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COMPANIONSHIP OF KNOTS AND
THE SMITH CONJECTURE

by

John Robert Myers

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIRED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's Signature

Houston, Texas

May, 1977
This thesis is dedicated
to my parents
Robert William Myers
and
Margaret Palmour Myers
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Introduction

The Smith Conjecture states that no periodic PL homeomorphism of $S^3$ can have a knotted simple closed curve as fixed point set. The theory of companionship of knots studies ways in which a given knot can be constructed from "smaller" knots in a manner generalizing the familiar process of composition of knots. This paper investigates the Smith Conjecture in terms of the companionship structure of knots. The main results relate the problem of whether a given knot can be the fixed point set of a periodic PL homeomorphism of $S^3$ to the same problem for the "smaller" knots of which it is constructed. For several well-known classes of knots the problem is completely solved.

We first discuss the history of the problem. In 1939 P. A. Smith [37] proved that the fixed point set $K$ of a periodic homeomorphism of $S^3$ is always homeomorphic to a sphere of some dimension $r$, $-1 \leq r \leq 2$. Thus if $r = 1$, $K$ is a simple closed curve and one can consider the placement problem for $K$ in $S^3$: Is $K$ knotted? In 1954 Montgomery and Zippin [26] constructed a topological involution $h$ with $K$ a wild knot, and in 1964 Bing [2] constructed topological homeomorphisms of arbitrary period with $K$ a wild knot. But if the homeomorphism is required to be piecewise linear (PL) then $K$ is tame and no examples are known with $K$ knotted. The Smith Conjecture says that there are none. In 1962 Moise [24] showed that, for $h$ PL,
K is unknotted if and only if h is conjugate to a rotation. A shorter proof was also given by Smith [38] in 1964. It follows from the earlier work of Smith that it suffices to consider only homeomorphisms of prime period.

One approach to the Smith Conjecture is to consider various classes of (tame) knots K and periods p and try to show that there is no PL action of period p having K as fixed point set. The first positive result using this approach was obtained in 1955 by Montgomery and Samelson [25], who proved the conjecture for K a 2-strand cable and p = 2. A quick sketch of their proof can be found in Fox [12]. In 1958 Kinoshita [21] and Fox [9], [12] found conditions on the Alexander polynomial of K which allowed them to rule out several combinations of K and p. In particular they obtained a new proof of the Montgomery-Samelson result and a partial solution for torus knots. In 1964 Giffen [14] proved the conjecture for K a torus knot and p arbitrary. A simpler, group theoretic proof of this theorem was given in 1966 by Fox [13]. The single most important advance was made in 1969 by Waldhausen [42], who proved the conjecture for K arbitrary and p even. In 1976 Cappell and Shaneson [6] proved the conjecture for K a 2-bridge knot and p arbitrary.

Waldhausen's methods do not seem to generalize to odd periods, while the proofs for torus and 2-bridge knots depend on properties special to those knots, so that a
general solution is not yet in sight. The Kishoshita-Fox conditions are the most general results known and are useful for ruling out particular knots and periods, but they are difficult to apply to large classes of knots; they fail, for example, to prove the Giffen and Cappell-Shaneson theorems because they give only partial information. For knots with trivial Alexander polynomial (e.g., untwisted doubled knots) they give no information at all.

We now discuss companionship of knots. Classical knot theory has several ways of constructing new knots from old ones, the most familiar being the construction of composite knots (Produktknoten). A knot is called prime if it is not composite. Every knot has a unique factorization as a product of finitely many prime knots. Thus the prime knots can be regarded as building blocks for knot theory. Other classical constructions include those of cabled knots (Schlauchknoten) and the doubled knots created by Whitehead [43] and investigated by Seifert [34]. The doubled knots are particularly interesting in that doubling a knot destroys its Alexander invariants. In a sequence of papers [30], [31], [32], [33] in the 1950's Schubert developed a general theory of "companionship" of knots which includes these constructions as special cases and introduced new examples, including a generalization of cabled knots called cable braids (Schlauchzöpfe). The general construction is roughly as follows. Take a simple closed curve \( L \) lying in an unknotted solid torus \( \mathbb{T}^3 \) in a non-trivial fashion.
Then tie $V$ into a knotted solid torus $W$ with core $K$ and let $J$ be the image of $L$ in $W$. Then $K$ is called a companion of $J$, denoted $K < J$. A knot with no companion is called simple. Companionship induces a partial ordering on the set of (non-trivial) knot types with the simple knot types as minimal elements. A given knot has only finitely many companions, each appearing with a finite "multiplicity". One might regard the simple knots as the basic building blocks of knot theory. They are more basic than prime knots since prime knots may have companions.

Our general program is to reduce the Smith Conjecture as far as is possible to the study of simple knots. Our first result in this direction is a reduction to the study of prime knots: If the Smith Conjecture is true for prime knots, then it is true for all knots (Corollary III.1.2). If we restrict our attention to fibered knots then the program succeeds completely: If the Smith Conjecture is true for simple fibered knots, then it is true for all fibered knots (Corollary V.4.4). Moreover, these statements hold for the "period $p$ Smith Conjecture" for each prime $p$ (Theorem III.1.1 and Theorem V.4.4). Throughout this paper we consider only prime periods, which, by the work of Smith, is no loss of generality.

Our general technique is, given an action with fixed point set $J$, to try "splitting" it into a cyclic action with fixed point set $K$ and a cyclic action with fixed point set $L$. If the multiplicity of $K$ is less than the
period this can always be done (Theorem III.5.1 and Theorem III.5.3). We say that the action with fixed point set $K$ is obtained by "Type I surgery". If the multiplicity of $K$ is greater than or equal to the period then the action cannot be split in the fashion described above, so we must use a more complicated process called "Type II surgery", which gives a new cyclic action with a new fixed point set, which is not necessarily $L$. We perform a sequence of these two surgeries as long as we have companions of the fixed point sets. In the case of fibered knots it is proven that the sequence terminates after a finite number of steps with a (non-trivial) fibered knot, which must then be simple.

For certain choices of $L$ in $V$ we can always split the action, no matter how we choose $K$. This enables us to prove the conjecture for several familiar classes of knots. The Smith Conjecture is true for all cabled knots (Theorem IV.1.3), doubled knots (Theorem IV.2.5), and cable braids (Theorem IV.3.2). In addition, we prove the Smith Conjecture for the class of knots, discovered by Lyon [22], with the property that they are non-fibered but have unique isotopy type of incompressible spanning surface (Theorem IV.2.2), thus giving a new proof for twist knots (Corollary IV.2.3). Finally, we prove the Smith Conjecture for all non-simple knots with bridge number less than five (Theorem V.5.2).
Chapter I: Preliminaries

1.1 Notations and conventions

Throughout this paper we shall work exclusively in the PL category. All manifolds and submanifolds are polyhedral. All maps are piecewise linear.

A manifold may or may not have a boundary; it is assumed compact unless stated otherwise or the non-compactness arises from some construction (e.g., coverings). All 3-manifolds are assumed orientable, while 2-manifolds are assumed orientable unless the contrary is explicitly stated.

A surface is a connected 2-manifold. The boundary and interior of a manifold $M$ are denoted by $\partial M$ and $\text{Int}(M)$, respectively. The topological boundary, interior, and closure of a subset $Y$ of a space $X$ are denoted by $\text{fr}_X(Y)$, $\text{int}_X(Y)$, and $\text{cl}_X(Y)$, respectively. When the context permits, reference to the ambient space $X$ will be suppressed. A codimension-one submanifold $F$ of a manifold $M$ is properly embedded if $F \cap \partial M = \partial F$.

Let $M$ be a 3-manifold and $F$ a surface. $F$ is a surface in $M$ if $F$ is properly embedded in $M$. $F$ is a surface in $\partial M$ if $F$ is a submanifold of $\partial M$.

If $Q$ and $M$ are manifolds and $g: Q \to M$ is an embedding, then we shall usually identify $Q$ with its image in $M$ and suppress reference to the embedding map $g$. Sometimes, however, we shall need to retain $g$ and the
distinction between \( Q \) and its image.

Let \( X \) and \( Y \) be spaces and \( A \) a subset of \( X \). Let \( f: A \to Y \) be a homeomorphism onto its image. Then \( X \cup Y \) denotes the space obtained by identifying each \( x \in A \) with \( f(x) \in Y \). If \( X \) is a space, \( A_1, A_2 \) disjoint subspaces of \( X \), and \( f: A_1 \to A_2 \) a homeomorphism, then \( X/f \) denotes the space obtained by identifying \( x \in A_1 \) with \( f(x) \in A_2 \).

Let \( P \) be a compact polyhedron in the manifold \( M \). A regular neighborhood \( N \) of \( P \) in \( M \) is the closed star of \( P \) in the second barycentric subdivision of some triangulation of \( M \) in which \( P \) is triangulated as a subcomplex. If \( P \cap \partial M \neq \emptyset \), then we require that \( N \cap \partial M \) be a regular neighborhood of \( P \cap \partial M \) in \( \partial M \).

Let \( F \) be a surface in a 3-manifold \( M \). The manifold \( M' \) obtained by splitting \( M \) along \( F \) has by definition the property that \( \partial M' \) contains two disjoint copies \( F_1 \) and \( F_2 \) of \( F \) such that \( M = M'/f \) for some homeomorphism \( f: F_1 \to F_2 \) and the image of \( F_1 \) and \( F_2 \) in \( M \) is \( F \). Note that \( M' \) is homeomorphic to \( M - \text{int}(N) \), where \( N \) is a regular neighborhood of \( F \) in \( M \).

Let \( I \) denote the unit interval \( [0,1] \).

Let \( X \) be a space and \( Y \) a subspace of \( X \). An isotopy of \( X \) rel \( Y \) is a map \( F: X \times I \to X \) such that, for \( f_t = F|(X \times \{t\}) \), \( f_t \) is a homeomorphism, \( f_t|Y \) is the identity map of \( Y \), and \( f_0 \) is the identity map of \( X \).
Two subspaces $Z_1$ and $Z_2$ of $X$ are isotopic rel $Y$ if there is an isotopy $F$ of $X$ rel $Y$ such that $f_1(Z_1) = Z_2$.

Let $M$ be a 3-manifold and $F, G$ surfaces either in $M$ or $\partial M$. $F$ is parallel to $G$ in $M$ if there is an embedding of $F \times I$ in $M$ such that $F \times \{0\} = F$ and $\partial (F \times I) - \text{Int}(F \times \{0\}) = G$. $F$ is parallel to $G$ mod $\partial M$ if there is an embedding of $F \times I$ in $M$ such that $F \times \{0\} = F$, $F \times \{1\} = G$, and $\partial F \times I \subseteq \partial M$. Note that these two relations are symmetric in $F$ and $G$.

An arc is a 1-cell; a disc is a 2-cell; a ball is a 3-cell. A simple closed curve is a 1-sphere.

We denote the $n$-sphere by $S^n$. We shall sometimes regard $S^1$ as the unit circle in the complex plane and a 2-cell as the unit disc $D^2$ in the complex plane. We denote $S^1 \times S^1$ by $T^2$.

A 3-manifold $V$ is a solid torus if there is a disc $D$ in $V$ such that the manifold $V'$ obtained by splitting $V$ along $D$ is a ball. $D$ is a meridional disc of $V$ and $\partial D$ a meridian of $V$. It is well known that a solid torus is homeomorphic to $S^1 \times D^2$. A simple closed curve $c \subseteq \text{Int}(V)$ is a core of $V$ if there is a product structure $S^1 \times D^2$ on $V$ such that $c = S^1 \times \{0\}$.

A 3-manifold $\Sigma^3$ is a homotopy 3-sphere if it is closed and simply connected. A 3-manifold $B$ is a homotopy 3-cell if $\partial B$ is a 2-sphere and $B$ is simply connected. A 3-manifold $W$ is a homotopy-solid-torus if
there is a disc \( D \) in \( W \) such that the manifold \( V' \) obtained by splitting \( W \) along \( D \) is a homotopy 3-cell. Any 3-manifold having torus boundary and infinite cyclic fundamental group is a homotopy solid torus (see, for example, [17, p. 59]). If \( B \) is a homotopy 3-cell which is not a 3-cell then \( B \) is a \textbf{fake 3-cell}.

Let \( X \) be a space with basepoint \( x_0 \). If \( \alpha \) is an oriented simple closed curve containing \( x_0 \), then \( \alpha \) determines an element in \( \pi_1(X,x_0) \) which we denote by \([\alpha]\). Generally, when the context permits, we shall suppress reference to the basepoint when dealing with fundamental groups. Unlabelled or unspecified homomorphisms between fundamental groups are induced by inclusion maps. Let \( Y \) be a connected subset of a space \( X \). \( Y \) is \textbf{injective} in \( X \) if \( \pi_1(Y) \to \pi_1(X) \) is monic; \( Y \) is \textbf{surjective} in \( X \) if \( \pi_1(Y) \to \pi_1(X) \) is epic.

If \( G \) is a group and \( S \) a set of elements of \( G \), then \( \text{gp}(S;G) \) and \( \text{nm}(S;G) \) denote, respectively, the subgroup and normal subgroup of \( G \) generated by \( S \). \( G' \) denotes the commutator subgroup of \( G \). Our notation for presentations of groups is the standard notation used in [23].

Let \( X \) be a space. A homeomorphism \( h \) of \( X \) is \textbf{periodic} if \( h^n \) is the identity map for some integer \( n \) and \( h \) is not itself the identity map. The least positive integer \( p \) for which \( h^p \) is the identity is the \textbf{period} of \( h \). We denote by \( \langle h \rangle \) the subgroup generated by \( h \)
in the group of all homeomorphisms of $X$, and call $\langle h \rangle$ a cyclic action of order $p$ on $X$. Two cyclic actions $\langle h_1 \rangle$ and $\langle h_2 \rangle$ are equivalent if they are conjugate in the group of all homeomorphisms of $X$. $X/\langle h \rangle$ denotes the space obtained by identifying $x$ and $h(x)$ for all $x$ in $X$. The map $q: X \rightarrow X/\langle h \rangle$ sending $x$ to its equivalence class is called the quotient map of $\langle h \rangle$. The set of points $x$ in $X$ such that $h^n(x) = x$ for all $n$ is called the fixed point set of $\langle h \rangle$ and is denoted $\text{Fix}(h)$. 
1. **Definition:** Give $S^3$ a fixed orientation. Let $K$ be an oriented simple closed curve in $S^3$. $K$ is **unknotted** and is called an **unknot** if $K$ is the boundary of some disc embedded in $S^3$. If $K$ is not the boundary of such a disc then $K$ is **knotted** and is called a **knot**. The knots $K_1$ and $K_2$ are **equivalent** or have the same **knot type** if there is an orientation preserving homeomorphism $f$ of $S^3$ such that $f(K_1) = K_2$ and $f$ takes the orientation of $K_1$ to that of $K_2$.

2. **Definition:** A solid torus $V$ in $S^3$ is **unknotted** if $S^3 - \text{int}(V)$ is a solid torus; otherwise $V$ is **knotted**. A simple closed curve $\mu$ on $\partial V$ is a **meridian** if $\mu$ bounds a disc in $V$ but not in $\partial V$; a simple closed curve $\lambda$ on $\partial V$ is a **longitude** if $\lambda$ bounds a surface in $S^3 - \text{int}(V)$ but not in $\partial V$. A **meridian-longitude pair** $(\mu, \lambda)$ consists of an oriented meridian $\mu$ and oriented longitude $\lambda$ which intersect transversely in a single point. If a core $c$ of $V$ is given an orientation, then a meridian-longitude pair $(\mu, \lambda)$ is **canonical** (with respect to $c$) if $\mu$ and $c$ have linking number $+1$ and $\lambda$ is homologous to $c$ in $V$. If $V$ and $W$ are solid tori in $S^3$ having oriented cores then a homeomorphism $f: V \to W$ is **faithful** if it takes a canonical meridian-longitude pair for $V$ to one for $W$.

3. **Definition:** Let $K$ be a knot in $S^3$ and $N$ a
regular neighborhood of $K$ in $S^3$. Then $C(K) = S^3 - \text{int}(N)$ is a knot space of $K$. A meridian-longitude pair of $C(K)$ is a meridian-longitude pair of the solid torus $N$. Any manifold homeomorphic to $C(K)$ is called a $K$-knot space.

4. **Remarks:** A solid torus $V$ in $S^3$ is knotted if and only if its core is knotted. Two oriented cores homologous in $V$ have the same knot type. Equivalent knots have knot spaces which are homeomorphic via a homeomorphism which preserves canonical meridian-longitude pairs. Conversely, given such a homeomorphism, the knots are equivalent. If a solid torus $W$ is attached to a knot space $C(K)$ in such a way that a meridian of $W$ is identified to a meridian of $C(K)$, then the resulting manifold is a 3-sphere and any properly oriented core of $W$ is equivalent to $K$.

5. **Definition:** A toral solid is a 3-manifold homeomorphic either to a solid torus or to a knot space.

6. **Lemma:** Let $T$ be a torus in $S^3$. Then $S^3 - T$ has two components whose closures $X$ and $Y$ are toral solids at least one of which is a solid torus.

**Proof:** The fact that, say, $Y$ is a solid torus is due to Alexander [1]. If a core $K$ of $Y$ is unknotted then $X$ is a $K$-knot space.

7. **Definition:** Let $K$ be a knot in $S^3$. A spanning surface $F$ of $K$ is a compact orientable surface embedded in $S^3$ such that $\partial F = K$. The genus of $K$, denoted $g(K)$, is the minimum of the genera of all the
spanning surfaces of $K$.

8. **Remarks:** Every knot has spanning surfaces [11], [28]; thus $g(K)$ is always defined; since $K$ is a knot $g(K) \neq 0$. Given any spanning surface $F$ of $K$ there is a regular neighborhood $N$ of $K$ in $S^3$ such that $F \cap N$ is an annulus $A$ with $\text{Int}(A) \subseteq \text{Int}(N)$ and $\partial A = K \cup \lambda$, $\lambda$ a longitude of $K$. $F' = F \cap C(K)$ is a surface in $C(K)$ with $\partial F'$ a longitude of $C(K)$. Given any such surface $F'$ it can be extended to a spanning surface $F$ of $K$.

9. **Definition:** Let $K$ be a knot in $S^3$ with knot space $C(K)$. Then $G(K) = \pi_1(C(K))$ is the **knot group** of $K$. If $(\mu, \lambda)$ is a meridian-longitude pair for $C(K)$, then set $x_0 = \mu \cap \lambda$ and let $m = [\mu], \ i = [\lambda]$. $(m, i)$ is a **meridian-longitude pair in** $G(K)$. $P(K) = \text{gp}(m, i; G(K))$ is the **peripheral subgroup** of $G(K)$.

10. **Notation:** If $K$ is a simple closed curve in $S^3$, then $\Delta_K(t)$ denotes the **Alexander polynomial** of $K$, normalized so that $\Delta_K(0)$ is positive.

11. **Remarks:** $\Delta_K(t)$ is a polynomial in $t$ with integer coefficients and degree $2h$. It is an invariant of knot type and has the properties that $\Delta_K(1) = \pm 1$ and $\Delta_K(t) = t^{2h}\Delta_K(1/t)$. If $K$ is an unknot then $\Delta_K(t) = 1$. For the definition of $\Delta_K(t)$ and proofs of these properties see [27] or [28].

12. **Definition:** A knot $K$ in $S^3$ is **fibered** if there is a fiber bundle map $f: S^3 - K \to S^1$ and a regular
neighborhood \( N \) of \( K \) in \( S^3 \) having a product structure \( S^1 \times D^2 \) such that \( (f|(N - \kappa))(e^{i\varphi}, re^{i\varphi}) = e^{i\varphi} \).

13. **Remarks:** If \( K \) is fibered then it has a knot space \( C(K) \) such that \( f|C(K) : C(K) \rightarrow S^1 \) is a fiber bundle with fiber a compact surface \( F \) with \( \partial F \) a longitude of \( K \). Conversely, given such a fibering of \( C(K) \) we can extend the fibration to \( N - K \) in such a way that \( K \) is a fibered knot. For a fibered knot \( (G(K))' = \pi_1(F) \) [28].

14. **Definition:** A link \( L \) in \( S^3 \) is the union of an ordered set of oriented disjoint simple closed curves \( K_1, \ldots, K_n \) in \( S^3 \). \( L \) is **trivial** if there are disjoint discs \( D_1, \ldots, D_n \) embedded in \( S^3 \) such that \( \partial D_j = K_j \), \( j = 1, \ldots, n \).

15. **Remarks:** If we replace \( S^3 \) by a homotopy 3-sphere \( \Sigma^3 \) in all of the preceding statements in this section, we immediately obtain, with some exceptions which we now discuss, true statements about knots in \( \Sigma^3 \). We modify Definition 2 so that a solid torus \( V \) is unknotted if \( \Sigma^3 - \text{int}(V) \) is a homotopy solid torus. We reserve the term **toral solid** in Definition 5 for the case \( \Sigma^3 = S^3 \). If \( W \) is a homotopy solid torus in \( \Sigma^3 \), then we define knottedness, meridians, longitudes, etc. exactly as in the case of a solid torus in \( \Sigma^3 \). However, if \( W \) is knotted, we do not call \( \Sigma^3 - \text{int}(W) \) a knot space; we call it a **knot manifold** and define its meridians and longitudes to be those of \( W \). Note that we cannot in general define a core of a homotopy solid torus, so that a knot manifold may not
be a knot space. We now give a replacement for Lemma 6.

16. Lemma: Let \( T \) be a torus in a homotopy sphere \( \Sigma^3 \). Then \( \Sigma^3 - T \) has two components with closures \( X \) and \( Y \) such that \( Y \) is a homotopy solid torus and \( X \) is either a homotopy solid torus or a knot manifold.

Proof: Since \( \pi_1(T) \to \pi_1(\Sigma^3) \) is trivial, the loop theorem implies the existence of a disc \( D \) in \( \Sigma^3 \) with \( D \cap T = \partial D \) a non-contractible simple closed curve in \( T \). We may suppose that \( D \subseteq Y \). Let \( N \) be a regular neighborhood of \( D \cup \partial Y \) in \( Y \) and let \( B = Y - \text{int}(N) \). By the Seifert-van Kampen theorem \( B \) is a homotopy 3-cell. It follows that \( Y \) is a homotopy solid torus. If \( Y \) is un-knotted \( X \) is a homotopy solid torus. If \( Y \) is knotted \( X \) is a knot manifold.
I.3 Companionship of knots

1. **Definition:** Let \( K_1 \) and \( K_2 \) be knots in \( S^3 \). Suppose \( S \) is a 2-sphere in \( S^3 \) and the closures of the components of \( S^3 - S \) are balls \( B_1 \) and \( B_2 \) with \( K_1 \subseteq B_1 \), \( K_2 \subseteq B_2 \) and \( K_1 \cap S = K_2 \cap S = \alpha \) an arc on which \( K_1 \) and \( K_2 \) induce opposite orientations. Then \( K = K_1 \cup K_2 - \text{Int}(\alpha) \) is the **composite** of \( K_1 \) and \( K_2 \). A knot which is not composite is **prime**.

2. **Lemma:** The composite of two knots is a knot. Every knot is either prime or the composite of a finite number of prime knots; this decomposition is unique up to the knot type and order of composition of the prime factors.

   **Proof:** These are the main results of [30].

3. **Definition:** Let \( V \) be a solid torus in \( S^3 \) with oriented core \( c \). If \( K \subseteq \text{int}(V) \) is an oriented simple closed curve, then \( K \) is homologous in \( V \) to some multiple of \( c \). The absolute value of this multiple is the **winding number** of \( K \) in \( V \) and is denoted by \( u_V(K) \).

   Each meridional disc \( D \) of \( V \) which is in general position with respect to \( K \) intersects \( K \) in a finite number of points. The minimum such number taken over all such discs is the **order** of \( V \) with respect to \( K \), denoted \( o_V(K) \). Note that \( u_V(K) \leq o_V(K) \).

   If \( K_1 \) and \( K_2 \) are two oriented simple closed curves in \( V \) then they are **equivalent** if there is a faithful
homeomorphism \( f: V \to V \) such that \( f(K_1) = K_2 \) and takes the orientation of \( K_1 \) to that of \( K_2 \).

4. **Definition:** Let \( J \) be a knot in \( S^3 \) and \( W \) a knotted solid torus in \( S^3 \) such that \( J \subseteq \text{int}(W) \). Let \( K \) be a core of \( W \) oriented so that \( J \) is homologous in \( W \) to a non-negative multiple of \( K \). If \( o_W(J) \neq 0 \) and \( J \) is not a core of \( W \), then \( K \) is a **companion** of \( J \), denoted \( K < J \). \( o_W(J) \) is the **order** of \( K \). A knot with no companions is **simple**.

Let \( J \) be a knot and \( W_0, \ldots, W_{m-1} \) a set of solid tori in \( S^3 \) with oriented cores \( K_0, \ldots, K_{m-1} \). This set is **subordinate** to \( J \) with order \( \alpha \) if for each \( j, j = 0, \ldots, m-1 \), \( K_j \) is a companion of \( J \) of order \( \alpha \) and \( S^3 - \text{int}(W_i) \subseteq \text{int}(W_j) \) for all \( i \neq j \). A companion \( K \) of \( J \) of order \( \alpha \) has **multiplicity** \( m \) if there is a set of \( m \) solid tori subordinate to \( J \) with order \( \alpha \) whose cores have the knot type of \( K \) but there is no such set of \( m+1 \) solid tori.

