^{r}}{\partial r^{2}} + \left( 2 \frac{\beta_{15}}{r} - 4 \beta_{33} \frac{r}{r_{0}} \right) \]

\[ \frac{\partial W^{r}}{\partial r} - \frac{2 \beta_{33}}{r_{0}} W^{r} + \beta_{48} \left( \frac{r}{r_{0}} \right) \frac{\partial^{2} W^{r}}{\partial r \partial z} \]

\[ + \frac{\beta_{49}}{r_{0}} \frac{\partial W^{r}}{\partial z} + \beta_{35} \frac{\partial^{2} W^{z}}{\partial \partial z} + \beta_{37} \left( \frac{r}{r_{0}} \right) \frac{\partial^{2} W^{z}}{\partial z^{2}} \]

\[ + \left[ \beta_{39} + \beta_{40} \left( \frac{r}{r_{0}} \right)^{2} \right] \frac{\partial^{2} W^{r}}{\partial z^{2}} - \frac{\partial \rho}{\partial r} \right] = 0 \]  

\[ \rho v_{0} \left[ 1 - \left( \frac{r}{r_{0}} \right)^{2} \right] \frac{\partial W^{\theta}}{\partial z} + \rho \sigma W^{\theta} = \sum_{p=1}^{\infty} K_{p}^{2} \left[ \beta_{39} \right. \]

\[ + \beta_{34} \left( \frac{r}{r_{0}} \right)^{2} \frac{\partial^{2} W^{\theta}}{\partial z^{2}} + \beta_{50} \left( \frac{r}{r_{0}} \right) \frac{\partial^{2} W^{\theta}}{\partial r \partial z} \]

\[ + \frac{\beta_{51}}{r_{0}} \frac{\partial W^{\theta}}{\partial z} + \beta_{15} \frac{\partial^{2} W^{\theta}}{\partial r^{2}} + \frac{\beta_{15}}{r} \frac{\partial W^{\theta}}{\partial r} \]
- \frac{\beta_{15}}{r^2} W^{<\theta>} = 0

\rho v_o \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \frac{\partial W^{<z>}}{\partial z} + \rho \alpha W^{<z>} = \sum_{p=1}^{\infty} K_p^2 \beta_{52} \left( \frac{r}{r_o} \right)

\cdot \frac{\partial^2 W^{<z>}}{\partial r \partial z} + \left[ \beta_{39} + \beta_{53} \left( \frac{r}{r_o} \right)^2 \right] \frac{\partial^2 W^{<z>}}{\partial r \partial z}

+ \left[ \frac{\beta_{39}}{r} + \beta_{54} \frac{r}{r_o} \right] \frac{\partial W^{<r>}}{\partial z} + \left[ \beta_{44} + \beta_{45} \left( \frac{r}{r_o} \right)^2 \right]

\cdot \frac{\partial^2 W^{<z>}}{\partial z^2} + \left[ \beta_{46} \left( \frac{r}{r_o} \right)^3 + \beta_{47} \left( \frac{r}{r_o} \right) \right] \frac{\partial^2 W^{<r>}}{\partial z^2}

+ \beta_{35} \frac{\partial^2 W^{<z>}}{\partial r^2} + \beta_{55} \frac{\partial W^{<r>}}{\partial r} + \beta_{38} \left( \frac{r}{r_o} \right) \frac{\partial^2 W^{<r>}}{\partial r^2}

+ 2 \frac{\beta_{37}}{r_o} \frac{\partial W^{<z>}}{\partial z} + \frac{\beta_{35}}{r} \frac{\partial W^{<z>}}{\partial r}

+ \frac{\beta_{36}}{r r_o} W^{<r>} - \frac{\partial P}{\partial z} \right] = 0 \quad 5.108

It is implied that \( W^{<p>} = W^{<p>}(r,z) \) and \( P = P(r,z) \).

The constants occurring in the differential equations above are defined in the parameter list at the end of the report.

The following dimensionless variables are not introduced:
\[ u^{<R>} = \frac{W^{<r>}}{v_o}, \quad u^{<\theta>} = \frac{W^{<\theta>}}{v_o}, \quad u^{<Z>} = \frac{W^{<Z>}}{v_o} \]

\[ \pi = \frac{p}{\rho v_o^2}, \quad R = \frac{r}{r_o}, \quad Z = \frac{z}{r_o} \]

The length of the section has not been introduced. It is felt that this quantity is not a physically well defined one. The length of the draw-down section on the other hand, is a constructive quantity of physical merits which should naturally occur in the mathematical model.

When the pressure is eliminated as it was for the draw-down section, the following field equations are obtained:

\[ \frac{1}{R} \frac{\partial}{\partial R} (Ru^{<R>}) + \frac{\partial u^{<Z>}}{\partial Z} = 0 \]  

\[ \sum_{p=1}^{\infty} \left\{ \frac{\partial^2 u^{<\theta>}}{\partial R^2} + \beta_{56} R \frac{\partial^2 u^{<\theta>}}{\partial R \partial Z} + (\beta_{57} + \beta_{58} R) \frac{\partial^2 u^{<\theta>}}{\partial Z^2} \right. \]

\[ + \frac{1}{R} \frac{\partial u^{<\theta>}}{\partial R} + \beta_{59} \frac{\partial u^{<\theta>}}{\partial Z} - \left( \frac{\text{Re}}{\beta_{15}} \right) \]

\[ + \frac{1}{R^2} ) u^{<\theta>} \} = 0 \]

provided

\[ |\beta_{15}| > 0 \]
\[
\left\{ \sum_{p=1}^{\infty} \left( \beta_{60} + \beta_{45} R^2 \right) \frac{a^3 u <z>}{aZ^2 aR} + 2 \beta_{45} \frac{a^2 u <z>}{aZ^2} + \beta_{52} R \frac{a^3 u <z>}{aR^2 aZ} + \beta_{61} \frac{a^2 u <z>}{aR aZ} + \beta_{35} \frac{a^3 u <z>}{aR^3} + \beta_{35} \frac{1}{R} \frac{a^2 u <z>}{aR^2} - \left( \frac{\beta_{35}}{R} + r \text{Re} \right) \frac{a u <z>}{aR} - \beta_{37} \frac{a^3 u <z>}{aZ^3} + \beta_{38} \frac{a^3 u <z>}{aR^3} + (\beta_{63} + \beta_{64} R^2) \frac{a^3 u <R>}{aR^2 aZ} + \beta_{68} \frac{a^2 u <R>}{aR^2} + (\beta_{46} R^2 + \beta_{62} R) \frac{a^3 u <R>}{aZ^2 aR} + \left( \frac{\beta_{63}}{R} + \beta_{67} R \right) \frac{a^2 u <R>}{aR aZ} + \frac{\beta_{36}}{R} \frac{a u <R>}{aR} + (3 \beta_{46} R^2 + \beta_{65}) \frac{a^2 u <R>}{aZ^2} + (\beta_{69} - \frac{\beta_{39}}{R^2}) \frac{a u <R>}{aZ} - \frac{\beta_{36}}{R^2} u <R> - (\beta_{39} + \beta_{40} R^2) \frac{a^3 u <R>}{aZ^3} \right\} = 0
\]

where

\[ \text{Re} = \rho v_o r_o / K_p^2, \text{ a Reynold's number} \]

\[ \tau = \sigma r_o / v_o, \text{ a non-dimensional eigenvalue} \]
5.6. **Normal Mode Analysis, Describing Ordinary Differential Equations**

Needless to say, generation of general solutions for the two sets of partial differential equations which were derived in the previous chapter is well beyond our capabilities. Alternatively, solving these equations numerically by means of high speed digital computers is equally well a formidable task. This situation is more typical than exceptional within hydrodynamical stability analysis. It is customary to apply the normal mode analysis in such cases.

The normal mode analysis assumes the solution vector to be decomposable into normal modes in all but one space coordinate:

\[ \mathbf{w}(x) = v(x_1) \exp\left(i \sum_{p=1}^{\infty} \nu_{2p} x_2 + \nu_{3p} x_3 \right) \]  \hspace{1cm} 5.112

The incentive of the normal mode assumption is the conversion of sets of partial differential equations into sets of ordinary differential equations, whose solutions may be found numerically or analytically with relative ease.

The resulting set of ordinary differential equations obtained when applying the normal mode assumption may be symbolically written as:

\[ L(w(x_1)) = 0 \]
where \( L() \) is a differential operator in one independent variable, \( x_1 \), say. It is clear that the constant in this operator contains the two wave numbers \( \nu_{2p} \) and \( \nu_{3p} \) in addition to the problem parameters in the original partial differential equations, as well as the steady state velocity field and its derivatives. The only boundary conditions permitted are those specifying \( W() \) and its derivatives at specific values of \( x_1 \).

It has been pointed out earlier (Chapter 5.1) that the normal mode assumption is a valid one for systems physically bounded in only one space coordinate direction. Difficulties are encountered when the system is bounded in more than one space coordinate direction. One wishes now to satisfy a larger set of boundary conditions. This is accomplished for certain combinations of the wavenumbers. It is, at least in principle, possible to find wavenumber combinations which will satisfy all the physical boundary conditions in a completely bounded system, provided that these boundary conditions do not possess discontinuities. The procedures by which to find such wavenumber combinations are, however, very difficult. Therefore, all wavenumbers are normally considered possible if not restricted for other reasons. If so, some of the physical boundary conditions must be ignored. In the present work, the normal mode assumption applies quite well since only a limited
number of boundary conditions are available. This will be discussed in a subsequent chapter.

5.6.1. Draw-down section

A considerable simplification of the mathematical complexity is achieved by assuming the perturbed velocity field to be rotationally symmetric, \( a/\theta = 0 \). It is easily shown that in such a case an arbitrary function \( \psi = \psi(R,Z) \) exists such that the continuity equation is identically satisfied. This function resembles very closely a stream function.

\[
\begin{align*}
  u^<R> &= -\frac{1}{R} \frac{\partial}{\partial Z} (R\psi(R,Z)), \quad u^<Z> = \frac{1}{AR} \frac{\partial}{\partial R} (R\psi(R,Z))
  
  \end{align*}
\]

5.113

The normal mode assumption is now applied for the \( Z \)-direction:

\[
\begin{align*}
  \psi(R,Z) &= -\hat{\psi}(R)e^{ivZ} \\
  u^<\theta>(R,Z) &= \hat{v}(R)e^{ivZ}
\end{align*}
\]

5.114

One realizes that this implies that the boundary conditions at \( Z = 0 \) and \( Z = 1 \) must be ignored if the wave-number is considered unrestricted. The boundary conditions at \( Z = 0 \), in particular, is not easily determined since the draw-down section at this point connects physically to the die-swell
region, and can only be found when the perturbed velocity field at the die-swell exit is known. The conditions at $Z = 1$ depend on the mode of operation of the extruder. The wind-up device may be run at constant speed or constant axial tension. The mode of constant speed is the most common for commercial extruders. In this case

$$v^<_Z> = \frac{v^<_Z>}{v^<_Z>} \text{ at } z = L$$

which implies that

$$v^<_Z> = 0 \Rightarrow u^<_Z> = 0 \text{ at } z = L$$

In terms of $\psi$-function

$$\frac{\partial \psi(R, Z)}{\partial R} + \frac{\psi(R, Z)}{R} = 0 \text{ at } z = L$$

It is clear that no additional boundary condition can be generated by using the continuity equation in conjunction with the last equation since the $\psi$-function satisfies the continuity equation identically. Thus, additional, non-physical conditions would have to be provided if one were to apply the normal mode assumption to the radial rather than the axial direction.

The conditions for the free surface are, on the other hand, well established. This will be discussed in detail in a subsequent chapter. It shall be shown that
the free surface conditions provide necessary and sufficient conditions for the problem at hand.

It should be noted that in earlier works on the stability of this section, the normal mode assumption was applied for the radial direction.

When equations 5.113 and 5.114 are substituted into equations 5.102, 5.103 the following ordinary differential equations emerge:

\[
\sum_{p=1}^{\infty} \left( \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} - \left[ F + \frac{1}{R^2} \right] \right) \hat{v}(R) = 0 \quad 5.115
\]

\[
\sum_{p=1}^{\infty} \left\{ \frac{d^4}{dR^4} + \frac{2}{R} \frac{d^3}{dR^3} + \left( B - C - \frac{2}{R^2} \right) \frac{d^2}{dR^2} + \frac{1}{R} \left( B + C + \frac{4}{R^2} \right) \frac{d}{dR} + \left( E - \frac{B}{R^2} - \frac{4}{R^4} \right) \right\} \hat{\psi}(R) = 0 \quad 5.116
\]

\[ R, Z \in [0,1], \hat{v}(R) \text{ and } \hat{\psi}(R) \text{ are complex} \]

One notes that the above equations are uncoupled. Coupling through the boundary conditions may take place, however. The constants are given in the parameter list at the end of the report.
5.6.2. **Metering section**

An arbitrary scalar function \( \psi = \psi(R,Z) \) satisfying the continuity equation identically exists.

\[
\begin{align*}
 u^{\langle R \rangle} &= -\frac{1}{R} \frac{3}{\partial R} (R \psi(R,Z)) , \\
 u^{\langle Z \rangle} &= \frac{1}{R} \frac{3}{\partial R} (R \psi(R,Z)) \tag{5.117}
\end{align*}
\]

Again, the normal mode assumption is applied for the axial direction:

\[
\psi(R,Z) = -i \hat{\psi}(R) e^{i\nu Z} , \quad u^{\langle \theta \rangle} = \hat{v}(R) e^{i\nu Z} \tag{5.114}
\]

Difficulties in determining the boundary conditions at the entrance and the exit of this flow section has been the justification for applying the normal mode assumption in the axial rather than in the radial coordinate direction. There boundary conditions are simply not available. There is no reason to believe that, for instance, the perturbed velocity field or components thereof, or any of their derivatives will vanish at the ends of the flow section. The conditions at the radial flow boundary are more readily available.

By equations 5.117, 5.114, 5.109 and 5.111:
\[ \sum_{p=1}^{\infty} \left\{ \frac{d^2}{dR^2} + \left( iv\beta_{56}R + \frac{1}{R} \right) \frac{d}{dR} \right\} \hat{v}(R) = 0 \]

\[ - \left( \frac{\text{Re}}{\beta_{15}} + v^2\beta_{57} - iv\beta_{59} + v^2\beta_{58}R + \frac{1}{R^2} \right) \hat{v}(R) = 0 \]

\[ \sum_{p=1}^{\infty} \left\{ \frac{d^4}{dR^4} + (\beta_{70}R + \frac{2}{R}) \frac{d^3}{dR^3} - (\beta_{72}R^2 + \beta_{71} + \frac{3}{R^2}) \frac{d^2}{dR^2} \right\} \]

\[ + \left( \beta_{76}R^3 + R\beta_{75} - \beta_{74} - \frac{\beta_{73}}{R} + \frac{3}{R^3} \right) \frac{d}{dR} \]

\[ + \left( \beta_{81}R^2 + \beta_{80}R + \beta_{79} + \frac{\beta_{78}}{R} + \frac{\beta_{77}}{R^2} - \frac{3}{R^4} \right) \hat{\psi}(R) = 0 \]

\[ R \in [0,1), \quad Z \in [0,\infty) \]

The constants are given in the parameter list at the end of the report. The above equations are uncoupled although coupling through the boundary conditions may take place.

5.7. Solutions of the Ordinary Differential Equations

Equations 5.115, 5.116, 5.118 and 5.119 are all singular at \( R = 0 \). Any singular solutions must be disregarded since \( R = 0 \) is included in the domain of interest. The number of regular solutions of any of the equations mentioned above must necessarily be less than the order of the equation in question. Possibly, no regular solutions
at all exist for any given equation. The only information one with certainty can provide at the boundary $R = 0$ is the existence of the solutions there. In fact, from the theory of ordinary differential equations (37) it is known that the only acceptable solutions at $R = 0$ are the rest solutions, or the trivial solutions, themselves, $\hat{\psi}(0) = \hat{v}(0) = 0$. The boundary conditions should consequently refer to the physical flow boundary in which case one is dealing with an initial value problem. Unique solutions exist through any specific set of initial value on account of the "fundamental theorem for linear, ordinary differential equations."

Numerical integration of the above equations is encumbered by the following facts:

i) The number of boundary conditions to provide is unknown since the number of independent solutions is unknown.

ii) As will be shown in a subsequent chapter, the boundary conditions are too complicated that expansion functions which will satisfy these boundary conditions identically, may be found.

Thus, an incentive is seen in developing analytic solutions for the equations at hand. One insists on solutions with the following properties:
\[ \psi(R), \ldots, \frac{d^4 \psi(R)}{dR^4}, \quad \hat{\psi}(R), \ldots, \frac{d^2 \hat{\psi}(R)}{dR^2} \]

shall all be continuous in \( R \in [0,1] \).

The solutions may be expressed as power series in \( R \),

\[ \hat{\psi}(R), \hat{\psi}(R) = R^s \sum_{n=0}^{\infty} c_n R^n \quad 5.120 \]

The Frobenius method was applied to determine \( s \) and the recursion formulae for the coefficients. The Frobenius method is well described in numerous texts on mathematics, and no point is seen in a detailed description of it in this thesis. The solution procedure is long and tedious, and the reader is referred to Appendix 2 for a more detailed treatment. The following results were obtained:

5.7.1. **Draw-down section**

Only one regular solution for equation 5.115 exists,

\[ \hat{\psi}(R) = c_o F^{-1/2} J_1(\text{iRF}^{1/2}) \quad 5.121 \]

where \( c_o \) is an arbitrary, complex constant and \( J_1() \) is the Bessel function of the first kind of order 1. When the Bessel function is decomposed into real and imaginary parts, ber and bei functions will result.

Two regular and independent solution functions were
found for equation 5.116:

\[ \hat{\psi}(R) = b_0 R(1-CR^2/8 + \sum_{n=4}^{\infty} b_n R^n) \]

\[ + b_1 R^2 (1-BR^2/20 + \sum_{n=5}^{\infty} b_n R^{n-1}) \]

The quantities \( b_0 \) and \( b_1 \) are arbitrary, complex constants.

The coefficient recursion formula is as follows:

\[ b_n = \frac{b_{n-2}(3C-2n(2C-B)-(B-C)n^2+b_{n-4}E)}{n(n-1)^2(n+2)} , \quad n \geq 4 \]

The first solution function of equation 5.122 is generated by the even terms of an infinite power series, the second solution function by the odd terms of the same power series. The functions are thus undoubtedly independent. It is clear that:

\[ F = F(Re', \Gamma, \kappa, E, \nu, \Lambda) \]

\[ b_n = b_n(Re', \Gamma, \kappa, E, \nu, \Lambda) \]

where \( \kappa = \lambda_2^{1/ \gamma} \) has the character of a Deborah number. The solutions thus span a Euclidian parameter space of dimension six:

\[ \mathbf{p}_\perp = [Re', \Gamma, \kappa, E, \nu, \Lambda]^T \]
5.7.2. **Metering section**

Only one regular solution of equation 5.118 exists,

\[ \hat{v}(R) = p_0 R (1 + p_2 R^2 + \sum_{n=3}^{\infty} p_n R^n) \]  \hspace{1cm} 5.126

where \( p_0 \) is an arbitrary, complex constant and

\[ p_2 = \frac{1}{8} \left\{ \frac{r \text{Re}}{\beta_{15}} + \nu^2 \beta_{57}^{-i} \nu \beta_{56} (8+4E)/(4+E) \right\} \]  \hspace{1cm} 5.127

The coefficients are determined by the following recursion formula:

\[ p_n = \frac{p_{n-2} \left[ r \text{Re}/\beta_{15} + \nu^2 \beta_{57}^{-i} \nu (n+2E)/(4+E) \right] + \nu^2 (2+E) \beta_{56} p_{n-3}}{n(n+2)} \]  \hspace{1cm} n \geq 3 \hspace{1cm} 5.128

Somewhat surprising, only one regular solution was found for equation 5.119:

\[ \hat{\psi}(R) = q_0 R^3 (1 + q_2 R^2 + q_3 R^3 + q_4 R^4 + q_5 R^5 + \sum_{n=6}^{\infty} q_n R^n) \]  \hspace{1cm} 5.129

where \( q_0 \) is an arbitrary, complex constant.

\[ q_2 = (6 \beta_{71} + 3 \beta_{73} - 6 \beta_{70} - \beta_{77})/192 \]

\[ q_3 = (3 \beta_{74} - \beta_{78})/525 \]

\[ q_4 = q_2 (20 \beta_{71} + 5 \beta_{73} - 60 \beta_{70} - \beta_{77}) + 6 \beta_{72} - 3 \beta_{75} - \beta_{79})/1152 \]
\[ q_5 = \{q_3(30\beta_{71} + 6\beta_{73} - 120\beta_{70} - \beta_{77}) + q_2(5\beta_{74} - \beta_{78}) - \beta_{74}/2\}/2205 \]

\[ q_n = \{q_{n-2}[\beta_{71}n(n+1) + \beta_{73}(n+1) - \beta_{70}n(n^2 - 1) - \beta_{77}] + q_{n-3}[n\beta_{74} - \beta_{78}] + q_{n-4}[(n-1)(n-2)\beta_{72} - (n-1)\beta_{75} - \beta_{79}] \} - q_{n-5} \frac{\beta_{74}}{2} - q_{n-6}[(n-3)\beta_{76} + \beta_{81}]/[n(n+2)^2(n+4)] \]

\[ n \geq 6 \]

It is easily verified that:

\[ p_n = p_n(Re, \Gamma, \kappa, \tau, E, \nu) \]

\[ q_n = q_n(Re, \Gamma, \kappa , \tau, E, \nu) \]

where

\[ \kappa = \lambda_2 p \nu_o / r_o \], a Weissenberg number

\[ \tau = \gamma r_o / \nu_o \], a non-dimensional extension rate

The solutions thus span a Euclidian parameter space of dimension six,

\[ Q = [Re, \Gamma, \kappa , \tau, E, \nu]^T \]
5.8. **Boundary Condition**

Since the number of independent functions comprising the solutions of the ordinary differential equations of Chapter 5.7, have now been established, one may proceed to specify the boundary conditions for the respective equations.

The boundary conditions will be of the form:

\[ \text{Real}(L(v'(t,r))) = 0 \text{ on boundary} \]

where \( L() \) is an ordinary, differential operator in \( r \).

When writing

\[ \sigma = \sigma_R + i\sigma_I, \ u(R) = u_R(R) + iu_I(R) \]

one obtains

\[ \text{Real}(e^{(i\sigma_I t + \nu Z)} L(u(R))) = 0 \]

or

\[ L(u_R(R)) - L(u_I(R)) \tan(\sigma_I t + \nu Z) = 0 \]

But since the boundary conditions must be satisfied for all times \( t \),

\[ L(u_R(R)) = L(u_I(R)) = 0 \implies L(u(R)) = 0 \]

on boundary.
5.8.1. **Draw-down section**

A complication for this section is the fact that perturbations in the velocity field induce excursions in the free surface. The shape of the free surface may be described by a partial differential equation which must be solved simultaneously with the field equations for the section. The boundary conditions must be applied at the perturbed rather than the steady state free surface.

