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STRUCTURAL ANALYSIS IN LINEAR SYSTEMS

by

Robert Wade Shields

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1. INTRODUCTION

1.1 General Introduction

The object of engineering analysis and design is to determine system characteristics and to develop design procedures to modify a system in such a way as to satisfy certain desired objectives. The usefulness of the resulting analysis depends on the ability to perform the computations required and on the degree the final design accomplishes the desired goals in a physical implementation. The more closely the basic assumptions used in an analysis corresponds to the actual environmental conditions encountered, the more likely the attainment of these practical objectives.

Over the years a major effort has been placed on the study of systems described by linear time invariant differential equations. In the analysis of a particular problem it is usual to progress through several stages of development. The problem is initially stated in terms of idealized conditions. That is, the assumptions are made that the linear model used to represent the system is exact and all operations on this data can be performed precisely. The problem is then analyzed using classical techniques, theoretical conditions obtained for the existence of a solution, and algorithms formulated for the theoretical determination of a solution. This type of analysis is exemplified in the area of controllability by Kalman[1], in
the solution of the decoupling problem by Wonham and Morse
[2] and in the solution of the servomechanism problem by
Wonham and Pearson[3]. Although certain solutions obtained
using this analysis technique exhibit additional desirable
characteristics, such as insensitivity to problem data
variations [4], the reasons for their occurance was not
understood[5].

Insensitivity of system characteristics and control
solutions to parameter variations is a desirable and
realistic property. System parameter uncertainty can arise
naturally from several sources. Firstly, it is unlikely that
the parameters of the assumed linear model are known
precisely, and in fact they are generally obtained using
some form of physical measurement with its inherent errors.
Also the actual physical parameters of the system can be
expected to change with time and ambient conditions. Finally
the use of computers to facilitate the analysis, as is
required of any reasonable sized system, introduces errors
inherent in number representation and data manipulation on a
digital machine. The effects of all these errors may be
equivalently considered as resulting from data uncertainty
[6]. The next logical step in the development of practical
analysis and design techniques is therefore the inclusion of
data uncertainty in the problem formulation. In the case of
system controllability this was first done by Lee and Markus
[7, Theorem 11, p.100] and later by Fabian and Wonham[8].
Considerable effort has been devoted to the analysis of the servomechanism problem, [3], under various assumptions concerning data uncertainty. In particular the concept of "robustness" was introduced to denote a closed loop system which continues to satisfy design objectives when subjected to a certain class of model data uncertainty [5], [9], [10], [11] [12]. In references [9] and [10] data uncertainty is incorporated into the problem formulation using a parameter space concept. In particular a parameter space of dimension equal to the number of uncertain elements in the model is associated with the system. Conditions are then obtained for which only atypical systems in the parameter space, do not admit a solution or yield unacceptable solutions. An alternative procedure to include parameter uncertainty is used in [11] and [12]. Here the problem of admissible perturbations; i.e., changes in parameter nominal values, in the feedback law is first formulated and solved. With this result the class of admissible system parameter perturbations is then characterised. All of the above methods consider, or result in, cases where only selected model parameters are allowed to perturb.

There is an additional characteristic of physical systems which must be considered if practical analysis techniques based on modern control theory results are to be developed. This characteristic is termed the structure of the system and results from the following considerations. In
the linear time invariant description of a system it often occurs that certain state variables are not directly related, do not directly affect the output or are not directly effected by certain system inputs. These situations result in the occurrence of zeros in the model matrices. This is especially evident in large sparse systems; e.g., large power systems. Because these zeros of the model are fixed by physical considerations, it is reasonable to assume they are not subject to perturbations. Thus structural considerations involve the assumption that only the nonzero parameters of a system are subject to perturbations.

As an example of the various stages of analysis described above, consider the problem of determining the rank of a matrix. The rank of the matrix;

\[
\begin{bmatrix}
5 & 4 \\
5 & 4
\end{bmatrix}
\]

is clearly one. However if parameter uncertainty is introduced into the problem the matrix takes the form;
\[
\begin{bmatrix}
5 + \varepsilon_1 & 4 + \varepsilon_4 \\
5 + \varepsilon_3 & 4 + \varepsilon_4
\end{bmatrix}
\]

It is well known, [8], that the rank of this matrix is almost always two depending on the values of the arbitrary perturbations \(\varepsilon_1, \varepsilon_3, \varepsilon_3\), and \(\varepsilon_4\). The inclusion of structural considerations complicates the problem by permitting a matrix whose nonzero entries are the only perturbable entries to always have less than full rank. For example the matrix;

\[
\begin{bmatrix}
a_1 & a_2 & 0 & 0 \\
a_3 & a_4 & 0 & 0 \\
a_5 & a_6 & 0 & 0 \\
a_i & a_i & a_i & a_i
\end{bmatrix}
\]

has rank less than four regardless of the values of \(a_i\), \(i=1,\ldots,10\).