5. **Remarks:** If \( J \) is a composite knot, then the factors of \( J \) are companions of \( J \) (Satz 2 on p. 198 of [31]), and the companions of \( J \) are, up to knot type, precisely the factors of \( J \) and their companions (Satz 2 on p. 232 of [31]).

6. **Construction:** Let \( K \) be a knot with regular neighborhood \( W \). Let \( V \) be an unknotted solid torus in \( S^3 \) having oriented core. Let \( L \subseteq \text{int}(V) \) be an oriented simple closed curve such that \( o_V(L) \neq 0 \) and \( L \) is not a
core of \( V \). Let \( f: V \to W \) be a faithful homeomorphism. Let \( J = f(L) \). Denote this construction by \( J = J(K,V,L) \).

7. **Remarks:** \( J \) is a knot (Satz 1 on p. 164 of [31]). The knot type of \( J \) depends only on the knot type of \( K \) and the equivalence class of \( L \) in \( V \). Every knot \( J \) having a companion \( K \) can be constructed in this manner.

8. **Definition:** Let \( K \) be a knot, \( V \) a solid torus, and \( L \subseteq \text{Int}(V) \) a simple closed curve. The pair \( (V,L) \) is \( K \)-**dependent** if \( V - L \) contains an injective \( K \)-knot space. Otherwise, \( (V,L) \) is \( K \)-**independent**.

9. **Lemma:** Let \( J = J(K,V,L) \) as in Construction 6. If \( K \) has multiplicity \( m > 1 \), then \( (V,L) \) is \( K \)-dependent.

**Proof:** If \( m > 1 \), then there is a solid torus \( W_1 \) with oriented core \( K_1 \) having the knot type of \( K \) such that \( K_1 \) is a companion of \( J \) and \( Q = S^3 - \text{int}(W_1) \subseteq \text{int} W \). \( Q \) is a \( K \)-knot space. Suppose \( Q \) is not injective in \( W - J \). Then, by the loop theorem, there is a disc \( D \) in \( W - J \) with \( D \cap \partial Q = \partial D \) a non-contractible curve in \( \partial Q \).

Since \( Q \) is a knot space, \( D \subseteq W - J \). But then \( D \) is a meridional disc of \( W_1 \) missing \( J \), hence \( \partial_{W_1}(J) \neq 0 \), contradicting the fact that \( K_1 \) is a companion of \( J \). Thus \( Q \) is injective in \( W - J \). It follows that \( f^{-1}(Q) \) is an injective \( K \)-knot space in \( V - L \).

10. **Lemma:** Let \( J = J(K,V,L) \), and suppose \( u_V(L) = \alpha \). Then \( \Delta_J(t) = \Delta_K(t^\alpha) \Delta_L(t) \).
Proof: This is Theorem II of [35].

11. Definition: Let \( K \) be a knot. Every regular projection of \( K \) has a certain number of overpasses. The minimum such number, taken over all projections of \( K \) is the bridge number of \( K \), denoted \( b(K) \).

12. Lemma: (1) \( b(K) \geq 2 \).
   (2) If \( b(K) = 2 \), then \( K \) is simple.
   (3) If \( K \) is the composite of \( K_1 \) and \( K_2 \), then \( b(K) = b(K_1) + b(K_2) - 1 \).
   (4) If \( K < J \) and has order \( \alpha \), then \( \alpha b(K) \leq b(J) \) and \( b(K) < b(J) \).
   (5) If \( K < J \) and has order \( \alpha \) and multiplicity \( m \), then \( \alpha m(b(K) - 1) \leq b(J) \).

Proof: (1), (2), (3) and (4) are Satz 1, Satz 5, Satz 7, and Satz 3 of [32]. (5) follows from Satz 2 of [32] and the definition of multiplicity.

13. Definition: Let \( K_1 \) be an oriented knot in \( S^3 \). Let \( V \) be an unknotted solid torus in \( S^3 \) with oriented core \( c \). Let \( K_2 \subseteq \text{int}(V) \) be a knot such that \( \partial_V(K_2) = 1 \) and \( K_2 \) is homologous to \( c \) in \( V \). (See Figure 1.) Then \( J = J(K_1,V,K_2) \) is the composite of \( K_1 \) and \( K_2 \).

14. Remarks: This is equivalent to our previous definition (Satz 2 on p. 198 of [31]). (See also Note 27.)

15. Definition: Let \( V' \) be an unknotted solid torus in \( S^3 \) with oriented core \( c \) and canonically oriented meridian-longitude pair \((\mu, \lambda)\). Let \((m,n)\) be a
relatively prime pair of integers and $T_{m,n}$ an oriented simple closed curve on $\partial V'$ homologous to $m\lambda + n\lambda$. (See Figure 2.) Then $T_{m,n}$ is called an $(m,n)$-curve on $\partial V'$. If $|m| \geq 1$ and $|n| \geq 2$, then $T_{m,n}$ is an $(m,n)$-cable. If $|m| \geq 2$ and $|n| \geq 2$, then $T_{m,n}$ is an $(m,n)$-torus knot. Now let $V$ be a regular neighborhood of $V'$ in $S^3$. $c$ is also an oriented core of $V$. Let $K$ be a knot in $S^3$ and $T_{m,n}$ a cable. Then $J = J(K,V,T_{m,n})$ is an $(m,n)$-cable knot with carrier $K$.

16. Remarks: $\Delta_{T_{m,n}}(t) = \frac{(t^mn-1)(t-1)}{(t^m-1)(t^n-1)}$. (See [27].)

$T_{m,n}$ is knotted if and only if it is a torus knot. Torus knots are simple (Satz 2 on p. 250 of [31]). If $J = J(K,V,T_{m,n})$, then $K$ is the unique maximal companion of $J$ (Satz 3 on p. 250 of [31]). Every cable knot is prime (Satz 4 on p. 250 of [31]). (Note that our terminology varies from standard usage in that torus knots are usually regarded as cable knots.)

17. Definition: Let $V$ be an unknotted solid torus in $S^3$ with oriented core $c$. Let $\rho$ and $\eta$ be integers, where $\eta = \pm 2$. Let $K(\rho,\eta)$ be one of the four oriented simple closed curves in Figure 3. $|\rho|$ is the number of full twists in the lower part of $K(\rho,\eta)$. (So these twists account for $2|\rho|$ of the total number $2|\rho| + 2$ of crossings.) If $\rho \neq 0$, then $K(\rho,\eta)$ is a $(\rho,\eta)$-twist knot. If $K$ is a knot in $S^3$, then $J = J(K,V,K(\rho,\eta))$ is a $(\rho,\eta)$-doubled knot with
diagonal $K$.

18. **Remarks:** $\Delta_{K}(\rho, \eta) = (\rho / |\rho|)(\rho - (2\rho + 1)t + \rho t^2)$ if $\rho \neq 0$ and 1 if $\rho = 0$. (See [43].) Note that by Lemma 10, $J$ has the same polynomial as $K(\rho, \eta)$. In particular, for $\rho = 0$ it has the same polynomial as the unknot. (Note again that our terminology varies from standard usage in that twist knots are usually regarded as doubled knots.)

19. **Definition:** Let $V$ be an unknotted solid torus in $S^3$ with oriented core $c$. Let $\hat{b} \subseteq \text{int}(V)$ be an oriented simple closed curve homologous to a positive multiple of $c$ such that for some product structure $S^1 \times D^2$ on $V$, $\hat{b}$ meets each $\{e^{i\theta}\} \times D^2$ transversely in precisely $n$ points. $\hat{b}$ is a **closed braid** in $V$. If $K$ is a knot and $n \geq 2$, then $J(K, V, \hat{b})$ is a **cable braid** with **carrier** $K$. (See Figure 4.)

20. **Remarks:** $\hat{b} \subseteq V$ is determined up to equivalence by the conjugacy class of an element $b$ in the Artin braid group $B_n$ which has image an $n$-cycle in the symmetric group $\Sigma_n$. (See [4].) Every cable knot is a cable braid. Note that $V - \hat{b}$ is a surface bundle over $S^1$ with fiber a disc with $n$ punctures. (See also Note 28.)

21. **Definition:** Let $V$ be an unknotted solid torus in $S^3$ having oriented core $c$ and canonical meridian-longitude pair $(\mu, \lambda)$. A homeomorphism $\tau : V \to V$ is a **simple twist** if $\tau(\mu)$ is homologous to $\mu$ and $\tau(\lambda)$ is homologous to $\mu + \lambda$ in $\partial V$. 
22. **Lemma:** Let $V$ be as above and $K \subseteq V$ an oriented simple closed curve. If $\tau$ and $\tau'$ are simple twists, then $\tau(K)$ and $\tau'(K)$ are equivalent in $V$.

**Proof:** $\tau'(K) = \tau'\tau^{-1}(\tau(K))$. $\tau'\tau^{-1}(\mu)$ is homologous to $\mu$ and $\tau'\tau^{-1}(\lambda)$ is homologous to $\lambda$, so $\tau'\tau^{-1}$ is faithful. Hence $\tau'(K)$ and $\tau(K)$ are equivalent.

23. **Definition:** Let $V$ be an unknotted solid torus with oriented core $c$, $K_1, K_2 \subseteq \text{int}(V)$ simple closed curves. $K_1$ and $K_2$ are **congruent along** $V$ if $K_2 = \tau^q(K_1)$ for some integer $q$ and simple twist $\tau$.

24. **Remarks:** The notion of congruence is due to R. H. Fox [10]. Lemma 22 shows that, up to equivalence, the definition does not depend on the choice of $\tau$. We now consider two examples of congruence which will be used in Chapter IV.

25. **Example:** Let $T_{m,n}$ be a cable in $V$ as in Definition 15. Let $q$ be an integer. Then $T_{m,n}$ is congruent along $V$ to $T_{m+qn,n}$.

26. **Example:** Let $K(\rho,\eta)$ be a twist curve as in Definition 17. Let $q$ be an integer. Then $K(\rho,\eta)$ is congruent along $V$ to $K(\rho + q,\eta)$.

27. **Note:** (to Remark 14) $K_1$ and $K_2$ are companions of $K$ of order one. Conversely, any companion of $K$ of order one is a factor of $K$ (Satz 1 on p. 198 of [31]).

28. **Note:** (to Remark 20) Let $J$ be a cable braid and $W = f(V)$ the solid torus containing $J$ as in Construction 6. Let $W'$ be a knotted solid torus with $J \subseteq \text{int}(W')$.
such that $o_{W'}(J) \neq 0$ and $J$ is not a core of $W'$. If $\partial W' \subseteq \text{int}(W)$, then $W' \subseteq \text{int}(W)$. (Hilfsatz 2 on p. 263 of [31].)
Figure 3a
Figure 3b
I.4 The Smith Conjecture

1. **Conjecture**: If \( \langle h \rangle \) is a cyclic action of order \( p \) on \( S^3 \) with \( \text{Fix}(\langle h \rangle) = K \) a simple closed curve, then \( K \) is unknotted.

2. **Remarks**: We shall be concerned with the following obviously equivalent statement which we call the **period** \( p \) Smith Conjecture.

3. **Conjecture**: Let \( K \) be a knot in \( S^3 \) and \( p > 1 \) an integer. Then there does not exist a cyclic action \( \langle h \rangle \) of order \( p \) on \( S^3 \) with \( \text{Fix}(\langle h \rangle) = K \).

4. **Lemma**: Let \( K_1 \) and \( K_2 \) be equivalent knots in \( S^3 \). If \( K_1 \) is a counterexample to the period \( p \) Smith Conjecture, then so is \( K_2 \).

**Proof**: By Definition 2.1 there is a homeomorphism \( f \) of \( S^3 \) such that \( f(K_1) = K_2 \). If \( \langle h \rangle \) is a period \( p \) action with \( \text{Fix}(\langle h \rangle) = K_1 \), then \( \langle fhf^{-1} \rangle \) is a period \( p \) action with \( \text{Fix}(fhf^{-1}) = K_2 \).

5. **Proposition**: (Moise) Regard \( S^3 \) as the set \( \{(z_1,z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1 \} \) and let \( \langle g \rangle \) be the cyclic action of order \( p \) on \( S^3 \) generated by

\[
g(z_1,z_2) = (e^{\frac{2\pi i q}{p}} z_1, z_2), \quad (p,q) = 1 \text{ (the standard orthogonal action)}.\]

Let \( \langle h \rangle \) be a cyclic action of order \( p \) with \( \text{Fix}(\langle h \rangle) = K \) a simple closed curve. Then \( K \) is unknotted if and only if \( \langle h \rangle \) is equivalent to \( \langle g \rangle \).

**Proof**: This is proven in [24]; see also [38].
6. **Lemma:** Let \( K \) be a simple closed curve in \( S^3 \).
If there is a cyclic action \( \langle h \rangle \) of order \( p \) on \( S^3 \) with \( \text{Fix}(h) = K \), then there is a cyclic action \( \langle h' \rangle \) of prime order \( p' \) on \( S^3 \) with \( \text{Fix}(h') = K \).

**Proof:** Write \( p = p'q \), where \( p' \) is a prime. Let \( h' = h^q \). Then \( \langle h' \rangle \) has order \( p' \). \( \text{Fix}(h') \) is a simple closed curve [37] and \( \text{Fix}(h) \subseteq \text{Fix}(h') \), hence \( \text{Fix}(h) = \text{Fix}(h') \).

7. **Remark:** This lemma shows that in order to prove the Smith Conjecture for a knot \( K \), it suffices to prove it for prime periods.

8. **Lemma:** Let \( \langle h \rangle \) be a cyclic action of order \( p \) on \( S^3 \) such that \( K = \text{Fix}(h) \) is a simple closed curve.
Let \( q : S^3 \to \Sigma^3 = S^3 / \langle h \rangle \) be the quotient map. Then
\begin{enumerate}
\item \( \Sigma^3 \) is a homotopy 3-sphere.
\item \( (q|S^3 - K) : S^3 - K \to \Sigma^3 - q(K) \) is a \( p \)-fold cyclic covering.
\item \( K \) is knotted if and only if \( q(K) \) is knotted.
\item There is a regular neighborhood \( N \) of \( K \) in \( S^3 \) and a product structure \( S^1 \times D^2 \) on \( N \) such that \( h(N) = N \) and \( (h|N)(e^{i\theta}, re^{i\varphi}) = (e^{i\theta}, re^{i\frac{\varphi + 2\pi q}{p}}) \), \( (p,q) = 1 \).
\end{enumerate}

**Proof:** Since \( h \) is periodic, there is a triangulation of \( S^3 \) with respect to which \( h \) is simplicial [29, p. 27], so by [38, p. 343], \( \Sigma^3 \) is a closed 3-manifold. By [13, p. 331], \( \Sigma^3 \) is simply connected.

(2) is trivial.
(3) If $K$ is unknotted, then by Proposition 5, $\langle h \rangle$ is equivalent to the standard orthogonal action and it follows that $q(K)$ is unknotted. If $q(K)$ is unknotted, then $q(K) = \partial D$, $D$ a disc embedded in $\Sigma^3$. By (2), $\text{Int}(D)$ lifts homeomorphically to an open disc in $S^3$ whose closure is a disc with boundary $K$. Hence $K$ is unknotted.

(4) follows from [38, pp. 344-345].

9. **Notation:** Let $\langle h \rangle$ be a cyclic action of order $p$ on $S^3$ with $\text{Fix}(h) = K$, a knot. Let $N$ be as in part (4) of Lemma 8. Let $C(K) = S^3 - \text{int}(N)$. $N$ is called an **invariant regular neighborhood** and $C(K)$ an **invariant knot space** of $K$. If $q: S^3 - \Sigma^3$ is the quotient map, then let $K^* = q(K)$, $N^* = q(N)$, $C^*(K^*) = q(C(K))$.

10. **Lemma:** Let $\langle h \rangle$ be a cyclic action of order $p$ on $S^3$ with $\text{Fix}(h) = K$, a knot. Let $C(K)$ be an invariant knot space. Then $(q|C(K)): C(K) \rightarrow C^*(K^*)$ is a $p$-fold cyclic covering. There exist meridian-longitude pairs $(\mu, \lambda), (\mu^*, \lambda^*)$ for $K, K^*$ respectively, such that $(q|\mu): \mu - \mu^*$ is a $p$-fold cyclic covering and $(q|\lambda): \lambda - \lambda^*$ is a homeomorphism, and each of these maps has positive degree.

**Proof:** The first statement follows from (4) of Lemma 8. If $\mu^*$ is any meridian of $K^*$, then part (4) of Lemma 8 implies that $\mu = q^{-1}(\mu^*)$ is connected. If $\lambda^*$ is any longitude of $K^*$, then $\lambda^* = \partial F^*$, $F^*$ a surface in
\( C^*(K^*) \). Since \( q|C(K) \) is a cyclic covering and \( \pi_1(F^*) \) maps into the commutator subgroup of \( \pi_1(C^*(K^*)) \), \( F^* \) lifts homeomorphically to a surface \( F \) in \( C(K) \) with \( \partial F = \lambda \) the desired longitude of \( K \). Now choose the orientations of \( \mu^*, \lambda^* \) to make the degrees positive.

11. **Proposition**: (Fox) Let \( \langle h \rangle \) be a cyclic action of order \( p \) on \( S^3 \) with \( \text{Fix}(h) = K \), a knot. Let \( w \) be a primitive \( p^{th} \) root of unity. Then

\[
(1) \quad \Delta_K(t^p) = \prod_{j=0}^{p-1} \Delta_K(w^j t).
\]

(2) If \( \Delta_K(t) = \frac{(tmn-1)(t-1)}{(tm-1)(t^n-1)} \) where \( m, n \) are relatively prime and \( |m|, |n| \geq 2 \), then \( p \) and \( mn \) are relatively prime.

**Proof**: See [9] or [12].

12. **Proposition**: (Giffen) The Smith Conjecture is true for torus knots.

**Proof**: See [14] or Fox [13].

13. **Proposition**: (Cappell and Shaneson) The Smith Conjecture is true for 2-bridge knots.

**Proof**: See [6].

14. **Lemma**: Let \( 1 \to G \to G^* \to \mathbb{Z}_p \to 1 \) be an exact sequence (in which we identify \( G \) with its image in \( G^* \)). Let \( m \in G \) such that \( G/\text{nm}(m;G) \cong \{1\} \) and \( m^* \in G^* \) such that \( (m^*)^p = m \). Then

\[
(1) \quad G^*/\text{nm}(m^*,G^*) \cong \{1\}
\]
(2) \((G^*)' = G'\).

**Proof:** See [13, p. 331].
I.5 Incompressible surfaces

1. **Definitions:** Let $M$ be a 3-manifold.

   A surface $F$ in $M$ or $\partial M$ is **compressible** in $M$ if there is a disc $D \subseteq M$ with $D \cap F = \partial D$ such that $\partial D$ is non-contractible in $F$. $D$ is a **compressing disc** for $F$. If $F$ is not compressible in $M$ then it is **incompressible** in $M$.

   $M$ is **irreducible** if every 2-sphere contained in $M$ bounds a ball in $M$. $M$ is **sufficiently large** if it is not a ball and there is an incompressible surface in $M$.

2. **Remarks:** $F$ is incompressible in $M$ if and only if $F$ is injective in $M$. (See [17, Lemma 6.1].) If $M$ is irreducible and $\partial M$ contains a torus then $M$ is sufficiently large. (See [17, Lemma 6.8].) A 3-manifold with an irreducible covering space is irreducible, by the sphere theorem. An irreducible knot manifold is sufficiently large and has incompressible boundary.

3. **Lemma:** Let $F$ be an incompressible surface in $M$ and $M'$ be the result of splitting $M$ along $F$.

   (1) $M'$ is irreducible if and only if $M$ is.

   (2) A surface $G$ in $M$ with $G \cap F = \emptyset$ is incompressible in $M$ if and only if it is incompressible in $M'$.

   **Proof:** See [39].

4. **Lemma:** Let $M,N$ be sufficiently large, $\partial M, \partial N$ incompressible, $\partial N$ connected, $f: (M, \partial M) \to (N, \partial N)$ a map
such that \( f_\ast : \pi_1(M) \to \pi_1(N) \) is monic. Then \( f \) is homotopic to a covering map.

**Proof:** This follows from Theorem 6.1 of [40].

5. **Lemma:** Let \( M \) be an irreducible 3-manifold and \( F \) an incompressible surface in \( M \). If the embedding map of \( F \) into \( M \) is homotopic rel \( \partial F \) to a map into \( \partial M \), then \( F \) is parallel to a surface \( F' \) in \( \partial M \).

**Proof:** This follows from Lemma 5.3 of [40].

6. **Lemma:** Let \( M = F \times I \) where \( F \) is a compact surface other than \( S^2 \). Let \( G \) be an incompressible surface in \( M \) with \( \partial G \subseteq F \times \{0\} \). Then \( G \) is parallel to a surface \( G' \) in \( F \times \{0\} \).

**Proof:** This is Corollary 3.2 of [40].

7. **Lemma:** Let \( M = F \times I \) where \( F \) is a compact surface other than \( S^2 \). Let \( G_1, \ldots, G_n \) be disjoint surfaces in \( M \) such that each \( G_j \) is either a disc intersecting \( \partial F \times I \) in two arcs of the form \( \{x\} \times I \) or an incompressible annulus having one boundary component in \( F \times \{0\} \) and the other in \( F \times \{1\} \). Then there is an isotopy \( f_t \) of \( M \) rel \( (F \times \{0\}) \cup (\partial F \times I) \) such that \( f_t(G_j) = (G_j \cap (F \times \{0\})) \times I \).

**Proof:** This is Lemma 3.4 of [40].

8. **Lemma:** Let \( V \) be a solid torus and \( A \) an annulus in \( \partial V \) which is injective in \( V \). Then \( A \) is parallel to \( \partial V \) - \( \text{int}(A) \) in \( V \) if and only if \( A \) is surjective in \( V \).

**Proof:** The proof is straightforward.

9. **Lemma:** Let \( V \) be a solid torus and \( A \) an
annulus in \( V \) such that the components \( c_1 \) and \( c_2 \) of \( \partial A \) are injective in \( \partial V \). Then \( A \) separates \( V \) into two components with closures \( V_1 \) and \( V_2 \) such that 
\[
\partial V_1 = A \cup B_1, \quad \partial V_2 = A \cup B_2, \quad B_1, B_2 \text{ the closures of the components of } \partial V - \partial A, \text{ such that }
\]

(1) If \( A \) is injective in \( V \), then \( V_1 \) and \( V_2 \) are solid tori and \( A \) is parallel to \( B_1 \) in \( V_1 \). If \( A \) is also surjective in \( V \) then \( A \) is also parallel to \( B_2 \) in \( V_2 \).

(2) If \( A \) is not injective in \( V \), then \( V_1 \) is a solid torus, \( V_2 \) is a toral solid, and \( c_1, c_2 \) are meridians of \( V_1 \). If \( V_2 \) is also a solid torus then \( A \) is parallel to \( B_2 \) in \( V_2 \).

Proof: (1) follows from Satz 1 on p. 207 of [31] and Lemma 7. The first part of (2) follows from Satz 2 on p. 211 of [31]. Suppose \( V_2 \) is a solid torus and \( A \) is not parallel to \( B_2 \) in \( V_2 \). Then by Lemma 7, either \( A \) is not injective in \( V_2 \) or \( A \) is injective but not surjective in \( V_2 \). The first possibility implies that 
\[ \pi_1(V) \cong \mathbb{Z} \times \mathbb{Z}, \] 
the second that \( \pi_1(V) \cong \mathbb{Z} \times \mathbb{Z}_m \) for some \( m > 1 \), both contradicting \( \pi_1(V) \cong \mathbb{Z} \).

10. Lemma: Let \( V \) be a solid torus, \( Q \subseteq \text{Int}(V) \) a knot space. Then \( Q \subseteq \text{int} B \subseteq \text{Int}(V) \), where \( B \) is a ball.

Proof: This follows from Lemma 1 of [31].

11. Lemma: Let \( C(K_1) \) and \( C(K_2) \) be knot spaces in \( S^3 \) such that \( A = C_1(K_1) \cap C(K_2) \) is an annulus injective in both \( C(K_1) \) and \( C(K_2) \). Then
$V = S^3 - \text{int}(C_1(K_1) \cup C(K_2))$ is a solid torus and the components of $\partial A$ are meridians. If the orientations of $A$ inherited from canonically oriented meridians of $K_1$ and $K_2$ are opposite, then an oriented core of $V$ is the composite of $K_1$ and $K_2$.

Proof: This follows from Lemma 2.1 of [36] and from Definition 3.1.

12. Lemma: Let $T$ be a torus in the irreducible 3-manifold $M$ such that $M - T$ has two components, with closures $X$ and $Y$. If $T$ compresses in $X$, then either $X$ is a solid torus or $Y$ is a toral solid.