Since the streamlines are nearly parallel (\( \gamma \) very small), the steady state surface may be determined from a macroscopic mass balance:

\[
\int_A \int \overline{v} \mathrm{d}A - \text{Constant, } A \text{ is extrudate cross section}
\]

After integration,

\[
\overline{R} = (\overline{v}_o + z)^{1/2}, \quad \overline{v}_o = v_o/(\gamma \ell) = (v_L/v_o - 1)^{-1}
\]

It is interesting to note that this is precisely the relation Spearot and Metzner (83) obtained by correlation of their experimental data. The important implication is that purely elongational flows of viscoelastic fluids are indeed physically realizable.

The locus of any point on the free surface is:

\[
m = m(t, \theta, z)
\]
One assumes that this function is sufficiently well-behaved that its total differential exists,

\[ \frac{dm}{dt} = \frac{\partial m}{\partial t} + \frac{\partial m}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial m}{\partial z} \frac{dz}{dt} \]  

5.135

Rotational symmetry has earlier been assumed, thus, \( \partial / \partial \theta = 0 \). Then,

\[ v^<r> (t, r, z)_{r=m} = \frac{\partial m}{\partial t} + v^<z> (t, r, z)_{r=m} \frac{\partial m}{\partial z} \]  

5.136

Letting \( m = a + \bar{r} \) and realizing that \( v^<r> = v'^<r> (\bar{v}^<r> = 0) \) and \( v^<z> = \bar{v}^<z> + v'^<z> \), the equation 5.136 then becomes after linearization around \( a = 0 \),

\[ v'^<r> (t, \bar{r}, z) = \frac{\partial a}{\partial t} + \bar{v}^<z> (z) \frac{\partial a}{\partial z} \]

\[ + v'^<z> (t, \bar{r}, z) \frac{\partial \bar{r}}{\partial z} \]  

5.137

It is now assumed that the function \( a = a(t, z) \) is separable in time with the same time dependence as found for the perturbed velocity field,

\[ \frac{\bar{a}}{\bar{r}_0} = \eta \sigma t, \eta = \eta(z) \]  

5.138

Substituting equations 5.113, 5.114 and 5.138 into equation 5.137 yield the following result,
\[
\frac{d\eta(Z)}{dZ} + \left( \frac{\Gamma}{Z+V_0} \right) \eta(Z) = \frac{e^{i\nu Z} \left[ i \frac{1}{R} \frac{d}{dR} (R\hat{\psi}(R)) \frac{dR}{dZ} - \nu \hat{\psi}(R) \right]_{\hat{R}=\hat{R}}}{\Lambda(Z+V_0)} \tag{5.139}
\]

where

\[
\frac{dR}{dZ} = -\frac{R^2}{V_0+Z} = -\frac{R^3}{2} \left( \frac{V_L}{V_0} - 1 \right) \tag{5.140}
\]

Equation 5.139 is solved subject to the boundary condition,

\[
\eta(Z) = 0 \text{ at } Z = 0 \tag{5.141}
\]

The solution is,

\[
\eta(Z) = \frac{\left[ i \frac{1}{R} \frac{d}{dR} (R\hat{\psi}(R)) \right]_{\hat{R}=\hat{R}} \frac{d\hat{R}}{dZ} - \nu \hat{\psi}(R) \int_0^Z e^{i\nu q (q+V_0)} (R-1) dq}{\Lambda(Z+V_0)^{\Gamma}} \tag{5.142}
\]

It is of interest to investigate the excursions of the free surface at the flow nodes, that is, where

\[
u^{<z>} = u^{<R>} = 0 \tag{5.143}
\]

By equations 5.113 and 5.114 this is equivalent to

\[
\frac{d\hat{\psi}(R)}{dR} = \hat{\psi}(R) = 0 \tag{5.144}
\]

By consulting equation 5.142 it is seen that this leads to
\( \eta(Z) = 0 \) at the flow nodes

This result is precisely what should be expected.

Since the surface tension effects have been neglected (see Chapter 4), the conditions at the free surface are:

i) The normal stresses must vanish locally.

ii) The tangential surface stresses or, equivalently, the surface traction vector must vanish locally.

The tangential stresses may be decomposed into their three directional components, it is necessary and sufficient that each of these vanish individually for the total tangential stresses to vanish. Expressed mathematically the boundary conditions are:

\[
\begin{align*}
T^{rr}n_r^2 + T^{\theta \theta}n_\theta^2 + T^{zz}n_z^2 + 2(T^{rr}n_r n_\theta + T^{rz}n_r n_z + T^{\theta z}n_\theta n_z) &= 0 \\
T^{rr}n_r + T^{\theta \theta}n_\theta + T^{zz}n_z &= 0 \\
T^{rz}n_r + T^{\theta z}n_\theta &= 0 \\
T^{\theta r}n_\theta + T^{z r}n_z &= 0
\end{align*}
\]

where \( \mathbf{n} \) is the outward surface normal.

The surface normal is readily found from:
\[ \phi = \frac{m}{r_0} - \left( \frac{r}{r_0} + e^{\sigma t} n(Z) \right) = 0 \]  

5.150

\[ n = \nabla \phi / |\nabla \phi|, \text{ a covariant vector} \]  

5.151

The results are:

\[ n_r = n^{<r>} = \frac{1}{\{1 + \Lambda^2 \left[ \frac{\sqrt{v_0}}{2} (v_0 + Z)^{-3/2} - e^{\sigma t} \frac{dn}{dz} \right]^2 \}^{1/2}} \]  

5.152

\[ n_z = n^{<z>} = \frac{\Lambda \left[ \frac{\sqrt{v_0}}{2} (v_0 + Z)^{-3/2} - e^{\sigma t} \frac{dn}{dz} \right]}{\{1 + \Lambda^2 \left[ \frac{\sqrt{v_0}}{2} (v_0 + Z)^{-3/2} - e^{\sigma t} \frac{dn}{dz} \right]^2 \}^{1/2}} \]  

5.153

\[ n_\theta = n^{<\theta>} \equiv 0 \]  

5.154

For \( \sigma_R = 0 \), the components of the normal may easily be linearized and subsequently decomposed into a steady state and a perturbed part, \( n = \overline{n} + n' \). This procedure is trivial.

Note that due to \( n_\theta = 0 \), equations 5.148 and 5.149 become identical. Using 5.154 and linearizing, the boundary conditions become (\( T^{rr} = T^{\theta \theta} = 0 \)),

\[ T'^{rr} n_r^{-2} + 2 T'^{zz} n_z^{-2} n_z' + T'^{z z} n_z^{-2} = 0 \]  

1.55
\[
T'_{r0}^{\bar{n}_r} + T'_{z0}^{\bar{n}_z} = 0
\]

5.156

\[
T'_{rz} = 0
\]

5.157

When substituting the stresses found in Chapter 5.2.1 and 5.4.1 into the last three equations, the following results are obtained:

\[
\sum_{p=1}^{\infty} \{ N_8^p \frac{\nu}{\Lambda} - 4N_8^p \left( \frac{\bar{v}_o}{\Lambda R^2} \right)^3 \left( \frac{\nu}{\bar{v}_o} \right) \\
+ 2i \frac{N_9^p}{\Lambda \bar{v}_o} \left( \frac{R^2 e^{i\nu Z}}{\bar{v}_o} - rI \right) \frac{d\psi(R)}{dR} + [N_8^p \left( \frac{\nu}{\Lambda} \right) ] \\
+ 4N_9^p \left( \frac{\nu}{\Lambda} \right) \frac{1}{R} ( e^{i\nu Z} - rI ) \\
+ 2i \frac{N_9^p}{\Lambda \bar{v}_o} \left( \frac{R^2 e^{i\nu Z}}{\bar{v}_o} - rI \right) \hat{\psi}(R) \} = 0
\]

5.158

\[
\sum_{p=1}^{\infty} \{ N_7^p \frac{d^2\psi(R)}{dR^2} + \frac{N_7^p}{R} \frac{d\psi(R)}{dR} \\
+ [N_6^p (\Lambda \nu)^2 - \frac{N_7^p}{R^2}] \hat{\psi}(R) \} = 0
\]

5.159
\[ \sum_{p=1}^{\infty} \left\{ N^2 \left( \frac{V_o + Z}{V_o} \right)^{3/2} \frac{d \hat{\nu}(R)}{dR} + \left[ \frac{i}{2} N^2 \frac{\Lambda^2 v}{V_o} \right] \right\} \hat{\nu}(R) = 0 \]  

at \( R = \hat{R} \)

\[ I = (Z + V_o)^{-\gamma} \int_0^Z e^{i \nu q} (q + V_o)^{(\gamma - 1)} dq \]

Since three independent solution functions were found for this section (Chapter 5.7), one for \( \hat{\nu}(R) \) and two for \( \hat{\psi}(R) \), it is clear that the above equations provide a sufficient and necessary number of boundary conditions. The following important assertion may therefore be made:

**Assertion 3.** A steady elongational flow of an incompressible, viscoelastic fluid suffering only axial stretching and subjected to no gravity effects and a free surface of no surface tension effects may be infinitesimally perturbed provided the perturbed velocity field is rotationally symmetric.

When the solution functions obtained in Chapter 5.7 are substituted into equation 5.158 through 5.160, the final results emerge:
\[ \sum_{p=1}^{\infty} \{ J_0(iR\sqrt{F}) + [i \frac{N_1}{N_p} \frac{\lambda^2 v R^3}{2V_0} - \frac{2}{R} \frac{1}{\sqrt{F}} J_1(iR\sqrt{F})] \} = 0 \]  
5.162

\[ \sum_{p=1}^{\infty} \{ [K_1 \frac{dS_2}{dR} + K_2 S_2] \frac{d^2 S_1}{dR^2} + \frac{1}{R} \frac{dS_1}{dR} + K_3 \frac{S_1}{R^2} \} \]

\[ - [K_1 \frac{dS_1}{dR} + K_2 S_1] \frac{d^2 S_2}{dR^2} + \frac{1}{R} \frac{dS_2}{dR} \]

\[ + K_3 \frac{S_2}{R^2} \} \} = 0 \quad \text{at } R = \bar{R} \]  
5.163

where

\[ K_1 = 1 - \frac{4}{R^6} \frac{N_1}{N_p} \left( \frac{V_0}{\lambda} \right)^2 \]

\[ + 2 \frac{iR^2}{\nu V_0} \frac{N_9}{N_p} \left( \frac{R^2 e^{i\nu Z}}{V_0} - \nu I \right) \]  
5.164

\[ K_2 = \{ 1 + 2 \frac{N_9}{N_p} e^{i\nu Z} (2 + \frac{iR^4}{\nu V_0^2}) \}

- 2\nu I (\frac{iR^2}{2\nu V_0} + 1) / R \]

5.165

\[ K_3 = (\lambda \nu)^2 \frac{N_6}{N_p} - \frac{1}{R^2} \]  
5.166
\[ S_1 = R(1 - CR^2/8 + \sum_{n=4}^{\infty} b_n R^n) \]

\[ S_2 = R^2(1 - BR^2/20 + \sum_{n=5}^{\infty} b_n R^{(n-1)}) \]

The recursion formula for \( b_n \) is given by equation 5.123.

Equations 5.162 and 5.163 represent eigenvalue problems in the sense that only certain combinations of parameters may provide solutions. This is most certainly expected. If all parameter combinations were possible, solutions of the above equations would always exist for negative Reynold's numbers since this quantity is one component of the parameter vector. Such a situation implies that all motions would be unstable.

One notes that equations 5.162 and 5.163 are uncoupled. This fact simplifies the ultimate numerical problem dramatically.

The boundary conditions introduced an additional parameter into the problem formulation, namely \( Z \). The parameter space to deal with is, thus, of dimension seven.

\[ \hat{\mathbf{p}} = [Re', \Gamma, \kappa, E, \nu, A, Z]^T \]
5.8.2. Metering section

The streamlines of this section as well are nearly parallel since \( \gamma \) must remain very small on account of compatibility. The radial flow boundary in steady state may then be obtained by employing the macroscopic mass balance,

\[
\int_A \nabla dA = \text{Constant, } A \text{ is cross section of channel} \quad 5.168
\]

The result is:

\[
\bar{R} = \sqrt{\zeta + 1 - \sqrt{(\zeta)^2 + 2\zeta}}
\]

\( \bar{R} \in [0,1] \) for \( Z \in [0,\infty] \) \quad 5.169

If the flow channel concurs exactly with the shape described by equation 5.169, only the primary flow exists. This is not likely, and both primary and secondary flows must be present (see Figure 2b). One would not expect the "no slip" condition to apply at the boundary separating the primary and secondary flow regimes. One remembers from Chapter 5.7.2 that only one regular solution function exists for each of the dependent variables \( \hat{v}(R) \) and \( \hat{\psi}(R) \). The "no slip" condition requires

\[
\hat{v}(R) = \hat{\psi}(R) = \frac{d\hat{\psi}(R)}{dR} = 0 \text{ at } R = \bar{R} \quad 5.170
\]
Fig. 3 METERING SECTION
that is, a total of three boundary conditions must be satisfied. The "no slip" condition can therefore not exist on the radial flow boundary, and the physical expectations have been borne out mathematically. This result is not surprising in view of the existence of axial slip in steady state.

Neither would one expect the conditions on the flow boundary to concur with those of a truly free surface. The existence of the rigid channel is of importance, creating, for instance, pressures on the flow boundary grossly different from those of a free surface. Furthermore, it has been shown in a previous chapter that three boundary conditions must be satisfied if the boundary may remain a free surface of no surface tension. Thus, it is mathematically impossible for the flow boundary to behave as a free surface.

The secondary flow region is stagnant in the sense that virtually no mass is exchanged with the primary flow region. The motion in the secondary flow region is purely rotational. The flow boundary may then be considered mass impenetrable,

\[ \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{at} \quad R = \bar{R} \]

This boundary condition applies equally well whether the flow channel concurs with the radial flow boundary or not.
One assumes that the flow boundary remains immune to the infinitesimal perturbations in the velocity field. That is, the radial flow boundary retains its steady state location also after the process is subjected to disturbances. When linearizing equations 5.171,

\[ \nabla' \cdot \overline{n} = 0 \quad \text{at} \quad R = \overline{R} \]

The outward normal, \( \overline{n} \), is easily found from equation 5.169 and the following:

\[ \phi = \overline{R} - \left( \tau Z + 1 - ((\tau Z)^2 + 2\tau Z)^{1/2} \right)^{1/2} = 0 \quad 5.172 \]

\[ \overline{n} = \nabla\phi / |\phi|, \quad \text{a covariant vector} \quad 5.173 \]

The results are:

\[ \overline{n}_r = \overline{n}^{<r>} = \frac{1}{2} \cdot \frac{1}{[1 + \frac{\tau R^2}{4(\tau Z + 1 - R^2)^2}]^{1/2}} \quad 5.174 \]

\[ \overline{n}_z = \overline{n}^{<z>} = \frac{1}{2} \cdot \frac{\tau R/(1 + \tau Z - R^2)}{[1 + \frac{\tau R^2}{4(\tau Z + 1 - R^2)^2}]^{1/2}} \quad 5.175 \]

\[ \overline{n}_\theta = \overline{n}^{<\theta>} = 0 \quad 5.176 \]

The boundary condition is then:

\[ u' <R> \overline{n}^{<R>} + u' <z> \overline{n}^{<z>} = 0 \quad \text{at} \quad R = \overline{R} \quad 5.177 \]
and the final result is:

\[ \sum_{p=1}^{\infty} \left\{ \tau \frac{dS_3(R)}{dR} + \frac{((\tau - 2i\nu(\tau Z + 1))/R + 2i\nu)S_3(R)}{R} \right\} = 0 \]

at \( R = \bar{R} \) \hspace{1cm} 5.178

where

\[ S_3(R) = \hat{\psi}(R)/q_0 \]

The function \( \hat{\psi}(R) \) is defined by equations 5.129 through 5.131.

Equation 5.178 represents, unfortunately, the only physical boundary condition one can specify with certainty for this flow section. A boundary condition for the rotational velocity component is still needed. It has been established earlier that the radial flow boundary is neither a rigid boundary in the sense of a vanishing velocity field (no slip) on the boundary nor a truly free surface in the sense of vanishing normal and tangential stresses. It seems reasonable to expect the conditions on the boundary to fall somewhere between the following limiting cases:

i) Rotational, perturbed velocity component vanishes on the boundary.

ii) Rotational component of the perturbed surface traction vector vanishes on the boundary.
That is, either

\[ u' \theta = 0 \quad \text{at} \quad R = \bar{R} \quad 5.179 \]

or

\[ T' r \theta / r + T' z \theta / z + T' \theta \theta / \theta \theta \quad \text{at} \quad R = \bar{R} \quad 5.180 \]

Both these conditions will be sequentially applied. Condition 5.179 will be denoted the "fixed surface" and 5.180 the "free surface" condition in a somewhat erroneous terminology. The final boundary conditions are:

either

\[ \sum_{p=1}^{8} S_4(R) = 0 \quad \text{at} \quad R = \bar{R} \quad 5.181 \]

or

\[ \sum_{p=1}^{\infty} \left\{ \frac{dS_4(R)}{dR} + ((1+1/2E)\beta_20/ \right\] \[ \beta_{15} \nu R^{-1/R} S_4(R) \} = 0 \quad \text{at} \quad R = \bar{R} \quad 5.182 \]

where \( S_4 = \hat{v}/P_0 \) and \( \hat{v} \) is defined by equations 5.126, 5.127 and 5.128. The equations 5.178, 5.181 and 5.182 represent uncoupled eigenvalue problems of the same nature as found for the draw-down section. The dimension of the parameter space is seven.

\[ \hat{Q} = [Re, \Gamma, \kappa, \tau, E, \nu, Z]^T \]
5.9. **Limiting Cases**

The parameter of primary importance in this work is the Reynold's number. One is, *a priori*, interested in generating conditions for neutral stability, that is, conditions in which $\Gamma$ is purely imaginary.

One notes from the dimensionless field equations that the Reynold's number appears in the combination $\Gamma \Re$. If solutions of the eigenvalue problems of the previous chapter are at all possible for very small $\Gamma_I$, the corresponding Reynold's number is then expected to be of large magnitude. If the problem permits solutions for very large $|\Gamma_I|$, then one should expect very small Reynold's numbers.

No success was achieved in an attempt to establish the feasibility of solutions for very small $\Gamma_I$'s, although the nature of the differential equations was found unaltered.

The case $|\Gamma_I| \rightarrow \infty$, on the other hand, turned out manageable. This case proved degenerate.

It is also of interest to investigate the behavior of the solution functions for large arguments. A large argument may only be realized by letting Reynold's number become large. This case proved degenerate as well.

5.9.1. **Draw-down section $|\Gamma_I| \rightarrow \infty$**

For $|\Gamma_I| \rightarrow \infty$, $\Gamma_R = 0$, the differential equations
became:

\[ \Lambda^2 \text{Re}'[\Gamma_I + \frac{\nu_0'}{\nu_L - \nu_0'}] \hat{\psi}(R) = 0 \]  

5.183

\[ \{ \frac{\nu_0'}{\nu_L - \nu_0'} + \Gamma_I \text{Re}'\} \{ \frac{d^2 \hat{\psi}(R)}{dR^2} + (\frac{1 + \Lambda \nu^2}{1 - \Lambda \nu^2}) \frac{1}{R} \frac{d\hat{\psi}(R)}{dR} \]

\[ + \left( \frac{\Lambda \nu^2}{1 - \Lambda \nu^2} - \frac{1}{1 - \Lambda \nu^2} \cdot \frac{1}{R^2} \right) \hat{\psi}(R) = 0 \]  

5.184

provided \( \Lambda \nu^2 \neq 1 \).

Equation 5.183 requires:

\[ \lim(\Gamma_I \text{Re}') \to 0 \implies \text{Re}' = \mathcal{O}(\Gamma_I^{-2}) \]

\[ |\Gamma_I| \to \infty \]

or

\[ \lim \text{Re}' \to 0 \text{ and } \hat{\nu}(R) \text{ remains arbitrary} \]  

5.185

The Frobenius method is employed in order to find the general solution of equation 5.184 represented as infinite power series. The result is:

\[ \hat{\psi}(R) = p_o R^1 \left( 1 + \sum_{n=1}^{\infty} (-1)^n p_n R^{2n} \right) \]

\[ + q_o R^2 \left( 1 + \sum_{n=1}^{\infty} (-1)^n q_n R^{2n} \right) \]  

5.186
where

\[ s_1 = \frac{-\Lambda \nu^2 + \sqrt{(\Lambda \nu^2)^2 - \Lambda \nu^2 + 1}}{1 - \Lambda \nu^2} \]

\[ s_2 = \frac{-\Lambda \nu^2 + \sqrt{(\Lambda \nu^2)^2 - \Lambda \nu^2 + 1}}{1 - \Lambda \nu^2} \]

\[ p_n = \frac{\Lambda \nu^2 p_{n-1}}{(1 - \Lambda \nu^2)(2n+s_1)(2n+s_1-1) + (1 + \Lambda \nu^2)(2n+s_1) - 1} \]

\[ q_n = \frac{\Lambda \nu^2 q_{n-1}}{(1 - \Lambda \nu^2)(2n+s_2)(2n+s_2-1)(1 + \Lambda \nu^2)(2n+s_2) - 1} \]

It is clear that \( s_1 < 0, \) \( s_2 > 0 \) if \( \Lambda \nu^2 < 1 \) and \( s_1 > 0, \) \( s_2 < 0 \) if \( \Lambda \nu^2 > 1. \) Thus, the general solution always comprises one regular and one singular solution function.

Since two boundary conditions must be satisfied on the \( \hat{\psi}(R) \) function if the free surface shall exist, the singular solution must be included. A necessary condition for equation 5.184 to be satisfied is thus,

\[ \lim_{|R_1| \to \infty} (R_1 \Re') = 0 \]

The sufficiency of this condition is readily established by L'Hospital's rule. The results from equation 5.185 are thus recovered.
Equation 5.184 reduces to the following for the case $\Lambda \nu^2 = 1$.

$$\frac{d\hat{\psi}(R)}{dR} + 1/2(1-1/R^2)\hat{\psi}(R) = 0$$  \hspace{1cm} (5.188)

Since this differential equation is of first order, the free surface can not exist unless $\hat{\psi}(R) \equiv 0$. Consequently, only rotational disturbances are possible.