Historically the field of combinatorial mathematics, in particular the theory of representations, has dealt with the effects of structural constraints in mathematical analysis [17], [14]. Early works having a direct relation to this thesis are the matrix structural results of Frobenius [15] and the matching or matrix covering results of König [16]. These
results, to be discussed in more detail, concern the conditions under which every term in the determinantal expansion of a matrix is zero. More recently Lin[17] included parameter uncertainty in the structural analysis of single input system controllability. This analysis, based on an extensive use of graph theory, is noteworthy in that it represents the first effort to include structural constraints in multivariable system analysis.

In this thesis an analysis technique is developed to permit the inclusion of structural constraints in the analysis of uncertain linear time invariant systems.
1.2 Thesis Outline

It is the object of this thesis to develop an analytical technique capable of analyzing systems having a given structure relative to fixed zero locations and whose nonzero parameters are imprecisely known. In addition, the analysis should be capable of being performed precisely on a digital computer and yield results insensitive to general classes of parameter variations. Precise calculation is taken to mean that there is no possibility of error resulting from the use of a digital computer to perform the required computations.

In Chapter II the concept of structure is made precise. A generic formulation used by Wonham [8] to include data uncertainty is then discussed and interpreted in such a way as to incorporate structural constraints. Based on this technique and the early results of König, a maximal generic matrix rank theorem is derived. It is shown that the maximal possible rank of a matrix is completely determined by certain structural constraints, and a computational algorithm is formulated to precisely obtain this rank.

Chapter III applies the results of Chapter II to formulate and solve the structural controllability problem for multi-input linear time invariant systems. It is shown that the results of Lin[17] for single input systems are also necessary and sufficient for the structural controllability of multi-input systems.
Chapter IV considers structural properties of the servomechanism problem[3]. No restrictions are placed on the system parameters other than those imposed by the structure of the model. The concept of a well-posed system[10] with respect to the servomechanism problem is introduced and discussed. Necessary and sufficient conditions for the well-posedness of a given system are then derived. Algorithmic procedures are formulated to precisely determine solvability of the servomechanism problem for a given system structure. The possibility of employing state space extension to yield a solution is then considered. An algorithmic procedure is formulated to determine if it is or is not possible to obtain a well-posed system using state space extension. The practical implications of this algorithm are then discussed. Finally some examples are given to demonstrate the analysis technique introduced in this chapter.

Chapter V summarizes the thesis and discusses the structural results obtained. Additional research areas are considered along with the practical implications of this work.
1.3 Notation and Conventions

Excepting the standard notation described below new symbols are defined as they occur. Previous results pertinent to the development and the relation of this thesis to these results are discussed in the text.

In the following, script letters $\mathcal{X}, \mathcal{Y}, \ldots, \mathcal{U}$ etc., denote finite dimensional vector spaces and $A, B$, etc., denote fixed linear maps as well as their matrix representations relative to some basis. The dimension of the vector space $\mathcal{X}$ is expressed as $d(\mathcal{X})$. Lower case letters $x, y, \text{etc.}$, represent vectors. If $F: \mathcal{X} \rightarrow \mathcal{Y}$ is a linear map, $\ker F$ is the kernel of $F$ and is a subspace of $\mathcal{X}$. If $B$ is a fixed map $\mathcal{B}$ or $\{B\}$ is the image of $B$. If $A: \mathcal{X} \rightarrow \mathcal{X}$, $B: \mathcal{U} \rightarrow \mathcal{X}$ and $d(\mathcal{X}) = n$, then

$$\langle A|B \rangle = \sum_{i=1}^{n} A^{-i}B$$

is the controllable space of the pair $(A, B)$. $\mathcal{X}(A)$ is the subspace of unstable modes of $A: \mathcal{X} \rightarrow \mathcal{X}$; i.e., let $\alpha(\lambda)$ be the minimal polynomial of $A$ and factor $\alpha = \alpha^+ \cdot \alpha^-$ into coprime factors where the roots of $\alpha^+(\lambda)$ are the closed right-half complex plane roots of $\alpha(\lambda)$. Then $\mathcal{X}(A) = \ker \alpha^+(A)$. $\mathbb{R}^i$ is the real vector space of dimension $i$, $\emptyset$ denotes the empty set and $\mathbb{C}$ the complex plane. $\triangle$ denotes the end of a proof, and $\mathcal{X}^T$ the transpose of $\mathcal{X}$. The complex spectrum of a matrix $A$ is expressed as $\sigma(A)$. $|k|$ denotes the absolute value of the scalar $k$ and $\|A\|$ denotes a suitably defined matrix norm of $A$. 
II. STRUCTURE AND THE MAXIMAL RANK OF A MATRIX

2.1 Introduction

The ability to determine the rank of a matrix is crucial in the analysis and design of control systems. For example, the determination of controllability (observability) [1], linear dependence of vectors in a vector space [18], solvability of the servomechanism problem in its various forms [3], [5], [11], [12] all require the evaluation of the rank of certain matrices.