Proof: Let $D$ be a compressing disc for $T$ in $X$, $N$ a regular neighborhood of $T \cup D$ in $X$. $\partial N = T \cup S$, $S$ a 2-sphere in $X$. $S$ bounds a ball $B$ in $M$. If $B \subseteq X$, then $X$ is a solid torus. If not, then $B = N \cup Y$ and so $Y$ is easily seen to be a toral solid.
I.6 Essential annuli and tori

1. Definitions: Let $M$ be a 3-manifold.

Let $A$ be an annulus. A map $f: (A, \partial A) \to (M, \partial M)$ is **essential** if $f_\#: \pi_1(A) \to \pi_1(M)$ and $f_\#: \pi_1(A, \partial A) \to \pi_1(M, \partial M)$ are monic. An annulus $A$ in $M$ is **essential** if its embedding map is essential.

Let $T$ be a torus. A map $f: T \to M$ is **essential** if $f_\#: \pi_1(T) \to \pi_1(M)$ is monic and $f$ is not homotopic to a map into $\partial M$.

2. Lemma: Let $M$ be an irreducible 3-manifold with $\partial M \neq \emptyset$ and incompressible.

   (1) (Annulus Theorem) If there is an essential map $f: (A, \partial A) \to (M, \partial M)$, then there is an essential embedding $g: (A, \partial A) \to (M, \partial M)$.

   (2) (Torus Theorem) If there is an essential map $f: T \to M$ then there is either an essential embedding $g: T \to M$ or an essential embedding $g: (A, \partial A) \to (M, \partial M)$.

   **Proof:** Both theorems were stated by Waldhausen in [41]. A proof of (1) may be found in Cannon and Feustel [5] and of (2) in Feustel [8]. They are also both corollaries of the results of Jaco and Shalen [18] and of Johannson [19].

3. Lemma: Let $M$ be an irreducible 3-manifold with $\partial M$ incompressible. An incompressible annulus $A$ or torus $T$ in $M$ is essential if and only if it is not parallel to a surface in $\partial M$. 
Proof: See [8].

4. **Lemma**: Let $M, N$ be irreducible 3-manifolds with non-empty boundaries and $f: M \to N$ a covering. If $M$ has an essential annulus then so does $N$. If $M$ has an essential torus then $N$ has either an essential torus or an essential annulus.

**Proof**: Suppose $(A, \partial A) \subseteq (M, \partial M)$ is essential. Then $(f|A)_\#$ is monic. If for some spanning arc $\alpha$ of $A$, $(f|\alpha)$ homotops rel $\partial \alpha$ into $\partial N$, then we can lift the homotopy to homotop the embedding of $\alpha$ in $M$ rel $\partial \alpha$ into $\partial M$, a contradiction. Thus $(f|A)$ is essential and Lemma 2 implies the result.

Suppose $T \subseteq M$ is essential. Then $(f|T)_\#$ is monic, and if $(f|T)$ homotops into $\partial N$, then we can lift the homotopy to homotop the embedding of $T$ into $\partial M$, a contradiction. Thus $(f|T)$ is essential and Lemma 2 implies the result.

5. **Lemma**: Let $M, N$ be irreducible 3-manifolds with incompressible boundaries and $f: M \to N$ a covering. If $A$ is an essential annulus in $N$, then each component of $f^{-1}(A)$ is an essential annulus in $M$. If $T$ is an essential torus in $N$, then each component of $f^{-1}(T)$ is an essential torus in $M$.

**Proof**: Let $A_0$ be a component of $f^{-1}(A)$ and $\alpha$ be a spanning arc of $A$. $\pi_1(A_0) \to \pi_1(M)$ is obviously monic. $\alpha$ lifts to a spanning arc $\alpha_0$ of $A_0$. If $\alpha_0$ homotops
rel $\partial x_0$ into $\partial M$ then we can project the homotopy via $f$ to homotop $a$ rel $\partial x$ into $\partial N$, a contradiction. Thus $A_0$ is essential in $M$.

Let $T_0$ be a component of $f^{-1}(T)$. $\pi_1(T_0) \to \pi_1(M)$ is obviously monic. If $T_0$ is inessential in $M$ then by Lemma 3 $T_0$ is parallel to a torus $T'$ in $\partial M$. So there is a submanifold $X$ of $M$ homeomorphic to the product of a torus and an interval having $\partial X = T_0 \cup T'$. $X$ covers a submanifold $Y$ of $N$ with $T \subseteq \partial Y$. $Y$ is thus either the product of a torus and an interval or a twisted $I$-bundle over a Klein bottle. The latter is impossible since $Y \cap \partial N \neq \emptyset$ and $T \cap \partial N = \emptyset$. Thus $T$ is parallel to a component of $\partial N$, a contradiction.

6. Lemma: Let $C(K)$ be a knot space.

(1) $K$ is composite if and only if $C(K)$ has an essential annulus whose boundary components are meridians.

(2) $K$ is cabled or a torus knot if and only if $C(K)$ has an essential annulus whose boundary components are not meridians.

Proof: This follows from Lemmas 2.1 and 2.2 of [36].

7. Lemma: (1) Let $K$ be a companion of the knot $J$, $W$ the solid torus in Definition 1.3.4, $C(J)$ a knot space of $J$ with $\partial W \subseteq \text{int } C(J)$. Then $\partial W$ is an essential torus in $C(J)$.

(2) Let $J$ be a knot, $C(J)$ a knot space, and $T$ an essential torus in $C(J)$. Let $W$ be the closed component of $S^3 - T$ containing $J$. Then $W$ is a solid torus
and a properly oriented core $K$ of $W$ is a companion of $J$.

Proof: (1) $W$ is knotted so $S^3 - \text{int}(W)$ is a knot space and $\partial W$ is incompressible in $S^3 - \text{int}(W)$. If $\partial W$ compresses in $W \cap C(J)$, then a compressing disc for $\partial W$ is a meridional disc of $W$ missing $J$. This contradicts $O_{\partial W}(J) \neq 0$. So $\partial W$ is incompressible in $C(J)$, hence injective in $C(J)$. If $\partial W$ is not essential in $C(J)$, then it is parallel to $\partial C(J)$ in $W$, implying that $J$ is a core of $W$, a contradiction.

(2) If $W$ is not a solid torus, then $S^3 - \text{int}(W) \subseteq C(J)$ is a solid torus and so $T$ compresses in $C(J)$, a contradiction. So $W$ is a solid torus. If $W$ is unknotted, then $S^3 - \text{int}(W)$ is a solid torus and we again get a contradiction. If $O_{\partial W}(J) = 0$, then there is a meridional disc of $W$ missing $J$, hence $\partial W$ compresses in $C(J)$, a contradiction. If $J$ is a core of $W$, then $\partial W$ is parallel to $\partial C(J)$ in $W \cap C(J)$, another contradiction. So the result follows.
Chapter II: Cyclic actions on solid tori

II.1 Standard actions

1. **Definition:** Let \( V \) be a solid torus, \( \langle h \rangle \) a cyclic action of order \( p \) on \( V \) with \( \text{Fix}(h) = K \subseteq \text{Int}(V) \) a simple closed curve. \( \langle h \rangle \) is **standard** if there is a product structure \( S^1 \times D^2 \) on \( V \) such that

\[
h(e^{i\theta}, re^{i\phi}) = (e^{i\theta}, re^{i(\phi + \frac{2\pi q}{p})})\]

where \( q \) is some integer relatively prime to \( p \).

2. **Lemma:** Let \( \langle h \rangle \) be a cyclic action on a compact 3-manifold \( M \) such that \( \text{Fix}(h) = K \subseteq \text{Int}(M) \) is a simple closed curve. Then there is a regular neighborhood \( N \) of \( K \) in \( M \) such that \( h(N) = N \) and \( \langle h|N \rangle \) is standard.

**Proof:** See pp. 344-345 of [38]. The proof as given there is only for \( \partial M = \emptyset \) but is easily modified to cover the case \( \partial M \neq \emptyset \).

3. **Lemma:** Let \( \langle h \rangle \) be a cyclic action of order \( p \) on the solid torus \( V \) with \( \text{Fix}(h) = K \subseteq \text{Int}(V) \) a simple closed curve. \( \langle h \rangle \) is standard if and only if \( K \) is a core of \( V \).

**Proof:** If \( \langle h \rangle \) is standard, then \( K \) is clearly a core of \( V \). Suppose \( K \) is a core of \( V \). Let \( N \) be the invariant regular neighborhood of Lemma 2. Let \( V^* = V/\langle h \rangle \), \( q: V \to V^* \) the quotient map, \( N^* = q(N) \). \( (q|V - \text{int}(N)) \) is a covering map, hence \( V^* - \text{int}(N^*) \) is either the
product of a torus and an interval or the twisted I-bundle over the Klein bottle. The latter is impossible since it has connected boundary. It then follows easily that \langle h \rangle is standard.

4. **Lemma**: Let \langle h \rangle be any cyclic action of order \( p \) on the boundary \( \partial V \) of a solid torus \( V \) such that \( h(\partial D) = \partial D \) for some meridional disc \( D \). Then \( h \) extends to a standard action on \( V \).

**Proof**: Split \( V \) along \( D \) to a ball \( B \). Give \( B \) a rotation about an unknotted arc joining the copies \( D_1 \) and \( D_2 \) of \( D \) in \( \partial B \) which agrees with \( h \) on \( \partial B - \text{int}(D_1 \cup D_2) \). Now glue back \( D_1 \) to \( D_2 \) so that the arc becomes a core of \( V \). By Lemma 3, the induced action \( \langle h \rangle \) on \( V \) is standard.
II.2 Arbitrary actions

1. **Notation:** Throughout this section \( \langle h \rangle \) is a cyclic action of prime order \( p \) on a solid torus \( V \) with \( \text{Fix}(h) = K \subseteq \text{Int}(V) \) a simple closed curve. \( V^* = V/\langle h \rangle \), \( q: V \to V^* \) is the quotient map, \( K^* = q(K) \).

2. **Lemma:** \( V^* \) is a homotopy solid torus.

**Proof:** By Lemma 2, there is an invariant regular neighborhood \( N \) of \( K \) on which \( \langle h \rangle \) is standard. Let \( N^* = q(N) \). Let \( G = \pi_1(V - \text{int}(N)) \), \( G^* = \pi_1(V^* - \text{int}(N^*)) \), \( q_0 = q|_V(V - \text{int}(N)) \). We have an exact sequence

\[
1 \to G \to G^* \to \mathbb{Z} \to 1. 
\]

Let \( m, m^* \) be elements of \( G, G^* \) represented by meridians of \( N, N^* \) respectively. Choosing orientations properly we have \( (q_0)_*(m) = (m^*)^p \). Identifying \( G \) with its image in \( G^* \) we can write \( (m^*)^p = m \).

\{1, m^*, \ldots, (m^*)^{p-1}\} is a set of coset representatives for \( G \) in \( G^* \).

Now \( G/\text{nm}(m; G) \cong \pi_1(V) \cong \mathbb{Z} \). Let \( t \in G \) represent a generator of \( \mathbb{Z} \) modulo \( \text{nm}(m; G) \). Then \( G = \text{gp}(t; G) \cdot \text{nm}(m; G) \), while \( G^* = G \cdot \text{gp}(m^*; G^*) \). So if \( g^* \in G^* \), then \( g^* = g(m^*)^k \), for some \( g \in G \), hence \( g^* = t^lx(m^*)^k \) for some \( x \in \text{nm}(m; G) \). Since \( \text{nm}(m; G) \subseteq \text{nm}(m^*; G^*) \) we see that \( g^* = t^ly \) for \( y = x(m^*)^k \in \text{nm}(m^*; G^*) \). Hence \( \pi_1(V^*) = G^*/\text{nm}(m^*; G^*) \) is cyclic.

Since \( (q|_{\partial V}): \partial V \to \partial V^* \) is a covering map, \( \partial V^* \) is a
torus. Hence \( H_1(V^\ast) \) is infinite [17, Lemma 6.7], so \( \pi_1(V^\ast) \cong \mathbb{Z} \). Thus \( V^\ast \) is a homotopy solid torus.

3. **Lemma:** Let \( \mu^\ast \) be a meridian of \( V^\ast \). Then \( \mu = q^{-1}(\mu^\ast) \) is a connected meridian of \( V \).

**Proof:** Choose basepoints \( x_0^\ast \in \mu^\ast \) and \( x_0 \in q^{-1}(x_0^\ast) \). Let \( \mu_0 \) be the component of \( q^{-1}(\mu^\ast) \) containing \( x_0 \).

Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_1(\partial V, x_0) & \xrightarrow{i_*} & \pi_1(V, x_0) \\
(q|\partial V)_\ast \downarrow & & \downarrow q_\ast \\
\pi_1(\partial V^\ast, x_0^\ast) & \xrightarrow{j_*} & \pi_1(V^\ast, x_0^\ast)
\end{array}
\]

Since \((q|\partial V)_\ast \) is a covering map, \((q|\partial V)_\ast \) is monic. Since \( V^\ast \) is a homotopy solid torus \( \text{im}(j_\ast) \neq \{1\} \). Since \( \text{im}(q|\partial V)_\ast \) has finite index \( p \) in \( \pi_1(\partial V^\ast, x_0^\ast) \), \( \text{im}(q_\ast i_\ast) = \text{im}(j_\ast(q|\partial V)_\ast) \neq \{1\} \). Thus \( \text{im } q_\ast \neq \{1\} \), so \( q_\ast \) is monic. Now \( q(\mu_0) = \mu^\ast \) implies \( q_\ast i_\ast([\mu_0]) = j_\ast(q|\partial V)_\ast([\mu_0]) = 1 \), so \( i_\ast([\mu_0]) = 1 \). So, by Dehn's lemma, \( \mu_0 \) bounds a disc in \( V \) and is thus a meridian. Since all the other components of \( q^{-1}(\mu^\ast) \) are parallel to \( \mu_0 \), they are also meridians.

Now suppose that \( q^{-1}(\mu^\ast) \) is not connected. Let \( f: \tilde{V} \rightarrow V \) be the universal cover of \( V \), with \( t \) a generator of the group of covering translations. Choose \( \tilde{x}_0 \in f^{-1}(x_0) \) and let \( \tilde{\mu}_0 \) be the component containing \( \tilde{x}_0 \). Since \( q^{-1}(\mu^\ast) \) is not connected, \((q|\partial V): \partial V \rightarrow \partial V^\ast \) is a \( p \)-fold cyclic covering whose group of covering transla-
tions is $\langle h|\partial V \rangle$. Let $g = (q|\partial V)(f|\tilde{\partial V})$: $\partial V \to \partial V^*$. $g$ is an infinite cyclic covering whose group of covering translations is generated by a homeomorphism $s$ such that $s^p = (t|\partial V)$. We may assume (by choosing a new generator for $\langle h \rangle$, if necessary) that $h$ lifts to a homeomorphism $\tilde{h}: \tilde{V} \to \tilde{V}$ with $(\tilde{h}|\partial V) = s$. Thus $\tilde{h}^p = t$.

Now choose $y \in \text{Fix}(h)$ and $\tilde{y} \in f^{-1}(y)$. Then $\tilde{h}(\tilde{y}) = t^m(\tilde{y})$ for some $m$. But $t(\tilde{y}) = \tilde{h}^p(\tilde{y}) = t^{mp}(\tilde{y})$, which implies $mp = 1$, contradicting $p > 1$. Thus $\mu = q^{-1}(\mu^*)$ is connected.

4. **Lemma**: There is a standard action $\langle h_1 \rangle$ of order $p$ on $V$ such that $h|\partial V = h_1|\partial V$.

**Proof**: By Lemma 2, $V^*$ is a homotopy solid torus, so by Lemma 3, there is an invariant meridian $\mu$ of $V$. Application of Lemma 1.4 completes the proof.
II.3 Closed braids as fixed point sets

1. **Proposition:** Let \( \langle h \rangle \) be a cyclic action of prime order \( p \) on a solid torus \( V \) such that \( \text{Fix} \langle h \rangle = \hat{\beta} \) is a closed braid in \( V \). Then \( \hat{\beta} \) is a core of \( V \) and \( \langle h \rangle \) is standard.

**Proof:** Let \( V^* = V/\langle h \rangle \), \( q: V \to V^* \) the quotient map, \( \hat{\beta}^* = q(\hat{\beta}) \). By Lemma 1.2 there is an invariant regular neighborhood \( N \) of \( K \) on which \( \langle h \rangle \) is standard. Let \( U = V - \text{int}(N) \), \( N^* = q(N) \), \( U^* = q(U) \).

By Lemma 2.2 \( V^* \) is a homotopy solid torus. Let \( D^* \) be a meridional disc in \( V^* \) which is in general position with respect to \( \hat{\beta}^* \) and chosen so that the number of components of \( D^* \cap \hat{\beta}^* \) is minimal. We may assume that \( D^* \cap N^* \) consists of meridional discs of \( N^* \). Let \( F^* = D^* \cap U^* \).

We claim that \( F^* \) is incompressible in \( U^* \). If not, then let \( D \) be a compressing disc for \( F^* \) in \( U^* \). Since \( D^* \) is a disc, \( \partial D = \partial D' \) for a disc \( D' \) in \( D^* \). Since \( \partial D \) is not contractible in \( F^* \), \( D' \) must contain \( n > 0 \) points of \( D^* \cap \hat{\beta}^* \). Let \( (D^*)' = (D^* - \text{int}(D')) \cup D \). Then \( (D^*)' \) is a meridional disc of \( V^* \) in general position with respect to \( \hat{\beta}^* \), and \( (D^*)' \cap \hat{\beta}^* \) has \( n \) fewer components than \( D^* \cap \hat{\beta}^* \), contradicting the choice of \( D^* \). This proves our claim.

Now choose basepoints \( x_0^* \in \partial D^* \), \( x_0 \in q^{-1}(x_0^*) \). \( q^{-1}(\partial D^*) \) is connected, by Lemma 2.3, so \( D = q^{-1}(D^*) \) and
\( F = q^{-1}(F^*) \) are connected. Since \((q|U)\) is a finite cyclic covering, \( F \) is compact. Since \( D^* \cap \hat{\beta}^* \) is finite and \( \langle h|N \rangle \) standard, \( D \) is also compact.

Since \( \hat{\beta} \) is a closed braid in \( V \) there is a product structure \( S^1 \times D^2 \) on \( V \) such that \( \hat{\beta} \) is transverse to each \( \{e^{i\theta}\} \times D^2 \) and \( \{e^{i\theta} \times D^2 \} \cap N \) consists of meridional discs of \( N \). Let \( E = \{1\} \times D^2 \) and \( G = E - \text{int}(E \cap N) \). Then \( U \) is a surface bundle over \( S^1 \) with fiber \( G \).

Now let \( f: \tilde{V} \to V \) be the universal covering space of \( V \) and let \( \tilde{x}_0 \in f^{-1}(x_0) \). Let \( \tilde{U} = f^{-1}(U) \) and set \( f_0 = (f|\tilde{U}) \). Then \( \tilde{V} \) is homeomorphic to \( E \times (-\infty, +\infty) \) and \( U \) is homeomorphic to \( G \times (-\infty, +\infty) \). Thus \( f_0: \tilde{U} \to U \) is the infinite cyclic covering of \( U \) corresponding to \( \pi_1(G, x_0) \).

Since \( \partial D \) is connected, \( D \) does not separate \( V \), hence \( \pi_1(D, x_0) \to \pi_1(V, x_0) \) is trivial. Therefore \( D \) lifts to a disc \( \tilde{D} \) in \( \tilde{V} \) containing \( \tilde{x}_0 \). Hence \( F \subseteq D \) lifts homeomorphically to \( \tilde{F} \) in \( \tilde{U} \) such that \( \tilde{x}_0 \in \tilde{F} \subseteq \tilde{D} \).

The following commutative diagram shows that \( F \) is incompressible in \( U \) and that \( \tilde{F} \) is incompressible in \( \tilde{U} \).
\[ \pi_1(\tilde{F}, x_0) \xrightarrow{(f_0|\tilde{F})_*} \pi_1(\tilde{U}, x_0) \xrightarrow{(f_0)_*} \]
\[ \pi_1(F, x_0) \xrightarrow{(q_0|F)_*} \pi_1(U, x_0) \xrightarrow{(q_0)_*} \]
\[ \pi_1(F^*, x_0^*) \xrightarrow{\pi_1(U^*, x_0^*)}. \]

Since \( \tilde{F} \) is compact, there is a number \( s \) such that \( \tilde{F} \subseteq G \times (-s, +s) \). Since the components of \( \partial G \times [-s, +s] \) are annuli there is an isotopy \( r_t \) of \( G \times [-s, +s] \) such that \( \partial r_1(\tilde{F}) \subseteq \text{int}(G \times \{-s\}) \). Note that the components of \( \partial r_1(\tilde{F}) \) are each parallel to a component of \( \partial G \times \{-s\} \).

Since \( \tilde{F} \) is incompressible in \( G \times (-\infty, +\infty) \), Lemma I.5.3 implies that \( \tilde{F} \) is incompressible in \( G \times [-s, +s] \), hence \( r_1(\tilde{F}) \) is incompressible in \( G \times [-s, +s] \). By Lemma I.5.6, \( r_1(\tilde{F}) \) is parallel to a surface \( H \subseteq G \times \{-s\} \). Since each component of \( \partial r_1(\tilde{F}) = \partial H \) is parallel to a component of \( \partial G \times \{-s\} \), \( H \) is homeomorphic to \( G \). It follows that \( F \) is homeomorphic to \( G \) and is thus a planar surface.

Let \( m \) be the number of components of \( D^* \cap \hat{\beta}^* \).

Then \( m \) is also the number of components of \( \partial F^* \cap \partial N^* \), hence \( m + 1 \) is the number of components of \( \partial F^* \). Since \( (h|N) \) is standard, each component of \( \partial F^* \cap \partial N^* \) has connected inverse image a disc under \( q \). Hence \( \partial F \) has \( (m + 1) \) components. Thus \( \partial G \) has \( (m + 1) \) components and so \( E \cap \hat{\beta} \) has \( m \) points. Now
\[ 1 - m = \chi(F) = p \chi(F^*) = p(1 - m), \]
which implies \( m = 1 \).
Thus $E \cap \mathring{V}$ consists of a single point. So $\mathring{V}$ is a core of $V$. By Lemma 1.3 $\langle h \rangle$ is standard.
Chapter III: Splitting theorems

III.1 Composite knots

1. **Theorem:** Let $K$ be the composite of the knots $K_1$ and $K_2$ and let $p$ be a prime. Then $K$ is a counterexample to the period $p$ Smith Conjecture if and only if $K_1$ and $K_2$ are counterexamples to the period $p$ Smith Conjecture.

**Proof:** Suppose $K_1$ and $K_2$ are counterexamples to the period $p$ Smith Conjecture. Then there are cyclic actions $\langle h_i \rangle$ of order $p$ on $S^3$ with $\text{Fix}(h_i) = K_i$, $i = 1,2$. Let $C(K_i)$, $N_i$ be an invariant knot space and regular neighborhood of $K_i$ (see I.4.9). Since $\langle h_i | N_i \rangle$ is standard, there is an annulus $A_i$ in $\partial N_i$ such that $h_i(A_i) = A_i$ and $A_i$ is oriented by a core which is a canonical meridian of $N_i$ and $I$-fibers which are intersections of $A_i$ with canonically oriented longitudes. Let $f: A_1 \rightarrow A_2$ be a homeomorphism which preserves the core orientations and reverses the $I$-fiber orientations. We can choose $f$ so that $f(h_1 | A_1) = (h_2^q | A_2)f$ for some $q$, $1 \leq q \leq p - 1$. Thus we get an induced cyclic action $\langle h_0 \rangle$ of order $p$ on $C = C(K_1) \bigcup_f C(K_2)$ such that the components of $\partial A$ are invariant, where $A$ denotes the image in $C$ of $A_1$ and $A_2$. Let $N$ be a solid torus and $g: \partial C \rightarrow \partial N$ a map such that $g(\partial A)$ is a pair of meridians for $N$. Let $M = C \cup_g N$. Then $M$ is $S^3$ and
by Lemma I.5.11 a properly oriented core of $N$ is equivalent to $K$. By Lemma II.1.4 we can extend $\langle h | \partial N \rangle$ to a standard action on $N$. Thus by Lemma I.4.4 $K$ is a counterexample to the period $p$ Smith Conjecture.

Now suppose $K$ is a counterexample to the period $p$ Smith Conjecture. By the first part of the theorem, it suffices to show that the prime factors $J_1, \ldots, J_n$ of $K$ are counterexamples, for $K_1$ and $K_2$ are composites of these factors.

We induct on the number $n$ of prime factors of $K$. If $n = 1$, then there is nothing to prove, so we assume that $n > 1$ and that the assertion is true for all knots having fewer than $n$ prime factors.

Let $\Sigma^3$, $q$, $K^*$, $C(K)$, $N$, $C^*(K^*)$ and $N^*$ be as in I.4.9. Let $(\mu, \lambda)$, $(\mu^*, \lambda^*)$ be as in Lemma I.4.10. $x_0 = \mu \cap \lambda$, $x_0^* = \mu^* \cap \lambda^*$. Let $m = [\mu]$, $\iota = [\lambda]$ in $\pi_1(C(K_1), x_0)$, $m^* = [\mu^*]$, $\iota^* = [\lambda^*]$ in $\pi_1(C^*(K^*), x_0^*)$. Letting $q_0 = (q|_C)$, $(q_0)^*(m) = (m^*)^p$ and $(q_0)^*(\iota) = \iota^*$.