5.9.2. Metering section, $\Gamma_1 \rightarrow \infty$

For $\Gamma_1 \rightarrow \infty$, $\Gamma_R = 0$, the differential equation becomes:

$$\text{Re}(\Gamma_1 + \nu)\hat{\nu}(R) = 0$$  \hspace{1cm} (5.189)

$$\left(\nu + \Gamma_1 \text{Re}\right)\left\{\frac{d^2\hat{\psi}(R)}{dR^2} + \frac{1}{R} \frac{d\hat{\psi}(R)}{dR}\right\} + \left(\nu^2 + \frac{1}{R^2}\right)\hat{\psi}(R) = 0$$  \hspace{1cm} (5.190)

It is obvious that solutions of equation 5.189 generate the results of equation 5.185. It is easily verified by the Forbenius method that equation 5.190 has the following general solution:
\[ \hat{\psi}(R) = p_o [\cos(lnR) + i \sin(lnR)] \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (\nu R)^{2n}}{2^{2n+1} n! (n+1)!} \right] + q_o [\cos(lnR) - i \sin(lnR)] \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (\nu R)^{2n}}{2^{2n+1} n! (n+1)!} \right] \] 5.191

where \( p_o \) and \( q_o \) are arbitrary, complex constants. Both independent solution functions are singular at \( R = 0 \). A necessary condition for satisfying equation 5.190 is obviously:

\[ \lim_{|R_I| \to \infty} (R_I \cdot \text{Re}) = 0 \]

and the results from 5.185 are recovered. Furthermore, the sufficiency of above condition is easily verified by L'Hospital's rule.

**Assertion 4.** Under the framework of assumptions of this work, wave-propagation speeds, \( \Gamma_I \), of sufficiently large magnitudes exist both for the draw-down and the metering sections, that the associated Reynold's numbers approach zero.

The result implies that any stretching of the extrudate in the draw-down section and any axial motion in the metering
section must lead to instabilities. The physical feasibility of an unbounded wave-propagation speed thus becomes a question of much importance. It will be discussed in a subsequent chapter.

5.9.3. **Draw-down section, \( \text{Re}' \to \pm \infty \)**

For very large absolute values of \( \text{Re}' \), the differential equations become:

\[
\hat{\nu}(R) = 0 \quad 5.192
\]

\[
\frac{d^2 \hat{\psi}(R)}{dR^2} + \frac{1}{R} \frac{d \hat{\psi}(R)}{dR} - (\Delta^2 - \frac{1}{R^2}) \hat{\psi}(R) = 0 \quad 5.193
\]

Since only regular solutions are of interest,

\[
\hat{\psi}(R) = q_0 J_1(\nu R \sqrt{\Delta}), \text{ where } q_0 \text{ is a complex constant} \quad 5.194
\]

Equation 5.192 can not be satisfied for large Reynold's numbers.

5.9.4. **Metering section, \( \text{Re} \to \pm \infty \)**

The differential equations become,

\[
\hat{\nu}(R) = 0 \quad 5.195
\]
\[
\frac{d^2 \hat{\psi}(R)}{dR^2} + \frac{1}{R} \frac{d\hat{\psi}(R)}{dR} + \frac{1}{R^2} \hat{\psi}(R) = 0
\]  

The last equation is Euler's equation whose general solution is:

\[\hat{\psi}(R) = p_o \cos(lnR) + q_o \sin(lnR),\]

\[p_o \text{ and } q_o \text{ are complex constant.}\]

Both solution functions of last equation are singular. Thus, no acceptable solutions exist for large \(Re\).

5.9.5. Limiting behavior as \(\Lambda \to 0\)

Considerable numerical difficulties were encountered when attempting to solve equation 5.163 for very small values of \(\Lambda\). This parameter appears in high powers in this equation and inaccuracies when adding or subtracting small numbers became a problem.

A small \(\Lambda\) depicts a situation in which the length of the molten extrudate is large compared to the diameter of the die. It represents a relevant industrial case. It was possible to derive an approximate equation for the limiting Reynolds number corresponding to neutral stability under the following assumptions:

i) \(\kappa \ll 1\), the fluid is only slightly elastic. The
two relaxation spectra for the fluid contain only one component or time constant each.

ii) The following truncations are valid, $b_6 R^6 = \mathcal{O}(0)$, $b_4 b_5 R^8 = \mathcal{O}(0)$. (See equations 5.166 and 5.167.)

It is obvious that as $\Lambda$ becomes very small, the diameter of the extrudate must necessarily tend to zero if any non-zero extension rate is maintained. The analysis is a lengthy, somewhat detailed account of it is given in Appendix 3.

The second assumption (ii) proved quite valid, inducing errors in Reynold's number of a percent or less in the parameter ranges of interest. The first assumption (i) is simply a restriction on the fluid for which the results are valid.

A Reynold's number different from the one naturally occurring in the differential equations was adopted:

$$\text{Re} = \text{Re}' \left( \frac{v_L}{v_0} - 1 \right) / \Lambda$$  

This Reynold's number is more instructive as will be discussed in the next chapter.

The result of the analysis is:
\[ \text{Re} = \frac{100\kappa \left[ \frac{1 + \frac{E\kappa}{2}}{1 + (\kappa \Gamma_i)^2} \right]^2 \cdot 0.2 - 11.2\kappa(1+E/2)}{1 + 2\kappa(1+E/2)} \left[ \bar{\nu}_o + \mathcal{Z} \right] \lambda[9.05 - 6.09 \frac{1 + \kappa E}{1 + (\kappa \Gamma_i)^2}] \]
CHAPTER 6

PROPERTIES OF THE SOLUTION FUNCTIONS

Elementary analysis reveal the following properties of the solution functions $J_1$, $S_1$, $S_2$, $S_3$, and $S_4$:

i) The solution functions converge absolutely for any finite argument.

ii) The solution functions do not converge uniformly for any argument.

iii) The solution functions are infinitely differentiable with respect to $R$.

iv) The real part as well as the imaginary part of the solution functions are alternating functions.

v) The extremal points for the real part of any given solution function do not occur, in general, at the same argument as the extremal point for the imaginary part.

The solution functions satisfying the boundary conditions of Chapter 5.8 have the character of eigenfunctions. One expects, in view of (iv), that a multitude of such eigenfunctions exists satisfying the boundary condition in

127
question. It is of interest to investigate the orthogonality properties of these eigenfunctions. One assumes
again that the relaxation spectra for the fluid contain only one exponential function each.

Consider equation 5.115 of the draw-down section, which may be written in the following form:

\[
\frac{d}{dR} \left( \frac{1}{R} \frac{d}{dR} (R\hat{v}(R)) \right) = F\hat{v}(R)
\]

Assume that at least two eigenfunctions exist,

\[
\frac{d}{dR} \left( \frac{1}{R} \frac{d}{dR} (R\hat{v}_n(R)) \right) = F_n \hat{v}_n(R)
\]

\[
\frac{d}{dR} \left( \frac{1}{R} \frac{d}{dR} (R\hat{v}_m(R)) \right) = F_m \hat{v}_m(R)
\]

Multiplying equation 6.1 by \( R\hat{v}_m(R) \) and 6.2 by \( R\hat{v}_n(R) \), subtracting the resulting equations and integrating between limits,

\[
\int_{o}^{R} \frac{d}{dR} \left( \frac{1}{R} \frac{d}{dR} (R\hat{v}_m(R)) \right) dR - \int_{o}^{R} \frac{d}{dR} \left( \frac{1}{R} \frac{d}{dR} (R\hat{v}_n(R)) \right) dR
\]

\[
= (F_n - F_m) \int_{o}^{R} \hat{v}_m(R) \hat{v}_n(R) RdR
\]

By integrating the left hand side of equation 6.4 by parts:
\[
\hat{v}_m(R) \frac{d\hat{v}_n(R)}{dR = R} - \hat{v}_n(R) \frac{d\hat{v}_m(R)}{dR = R} = \frac{F_n - F_m}{R} \int_{0}^{R} \hat{v}_m(R) \hat{v}_n(R) RdR
\]

6.5

The well known theorem from the theory of linear, ordinary differential equations stating that the only permissible solutions at the singular point are the zero solutions, has been used. The quantity F may be considered an eigenvalue. The left hand side of the last equation vanishes if either

\[
\hat{v}(R) = 0, \quad \frac{d\hat{v}(R)}{dR = R} = 0 \quad \text{or} \quad p\hat{v}(R) + q \frac{d\hat{v}(R)}{dR = R} = 0
\]

where q and q is constants not containing the eigenvalue. The eigenfunctions are in such cases orthogonal with respect to R. By consulting equation 5.160 it is seen that none of the above orthogonality conditions are satisfied.

Considering next equation 5.163 of the draw-down section:

\[
\frac{d^4\hat{\psi}(R)}{dR^4} + \frac{2}{R} \frac{d^3\hat{\psi}(R)}{dR^3} + \eta_1 \frac{d^2\hat{\psi}(R)}{dR^2} + \eta_2 \frac{d\hat{\psi}(R)}{dR} + \eta_3 \hat{\psi}(R) = 0
\]

6.6
where
\[ \eta_1 = B - C - \frac{2}{R^2}, \quad \eta_2 = \frac{B+C + \frac{4}{R^2}}{R}, \]
\[ \eta_3 = E - \frac{B}{R^2} - \frac{4}{R^4} \]

One assumes again the existence of at least two eigenfunctions:

\[
\frac{d^4 \hat{\psi}_n(R)}{dR^4} + \frac{2}{R} \frac{d^3 \hat{\psi}_n(R)}{dR^3} + \eta_{1n} \frac{d^2 \hat{\psi}_n(R)}{dR^2} + \eta_{2n} \frac{d\hat{\psi}_n(R)}{dR} + \eta_{3n} \hat{\psi}_n(R) = 0 \tag{6.7}
\]

\[
\frac{d^4 \hat{\psi}_m(R)}{dR^4} + \frac{2}{R} \frac{d^3 \hat{\psi}_m(R)}{dR^3} + \eta_{1m} \frac{d^2 \hat{\psi}_m(R)}{dR^2} + \eta_{2m} \frac{d\hat{\psi}_m(R)}{dR} + \eta_{3m} \hat{\psi}_m(R) = 0 \tag{6.8}
\]

Equations 6.7 and 6.8 are multiplied by \( \hat{\psi}_m(R)R^3 \) and \( \hat{\psi}_n(R)R^3 \), respectively, and then subtracted. The resulting equation is integrated between limits. After a series of integrations by parts, the following result is obtained:
\[ \mathbb{R}^3 [ \hat{\psi}_n(R) \frac{d^3 \hat{\psi}_m(R)}{dR^3} - \hat{\psi}_m(R) \frac{d^3 \hat{\psi}_n(R)}{dR^3} + \frac{d^2 \hat{\psi}_n(R)}{dR^2} \frac{d\hat{\psi}_m(R)}{dR} ]_{R=\bar{R}} - \mathbb{R}^3 [(B_m - C_m) \hat{\psi}_n(R) \frac{d\hat{\psi}_m(R)}{dR}]_{R=\bar{R}} \]

\[ + 4 \int_0^{\bar{R}} \mathbb{R} R^2 [ \frac{d^2 \hat{\psi}_m(R)}{dR^2} \frac{d\hat{\psi}_n(R)}{dR} - \frac{d^2 \hat{\psi}_n(R)}{dR^2} \frac{d\hat{\psi}_m(R)}{dR} ] dR \]

\[ + 4 \int_0^{\bar{R}} [ \hat{\psi}_n(R) \frac{d\hat{\psi}_m(R)}{dR} - \hat{\psi}_m(R) \frac{d\hat{\psi}_n(R)}{dR} ] dR \]

\[ - [(B_m - C_m) - (B_n - C_n)] \int_0^{\bar{R}} \mathbb{R} R^3 \frac{d\hat{\psi}_n(R)}{dR} \frac{d\hat{\psi}_m(R)}{dR} dR \]

\[ - 2 (B_m - C_m) \int_0^{\bar{R}} R^2 \hat{\psi}_n(R) \frac{d\hat{\psi}_m(R)}{dR} dR \]

\[ + 2 (B_n - C_n) \int_0^{\bar{R}} R^2 \hat{\psi}_m(R) \frac{d\hat{\psi}_n(R)}{dR} dR \]

\[ = (E_n - E_m) \int_0^{\bar{R}} R \hat{\psi}_n(R) \hat{\psi}_m(R) dR \]

\[ - (B_n - B_m) \int_0^{\bar{R}} R^3 \hat{\psi}_n(R) \hat{\psi}_m(R) dR \]

6.9
The right hand side of this equation indicates the existence of two independent solution functions. If the left hand side of the equation vanishes, two sets of orthogonal eigenfunctions exist, one set orthogonal with respect to \( R \) and the other orthogonal with respect to \( R^3 \). The boundary conditions of Chapter 5.8.1 do not result in a vanishing left hand side.

Equations 5.118 and 5.119 of the metering section were subjected to similar analysis. Results identical to the above were found, that is, the eigenfunctions are not orthogonal. The lengthy but in principle straightforward analysis has been omitted in the report.

The above results introduce a certain non-uniqueness into the system. Suppose the parameter combination \( p_1^1 \ldots p_N^1 \) solves any given eigenvalue problem. When the eigenfunctions are not orthogonal, the parameters \( p_1^1 \ldots p_{M}^1, p_{M+1}^2 \ldots p_N^2 \) may also solve the same eigenvalue problem.
CHAPTER 7

NUMERICAL ANALYSIS

The Reynold's number occurring naturally in the differential equations for the draw-down section is not a descriptive one. One feels that this quantity should reflect the throughput or production capabilities of the extrusion system. The following is preferred:

\[ \text{Re} = \frac{\rho v_0 r_0}{K_p}, \text{ hence } \text{Re} = \frac{\text{Re}'(v_L/v_o - 1)}{\lambda} \]

The Reynolds numbers for the two flow sections are now essentially identical and directly comparable.

One has throughout the numerical analysis assumed that the two relaxation spectra for the fluid contain only one component or time constant each, \( \lambda_{2p} = \lambda_2, \lambda_{1p} = \lambda_1 \).

One is interested in obtaining the relationships between Reynold's number and the other parameters at neutral stability conditions, symbolically expressed as

\[ \text{Re} = \phi_1(p_1, \ldots, p_N) |_{p_R=0} \]

where \( p_1, \ldots, p_N \) are the basic problem parameters save.
Reynold's number. All the eigenfunctions solving any of the eigenvalue-problems of Chapter 5 should ideally be investigated in order to obtain the lowest or critical value. Since, most probably, an infinite number of eigenfunctions exist, this becomes, needless to say, a very ambitious task.

The relationships 7.1 may describe curves of a variety of shapes, such as closed or multivalued curves, to quote a couple of the more unusual results obtained in hydrodynamical stability analysis.

It is, furthermore, necessary to determine the domains of stability and instability, that is, to obtain the following relationships,

\[ \text{Re} = \phi_2(p_1, \ldots, p_N) \Gamma_R > 0 \]

\[ \text{Re} = \phi_3(p_1, \ldots, p_N) \Gamma_R < 0 \]  

7.2

If equation 7.1 describes unique relationships, the following results are normally found:

\[ \text{Re}_{\Gamma_R > 0} < \text{Re}_{\Gamma_R = 0} < \text{Re}_{\Gamma_R > 0} \]  

7.3

Quite often \( \text{Re}_{\Gamma_R = 0} \) is obtained by interpolation between \( \text{Re}_{\Gamma_R > 0} \) and \( \text{Re}_{\Gamma_R < 0} \).

Every eigenvalue problem of Chapter 5.8 represents
in reality two equations since both the real and the imaginary parts must vanish for any given eigenfunction. It is, consequently, not possible to vary only one on the parameters \( p_1 \) through \( p_N \) in equation 7.1 while keeping the other \( N-1 \) parameters constant. One of the \( N-1 \) parameters must be permitted to take on any value needed in order to solve the eigenvalue problem in question. This particular parameter can not suffer physical restrictions. The parameter vectors for the two flow sections contains two components with the described properties, \( \Gamma \) and \( \nu \). The parameter \( \Gamma \) is somewhat special since the Reynolds number appears in the combination \( \Gamma \Re \). Thus, the wave-number is chosen as the parameter over which one may exercise no physical control.

Each flow section is described by two independent eigenvalue problems. It is easily verified that the equations may be formulated as

\[
\sum_{n=0}^{\infty} a_n \Re^n = 0 + i0 \quad 7.4
\]

\[
\sum_{n=0}^{\infty} b_n \Re^n = 0 + i0, \quad a_n \text{ and } b_n \text{ are complex} \quad 7.5
\]

The coefficients \( a_n \) and \( b_n \) are complicated algebraic and integral functions of all problem parameters save Reynolds's number. Their recursion formulae need not be known for
the subsequent argumentation. Each of the above equations produces two coupled equations when written in terms of real quantities. The problem thus becomes one of finding parameter combinations which lead to coinciding roots of two polynomials of infinite degree.

The "fundamental theorem of algebra" guarantees the existence of at least one root of any \( n \)th degree polynomial with real coefficients. There is, however, no reason to believe that this root will solve any other polynomial of \( n \)th degree. Thus, existence of solutions can not be proved.

If one insists on a three dimensional perturbation field, then, by physical arguments, parameter combinations which solve equations 7.4 and 7.5 simultaneously must be found. Search over a four-dimensional surface would be necessary in order to find such parameter combinations, if they exist at all. Considering the complicated equations of Chapter 5.8, it takes little speculation to conclude that such procedures would require enormous computer efforts. Fortunately, this case is of little or no interest when regarding the stability of the systems. This will now be discussed. Assume the parameter vectors \( A \) and \( B \) both solve equations 7.4 and 7.5, respectively.

\[
A = [p_1^1, p_2^1, \ldots, p_N^1, \text{Re}_1]^T
\]

7.6
\[ B = [p_1^2, p_2^2, \ldots, p_N^2, Re_2]^T \]

A three dimensional disturbance field may only exist if

\[ A = B \]

Obviously, all possible solutions of equation 7.8 may be obtained from the totalities of parameter combinations \( A \) and \( B \). Needless to say, not all vectors \( A \) and \( B \) qualify as solutions of equation 7.8.

The physical implication of \( A \neq B \) is that either \( \hat{v}(R) \) or \( \hat{\psi}(R) \) must vanish identically, equivalently, either \( v'\theta(t, x) = 0 \) or \( v'^R(t, x) = v'^Z(t, x) = 0 \). One naturally assumes existence of the velocity component or components most critical to stability.

**Assertion 5.** For both flow sections considered and within the framework of assumptions of this work, the following disturbance fields may exist:

i) Perturbed velocity field one-dimensional,
\[ v'\theta(t, x) \neq 0, v'^R(t, x) = v'^Z(t, x) = 0. \]

ii) Perturbed velocity field two-dimensional,
\[ v'\theta(t, x) = 0, v'^R(t, x) \text{ and } v'^Z(t, x) \neq 0. \]

iii) Perturbed velocity field three-dimensional,
\[ v'\theta(t, x), v'^R(t, x) \text{ and } v'^Z(t, x) \neq 0. \]
The three-dimensional perturbations can never be less stable than one or two-dimensional perturbations.

In view of the above assertion, it is sufficient to solve equations 7.4 and 7.5 individually. This may, in general, be accomplished by an iterative search on the parameters Re and ν.

An attempt was made to solve the eigenvalue problems of Chapter 5.8 without additional assumptions. The problems proved extremely ill-behaved for numerical treatment, in particular for large Reynold's numbers. Although the formulation of equations 7.4 and 7.5 were not used in this case, it may conveniently be adopted for the sake of argumentation.

Any given eigenvalue problem of Chapter 5.8 may be expressed as

\[ \sum_{n=0}^{\infty} a_n \text{Re}^n = \epsilon_1 + i\epsilon_2 \]  

7.9

A solution is said to be found if |ε₁|, |ε₂| < specified error limits. The following facts came to light:

i) The error sensitivity was very large, in particular for large Reynold's numbers. That is
$$\frac{\partial \varepsilon_1}{\partial v}, \frac{\partial \varepsilon_2}{\partial v}, \frac{\partial \varepsilon_1}{\partial \text{Re}}, \frac{\partial \varepsilon_2}{\partial \text{Re}} \varepsilon_1, \varepsilon_2 \gg 1$$

ii) Several solutions may be clustered very closely together.

iii) Large computer efforts were required to obtain solutions.

Typical results are depicted in Figure 4.

![Graph of $\varepsilon_1, \varepsilon_2$ against $v, \text{Re}$]

Fig.4

Aside from finding the solutions themselves, it became a problem to ensure that the correct solution was traced as any of the remaining parameters were altered.

It became apparent that only insufficient amounts of results could be compiled this way within the framework of reasonable computer efforts. Since such a large number of parameters is involved, substantial quantities of results are needed in order to uncover the behavior of the two flow systems.

It was found possible to reduce the numerical work
dramatically provided one is willing to accept slight errors in the final results. For realistic ranges of parameters a truncation of equations 7.4 and 7.5 was found justified.

\[ a_0 + a_1 \text{Re} + A_2 \text{Re}^2 + a_3 \text{Re}^3 = 0 + i \cdot 0, \]

\[ |a_4| \ll |a_3| \quad 7.10 \]

The right hand side of the last equation indicates that both the real and the imaginary part of equation 7.10 must vanish. When written in real quantities and slightly rearranged to make a point:

\[ a_0 + a_1 \text{Re} + a_2 \text{Re}^2 + a_3 \text{Re}^3 = 0 \quad 7.11 \]

\[ \sum_{n=0}^{N} \bar{a}_n \nu^n = 0, \quad N > 3 \quad 7.12 \]

where \( \alpha_p = \alpha_p(\kappa, \Gamma, \Lambda, E, v_L/v_o, Z, \nu) \) and \( \bar{\alpha}_p = \bar{\alpha}_p(\kappa, \Gamma, \Lambda, E, v_L/v_o, Z, \text{Re}) \) for the draw-down section, and

\[ \alpha_p = \alpha_p(\kappa, \Gamma, \tau, Z, E, \nu) \] \[ \bar{\alpha}_p = \bar{\alpha}_p(\kappa, \Gamma, \tau, Z, E, \text{Re}) \]

for the metering section.

The incentive is immediately obvious. The Reynold's number may now be found analytically since polynomials of degree up to and including three possess analytic solutions.
The degree of the polynomial in \( \nu \) is much larger than three. Consequently, the wave-number must be determined by iterative techniques.

The assumption leading to equation 7.10 corresponds to truncations of the solution functions \( J_1, S_1, S_2, S_3 \) and \( S_4 \) and their various derivatives. The validity of these truncations was always checked carefully throughout the numerical analysis.

Conversion of the eigenvalue problems of Chapter 5.8 into the form of equation 7.10 was lengthy and tedious but in principle straightforward.

A combination of the False Position and Newton's methods proved efficient as iterative formalisms. The numerical integration was accomplished by means of a 12 point Gaussian quadrature routine with interval division determined by the estimated periodicity of the integrand.

Only a reduced, but what is felt sufficient, number of spot checks was performed in order to determine the domains of stability and instability. The results were all found in compliance with equation 7.3.

All programs were written in double precision and executed on an IBM 370/155 computer, whose normal word-length was 32 bits.