The linear time invariant system matrix descriptions considered in this thesis are assumed to possess those characteristics discussed in Chapter I. In particular, the nonzero entries are imprecisely known while the zero entries are assumed fixed by coordinatization and/or physical properties of the system. Thus all matrices are assumed to consist of perturbable entries and fixed zeros.

Definition 2.1.1 A structured system is composed of matrices having fixed zeros in certain locations and arbitrary entries in the remaining locations.

Conceptually a structured system is described by matrices composed of zeros and indeterminates. Inherent in Definition 2.1.1 is the assumption that the nonzero entries in a
structured system are not related and are therefore arbitrary. This definition is consistent with practical considerations as discussed in Chapter I, and with digital machine capabilities as regards exact zero representation and manipulation.

Definition 2.1.2 Two systems, having the same number and order of defining matrices, are structurally equivalent if there is a one to one correspondence between the locations of the fixed zeros and nonzero entries of the corresponding matrices of each system.

Classical methods of numerical analysis have long been employed in studying practical computational problems,[6], [19], [20]. This analysis attempts to correlate the effects of imprecise data and the error in the computed solution. Successful results are obtained via classical methods in the analysis of certain types of problems; eg., matrix eigenvalue location.

In the case of the determination of the eigenvalues of a matrix it is well known that the eigenvalues are continuous functions of the matrix entries. Thus given a matrix of imprecise data it is possible to obtain theoretical bounds on the locations of the eigenvalues which are dependent on the computational methods used and the
sensitivity or condition number of the matrix. The major
preoccupation of researchers in this area is to develop
improved numerical algorithms to yield stable and more
accurate results.

The determination of the rank of a matrix however is a
fundamentally different type of numerical problem unsuited
to the classical techniques of analysis referred to above.
This difference results because the rank of a matrix is not
a continuous function of the matrix entries. In fact, as
shown in Chapter I, it is possible for infinitesimal changes
in selected entries of a matrix to alter the rank, whereas
arbitrary changes in other entries have no effect. Use of
classical techniques in the analysis of this type of problem
is generally attempted by employing some form of subjective
criterion based on practical computational experience. This
usually involves the creation of a "reasonable" criterion
for recognizing when an entry has become zero in a matrix
reduction procedure; e.g., Gaussian elimination\[21\]. There
are many examples of this technique in the literature,\[22\],
\[23\]. The possibility of error in making these subjective
decisions and the means of choosing the tolerance criteria
are just two of the weaknesses of this method. However, the
major shortcoming is that this analysis technique does not
come to grips with the basic nonlinearity inherent in the
problem.

The need for a more realistic method of handling
problems of the rank determination type has been recognized, [8], [24]. Although no provision is made for structural considerations these researchers were among the first to include data uncertainty into the initial formulation of linear control and system analysis problems. The basis of their approach is the recognition that the property under investigation is, in a sense to be made clear later, independent of parameter values. Typically this analysis technique shows that all parameter values, except for an atypical set yields the same analytical result. The technique of generic analysis employed by Wonham [8], is used in this thesis to provide a mathematical basis for this type of analysis. In the following section the basis of generic analysis is introduced and interpreted in such a way as to include problem structural constraints. Section 2.3 presents the fundamental work of König and derives a certain matrix structural result needed in the analysis to follow. Section 2.4 combines Section 2.3 and Section 2.4 to formulate a realistic interpretation of the rank of a structured matrix.
2.2 Generic Analysis

Much of the nomenclature of this section is taken from [8] and Chapter 0 of [25].

Let $A, B, \ldots$ be structured matrices with elements in $\mathbb{R}$. If $N$ is the number of perturbable entries in $A, B, \ldots$ then associated with this system is the parameter space $\mathbb{R}^N$. Every possible set of $N$ values represents a data point $p \in \mathbb{R}^N$. Thus every particular system structurally equivalent to $A, B, \ldots$ is represented by a data point $p$. $A, B, \ldots$ can be considered as matrices with entries from the ring of polynomials in $N$ variables; $\mathbb{R}[\lambda]$ where $\lambda = (\lambda_1, \ldots, \lambda_N)$ is simply a list representing the $N$ perturbable entries of the system matrices.

The generic approach expresses properties of the structured matrices $A, B, \ldots$ in terms of hypersurfaces in the parameter space $\mathbb{R}^N$. Let $\Pi(A, B, \ldots)$ be a property that may be asserted about the matrices $A, B, \ldots$. Most properties of interest will turn out to be true for all data points except those which lie on an algebraic hypersurface in $\mathbb{R}^N$. More precisely consider polynomials $\Psi, \in \mathbb{R}[\lambda]$.