Since $K$ is composite, $C(K)$ contains an essential annulus, by Lemma I.6.6. By Lemma I.6.4, $C^*(K^*)$ also has an essential annulus $A^*$; let $c^*$ be an oriented component of $\partial A^*$. We may assume $x_0^* \in c^*$. Then $[c^*] = (m^*)^a(\iota^*)^b$ for some $a, b$ such that $(a, b) = 1$.

Suppose $q^{-1}(A^*)$ is not connected. Then it has $p$ components $A_0, A_1, \ldots, A_{p-1}$. Let $A_0$ be the component containing $x_0$ and $c_0$ the component of $\partial A_0$ containing $x_0$. Then $q(c_0) = c^*$ and preserves orientation for $c_0$.
properly oriented. \([c_0] = m^r \ell^s\) in \(\pi_1(C;x_0)\). \((q|A_0)\) is a homeomorphism, so \((m^*)^a(\ell^*)^b = [c^*] = (q_0)^*[c_0]\) = \((q_0)^*_\ast(m^r \ell^s) = (m^*)^p \ell^s\) implying that \(s = b\) and \(ps = a\). Thus if \(b \neq 0\), \(s \neq 0\). By Lemma I.6.5 \(A_0\) is essential in \(C(K)\) and so by Lemma I.6.6 \(K\) is a cable knot. But this contradicts the fact that cable knots are prime.

Therefore \(A = q^{-1}(A^*)\) is connected. Let \(c\) be the component of \(\partial A\) containing \(x_0\), and orient \(c\) so the degree of \(q|c\) is positive. Then \([c] = m^r \ell^s\) for some \((r,s) = 1\). \((q|A)\) is a \(p\) - fold cyclic covering, so \((m^*)^p \ell^s = (m^*)^a(\ell^*)^b = (\ell^*)^p = (q_0)^*[c]\) = \((q_0)^*_\ast(m^r \ell^s) = (m^*)^p \ell^s\). This implies \(a = r\) and \(pb = s\).

If \(b \neq 0\), then \(s \neq 0\). Since by Lemma I.6.5 \(A\) is essential in \(C(K)\), Lemma I.6.6 implies that \(K\) is a cabled knot again contradicting the fact that cabled knots are prime. So \(s = b = 0\) and \(a = r = \pm 1\). Thus \(A\) is an invariant annulus with \(\partial A\) a pair of meridians. The closures of the components of \(C(K) - A\), \(C'\) and \(C''\), are knot spaces invariant under \(h\). \(W' = S^3 - \text{int}(C')\) and \(W'' = S^3 - \text{int}(C'')\) are solid tori whose cores \(K', K''\) are such that \(K' \neq K''\) is equivalent to \(K\). By Lemma II.1.4, we can extend \(\langle h|C'\rangle\) and \(\langle h|C''\rangle\) to actions \(\langle h'|\rangle\) and \(\langle h''\rangle\) on \(W'\) and \(W''\) respectively, with \(\text{Fix}(h'|) = K',\) \(\text{Fix}(h'') = K''\). Since \(K'\) and \(K''\) each have fewer than \(n\) prime factors the result follows by induction.
2. **Corollary:** If the Smith Conjecture is true for all prime knots, then it is true for all knots.

   **Proof:** This follows immediately from Theorem 1 and Lemma I.4.6.
III.2 Some special manifolds

1. **Definition**: Let $T_{m,n} \subseteq \partial V'$ be the torus knot in Definition I.3.15. Let $N$ be a regular neighborhood of $T_{m,n}$ in $S^3$ such that $N \cap \partial V'$ is an annulus. Any manifold homeomorphic to $S^3 - \text{int}(N)$ is a **torus knot space**.

2. **Lemma**: Let $X$ be a torus knot space. Then $X = V_1 \cup V_2$, where $V_1$ and $V_2$ are solid tori, $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = A$, an annulus injective in both $V_1$ and $V_2$ but surjective in neither.

   **Proof**: It suffices to show that $S^3 - \text{int}(N)$ has this structure. Let $V_1 = V' - \text{int}(N \cap V')$, $V_2 = (S^3 - \text{int}(V')) - \text{int}(N \cap (S^3 - \text{int}(V')))$, $A = \partial V' - \text{int}(N \cap \partial V')$. The properties of $A$ are easily verified.

3. **Lemma**: Let $X = V_1 \cup V_2$ where $V_1$ and $V_2$ are solid tori, $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = A$, an annulus injective in both $V_1$ and $V_2$ but surjective in neither. If $X \subseteq \Sigma^3$, a homotopy 3-sphere, then $X$ is a torus knot space.

   **Proof**: This follows from Exercise 13.11 of [17].

4. **Lemma**: Let $X \subseteq S^3$ be a torus knot space with the structure given in Lemma 2. Let $B_1 = \partial V_1 - \text{int}(A)$, $B_2 = \partial V_2 - \text{int}(A)$. Then $B_1$ is parallel to $B_2$ in $S^3 - \text{int}(X)$.

   **Proof**: Let $f: S^3 - \text{int}(N) \to X$ be a homeomorphism. By [16], $f$ extends to a homeomorphism $f: S^3 - S^3$. Thus it suffices to prove the result for $S^3 - \text{int}(N)$. Since
\( N \cap \partial V' \) contains a core of \( N \), \( N \cap \partial V' \) is injective and surjective in \( N \), so the result follows by Lemma I.5.9.

5. **Lemma:** Let \( X \) be a torus knot space. Then \( X \) contains no essential tori.

**Proof:** If \( X \) contained an essential torus, then Lemma I.6.7 would imply that \( T_{m,n} \) has a companion, contradicting the fact that torus knots are simple. (See Remark I.3.16.)

6. **Remark:** If \( X \) is a torus knot space given the structure in Lemma 2, then \( A \) is essential in \( X \).
Conversely, it can be shown that any essential annulus in \( X \) is isotopic to \( A \). However, we shall not need this result.

7. **Definition:** Let \( T_{m,n} \subseteq \partial V' \) be a cable as in Definition I.3.15. Let \( N \) be a regular neighborhood of \( T_{m,n} \) in \( V \) such that \( N \cap \partial V' \) is an annulus. Any manifold homeomorphic to \( V - \text{int}(N) \) is a cable space.

8. **Lemma:** Let \( X \) be a cable space. Then \( X = S \cup V \) where \( S = T^2 \times I \), \( V \) is a solid torus, and \( S \cap V = (T^2 \times I) \cap \partial V = A \), an annulus injective in both \( S \) and \( V \) but not surjective in \( V \).

**Proof:** To avoid ambiguity, write \( W,W' \) for the \( V,V' \) appearing in I.3.15. It suffices to show that \( W - \text{int}(N) \) has the above structure. Let \( V = V' - \text{int}(N \cap W') \), \( S = (W - \text{int}(W')) - (N \cap (W - \text{int}(W'))) \), and \( A = \partial W' - \text{int}(N \cap \partial W') \). The properties of \( A \) are easily verified.
9. Lemma: Let \( X = S \cup V \) where \( S = T^2 \times I \), \( V \) is a solid torus, and \( S \cap V = (T^2 \times 1) \cap \partial V = A \), an annulus injective in both \( S \) and \( V \) but not surjective in \( V \). Then \( X \) is a cable space.

Proof: Orient the core of \( V \) and let \((\mu, \lambda)\) be a canonical meridian-longitude pair. Then the core of \( \partial V - \text{int}(A) \) is homologous in \( \partial V \) to \( m\mu + n\lambda \) for some \( |m| \geq 1, |n| \geq 2 \). Let \( T_{m, n} \subseteq \partial W' \) as in I.3.15, \( N \) as in Definition 7. Let \( f: V \to W' - \text{int}(N \cap W') \) be a homeomorphism such that \( f(A) = \partial W' - \text{int}(N \cap \partial W') \) and \( f(\partial V - \text{int}(A)) = \partial N - \text{int}(W') \). Extend \( f \) to a homeomorphism \( f: V \cup (T^2 \times 1) \to (W' - \text{int}(N \cap W')) \cup (\partial N \cap (W - \text{int} W')) \) and then use the product structure to extend \( f \) to \( f: X \to W - \text{int}(W') \).

10. Lemma: Let \( X \) be a cable space given the structure \( S \cup V \) in Lemma 8. Let \( T_0 = T^2 \times 0 \), \( T_1 = \partial X - T_0 \). Then there is another structure \( S' \cup V' \) on \( X \), with \( S' = T^2 \times I \), \( V' \) a solid torus, \( S' \cap V' = (T^2 \times 1) \cap \partial V' = A' \), an annulus injective in both \( S' \) and \( V' \) but not surjective in \( V' \), where if we let \( T_0' = T^2 \times 0 \), \( T_1' = \partial X - T_0' \), we have \( T_0' = T_1 \), \( T_1' = T_0 \).

Proof: Let \( Q \) be a regular neighborhood of \( \partial V - \text{int}(A) \) in \( V \) such that \( Q \cap A \) is a regular neighborhood of \( \partial A \) in \( A \). Let \( a_1, a_2 \) be the components of \( \partial (Q \cap A) \cap \text{int}(A) \). Let \( A_1 = a_1 \times I \), \( A_2 = a_2 \times I \) in
\( T^2 \times I \). Let \( A_0 = \text{cl}(\partial Q \cap \text{int}(V)) \). Let
\[ A' = A_0 \cup A_1 \cup A_2 . \]
Then \( A' \) is an annulus with \( \partial A' \) in \( T_0 \). Let \( V' = (V - \text{int}(Q)) \cup ((A - \text{int}(Q \cap A)) \times I) \) and \( S' = Q \cup ((T_1 - \text{int}(T_1 \cap \partial V)) \cup (Q \cap A)) \times I) \). \( S' \) is a product \( T^2 \times I \) with \( T^2 \times 0 = T_1 \) and \( T^2 \times 1 = A' \cup (T_0 - \text{int}(T_0 \cap V')) \). \( V' \) is a solid torus.

Since \( A_0 \) is parallel to \( \partial V - \text{int}(A) \mod \partial V \), it is injective but not surjective in \( V - \text{int}(Q) \). It follows that \( A' \) is injective but not surjective in \( V' \). Since \( X = S' \cup V' \) and \( S' \cap V = A' \), this completes the proof.

**11. Lemma:** Let \( X \) be a cable space. Then \( X \) contains no essential tori.

**Proof:** We may assume that \( X = V - \text{int}(N) \) as in Definition 7. Let \( K \) be any knot and let \( J = J(K, V, T_{m,n}) \).
Recall the construction of \( J \). \( W \) is a regular neighborhood of \( K \) and \( f: V \to W \) is a faithful homeomorphism.
\( J = f(T_{m,n}) \). If \( X \) contains an essential torus \( T \), then \( T' = f(T) \) is an essential torus in \( W - \text{int} f(N) \). \( T' \) bounds a solid torus \( W' \) in \( S^3 \) with core \( K' \). Since \( T' \) is injective in \( C(J) \), \( W' \) contains \( f(N) \) and \( W' \) is knotted. Hence \( K' \) is a knot with \( C(K') = S^3 - \text{int}(W') \).
So \( \partial W' \) is essential in \( C(J) \). Lemma I.6.7 implies that \( K' \) is a companion of \( J \). It follows from Hilfsatz 2 on p. 263 of [31] that \( \partial W \subseteq C(K') \). \( \partial W \) is essential in \( C(K') \), so by Lemma I.6.7, \( K \) is a companion of \( K' \).
This contradicts the fact that \( K \) is a maximal companion of \( J \). (See Remark I.3.16.)
12. **Remark:** Let $X$ be a cable space with the structure given in Lemma 8. Let $A'' = \alpha \times I$ in $S$, where $\alpha$ is a core of $(T^2 \times 1) - \text{int}(A)$. Let $A'$ be as in Lemma 10. Then $A, A', A''$ are essential annuli in $X$. It can be shown that any essential annulus in $X$ is isotopic to either $A, A'$ or $A''$, but we shall not need this result in the sequel.

13. **Definition:** Let $F$ be a compact surface of genus zero having three boundary components $J_0, J_1, J_2$ (a disc with two holes). Any manifold homeomorphic to $F \times S^1$ is a **composing space**.

14. **Lemma:** Let $X$ be a composing space. Then $X = S_1 \cup S_2$ where $S_i = T^2_i \times I$, $i = 1, 2$, and $S_1 \cap S_2 = \partial S_1 \cap \partial S_2 = A$, an annulus injective in both $S_1$ and $S_2$.

**Proof:** Let $\alpha$ be an arc in $F$ with $\partial \alpha$ in $J_0$ separating $J_1$ from $J_2$. Then $A = \alpha \times S^1$ is the required annulus.

15. **Lemma:** Let $X = S_1 \cup S_2$ where $S_i = T^2_i \times I$, $i = 1, 2$, and $S_1 \cap S_2 = \partial S_1 \cap \partial S_2 = A$ an annulus injective in both $S_1$ and $S_2$. Then $X$ is a composing space.

**Proof:** We can put a product structure $B_1 \times S^1$ on $S_1$, $B_1$ an annulus, such that $A = \alpha \times I$ for an arc in $\partial B_1 \cap \partial B_2$. Then $B_1 \cup B_2$ is a disc with two holes and we can match product structures so that $X = (B_1 \cup B_2) \times S^1$.

16. **Lemma:** Let $X = F \times S^1$ be a composing space embedded in a knot space $C(K)$ in $S^3$ such that the
components $T_i = J_i \times S^1$ of $\partial X$ are incompressible in $C(K)$. Then each $S^1$-fiber of $T_i$ is a meridian of the solid torus $W_i$ in $S^3$ bounded by $T_i$.

**Proof:** $S^3 - \text{int}(X)$ consists of three disjoint toral solids $C_0, C_1, C_2$ with $\partial C_i = T_i$. By renumbering, if necessary, we have $C_0 = W_0$. $T_0$ incompressible in $C(K)$ implies $X \cup C_1 \cup C_2 \subseteq C(K)$. $T_1, T_2$ incompressible in $C(K)$ implies that $C_1$ and $C_2$ are knot spaces.

Let $\alpha$ be an arc in $F$ with $\partial \alpha$ in $J_0$ which separates $J_1$ from $J_2$. Let $A = \alpha \times S^1$, and let $C_1', C_2'$ be the closures of the components of $X - A$ containing $C_1$ and $C_2$ respectively. Then $C_1', C_2'$ are homeomorphic to $C_1, C_2$ respectively and are thus knot spaces. The annulus $A$ is injective in both $C_1'$ and $C_2'$, so by Lemma I.5.11, the components of $\partial A$ are meridians of $W_0$. Thus any $S^1$-fiber of $T_0$ is a meridian of $W_0$. Since we can join any $S^1$-fiber of $T_0$ to any $S^1$-fiber of $T_i$ by annulus in $X$, it follows that any $S^1$-fiber of $T_i$ is a meridian of $W_i, i = 1, 2$.

17. **Lemma:** Let $X$ be a composing space. Then $X$ contains no essential tori.

**Proof:** This follows from Satz 2.8 of [39].

18. **Remark:** If $\alpha$ is an arc joining $J_i$ and $J_\kappa$, $i \neq \kappa$, or an arc with $\partial \alpha$ in $J_i$ which separates $J_j$ from $J_\kappa$, $i \neq j \neq \kappa \neq i$, then $\alpha \times S^1$ is an essential annulus in $X$. It also follows from Satz 2.8 of [39] that any essential annulus in $X$ can be isotoped to one of this
form. We shall need this result but must be careful about our choice of isotopy. This choice is explained in the next lemma.

19. Lemma: Let \( X \) be a composing space with a fixed product structure \( F \times S^1 \). Let \( A = A_1 \cup \ldots \cup A_n \), where the \( A_i \) are disjoint essential annuli in \( X \). Then there is an isotopy \( r_t \) of \( X \) such that for each \( i \), \( r_1(A_i) = \alpha_i \times S^1 \) for an arc \( \alpha_i \) in \( F \). Note that \( \partial A \) consists of \( S^1 \)-fibers of \( \partial X \). If some of the components of \( \partial X \) have the property that all the components of \( \partial A \) contained in them are already \( S^1 \)-fibers, then we can choose the isotopy so that it remains fixed on those components.

Proof: Let \( F \) be a typical fiber of \( F \times S^1 \). Let \( X' = F \times I \) be the manifold obtained by splitting \( X \) along \( F \) and \( g: X' \to X \) the regluing map. \( g \) takes each \( I \)-fiber of \( X' \) to an \( S^1 \)-fiber of \( X \). Let \( f_t \) be an isotopy of \( X \) such that \( f_1(A) \cap F \) is minimal and \( f_t \) is fixed on those components of \( \partial X \) all of whose intersections with \( A \) are \( S^1 \)-fibers.

Claim 1: \( f_1(A_i) \cap F \neq \emptyset \) for all \( i \). If not, then \( g^{-1}f_1(A_i) \) is an incompressible annulus in \( X' = F \times I \) with boundary in \( (\partial F \times \text{int}(I)) \). It follows from Lemma I.5.6 that \( g^{-1}(f_1(A_i)) \) is parallel to an annulus in \( (\partial F \times I) \cup (F \times 0) \); clearly this annulus lies in \( \partial F \times \text{int}(I) \). This implies that \( f_1(A_i) \), and hence \( A_i \), is parallel to an annulus in \( \partial X \), contradicting the fact
that \( A_i \) is essential.

**Claim 2:** No component of \( f_1(A) \cap F \) is a simple closed curve contractible in \( f_1(A) \). For, suppose \( J \) in \( f_1(A_i) \) is such a curve. Then \( J \) bounds a disc \( E' \) in \( f_1(A_i) \), which we may assume contains no other intersection points in its interior. Since \( F \) is incompressible in \( X \), \( J = \partial E \), \( E \) a disc in \( F \). \( E \cup E' \) is a 2-sphere which, by the irreducibility of \( X \), bounds a ball \( B \) in \( X \). We can isotop \( f_1(A) \) so as to move \( E' \) across \( B \) to the other side of \( E \), removing \( J \) (and possibly other components) from the intersection. This can be done by an isotopy fixed outside a small regular neighborhood of \( B \), so that no new intersections are introduced and \( \partial X \) remains fixed. Thus we have contradicted our minimality assumption on \( f_t \).

**Claim 3:** \( f(A) \cap F \) does not contain an arc parallel in \( f(A) \) to an arc in \( \partial f(A) \). For, suppose \( \alpha \) in \( f(A_i) \) is an arc parallel to an arc \( \beta \) in \( \partial f_1(A_i) \). Then \( \alpha \cup \beta = \partial E' \), \( E' \) a disc in \( f_1(A_i) \). We may assume that \( F \cap \text{int}(E') = \emptyset \). \( g^{-1}(E') \) is a disc in \( X' \) with boundary in, say, \( (\partial F \times I) \cup (F \times 0) \). Since the latter surface is incompressible in \( X' \), \( \partial g^{-1}(E') \) bounds a disc \( D \) in \( \partial X \) containing an arc \( \gamma \) in \( \partial F \times 0 \) which separates \( D \) into a disc \( D' \) in \( \partial F \times I \) and a disc \( D'' \) in \( F \times 0 \). So \( g(D') \) is a disc in \( \partial X \) with \( \partial g(D') = \beta \cup g(\gamma) \) and \( g(D'') \) is a disc in \( F \) with \( \partial g(D'') = \alpha \cup g(\gamma) \).

\( E \cup g(D') \cup g(D'') \) is a 2-sphere which, since \( X \) is
irreducible, bounds a ball $B$. We can isotop $f_1(A)$ so that $E'$ moves across $B$ to the other side of $g(D')$, removing $\alpha$ (and possibly other components) from the intersection. The isotopy can be chosen fixed outside a small regular neighborhood of $B$, so that no new intersections are introduced and the components of $\partial X$ other than the one containing $\beta$ remain fixed. This contradicts minimality.

Claim 4: $f_1(A) \cap F$ contains no simple closed curves which are non-contractible in $f_1(A)$. For, suppose such an $\alpha$ in $f(A_1)$ exists. Then we may assume there is an annulus $G'$ in $f_1(A_1)$ with $\partial G' = \alpha \cup \alpha'$, $\alpha'$ a component of $\partial f_1(A_1)$ and $F \cap \text{int}(G') = \emptyset$. Then $g^{-1}(G')$ is an incompressible annulus in $X' = F \times I$ with one boundary component in $\partial F \times \text{int}(I)$ and the other in, say, $F \times 0$. By Lemma I.5.6, $g^{-1}(G')$ is parallel to an annulus $G$ in $(\partial F \times I) \cup (F \times 0)$. There is a component $\gamma$ of $\partial F \times 0$ in $G$ which separates $G$ into an annulus $H'$ in $\partial F \times I$ and an annulus $H''$ in $F \times 0$. Thus $G' \cup g(H') \cup g(H'')$ bounds a solid torus $W$ in $X$ in which $G$ and $g(H') \cup g(H'')$ are parallel. It follows that we can isotop $f_1(A)$ to move $G$ across $W$ and to the other side of $g(H'')$, removing $\alpha$ (and possibly other components) from the intersection. This can be done by an isotopy fixed outside a small regular neighborhood of $W$, so that no new intersections are introduced and the components of $\partial X$ not containing $\alpha'$ are left fixed.
Claim 5: \( g^{-1}(f_1(A) \cap \partial X) \) contains no arc with its boundary in a single component of \( F \times \partial I \). For suppose such an arc \( \beta \) exists, with \( \partial \beta \) in, say, \( F \times 0 \). Then there is a disc \( E' \) in \( f_1(A_1) \) with 
\[ \partial E' = \alpha \cup \alpha' \cup g(\beta) \cup g(\beta') \]
where \( \alpha \) and \( \alpha' \) are spanning arcs of \( f_1(A_1) \) in \( f_1(A_1) \cap F \) and \( \beta' \) is another arc in \( \partial F \times I \) with \( \partial \beta' \) in \( F \times 0 \). \( g^{-1}(E') \) is a disc in \( X' \) with its boundary in \( (\partial F \times I) \cup (F \times 0) \). Since the latter surface is incompressible in \( X' \), \( \partial g^{-1}(E') \) bounds a disc \( D \) in \( (\partial F \times I) \cup (F \times 0) \). \( D \) intersects \( \partial F \times 0 \) in two arcs \( \gamma, \gamma' \) dividing \( D \) into three discs \( D', D'', D''' \) such that \( D', D'' \) are in \( \partial F \times I \) and \( D'' \) in \( F \times 0 \), with 
\[ \partial D' = \beta \cup \gamma, \quad \partial D'' = g^{-1}(\alpha) \cup g^{-1}(\alpha') \cup \gamma \cup \gamma', \quad \partial D''' = \beta' \cup \gamma'. \]
\( g(D) \cup E' \) is a 2-sphere which, by the irreducibility of \( X \), bounds a ball \( B \). We can isotop \( f(A) \) to move \( E' \) across \( B \) to the other side of \( g(b'') \), removing \( \alpha \) and \( \alpha' \) (and possibly other components) from the intersection. This can be done by an isotopy fixed outside a small regular neighborhood of \( B \), so that no new intersections are introduced and the isotopy is fixed on those components of \( \partial X \) not containing \( \beta \cup \beta' \). This contradicts minimality.

The above claims show that each component of \( g^{-1}(A) \) is a disc \( D \) intersecting \( \partial F \times I \) in a pair of arcs joining \( F \times 0 \) and \( F \times 1 \). If, for some component \( T_j \) of \( \partial X \), \( T_j \cap f_1(A) \) does not consist of \( S^1 \)-fibers, then
\( g^{-1}(f_1(A) \cap g^{-1}(T_j)) \) does not consist of \( I \) - fibers.

There is an isotopy \( h_t' \) of \( X \) rel 
\[(F \times \partial I) \cup (\partial F \times I - g^{-1}(T_j)) \]
such that
\[h_1'(g^{-1}(f_1(A)) \cap g^{-1}(T_j)) \text{ consists of } I \text{- fibers.} \]
\( h_t' \) induces an isotopy \( h_t \) of \( X \) rel \((F \cup (\partial X - T_j)) \) such that
\[h_1 f_1(A) \cap T_j \text{ consists of } S^1 \text{- fibers.} \]
We do this for each such \( T_j \) and denote the composite isotopy again by \( h_t \).

Now each component of \( g^{-1}(h_1 f_1(A)) \) is a disc \( D \) meeting \( \partial F \times I \) in two arcs which are \( I \) - fibers of \( X' \).

By Lemma I.5.7, there is an isotopy \( \kappa_t' \) of \( X' \) rel 
\[(F \times 0) \cup (\partial F \times I) \]
such that \( \kappa_1'(D) = \alpha' \times I \) for an arc \( \alpha' \) in \( F \times 0 \). Note that \( \kappa_1(D \cap (F \times 1)) = D \cap (F \times 1) \).

It follows that \( \kappa_t' \) induces an isotopy \( \kappa_t \) of \( X \) rel \( \partial X \) such that \( \kappa_1 h_1 f_1(A_i) = \alpha_i \times S^1 \) for \( \alpha_i \) an arc in \( F \).