The resulting Reynold's numbers suffer from the following estimated errors:
i) Draw-down section, rotational velocity component, less than 1 percent.

ii) Draw-down section, \( \psi(R) \)-function, as high as 3 percent but usually less than 1 percent.

iii) Metering section. Rotational velocity component, "fixed boundary." As high as 7 percent but usually less than 3 percent.

iv) Metering section. Rotational velocity component, "free boundary." As high as 5 percent but usually less than 2 percent.

v) Metering section, \( \hat{\psi}(R) \)-function. As high as 5 percent but usually less than 3 percent.

The errors were found in the following way. The values for \( Re \) and \( \nu \) found by solving equations 7.11 and 7.12 are slightly in error due to the truncation, and more accurate quantities would be found if one considers the following equations:

\[
\alpha_0 + \alpha_1 Re + \alpha_2 Re^2 + \alpha_3 Re^3 = -\alpha_4 Re^4 \quad 7.13
\]

\[
\sum_{n=0}^{N} \frac{-\alpha_n \nu^n}{\alpha_{N+1} \nu^{N+1}} = -\frac{\alpha_{N+1} \nu^{N+1}}{\alpha_{N+1} \nu^{N+1}} \quad 7.14
\]

Writing \( Re = \hat{Re} + \Delta Re, \nu = \hat{\nu} + \Delta \nu, \alpha_p = \hat{\alpha}_p + \Delta \alpha_p, \bar{\alpha}_p = \hat{\bar{\alpha}}_p + \Delta \bar{\alpha}_p \), where \( \hat{Re}, \hat{\nu}, \hat{\alpha}_p \) and \( \hat{\bar{\alpha}}_p \) are solutions of equations 7.11 and 7.12 and \( \Delta Re/\hat{Re}, \Delta \nu/\hat{\nu}, \Delta \alpha_p/\hat{\alpha}_p, \Delta \bar{\alpha}_p/\hat{\bar{\alpha}}_p \ll 1 \), it is
easily shown that equation 7.13 becomes

\[ \hat{a}_o + \hat{a}_1 \hat{R} + \hat{a}_2 \hat{R}^2 + \hat{a}_3 \hat{R}^3 + \Delta \hat{R} \sum_{n=1}^{4} n \hat{\alpha}_n \hat{R}^{(n-1)} \]

\[ + \sum_{n=1}^{4} \Delta \alpha_n \hat{R}^n = -\hat{a}_4 \hat{R}^4 \] 7.15

Higher order error terms have been ignored. But by equation 7.11,

\[ \hat{a}_o + \hat{a}_1 \hat{R} + \hat{a}_2 \hat{R}^2 + \hat{a}_3 \hat{R}^3 = 0 \]

Equation 7.15 then reduces to

\[ \Delta \hat{R} = -\hat{a}_4 \hat{R}^4 / \left( \sum_{n=1}^{4} n \hat{\alpha}_n \hat{R}^{(n-1)} + \sum_{n=1}^{4} \Delta \alpha_n \hat{R}^n \right) \] 7.16

It is clear that the last sum in the denominator is of much smaller absolute value than the first, and may safely be ignored. The final result is then:

\[ \Delta \hat{R} = -\hat{a}_4 \hat{R}^4 / \sum_{n=1}^{4} n \hat{\alpha}_n \hat{R}^{(n-1)} \] 7.17
CHAPTER 8

RESULTS AND DISCUSSION

This chapter will be limited primarily to a presentation and discussion of the numerical results. The analytical results obtained earlier in the report, most of which are expressed in the form of assertions, shall not be restated in this chapter.

One remembers from Chapters 5.4.1 and 5.4.2 that unbounded stresses were attained for both flow sections in a perturbed state when \( \lambda_{2p} \dot{\gamma} \geq 1/3 \). This result contrasts the steady state case in which stress unboundedness occurred when \( \lambda_{2p} \dot{\gamma} \geq 1/2 \). No physical system remains immune to infinitesimal disturbances. It is, thus, realistic to expect a pathological stress behavior when \( \lambda_{2p} \dot{\gamma} \) approaches 1/3 from below.

The "principle of Exchange of Stabilities" (see Chapter 5.1) does not apply for either flow section. This principle requires \( \Gamma = \Gamma_R + i\Gamma_I = 0 + i \cdot 0 \) when the systems are in the state of neutral stability. There is no reason to believe that solutions of the various eigenvalue problems do not exist in this case. This situation implies
however, that the Reynold's number may become undefined, at least as long as it is bounded, since this quantity appears in the combination $\Gamma \Re$. This case is therefore of no interest when regarding the stability of the systems. Since $\Gamma_I \neq 0$, the disturbances at neutral stability must be purely oscillatory.

"Squire's Theorem" has been disproved for the flows considered in this work. This is indicated by the numerical results. This theorem applies only in parallel flows. It has been shown that its validity is closely associated with the self adjointness of the differential operators. None of the differential operators in this work is self adjoint.

It was shown in Chapter 5.9 that

$$\lim(\Re) \to 0 \text{ when } \text{abs}(\Gamma_I) \to \infty, \Gamma_R = 0$$

8.1

The numerical results seem to indicate that this limit is sometimes approached smoothly. It is remembered from Chapter 5 that the mathematical models for the two flow sections include only incomplete inertial effects, of the form $v_0 \frac{\partial y'(t,x)}{\partial z}$. The linear theory does not permit the existence of general inertial effects, that is, inertial effects associated with the entire three-dimensional perturbation velocity field, on account of compatibility with respect to the equations of motion. It is possible but not likely that the incomplete inertia effects accounts for the
behavior of the Reynold's number at high wave propagation speeds. One certainly associates the inertial effects, in any context, with the ability to attenuate more efficiently the higher frequency of the disturbances. Whether such effects would ultimately enhance the stability of the system or not, is a question which can not be answered a priori.

In the case of neutral stability, equation 5.16 may be written as follows after some minor rearrangements,

\[ y'(t, x) = \sum_{n=1}^{\infty} \exp(\pm in\Gamma I t) \hat{W}(x) \] \hspace{2cm} 8.2

An infinite number of terms in the above expansion is only needed for functions which exhibit discontinuities in the time domain. Such discontinuities are only acceptable as initial values. Besides, it is necessary that the Dirichlet conditions are satisfied for expansion 8.2 to be possible at all. It was shown in Chapters 5.9.1 and 5.9.2 that \( \hat{W}(x) \) exists for \( n\Gamma I \) tends to \( \pm \infty \). In fact, \( \hat{V}(R) \) could be an arbitrary function. The Dirichlet conditions are clearly violated when \( n\Gamma I \) tends to \( \pm \infty \). An expansion which does satisfy these conditions and which, furthermore, represents continuous functions is:

\[ y'(t, x) = \sum_{n=1}^{N} \exp(\pm in\Gamma I t) \hat{W}(x) \] \hspace{2cm} 8.3

where \( N \) may be a large number but never infinite. The case
of infinite wave propagation speeds is then dismissed for mathematical as well as physical reasons.

The models have been developed under a number of assumptions, most of which are in good accord with reality. Some attention should be focused on the assumption stated in equation 5.66, which, in the author's opinion, deserves most criticism. It is immediately obvious that the assumption is inherent in the linear theory, that is, a linear system of differential equations will not be obtained unless this assumption is made. But in spite of its mathematical necessity, it may not be physically justified. The assumption states that for a velocity perturbation field of infinitesimal magnitude, it is sufficient to consider the history of this field only in a very small neighborhood around its steady state history. The assumption is certainly valid for fluids which represent first order approximations to a Newtonian fluid (second order fluid). A rigorous formulation of its applicability to viscoelastic fluids has not been possible, and ought to constitute an interesting problem in future research. It is thought that the exponential memory function in Bird-Carreau's model fades sufficiently fast that assumption 5.66 is reasonably valid.

It is emphasized that the numerical results by no means meant to be exhaustive. The solution functions have been truncated and emphasis thus placed on low Reynold's
numbers. Although low Reynold's numbers are expected on physical grounds, one should keep in mind that other solutions yielding Reynold's numbers of larger magnitudes may exist. For this reason and furthermore in accord with the linear theory as such, the results obtained must be regarded as sufficient conditions for instability.

8.1. **Draw-Down Section**

It is encouraging to note that equation 5.133 is precisely the equation Spearot and Metzner (83) found when correlating their experimental data. This equation is furthermore in very good agreement with the experimental results of Cruz-Saenz et al. (26). The question of the physical existence of purely elongational flow fields should then be laid to rest for good.

The marginal rather than the critical Reynold's numbers were found for this section. The marginal Reynold's number is one which solves the considered eigenvalue problem for specified values of the remaining parameters, $\kappa_1$, $(v_L/v_0)_1$, $Z_1$, $E_1$, $\Lambda_1$, $\Gamma_{\text{II}}$, say. The lowest marginal Reynold's number obtained by letting $\Gamma_1$ assume all possible values, $\Gamma_1 \in [-\infty, +\infty]$, is defined as the critical Reynold's number. The wave propagation speed is, of course, a parameter which can take on any value except infinity. Besides
being very time consuming, this procedure was judged pointless since, as indicated by the numerical results, the marginal Reynold's numbers became consistently smaller as the wave propagation speeds approached large absolute values.

For the rotational velocity component, only the base or first mode, that is, the wavenumber of lowest magnitude providing solution of the eigenvalue problem, produced good convergence of the associated eigenfunction and its derivatives. The relative error in the Reynold's number proved essentially independent of the remaining elements of the parameter vector.

Several modes provided good convergence for the eigenfunctions $S_1$ and $S_2$ which constitute the $\hat{\phi}(R)$ function. The relative error in the result varied somewhat with the numerical values of the remaining parameters. This eigenvalue problem proved extremely ill-behaved and time consuming to solve. This accounts for the relatively few results compiled.

Most marginal Reynold's numbers found are of sufficiently small magnitudes that, according to Shah and Pearson (80), inertial effects in all probability should be negligible, at least for reasonable wave numbers and wave propagation speeds.

The number varying parametrically along the curves
are the marginal wave numbers. Spot checks showed, as ex-
expected, that $\text{Re} > \text{Re}_{\text{marg}}$ defined the unstable region.

**Effects of Fluid Elasticity**

The elastic properties of the fluid is described by
the group $\lambda_2 p \dot{\gamma} = \kappa$. This quantity exhibits the character
of a Deborah number, which is the ratio of a characteris-
tic fluid time (relaxation time) and a characteristic
process time as a material particle moves along a pathline.

The results are given in Figures D1, D2 and D3 for
the rotational velocity component and D20 and D21 for the
$\hat{\psi}(R)$ function. All results showed the same basic trends in
the sense that the curves obtained exhibit a maximum at
some intermittent value within the mathematically permis-
sible range of the Deborah number. This implies that
neither the slightly elastic nor the highly elastic fluids
will favor a stable extrusion. One notes that negative
Reynold's numbers were found in some cases, both for very
small and very high Deborah numbers, see D1 and D3. Neg-
ative Reynold's numbers correspond, of course, to an unstable
system. It is possible that the high Deborah numbers are
not physically realistic. The location of the point of the
highest marginal Reynold's number depends on the param-
ters. The results indicate, for instance, that this maxi-
mum occurs at lower Deborah numbers the higher the magnitude
of the wave propagation speeds are. One finds the maxima at Deborah numbers of 0.12 and 0.40 for wave propagation speeds of 35 and 10, respectively. Results for both the first and second marginal mode were obtained in the case of the \( \hat{\psi}(R) \) function. The second mode was somewhat less stable than the first. The marginal modes were, in general, remarkably well defined (insensitive to parameter variations).

Zeichner (99) and Fisher and Denn (32) found that elastic effects promote stability in the context of higher stable draw ratios. Shah and Pearson (82) found that for Power-Law fluids the critical draw ratio increases with the Power-Law constant. Cruz-Saenz et al. have argued that "elastic response would be manifested by an apparent extension-thickening behavior, the conclusion regarding such behavior and stability is the same as that for inelastic fluids." Although this statement is an oversimplification, it does explain the principle coherence between Shah and Pearson's and Fisher and Denn's results. The present results agree principally with those of the above-mentioned authors for small Deborah numbers, but show destabilizing effects of elasticity for highly elastic fluids.
Effects of Stretching

The parameter describing threadline stretching is $v_L/v_0$, the ratio between the terminal and the initial axial velocities. The results are given in Figures D4 through D8 for the rotational velocity component and in Figures D22, D23 and D24 for the $\hat{\psi}(R)$ function. All curves with the exception of D24 showed the same basic trends and indicate that instabilities can never develop for draw ratios close to unity. This result is expected on physical grounds. Both inertia and stresses will vanish as the draw ratio approaches one. Thus, there is no mechanism left which can cause instabilities to develop. Figure D24 shows a peculiar behavior which will be discussed as a special case later.

All curves (except D24) exhibit a minimum at some intermittent draw ratio. This implies the existence of a stable region at high draw ratios. Such results have been obtained by Fisher and Denn (32) for a generalized Maxwell fluid. Experimental results of some polymer systems seem to support these results (70). The locus of the minimum point depends on the parameters, as is obvious from curves D4 through D8. One notes, in particular, that wave propagation speeds, $\Gamma_I$, of increasing magnitudes lead to a more unstable system besides shifting the curve minima toward unity draw ratio. These results suggest that for
sufficiently large $|\Gamma_\|'$s, instabilities may always develop. The eigenvalue problem associated with the rotational velocity component was relatively well-behaved, and permitted some further investigation into this matter. Figures D18 and D19 were compiled where $\Gamma_{IC}$ is the wave propagation speed of lowest magnitude which corresponds to a zero marginal Reynolds's number. Figure D18, which is of most interest, suggests the following limiting results:

$$\lim_{v_L/v_0 \to 1} (\Gamma_{IC}) \to \infty, \quad \lim_{v_L/v_o \to \infty} (\Gamma_{IC}) \to 0$$

One must remember that only draw ratios close to unity is permitted on account of compatibility. Figure D19 shows that $\Gamma_{IC}$ decreases somewhat as the fluid becomes more elastic, but no asymptotes are obvious in this case.

Figure D24 showed an unexpected behavior. These results are not physical for reasons mentioned above, and must be disregarded altogether. The equations of motions in the limit as $v_L/v_0 \to 1$ are

$$\frac{\partial v'(t,x)}{\partial t} = 0$$

with the obvious solution $v' = f(x)$ where $f()$ is any function. The Reynolds's number becomes undefined, in which case the possibility for branching exists.

The present results indicate the existence of
significantly lower critical draw ratios than predicted by Fisher and Denn (32) and Shah and Pearson (82). Recent experimental results, e.g., Cruz-Saenz et al. (26), seem to be more in accord with the results of this work than with those of the above mentioned authors.

**Positional Effects**

The parameter describing these effects is the axial space coordinate, Z. The appearance of this quantity as a relevant parameter in the eigenvalue problems means that any given fluid particle may be subjected to different stability conditions as it travels the distance from the die to the solidification zone. It implies, furthermore, that draw resonance may be a phenomenon of local rather than global nature. Many researchers apparently believe that draw resonance is global in character.

The results are shown in Figures D9, D10 and D11 for the rotational velocity component and D25 and D26 for the \( \hat{\psi}(R) \) function. The rotational velocity component produced monotone curves, indicating that the critical position is at the section entrance. The \( \hat{\psi}(R) \) function produced curves of more complicated patterns, indicating the existence of several threadline segments critical to stability. The general impression, however, is one of stability enhancement as the solidification zone is approached.
The complicated pattern of these curves is attributed to the existence of a perturbed free surface, which, on account of the rotational symmetry, effects the \( \psi(R) \) function only.

No theoretical or experimental results are available for comparison.

**Effects of the Weighing Factor E**

The parameter \( E \) is generally a function of the second invariant of the rate of deformation tensor, but has in this work been assumed constant for reasons given in Chapter 3. Although this quantity carries physical significance in steady shear flow in which case \( E/2 = \frac{(T<22> - T<33>)}{(T<11> - T<22>)} \), it must for general flows be interpreted merely as a weighing factor determining whether the emphasis is focused on the contravariant or the covariant strain.

The results are given in Figures D12 and D13 for the rotational velocity component and D27 and D28 for the \( \psi(R) \) function. The range over which \( E \) has been allowed to vary reflects a preference for the contravariant strain. This is in accord with results from the network theories.

The results uncover a complicated dependence on the parameters. For the rotational velocity component an increasing \( E \) enhanced the stability of modestly elastic but
hindered the stability of highly elastic fluids. Results for two modes were obtained in the case of the \( \psi(R) \) function. The two modes produced opposing trends but a very weak correlation.

No theoretical or experimental results are available for comparison.

**Effects of the Wave Propagation Speed**

The results for the rotational velocity component are given in Figures D14, D15 and D16, for the \( \psi(R) \) function in Figures D29 through D33.

The results suggest the existence of the following asymptotes:

\[
\lim_{R_i \to 0, |R_i| \to \infty} \text{Re} \to 0, \quad \lim_{R_i \to 0} |\text{Re}| \to \infty
\]

This is an expected result (see Chapter 5.9) and indicates that the limit derived in Chapter 5.9.1 is approached smoothly. The feasibility of an unbounded wave propagation speed has been discussed earlier.

For the rotational velocity component Reynold's number approaches the same asymptote whether \( R_i \) tends to zero from above or below. The curves are not monotone, indicating the existence of a bounded, critical wave propagation speed which always seemed to correspond to very small
but negative Reynolds's numbers. It was assured that the minute excursions into the domain of negative Reynolds's numbers were much larger than the possible errors. The behavior of the locus to the critical wave propagation speed has been discussed earlier in this chapter.

The results for the \( \hat{\psi}(R) \) function uncover a discontinuity at \( R^* = 0 \) (see D29, D30 and D31, D32), indicating that small negative wave propagation speeds are critical to stability. Figures D29 and D30 show how one mode may branch off from another. Again, the marginal modes are remarkably well-defined. The alternating curves of Figures D32 and D33 show the impact of the perturbed free surface.

**Effects of the Scaling Factor, \( \Lambda \)**

The results for the rotational velocity component are given in Figure D17 and for the \( \hat{\psi}(R) \) function in Figures D34 and D35. Figure D17 indicates that stability is enhanced as the length of the draw-down section increases. Although Figures D34 and D35 show that finite critical lengths exist, the overall impression is also here one of promoted stability as the length of the section increases.

The results were obtained for constant ratios, \( v_\ell/v_o \). It is then clear that the axial extension rate must
change as \( \Lambda \) is varied. The extension rate is determined by:

\[
v_L - v_o = \dot{\gamma}L \quad \text{or} \quad v_o(v_L/v_o - 1) = \dot{\gamma}L \quad 8.7
\]

The only parameter containing the inlet velocity \( v_o \) is the Reynold's number. Thus,

\[
\frac{\rho v_0 r_o}{K_p} (v_L/v_o - 1) = \text{Re}(v_L/v_o - 1)
\]

\[
= \dot{\gamma}L \frac{1+2(\lambda_1 p \dot{\gamma})^2}{\eta_p} \rho r_o \quad 8.8
\]

The quantities \( \rho, \eta_p \) and \( r_o \) are usually kept constant. Then:

\[
(1+2(\lambda_1 p \dot{\gamma})^2) = \text{KA}(v_L/v_o - 1)\text{Re}, \quad K = \rho r_o^2/\eta_p \quad 8.9
\]

If \((\lambda_1 p \dot{\gamma})^2 \ll 1\) and neutral stability is considered:

\[
\dot{\gamma} = \text{KA}(v_L/v_o - 1)\text{Re}_M \quad 8.10
\]

The extension rates pertaining to Figures D17 (rotational velocity component) have been calculated using equation 8.10.
Rotational Velocity Component

\[ E = 0.2 \]
\[ \kappa = 0.1 \]
\[ Z = 1.0 \]
\[ \nu_L/\nu_o = 2 \]

<table>
<thead>
<tr>
<th>$1/\Lambda$</th>
<th>$\Gamma = 35$</th>
<th>$\Gamma = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.30</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.75</td>
<td>0.37</td>
</tr>
<tr>
<td>30</td>
<td>0.92</td>
<td>0.42</td>
</tr>
<tr>
<td>40</td>
<td>1.01</td>
<td>0.48</td>
</tr>
<tr>
<td>50</td>
<td>1.10</td>
<td>0.52</td>
</tr>
<tr>
<td>60</td>
<td>1.15</td>
<td>0.56</td>
</tr>
</tbody>
</table>

An approximation expression for the eigenvalue problem associated with the $\hat{\Psi}(R)$ function was obtained for reasons given in Chapter 5.9.6. The result is given in equation 5.199. By examining this expression it is seen that it predicts precisely the same trends as have been obtained numerically.

The experimental results of Ishihara and Kase (39) seem to be in disagreement with the above predictions.
A Few Summarizing Remarks on the Draw-Down Section

The overall impression of the numerical results is one of poor stability. Within the maze of available experimental data one can find support for almost any set of theoretical results. When judged on the results of this work, it appears that the rotational velocity component may be the more critical one. Relatively high wavenumbers were found. Instabilities of such a nature should manifest themselves physically through the occurrence of short wavelength surface irregularities like a "matte" phenomenon. But the "matte" phenomenon is thought to be more typical for melt fracture instabilities than for draw-resonance instabilities. The draw-resonance instabilities are characterized by a relatively long wavelength. Within the context of the obtained numerical results, the present theory seems to fail in predicting the draw-resonance phenomenon.

8.2. Metering Section

The correlation between the marginal Reynolds's number and the wave propagation speed, $\Gamma_1$, in the case of the rotational velocity component when subjected to a "fixed boundary" turned out somewhat surprising (see Figure M23). The marginal Reynolds number did approach, as expected, very large positive values as $|\Gamma_1|$ became small. But an
intermittent, critical wave propagation speed existed at relatively low values of $|\Gamma_I|$, beyond which the truncation errors of equation 7.10 increased significantly. It appeared necessary to locate the critical wave propagation speed to ensure acceptably low truncation errors. Thus, a critical Reynold's number was found.

The situation is similar for the rotational velocity component when subjected to a "free boundary." The Reynold's number changed very slowly around its critical value, however, and it appeared sufficient to estimate $\Gamma_{IC}$ rather than to search for it iteratively. Although the Reynold's numbers found are the marginal Reynold's numbers, they are all close to their critical values.

For the $\hat{\psi}(R)$ function, it was found that the marginal Reynold's numbers approached zero from above as $|\Gamma_I|$ became very large. For wave propagation speeds of magnitudes below a threshold, and within the restriction of the truncation of equation 7.10, no solutions at all were found. Thus, a maximum marginal Reynold's number exists. It may be of some merit to find this.

The flow in this section is non-viscometric on account of the elongational component in the steady state flow field. Physical restrictions have been necessary to accommodate compatibility resulting in special forms of the equations of motion. It is true that a viscometric flow situation is generated when the extension rate is permitted
to vanish. The result is, however, generally different from what would be obtained if the problem was posed as viscometric at the outset since the field equations are still of the same form as in the non-viscometric case. All the earlier works pertaining to this section have dealt with viscometric flows \(16, 25, 42, 51, 52, 74, 87, 90\). In fact, most authors have studied flow between infinite, parallel plates, a situation of pronounced advantages. This flow is controllable, that is, the equations of motion are satisfied in steady state no matter what the first and second normal stress differences are. Consequently, the literature provides no results with which the results of this work may be directly compared.