Definition 2.2.1 A variety $V \subset \mathbb{R}^N$ is the set of common zeros of a finite number of polynomials $\Psi_1, \ldots, \Psi_k$:

$$V = \{ p \in \mathbb{R}^N : \psi_i(p) = 0, \ i = 1, \ldots, k \}.$$ 

$V$ is proper if $V \cap \mathbb{R}^N$ and nontrivial if $V \neq \emptyset$. 

Definition 2.2.2 A data point \( p \in \mathbb{R}^N \) is typical relative to the variety \( V \) if \( p \in V^c \); the complement of \( V \).

Definition 2.2.3 A property \( \pi \) is a function from \( \mathbb{R}^N \) to the set \( \{0,1\} \), where \( \pi(p) = 0 \) (or 1) means \( \pi \) fails (or holds) at \( p \).

Definition 2.2.4 The property \( \pi \) is generic relative to the proper variety \( V \) if \( \ker \pi \subset V \); and \( \pi \) is generic if such a \( V \) exists.

In the sequel the following conventions will be employed. A structured system refers to a system having a given fixed zero structure and whose nonzero entries are considered as indeterminates; i.e., the set of all structurally equivalent systems. A data point in the associated parameter space is a specific set of parameter values for the system. The useful implications of generic analysis follow from the following lemma.

Lemma 2.2.1

If \( V \subset \mathbb{R}^N \) is a proper nontrivial variety then \( V \) is a closed set.
Proof:
It is sufficient to show that \( V^c \) is an open set. Let \( \Psi_i \) 
\[ i=1, \ldots, k \]
be a defining set of polynomials for \( V; \ i.e., \]
\[ V = \{ p: \Psi_i(p) = 0; \ i=1, \ldots, k \} \]
Then
\[ V^c = \{ p: \Psi_i(p) \neq 0 \text{ for some } i; \ i=1, \ldots, k \} \]
Let \( p' \in V^c \); then for some \( i, i=1, \ldots, k \),
\[ \Psi_i(p') \neq 0. \]
Since \( \Psi_i \) is a continuous function it follows that for 
every \( \gamma > 0 \) there exists a \( \delta > 0 \) such that
\[ |\Psi_i(p) - \Psi_i(p')| < \gamma \text{ whenever } \|p - p'\| < \delta. \]
So for \( p \) close to \( p' \), \( \Psi_i(p) \neq 0 \) and so \( p \in V^c \). Therefore \( V^c \) is 
an open set.
\[ \square \]

Therefore if a property is generic relative to \( V \), the 
property holds at any data point \( p' \in V^c \) and in a 
sufficiently small neighborhood about \( p' \). In addition it is 
clear from the lemma that if \( p' \in V \), with \( V \) nontrivial and 
proper, then every neighborhood of \( p' \) contains data points 
\( p \in V^c \). Otherwise each defining polynomial \( \Psi_i \) of \( V \) vanishes 
identically in some neighborhood of \( p' \), hence vanishes on \( \mathbb{R}^n \) 
[35]. This contradicts the assumption that \( V \) is proper.

All data points which cause a given generic property to
fail are atypical in the sense they lie on a hypersurface in the parameter space. In addition any such data point may be suitably perturbed an arbitrarily small amount and cause the given property to hold. Therefore system properties which can be shown to be generic, can be expected to hold for practically every set of parameter values; i.e., all typical data points.

As an example of the application of the generic analysis technique, consider the generic rank of a matrix $A$ of order $(n \times m)$ consisting totally of perturbable entries. Assume $n \ll m$ and define the property $\Pi$ as:

$$\Pi(p) = \begin{cases} 1 & \text{if the rank of } A \text{ is } n \\ 0 & \text{if the rank of } A \text{ is less than } n \end{cases}$$

where $p \in \mathbb{R}^{n \times m}$. Consider the polynomial $\Psi$ in $nm$ indeterminates defined as the sum of the squares of all possible $n^{th}$ order minors of $A$. $\Psi$ satisfies the property that $p \in \ker \Pi$ implies $\Psi(p) = 0$. Thus to prove that the property defined above is generic it only remains to show that the variety $V$ defined by $\Psi$,

$$V = \{ p : \Psi(p) = 0 \},$$

is proper. Since it has been assumed that $A$ has no structure it is trivial to select a $p' \in \mathbb{R}^{n \times m}$ for which $\Psi(p') \neq 0$. This proves the following well known result.
Lemma 2.2.2

The maximal rank of a matrix having no specified structure is equal to the minimum dimension of the matrix.