Thus the product of the isotopies \( f_t, h_t \) and \( \kappa_t \) is the desired isotopy \( r_t \).
III.3 Haken systems of tori

1. **Definition:** Let \( C \) be a knot space in a homotopy 3-sphere. A **Haken system of tori** in \( C \) is a maximal collection \( \mathcal{J} = \{T_0, \ldots, T_n\} \) of incompressible, pairwise disjoint, mutually non-parallel tori in \( C \), one of which is \( \partial C \).

2. **Lemma:** Every knot space contains a Haken system of tori.

   **Proof:** This follows from Theorem 4 of Haken [15].

3. **Proposition:** Let \( \mathcal{J} = \{T_0, \ldots, T_n\} \) be a Haken system of tori in the irreducible knot space \( C \) in the homotopy 3-sphere \( \Sigma^3 \). Let \( X_1, \ldots, X_m \) be the closures of the components of \( C - (T_0 \cup \ldots \cup T_n) \). Then

   (i) no \( X_i \) contains an essential torus, and

   (ii) if some \( X_i \) contains an essential annulus \( A \), then either

   (a) \( X_i \) is a torus knot space with a structure \( X_i = V_1 \cup V_2 \) as in Lemma 2.2 with \( A \) the annulus appearing there, or

   (b) \( X_i \) is a cable space with a structure \( X_i = S \cup V \) as in Lemma 2.8 such that \( A \) is either the annulus \( A_1 \) appearing there or the annulus \( A'' \) defined in Remark 2.12, or

   (c) \( X_i \) is a composing space and \( A \) is one of the annuli in Remark 2.18.
Proof: (i) Suppose $\partial X_i = T_0 \cup \ldots \cup T_\kappa$ for some $\kappa$, $0 \leq \kappa \leq n$. Let $T$ be an essential torus in $X_i$. Then $T$ is contained in $\text{int}(X_i)$ and so is disjoint from $T_0 \cup \ldots \cup T_n$. Since $T$ is incompressible and $X_i$ is a Haken system, $T$ must be parallel to some $T_j$. If $0 \leq j \leq \kappa$, then $T$ is boundary parallel in $X_i$ and hence inessential in $X_i$. Thus $j > \kappa$. There is an embedded $T^2 \times I$ in $C$ with $T^2 \times 0 = T$ and $T^2 \times 1 = T_j$. But since $T_j$ is contained in $C - X_i$, there must be some $T_\ell$, $0 \leq \ell \leq \kappa$ with $T_\ell$ in $T^2 \times \text{int}(I)$. By Lemma I.5.6 $T_\ell$ is parallel to $T$ in $T^2 \times I$, so $T$ is again inessential in $X_i$.

(ii) Let $X = X_i$. Let $c_0$ and $c_1$ be the components of $\partial A$.

Case 1: $c_0$ and $c_1$ are in the same component of $\partial X$, say $T_0$.

Let $A'$ and $A''$ be the closures of the components of $T_0 - \partial A$. Let $T' = A \cup A'$, $T'' = A \cup A''$. $T'$ and $T''$ each separate $C$ into two components, hence separate $X$ into two components. Thus $A$ separates $X$ into two components, the closures of which we denote by $X'$ and $X''$, so that $\partial X' = T'$, $\partial X'' = T''$. Let $N$ be a regular neighborhood of $T_0 \cup A$ in $X$. Let $U' = \partial N \cap \text{int}(X')$, $U'' = \partial N \cap \text{int}(X'')$. Let $Y = \text{cl}(X' - (N \cap X'))$ and $Y'' = \text{cl}(X'' - (N \cap X''))$. Note that $N \cap X'$ and $N \cap X''$ are regular neighborhoods of $T'$ and $T''$ in $X'$ and $X''$, respectively. Hence they are each homeomorphic to $T^2 \times I$.
They intersect in the injective annulus \( A \) in their boundaries, so by Lemma 2.15 \( N \) is a composing space.

(a) Suppose \( U' \) is compressible in \( X \). Let \( D \) be a compressing disc. Then either \( D \subseteq Y' \) or \( D \subseteq Y'' \cup N \). Suppose \( D \subseteq Y'' \cup N \). Put \( D \) in general position with respect to \( A \) and such that \( D \cap A \) is minimal. If \( D \cap A \neq \emptyset \), then there is a subdisc \( E \) of \( D \) with \( E \cap A = \emptyset \). If \( \partial E = \emptyset F \), \( F \) a subdisc of \( A \), then \( E \cup F \) bounds a ball in \( X \) and so \( \partial E \) can be removed by an isotopy. So \( E \) is a compressing disc for \( A \), contradicting the fact that \( A \) is injective. So \( D \cap A = \emptyset \). But then \( D \subseteq N \cap X' \) and so compresses \( U' \) in \( N \cap X' \), contradicting the fact that \( U' \) is incompressible in the product space \( N \cap X' \). Thus \( D \subseteq X' \). By Lemma 1.5.12 either \( X' \) is a solid torus or \( Y'' \cup N \) is a toral solid. The latter is impossible since \( Y'' \cup N \) has at least two boundary components \( U' \) and \( T_0 \). So \( Y' \) is a solid torus. \( X' \) is homeomorphic to \( Y' \) and so is also a solid torus. \( A \) is injective in \( X' \) since it is injective in \( X \). Suppose \( A \) is surjective in \( X \). Then \( A \) is parallel to \( A' \) in \( X' \) and so is inessential. Hence \( A \) is not surjective in \( X' \).

Suppose \( U'' \) is also compressible in \( X \). By arguments precisely the same as those above, we get that \( X'' \) is a solid torus in which \( A \) is injective but not surjective. Let \( V_1 = X' \), \( V_2 = X'' \). Then by Lemma 2.3 \( X \) is a torus knot space and has the desired structure.
Now suppose $U''$ is incompressible in $X$. By (i), $X$ has no essential tori, so $U''$ is inessential in $X$. By Lemma I.6.3, $U''$ is parallel to a component of $\partial X$.

Suppose $U''$ is parallel to $T_0$. Then $N \cup Y' = T^2 \times I$ with $T_0 = T^2 \times 0$ and $U'' = T^2 \times I$. $A$ is an incompressible annulus in $N \cup Y'$ with $\partial A \subseteq T_0$, so by Lemma I.5.6 is parallel to an annulus in $T_0$, contradicting its essentiality. So $U''$ must be parallel to a component $T_1$ of $\partial X$ in $Y''$. Thus $Y'' = T^2 \times I$ and so $X'' = T^2 \times I$. $A$ is injective in $X'$ and $X''$ but not surjective in $X'$. Let $V = X'$ and $S = X''$. Then by Lemma 2.9, $X$ is a cable space and has the desired structure.

(b) Suppose $U'$ is incompressible in $X$.

If $U''$ is compressible in $X$, then by arguments precisely the same as those above we get that $X$ is a cable space with the desired structure.

Suppose $U''$ is incompressible in $X$. By (i), $X$ contains no essential tori, so $U'$ and $U''$ are parallel to components of $\partial X$. Suppose $U'$ is parallel to $T_0$. Then exactly as in subcase (a) we get that $A$ is inessential in $X$. So $U'$ is not parallel to $T_0$, and by the same argument neither is $U''$. Suppose $U'$ is parallel to a component $T_1$ of $\partial X$ in $Y''$. Then $N \cup Y'' = T^2 \times I$, but this is impossible since $N \cup Y''$ has at least three boundary components $U'$, $T_0$, and $T_1$. So $U'$ must be parallel to a component $T_1$ of $\partial X$ in $Y'$. Hence $Y' = T^2 \times I$ and so $X' = T^2_1 \times I$. Similarly, $X''$ is
homeomorphic to $N \cap X$". So $X$ is homeomorphic to $N$ and is thus a composing space which is easily seen to have the desired structure.

**Case 2:** Suppose $c_0 \subseteq T_0$ and $c_1 \subseteq T_1$, where $T_0$ and $T_1$ are different components of $\partial X$.

Let $N_0$ be a regular neighborhood of $T_0$ in $X$, $N_1$ a regular neighborhood of $T_1$ in $X$, and $N_2$ a regular neighborhood of $\text{cl}(A - (A \cap (N_0 \cup N_1)))$ such that $N = N_0 \cup N_1 \cup N_2$ is a regular neighborhood of $T_0 \cup \text{AUT}_1$ in $X$. Note that $N_0$ and $(N_1 \cup N_2)$ are homeomorphic to $T^2 \times I$ and intersect in the injective annulus $N_0 \cap N_2$ in their boundaries. By Lemma 2.15 $N$ is a composing space. Let $U = \partial N \cap \text{int}(X)$ and $Y = \text{cl}(X - N)$.

(a) Suppose $U$ is compressible in $X$. Let $D$ be a compressing disc. Then either $D \subseteq N$ or $D \subseteq Y$. Suppose $D \subseteq Y$. Then $D$ compresses $U$ in the product $N = T^2 \times I$, which is impossible. So $D \subseteq Y$. By Lemma I.5.12 either $Y$ is a solid torus or $N$ is a toral solid. The latter is impossible since $N$ has three boundary components. So $Y$ is a solid torus. $N_0 = T^2 \times I$ with $T_0 = T^2 \times 0$ and $\partial N_0 - T_0 = T^2 \times 1$. Let $\alpha$ and $\beta$ be the components of $\partial(N_2 \cap N_0)$. Let $F$ be the annulus $\alpha \times I$ in $N_0$ and $G$ the annulus $\beta \times I$ in $N_0$. Let $Q$ and $R$ be the closures of the components of $N_0 - (F \cup G)$, with $Q = (N_2 \cap N_0) \times I$ and $R = \text{cl}(N_0 - Q)$. Let $V = Y \cup R$. $V$ is a solid torus. Let $S = N_1 \cup N_2 \cup Q$. $S = T^2 \times I$. 
$S \cap V$ is the annulus $B = F \cup G \cup (\partial N_2 - (N_2 \cap (N_0 \cup N_1)))$
$\cup \partial \overline{N_1} - (N_1 \cap N_2)$. Let $B' = R \cap T_0$. Suppose $B$ is compressible in $V$ or in $S$ with compressing disc $D$.
Then there is a subannulus $H$ of $B$ with
$\partial H = \partial D \cup (\alpha \times 0)$. Let $E = D \cup H$. Since $\alpha \times 0$ is parallel to $c_0$ in $T_0$, $\alpha \times 0$ is non-contractible in $T_0$. Thus $E$ compresses $T_0$, a contradiction. So $B$ is injective in both $V$ and in $S$. If $B$ is surjective in $V$, then it is parallel to $B'$ in $V$. But this implies that $V \cup S$ is homeomorphic to $S$, hence $X = T^2 \times I$
with $T_0 = T^2 \times 0$, $T_1 = T^2 \times 1$, so $T_0$ and $T_1$ are parallel, contradicting the fact that they are members of a Haken system. So $B$ is not surjective in $V$. Thus by
Lemma 2.9, $X$ is a cable space with $B$ taking the role of the annulus $A$ appearing there. It is easy to see that $A$
is the annulus $A'$ in Remark 2.12.

(b) Suppose $U$ is incompressible in $X$. By (i), $X$
contains no essential tori. So $U$ is parallel to a compon-
ent of $\partial X$. If $U$ is parallel to $T_0$, then $N = T^2 \times I$
which is impossible since it has at least three boundary
components. So $U$ must be parallel to a component $T_2$ of
$\partial X$ in $Y$. Hence $Y = T^2 \times I$ with $U = T^2 \times 0$ and
$T_2 = T^2 \times 1$. Thus $X = Y \cup N$ is homeomorphic to $N$ and
is therefore a composing space which has the desired
structure.

4. **Lemma**: Let $C \subseteq S^3$ be a knot space, $T_0$ an
incompressible torus in $C$ separating $S^3$ into the solid
torus \( W' \) and the knot space \( W'' \), and \( R \) a torus in \( C \) such that \( R \cap W' = F \), an annulus separating \( W' \) into components with closures \( W'_1 \) and \( W'_2 \) such that \( W'_1 \subseteq C \) is a solid torus in which \( F \) is injective but not surjective. Let \( F' = \partial W'_2 - \text{int}(F) \), \( G' = \partial W'_1 - \text{int}(F) \), \( G = R \cap W'' \). Let \( W''_1 \), \( W''_2 \) be the closures of the components of \( W'' - G \), numbered so that \( \partial W''_1 = G \cup G' \) and \( \partial W''_2 = G \cup F' \). Then either \( G \) is parallel to \( G' \) in \( W''_1 \) or \( G \) is parallel to \( F' \) in \( W''_2 \).

**Proof:** \( W'_1 \), \( W''_1 \), and \( W''_2 \) are toral solids. \( W'_2 \), \( W''_1 \), and \( W''_2 \) are contained in \( C \). \( F \) is an annulus in \( W' \) with \( \partial F \) injective in \( \partial W' \). Since \( F \) is not parallel to \( G' \) in \( W'_1 \), part (2) of Lemma I.5.9 implies that \( F \) is injective in \( W' \) and \( W'_2 \) is a solid torus. By part (1) of I.5.9, \( F \) is injective in \( W'_2 \), and since \( F \) is not surjective in \( W'_1 \), it must be surjective in \( W'_2 \). Hence \( F \) is parallel to \( F' \) in \( W'_2 \). It follows that \( G' \) and \( F' \) are injective but not surjective in \( W' \).

Let \( \bar{W}_1 = W'_1 \cup W''_1 \), \( \bar{W}_2 = W'_2 \cup W''_2 \).

Suppose \( W_2 \) is a solid torus. Since \( F' \) is parallel to \( F \) in \( W'_2 \), \( W''_2 \) is also a solid torus. Since \( W'_2 \subseteq C \) and \( T_0 \) is incompressible in \( C \), \( F' \) is injective in \( W''_2 \). If \( F' \) is also surjective in \( W''_2 \), then \( F' \) is parallel to \( G \) in \( W''_2 \) and we are done. If \( F' \) is not surjective in \( W''_2 \), then let \( V_1 = W' \), \( V_2 = W''_2 \) and \( A = F' \). By Lemma 2.3, \( W' \cup W''_2 = V_1 \cup V_2 \) is a torus knot space. By Lemma 2.4, \( B_1 = G' \) is parallel to
Suppose \( W_2 \) is a knot space. Then \( W_1 \) is a solid torus containing an annulus \( G' \) with \( \partial G' \) injective in \( \partial W_1 \). Since \( F \) is injective in \( W_1' \), \( G' \) is injective in \( W_1' \). So, by part (2) of Lemma I.5.9, \( W_1'' \) is a solid torus, and since \( F \), and hence \( G' \), is not surjective in \( W_1' \), \( F \) is not parallel to \( G' \) in \( W_1' \), which implies that \( G' \) must be injective in \( W_1 \). Thus, by part (1) of Lemma I.5.9, \( G' \) is injective in \( W_1'' \) and hence parallel to \( G \) in \( W_1'' \), so we are done.

5. **Proposition:** Let \( C \) be a knot space in \( S^3 \) and \( \mathcal{S} = \{ T_0, \ldots, T_n \} \) a Haken system of tori in \( C \). Let \( X_1, \ldots, X_m \) be the closures of the components of \( C - (T_0 \cup \ldots \cup T_n) \). Let \( T \) be an arbitrary essential torus in \( C \). Then there is an isotopy \( f_t \) of \( C \) rel \( \partial C \) such that either

(i) \( f_1(T) \) is a member of \( \mathcal{S} \), or

(ii) there is a collection \( X_1, \ldots, X_s \) of composing spaces such that \( f_1(T) \subseteq (X_1 \cup \ldots \cup X_s) \), and each component of \( T \cap X_i \) is an essential annulus in \( X_i \).

**Proof:** Choose \( f_t \) so that \( T' = f_1(T) \) is in general position with respect to \( T_0 \cup \ldots \cup T_n \) and \( T' \cap (T_0 \cup \ldots \cup T_n) \) is minimal.

If \( T' \cap (T_0 \cup \ldots \cup T_n) = \emptyset \), then \( T' \) is parallel to a member \( T_i \) of \( \mathcal{S} \), since \( \mathcal{S} \) is a Haken system.

Since \( T' \) is essential, \( T_i \not\in \partial C \), so there is a further isotopy rel \( \partial C \) taking \( T' \) to \( T_i \). We denote the
composition of this new isotopy and $f_t$ again by $f_t$ and we are done.

Suppose $T' \cap (T_0 \cup \ldots \cup T_n) \neq \emptyset$. Since $C$ is irreducible and $T'$, $T_i$ are incompressible, arguments similar to those in Claim 2 of the proof of Lemma 2.19 show that the intersection contains no simple closed curves contractible in $T'$ or $T_i$. Thus the intersection consists entirely of simple closed curves which are non-contractible, and hence parallel, in $T'$ and in those $T_i$ in which they occur.

Let $X = X_i$ be a component with $X \cap T' \neq \emptyset$. Each component $A$ of this intersection is an annulus in $X$. If $A$ is not essential in $X$, then $A$ is parallel to an annulus $A'$ in $\partial X$ in some solid torus $W$ in $X$. We can isotop $T'$ to move $A$ across $W$ to the other side of $A'$, removing $\partial A$ (and possibly other components) from the intersection. The isotopy can be chosen fixed outside a small regular neighborhood of $W$, so that no new intersections are introduced and the isotopy is fixed on $\partial C$.

Thus $A$ is an essential annulus in $X$. By Proposition 3, $X$ is either a torus knot space, cable space, or composing space and has the structure with respect to $A$ as stated there. If we can show that $X$ is neither a torus knot space nor a cable space, then $X$ is a composing space and we are done.

**Case 1:** Suppose $X$ is a torus knot space. Then
\( X = V_1 \cup V_2 \) with \( V_1, V_2 \) solid tori and \( V_1 \cap V_2 = A \) as in Lemma 2.2. Let \( A_1 \) be a component of 
\( T' \cap (S^3 - \text{int}(X)) \) having a common boundary component with A. By Lemma 2.4, \( \partial V_1 - \text{int}(A) \) is parallel to 
\( \partial V_2 - \text{int}(A) \) in \( S^3 - \text{int}(X) \), so the components of \( \partial A \) 
are injective and surjective in \( S^3 - \text{int}(X) \). Thus \( A_1 \) 
is injective and surjective in \( S^3 - \text{int}(X) \) and, by Lemma 
I.5.9, divides it into two components \( W_1 \) and \( W_2 \) in which 
it is parallel to \( \partial W_1 - \text{int}(A_1) \) and \( \partial W_2 - \text{int}(A_1) \) 
respectively. One of them, say \( W_1 \), is contained in C. 

We can isotop \( T' \) to move \( A_1 \) across \( W_1 \) to the other 
side of \( \partial W_1 - \text{int}(A_1) \) thus removing \( \partial A_1 \) (and possibly 
other components) from the intersection. This can be done 
by an isotopy fixed outside a small regular neighborhood 
of \( W_1 \), so that no new intersections are introduced and 
\( \partial C \) is left fixed. This contradicts minimality. So \( X \) 
is not a torus knot space.

**Case 2:** Suppose \( X \) is a cable space.

(a) Suppose \( X = S \cup V, S = T^2 \times I, V \) a solid 
torus, \( S \cap V = (T^2 \times I) \cap \partial V = A \). \( X \) separates \( S^3 \) into 
two toral solids \( Y \) and \( Z \) with \( \partial Y = T_0 = T^2 \times 0 \) and 
\( \partial Z = T_1 = \partial X - T_0 \). One is a knot space and the other is 
a solid torus.

Suppose \( Y \) is a knot space and \( Z \) a solid torus. 
Then \( V \cup Z \) is also a solid torus. Since \( A \) is not 
parallel to \( \partial V - \text{int}(A) \) in \( V \), \( \partial V - \text{int}(A) \) is parallel 
to \( T_1 \) - \( \text{int}(\partial V \cap T_1) \) by Lemma I.5.9. Let \( A_1 \) be a
component of \((S^3 - \text{int}(X)) \cap T'\). Then \(A_1 \subseteq Z\) and it follows as in Case 1 that we can remove \(\partial A_1\) from the intersection by an isotopy rel \(\partial C\) which introduces no new intersections, thus contradicting minimality.

Suppose \(Y\) is a solid torus and \(Z\) a knot space. Let \(A_1\) be a component of \(T' \cap Z\) having a boundary component \(c_0\) in common with \(A\). Let \(c_1\) be its other boundary component. We consider three cases.

i) \(c_1\) is the other boundary component of \(A\). Then \(T' = A \cup A'\). We now apply Lemma 4 with \(R = T'\), \(T = T_1\), \(W_1' = V\), \(F = A\), \(F' = T_1 - \text{int}(T_1 \cap \partial V)\), \(G = A_1\), and \(G' = \partial V - \text{int}(A)\) to conclude that either \(A_1\) is parallel to \(G'\) in \(Z\) or \(A_1\) is parallel to \(F'\) in \(Z\). In either case we can isotop \(T'\) to remove \(\partial A\) (and possibly other components) from the intersection. This can be done keeping \(\partial C\) fixed and introducing no new intersections. This contradicts minimality.

ii) \(c_1\) is in \(\partial V - A\). Let \(H\) be the annulus in \(\partial V - \text{int}(A)\) joining \(c_1\) and the component \(d\) of \(\partial A\) not in \(\partial A_1\). Let \(N = T_1 \times I\) be a regular neighborhood of \(T_1\) in \(Z\) with \(T_1 = T_1 \times 0\), such that \(N \cap T'\) consists of product annuli. Let \(G = (A_1 - (c_1 \times I)) \cup (H \times I) \cup (d \times I)\). Let \(G' = \partial V - \text{int}(A)\), \(F = A\), \(F' = T_1 - \text{int}(T_1 \cap \partial V)\), \(W_1' = V\), \(R = A \cup G\), and \(T = T_1\). Apply Lemma 4 to conclude that either \(G\) is parallel to \(G'\) or to \(F'\) in \(Y\). In the first case, it follows that \(A_1\) is parallel to \(G' - \text{int}(H)\) in \(Y\), so
that $\partial A_1$ (and possibly other components) can be removed from the intersection by an isotopy. In the second case, let $A_2$ be the component of $T' \cap Y$ having $d$ in its boundary. If the other component $e$ of $\partial A_2$ lies in $F'$, then by Lemma I.5.6, $A_2$ is parallel to an annulus in $F'$ so that $\partial A_2$ can be removed by an isotopy. If $e$ is not in $F'$, then it is in $H$ so that $A_2$ intersects $H \times I$ in a product annulus, and it easily follows that $A_2$ is parallel to an annulus in $H$ and so $\partial A_2$ can be removed by an isotopy. As before, we contradict minimality.

iii) $c_1$ is contained in $T_1 - (T_1 \cap \partial V)$. Let $F' = T_1 - \text{int}(T_1 \cap \partial V)$, $F = A$, and $W_1' = V$. Let $H$ be the annulus in $F'$ joining $c_1$ to the other component $d$ of $\partial A$. Let $N = T_1 \times I$ be a regular neighborhood of $T_1$ in $Y$ such that $T_1 = T_1 \times 0$ and $T' \cap N$ consists of product annuli. Let $G = (d \times I) \cup (H \times I) \cup (A_1 - (c_1 \times I))$. Let $R = A \cup G$, $T = T_1$ and apply Lemma 4 to conclude that either $G$ is parallel to $G'$ or to $F'$ in $Y$. In the first case, let $A_2$ be the component of $T' \cap Y$ containing $d$ in its boundary. The other component $e$ of $\partial A_2$ lies either in $G'$ or in $F'$. It follows from Lemma I.5.6 that $A_2$ is parallel to an annulus either in $G'$ or in $F'$, so $\partial A_2$ can be removed from the intersection by an isotopy. In the second case, $A_2$ is parallel to an annulus in $F'$ so $\partial A_2$ can be removed by an isotopy. As before, we contradict minimality.

This completes subcase (a).
(b) Suppose \( X = S \cup V \), \( S = T^2 \times I \), \( V \) a solid torus, \( X \cap S = (T^2 \times I) \cap \partial V = B \), \( T_0 = T^2 \times 0 \), \( T_1 = \partial X - T_0 \), and \( A = \alpha \times I \), \( \alpha \) a core of \( T_1 = \text{int}(T_1) \cap \partial V \). \( S^3 = \text{int}(X) \) has two components \( Y \) and \( Z \) with \( \partial Y = T_0 \) and \( \partial Z = T_1 \), one of which is a knot space, the other a solid torus. By Lemma 2.10, we may assume that \( Y \) is the knot space and \( Z \) the solid torus.

Let \( A_1 \) be the component of \( T' \cap Z \) having \( \alpha \) in its boundary. \( Z \cup V \) is a solid torus, and since \( \partial V - \text{int}(A) \) is injective but not surjective in \( V \), Lemma I.5.9 implies that \( \partial V - \text{int}(A) \) is parallel to \( T_1 = \text{int}(T_1) \cap \partial V \) in \( Z \). So \( \alpha \), and hence \( A_1 \), is both injective and surjective in \( Z \). So \( A_1 \) divides \( Z \) into two solid tori \( W_1 \) and \( W_2 \) with \( A_1 \) parallel to \( \partial W_1 - \text{int}(A_1) \) in \( W_1 \) and \( A_1 \) parallel to \( \partial W_2 - \text{int}(A_1) \) in \( W_2 \). One of these, say \( W_1 \), is contained in \( C \). So we can isotop \( T \) to remove \( \partial A_1 \) from the intersection. As usual, this contradicts minimality.