Reynold's numbers of sufficient magnitudes that inertia may play a role were found. Spot checks showed, as expected, that the unstable domain is defined as \(\text{Re} > \text{Re}_M\).

**Effects of Fluid Elasticity**

The effects of fluid elasticity is described by \(\kappa = \lambda_2 p (v_0 / r_0)\), which is the classical Weissenberg number for shear flows. The results are given in Figures M1, M2 and M3 and in Figures M13 through M16 for the rotational velocity component subjected to "free" and "fixed" boundaries, respectively, and in Figures M24 and M25 for the
\( \hat{\psi}(R) \) function.

The results in Figures M1, M2 and M3 uncover a very complicated behavior pattern. Two regions of fluid elasticity favorable to stability are indicated. Newtonian or slightly elastic fluids, \( 0 \leq \kappa \leq 0.3 \), are the most stable, but also fluids with more pronounced elastic response, \( \kappa \leq 1.8 \), provide favorable stability conditions. Highly elastic fluids, \( \kappa > 2.3 \), are unstable in any motion. An intermittent, critical region of elasticity, \( 0.4 \leq \kappa \leq 1.2 \), should be avoided.

The results in Figures M13 and M15 indicate the existence of two regions of the Weissenberg number in favor of stability. Newtonian or slightly elastic fluids as well as highly elastic fluids are the most stable. The critical Weissenberg number obviously depends on the remaining elements of the parameter vector, \( E, \tau \) and \( Z \). Figure M13 indicates that the critical Weissenberg number shifts toward higher values as \( Z \) is increased. The dependence is slight. Figure M15 indicates that the critical Weissenberg number increases as the dimensionless extension rate, \( \tau = \dot{\gamma}r_0/v_0 \), decreases. The dependence is strong. Most of the critical Reynolds numbers are probably above what may be expected for typical, molten polymers. Figures M14 and M16 show the relation between the critical wavenumber and the critical wave propagation speed.

Figure M24 shows a trend which deviates from those
discussed above. These results indicate that system stability is always promoted as the fluid becomes increasingly elastic. It should be emphasized, however, that only the range from slightly elastic to modestly elastic fluids is covered by these results. In the above discussed cases, on the other hand, the whole range from slightly elastic to highly elastic fluids were investigated.

It is clear that the results for the $\hat{\psi}(R)$ function and the two cases of the rotational velocity component constitute conflicting results, one would suggest an optimal Weissenberg number of about 1.8.

Many of the earlier works pertaining to viscoelastic fluids in viscometric flows gave critical Reynolds's numbers orders of magnitudes larger than those of this work (16, 42, 74, 87). All these researchers report that fluid elasticity had a destabilizing effect. These results are hardly relevant to the melt fracture problem. Bonnett and McIntire (13) found a rheological instability in plane, isothermal Couette flow at a Weissenberg number of unity. The critical value of the Weissenberg number seems to be nearly independent of the Reynolds's number.

**Effects of Stretching**

The parameter describing the stretching effects is the dimensionless extension rate, $\tau = \dot{\gamma}(r_0/v_0)$. The
results are given in Figures M4, M5 and in M17, M18 for the rotational velocity component subjected to "free" and "fixed" boundaries, respectively, and in Figures M26 and M27 for the \( \hat{\psi}(R) \) function.

The results in Figures M4 and M5 indicate the existence of an intermittent, critical stretch range for sufficiently elastic fluids, \( 0.02 \leq \tau \leq 0.1 \) for \( \kappa = 0.5 \) (Figure M4). The results show, furthermore, that this critical range shrinks as the elastic property of the fluid decreases such that the effect of stretching on slightly elastic fluids is virtually one of stabilization. The locus of the point at which the stabilizing effects of stretch initiate is parameter dependent. It occurs for instance, at \( \tau = 0.02 \) for \( \kappa = 0.25 \) (Figure M5).

Figure M4 shows one of the many examples of the non-uniqueness in the results caused by the non-orthogonality of the eigenfunctions and discussed at the end of Chapter 6.

Figure M17 shows a monotone relationship between the critical Reynold's number and the extension rate. The stability is dramatically improved as the extension rate increases. The critical Reynold's numbers are sufficiently large beyond approximately \( \tau = 0.2 \) that, in the case of melt flow, instabilities should never develop. Small but finite Reynold's numbers were found for the smallest extension rates. The truncation error was at its maximum in
this region. Figure M18 shows the relationship between the critical wave numbers and the wave propagation speeds.

Figure M26 shows essentially the same trend. The dependence on the extension rate is sufficiently weak, however, that for reasonably small extension rates this case is virtually independent of the stretch.

No theoretical or experimental results are available for comparison.

**Positional Effects**

The parameter describing these effects is the axial space coordinate, \( Z = z/r_0 \). The results are given in Figures M6, M7, M8 and M19, M20 for the rotational velocity component subjected to "free" and "fixed" boundaries, respectively, and in Figures M28 and M29 for the \( \hat{\psi}(R) \) function.

The results of Figures M6, M7 and M8 indicate the existence of a channel section critical to stability. The length of the critical section is substantial for a modestly elastic fluid, \( 1 \leq Z \leq 20 \) for \( \kappa = 0.5 \) (Figure M6), but decreases and shifts toward the entrance as the fluid becomes less elastic, \( 0 \leq Z \leq 2 \) for \( \kappa = 0.25 \) (Figure M8). These results further suggest that the stability is slowly impaired toward the exit for slightly elastic fluids if the channel is sufficiently long.
Figure M19 shows a monotone relation between the critical Reynolds' number and the axial position. The stability improves as the fluid travels the distance from the channel entrance to the exit, the critical position is the entrance. This stability improvement is increased as the fluid becomes more elastic. The relation between critical wave number and wave propagation speed is given in Figure M20.

The trend is principally the same for the results of Figure M28. These results, however, show in addition that stability is virtually independent of axial position if the fluid is only slightly elastic.

When comparing the three cases discussed above, the \( \dot{\psi}(R) \) function and the two cases of the rotational velocity component, one would conclude that instabilities are most likely to initiate at or in the proximity of the channel entry. A similar result was obtained for the draw-down section.

**Effects of the Weighing Factor \( E \)**

This parameter carries physical significance in steady shear flow (see Chapter 3). Most researchers seem to agree that \( E \) is small and negative in such flows. The range over which this quantity was permitted to vary in this work is thought to be sufficient if not necessarily
realistic. The results are given in Figures M9, M10, and M21, M22 for the rotational velocity component subjected to "free" and "fixed" boundaries, respectively, and Figures M30 and M31 for the $\hat{\psi}(R)$ function.

All the results show identical trends, and a monotone relationship between the Reynold's number and $E$ indicated. The effect is one of consistent stabilization as $E$ varies from large positive to large negative values. The correlation is very strong for large negative and very weak for large positive values of $E$. The Reynold's number is essentially independent of $E$ for values above approximately unity.

The impact of the quantity on the stability of several viscometric flows of viscoelastic fluids have been studied. Most of the results show trends in complete agreement with this work.

**Effects of the Wave Propagation Speed**

The parameter describing these effects in the dimensionless quantity $\Gamma_I = \sigma_I V_0/r_0$. The results are given in Figures M11, M12 and in M23 for the rotational velocity component subjected to "free" and "fixed" boundaries, respectively, and in Figure M32 for the $\hat{\psi}(R)$ function. These results have been discussed to some extent earlier in this chapter.
The results show that a finite, critical wave propagation speed exists both for the "free" and the "fixed" boundary case of the rotational velocity component, and give some indication as to how the critical value varies when the parameters $E$, $\kappa$ and $\tau$ are changed (Figures M11, M12 and M23).

A monotone relationship was found for the $\hat{\psi}(R)$ function, suggesting a very large critical wave propagation speed (Figure M32). These results indicate that the limiting value derived in Chapter 5.9.2 is approached smoothly.

A Few Summarizing Remarks on the Metering Section

Most of the marginal and critical Reynold's numbers obtained are of sufficiently small magnitudes that the results are relevant in the context of melt flows. The rotational velocity component, "fixed" boundary case, yielded for most parameter combinations higher Reynold's numbers than the other two cases. In fact, they were sufficiently high that the case may be considered generally stable when regarding melt flows. No negative Reynold's numbers were found. The $\hat{\psi}(R)$ function gave very low Reynold's numbers, which may possibly approach zero as the wave propagation speed tends to very large absolute values. No negative Reynold's numbers were found. Thus, axial and radial disturbances may become unstable in melt flows. The rotational
velocity component, "free" boundary case, gave small
Reynold's numbers. Negative values were found. This case
appears to be most unstable. Thus, rotational disturbances
may become unstable in melt flows.

One notes that for the $\hat{\psi}(R)$ function very high
marginal wave propagation speeds, which are those most
likely to cause instabilities, result in very low marginal
wavenumbers. For both cases of the rotational velocity
component, on the other hand, relatively low wave propaga-
tion speeds and very high wavenumbers were found. This
fact gives rise to two instabilities of distinctly dif-
f erent nature. One instability characterized by an ex-
tremely rapid oscillation in time and a long wavelength in
space originating in the growth of either the radial or
the axial components of the perturbed field or in both.
This instability will show no rotational spiralling. The
other instability is characterized by a slow oscillation in
time and a very short wavelength in space originating in
the growth of the rotational component of the perturbed
field. This instability will give a spiralling appearance.
One should expect this latter instability to occur in physi-
cal systems since it was established above that the rota-
tional velocity component is the more critical to stability.
The frequently observed short wavelength "matte" on the
free surface of extrudates can be interpreted as a
confirmation of this. Spiralling instabilities have also often been observed experimentally.
$E = 0.2$
$Z = 1.0$
$\Lambda = 0.1$
$V_L/V_0 = 2.0$

**FIG. D1. DRAW-DOWN SECTION**

ROTATIONAL VELOCITY COMPONENT
FIG. D2. DRAW-DOWN SECTION
ROTATIONAL VELOCITY COMPONENT
$E = -0.2$
$Z = 1.0$
$\Lambda = 0.1$
$V_L/V_0 = 2.0$

FIG. D3. DRAW-DOWN SECTION
ROTATIONAL VELOCITY COMPONENT
\( E = 0.2 \)
\( \gamma = 1.0 \)
\( \delta = 0.1 \)
\( \Lambda = 0.1 \)

**Fig. D4. Draw-Down Section**

**Rotational Velocity Component**
FIG. D5. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT
FIG. D6. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT

\( E = 0.2 \)
\( Z = 1.0 \)
\( \Lambda = 0.4 \)
\( \Lambda = 0.1 \)
FIG. D7. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT
FIG. D8. DRAW-DOWN SECTION
ROTATIONAL VELOCITY COMPONENT
FIG. D9. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT

$E = 0.2$
$\mathcal{H} = 0.1$
$\Lambda = 0.1$
$V_L / V_0 = 2.0$
FIG. D10. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT
FIG. D11. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT

$E = -0.2$
$\delta = 0.1$
$\Lambda = 0.1$
$\nu_L/\nu_s = 2.0$
FIG. D12. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT

\( \Theta = 0.4 \)
\( Z = 1.0 \)
\( \Lambda = 0.1 \)
\( v_L/v_o = 2.0 \)
FIG. D13. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT
Fig. D14. Draw-down region

Rotational velocity component

$E = 0.2$
$Z = 1.0$
$\mu = 0.2$
$\Lambda = 0.1$
FIG.D15. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT
FIG. D16. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT
FIG. D17. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT

$E = 0.2$
$Z = 1.0$
$\beta = 0.1$
$V_{x}/V_{o} = 2.0$

$\Gamma_{i} = 35$

$\Gamma_{i} = 50$

$Re_{M}$

$1/\Lambda$
$E = 0.2$
$Z = 1.0$
$\lambda = 0.2$
$\Lambda = 0.1$
$Re = 0.0$

FIG. D18. DRAW-DOWN SECTION
ROTATIONAL VELOCITY COMPONENT
FIG. D19. DRAW-DOWN SECTION

ROTATIONAL VELOCITY COMPONENT

E = 0.2
Z = 1.0
Λ = 0.1
V_e/V_0 = 1.5
Re = 0.0
FIG. D20, DRAW-DOWN SECTION

$\psi$ - FUNCTION
FIG. D22, DRAW-DOWN SECTION

\psi - FUNCTION
FIG.D23. DRAW-DOWN SECTION

\psi - FUNCTION
\[ \phi = 0.1 \quad E = 0.2 \quad \Lambda = 0.1 \quad F_t = -10.0 \]
\[ \psi = \alpha \frac{1}{2} \rho \frac{V^2}{\Lambda} \]

\[ \Lambda = 20.0 \]

\[ \psi_0 = 2.0 \]

Points:
- (13.438)
- (18.406)
- (19.737)
- (21.285)
- (23.313)
- (25.244)
- (27.963)
\[ \psi = 0.1 \]
\[ Z = 0.25 \]
\[ \Lambda = 0.1 \]
\[ \Gamma = 20 \]
\[ \nu_L/\nu_o = 2.0 \]

**FIG. D27. DRAW-DOWN SECTION**

\[ \psi - FUNCTION \]
\[ \text{FIG. D29, DRAW-DOWN SECTION} \]

\[ \psi - \text{FUNCTION} \]
$d = 0.1$
$E = -0.2$
$z = 1.0$
$\Lambda = 0.1$
$V_L/V_0 = 3.0$

FIG. D30. DRAW-DOWN SECTION

$\psi$ - FUNCTION
$\psi$ - FUNCTION

FIG. D31. DRAW-DOWN SECTION

$\psi = 0.1$
$E = -0.2$
$Z = 1.0$
$\Lambda = 0.1$
$V_L/V_o = 2.0$
$\varepsilon = 0.1$

$E = -0.2$

$Z = 1.0$

$\Lambda = 0.1$

$V_L/V_0 = 3.0$

FIG. D32. DRAW-DOWN SECTION

$\psi$ - FUNCTION

SECOND NEUTRAL MODE
$\phi = 0.1$

$E = -0.2$

$\zeta = 0.1$

$\Lambda = 0.1$

$V_t/V_o = 3.0$

Fig. D.3. Drawdown section

$\psi$ - function

First neutral mode
FIG. D35. DRAW-DOWN SECTION

$\psi$ - FUNCTION

$Re_m$

$1/\lambda$

$E = -0.2$

$\epsilon = 0.1$

$V_l/V_0 = 2.0$

$Z = 1.0$

$\Gamma_1 = 20.0$
FIG. M1. METERING SECTION

ROTATIONAL VELOCITY COMPONENT
"FREE" BOUNDARY
FIG. M.6. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FREE" BOUNDARY

$E = 0.5$
$t = 0.1$
$\delta = 0.5$

Re$_M$

$\Gamma_t = 50$
$\Gamma_t = 100$

(-0.576)
(-1.41)
(-2.48)
(-3.69)
(-4.87)
(-5.94)
(-6.91)
(-7.75)
(-8.50)
(-8.81)
(-47.5)
(-60.5)
(-65.0)
(-68.81)

5
10
15
20
25
30
35
40

Z
$E = 0.5$
$\tau = 0.1$
$\gamma = 0.5$

FIG. M7. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FREE" BOUNDARY
FIG. M8. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FREE" SBOUNDARY
FIG. M9. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FREE" BOUNDARY
Fig. M10. Metering Section
Rotational Velocity Component
"Free" Boundary
FIG. M11. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FREE" BOUNDARY
FIG. M12. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FREE" BOUNDARY
FIG. M13. METERING SECTION

ROTATIONAL VELOCITY COMPONENT

"FIXED" BOUNDARY
FIG. M14. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FIXED" BOUNDARY
FIG. M15. METERING SECTION

ROTATIONAL VELOCITY COMPONENT

"FIXED" BOUNDARY
FIG. M16. METERING SECTION
ROTATIONAL VELOCITY COMPONENT
"FIXED" BOUNDARY
\[ E = 0.5 \]
\[ \beta = 1.5 \]

**LEGEND:**
- NUMBERS IN ( ) ARE \( v_{cr} \)
- NUMBERS IN [ ] ARE \( \Gamma_{rc} \)

**Fig. M17. Metering Section**
- Rotational Velocity Component
- "Fixed" Boundary
**FIG. M18. METERING SECTION**

**ROTATIONAL VELOCITY COMPONENT**

"FIXED" BOUNDARY
FIG. M19. METERING SECTION

ROTATIONAL VELOCITY COMPONENT

"FIXED" BOUNDARY
FIG. M20, METERING SECTION

ROTATIONAL VELOCITY COMPONENT

"FIXED" BOUNDARY
**FIG. M21. METERING SECTION**

**ROTATIONAL VELOCITY COMPONENT**

"FIXED" BOUNDARY
FIG. M22. METERING SECTION

ROTATIONAL VELOCITY COMPONENT

"FIXED" BOUNDARY
FIG. M23. METERING SECTION

ROTATIONAL VELOCITY COMPONENT

"FIXED" BOUNDARY
FIG. M24. METERING SECTION
Ψ - FUNCTION
FIG. M25. METERING SECTION
Ψ - FUNCTION

NO SOLUTION

E = 0.5
τ = 0.1
\[ E = 0.5 \]
\[ \tau = 0.2 \]

\[ \text{LEGEND:} \]
\[ \text{NUMBERS IN ( ) ARE } \nu_M \]
\[ \text{NUMBERS IN [ ] ARE } \Gamma_M \]

\[ \phi = 0.4 \]
\[ \phi = 0.1 \]

\[ \text{FIG. M28. METERING SECTION} \]
\[ \text{Ψ - FUNCTION} \]
Fig. M29. Metering Section
ψ - Function
$d = 0.1$
$\tau = 0.2$

**LEGEND:**
Numbers in ( ) are $\nu_m$
Numbers in [ ] are $\Gamma_m$

**Fig. M30.** Metering section $\psi$ - function
FIG. M52. METERING SECTION
Ψ - FUNCTION
CHAPTER 9

CONCLUDING REMARKS

Three-dimensional disturbances, \( v'_r(t,x) \), \( v'_\theta(t,x) \), \( v'_z(t,x) \) \( \neq 0 \), are never more unstable than one-dimensional disturbances, \( v'_\theta(t,x) \) \( \neq 0 \), \( v'_r(t,x) = v'_z(t,x) = 0 \), or two-dimensional disturbances, \( v'_\theta(t,x) = 0 \), \( v'_r(t,x) \), \( v'_z(t,x) \) \( \neq 0 \). Critical and marginal Reynold's numbers sufficiently small to be relevant for melt flows of polymers were found for both flow sections considered.

9.1. Draw-Down Section

A uniform, time independent, purely elongational and two-dimensional flow of a viscoelastic fluid is in general incompatible with respect to the equations of motion. An approximately compatible system is obtained when the extension rate becomes very small. A steady state in which inertia forces dominate the stress forces is not possible for any two-dimensional flow (rotational velocity component vanishes) of any fluid. The conditions characterizing a free surface furnish a necessary and sufficient number of boundary conditions for the differential equations in the
perturbed variables.

The marginal Reynold's numbers depend on the following parameters: \( \kappa, \nu_L/\nu_0, Z, E, \Gamma, \Lambda \). With a few exceptions the same basic trends were predicted by the rotational velocity component and the \( \hat{\psi}(R) \) function.

i) Modestly elastic fluids were in general found the most stable and slightly elastic and highly elastic fluids less stable although parameter combinations exist for which stability was enhanced consistently as the fluid became more elastic.

ii) A finite and relatively low critical draw-ratio was identified for the rotational velocity component. Two finite and relatively low critical draw-ratios were identified in the case of the \( \hat{\psi}(R) \) function.

iii) Although the \( \hat{\psi}(R) \) function indicates the existence of critical axial positions, the overall impression is one of stability improvement as the solidification zone is approached.

iv) The correlation between the marginal Reynold's number and the weighing factor \( E \) is rather complex, and caution must be exercised when drawing general conclusions. For the rotational velocity component, stability is enhanced as \( E \) increases.
if the fluid is slightly elastic and impaired
if the fluid is highly elastic. Two critical
regions of E may exist. For the \( \hat{\psi}(R) \) function,
the marginal Reynold's number is only slightly
dependent on E.

v) Finite but high critical wave propagation speeds
were identified for the rotational velocity com-
ponent. The correlation in the case of the
\( \hat{\psi}(R) \) function is very complex.

vi) Although finite critical values of the parameter
A are identified, the overall impression is one
of stability improvement as the length of the
draw-down section increases.

The general impression of the draw-down section is
one of poor stability. Unstable draw-ratios significantly
lower than those predicted by earlier theories were found.
The instabilities predicted in this work are characterized
by very short axial wavelengths (high wave numbers) and
rapid oscillations in time (very high wave propagation
speeds). The "draw-resonance" phenomenon is often thought
to be characterized by relatively long axial wavelengths.
Instabilities of such a nature were not found in this work.

9.2. Metering Section

A two-dimensional, steady combination flow
(rotational velocity component vanishes) consisting of pure Poiseuille flow with a superimposed uniform elongational component is in general incompatible with respect to the equations of motions. An approximately compatible system is obtained when the extension rate becomes very small. The radial flow boundary is neither a rigid boundary in the sense of no slip or a free surface in the sense of vanishing normal stress and tangential surface traction vector.

The marginal Reynolds's numbers depend on the parameters $\kappa$, $\tau$, $Z$, $E$, $\Gamma_I$ and the critical Reynolds's numbers depend on the same parameters save $\Gamma_I$. The trends predicted by the $\psi(R)$ function and the two cases of the rotational velocity component were often quite different.

i) Critical values of the elasticity parameter, $\kappa$, exist. It appears that modestly to highly elastic fluids favor stability most.

ii) Critical extension rates exist. For slightly elastic fluids, however, stability is, in essence, consistently improved as the extension rate increases.

iii) Critical axial positions at or close to the channel entry have been identified.

iv) Stability is consistently impaired as $E$ increases from large negative to large positive values.
v) Finite critical wave propagation speeds exist for both cases of the rotational velocity component. The critical wave propagation speeds for the \( \hat{\psi}(R) \) function are of very large magnitudes.

Two instabilities of distinctly different nature are predicted. If the instability is created by the growth of the radial or axial components of the velocity perturbation field or both, then the instability is characterized by an extremely rapid oscillation in time and a very long axial wavelength. If the instability is created by the growth of the rotational velocity component, on the other hand, then the instability is characterized by a relatively slow oscillation in time but a very short axial wavelength. The latter instability has often been observed within the fiber spinning industry as "matte" on extrudate surface, and is often used as a criterion for the initiation of instability.

9.3. Suggestions for Future Work

It has been shown earlier in this thesis that when elastic fluids are considered, the linear theory not only limits the magnitudes of the instantaneous perturbations to infinitesimal values, but furthermore places restrictions on the flow history. It is necessary to establish
rigorously what type of flow histories the linear theory may be applied to.