The inclusion of structure into the problem makes it possible for matrices to have less than full rank independently of parameter values. For this case, \( \Psi(p) = 0 \) for every \( p \in \mathbb{R}^{n-m} \) and so \( V \) is not proper in the above proof. The additional analysis required to handle this situation is presented in the remainder of this chapter.
2.3 Preliminary Matrix Structural Results

The first investigations into the structural properties of matrices were performed by Frobenius in 1912 [30] and in a more general form by König in 1931 [27]; see also [15], [16], [31], [32]. Frobenius' result concerns the structural conditions necessary and sufficient for the vanishing of every term in the determinantal expansion of a square matrix. His proof employed an induction on the matrix order. König's results however were much more general and represent a fundamental mathematical achievement applicable in many areas. His work is variously referred to as "duality theorems" [28] or "matching theorems" [29]. Although König originally proved his main result using graph theoretic arguments, it has since been proven numerous times using various techniques, [15], [29], [31], [32]. His theorem is stated below in a form most suited to matrix analysis. The result is not proven and the interested reader is referred to the above references.

Let $A$ be an $(n \times m)$ matrix composed of two types of entries; zeros and indeterminates.

Definition 2.3.1 A line of the matrix $A$ refers to either a row or column.

Definition 2.3.2 A set of independent entries of $A$ is a finite set of entries no two of which lie
on the same line.

Thus if \( n \leq m \), the maximum possible number of independent entries of \( A \) is \( n \). Note that the maximum number of independent nonzero entries depends on the structure of the matrix.

\[ \text{Definition 2.3.3 } M \text{ denotes the cardinality of the largest set of nonzero independent entries of } A. \]

\[ \text{Definition 2.3.4 } M^* \text{ denotes the minimum number of lines of } A \text{ which contain (cover) all the nonzero entries of } A. \]

In the following example \( M^* = 2 \) since the first column and second row contain all the nonzero entries and \( M^* = 2 \) by considering the set containing the \((1,1)\) and \((2,2)\) entries:

\[
\begin{bmatrix}
x & 0 & 0 \\
x & x & x \\
x & 0 & 0
\end{bmatrix}
\]

Using the nomenclature introduced in the above definitions, König's classical result takes the following form.

\[ \text{Theorem 2.3.1} \]

If \( A \) is a finite rectangular matrix consisting of zero
and nonzero entries then

\[ M^* = M_+ \]

The relation between König's theorem and the structural properties of the matrix A is made clear by the following lemma. Assume \( n \leq m \).

Lemma 2.3.1

Every term in the determinantal expansion of every \( t^{th} \) order submatrix of A \( 1 \leq t \leq n \), vanishes if and only if for some \( k \) in the range \( m-t < k \leq m \), A contains a zero submatrix of order \( (n+m-t-k+1) \times k \).

Proof:

(Necessity)

Assume every term in the determinantal expansion of every \( t^{th} \) order submatrix of A vanishes for some \( t \) in the range \( 1 \leq t \leq n \). Then since every term in such an expansion consists of the product of \( t \) independent entries of A, it follows from Theorem 2.3.1 that

\[ M^* = M_+ \leq t-1. \]

Otherwise there would exist a \( t^{th} \) order minor of A having a nonzero term in its expansion. Thus there exist \( r \) rows and \( s \) columns of A covering all nonzero entries such that

\[ r + s = M_. \]
Let \( t, (t_i) \) designate any rows (columns) such that

\[(r + t_i) + (s + t_i) = t-1\]

where \( t, t_i \geq 0 \). Then there exists a \( n-(r+t_i) \times m-(s+t_i) \) zero submatrix of \( A \). With \( k=m-(s+t_i) \) there are

\[n-(r+t_i) = n+m-(r+t_i)-m\]

\[= n+m-t+(s+t_i)+1-m\]

\[= n+m-t-k+1\]

rows and \( k \) columns which define a zero submatrix of \( A \).

(Sufficiency)

By assumption there exists a \( (n+m-t-k+1) \times k \) zero submatrix of \( A \) for some \( k \) in the range \( m-t<k \leq m \). Using permutation operations * \( A \) can be brought to the form

\[
\begin{bmatrix}
A_1 & \Phi \\
A_2 & A_3
\end{bmatrix}
\]

where \( A_1 \) is of order \( (n+m-t-k+1) \times (m-k) \) and \( A_3 \) is of order \( (t-m+k-1) \times k \). Assume there exists a \( t \)th order submatrix of \( A \) having a nonzero term in its determinantal expansion. Then there exists a set of \( t \) nonzero independent entries in \( A \).