Thus \( X \) is not a cable space.
III.4 The Main Theorem

1. Notation: Throughout this section \(<h>\) is a cyclic action of prime order \(p\) on \(S^3\) having fixed point set a knot \(K\). \(N\) is an invariant regular neighborhood of \(K\) and \(C(K) = C\) the corresponding invariant knot space. \(\Sigma^3 = S^3/<h>\), with \(q: S^3 \to \Sigma^3\) the quotient map. \(K^* = q(K)\), \(N^* = q(N)\), \(C^* = q(C)\).

2. Lemma: Let \(X^* \subseteq C^*\) be a torus knot space such that either \(\partial X^* = \partial C^*\) or \(\partial X^*\) is an incompressible torus in the interior of \(C^*\). Then each component of \(q^{-1}(X^*)\) is a torus knot space.

Proof: If \(q^{-1}(X^*)\) is not connected then, since \(p\) is prime, it consists of \(p\) components each of which is homeomorphic to \(X^*\) and so they are torus knot spaces.

Suppose \(X = q^{-1}(X^*)\) is connected. By Lemma 2.2, \(X^* = V_1^* \cup V_2^*\), \(V_1^*\), \(V_2^*\) solid tori, \(V_1^* \cap V_2^* = A^*\) an annulus injective in both \(V_1^*\) and \(V_2^*\) but surjective in neither.

Suppose \(A = q^{-1}(A^*)\) is connected. Then both \(V_1 = q^{-1}(V_1^*)\) and \(V_2 = q^{-1}(V_2^*)\) are connected. Thus \(X = V_1 \cup V_2\), \(V_1\), \(V_2\) solid tori, \(V_1 \cap V_2 = A\), an annulus. By Lemma I.6.5, \(A\) is essential in \(X\). Hence \(A\) must be injective in both \(V_1\) and \(V_2\). If \(A\) is surjective in either \(V_1\) or \(V_2\) then it must be parallel to either \(\partial V_1 - \text{int}(A)\) or \(\partial V_2 - \text{int}(A)\), contradicting essentiality. Thus, since \(X \subseteq S^3\), \(X\) is a torus knot.
space, by Lemma 2.3.

Suppose \( q^{-1}(A^*) \) is not connected. Then it has \( p \) components \( A_0, \ldots, A_{p-1} \), each of which is an essential annulus in \( X \) by Lemma I.6.5. If neither \( q^{-1}(V_1^*) \) nor \( q^{-1}(V_2^*) \) is connected, then neither is \( X \), a contradiction. So at least one of them, say \( V_1 = q^{-1}(V_1^*) \), is connected.

Suppose \( q^{-1}(V_2^*) \) is not connected. Then it has \( p \) components \( U_0, \ldots, U_{p-1} \). Thus \( X = V_1 \cup U_0 \ldots \cup U_{p-1} \), where \( V_1 \cap U_i = A_i \). Since \( A_i \) is essential in \( X \) it is injective in \( V_1 \) and injective but not surjective in \( U_i \).

Suppose \( A_0 \) is not surjective in \( V_1 \). Then by Lemma 2.3, \( V_1 \cup U_0 \) is a torus knot space. By Lemma 2.4, \( \partial U_0 - \text{int}(A_0) \) is parallel to \( \partial V_1 - \text{int}(A_0) \) in the solid torus \( S^3 - \text{int}(V_1 \cup U_0) \). But for \( i > 0 \), \( \partial U_i - \text{int}(A_i) \) is an annulus in \( S^3 - \text{int}(V_1 \cup U_0) \) with its boundary in \( \partial V_i - \text{int}(A_0) \). Thus by Lemma I.5.6, \( \partial U_i - \text{int}(A_i) \) is parallel to \( A_i \) in \( U_i \), contradicting the fact that \( A_i \) is essential in \( X \).

So assume \( A_0 \) surjective in \( V_1 \). Renumber the \( A_i \) so that \( A_i = h^i(A_0) \). \( h(V_1) = V_1 \), so we have the commutative diagram

\[
\begin{array}{ccc}
\pi_1(V_1) & \longrightarrow & \pi_1(V_1) \\
\uparrow & & \uparrow \\
\pi_1(A_0) & \longrightarrow & \pi_1(A_i) \\
\downarrow & & \downarrow \\
\pi_1(U_0) & \longrightarrow & \pi_1(U_i)
\end{array}
\]
where the horizontal maps are induced by the appropriate restrictions of $h$. The diagram shows that each $A_i$ is surjective in $V_1$ and that a generator of $\pi_1(A_i)$ maps to the same power $m \neq 0$, $\pm 1$ of a generator of $\pi_1(U_i)$ for each $i$. Therefore $\pi_1(X)$ has the presentation

$$\langle t_0, \ldots, t_{p-1} : t_0^m = \cdots = t_{p-1}^m \rangle,$$

so that $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}_m \oplus \cdots \oplus \mathbb{Z}_m$. But since $\partial X$ is a torus, $X$ is a toral solid and so has $H_1(X) \cong \mathbb{Z}$, a contradiction.

Thus $V_2 = q^{-1}(V_2^*)$ is connected. So $X = V_1 \cup V_2$, $V_1$, $V_2$ solid tori, and $V_1 \cap V_2 = A_0 \cup \ldots \cup A_{p-1}$. Let $B_0', \ldots, B_{p-1}'$ be the components of $\partial V_1 - \text{int}(A_0 \cup \ldots \cup A_{p-1})$ and $B_0'', \ldots, B_{p-1}''$ be the components of $\partial V_2 - \text{int}(A_0 \cup \ldots \cup A_{p-1})$.

We claim that (after renumbering) $\partial B_k' = \partial B_k''$. $\partial B_k' \subseteq \partial V_1$, which separates $S^3$ into two components with closures $Y_1$ and $Y_2$, with $V_1 = Y_1$, say. $\partial B_k'$ separates $\partial V_2$ into two components whose closures are annuli $F_k$ and $G_k$, one of which, say $G_k$, contains the annuli $A_j$ and $A_\ell$ adjacent to $B_k'$ on $\partial V_1$. If $\text{int}(F_k) \cap V_1 = \phi$, then we are done; simply set $B_k'' = F_k$. If $\text{int}(F_k) \cap V_1 \neq \phi$, then $F_k$ contains two subannuli $B_j'$ and $B_\ell''$, each of which shares a boundary component with $B_k'$, and hence (by numbering correctly) $B_j''$ shares a boundary component with $A_j$, and $B_j''$ shares a boundary component with $A_\ell$. Thus $B_j''$ separates $Y_2$ into two components with closures $Z'$ and $Z''$, one of which, say
Z', contains A_j. The other, Z'', then contains B_{L''}. But B_{L''} separates Z'' into two components with closures W', W'', one of which, say W'', contains A_L. We observe that Z' ∩ W'' = ∅. But since A_j ⊆ Z' and A_L ⊆ W'', we have V_2 ⊆ Z' and V_2 ⊆ W'', which is impossible. Therefore ∂B_{K} = ∂B_{K'}.

Now renumber so that h^K(B_0') = B_{K'}. It follows that h^K(B_0'') = B_{K''}. Since ∂X has p components B_{K'} ∪ B_{K''}, k = 0, ..., p-1, and every torus in S^3 separates, S^3 - int(X) has p components W_0, ..., W_{p-1}, which we number so that ∂W_k = B_{K'} ∪ B_{K''}. Since h^K(∂W_0) = ∂W_k, and h(X) = X, it follows that h^K(W_0) = W_k. So h cyclically permutes W_0, ..., W_{p-1}.

But since X ⊆ C, K ⊆ W_k for some k, so that ⟨h|W_k⟩ has a fixed point, contradicting the fact that the W_k are disjoint and permuted by h.

Thus q^{-1}(A^*) must have been connected and X a torus knot space.

3. **Lemma**: Let X^* ⊆ C* be a cable space such that ∂X^* consists of incompressible tori in the interior of C*, or perhaps one component equal to ∂C* and the other in the interior. Then each component of q^{-1}(X^*) is a cable space.

**Proof**: If q^{-1}(X^*) is not connected, then, since p is prime, it has p components, each of which is homeomorphic to X^*; thus they are cable spaces.

Suppose X = q^{-1}(X^*) is connected. By Lemma 2.8,
$X^* = S^* \cup V^*$, $S^* = T^2 \times I$, $V^*$ a solid torus, $S^* \cap V^* = (T^2 \times 1) \cap \partial V^* = A^*$, an annulus injective in both $S^*$ and $V^*$, but not surjective in $V^*$. Let $T_0^* = T^2 \times 0$, $U^* = T^2 \times 1$, $T_1^* = \partial X^* - T_0^*$. $\Sigma^3 - \text{int}(X^*)$ has two components $Y^*$ and $Z^*$ with $\partial Y^* = T_0^*$, $\partial Z^* = T_1^*$, one of which is a knot manifold contained in $C^*$, the other a homotopy solid torus containing $N^*$. By Lemma 2.10, we may assume that $Y^*$ is the knot manifold and $Z^*$ the homotopy solid torus.

Suppose $A = q^{-1}(A^*)$ is connected. Then both $S = q^{-1}(S^*)$ and $V = q^{-1}(V^*)$ are connected. $S = T^2 \times I$ and $V$ is a solid torus. By Lemma I.6.5, $A$ is essential in $X$. Therefore it must be injective in both $S$ and $V$, but not surjective in $V$. Thus by Lemma 2.9, $X$ is a cable space.

Suppose $q^{-1}(A^*)$ is not connected. Then $q^{-1}(A^*)$ has $p$ components $A_0, \ldots, A_{p-1}$, each of which is an essential annulus in $X$. If neither $q^{-1}(S^*)$ nor $q^{-1}(V^*)$ is connected, then $q^{-1}(X^*)$ is not connected, a contradiction.

We first suppose that $S = q^{-1}(S^*)$ is connected. Then $T_0 = q^{-1}(T_0^*)$, $U = q^{-1}(U^*)$, and $Y = q^{-1}(Y^*)$ are connected. $U$ separates $S^3$ into a knot space $Q = Y \cup S$ and a solid torus $R = q^{-1}(V^* \cup Z^*)$.

Suppose $q^{-1}(V^*)$ is not connected. Then it has $p$ components $V_0, \ldots, V_{p-1}$ with $V_k \cap S = A_k$. Since $A_k$ is
essential in \( X \), it is injective in both \( S \) and \( V_\kappa \) but not surjective in \( V_\kappa \). Let \( B_\kappa = \partial V_\kappa - \text{int}(A_\kappa) \). \( B_0 \) is an annulus in the solid torus \( R \) such that \( \partial B_0 \) is injective in \( \partial R \). Since \( B_0 \) is not parallel to \( A_0 \) in \( V_0 \), Lemma I.5.9 implies that \( R - \text{int}(V_0) \) is a solid torus in which \( B_0 \) is parallel to \( \partial R - \text{int}(A_0) \). But \( B_1, \ldots, B_{p-1} \) are annuli in \( R - \text{int}(V_0) \) with their boundaries in \( \partial R - \text{int}(A_0) \). It follows from Lemma I.5.6 that \( B_\kappa \) is parallel to \( A_\kappa \) in \( V_\kappa \) for \( \kappa > 0 \), contradicting the fact that \( A_\kappa \) is essential in \( X \).

Therefore \( \text{q}^{-1}(V^*) \) is connected. Let \( B_0', \ldots, B_{p-1}' \) be the components of \( \partial V - \text{int}(A_0 \cup \ldots \cup A_{p-1}) \) and \( B_0'', \ldots, B_{p-1}'' \) the components of \( \partial V - \text{int}(A_0 \cup \ldots \cup A_{p-1}) \). We claim that (after renumbering) \( \partial B_\kappa' = \partial B_\kappa'' \). Each \( B_\kappa' \) is an annulus in the solid torus \( R \) which divides \( R \) into two components. There must be an innermost such annulus \( B_0' \), i.e., \( \partial B_0' = \partial F_0 \), \( F_0 \) an annulus in \( U \) such that \( F_0 \cap (B_1' \cup \ldots \cup B_{p-1}') = \emptyset \). So \( F_0 \) must either be an \( A_j \) for some \( j \) or a \( B_\kappa'' \) for some \( \kappa \). If \( F_0 = A_j \), then \( A_j \cup B_0' = \partial V \) since \( A_j \subseteq \partial V \). Thus \( p = 1 \), a contradiction. Therefore \( F_0 = B_\kappa'' \). Renumber so that \( F_0 = B_0'' \). Now let \( B_\kappa' = h^K(B_0) \) and \( B_\kappa'' = h^K(B_0'') \). It follows that \( \partial B_\kappa' = \partial B_\kappa'' \).

Now \( B_\kappa' \cup B_\kappa'' \) divides \( S^3 \) into two toral solids, one of which, call it \( W_\kappa \), lies in \( R \). Observe that \( W_i \cap W_j = \emptyset \) for \( i \neq j \), since \( \partial W_\kappa = B_\kappa' \cup B_\kappa'' \). Now \( h(S) = S \) implies \( h(R) = R \); hence \( h^K(W_0) = W_\kappa \), so h
cyclically permutes the $W_K$. But $R = V \cup W_0 \cup \ldots \cup W_{p-1}$
and $V \subseteq C$ so that $K \subseteq W_K$ for some $K$, so that $h$ has
fixed points in $W_K$, a contradiction.

We now suppose that $V = q^{-1}(V^*)$ is connected. Then
$q^{-1}(S^*)$ is not connected and so has $p$ components
$S_0, \ldots, S_{p-1}$ with $S_i \cap V = A_i$. Since $A_i$ is essential
in $X$ it is injective in $S_i$ and $V_i$. It may possibly
also be surjective in $V_i$. $q^{-1}(Y^*)$ has $p$ components
$Y_0, \ldots, Y_{p-1}$ intersecting $X$ in the components
$T_{0,0}, \ldots, T_{0,p-1}$ of $q^{-1}(T_0^*)$ contained in $S_0, \ldots, S_{p-1}$,
respectively. $Z = q^{-1}(Z^*)$ is connected.

Consider the manifold $M = (S_1 \cup \ldots \cup S_{p-1}) \cup$
$(Y_1 \cup \ldots \cup Y_{p-1}) \cup V$. $M$ is a toral solid contained in
$C$. Suppose it is a solid torus. Then by Lemma I.5.10,
each $Y_i$ is contained in a ball $B_i$ in $M$. But this
implies that $\partial Y_i = T_{0,i}$ is not injective in $C$, con-tradicting the fact that it is the lifting of an injective
torus $T_0^*$ in $C^*$. Thus $M$ is a knot space.

$A_0$ is injective in $(Y_0 \cup S_0)$ and in $M$, both of
which are knot spaces. Therefore by Lemma I.5.11, the
components of $\partial A_0$ are meridians of the solid torus $Z$.
It follows that the components of $\partial A^*$ are meridians of
the homotopy solid torus $Z^*$. Given one such meridian $\mu^*$
of $Z^*$ we see that $q^{-1}(\mu^*)$ has $p$ components
$\mu_0, \ldots, \mu_{p-1}$, contained in the boundaries of $A_0, \ldots, A_{p-1}$,
respectively. But $Z^* = Z/\langle h \rangle$, and so by Lemma II.2.3,$q^{-1}(\mu^*)$ is connected. This contradiction completes the
proof.

4. Lemma: Let $X^* \subseteq C^*$ be a composing space such that $\partial X^*$ consists of incompressible tori in the interior of $C^*$ (or possibly one component equal to $\partial C^*$ and the others in the interior). Then each component of $q^{-1}(X^*)$ is a composing space. If $X = q^{-1}(X^*)$ is connected, then there is a product structure $F \times S^1$ on $X$ such that $h(x, e^{i\varphi}) = (x, e^{i(\varphi + 2\pi q/p)})$, for some $q$, $1 \leq q \leq p-1$.

Proof: If $q^{-1}(X^*)$ is not connected, then it has $p$ components, each of which is homeomorphic to $X^*$ and is therefore a composing space.

Suppose $X = q^{-1}(X^*)$ is connected. $X^* = F^* \times S^1$, $F^*$ a disc with two holes. Let $J_0^*$, $J_1^*$, $J_2^*$ be the components of $\partial F^*$, $T_0^* = J_0^* \times S^1$, $T_1^* = J_1^* \times S^1$, $T_2^* = J_2^* \times S^1$ the components of $\partial X^*$. Since $X$ is connected and $p$ prime, we either have $X = F \times S^1$ with $q$ the identity on $F$ and a $p$-fold covering on $S^1$, or $X = \widetilde{F} \times S^1$, with $q$ a $p$-fold cyclic covering on $\widetilde{F}$ and the identity on $S^1$. In the first case, we are done since $X$ is a composing space with the desired product structure. So we show that the second case cannot occur.

Suppose $X = \widetilde{F} \times S^1$, with $q$ a $p$-fold cyclic covering on $\widetilde{F}$, the identity on $S^1$. We claim that $\widetilde{F}$ is a planar surface. Let $b$ be the number of components of $\partial \widetilde{F}$. Then $b$ is the number of components of $\partial X$. Thus $S^3 - \text{int}(X)$ has $b$ components, each of which is a
toral solid. Therefore $H_1(S^3 - X)$ is free abelian of rank $b$. From the exact sequence of the pair $(S^3, S^3 - X)$ we obtain $H_2(S^3, S^3 - X) \cong H_1(S^3 - X)$. By Alexander duality $H^1(X) \cong H_2(S^3, S^3 - X)$ and so is free abelian of rank $b$. Let $g$ be the genus of $\tilde{F}$. Then

$$\chi(\tilde{F}) = 2 - 2g - b,$$

so $H^1(\tilde{F})$ is free abelian of rank $l - \chi(\tilde{F}) = 2g + b - 1$. By the universal coefficient theorem, $H^1(X) \cong \mathbb{Z} \otimes H^1(\tilde{F})$ and so is free abelian of rank $2g + b$. Thus $g = 0$ and $\tilde{F}$ is planar. $\chi(F) = -1$, so $\chi(\tilde{F}) = p\chi(F)$ implies $2 - b = -p$, hence $b = p + 2$.

This implies that the inverse images of two of the components of $\partial F$ are connected, while that of the other has $p$ components. Number so that $J_1 = q^{-1}(J_1^*)$ and $J_2 = q^{-1}(J_2^*)$ are connected, and $q^{-1}(J_0^*)$ has components $L_0, \ldots, L_{p-1}$. Let $Y_0^*, Y_1^*, Y_2^*$ be the components of $S^3 - \text{int}(X^*)$ so that $\partial Y_1^* = T_1^*$. $Y_1 = q^{-1}(Y_1^*)$ and $Y_2 = q^{-1}(Y_2^*)$ are connected, whereas $q^{-1}(Y_0^*)$ has $p$ components $Z_0, \ldots, Z_{p-1}$ which are cyclically permuted by $h$. Hence $K$ must lie either in $Y_1$ or $Y_2$, say $Y_1$.

Then $Y_2 \cup X \cup q^{-1}(Y_0^*)$ lies in $C$ and has boundary $\partial Y_1$, which is incompressible in $C$. Therefore it must be a knot space. Hence $Y_1$ is a solid torus, and

$$Y_1^* = Y_1/\langle h|Y_1 \rangle$$

its orbit space. Let $\mu$ be an $S^1$ fiber of $\partial Y_1$. It follows as in the proof of Lemma 2.16 that $\mu$ is a meridian of $Y_1$. Thus $\mu^* = q(\mu)$ is a meridian of $Y_1^*$, and $q^{-1}(\mu^*)$ has $p$ components, since $q^{-1}(J_1)$ is connected. But this contradicts Lemma II.2.3 which says
that \( q^{-1}(\mu^*) \) must be connected. This contradiction completes the proof.

5. **Proposition**: Let \( \mathcal{J}^* \) be a Haken system of tori in \( C^* \). Then the set \( \mathcal{J} \) consisting of the components of the inverse images of the elements of \( \mathcal{J}^* \) under \( q \) is a Haken system of tori in \( C^* \).

**Proof**: Obviously the elements of \( \mathcal{J} \) are pairwise disjoint and include \( \partial C \), and it is easily checked that they are injective, hence incompressible, in \( C \). Suppose \( T_i \) and \( T_j \) are elements of \( \mathcal{J} \) which are parallel in \( C \). Then there is a submanifold \( M = T^2 \times I \) of \( C \) with \( T^2 \times 0 = T_i \) and \( T^2 \times 1 = T_j \). If \( T_i \) and \( T_j \) are in the inverse images of two different components \( T_i^* \) and \( T_j^* \) of \( \mathcal{J}^* \), then \( T_i^* \) and \( T_j^* \) cobound a submanifold \( M^* \) of \( C^* \), which is easily seen to be homeomorphic to \( T^2 \times I \). This contradicts the fact that \( \mathcal{J}^* \) is a Haken system. If \( T_i \) and \( T_j \) are in the inverse image of a single element \( T^* \) of \( \mathcal{J}^* \), then \( T^* \subseteq \partial M^* \), \( M^* \) a manifold homeomorphic to the twisted I-bundle over the Klein bottle. But such a manifold cannot embed in a homotopy sphere, since \( H_1(M^*) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \) and \( \partial M^* \) is connected, so that \( H_1(M^*) \) must be \( \mathbb{Z} \).

So the only way \( \mathcal{J} \) can fail to be a Haken system is for \( C \) to contain an incompressible torus \( T \) which neither intersects nor is parallel to any of the members of \( \mathcal{J} \). If such a torus exists, then there is a component of \( C^* - \mathcal{J}^* \) with closure \( X^* \) such that \( T \) is contained in the interior of a component \( X \) of \( q^{-1}(X^*) \). Since \( \partial X \)
consists of elements of $\mathcal{J}$, T must be essential in $X$. By Lemma I.6.4, $X^*$ must have either an essential torus or an essential annulus. By Proposition 3.3, $X^*$ contains no essential torus, hence must be either a torus knot space, cable space, or composing space. By Lemmas 2, 3, and 4, $X$ must be either a torus knot space, cable space, or composing space. By Lemmas 2.5, 2.11, and 2.17, $X$ contains no essential tori, a contradiction.

Therefore such a torus cannot exist, and thus $\mathcal{J}$ is a Haken system.

6. **Theorem:** Let $T$ be an arbitrary essential torus in $C$. Then there is an isotopy $f_t$ of $C$ rel $\partial C$ such that, for $T' = f_1(T)$, either $h(T') = T'$ or $h^i(T') \cap h^j(T') = \emptyset$ for $i \neq j$. If $Q$ is the knot space in $C$ bounded by $T$, then, letting $Q' = f_1(Q)$, either $h(Q') = Q'$ or $h^i(Q') \cap h^j(Q') = \emptyset$ for $i \neq j$.

**Proof:** Let $\mathcal{J}$ be a Haken system of tori in $C^*$. By Proposition 5, the set $\mathcal{J}$ is a Haken system of tori in $C$. By Proposition 3.5, there is an isotopy of $C$ rel $\partial C$ taking $T$ to a torus $T''$ which is either a member of $\mathcal{J}$ or is contained in $(X_1 \cup \ldots \cup X_s)$, where each $X_i$ is a composing space which is the closure of one of the components of $C - \mathcal{J}$, and whose intersections with $T''$ consist of essential annuli. In the first case, we are done, since the members of $\mathcal{J}$ have the required property. So we consider the second case.
$X_i$ is a component of $q^{-1}(X_i^*)$ for some $X_i^*$ among the closures of the components of $C^* - J^*$. Thus either $h(X_i) = X_i$ or $h^j(X_i) \cap h^K(X_i) = \emptyset$ for $j \neq k \mod (p)$. If $h(X_i) = X_i$, then $X_i = q^{-1}(X_i^*)$ for some $X_i^*$ in $C^*$.

By Lemma 1.6.3, Proposition 3.3, and Lemmas 2 and 3, $X_i^*$ is also a composing space. By Lemma 4, $X_i$ has a product structure $F \times S^1$ for which $h(x, e^{i\phi}) = (x, e^{i(\phi + 2\pi a/p)})$.

Thus each boundary component of $X_i$ is invariant under $h$.

We assume that $T''$ meets each $X_i$, $i = 1, \ldots, s$.

It follows from the above remarks that either $h(X_i) = X_i$ for all $i$, or that $h^j(X_i) \cap h^K(X_i) = \emptyset$ for $j \neq k \mod (p)$ for all $i$. Let $X = X_1 \cup \ldots \cup X_s$. We claim that either $h(X) = X$ or $h^j(X) \cap h^K(X) = \emptyset$ for $j \neq k \mod (p)$. If each $X_i$ is invariant, then clearly $h(X) = X$. So suppose each $X_i$ is disjoint from its translates. Let $Y = X \cup h(X) \cup \ldots \cup h^{p-1}(X)$. We associate to $Y$ a graph $G$ consisting of one vertex for each submanifold of $Y$ equal to some $h^j(X_i)$ and one edge for each boundary component of the submanifold, so that the vertices corresponding to two submanifolds having a common boundary component are joined by the edge corresponding to that component. $h$ induces a periodic simplicial map on $G$.