The ultimate problem of non-isothermal extrusion of viscoelastic fluids is certainly one of vast mathematical complexity, but, nevertheless, should constitute a future research ambition.
BIBLIOGRAPHY


LIST OF PARAMETERS

Latin Characters

\[ B = \lambda^2 \left\{ (1-2 \frac{N_p^8}{N_p^7}) \nu^2 - \text{Re}'(1+\nu \frac{v_o}{v_{L-o}}) \right\}/N_p^7 \]

\[ c = (\lambda \nu)^2 (2N_p^1 - N_p^6)/N_p^7 \]

\[ E = \{ \text{Re}'\lambda^4 \nu^2 (1+\nu \frac{v_o}{v_{L-o}}) - N_p^6 \nu^4 \lambda^4 \}/N_p^7 \]

\[ F = \{ N_p^6 (\lambda \nu)^2 + \lambda^2 \text{Re}'(1+\nu \frac{v_o}{v_{L-o}}) \}/N_p^1 \]

\[ K_p^1 = \frac{\eta_p}{[1+2(\lambda_1 \nu)^2]} \]

\[ K_p^2 = \frac{\eta_p}{[1+2(\nu^2 + 2(\frac{v_o T}{r_o})^2 \lambda_1 \nu)]} \]

\[ N_p^1 = 1/[1+\lambda_2 \nu] \]

\[ N_p^2 = 1/[(1+\lambda_2 (\sigma+\dot{\gamma}))(1+2\lambda_2 \lambda_2 \dot{\gamma})] \]

\[ N_p^3 = 1/[1+\lambda_2 (\sigma-\dot{\gamma})] \]

\[ N_p^4 = 1/[(1+\lambda_2 (\sigma-2\dot{\gamma}))(1-2\lambda_2 \dot{\gamma})] \]

\[ N_p^5 = 1/[(1+\lambda_2 (\sigma-3\dot{\gamma}))(1-2\lambda_2 \dot{\gamma})] \]
\[ N_p^6 = (1+E/2)N_p^4 - \frac{E}{2} N_p^1 \]

\[ N_p^7 = (1+E/2)N_p^3 - \frac{E}{2} N_p^2 \]

\[ N_p^8 = (1+E/2)N_p^5 - \frac{E}{2} N_p^2 \]

\[ N_p^9 = [1+2(1+E)^\gamma \lambda_2 p]/[1-2(\lambda_2 p)^\gamma]^2 \]

**Greek Characters**

\[ \alpha_1 = \frac{e^{\sigma t}}{\sigma} (e^{-\sigma s}-1) \]

\[ \sigma_2 = \frac{e^{\sigma t}}{\sigma-\gamma} (e^{-\sigma s}-e^{-\gamma s}) \]

\[ \beta_1 = e^{-\gamma s} \]

\[ \beta_2 = \frac{\nu_0}{r_0^{1/2}} \frac{r_0^{1/2}}{r_0^{1/2}} (1-e^{-\gamma s}) \]

\[ \beta_3 = 2 \frac{\nu_0}{r_0^{1/2}} \frac{e^{\sigma t}}{\sigma-\gamma} \left( \frac{1-e^{-\sigma s}}{\sigma} - \frac{1-e^{-\gamma s}}{\gamma} \right) \]
\[ \beta_4 = \beta_3 - 2\beta_2 \theta_1 \]
\[ \beta_5 = \frac{\beta_3 + 2\beta_2 \theta_1}{\beta_1} \]
\[ \beta_6 = \beta_3 r/\beta_1 + r\theta_1 \theta_{10} \]
\[ \beta_7 = r^2 \theta_9 \theta_{10} + r\theta_1 \theta_{10} + \beta_1/\beta_1^2 \]
\[ \beta_8 = \theta_1 (r\theta_{10} + \frac{1}{\beta_1^2}) \]
\[ \beta_9 = \beta_3/\beta_1 - 2\beta_2 \theta_1/\beta_1 \]
\[ \beta_{10} = -2\beta_2/\beta_1 \]
\[ \beta_{11} = r\theta_1 \theta_{10} (1/\beta_1 - 1) \]
\[ \beta_{12} = r\theta_3/\beta_1 + r\theta_1 \theta_{10}/\beta_1 + r\theta_1 \theta_{10} \]
\[ \beta_{13} = r^2 \theta_2 \theta_2 /\beta_1 + \theta_2 /\beta_1^2 \]
\[ \beta_{14} = r^3 \theta_2 /\beta_1 + r\theta_3 /\beta_1 + r\theta_1 \theta_{10} /\beta_1^2 \]
\[ \beta_{15} = 1/[1 + \lambda_{2P} \sigma] \]
\[ \beta_{16} = -2 \left( \frac{v_o}{r_o} \right)^2 \frac{r_o^2}{(\sigma - \gamma) \gamma} \right) \]
\[ \left\{ \frac{2\lambda_{2P} \gamma}{(1+\lambda_{2P} \gamma)(1+2\lambda_{2P} \gamma)} \right\} \]
\[ + \frac{1}{(1+\lambda_{2P} \gamma)(1+\lambda_{2P}(\sigma + \gamma))} - \frac{1}{1+\lambda_{2P} \sigma} \} \]}
$$\beta_{17} = - \left( \frac{v_0}{r_o} \right) \frac{r_o}{\gamma} \left\{ \frac{1}{(1+\lambda_{2p} \sigma)(1+\lambda_{2p} \hat{\gamma})} \right. \\
\left. - \frac{1}{(1+\lambda_{2p} (\sigma-\gamma))(1+2\lambda_{2p} \hat{\gamma})} \right\}$$

$$\beta_{18} = \frac{1}{(1+\lambda_{2p} (\sigma+\gamma))(1+2\lambda_{2p} \hat{\gamma})}$$

$$\beta_{19} = 2 \left( \frac{v_0}{r_o} \right) \frac{\lambda_{2p}}{[1+2\lambda_{2p} \gamma][1+\lambda_{2p} \gamma][1+\lambda_{2p} (\sigma+\gamma)]}$$

$$\beta_{20} = 2 \left( \frac{v_0}{r_o} \right) \frac{1}{\gamma} \left\{ \frac{1}{(1+\lambda_{2p} \sigma)} - \frac{1}{[1+\lambda_{2p} (\sigma-\gamma)][1-\lambda_{2p} \hat{\gamma}]} \right\}$$

$$\beta_{21} = \frac{1}{1+\lambda_{2p} (\sigma-\gamma)}$$

$$\beta_{22} = 2 \left( \frac{v_0}{r_o} \right) \frac{\lambda_{2p}}{(1-\lambda_{2p} \gamma)(1+\lambda_{2p} (\sigma-\gamma))}$$

$$\beta_{23} = 2 \left( \frac{v_0}{r_o} \right) \frac{1}{\sigma-\gamma} \left\{ \frac{1}{[1+\lambda_{2p} (\sigma-2\gamma)][1+\lambda_{2p} (\sigma-\gamma)]} - \frac{1}{1-\lambda_{2p} \gamma} \right\}$$

$$\beta_{24} = 2 \left( \frac{v_0}{r_o} \right) \left\{ \frac{1}{[1-\lambda_{2p} \gamma][1+\lambda_{2p} (\sigma-\gamma)]} \right. \\
\left. + \frac{2}{\gamma} \left( \frac{1}{1+\lambda_{2p} \sigma} - \frac{1}{[1+\lambda_{2p} (\sigma-\gamma)][1-\lambda_{2p} \gamma]} \right) \right\}$$

$$\beta_{25} = \frac{1}{[1+\lambda_{2p} (\sigma-2\gamma)][1-2\lambda_{2p} \hat{\gamma}]}$$
\begin{align*}
\beta_{26} &= \left( \frac{2 \nu_0}{r_0 \gamma} \right)^2 \frac{r_0^2}{\gamma} \left\{ \frac{1}{\sigma - \gamma} \cdot \left[ \frac{1}{1 - 2 \lambda_{2p} \gamma} \right] \frac{2 \lambda_{2p} \gamma}{\left[ 1 + \lambda_{2p} (\sigma - 2 \gamma) \right] \left[ 1 - \lambda_{2p} \gamma \right]} 
&\quad - \frac{1}{\left[ 1 + \lambda_{2p} (\sigma - 2 \gamma) \right] \left[ 1 - \lambda_{2p} \gamma \right]} 
&\quad + \frac{2 \lambda_{2p} \gamma}{\left[ 1 - 2 \lambda_{2p} \gamma \right] \left[ 1 - \lambda_{2p} \gamma \right]} \right\} \\
\beta_{27} &= \left( \frac{2 \nu_0}{r_0^2} \right)^2 \frac{r_0^2}{\gamma} \left\{ \frac{1}{\gamma} \cdot \left[ \frac{1}{1 - 2 \lambda_{2p} \gamma} \right] \frac{1}{\left[ 1 + \lambda_{2p} (\sigma - 2 \gamma) \right]} 
&\quad - \frac{1}{\left[ 1 + \lambda_{2p} (\sigma - 2 \gamma) \right] \left[ 1 - \lambda_{2p} \gamma \right]} 
&\quad + \frac{2 \lambda_{2p} \gamma}{\left[ 1 - 2 \lambda_{2p} \gamma \right] \left[ 1 - \lambda_{2p} \gamma \right]} \right\} 
\end{align*}
\[ \beta_{28} = \left(2 \frac{v_0}{r_0^2} \right)^2 \frac{r_0^2}{\gamma (\sigma - \gamma)} \left\{ \frac{1}{[1 - 2 \lambda_2 \gamma \dot{\gamma}] [1 + \lambda_2 (\sigma - 2 \gamma)]} \right. \\
- \frac{1}{[1 - \lambda_2 \gamma \dot{\gamma}] [1 + \lambda_2 (\sigma - \gamma)]} - \frac{2 \lambda_2 \gamma}{[1 - 2 \lambda_2 \gamma \dot{\gamma}] [1 - \lambda_2 \gamma \dot{\gamma}]} \right\} \]

\[ \beta_{29} = \frac{1}{[1 + \lambda_2 (\sigma - 3 \gamma)] [1 - 2 \lambda_2 \gamma \dot{\gamma}]} \]

\[ \beta_{30} = 2 \left(2 \frac{v_0}{r_0^2} \right)^2 \frac{r_0^2}{\gamma \dot{\gamma}} \left\{ \frac{1}{[1 + \lambda_2 (\sigma - 2 \gamma)] [1 + \lambda_2 (\sigma - \gamma)] [1 + \lambda_2 \sigma]} \right. \\
+ \frac{\lambda_2 \gamma}{1 - (\lambda_2 \gamma \dot{\gamma})^2} \right\} \]

\[ \beta_{31} = \left(\frac{2v_0}{r_0^2} \right)^3 \frac{1}{\gamma^2} \left\{ \frac{1}{\gamma \dot{\gamma}} \right. \left[ \frac{1}{[1 - 2 \lambda_2 \gamma \dot{\gamma}] [1 + \lambda_2 (\sigma - 2 \gamma)]} \right. \\
- \frac{1}{[1 - 3 \lambda_2 \gamma \dot{\gamma}] [1 + \lambda_2 (\sigma - 3 \gamma)]} \right. \\
- \frac{1}{[1 + \lambda_2 (\sigma - \gamma)] [1 - 2 \lambda_2 \gamma \dot{\gamma}]} + \frac{3}{1 - 3 \lambda_2 \gamma \dot{\gamma}} \right. \\
- \frac{3}{[1 - 2 \lambda_2 \gamma \dot{\gamma}] [1 - \lambda_2 \gamma \dot{\gamma}]} + \frac{3}{\sigma} \left[ \frac{1}{[1 + \lambda_2 (\sigma - 2 \gamma)] [1 + \lambda_2 \sigma]} \right. \\
- \frac{1}{[1 + \lambda_2 (\sigma - 2 \gamma)] [1 + \lambda_2 (\sigma - \gamma)]} \right. \\
+ \frac{1}{[1 - 2 \lambda_2 \gamma \dot{\gamma}] [1 - \lambda_2 \gamma \dot{\gamma}]} - \frac{1}{1 - 3 \lambda_2 \gamma \dot{\gamma}]} \right\} \} \]
\[ \beta_{32} = \left( \frac{\nu_o}{r_o} \right) \left\{ \frac{1}{[1-3\lambda_{2p}\gamma][1-2\lambda_{2p}\gamma][1+\lambda_{2p}(\sigma-3\gamma)]} \right. \\
+ \frac{2}{\sigma} \left[ \frac{1}{[1+\lambda_{2p}(\sigma-3\gamma)][1+\lambda_{2p}(\sigma-2\gamma)]} \right] \\
\left. - \frac{1}{[1-3\lambda_{2p}\gamma][1-2\lambda_{2p}\gamma]} \right\} \}

\beta_{33} = \beta_{16E}^E \\
\beta_{34} = (1+E/2)\beta_{20} \\
\beta_{35} = (1+E/2)\beta_{21} - E/2 \beta_{18} \\
\beta_{36} = (1+E/2)\beta_{22} - E/2 \beta_{19} \\
\beta_{37} = (1+E/2)\beta_{23} - E \beta_{17} \\
\beta_{38} = (1+E/2)\beta_{24} - E/2 \beta_{19} \\
\beta_{39} = (1+E/2)\beta_{25} - E/2 \beta_{15} \\
\beta_{40} = (1+E/2)\beta_{26} - E \beta_{16} \\
\beta_{41} = (1+E/2)\beta_{23} \\
\beta_{42} = 2(1+E/2)\beta_{27} \\
\beta_{43} = 2(1+E/2)\beta_{28} \]
\[ \beta_{44} = 2(1+E/2)\beta_{29} - \overline{E}\beta_{18} \]

\[ \beta_{45} = 2(1+E/2)\beta_{30} \]

\[ \beta_{46} = 2(1+E/2)\beta_{31} \]

\[ \beta_{47} = 2(1+E/2)\beta_{32} - \overline{E}\beta_{19} \]

\[ \beta_{48} = 2\beta_{34} + \beta_{36} \]

\[ \beta_{49} = 4\beta_{34} + \beta_{36} \]

\[ \beta_{50} = \beta_{20} + \beta_{34} \]

\[ \beta_{51} = 3\beta_{34} - \beta_{20} \]

\[ \beta_{52} = \beta_{41} + \beta_{37} \]

\[ \beta_{53} = \beta_{40} + \beta_{42} \]

\[ \beta_{54} = \beta_{43} + 3\beta_{40} \]

\[ \beta_{55} = \beta_{36} + 2\beta_{38} \]

\[ \beta_{56} = \beta_{50}/\beta_{15} \]

\[ \beta_{57} = \beta_{39}/\beta_{15} \]

\[ \beta_{58} = \beta_{34}/\beta_{15} \]

\[ \beta_{59} = (\beta_{51} + i\text{Re})/\beta_{15} \]
\[ \beta_{60} = \beta_{44} - \beta_{35} \]
\[ \beta_{61} = \beta_{52} + 2\beta_{37} \]
\[ \beta_{62} = \beta_{47} - \beta_{48} \]
\[ \beta_{63} = \beta_{39} - 2\beta_{15} \]
\[ \beta_{64} = \beta_{53} + \beta_{33} \]
\[ \beta_{65} = \beta_{47} - \beta_{49} + \text{Re} \]
\[ \beta_{67} = 2\beta_{53} + \beta_{54} + 4\beta_{33} - \frac{\text{Re}}{R} \]
\[ \beta_{68} = \beta_{38} + \beta_{55} \]
\[ \beta_{69} = \beta_{54} + \text{Re} + 2\beta_{33} \]
\[ \beta_{70} = \frac{i\nu}{\beta_{35}} (\beta_{52} - \beta_{38}) \]
\[ \beta_{71} = (\text{Re} + \nu^2 \beta_{60} + i\nu \beta_{68} - i\nu \beta_{52} - i\nu \beta_{61} - \nu^2 \beta_{63})/\beta_{35} \]
\[ \beta_{72} = \nu^2 (\beta_{45} - \beta_{64})/\beta_{35} \]
\[ \beta_{73} = (\text{Re} + 2i\nu \beta_{52} + \nu^2 \beta_{60} + i\nu \beta_{36} - i\nu \beta_{61} - \nu^2 \beta_{63})/\beta_{35} \]
\[ \beta_{74} = 2\nu^2 \beta_{45}/\beta_{35} \]
\[ \beta_{75} = (i\nu^3 \beta_{62} + \nu^2 \beta_{67} - \nu^2 \beta_{45} + i\nu^3 \beta_{37})/\beta_{35} \]
\[ \beta_{76} = i\nu^3 \beta_{46}/\beta_{35} \]
\[ \beta_{77} = (rRe + 2i\nu\beta_{52} - i\nu\beta_{61} - \nu^2\beta_{39})/\beta_{35} \]

\[ \beta_{78} = \nu^2(\beta_{60} - 2\beta_{45})/\beta_{35} \]

\[ \beta_{79} = (i\nu^3\beta_{65} + i\nu\beta_{36} + \nu^2\beta_{69} + i\nu^3\beta_{37} + \nu^4\beta_{39})/\beta_{35} \]

\[ \beta_{80} = \beta_{74}/2 \]

\[ \beta_{81} = (3i\nu^3\beta_{46} + \nu^4\beta_{40})/\beta_{35} \]
APPENDIX 1

DETERMINATION OF THE PERTURBED CONTRAVARIANT STRAIN TENSORS

This appendix explains in some detail the derivation of the perturbed contravariant strain tensors. These tensors were determined by inverting the matrices of the deformation gradient tensors.

Draw-Down Section

\[
(F_t)_{ij} = \frac{\partial \xi_i}{\partial x_j} =
\]

\[
\begin{bmatrix}
1 + \alpha_1 \frac{\partial w^r}{\partial r}, & \alpha_1 \frac{\partial w^r}{\partial \theta}, & \alpha_1 \frac{\partial w^r}{\partial z} \\
\alpha_1 \frac{\partial w^\theta}{\partial r}, & 1 + \alpha_1 \frac{\partial w^\theta}{\partial \theta}, & \alpha_1 \frac{\partial w^\theta}{\partial z} \\
\alpha_2 \frac{\partial w^z}{\partial r}, & \alpha_2 \frac{\partial w^z}{\partial \theta}, & e^{-\gamma s} + \alpha_2 \frac{\partial w^z}{\partial z}
\end{bmatrix}_{Al}
\]

The determinant of the matrix Al is,
\[
\det(F^T_t) = \det(F_t) = \\
[1+\alpha_1(\frac{\partial w}{\partial \theta} + \frac{\partial w^r}{\partial r})]e^{-\gamma_s} + \alpha_2 \frac{\partial w^z}{\partial z} \\
+ \text{higher order terms.}
\]

By evaluating the cofactors of the matrix A1, the following results emerge:

\[
\frac{\partial x^i}{\partial \xi^j} = \\
\begin{bmatrix}
(1+\alpha_1 \frac{\partial w}{\partial \theta})e^{-\gamma_s} + \alpha_2 \frac{\partial w^z}{\partial z}, & -\alpha_1 \frac{\partial w^r}{\partial \theta} e^{-\gamma_s}, & -\alpha_1 \frac{\partial w^r}{\partial z}

-\alpha_1 \frac{\partial w^r}{\partial r} e^{-\gamma_s}, & (1+\alpha_1 \frac{\partial w^r}{\partial \theta})e^{-\gamma_s} + \alpha_2 \frac{\partial w^z}{\partial z}, & -\alpha_1 \frac{\partial w^r}{\partial z}

-\alpha_2 \frac{\partial w^z}{\partial r}, & -\alpha_2 \frac{\partial w^z}{\partial \theta}, & 1+\alpha_1(\frac{\partial w^r}{\partial \theta} + \frac{\partial w^r}{\partial r})
\end{bmatrix}
\]

The higher order terms of the determinant of A1 have been disregarded, or equivalently, this quantity has been linearized. The equation of continuity for the vector components of the velocity field,

\[
\frac{\partial (rw^r)}{\partial t} + \frac{\partial w^r}{\partial \theta} + \frac{\partial w^z}{\partial z} = 0
\]

is now used in order to convert matrix A2 into a more suitable form. After some manipulations, the following results are obtained:
\[
\frac{\partial x^i}{\partial \xi^j} = \\
\begin{bmatrix}
1 - \frac{\alpha_1}{D} \frac{\partial w^r}{\partial r}, & - \frac{\alpha_1}{D} \frac{\partial w^r}{\partial \theta}, & - \frac{1}{D} \frac{\partial}{\partial z} e^y \frac{\partial w^r}{\partial z} \\
- \frac{\alpha_1}{D} \frac{\partial w^\theta}{\partial r}, & 1 - \frac{\alpha_1}{D} \frac{\partial w^\theta}{\partial \theta}, & - \frac{1}{D} \frac{\partial}{\partial z} e^y \frac{\partial w^\theta}{\partial z} \\
- \frac{\alpha_2}{D} \frac{\partial}{\partial r} e^{\gamma_y} \frac{\partial w^z}{\partial r}, & - \frac{\alpha_2}{D} \frac{\partial}{\partial \theta} e^{\gamma_y} \frac{\partial w^z}{\partial \theta}, & e^{\gamma_y} - \frac{\alpha_2}{D} e^{2\gamma_y} \frac{\partial w^z}{\partial z}
\end{bmatrix}
\]

where

\[
D = 1 + \alpha_1 (\frac{\partial w^r}{\partial r} + \frac{\partial w^\theta}{\partial \theta}) + \alpha_2 e^{\gamma_y} \frac{\partial w^z}{\partial z}
\]

\[
= 1 + \frac{e^s}{\sigma} \{(e^{-\sigma s} - 1)(\frac{\partial w^r}{\partial r} + \frac{\partial w^\theta}{\partial \theta})
\]

\[
+ \left( \frac{1}{1 - \gamma_y \sigma} \right) (e^{-(\sigma - \gamma_y)s} - 1) \frac{\partial w^z}{\partial z} \}
\]

One remembers that the extension rate, \( \gamma \), must remain very small on account of compatibility. The following approximations should accordingly be appropriate:

\[
1/(1 - \gamma_y / \sigma) \approx 1, \quad e^{\gamma_y} \approx 1.
\]

Then, \( D \approx 1 - \frac{e^s}{\sigma} (e^{-\sigma s} - 1) \frac{w^r}{r} \).
The validity of above assumption is obvious for all values of s except infinity. At this point, the approximation may or may not be good.

An expression of 1/D may now be obtained.

\[ \frac{1}{D} \approx 1 + \frac{e^{\sigma t}}{\sigma} (e^{-\sigma s} - 1)w^r/r \]  \hspace{1cm} (A6)

provided \( \text{Real}(\sigma) > 0 \) and \( w^r/r \) remains small at \( r = 0 \). It is shown elsewhere in this report that \( w^r/r = 0 \) at \( r = 0 \).