* A permutation matrix is a square matrix which in each row and each column has some one entry unity, all others zero. Permutation operations have the property that they preserve structure.
If \( m-k \) such entries are chosen from the first \( m-k \) columns of the above matrix, \( t-(m-k) \) remain to be selected from the remaining \( k \) columns. However from the size of the submatrix \( A_y \), the maximum number of nonzero independent entries possible is \( t-(m-k)-1 \). This contradicts the assumption that there exists a set of \( t \) nonzero independent entries in \( A \). Therefore every term in the expansion of every \( t \text{th} \) order submatrix of \( A \) vanishes. △

By setting \( m=n=t \) Lemma 2.3.1 reduces to the result first published by Frobenius[15] and[30]. It is the above lemma which leads to the maximal rank theorem of the next section.
2.4 Maximal Rank of a Matrix

Lemma 2.3.1 is only sufficient to determine a bound on the rank of a matrix, since every term in the determinantal expansion of a matrix need not vanish for the determinant to be zero. The above form of analysis is not capable of taking into consideration the effect of the nonzero entries of $A$ on the rank of the matrix.

The additional assumption that $A$ is a structured matrix, see Definition 2.1.1, makes applicable the generic analysis technique introduced above and yields a solution to the maximal matrix rank problem. Using the nomenclature and definitions of Section 2.2, let $A$ be a structured matrix with associated parameter space $R^N$; where $N$ is the number of nonzero entries in $A$.

Definition 2.4.1 The matrix $A$ has generic rank equal to $t$ (denoted as rank($A$) = $t$ (g!)) if there exists a proper variety $V \subset R^N$ such that all data points for which rank($A$) $\neq t$ lie on $V$.

This definition simply restates what is meant by a generic property and is included only to clarify this concept. In the sequel all generic properties should be interpreted in terms of this definition. Assume $A$ is of order (nxm) with $n \leq m$. 
Definition 2.4.2 A matrix $A$ has Form($t$) for some $t$, $1 \leq t \leq n$, if for some $k$ in the range $m-t < k \leq m$, $A$ contains a zero submatrix of order $(n+m-t-k+1) \times k$.

Consider the following two matrices with $n=m=5$. Each has Form(4), but $k=5$ for the first and $k=4$ for the second.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
x & x & x & x & x \\
x & x & x & x & x \\
x & x & x & x & x \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 \\
x & x & x & x & x \\
x & x & x & x & x \\
\end{bmatrix}
\]

Also note that if $A$ is of Form($t$), then $A$ is of Form($j$), $t \leq j \leq n$. Form($t$) implies the existence of a zero submatrix of order $(n+m-t-k+1) \times k$ for some $k$ in the range $m-t < k \leq m$. But since $j > t$, there exists a zero submatrix of order $(n+m-j-k+1) \times k$ for the same $k$. 
Theorem 2.4.1

For any \( t, 1 \leq t \leq n \), \( \text{rank} \left( A \right) < t \) for every \( p \in \mathbb{R}^N \) if and only if \( A \) has Form\((t)\).

Proof:

(Sufficiency)

By Lemma 2.3.1 the result is trivial.

(Necessity)

Assume \( A \) does not have Form\((t)\). Then by Lemma 2.3.1 there exists at least one nonzero term in the determinantal expansion of at least one \( t \)-order submatrix of \( A \). Let this term be represented by

\[
a_{i_1 i_2} \ldots a_{i_{t-1} i_t}
\]

Choose \( a_{i_1 i_2} = \ldots = a_{i_{t-1} i_t} = 1 \) and all other entries of \( A \) to be zero. Therefore there exists a \( \mathbf{p} \in \mathbb{R}^N \) such that \( \text{rank} \left( A \right) \geq t \).

The main result of this chapter is contained in the following theorem.

Theorem 2.4.2

\[
\text{rank} \left( A \right) = t \quad (g)
\]

for \( t=n \) if and only if \( A \) is not of Form\((n)\);

for \( 1 \leq t \leq n \) if and only if \( A \) is of Form\((t+1)\) but not of Form\((t)\).
Proof:

(Necessity)

Consider the case where \( t = n \). By assumption there exists a \( p \in \mathbb{P}^N \) such that the rank(A) = n. Then by Theorem 2.4.1 this implies A is not of Form(n).

Consider the case where \( 1 \leq t < n \). Again by Theorem 2.4.1 and the necessity assumption, A is not of Form(t).

Assume A is not of Form(t+1) and define

\[
\Pi = \begin{cases} 
1 & \text{if the rank}(A) \geq t+1 \\
0 & \text{if the rank}(A) \leq t
\end{cases}
\]

Also define the polynomial \( \Psi \in \mathbb{P}[\Lambda] \) as the sum of the squares of all minors of order greater than or equal to t+1. Then by Theorem 2.4.1 there exists a \( p' \in \mathbb{P}^N \) such that rank(A) \( \geq t+1 \) and therefore the variety defined by \( \Psi \) is proper. Since \( p \in \ker \Pi \) implies \( \Psi(p) = 0 \), there results

\[
\text{rank}(A) \geq t+1 \quad (g)
\]

Since the generic rank of A cannot be both t and greater than t, A must have Form(t+1). This establishes the necessity proof.