Since each torus in $S^3$ separates, $G$ is a tree. If $G$ is connected, then $h'$ must have a fixed point, so that there is some $X_i$ for which $h(X_i) = X_i$, a contradiction. So $G$ is not connected, and, since $p$ is prime, must have
p components which are cyclically permuted by \( h' \). These components correspond to the \( h^j(X) \), \( j = 0, \ldots, p-1 \), so that \( h^j(X) \cap h^k(X) = \emptyset \) for \( j \neq k \) (mod \( p \)), as claimed.

Suppose \( h(X) = X \). Give \( X_1 \) the product structure \( F \times S^1 \) of Lemma 4. By Lemma 2.19, there is an isotopy of \( X_1 \) taking the components of \( X_1 \cap T'' \) to product annuli. It can be chosen so that it is constant on those components of \( \partial X_1 \) not meeting \( T'' \). We extend the isotopy to an isotopy of \( C \rel \partial C \) by constructing an isotopy constant outside a small regular neighborhood of \( \partial X \). Suppose \( X_2 \) shares a boundary component \( U \) with \( X_1 \). Give \( X_2 \) the product structure \( F \times S^1 \) of Lemma 4. Since \( U \) is invariant and the components of \( T'' \cap U \) are \( S^1 \)-fibers in the product structure of \( X_1 \), they are also \( S^1 \)-fibers in the product structure of \( X_2 \). Thus there is an isotopy of \( X_2 \) taking the components of \( X_2 \cap T'' \) to product annuli which remains constant on \( U \) and on those components of \( \partial X_2 \) not meeting \( T'' \). We extend the isotopy to one of \( C \rel \partial C \) as before, keeping it constant on \( X_1 \). We continue this process until all the components of \( T'' \cap X_i \) are product annuli for each \( i = 1, \ldots, s \). Note that since each torus separates, we never come back to an \( X_i \) after performing an isotopy in it. Since \( T'' \cap \partial C = \emptyset \), we have kept the isotopies constant on \( \partial C \). Denote the image of \( T'' \) under the composition of these isotopies by \( T' \).

Since the product annuli in each \( X_i \) are invariant under \( h \), \( T' \) is invariant under \( h \) and we are done.
Suppose $X$ is disjoint from its translates under $h$. Then since $T'' \subset X$, we let $T' = T''$ and we are done.

We now prove the second part of the theorem. Let $Q$ be the knot space bounded by $T$ in $C$, $Q' = f(Q)$. If $h(T') = T'$, then either $h(Q') = Q'$ or $h(Q') = S^3 - \text{int}(Q)$. The latter is impossible since $S^3 - \text{int}(Q)$ contains $K = \text{Fix}(h)$. If $T'$ is disjoint from its translates and $Q'$ is not, then either $Q' \subseteq h(Q')$ and $Q' \neq h(Q')$, or $h(Q') \subseteq Q'$ and $Q' \neq h(Q')$. We may assume the former. Then $Q' \subseteq h(Q') \subseteq \ldots \subseteq h^p(Q') = Q'$, where all the inclusions are proper, which is absurd. Therefore $Q'$ is disjoint from its translates.
III.5 General splitting theorems

1. Theorem: Let $J = J(K, V, L)$ as in Construction I.3.10, where $K$ has order $a$ and multiplicity $m$. Let $p$ be a prime. If $J$ is a counterexample to the period $p$ Smith Conjecture and $m < p$, then $K$ is a counterexample to the period $p$ Smith Conjecture.

Proof: Let $\langle h \rangle$ be a cyclic action of order $p$ on $S^3$ with $\text{Fix}(h) = J$, $N$ and $C = C(J)$ an invariant regular neighborhood and knot space of $J$. We have a solid torus $W$ in $S^3$ with core $K$ such that $N \subseteq \text{int}(W)$ and a faithful homeomorphism $f: V \to W$ with $f(L) = J$. Let $T = \partial W$ and $Q = S^3 - \text{int}(W)$. By Lemma I.6.7, $T$ is essential in $C$. By Theorem 4.6, there is an isotopy $g_t$ of $C$ rel $\partial C$ such that $T' = g_1(T)$ and $Q' = g_1(Q)$ are either invariant or disjoint from their translates. Suppose the latter. Then let $W' = g_1(W)$ and $K' = g_1(K)$. $W', h(W'), \ldots, h^{p-1}(W')$ is a set of solid tori whose cores $K', h(K'), \ldots, h^{p-1}(K')$ are equivalent to $K$ and are companions of $J$ of order $a$. Since $h^i(Q') \subseteq \text{int} h^j(W')$ for $i \neq j \pmod{p}$, we see that $K$ has multiplicity at least $p$, a contradiction. Therefore $T'$ and $Q'$ are invariant under $h$. $\langle h|W' \rangle$ is a cyclic action of order $p$ on $W'$ with $\text{Fix}(h|W') = J$. So by Lemma II.2.4, we can extend $\langle h|\partial W' \rangle$ to a standard action $\langle h_0 \rangle$ on $W'$ with $\text{Fix}(h_0) = K'$. Let $h' = h$ on $Q'$ and $h_0$ on $W'$. Then $\text{Fix}(h') = K'$. Since $K'$ is equivalent to $K$,
Lemma I.4.4 implies that $K$ is a counterexample to the period $p$ Smith Conjecture.

2. **Corollary:** Let $J = J(K,V,L)$, $p$ a prime. Suppose $(V,L)$ is $K$-independent. If $J$ is a counterexample to the period $p$ Smith Conjecture, then so is $K$.

**Proof:** By Lemma I.3.9, $K$ has multiplicity 1, so the result follows from Theorem 1.

3. **Theorem:** Let $J = J(K,V,L)$, where $K$ has order $\alpha$ and multiplicity $m$. Let $p$ be a prime. If $J$ is a counterexample to the period $p$ Smith Conjecture, then so is each knot congruent to $L$ along $V$.

**Proof:** With notation as in the proof of Theorem 1, we obtain, exactly as we did there, that $T'$ and $Q'$ are invariant. Let $h_0 = f^{-1}(g_1^{-1}(W')(h|W')(g_1|W)f$. $h_0$ generates a cyclic action $\langle h_0 \rangle$ of order $p$ on $V$ with $\text{Fix}(h_0) = L$. By Lemma II.2.3, there is a meridian $\mu$ of $V$ which is invariant under $h_0$. It follows that any simple closed curve on $\partial V$ which meets $\mu$ transversely in a single point is isotopic to one which is disjoint from its translates under $h_0$. In particular, we can find a longitude $\lambda_0$ of $V$ disjoint from its translates under $h_0$. Since $\lambda_0$ bounds a meridianal disc of $S^3 - \text{int}(V)$, we can extend $(h_0|\partial V)$ to a free cyclic action of order $p$ on the solid torus $S^3 - \text{int}(V)$. We denote the resulting homeomorphism defined on $S^3$ by $h_0'$. $\text{Fix}(h_0') = L$.

Now suppose $L'$ is congruent to $L$ along $V$. Then for some simple twist $\tau$ and integer $q$, $L' = \tau^q(L)$. 
Let $h_q = \tau^q h_0 \tau^{-q}$. Then $h_q$ generates a cyclic action $\langle h_q \rangle$ on $V$ with $\text{Fix}(h_q) = L'$. Since $\tau$ is a simple twist, $\tau^q(\mu)$ is an invariant meridian of $V$. As before we can find a longitude $\lambda_q$ of $V$ disjoint from its translates under $h_q$, and so can extend $\langle h_q \rangle \lhd V$ to a free cyclic action on $S^3 - \text{int}(V)$, denoting the resulting action on $S^3$ by $\langle h_q' \rangle$. Then $\langle h_q' \rangle$ is a cyclic action of order $p$ on $S^3$ with $\text{Fix}(h_q') = L'$. So if $L'$ is a knot, it is a counterexample to the period $p$ Smith Conjecture.

4. **Corollary:** Let $J = J(K,V,L)$, $p$ a prime. Suppose $(V,L)$ is $K$-independent. If $J$ is a counterexample to the period $p$ Smith Conjecture, then so is each knot congruent to $L$ along $V$.

**Proof:** This follows exactly as in Corollary 2 and Theorem 3.
Chapter IV: Applications

IV.1 Cabled knots

1. **Remark**: Using Corollary 5.4, we shall give two proofs that the Smith Conjecture is true for cabled knots. One of them uses Giffen's result that the Smith Conjecture is true for torus knots (I.4.12). The other uses only Fox's earlier partial result in terms of Alexander polynomials (I.4.11).

2. **Lemma**: Let \( T_{m,n} \subseteq V \) be a cable as in Definition I.3.15. Then \( (V,T_{m,n}) \) is \( K \) - independent for all knots \( K \).

**Proof**: If not, then \( V - T_{m,n} \) would contain an injective \( K \) - knot space \( Q \), which we may assume lies in \( V - \text{int}(N) \), \( N \) a regular neighborhood of \( T_{m,n} \) in \( V \). But \( V - \text{int}(N) \) is a cable space, and so by Lemma IV.2.11, contains no essential tori. Thus since \( \exists Q \) is injective in \( V - \text{int}(N) \) it must be parallel to a component of \( \exists (V - \text{int}(N)) \), but since there are two components, this is impossible. Therefore \( (V,T_{m,n}) \) is \( K \) - independent.

3. **Theorem**: The Smith Conjecture is true for all cabled knots.

**Proof**: Let \( J = J(K,V,T_{m,n}) \). By Lemma I.4.6, it suffices to prove the period \( p \) Smith Conjecture for each prime \( p \). If it is false for some prime \( p \), then by Corollary 5.4 and Lemma 2, each knot congruent to
along $V$ is a counterexample to the period $p$ Smith Conjecture. We now give two different proofs that this is not so.

**Proof 1:** If $|m|, |n| \geq 2$, then $T_{m,n}$ is a torus knot and so, by Proposition I.4.12, cannot be a counterexample. If $|m| = 1$ and $|n| \geq 2$, then $T_{m,n}$ is an unknot. Choose an integer $q$ such that $|m + nq| \geq 2$, say $q \geq (2 + |m|)/|n|$. Then $T_{m+nq,n}$ is a torus knot congruent to $T_{m,n}$ along $V$ and so would be a counterexample. But, by Proposition I.4.12, it is not.

**Proof 2:** As in Proof 1, we may assume that $|m|, |n| \geq 2$. If $p$ divides $mn$ then, by Proposition I.4.11, $T_{m,n}$ cannot be a counterexample. If $p$ does not divide $mn$, then $n$ is a unit modulo $p$ so that the equation $mn + n^2q = 0 \mod(p)$ can be solved for $q$. It follows that $p$ divides $(m + nq)n$. But $T_{m+nq,n}$ is congruent to $T_{m,n}$ along $V$ so that it must be a counterexample. But, by Proposition I.4.11, it is not.
IV.2 Doubled knots

1. Remarks: Using Corollary 5.4, we shall give two proofs that the Smith Conjecture is true for doubled knots. One of them uses the result that the Smith Conjecture is true for twist knots. This follows from Cappell and Shaneson's theorem that the conjecture is true for 2-bridge knots; but we shall give an independent proof based on a new result which seems interesting in its own right. The second proof uses only Fox's Alexander polynomial conditions.

2. Theorem: Let $K$ be a non-fibered knot all of whose incompressible spanning surfaces are isotopic. Then the Smith Conjecture is true for $K$.

Proof: Let $q : S^3 \to \Sigma^3 = S^3/\langle h \rangle$ where $\langle h \rangle$ is some cyclic action of order $p$ on $S^3$ with $\text{Fix}(h) = K$. Let $C$ be an invariant knot space of $K$ and $C^* = q(C)$. Let $F^*$ be an incompressible spanning surface for $K^* = q(K)$ and let $G^* = F^* \cap C^*$. Then $q^{-1}(F^*)$ has $p$ components $F_0, \ldots, F_{p-1}$ with $G_i = F_i \cap C$ the components of $q^{-1}(G^*)$. Since all the $F_i$ are incompressible, they are all isotopic. It follows that all the $G_i$ are isotopic. By Lemma I.5.5, all the $G_i$ are parallel. It follows that $K$ is fibered, a contradiction.

3. Corollary: The Smith Conjecture is true for all twist knots.

Proof 1: Twist knots are 2-bridge knots, so the result follows from Proposition I.4.13.
Proof 2: The knots $K(\rho, \eta)$ with $|\rho| > 1$ are non-fibered and have unique isotopy type of incompressible spanning surface by Lyon [22]. The knots $K(\rho, \eta)$ with $|\rho| = 1$ are fibered, but one is the trefoil and is thus ruled out since it is a torus knot; the other is the figure eight knot, having Alexander polynomial $1 - 3t + t^2$. The only possibilities for the polynomial $\Delta_{K^*}(t)$ in Proposition I.4.11 are $1 - 3t + t^2$ and $1 - t + t^2$, neither of which satisfy equation (1).

4. Lemma: Let $K(\rho, \eta) \subset V$ as in Definition I.3.17. Then $(V, K(\rho, \eta))$ is $K$-independent for all knots $K$.

Proof: If not, then $V - K(\rho, \eta)$ contains an injective $K$-knot space $Q$ which we may assume lies in $V - \text{int}(N)$, $N$ a regular neighborhood of $K(\rho, \eta)$ in $V$. $S^3 - \text{int}(Q)$ is a solid torus $W$ with its boundary in $\text{int}(V)$ and $K(\rho, \eta)$ in its interior. Obviously $\exists W = \exists Q$ cannot be parallel to a component of $\exists(V - \text{int}(N))$, since there are two components. $O_W(K(\rho, \eta)) \neq 0$, since $\exists Q$ is injective in $V - K(\rho, \eta)$. Thus it follows from Hilfsatz 3 on p. 238 of [31], that $W$ must be contained in $V$, which is impossible.

5. Theorem: The Smith Conjecture is true for all doubled knots.

Proof: Let $J = J(K, V, K(\rho, \eta))$. By Lemma I.4.6, it suffices to prove the period $p$ Smith Conjecture for each prime $p$. If it is false for some prime period $p$, then by Corollary 5.4 and Lemma 4, each knot congruent to
$K(\rho, \eta)$ along $V$ is a counterexample to the period $p$ Smith Conjecture. We give two proofs that this is not so.

**Proof 1:** If $|\rho| \neq 0$, then $K(\rho, \eta)$ is a twist knot and so, by Corollary 3, is not a counterexample. If $\rho = 0$, then $K(\rho, \eta)$ is an unknot. Choose an integer $q \neq 0$. Then $K(\rho+q, \eta)$ is a twist knot and is congruent to $K(\rho, \eta)$ along $V$, and so must be a counterexample. But it cannot be, by Corollary 3.

**Proof 2:** Since $K(\rho, \eta)$ is congruent along $V$ to $K(\rho+q, \eta)$ for any integer $q$, every $K(\rho, \eta)$ is congruent to the stevedore's knot $K(2, \eta)$, which has polynomial $2 - 5t + 2t^2$. By Proposition I.14.11, the leading coefficient of $\Delta_{K(\rho, \eta)}(t)$ must be a $p^{th}$ power, which 2 is not. So $K(2, \eta)$ cannot be a counterexample, a contradiction.
IV.3 Cable braids

1. **Lemma:** Let \( \hat{\beta} \subseteq V \) be as in Definition I.3.19. Then \((V, \hat{\beta})\) is \(K\)-independent for all knots \(K\).

   **Proof:** If not, then \(V - \hat{\beta}\) contains an injective \(K\)-knot space \(Q\), which we may assume lies in \(V - N\), \(N\) a regular neighborhood of \(\hat{\beta}\) in \(V\). Let \(K'\) be any knot and \(J = (K', V, \hat{\beta})\). Then \(W\) contains an injective \(K\)-knot space \(Q'\). Let \(W' = S^3 - \text{int}(Q')\). Then \(W'\) is a knotted solid torus which contains \(J = f(\hat{\beta})\) in its interior and has its boundary in \(\text{int}(W)\). Since \(Q'\) is injective in \(W' = f(\text{int} N)\), \(\partial W' = \partial Q'\) is also. Hence \(O_{W'}(J) \neq 0\). \(J\) is not a core of \(W'\), for if it were then \(\partial W'\) would be parallel to \(\partial f(N)\) in \(C(J)\); thus \(C(J) - \text{int}(Q) = T^2 \times I\). By I.6.7, \(W\) is an incompressible torus in this product and so, by Lemma I.5.6, is parallel to \(\partial Q\) in \(W - \text{int}(N)\). But this is impossible since \(W - \text{int}(N \cup Q)\) has three boundary components. Therefore it follows by Note I.3.28 that \(W'\) is contained in \(\text{int}(W)\) so that \(Q'\) cannot be. This shows that \(V - \hat{\beta}\) cannot contain \(Q\).

2. **Theorem:** The Smith Conjecture is true for all cable braids.

   **Proof:** Let \(J = (K, V, \hat{\beta})\). By Lemma I.4.6, it suffices to prove the period \(p\) Smith Conjecture for each prime \(p\). It follows from Lemma 1 and the proof of Corollary 3.4 that there is a cyclic action \(\langle h_0 \rangle\) of
order \( p \) on \( V \) with fixed point set \( \hat{\beta} \). By Proposition II.3.1, \( \hat{\beta} \) must be a core of \( V \), hence \( n = 1 \), a contradiction.
Chapter V: Reduction theorems

V.1 Type I Surgery

1. **Construction:** Let \( \langle h_0 \rangle \) be a cyclic action on \( S^3 \) of prime order \( p \) with \( \text{Fix}(h_0) = K_0 \), a knot. Let \( C_0 = C(K_0) \) be an invariant knot space. Suppose \( T \) is an invariant essential torus in \( C_0 \). Then \( T \) divides \( S^3 \) into a knot space \( C_1 \) and a solid torus \( N_1 \) containing \( K_0 \). By Lemma II.2.4, there is a cyclic action \( \langle h_0' \rangle \) on \( N_1 \) having \( \text{Fix}(h_0') = K_1 \), a core of \( N_1 \), such that \( h_0' = h_0 \) on \( T \). Define a new action \( h_1 \) on \( S^3 \) by letting \( h_1 = h_0 \) in \( C_1 \) and \( h_0' \) in \( N_1 \). Then \( \text{Fix}(h_1) = K_1 \). By Lemma I.6.7, \( K_1 \) is a companion of \( K_0 \).

2. **Definition:** We say that \( (K_1, \langle h_1 \rangle) \) has been obtained from \( (K_0, \langle h_0 \rangle) \) by **Type I surgery** on \( T \).

3. **Proposition:** Let \( K_0, K_1, K_2, \ldots \) be a sequence of knots and \( \langle h_0 \rangle, \langle h_1 \rangle, \langle h_2 \rangle, \ldots \) be a sequence of cyclic actions on \( S^3 \) with \( \text{Fix}(h_i) = K_i \), where \( (K_{i+1}, \langle h_{i+1} \rangle) \) is obtained from \( (K_i, \langle h_i \rangle) \) by Type I surgery. Then the sequence terminates with a knot \( K_n \) such that \( C_n = C(K_n) \) has no \( h_n \)-invariant essential tori.

**Proof:** By Lemma I.3.12, \( b(K_{i+1}) < b(K_i) \) so the sequence terminates. \( K_n \) is a knot since \( C_n \) is a knot space. \( C_n \) has no invariant essential torus, for if it did we could perform Type I surgery on it and continue the sequence.
V.2 Type II surgery

1. Construction: Let \( \langle h_0 \rangle \) be a cyclic action of prime order \( p \) on \( S^3 \) with \( \text{Fix}(h_0) = K_0 \), a knot. Let \( C_0 = C(K_0) \) be an invariant knot space. Suppose \( T \) is an essential torus in \( C_0 \) which is disjoint from its translates under \( \langle h_0 \rangle \). \( T \) separates \( S^3 \) into a knot space \( Q \) and solid torus \( N \). As in the proof of Theorem III.4.6, we see that \( Q \) is disjoint from its translates under \( h_0 \).

Let \( T^K_0 = h^K_0(T) \), \( Q^K_0 = h^K_0(Q) \). If \( (\mu, \lambda) \) is a meridian-longitude pair for \( Q \), then \( \mu^K = h^K_0(\mu) \) and \( \lambda^K = h^K_0(\lambda) \) form a meridian-longitude pair for \( Q^K_0 \).

Let \( V = V_0 \cup \ldots \cup V_{p-1} \) be a disjoint union of solid tori and \( h_0' : V \to V \) be a homeomorphism of period \( p \) taking \( V^K_k \) to \( V^K_{k+1} \) (mod \( p \)). Choose a meridian-longitude pair \( (\alpha_0, \beta_0) \) for \( V_0 \) and let \( (\alpha^K_k, \beta^K_k) = (h^K_0)'(\alpha_0, \beta_0) \) be a meridian-longitude pair for \( V^K_k \). Let \( \varphi_0 : \partial V_0 \to T_0 \) be a homeomorphism taking \( \alpha_0 \) to \( \lambda_0 \) and \( \beta_0 \) to \( \mu_0 \).

Define \( \varphi^K_k : \partial V^K_k \to T^K_k \) by \( \varphi^K_k = (h^K_0|T_0)^K(\varphi_0)(h^K_0)'^{K-K} \). Let \( \varphi = \varphi_0 \cup \ldots \cup \varphi_{p-1} : \partial V \to \partial R \), where \( R = S^3 - \text{int}(Q_0 \cup \ldots \cup Q_{p-1}) \).

Let \( M = V \cup R \). Obviously \( \varphi \) commutes with \( (h_0|R) \), so \( h_0' \) and \( (h_0|R) \) induce a cyclic action \( \langle h_1 \rangle \) on \( M \) with \( \text{Fix}(h_1) = K_1 \), the image of \( K_0 \) in \( M \).

2. Claim: There exist disjoint meridional discs \( D_0, \ldots, D_{p-1} \) for \( Q_0, \ldots, Q_{p-1} \) such that each is contained in \( R \).
Proof: We first claim there is a meridional disc \( D \) of \( Q_0 \) missing \( Q_1 \cup \ldots \cup Q_{p-1} \). Let \( D \) be any meridional disc of \( Q_0 \) in general position with respect to the boundaries of \( Q_1, \ldots, Q_{p-1} \) having minimal intersection with them. If the intersection is empty, we are done. If not, then there is a subdisc \( E \) of \( D \) with \( \partial E \) in the intersection (with \( \partial Q_j \), say) whose interior misses the intersection. If \( E \) is in \( Q_j \), then there is a disc \( E' \) in \( \partial Q_j \) with \( \partial E' = \partial E \), since \( Q_j \) has incompressible boundary. Replace \( E \) by \( E' \) and push it off \( Q_j \) to get a disc \( D' \) with fewer intersections, a contradiction. If \( E \) is in \( R \), then \( Q_j \cup X \), \( X \) a regular neighborhood of \( E \) in \( R \), is a ball and so we can replace \( D \) by a disc \( D' \) missing it and having no more intersections with the other \( \partial Q_i \) than did \( D \), a contradiction.

Let \( D_0 = D \) and suppose \( D_0, \ldots, D_{k-1} \) have been chosen satisfying the claim. Exactly as above, we have a meridional disc \( D_k \) for \( Q_k \) contained in \( R \). Assume it is in general position with respect to \( (D_0 \cup \ldots \cup D_{k-1}) \) and has minimal intersection. If the intersection is empty, then we are done. If not, then there is a disc \( E' \) on some \( D_j \) with \( \partial E' \) in the intersection and \( \text{int}(E') \) missing it. \( \partial E' = \partial E \), \( E \) a disc in \( D_k \). Replace \( E \) by \( E' \) and move to one side to get a new disc having fewer intersections.

The proof is completed by induction.

3. **Claim:** \( M \) is homeomorphic to \( S^3 \).
Proof: Consider the sequence of manifolds
\[ S^3 = M_0, M_1, \ldots, M_p = M, \text{ where } M_{k+1} = V_k \cup (M_k - \text{int}(Q_k)). \]

We show that \( M_k = S^3 \) by induction on \( k \). Assume \( M_k \) is \( S^3 \). Then \( M_k - \text{int}(Q_k) \) is a solid torus. Let \( D_{k+1} \) be the meridional disc for \( Q_{k+1} \) given in Claim 2. Then \( D_k \subseteq R \subseteq M_k - \text{int}(Q_k) \) and so is a meridional disc of \( M_k - \text{int}(Q_k) \). It follows from the definition of \( \varphi_k \) that \( \varphi_k \) defines a genus one Heegaard splitting of \( S^3 \).

4. Definition: We say that \((K_1, \langle h_1 \rangle)\) has been obtained from \((K_0, \langle h_0 \rangle)\) by **Type II surgery** on \( T \).

5. Remarks: There are two fundamental problems with Type II surgery. First, \( K_1 \) is not a companion of \( K_0 \), so that we cannot use bridge numbers to conclude that any sequence of such surgeries must terminate. Secondly, even if such a sequence terminates, the final result \( K_n \) might be unknotted. In the case of fibered knots, both these difficulties can be overcome, as will be shown in Section 4. The second difficulty raises interesting possibilities, which are the subject of the next section.
V.3 Symmetric trivial links

1. Notation: Let $p$ be a prime and $t$ a standard rotation of $S^3$ of period $p$ about the unknotted simple closed curve $A$.