Then, linearizing matrix A3 using equation A6,

\[ \frac{\partial x^i}{\partial \xi^j} = \]

\[ \begin{bmatrix}
1 - \alpha_1 \frac{\partial w^r}{\partial r}, & -\alpha_1 \frac{\partial w^r}{\partial \theta}, & -\alpha_1 e^\gamma s \frac{\partial w^r}{\partial z} \\
-\alpha_1 \frac{\partial w^\theta}{\partial r}, & 1 - \alpha_1 \frac{\partial w^\theta}{\partial \theta}, & -\alpha_1 e^\gamma s \frac{\partial w^\theta}{\partial z} \\
-\alpha_2 e^\gamma s \frac{\partial w^z}{\partial r}, & -\alpha_2 \frac{\partial w^z}{\partial \theta} e^\gamma s, & e^\gamma s - \alpha_2 e^\gamma s \frac{\partial w^z}{\partial z}
\end{bmatrix} \]  \hspace{1cm} (A7)

The contravariant strains are now readily found from equations 5.88 and A7. The results are,
This tensor may logically be decomposed into the sum of two tensors.

\[
\mathbf{b}^{ij} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & e^{2\gamma_s-1}
\end{bmatrix} +
\begin{bmatrix}
-2\alpha_1 \frac{\partial w^r}{\partial r}, & -\alpha_1 \frac{\partial w^\theta}{\partial r}, & -e^{\gamma_s}(\alpha_2 \frac{\partial w^z}{\partial r}) \\
+ \frac{1}{r^2} \frac{\partial w^r}{\partial \theta}, & -2\alpha_1 \frac{\partial w^\theta}{\partial \theta}, & -e^{\gamma_s}(\alpha_2 \frac{\partial w^\theta}{\partial z}) \\
+ \frac{1}{r^2} \frac{\partial w^r}{\partial \theta}, & + \frac{1}{r^2} \frac{\partial w^\theta}{\partial \theta}, & + \frac{\partial w^z}{\partial \theta} \\
-e^{\gamma_s}(\alpha_2 \frac{\partial w^z}{\partial r}), & -e^{\gamma_s}(\alpha_2 \frac{\partial w^z}{\partial z}), & (e^{2\gamma_s-1}
\end{bmatrix}
\]
\[
\begin{bmatrix}
-2\alpha_1 \frac{\partial w^r}{\partial r} & -\alpha_1 \frac{\partial w^\theta}{\partial r} & -e^{-\gamma^s}(\alpha_2 \frac{\partial w^z}{\partial r}) \\
+ \frac{1}{r^2} \frac{\partial w^r}{\partial \theta} & -2\alpha_1 \frac{\partial w^\theta}{\partial \theta} & -e^{\gamma^s}(\alpha_1 e^{\gamma^s} \frac{\partial w^\theta}{\partial z}) \\
-\alpha_1 \frac{\partial w^\theta}{\partial r} & -2\alpha_1 \frac{\partial w^\theta}{\partial \theta} & -e^{\gamma^s}(\alpha_1 e^{\gamma^s} \frac{\partial w^\theta}{\partial z}) \\
+ \frac{1}{r^2} \frac{\partial w^r}{\partial \theta} & + \frac{\alpha_2}{r^2} \frac{\partial w^z}{\partial \theta} & -e^{\gamma^s}(\alpha_1 e^{\gamma^s} \frac{\partial w^\theta}{\partial z}) \\
+ e^{\gamma^s}(\alpha_1 e^{\gamma^s} \frac{\partial w^r}{\partial z}) & + \frac{\alpha_2}{r^2} \frac{\partial w^z}{\partial \theta} & -2\alpha_2 e^{3\gamma^s} \frac{\partial w^z}{\partial z}
\end{bmatrix}
\]

By consulting equation 5.27, it is seen that the steady-state contravariant strain tensor is now recovered. This is, of course, to be expected. Thus,

\[
B^{' ij} = B^{ij} + B^{' ij}
\]

The matrix for the perturbed, contravariant strain tensor is therefore
\[ B_{ij}^{\prime} = \]
\[
\begin{bmatrix}
-2\alpha_1 \frac{\partial w^r}{\partial r}, & -\alpha_1 \left( \frac{\partial w^\theta}{\partial r} \right), & -\dot{\gamma}_s (\alpha_2 \frac{\partial w^z}{\partial r}) \\
+ \frac{1}{r^2} \frac{\partial w^r}{\partial \theta}, & -2\alpha_1 \frac{\partial w^\theta}{\partial \theta}, & -\dot{\gamma}_s (\alpha_1 \dot{\gamma}_s \frac{\partial w^\theta}{\partial z}) \\
+ \frac{1}{r^2} \frac{\partial w^r}{\partial \theta}, & -\dot{\gamma}_s (\alpha_2 \frac{\partial w^z}{\partial r}), & + \frac{\alpha_2}{r^2} \frac{\partial w^z}{\partial \theta} \\
+ \alpha_2 \frac{\partial w^r}{\partial z}, & + \frac{\alpha_2}{r^2} \frac{\partial w^z}{\partial \theta} \\
\end{bmatrix}
\]

\[ An 1 \]

\textbf{Métering Section}

The matrix for the deformation gradient tensor for this flow section is as follows:

\[ F_{ij}^{T_i} = \frac{\partial \xi_i}{\partial x^j} = \]
\[
\begin{bmatrix}
1+\beta_1 \frac{\partial w}{\partial r}, & \beta_1 \frac{\partial w}{\partial \theta}, & \beta_1 \frac{\partial w}{\partial z} \\
\beta_1 \frac{\partial w}{\partial r}, & 1+\beta_1 \frac{\partial w}{\partial \theta}, & \beta_1 \frac{\partial w}{\partial z} \\
2r\beta_2 + \beta_2 \frac{\partial w}{\partial r}, & \beta_2 \frac{\partial w}{\partial \theta}, & \beta_1 + \beta_2 \frac{\partial w}{\partial z} \\
+\beta_3 \frac{\partial (rw)}{\partial r}, & +\beta_3 r \frac{\partial w}{\partial \theta}, & +\beta_3 r \frac{\partial w}{\partial z}
\end{bmatrix}
\]

The determinant of the matrix All is,

\[
\det(F_t^T) = \det(F_t) = \beta_1 + \beta_2 \frac{\partial w}{\partial z} + \beta_4 r \frac{\partial w}{\partial z}
\]

\[
+ \beta_1 \beta_1 \frac{\partial w}{\partial \theta} + \beta_1 \beta_1 \frac{\partial w}{\partial r}
\]

By evaluating the co-factors of matrix All,
\[
\frac{\partial x^i}{\partial \xi^j} =
\begin{bmatrix}
\beta_1 + \beta_2 \frac{\partial w^z}{\partial z} \\
\beta_3 r \frac{\partial w^r}{\partial z} \\
-\beta_1 \frac{\partial w^\theta}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
-\beta_1 \frac{\partial w^r}{\partial \phi} \\
-\beta_1 \frac{\partial w^r}{\partial z} \\
+\beta_1 \frac{\partial w^\theta}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
-\beta_1 \frac{\partial w^r}{\partial z}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta_1 (2\beta_2 r \frac{\partial w^\theta}{\partial z}) \\
-\beta_1 \frac{\partial w^\theta}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
\beta_1 + \beta_2 \frac{\partial w^z}{\partial z} \\
\beta_3 r \frac{\partial w^r}{\partial z} \\
+\beta_1 \frac{\partial w^r}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
-\beta_1 \frac{\partial w^\theta}{\partial z}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-(2\beta_2 r + \beta_2 \frac{\partial w^z}{\partial z}) \\
+\beta_3 \frac{\partial (r w^r)}{\partial \phi} \\
+2\beta_2 \beta_1 r \frac{\partial w^\theta}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
-\beta_2 \frac{\partial w^z}{\partial \phi} \\
-\beta_3 r \frac{\partial w^r}{\partial \phi} \\
+2\beta_2 \beta_1 r \frac{\partial w^r}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
1 + \beta_1 \frac{\partial w^r}{\partial \phi} \\
+\beta_1 \frac{\partial w^\theta}{\partial \phi}
\end{bmatrix}
\]

\[
\beta_1 + \beta_2 \frac{\partial w^z}{\partial z} + \beta_2 r \frac{\partial w^r}{\partial z} + \beta_1 \beta_1 \frac{\partial w^\theta}{\partial \phi} + \beta_1 \beta_1 \frac{\partial w^r}{\partial \phi}
\]

The equation of continuity for the vector components of the velocity field is again used to convert matrix A12 into a
more suitable form. After numerous manipulations, such as adding and subtracting equal terms in order to simplify via the equation of continuity, the following result is obtained:

\[
\frac{\partial x_i}{\partial \xi^j} =
\begin{bmatrix}
1 - \frac{1}{D} \left( \beta_1 \beta_1 \frac{\partial w_r}{\partial r} \right) & -\frac{\beta_1}{D} \frac{\partial w_r}{\partial \theta} & -\frac{1}{D} \frac{\partial w_r}{\partial z} \\
-2r \beta_2 \beta_1 \frac{\partial w_r}{\partial z} & \frac{1}{D} \left( \frac{2\beta_2 \beta_1}{\beta_1} r \frac{\partial w_\theta}{\partial z} - \beta_1 \frac{\partial w_\theta}{\partial z} \right) & 1 - \frac{1}{D} \beta_1 \frac{\partial w_\theta}{\partial \theta} - \frac{1}{D} \frac{\partial w_\theta}{\partial z} \\
- \frac{1}{D} \left( \frac{2\beta_2 \beta_1 r}{\beta_1} + \frac{\beta_2 \beta_1}{\beta_1} \frac{\partial w_z}{\partial r} + \frac{\beta_3}{\beta_1} \frac{\partial (r w_r)}{\partial r} \right) & \frac{1}{D} \left( -\frac{\beta_2}{\beta_1} \frac{\partial w_z}{\partial \theta} \right) - \frac{\beta_3}{\beta_1} r \frac{\partial w_r}{\partial \theta} + \frac{\beta_3}{\beta_1} r \frac{\partial w_r}{\partial z} & 1 - \frac{1}{D} \left( \frac{\beta_2}{\beta_1} \frac{\partial w_z}{\partial \theta} \right)
\end{bmatrix}
\]

where
\[ D = 1 + \left( \frac{\beta_2}{\beta_1} - \beta_1 \right) \frac{\partial W^z}{\partial z} + \frac{\beta_4}{\beta_1^2} \frac{\partial W^r}{\partial z} - \frac{\beta_1}{r} W^r \quad \text{A14} \]

It is easily verified that \( \beta_1, \beta_2/\beta_1 \) and \( \beta_4/\beta_1 \) all remain bounded in \( 0 \leq s \leq \infty \) provided

\[ \text{Real}(\sigma) > \dot{\gamma} \quad \text{A15} \]

But since \( \dot{\gamma} \) must remain very small, the condition A15 thus read,

\[ \text{Real}(\sigma) \geq 0 \quad \text{A16} \]

Since the linear theory requires the perturbation field and its partial derivatives to remain vanishing small, the quantity \( 1/D \) may then be found by a Taylor series expansion around \( W(x) = 0 \). The result is

\[ \frac{1}{D} \approx 1 - \left( \frac{\beta_2}{\beta_1} - \beta_1 \right) \frac{\partial W^z}{\partial z} - \frac{\beta_4}{\beta_1^2} \frac{\partial W^r}{\partial z} + \frac{\beta_1}{r} W^r \quad \text{A17} \]

It is, of course, necessary that \( W^r/r \) is small. This is shown elsewhere in the report.

The point \( s = \infty \) is again a difficult one. The above expansion may not be valid there. An attempt to treat this as a special case did not produce, and one is forced to assume equation A17 valid for \( s \in [0, \infty] \).

The matrix A13 is now linearized using equation A17. The following result was obtained.
\[
\frac{a x^i}{a \xi^j} = \\
\begin{bmatrix}
1 - (\beta_2 \beta_1 \frac{a W^r}{a r} & -\beta_1 \frac{a W^r}{a \theta} & -\beta_1 \frac{a W^r}{a z} \\
-2r \beta_2 \beta_1 \frac{a W^r}{a z} \\
2 \frac{\beta_2 \beta_1}{\beta_1} r \frac{a W^\theta}{a z} \\
-\beta_1 \frac{a W^\theta}{a z} & 1 - \beta_1 \frac{a W^\theta}{a \theta} & -\frac{\beta_1}{\beta_1} \frac{a W^\theta}{a z} \\
-\left(2 \frac{\beta_2}{\beta_1} \frac{a W^z}{a r} + \frac{\beta_2}{\beta_1} \frac{a W^z}{a \theta} + \frac{\beta_3}{\beta_1} \frac{a (r W^r)}{a r} + 2 \frac{\beta_2 \beta_1}{\beta_1} r \frac{a W^\theta}{a \theta}\right) \\
+ \frac{\beta_2 \beta_1}{\beta_1} \frac{a W^\theta}{a \theta} & \frac{\beta_2 \beta_1}{\beta_1} \frac{a W^\theta}{a \theta} & -2 \frac{\beta_2 \beta_1}{\beta_1} \frac{a W^\theta}{a z}
\end{bmatrix}
\]

A18

The contravariant strain tensor is now found by means of matrix A18 and equation 5.88. The result is, after a great deal of simplifying manipulations involving the continuity equation,
\[ B_{ij} = \]

\[
\begin{align*}
-2\beta_1 & \frac{\partial W^r}{\partial r} \\
\beta_1 & 10^r \frac{\partial W^r}{\partial z} \\
\frac{\partial W^\theta}{\partial r} & 1 \frac{\partial W^r}{\partial \theta} \\
-\beta_1 & (\beta 10^r \frac{\partial W^\theta}{\partial z} \\
+ & \frac{\partial W^\theta}{\partial r} \frac{1}{r^2} \frac{\partial W^r}{\partial \theta}) \\
-\beta_1 & (\beta 10^r \frac{\partial W^\theta}{\partial z} \\
+ & \frac{\partial W^\theta}{\partial r} \frac{1}{r^2} \frac{\partial W^r}{\partial \theta}) \\
\frac{\partial W^z}{\partial r} & \frac{\beta_2}{\beta_1} \frac{\partial W^r}{\partial \theta} \\
+ & \frac{\partial W^r}{\partial r} \frac{\beta_2}{\beta_1} \frac{\partial W^z}{\partial \theta} \\
+ & 2\beta_2 r/\beta_1 \\
\end{align*}
\]
This tensor may now be decomposed into a sum of two tensors,

\[ b_{ij} = B^i_{ij} + B'^{ij}, \]

where

\[ B'^{ij} = \]

\[ \begin{bmatrix}
-2\beta_1 \frac{aW_r}{r} & -\beta_1 (r\beta_{10} \frac{aW^\theta}{az}) & -\left( \frac{-\beta_2 aW_z + \beta_3 W_r}{\beta_1} \right) \\
+\beta_1 \beta_{10} \frac{aW_r}{az} & + \frac{aW^\theta}{ar} + \frac{1}{r^2} \frac{aW_r}{az} & + \frac{\beta_2}{\beta_1} \frac{aW^\theta}{ar} \\
-\beta_1 (\beta_{10} \frac{aW^\theta}{az}) & + \frac{aW^\theta}{ar} + \frac{1}{r^2} \frac{aW_r}{az} & + \left( \frac{-\beta_2 aW_z + \beta_3 W_r}{r} \right) \\
+ \frac{aW^\theta}{ar} + \frac{1}{r^2} \frac{aW_r}{az} & -2 \frac{\beta_1}{\beta_2} \frac{aW^\theta}{ar} & + \beta_1 \beta_{10} \frac{aW^\theta}{ar} \\
-\left( \frac{-\beta_2 aW_z + \beta_3 W_r}{\beta_1} \right) & + \frac{\beta_2}{\beta_1} \frac{aW^\theta}{ar} & + \left( \frac{-\beta_2 aW_z + \beta_3 W_r}{r} \right) \\
\end{bmatrix}. \]
and

\[
\bar{\mathbb{B}}^{ij} = \begin{bmatrix}
0 & 0 & -\frac{2\beta_2 r}{\beta_1} \\
0 & 0 & 0 \\
-\frac{2\beta_2 r}{\beta_1} & 0 & \left(\frac{2\beta_2 r}{\beta_1}\right)^2 + \frac{1}{\beta_1^2} - 1
\end{bmatrix}
\]

By consulting equation 5.55, it is confirmed that the steady state contravariant tensor is properly recovered. The matrix of the contravariant strain tensor is, consequently, given by equation A19.
APPENDIX 2

SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

This appendix explains in some detail how the solutions for the describing, ordinary, differential equations of Chapter 5.6 were obtained. The Frobenius method was employed. Solutions were sought in the form of infinite power series,

\[ \hat{\psi}(R), \nu(R) = R^s \sum_{n=0}^{\infty} C_n R^n \]

**Draw-Down Section**

**Rotational velocity component**

When substituting equation A20 into equation 5.115 and arranging terms of equal powers, the following equation results:

\[ C_0 \{ s(s-1)+(s-1) \} R^s + C_1 \{ (s+1)s+s \} R^{s+1} \]

\[ + \sum_{n=2}^{\infty} \{ C_n (s+n-1)(s+n+1)-F_{n-2} \} R^{s+n} = 0 \]
The roots of the indicial equation are:

\[ s = 1, -1; \]

for which the constant \( C_0 \) may remain arbitrary. Furthermore

\[ C_n = C_{n-2}/[n(n-2)] \quad \text{(A22)} \]

**Case \( s=1 \)**

From equation A21 it is seen that \( C_1 \) must be identically zero. Then, all the odd coefficients in the series must vanish. Consequently,

\[
\hat{v}_1(R) = R(C_0 + \sum_{n=2}^{\infty} C_n R^n) \quad \text{n-even}
\]

\[
= \frac{C_0}{i\sqrt{F}} \frac{iR\sqrt{F}}{\sqrt{F}} \left( \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(iR\sqrt{F})^{2n}}{2^{2n+1} \cdot n! \cdot (n+1)!} \right)
\]

\[
= \frac{C_0}{\sqrt{F}} J_1(iR\sqrt{F}) \quad \text{(A23)}
\]

**Case \( s=-1 \)**

The solution must be of the following form:
\[
\hat{\nu}_2(R) = k \hat{\nu}_1(R) \log R + \sum_{n=0}^{\infty} C'_n R^{n-1}
\]

But \( k = 0 \) since the solution must remain bounded at \( R = 0 \).

Thus,

\[
\hat{\nu}_2(R) = \sum_{n=0}^{\infty} C'_n R^{n-1}
\]

From equation A21 it is clear that \( C'_1 = 0 \). However,

\[
C'_2 = \frac{C'_0}{4-4} \implies C'_0 = 0
\]

The only possible solution for this root is then the trivial solution itself. Equation 5.115 has therefore only one regular solution, which is

\[
\hat{\nu}(R) = \hat{\nu}_1(R) = \frac{C_0}{\sqrt{F}} J_1(iR\sqrt{F})
\]

The function \( \hat{\psi}(R) \)

Substitution of A20 into equation 5.116 yields after some algebra:

\[
b_0(s-1)(s-2)2(s+1)R^s + b_1 s(s-1)^2(s+2)R^{s+1}
\]

\[+
\sum_{n=2}^{\infty} b_{n-2} \{(B-C)(s+n-2)(s+n-3)\}
\]
\[ (B+C)(s+n-2)-B) R^{s+n} + \sum_{n=4}^{\infty} b_{n-4} R^{s+n} = 0 \quad \text{A28} \]

The roots of the indicial equation are:

\[ s=1, 2, 2, -1 \]

**Case s=1**

The constant \( b_0 \) may, obviously, remain arbitrary. Furthermore, \( b_1 \) may remain arbitrary as well since

\[ b_1 s(s-1)^2(s+2) = 0, \quad |b_1| < \infty \quad \text{A29} \]

The following results are obvious:

\[ b_2 = -b_0 C/8 \]

\[ b_3 = -b_1 B/20 \]

\[ b_n = \frac{b_{n-2}[3C-2n(2C-B)-(B-C)n^2]-b_{n-4}E}{n(n-1)^2(n+2)}, \quad n \geq 4 \quad \text{A30} \]

The associated solution is:

\[ \hat{\psi}(R) = R(b_0[1 - \frac{C}{8} R^2 + \sum_{n=4}^{\infty} b_n R^n]_{n\text{ even}} \]
\[ + b_1 \left[ R - \frac{B}{20} R^2 + \sum_{n=5}^{\infty} b_n R^n \right] \]
\[ n \text{-odd} \]
\[ = R b_0 \left[ 1 - \frac{C}{8} R^2 + \sum_{n=4}^{\infty} b_n R^n \right] \]
\[ n \text{-even} \]
\[ + R^2 b_1 \left[ 1 - \frac{B}{20} R^2 + \sum_{n=5}^{\infty} b_n R^{n-1} \right] \]

It is clear that the above solution comprises two independent functions, one is generated by the even and the other by the odd terms of the same power series.

**Case \( s = -1 \)**

It is easily verified that

\[ b'_0 = b'_1 = 0 \text{ while } b'_2 \text{ and } b'_3 \text{ may remain arbitrary.} \]

The coefficient recursion formula is in this case:

\[ b'_n = \frac{b'_{n-2} \{(B-C)(n-3)(n-4)+(B+C)(n-3)\}+b'_{n-4}E}{n(n-2)(n-3)^2}, \]
\[ n \geq 4 \]

Furthermore,

\[ b'_4 = -b'_2 C/8, \quad b'_5 = -b'_3 B/20 \]
The solution is then,

\[ \hat{\psi}_2(R) = R^{-1} \{ b_2' R^2 - b_2 \frac{C}{8} R^4 + \sum_{n=6}^{\infty} b_n' R^n \} + \sum_{n=7}^{\infty} b_n' R^n \} \]  

\[ n\text{-even} \]

\[ n\text{-odd} \]

By comparing equations A32 and A30,

\[ b_n' = b_{n-2} \]  

\[ A34 \]

When this result is substituted into equation A33,

\[ \hat{\psi}_2(R) = b_2' R \{ 1 - \frac{C}{8} R^2 + \sum_{n=4}^{\infty} b_n R^n \} \]  

\[ n\text{-even} \]

\[ A35 \]

+ \[ b_3' R^2 \{ 1 - \frac{B}{20} R^2 + \sum_{n=5}^{\infty} b_n R^{n-1} \} \]  

A comparison of equations A35 and A31 reveals that since \( b_0', b_1', b_2' \), and \( b_3' \) are arbitrary constants, no new solution function is generated by this root.