(Sufficiency)

Consider the case where \( t = n \). Define the property
\[ \Pi = \begin{cases} 
1 & \text{if the rank}(A) = n \\
0 & \text{if the rank}(A) < n 
\end{cases} \]

Also define the polynomial \( \psi \in \mathbb{R}[x] \) as the sum of the squares of all minors of \( A \) of order \( n \). Now since \( A \) is not of Form(\( n \)), by Theorem 2.4.1 there exists a \( p' \in \mathbb{R}^n \) for which

\[ \text{rank}(A) = n. \]

Therefore \( \psi \) defines a proper variety having the property that \( p \in \ker \Pi \) implies \( \psi(p) = 0 \). Therefore

\[ \text{rank}(A) = n \quad (g) \]

Consider the case where \( 1 \leq t < n \). By assumption, \( A \) is of Form(\( t+1 \)) which implies by Theorem 2.4.1 that \( \text{rank}(A) < t+1 \) for every \( p \in \mathbb{R}^n \). Therefore define

\[ \Pi = \begin{cases} 
1 & \text{if the rank}(A) = t \\
0 & \text{if the rank}(A) < t 
\end{cases} \]

and the polynomial \( \psi \in \mathbb{R}[x] \) as the sum of the squares of all minors of order \( t \). Since \( A \) is not of Form(\( t \)) then by Theorem 2.4.1 there exists a \( p' \in \mathbb{R}^n \) for which

\[ \text{rank}(A) = t \]

Therefore \( \psi \) defines a proper variety having the property that \( \Pi(p) = 0 \) implies that \( \psi(p) = 0 \). Thus

\[ \text{rank}(A) = t \quad (g) \]

\[ \Delta \]
The above theorem makes precise what is meant by the rank of a structured matrix. That is, assuming a structured matrix $A$ satisfies the conditions of the theorem, all those matrices structurally equivalent to $A$ having rank different from $t$ lie on a proper variety in $\mathbb{R}^N$. This is a realistic interpretation of rank consistent with practical and computational considerations.

The actual determination of the generic rank of a structured matrix involves the ability to computationally recognize the existence and nonexistence of $\text{Form}(t)$, for any $t$. This requirement necessitates searching the given matrix for certain patterns of fixed zeros. Since it is possible to precisely represent and recognize a fixed zero using a digital computer it is a straightforward task to implement an algorithm to determine generic rank. Because the basis of this algorithm depends on this ability to recognize fixed zeros, the program is given the designation "fixed zero rank finder" and is denoted by $\text{FZRF}$. The details of $\text{FZRF}$ are given in Appendix I.
III. STRUCTURAL CONTROLLABILITY

3.1 Introduction

The concepts of Chapter II may be used to investigate the genericity of certain linear time invariant system characteristics. As mentioned previously, an important characteristic, expressible in terms of a matrix rank check, is the controllability of a system.

Consider the linear time invariant dynamical system described by

\[ \dot{x} = Ax + Bu \]

where \( x(t) \in \mathcal{X} \) is the \( n \) dimensional state vector and \( u(t) \in \mathcal{U} \) the \( r \) dimensional input vector. Assume the system (denoted by the pair \((A,B)\)) is described by structured matrices; i.e., \((A,B)\) is a structured system. Similarly structural equivalence of systems is interpreted as defined in Chapter II.

Associate with the structured pair \((A,B)\) the parameter space \( \mathbb{R}^{N \times M} \) where \( N(M) \) is the number of perturbable entries in \( A(B) \). Thus \( \mathbb{R}^N, \mathbb{R}^M, \mathbb{R}^{N \times M} \) are the parameter spaces associated with the structured matrices \( A \) and \( B \) and the structured system \((A,B)\).

Definition 3.1.1 A system pair \((A,B)\) is structurally controllable if there exists at least one
controllable system structurally equivalent to $(A, B)$.

Lemma 3.1.1
The pair $(A, B)$ is structurally controllable if and only if all uncontrollable pairs structurally equivalent to $(A, B)$ lie on a proper variety in $\mathbb{R}^{N+M}$.

Proof:
(Necessity)
Define the polynomial $\Psi \in \mathbb{P}[\lambda, \ldots, \lambda_{N+M}]$ as the sum of the squares of all maximal order minors of the controllability matrix

$$[B \ AB \ldots \ A^{n-1} B].$$

By assumption there exists a $p' \in \mathbb{R}^{N+M}$ for which the associated system pair is controllable. This implies that $\Psi$ defines a proper variety.