2. Definition: Let $c_0$ be an unknotted simple closed curve in $S^3 - A$ such that if $c_k = t^k(c_0)$, then $c_i \cap c_j = \emptyset$ for $i \neq j \pmod{p}$ and the resulting link $L(c_0) = c_0 \cup \ldots \cup c_{p-1}$ is trivial. Then $L(c_0)$ is a **symmetric trivial link** of order $p$. $L(c_0)$ is **standard** if $L(c_0) \cup A$ is a trivial link. Otherwise $L(c_0)$ is **non-standard**.

3. Proposition: Suppose $(K_1, \langle h_1 \rangle)$ is obtained from $(K_0, \langle h_0 \rangle)$ by Type II surgery and $K_1$ is unknotted. Let $c_0$ be a core of $V_0$. Then $c_0 \cup h_1(c_0) \cup \ldots \cup h_1^{p-1}(c_0)$ is a non-standard symmetric trivial link.

**Proof:** Since $K_1$ is unknotted, Proposition I.4.5 implies that $h_1$ is equivalent to the standard rotation $t$ about $A = K_1$. Obviously $c_k = h_1^k(c_0)$ is a core of $V_k$ and $c_i \cap c_j = \emptyset$ for $i \neq j \pmod{p}$. By Claim 2.2, there are disjoint discs $D_k$ in $R$ with $\partial D_k$ a meridian of $Q_k$. Since $\varphi_k(\beta_k) = \mu_k$, $\partial D_k$ is also a longitude of $V_k$. Join $\partial D_k$ to $c_k$ by an annulus in $V_k$ to get a disc $E_k$ in $M$ with $\partial E_k = c_k$. Since the $E_k$ are disjoint, the union of the $c_k$ is a trivial link. It is obviously symmetric of order $p$. If it were standard, then there would be a disc $H$ in $M$ with $\partial H = A$ and missing
\( E_0 \cup \ldots \cup E_{p-1} \). We may assume then that \( H \) misses \( V_0 \cup \ldots \cup V_{p-1} \). But then we have \( H \) in \( R \), so that we can regard it as a disc in \( M_0 = S^3 \) with \( \partial H = K_0 \), contradicting the fact that \( K_0 \) is a knot. So \( L(c_0) \) is non-standard.

4. **Proposition:** Suppose \( L(c_0) \) is a non-standard symmetric trivial link of order \( p \). Then there exist infinitely many knots which are counterexamples to the period \( p \) Smith Conjecture.

**Proof:** Let \( V_0, \ldots, V_{p-1} \) be disjoint regular neighborhoods of \( c_0, \ldots, c_{p-1} \) in \( S^3 \). A chosen so that \( t^K(V_0) = V_1 \). Let \( (\alpha_0, \beta_0) \) be a meridian-longitude pair for \( V_0 \) and let \( (\alpha_1, \beta_1) = t^K(\alpha_0, \beta_0) \). We now simply reverse the Type II surgery of Section 2. Let \( Q_0 \) be any knot space and \( Q = Q_0 \cup \ldots \cup Q_{p-1} \) the disjoint union of \( p \) copies of \( Q_0 \). Let \( h' : Q \to Q \) be a period \( p \) homeomorphism taking \( Q_k \) to \( Q_{k+1} \mod(p) \). Let \( (\mu_0, \lambda_0) \) be a meridian-longitude pair for \( Q_0 \) and let \( (\mu_1, \lambda_1) = (h')^K(\mu_0, \lambda_0) \). Let \( \varphi_0 : \partial Q_0 \to \partial V_0 \) be a homeomorphism taking \( \mu_0 \) to \( \beta_0 \) and \( \lambda_0 \) to \( \alpha_0 \). Define \( \varphi_K : \partial Q_k \to \partial V_k \) by \( \varphi_K = (t|\partial V_0)^K(\varphi_0)(h'|Q_0)^{-K} \). Let \( X = Q \cup R \). Since \( t\varphi = \varphi h' \), we get a cyclic action \( \langle h \rangle \) on \( X \) with \( \text{Fix}(h) = K \), the image of \( A \) in \( X \).

**Claim 1:** \( X \) is homeomorphic to \( S^3 \). Consider the sequence of manifolds \( S^3 = X_0, X_1, \ldots, X_p = X \) defined by \( X_{k+1} = Q_k \cup_{\varphi_K} (X_k - \text{int}(V_k)) \). Since \( L(c_0) \) is trivial,
there are disjoint discs $D_0, \ldots, D_{p-1}$ in $X_0$ with
\[ \partial D_k = \beta_k = \mu_k. \]
Thus they are contained in $R$. Suppose
inductively that $X_{k-1}$ is homeomorphic to $S^3$. $V_k$ is an
unknotted solid torus in $X_{k-1}$, since $\partial D_k$ is non-
contractible in $\partial V_k$. Thus $X_{k-1} - \text{int}(V_k)$ is a solid
torus. Since $\varphi_k(\mu_k) = \beta_k = \partial D_k$,
$X_k = Q_k \cup (X_{k-1} - \text{int}(V_k))$ is homeomorphic to $S^3$.

Claim 2: $K$ is knotted in $X$. If not, then there is
a disc $D$ in $X$ with $\partial D = K$. As in Claim 2.2, we can
choose $D$ so that it misses $Q_0, \ldots, Q_{p-1}$ and is therefore
contained in $R$. Thus we regard it as a disc in $X_0$ with
boundary $A$. Since $D$ is in $R$ it misses $L(c_0)$. Thus
we can analyze the intersections of $D$ with $D_0, \ldots, D_{p-1}$
as in Claim 2.2 to get that $D$ is disjoint from them.
Hence $L(c_0) \cup A$ is trivial, a contradiction. Therefore
$K$ is knotted. Letting $Q_0$ range over all knot spaces and
noting that a knot has only finitely many companions
completes the proof.

5. Remark: Every symmetric trivial link of order two
is standard since the period 2 Smith Conjecture is true
[42]. A direct proof can be given using equivariant
surgery techniques similar to those used by Kim and
Tollefson in [20].

6. Lemma: Let $L(c_0)$ be a symmetric trivial link of
order $p$ with axis $A$. Choose a basepoint $a_0$ on $A$.
Then $F = \pi_1(S^3 - (L(c_0)))$ is a free group on $p$ genera-
tors \( b_0, b_1, \ldots, b_{p-1} \) such that \( \alpha(b_i) = g_i b_{i+1} g_2^{-1} \), where \( \alpha \) is the automorphism induced by \( t|_{(S^3 - L(c_0))} \).

**Proof:** Let \( V_0 \) be a regular neighborhood of \( c_0 \) in \( S^3 \). A such that if \( V_0 = t^k(V_0) \), then the \( V_k \) are pairwise disjoint. Let \( \gamma_0 \) be a meridian of \( V_0 \) based at a point \( y_0 \) in \( \partial V_0 \). Let \( \gamma_k = t^k(\gamma_0) \), \( \gamma_k = t^k(y_0) \).

Since \( L(c_0) \) is the trivial link, there are arcs \( \varepsilon_k \) from \( a_0 \) to \( y_k \) such that \( F \) is the free group on the generators \( b_k = [\varepsilon_k \gamma_k \varepsilon_k^{-1}] \). Let \( \delta_0 = \varepsilon_0 \) and \( \delta_k = t^k(\delta_0) \).

Let \( g_k = [t(\varepsilon_k)^{\gamma_k \varepsilon_k} \varepsilon_k^{-1}] \). Then \( \alpha(b_k) = [t(\varepsilon_k \gamma_k \varepsilon_k)^{-1}] = [t(\varepsilon_k)^{t(\gamma_k) t(\varepsilon_k^{-1})}] = [t(\varepsilon_k)^{t(\gamma_k) t(\varepsilon_k^{-1})}] = [t(\varepsilon_k)^{\gamma_k \varepsilon_k} t(\varepsilon_k)^{-1}] = [t(\varepsilon_k)^{\gamma_k \varepsilon_k} t(\varepsilon_k)^{-1}] = [t(\varepsilon_k)^{\gamma_k \varepsilon_k} t(\varepsilon_k)^{-1}] = g_k b_k g_k^{-1} \).

7. **Lemma:** (Dyer and Scott) Let \( F \) be a free group and \( \alpha \) an automorphism of \( F \) of prime order \( p \). Then \( F = F^{\langle \alpha \rangle} * (\star_{i \in I} F_i)^{\star_{\lambda \in \Lambda} F_\lambda} \) where each factor is \( \alpha \)-invariant and

   (i) for each \( i \in I \), \( F_i \) has a basis \( x_i, 1, \ldots, x_i, p \) such that \( \alpha(x_{i, r}) = x_{i, r+1} \) (mod \( p \));

   (ii) for each \( \lambda \in \Lambda \), \( F_\lambda \) has a basis \( x_\lambda, 1, \ldots, x_\lambda, p-1 \), \( \{y_j | j \in J_\lambda \} \) such that \( \alpha(x_{\lambda, r}) = (x_{\lambda, r+1}) \), \( r = 1, \ldots, p-2 \), \( \alpha(x_{\lambda, p-1}) = (x_{\lambda, 1} \ldots x_{\lambda, p-1})^{-1} \), \( \alpha(y_j) = x_{\lambda, 1}^{-1} y_j x_{\lambda, 1} \), \( j \in J_\lambda \);

   (iii) \( F^{\langle \alpha \rangle} \) is the fixed point set of \( \alpha \).

**Proof:** This is Theorem 3 of [7].
8. **Proposition:** Let \( L(c_0) \) be a symmetric trivial link of order \( p \) with axis \( A \). Then \( A \) is contractible in \( S^3 - L(c_0) \).

**Proof:** It suffices to show that \([A] = 1\) in \( F\). Note that \( t(A) = A \) implies that \([A]\) is a fixed point of \( \alpha \). By Lemma 7, there are two possibilities for the form of \( \alpha \):

(i) there is a basis \( x_0, x_1, \ldots, x_{p-1} \) for \( F \) such that \( \alpha(x_k) = x_{k+1} \pmod{p} \), or

(ii) there is a basis \( x_0, x_1, \ldots, x_{p-1} \) for \( F \) such that \( \alpha(x_0) = x_0 \), \( \alpha(x_k) = x_{k+1} \) for \( 1 \leq k \leq p-2 \), \( \alpha(x_{p-1}) = (x_1 x_2 \ldots x_{p-1})^{-1} \).

Suppose \( \alpha \) has form (i). Then clearly \( \alpha \) has no non-trivial fixed points, so \([A] = 1\).

Suppose \( \alpha \) has form (ii). Let \( G \) be the split extension of \( F \) by \( \mathbb{Z}_p \) given by \( \mathbb{Z}_p = \langle \alpha \rangle \to \text{Aut}(F) \). \( G \) has the presentation

\[
\langle f_0, f_1, \ldots, f_{p-1}, s : sf_k s^{-1} = \alpha(f_k), s^p = 1 \rangle
\]

where \( f_0, f_1, \ldots, f_{p-1} \) is any basis for \( F \).

Choosing the basis \( x_0, x_1, \ldots, x_{p-1} \) given in (ii), we get the presentation

\[
\langle x_0, x_1, \ldots, x_{p-1}, s : sx_0 s^{-1} = x_0, sx_k s^{-1} = x_{k+1}, 1 \leq k \leq p-2, sx_{p-1} s^{-1} = (x_1 x_2 \ldots x_{p-1})^{-1}, s^p = 1 \rangle.
\]

So, suppressing the commutator relations, \( G/G' \) has the presentation

\[
\langle x_0, x_1, \ldots, x_{p-1}, s : x_1 = x_2 = \ldots = x_{p-1} = (x_1 x_2 \ldots x_{p-1})^{-1}, s^p = 1 \rangle
\]

which is equivalent to \( \langle x_0, x_1, s : x_1^p = 1, s^p = 1 \rangle \).
Thus \( G/G' \cong \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \).

Choosing the basis \( b_0, b_1, \ldots, b_{p-1} \) from Lemma 6, we get the presentation \( \langle b_0, b_1, \ldots, b_{p-1}, u: ub_k u^{-1} = g_k b_{k+1} g_k^{-1}, 0 \leq k \leq p-1 \text{ (mod } p) \rangle \). Suppressing the commutator relations we see that \( G/G' \) has the presentation
\[
\langle b_0, b_1, \ldots, b_{p-1}, u: b_0 = b_1 = \ldots = b_{p-1}, u^p = 1 \rangle
\]
which is equivalent to \( \langle b_0, u: u^p = 1 \rangle \), so that \( G/G' \cong \mathbb{Z} \oplus \mathbb{Z}_p \).

This contradiction shows that \( \alpha \) cannot have form (ii). Therefore \( \lbrack A \rbrack = 1 \).
V.4 A reduction theorem for fibered knots

1. Lemma: Let \( K \) be a fibered knot with regular neighborhood \( N \) and knot space \( C = C(K) \). Let \( T \) be an incompressible torus in \( \text{int}(C) \). Then we can choose the fibering of \( S^3 - K \) so that if \( F \) is a typical fiber of the induced fibering of \( C(K) \), \( C' = F \times I \) the manifold obtained by splitting \( C \) along \( F \) and \( g: C' \rightarrow C \) the gluing back map, then \( g^{-1}(T) \) consists of incompressible annuli \( G_1, \ldots, G_m \) such that \( G_i = \alpha_i \times I \) for some simple closed curve in \( F \times 0 \).

Proof: Take any fibering with \( F \) a typical fiber of \( C(K) \). By an isotopy of \( C \) rel \( \partial C \) we put \( F \) into general position with respect to \( T \) and intersecting it minimally.

Claim 1: \( T \cap F \neq \emptyset \).

Suppose not. Then \( G = g^{-1}(T) \) is a torus in \( \text{int}(F \times I) \) and is incompressible in \( C' \) since \( T \) is incompressible in \( C \). By Lemma I.5.6, \( T \) is parallel to a surface in \( F \times 0 \), which is impossible since \( F \) is not a torus.

Claim 2: \( g^{-1}(T) \) consists of annuli.

Suppose not. Then some component of \( g^{-1}(T) \) is a disc \( G_1 \) with boundary in, say, \( F \times 0 \). \( \partial G_1 = \partial G_1' \), \( G_1' \) a disc in \( F \times 0 \). \( G_1 \cup G_1' \) bounds a ball in \( C' \) so \( g(G_1 \cup G_1') \) bounds a ball in \( C \). Isotop \( g(G_1) \) across the ball to the other side of \( g(G_1') \) to reduce the intersection, thereby contradicting minimality.
Claim 3: Each component of $g^{-1}(T)$ is incompressible in $C'$. Suppose not. Let $D$ be a compressing disc for a component $G_1$ of $g^{-1}(T)$. $\partial D$ divides $G_1$ into two components $G_1'$ and $G_1''$, both annuli. Since $T$ is incompressible, $\partial g(D) = \partial E$, $E$ a disc in $T$. Either $g(G_1')$ or $g(G_1'')$ is contained in $E$ so that $F \cap \text{int}(E) \neq \emptyset$. Let $J$ be an innermost component in $E$ of this intersection. Then $J = gE'$, $E'$ a disc in $T$ with $E' \cap F = \partial E'$. This implies that $g^{-1}(E')$ is a disc component of $g^{-1}(T)$ contradicting Claim 2.

Claim 4: Each component of $g^{-1}(T)$ has one boundary component in $F \times 0$ and the other in $F \times 1$. Suppose not. Then there is a component $G_1$ of $g^{-1}(T)$ with $\partial G_1$ in, say, $F \times 0$. Since $G_1$ is incompressible, Lemma I.5.6 implies that $G_1$ is parallel to an annulus $G_1'$ in $F$, hence $g(G_1)$ is parallel to an annulus $g(G_1')$ in $F \subseteq C$. Isotop $G_1$ across the solid torus bounded by $g(G_1 \cup G_1')$ to remove $\partial g(G_1)$ from the intersection, contradicting minimality.

By Lemma I.5.7, there is an isotopy $f_t$ of $F \times I$ rel $(F \times 0) \cup (\partial F \times I)$ such that for each component $G_i$ of $g^{-1}(T)$, $f_t(G_i) = a_i \times I$ for a simple closed curve $a_i$ in $F \times 0$. Let $g' = g_f^{g^{-1}}$. Taking $g'$ as a new gluing back map we get the desired fibration of $C$.

2. Proposition: Let $J = J(K,V,L)$ be a fibered knot. Let $c$ be a core of $S^3 - \text{int}(V)$. Then

(i) $K$ is a fibered knot with $g(K) < g(J)$,
(ii) $L$ is either unknotted or a fibered knot with $g(L) < g(J)$,

(iii) the linking number of $c$ and $K$ is non-zero.

Proof: We use the notation of Construction I.3.6. Let $Q = S^3 - \text{int}(W)$. By Lemma I.6.7, $T = \partial W$ is essential in $C(J)$. Let $C(J)$ have the fibering constructed in Lemma 1, with $g: F \times I \to C(J)$. Let $R_1, \ldots, R_n$ be the components of $g^{-1}(Q) \cap (F \times 0)$, $X_1, \ldots, X_n$ the components of $g^{-1}(Q)$. Then $X_j = R_j \times I$. Let $\varphi = (g| (F \times 0))^{-1}(g| (F \times 1))$. Number so that $\varphi(R_j \times 1) = R_{j+1}$ (mod n). Let $\varphi_j = \varphi(R_j \times 1)$. Then let $X = X_1 \cup \varphi_1 X_2 \cup \ldots \cup \varphi_{n-1} X_n$ and $g_0: X \to Q$ the map induced by $g| g^{-1}(Q)$. $g_0$ is well defined and gives $Q$ the structure of a surface bundle over $S^1$ with fiber a copy $R$ of $R_j$. Let $1 \to \pi_1(R) \to \pi_1(Q) \to \pi_1(S^1) \to 1$ be the exact sequence of this fibration. Since $H_1(Q) \cong \mathbb{Z}$, it is easily checked that $\pi_1(R)$ is the commutator subgroup of $\pi_1(Q)$. Thus the covering space of $Q$ corresponding to $\pi_1(R)$ is equivalent to the universal abelian covering of $Q$; hence $R$ has connected boundary which is a longitude of $Q$. Extending the fibering to $W - K$ we see that $K$ is a fibered knot. Since $T$ is essential, $\partial R_j$ is not parallel to $\partial F$ in $F \times 0$. Hence $g(R) < g(F)$, so $g(K) < g(J)$. This proves (i).

Now let $Y = F \times I - \text{int} g^{-1}(Q)$ and $P = Y \cap (F \times 0)$. Note that $Y = P \times I$ and, since $\partial R$ is connected, $P$ and $Y$ are connected. Attach $n$ disjoint balls $E_1, \ldots, E_n$ to
Y so that $E_j \cap Y = \partial R_j \times I$. Let $D_{j0}, D_{j1}$ be the closures of the components of $\partial E_j - (\partial R_j \times I)$ with $\partial D_{jk} = \partial R_j \times \{k\}, \ k = 0, 1$. Call the resulting space $Z$. $Z$ is homeomorphic to $S \times I$ where $S = P \cup D_{10} \cup \ldots \cup D_{n0}$. Let $g_1: Y \to V - \text{int}(N)$ be given by $g_1 = (f| (V - \text{int}(N)))^{-1}l$. Since $g(\partial R_1)$ is a longitude of $Q$, $g_1(\partial R_1)$ is a longitude of $V$. Thus $g_1(\partial R_j)$, $j = 1, \ldots, n$, is a set of parallel simple closed curves in $\partial V$, each of which bounds a disc $D_j$ in $S^3 - \text{int}(V)$. We use these discs to extend $g_1$ to a map $g_1: Z \to S^3 - N$, $N$ the regular neighborhood of $L$ in $V$ with $f(\partial N) = \partial C(J)$. $g_1$ gives $S^3 - N$ the structure of a surface bundle over $S^1$ with fiber $S$. Extending the fibering to $N - L$, we see that $L$ is fibered. If $g(P) = 0$, then $L$ is unknotted. Since $T$ is essential in $C(J)$, $R_1, \ldots, R_n$ are not discs, so $g(S) = g(P) < g(F)$, hence $g(L) < g(J)$. This proves (ii).

We can choose oriented line segments $d_j$ in $E_j$ joining $D_{j0}$ to $D_{j1}$ such that $c = g_1(d_1 \cup \ldots \cup d_n)$ is a core of $S^3 - \text{int}(V)$. Clearly $c$ has linking number $n \neq 0$ with $L$. This proves (iii).

3. **Theorem**: Let $p$ be a prime. If the period $p$ Smith Conjecture is true for all simple fibered knots, then it is true for all fibered knots.

**Proof**: Let $K_0$ be a fibered knot and $\langle h_0 \rangle$ be a cyclic action of order $p$ with $\text{Fix}(h_0) = K_0$. Choose an
invariant knot space \( C_0 = C(K_0) \). If \( C_0 \) contains no essential tori, then by Lemma I.6.7 \( K_0 \) is simple and we are done. If \( C_0 \) contains an essential torus \( T \), then using Theorem III.4.6 we isotop it to a torus \( T' \) which is either invariant or disjoint from its translates. In the first case we do Type I surgery on \( T' \); in the second case we do Type II surgery. Either way we obtain a new simple closed curve \( K_1 \) and a cyclic action \( \langle h_1 \rangle \) of order \( p \) with \( \text{Fix}(\langle h_1 \rangle) = K_1 \).

We repeat the surgery as long as our knot spaces contain any essential tori, so that we get a sequence \((K_0, \langle h_0 \rangle), (K_1, \langle h_1 \rangle), (K_2, \langle h_2 \rangle), \ldots\), each pair being obtained from its predecessor by Type I or II surgery. The sequence must terminate, since \( g(K_{i+1}) < g(K_i) \) by Proposition 2. Let \( K_n \) be the final curve. Then \( K_n \) is either a simple knot or an unknot. If we show the latter is impossible, we are done.

Suppose \( K_n \) is unknotted. Let \( L(c_0) \) be the symmetric trivial link with axis \( A = K_n \) given by Proposition 3.3. By Proposition 2, the linking number of \( c_0 \) and \( A \) is non-zero. Hence \( A \) is not contractible in \( S^3 - L(c_0) \). But this contradicts Proposition 3.8, which says that \( A \) is contractible in \( S^3 - L(c_0) \). This completes the proof.

4. **Corollary:** If the Smith Conjecture is true for all simple fibered knots, then it is true for all fibered knots.
**Proof:** This follows immediately from Theorem 3 and Lemma I.4.6.
V.5 Other reduction theorems

1. **Theorem:** Let \( p \) be a prime. If the period \( p \) Smith Conjecture is true for all simple knots \( K \) with \( b(K) \leq p-1 \), then it is true for all non-simple knots \( J \) with \( b(J) \leq 2p-1 \).

**Proof:** Suppose \( J \) is a counterexample to the period \( p \) Smith Conjecture, \( b(J) \leq 2p-1 \) and \( J \) has a companion \( K \) of order \( \alpha \) and multiplicity \( m \). We shall use the relations stated in Lemma I.3.12. If \( \alpha = 1 \), then \( J \) is the composite of \( K \) and some knot \( K' \), so \( b(J) = b(K) + b(K') - 1 \). Thus either \( b(K) \leq p-1 \) or \( b(K') \leq p-1 \), say \( b(K) \leq p-1 \). By Theorem III.1.1, \( K \) is a counterexample to the period \( p \) Smith Conjecture. If \( K \) is simple, we are done. If \( K \) is non-simple, we repeat the argument with \( K \) in place of \( J \). If \( \alpha \geq 2 \), then \( b(K) \leq \frac{1}{2} b(J) \leq p-\frac{1}{2} \), so that \( b(K) \leq p-1 \). If \( m \geq p \), then \( b(J) \geq \alpha m (b(K) - 1) \geq 2p \), a contradiction. Thus \( m < p \) and, by Theorem III.5.1, \( K \) is a counterexample to the period \( p \) Smith Conjecture. If \( K \) is simple, then we are done. If \( K \) is non-simple, then we repeat the argument with \( K \) in place of \( J \). In this way, we get a sequence of counterexamples of decreasing bridge number, so the sequence terminates in a simple knot \( L \) with \( b(L) \leq p-1 \) and we are done.

2. **Theorem:** The Smith Conjecture is true for all non-simple knots \( J \) with \( b(J) \leq 4 \).
Proof: We prove the period $p$ Smith Conjecture for all primes $p$ and apply Lemma I.4.6. Let $J$ be a counterexample with $b(J) \leq 4$ and having a companion $K$ with order $\alpha$ and multiplicity $m$. If $\alpha = 1$, then $J$ is the composite of $K$ and some knot $K'$. Since $b(J) = b(K) + b(K') - 1$, either $b(K) = 2$ or $b(K') = 2$, say the former. By Theorem III.1.1, $K$ is a counterexample. But this contradicts Proposition I.4.13. If $\alpha \geq 2$, then $b(K) \leq \frac{1}{2} b(J)$, so $b(K) = 2$. If $m \geq p$, then $4 \geq b(J) \geq \alpha m (b(K) - 1) \geq 2p$, so $p = 2$, in which case the Smith Conjecture is true for all knots [42]. So $m < p$ and, by Theorem III.5.1, $K$ is a counterexample, again contradicting Proposition I.4.13.
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