Cases = 2, double root

One finds that \( b_1'' = 0, b_3'' = 0, b_2'' = -b_0'B/20 \) while
\( b_0'' \) may remain arbitrary. The coefficient recursion formula is

\[
b_n'' = \frac{b_{n-2}'' [(B-C)n(n-1)+(B-C)n-B] + b''_{n-4}E}{n^2(n+1)(n+3)}, \quad n \geq 4 \quad \text{A36}
\]

All the odd terms in the series vanish on account on \( b_1'' = 0 \). Furthermore, \( b_n'' = b_{n+1}'' \), and the solution is

\[
\hat{\psi}_3(R) = b_0'' R^2 \left(1 - \frac{B}{20} R + \sum_{n=4 \text{ even}}^{\infty} b_n R^{n-1} \right) \quad \text{A37}
\]

Due to the double root in the indicial equation, a second solution may exist

\[
\hat{\psi}_3'(R) = k \hat{\psi}_3(R) \log R + \sum_{n=0}^{\infty} b_n'' R^{n+2} \quad \text{A38}
\]

But \( k \equiv 0 \) since the solution must remain bounded at \( R = 0 \). Thus,

\[
\hat{\psi}_3'(R) = \sum_{n=0}^{\infty} b_n'' R^{n+2} = \hat{\psi}_3(R) \quad \text{A39}
\]

The function \( \hat{\psi}_3(R) \) is identical with one of the independent functions in \( \hat{\psi}_1(R) \). Therefore, only two independent functions exist which solve equation 5.116, the solution is given in A31.
Metering Section

Rotational velocity component

Substituting equation A20 into equation 5.118 gives:

\[
p_0(s-1)(s+1)R^s + p_1s(s+2)R^{s+1} + p_2(s+1)(s+3)R^{s+2}
\]

\[+ p_0\left(i\nu \left(s + \frac{4+3E}{4+E}\right)\beta_{56} - \nu^2 \beta_{57} - \frac{\Gamma Re}{\beta_{15}}\right)R^{s+2}\]

\[+ \sum_{n=3}^{\infty} \{p_n(n+s-1)(n+s+1)+p_{n-2}[i\nu(n+s-2+\frac{4+3E}{4+E})\beta_{56} - \nu^2 \beta_{57} - \frac{\Gamma Re}{\beta_{15}}] \}
\]

\[R^{n+s} = 0 \quad A40\]

The indicial equations have two roots:

s=1,-1

**Case s=1**

For this root \( p_1 = 0 \) while \( p_0 \) may remain arbitrary.

The associated solution is

\[
\hat{v}_1(R) = p_0R(1+p_2R^2 + \sum_{n=3}^{\infty} p_nR^n) \quad A41
\]

where
\[ p_2 = \frac{1}{8}[\Re e/\beta_{15} + \nu^2 \beta_{57} - i \nu \beta_{56}(8+4E)/(4+E)] \] 

and the coefficient recursion formula is

\[
p_n = \frac{\{\frac{\Re e}{\beta_{15}} + \nu^2 \beta_{57} - i \nu (n+2E/4+E) \beta_{56}\} p_{n-2} + \nu^2 (2+4E) \beta_{56} p_{n-3}}{n(n+2)}
\]

for \( n > 3 \)

Case \( s = -1 \)

In this case \( p_0' = p_1' = 0 \) while \( p_2' \) may remain arbitrary. The associated solution is:

\[
\hat{v}_2(R) = p_2'(R + \sum_{n=3}^{\infty} p_n' R^{n-1})
\]

\[
p_n' = \frac{p_{n-2}'\{\frac{\Re e}{\beta_{15}} + \nu^2 \beta_{57} - i \nu (n-2+2E/4+E) \beta_{56}\} + \nu^2 (2+4E) \beta_{56} p_{n-3}'}{n(n-2)}
\]

Furthermore,

\[ p_3' = 0, \quad p_4' = p_2' \quad \text{and} \quad p_n' = p_{n-2}' \]

Then,

\[
\hat{v}_2(R) = p_2' R(1+p_2 R + \sum_{n=4}^{\infty} p_{n-2} R^{n-2}) =
\]
Obviously, $\hat{v}_2(R) = \hat{v}_1(R)$ since $p_2'$ and $p_1'$ are arbitrary constants. Consequently, only one regular solution of equation 5.118 exists,

$$\hat{v}(R) = \hat{v}_1(R)$$

The function $\hat{\psi}(R)$

By substituting equation A20 into 5.119:

$$\sum_{n=0}^{\infty} q_n \{(n+s)(n+s-1)(n+s-2)(n+s-3)
\begin{align*}
&+ 2(n+s)(n+s-1)(n+s-2) - 3(n+s)-3\}R^{n+s} \\
&+ \sum_{n=0}^{\infty} q_n \{\beta_{70}(n+s)(n+s-1)(n+s-2) \\
&- \beta_{71}(n+s)(n+s-1) - \beta_{73}(n+s)+\beta_{77}\}R^{n+s+2} \\
&+ \sum_{n=0}^{\infty} q_n \{\beta_{78}-(n+s)\beta_{74}\}R^{n+s+3} \\
&+ \sum_{n=0}^{\infty} q_n \{\beta_{79}+\beta_{75}(n+s)-\beta_{72}(n+s)(n+s-1)\}R^{n+s+4}
\end{align*}$$
\[ + \sum_{n=0}^{\infty} q_n \frac{\beta_{74}}{2} R^{n+s+5} \]

\[ + \sum_{n=0}^{\infty} q_n \{\beta_{76}(n+s)+\beta_{81}\} R^{n+s+6} = 0 \quad \text{A48} \]

The indicial equation may easily be shown to be

\[(s-1)^2(s-3)(s+1) = 0 \quad \text{A49} \]

with the following roots:

\[ s = 1, 1, 3, -1 \]

**Case s=3**

The solution has the form:

\[ \hat{\psi}_1(R) = \sum_{n=0}^{\infty} q_n R^{n+3} \]

For this root, equation A48 becomes,

\[ q_o R^3 + 45q_1 R^4 + \{192q_2 + q_o (6\beta_{70} - 6\beta_{71} - 3\beta_{73} + \beta_{77})\} R^5 \]

\[ + \{525q_3 + q_1 (24\beta_{70} - 12\beta_{71} - 4\beta_{73} + \beta_{77})\} R^6 \]

\[ + q_o (\beta_{78} - 3\beta_{74}) R^7 \]
\[
+ \{1152q_4+q_2(6\beta_{70}-20\beta_{71}-5\beta_{73}-\beta_{77})\}
+ q_1(\beta_{78}-4\beta_{74})+q_o(\beta_{79}+3\beta_{75}-6\beta_{72})\}R^7
+ \{2205q_5+q_3(120\beta_{70}-30\beta_{71}-6\beta_{73}+8\beta_{77})\}
+ q_2(\beta_{78}-5\beta_{74})+q_1(\beta_{79}+4\beta_{75}-12\beta_{72})+\frac{q_o}{2}\beta_{74}\}R^8
+ \sum_{n=6}^{\infty} \{q_n(n+2)(n^3+4n^2+5n+5)+q_{n-2}[\beta_{70}n(n+1)(n-1)
- \beta_{71}n(n+1)+\beta_{77}]+q_{n-3}(\beta_{78}+n\beta_{74})
+ q_{n-4}[\beta_{79}+(n-1)\beta_{75}-(n-1)(n-2)\beta_{72}]+q_{n-5}\frac{\beta_{74}}{2}\}
+ q_{n-6}[\beta_{76}(n-3)+\beta_{81}]\}R^{n+3} = 0. \quad A50
\]

Non trivial solutions exist only if:

\( q_0 \) remains arbitrary,

\( q_1 = 0 \)

\( q_2 = q_o(6\beta_{71}+3\beta_{73}-6\beta_{70}-\beta_{77})/192 \)

\( q_3 = q_o(3\beta_{74}-\beta_{78})/525 \)

\( q_4 = \{q_2(20\beta_{71}+5\beta_{73}-60\beta_{70}-\beta_{77})+q_o(6\beta_{72}+3\beta_{75}-\beta_{79})\}/1152 \)
\[ q_5 = \{q_3(30\beta_{71} + 6\beta_{73} - 120\beta_{76} - \beta_{77}) + q_2(5\beta_{74} - \beta_{78}) - q_0 \frac{\beta_{74}}{2}\}/2205 \]

The coefficient recursion formula is:

\[ q_n = \{[(\beta_{71}n(n+1) + \beta_{73}(n+1) - \beta_{70}n(n^2 - 1) - \beta_{77})q_{n-2} + [n\beta_{74} - \beta_{78}]q_{n-3} + [(n-1)(n-2)\beta_{72}

- (n-1)\beta_{75} - \beta_{79}]q_{n-4} - \frac{\beta_{74}}{2} q_{n-5}

- [(n-3)\beta_{78} + \beta_{81}]q_{n-6}\}/[n(n+2)^2(n+4)] \]

It is clear that only one arbitrary constant is allowed. Thus, only one independent solution function is generated.

**Case s=1, double root**

Since the root \( s = 1 \) occurs twice in the indicial equation, it may generate two independent solution functions. They must, if they exist at all, be of the following form:

\[ \hat{\psi}_2(R) = k \log R \hat{\psi}_2^1(R) + \hat{\psi}_2^1(R) \]

where
\[ \hat{\psi}_2^1(R) = \sum_{n=0}^{\infty} q_n^{'R(n+1)} \] \hspace{1cm} A54

One solution function is generated by \( \hat{\psi}_2^1(R) \) and the other by \( k \log \hat{\psi}_2^1(R) \). They are obviously independent. However, since one insists on boundedness at \( R = 0 \), \( k = 0 \), and this root may, consequently, contribute only one additional solution function at the most.

\[ \hat{\psi}_2(R) = \hat{\psi}_2^1(R) = \sum_{n=0}^{\infty} q_n^{'R^{n+1}} \] \hspace{1cm} A55

The equation to satisfy is as follows:

\[ q_0^{'R\cdot0} - 3q_1^{'R^2} + q_2^{'R^3\cdot0} + 45q_3^{'R^4} + 192q_4^{'R^5} + 525q_5^{'R^6} + q_0^{'[\beta_{77}-\beta_{73}]R^3} + q_1^{'[\beta_{77}-2\beta_{73}-2\beta_{71}]R^4} + q_2^{'[\beta_{77}-3\beta_{73}-6\beta_{71}+6\beta_{70}]R^5} + q_3^{'[\beta_{77}-4\beta_{73}-12\beta_{71}-24\beta_{70}]R^6} + q_0^{'[\beta_{78}-\beta_{74}]R^4} + q_1^{'[\beta_{78}-2\beta_{74}]R^5} + q_2^{'[\beta_{78}-3\beta_{74}]R^6} + q_0^{'[\beta_{79}+\beta_{75}]R^5} + q_1^{'[\beta_{79}+2\beta_{75}-2\beta_{72}]R^6} + q_1^{'\frac{\beta_{74}}{2}R^6} \]
\[ + \sum_{n=6}^{\infty} q_n^2(n-2)(n+2) - q_{n-2}^0 \beta_7_0(n-1)(n-2)(n-3) \]

\[ - \beta_7_1(n-1)(n-2) - \beta_7_3(n-1) + \beta_7_7 \]

\[ + q_{n-3}^1 \beta_7_8(n-2) + q_{n-4}^1 \beta_7_9 + \beta_7_5(n-3) \]

\[ - \beta_7_2(n-3)(n-4) + q_{n-5}^1 \frac{\beta_7_4}{2} \]

\[ + q_{n-6}^1 \beta_7_6(n-5) + \beta_8_1 \}\mathbb{R}^{n+1} = 0 \]

After some reflections, one finds that the above equation requires the following conditions fulfilled for a solution to exist:

\[ q_0^1 = q_1^1 = q_3^1 = 0 \]

while \( q_2^1 \) may remain arbitrary. It is furthermore, easily verified that

\[ q_4^1 = q_2 q_2^1 / q_0, \quad q_5^1 = q_3 q_2^1 / q_0, \quad q_6^1 = q_4 q_2^1 / q_0, \]

\[ q_7^1 = q_5 q_2^1 / q_0 \]

and in general

\[ q_{n+2}^1 = q_n q_2^1 / q_0, \quad n \geq 6 \]

Thus
\[ \hat{\psi}_2(R) = \hat{\psi}_1(R) \text{ since } q_2^1 \text{ and } q_0 \text{ are arbitrary constants.} \]

No new solution functions are generated by this root.

**Case s = -1**

The solution must be of the form:

\[ \hat{\psi}_3(R) = \sum_{n=0}^{\infty} q_n^" R^{n-1} \]

The equation to satisfy is:

\[
q_o^" R^{-1.0} - 3q_1^" + q_2^" R^{0.0} - 3q_3^" R^2 + q_4^" R^{3.0} \\
+ q_5^" 95 R^4 + q_o^" [\beta^7_7 + \beta^7_3 - 2 \beta^7_1 + \beta^7_9 - 6] R \\
+ q_1^" \beta^7_7 R^2 + q_2^" [\beta^7_7 - \beta^7_3] R^3 \\
+ q_3^" [\beta^7_7 - 2 \beta^7_3 - 2 \beta^7_1] R^4 \\
+ q_4^" [\beta^7_8 + \beta^7_4] R^2 + q_1^" \beta^7_8 R^3 + q_2^" [\beta^7_8 - \beta^7_4] R^4 \\
+ q_o^" [\beta^7_9 - \beta^7_5 - 2 \beta^7_2] R^3 + q_1^" \beta^7_9 R^4 + q_o^" \beta^7_4 R^4 \\
+ \sum_{n=6}^{\infty} \left( q_n^" \right)^2 (n-2)(n-4) + q_2^" [\beta^7_0 (n-3)(n-4)(n-5) + q_3^" (n-2)(n-4)(n-5)] R^{n-5} \]
\[-\beta_{71}(n-3)(n-4) - \beta_{73}(n-3) + \beta_{77} + q_{n-3}''[\beta_{78} - (n-4)\beta_{74}] \]

\[+ q_{n-4}''[\beta_{79} + \beta_{75}(n-5) - \beta_{72}(n-5)(n-6)] + q_{n-5}'' \frac{\beta_{74}}{2} \]

\[+ q_{n-6}''[\beta_{76}(n-7) + \beta_{81}] \] \[R^{n-1} = 0 \] \[\text{(A58)} \]

The conditions for non-trivial solutions of the above equation are:

\[q_0'' = q_1'' = q_2'' = q_3'' = q_5'' = 0\]

while only \(q_4''\) may remain arbitrary.

The coefficient recursion formula is for this case:

\[q_n'' = \{q_{n-2}''[\beta_{73}(n-3) - \beta_{71}(n-3)(n-4) - \beta_{70}(n-3)(n-4)(n-5) - \beta_{77}] \]

\[+ q_{n-3}''[\beta_{74}(n-4) - \beta_{78}] + q_{n-4}''[\beta_{72}(n-5)(n-6)] \]

\[- \beta_{75}(n-5) - \beta_{79} - q_{n-5}'' \frac{\beta_{74}}{2} \]

\[- q_{n-6}''[\beta_{76}(n-7) + \beta_{81}] / [n(n-2)^2(n-4)] \]

\[n \geq 6.\]

It is easily proved that
\[ q_6'' = q_4''q_2/q_0, \quad q_7'' = q_4''q_3/q_0, \quad q_8'' = q_4''q_4/q_0, \]

\[ q_9'' = q_4''q_5/q_0 \]

and in general,

\[ q_{n+4}'' = q_4''q_n/q_0 \]

Thus,

\[ \hat{\psi}_3(R) = \hat{\psi}_1(R) \] since \( q_0 \) and \( q_4'' \) are arbitrary constants.

No additional solution functions are generated by this root.
APPENDIX 3

LIMITING RESULTS FOR THE FUNCTION $\hat{\psi}(R)$ OF
THE DRAW-DOWN SECTION AS $\Lambda$ TENDS TO
VERY SMALL VALUES

The case describes a situation in which $L >> R_0$, $R_0 \neq 0$. When $L$ becomes very large, the diameter of the extrudate must necessarily tend to very small values, if the extension rate remains non-zero. By equation 5.133

$$\bar{R} = \mathcal{O}(L^{-1/2})$$ \hspace{1cm} A59

Considerations on convergence of the solution functions $S_1$ and $S_2$ of equation 5.167 suggests the following postulate:

$$(\Lambda \nu)^2 = \mathcal{O}(L)$$ \hspace{1cm} A60

An order of magnitude analysis provides the following results,

$$K_1 = \mathcal{O}(L^3)$$

$$K_2 = \mathcal{O}(L^{1/2})$$

$$K_3 = \mathcal{O}(L)$$ \hspace{1cm} A61
The definitions of the quantities $K_1$, $K_2$ and $K_3$ are given by equations 5.164, 5.165 and 5.166. When dividing equation 5.163 by $K_1$ using the limit

$$\lim_{\Lambda \to 0} \left( \frac{K_2}{K_1} \right) = 0$$

one obtains

$$\frac{dS_2}{dR} \left( \frac{d^2S_1}{dR^2} + \frac{1}{R} \frac{dS_1}{dR} + K_3 \frac{S_1}{R^2} \right)$$

$$- \frac{dS_1}{dR} \left( \frac{d^2S_2}{dR^2} + \frac{1}{R} \frac{dS_2}{dR} + K_3 \frac{S_2}{R^2} \right) = 0$$

$$R = \overline{R} \quad \text{A62}$$

It is now assumed that the solution functions converge sufficiently fast that

$$b_6 \overline{R}^6 = \mathcal{O}(0) \quad \text{A63}$$

which implies that

$$S_1 = R(1+b_2R^2+b_4R^4), \quad S_2 = R^2(1+b_3R^2+b_5R^4) \quad \text{A64}$$

Combining equations A64 and A62,
\[
[1, b_2 \bar{R}^2, b_4 \bar{R}^4] \begin{bmatrix} 1 + K_3 \bar{R}^2 \\ 9 + K_3 \bar{R}^2 \\ 25 + K_3 \bar{R}^2 \end{bmatrix} \quad [2, 4, 6] \begin{bmatrix} 1 \\ b_3 \bar{R}^2 \\ b_5 \bar{R}^4 \end{bmatrix} = \\
[1, b_2 \bar{R}^2, b_4 \bar{R}^4] \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad [4+K_3 \bar{R}^2, 16+K_3 \bar{R}^2, 36+K_3 \bar{R}^{-4}] \begin{bmatrix} 1 \\ b_3 \bar{R}^2 \\ b_5 \bar{R}^4 \end{bmatrix}
\]

where the quantities in brackets represent vectors.

It is, in view of equation A63, reasonable to assume that

\[
b_4 b_5 \bar{R}^8 = \mathcal{O}(0)
\]

Equation A65 then becomes:

\[
(-2+K_3 \bar{R}^2) + b_2 (6-K_3 \bar{R}^2) \bar{R}^2 - (16+K_3 \bar{R}^2) \bar{R}^2 b_3 + (30-3K_3 \bar{R}^2) \bar{R}^4 b_4 - (36+K_3 \bar{R}^2) \bar{R}^4 b_5 + b_2 \{b_3 (-12+K_3 \bar{R}^2) \bar{R}^4 + b_5 (-54+3K_3 \bar{R}^2) \bar{R}^6 \} + b_3 b_4 (20-K_3 \bar{R}^2) \bar{R}^6 = 0
\]

A67
By order of magnitude considerations,

\[ b_2 R^2 = \mathcal{O}(L^0) \]

\[ b_3 R^2 = \mathcal{O}(L^{-4}) \]

\[ b_4 R^4 = \mathcal{O}(L^0) \]

\[ b_5 R^4 = \mathcal{O}(L^{-3} \text{Re}) \] A68

The quantities \( N_p^1 \), \( N_p^6 \) and \( N_p^7 \) are complicated functions of \( \kappa \), \( E \) and \( \Gamma_I \). Considerable simplifications are achieved if one restricts the analysis to cases in which

i) \( \kappa \ll 1 \)

ii) \( |\Gamma_I| \gg 1 \)

The following reduction of typical relationships are then possible:

\[
\frac{1}{(1 + a \kappa)(1 + b \kappa + i \kappa \Gamma_I)} = \frac{1 + a \kappa}{1 + i \kappa \Gamma_I} \]

A69

\[ a = \pm 1 \text{ or } \pm 2, \]

\[ b = \pm 1 \text{ or } \pm 2 \]

\[ 2N_p^1 - N_p^6 = \frac{1 + 2 \kappa (1 + \frac{E}{2}) - i \kappa \Gamma_I (1 + 2 \kappa (1 + \frac{E}{2}))}{1 + (\kappa \Gamma_I)^2} \]
\[
N_p^6 = \frac{[1+2 \kappa (1+E/2)][1-i \kappa E]}{1+(\kappa \Gamma_1)^2}
\]

\[
N_p^7 = \frac{1+\kappa E-i \kappa \Gamma_1(1+\kappa E)}{1+(\kappa \Gamma_1)^2}
\]

By order of magnitude considerations it is clear that the following terms in equation A67 are of negligible magnitudes when compared to the other terms,

\[
b_3 R^4 = b_2 b_3 R^4 = b_2 b_5 R^6 = b_3 b_4 R^6 = \mathcal{O}(0)
\]

Equation A67 is now reduced to

\[
-2 + K_3 R^2 + b_2 (6-K_3 R^2) R^2 + (30-3K_3 R^2) R^4 b_4

- (36+K_3 R^2) R^4 b_5 = 0
\]

Equation A72 comprises a real and an imaginary part since the quantities $K_3$, $b_2$ and $b_5$ are all complex. The real as well as the imaginary parts must vanish for the equation to be satisfied. Consequently, one parameter besides Reynolds's number must remain restricted. The wavenumber, $\nu$, was chosen for reasons given in Chapter 6.

The real part of equation A72 is clearly dominated by the first and second terms and vanishes approximately if the following equation is satisfied:
\[ x^4 + x^2 - 24 = 0 \]

where

\[ x = \Lambda \sqrt{\frac{1 + 2 \kappa \frac{1 + E/2}{1 + \kappa E}}{1 + 2 \kappa (1 + E/2)}} \]

The solution of equation A73 is:

\[ (\Lambda \nu)^2 = \frac{4.35}{R^2} \cdot \frac{1 + \kappa E}{1 + 2 \kappa (1 + E/2)} \]

This equation provides always a solution since, by assumption, \( \kappa \ll 1 \) and for physical systems \( |E| \ll 1 \). When using equation A59, it is seen that equation A75 predicts

\[ (\Lambda \nu)^2 = \mathcal{O}(L) \Rightarrow \lim_{\Lambda \text{ very small}} \nu^2 \to \infty \]

which is precisely the postulate made at the outset of this analysis. It is noted that equation A75 does not discriminate the sign of the wave-number. This is a reasonable result, often found in hydrodynamical and rheological stability analysis.

Substituting equation A75 into A72 and equating the imaginary part to zero, yields the following equation:

\[ \left[ -\frac{3(1 + \kappa E)}{1 + (\kappa \tau_I)^2} + \frac{4.35(1 + \kappa E)}{1 + (\kappa \tau_I)^2} \right] \left[ \frac{\kappa \tau_I(1 + \kappa E)}{1 + (\kappa \tau_I)^2} \right] \]
Employing the redefined Reynolds's number of equation 5.198, one obtains the final result,

\[
\text{Re} = \frac{100 \left[ \frac{1+\kappa E}{1+(\kappa \Gamma_I)^2} \right]^2 \left[ \frac{0.2-11.2 \kappa (1+E/2)}{1+2 \kappa (1+E/2)} \right] [\overline{V}_o + 2]}{\Lambda \left[ 9.05 - 6.09 \frac{1+\kappa E}{1+(\kappa \Gamma_I)^2} \right]}
\]

A78