(Sufficiency)
By the definition of a proper variety there exists a $p \in \mathbb{R}^{N+M}$ representing a controllable pair. ▲

As usual the interpretation of this concept is that all uncontrollable system pairs, structurally equivalent to a structurally controllable system are intuitively atypical. This lemma represents a generalization of the controllability results of Lee and Markus[7] to structured
systems. Lemma 3.1.1 characterizes structural controllability but does not yield any definitive conditions for determining structural controllability.

The ability to recognize a structurally controllable system is not a simple application of FZRF to the controllable matrix. For example, consider the system pair

\[
\begin{bmatrix}
a_1 & a_2 & a_3 & a_4 \\
a_2 & a_3 & a_4 & a_5 \\
a_3 & 0 & 0 & 0 \\
a_4 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
h_1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
b_1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\]

In general the formation of the controllability matrix destroys the structure inherent in the system:

\[
\begin{bmatrix}
b_1 & 0 & a_1 & a_2 & a_3 & a_4 \\
0 & b_1 & a_2 & b_2 & a_3 & 0 \\
0 & 0 & a_3 & b_2 & a_4 & 0 \\
0 & 0 & 0 & a_4 & b_2 & a_5
\end{bmatrix}
\begin{bmatrix}
h_1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\]

where the \(X\)'s denote general nonzero terms.

FZRF is not applicable to this matrix since the nonzero entries are not allowed to perturb independently. Thus although the system is obviously uncontrollable for any set of parameter values due to the dependency of the last two rows, a naive application of FZRF would give the generic rank as four, indicating a structurally controllable system. The analysis of structural controllability of multi-input systems requires a more involved investigation
of matrix structural properties than yet considered. As mentioned in Chapter I, Lin [17], has solved the problem of determining structural controllability for single input systems. Use is made of his result in the extension of structural controllability to multi-input systems.

Lin's result for single-input systems is most conveniently stated using two canonical forms of the partitioned matrix \([A \ B]\). One of these is Form (n) of Definition 2.4.2 and the second is defined as follows:

Definition 3.1.2 The matrix \([A \ B]\) is of Form I if there exists a permutation matrix \(P\) satisfying

\[
\begin{bmatrix}
A & B & 0
\end{bmatrix}
\begin{bmatrix}
P & 0 & 0 \\
0 & I
\end{bmatrix}
= \begin{bmatrix}
A_1 & 0 & 0 \\
0 & A_2 & A_3 & B_2
\end{bmatrix}
\]

with \(A_t\) of order \((t \times t)\), \(1 \leq t \leq n\).

Lin's result may now be stated as;

Lemma 3.1.2

A structured system pair \((A, B)\) is structurally uncontrollable* if and only if the matrix \([A \ B]\) is of Form I or Form (n) where \(B\) is of order \((n \times 1)\).

* A pair \((A, B)\) is structurally uncontrollable if every system structurally equivalent to \((A, B)\) is uncontrollable in the usual sense.
The following result concerning system controllability and observability is used throughout the thesis. For the proof see Rosenbrock [33].

Lemma 3.1.3 [33]

Given \((A,B,C)\) where \(A\) is an \(nxn\) matrix

i) \((A,B)\) is controllable if and only if

\[
\text{rank}(A-\text{sl} B) = n
\]

for all \(s \in \mathbb{C}\).

ii) \((A,C)\) is observable if and only if

\[
\text{rank} \left[ \begin{bmatrix} A-\text{sl} \\ C \end{bmatrix} \right] = n
\]

for all \(s \in \mathbb{C}\).

In the next section it is shown that Lin's single-input conditions are also necessary and sufficient for the structural controllability of multi-input structured systems. In Section 3.3 the problem is reformulated in such a way as to yield a single matrix structure condition which can be checked using FZRF. This is accomplished by obtaining a generic rank check condition which is equivalent to the two forms of Definition 3.1.2 and Definition 2.4.2.
3.2 Structural Controllability of Multi-input Systems

Consider the structured pair \((A, B)\) with associated parameter spaces \(\mathbb{R}^N, \mathbb{R}^M, \mathbb{R}^{N+M}\), where \(N(M)\) is the number of nonzero entries in \(A\) (\(B\)). The main result of this chapter is contained in the following theorem.

Theorem 3.2.1

The system pair \((A, B)\) is structurally uncontrollable if and only if the matrix \([A \ B]\) is of Form 1 or Form \((n)\).

This theorem is proved using several lemmas concerning the structural properties of the matrices \(A\) and \(B\). The first two lemmas concern the eigenvectors of a structured matrix.

Definition 3.2.1 If \(s\) is an eigenvalue of \(A\), the corresponding left eigenspace is the set of all vectors \(x\) satisfying \(x^T(A-sl) = 0\). Any such \(x\) is a left eigenvector of \(A\).

Define \(\Theta\) as the set of data points in the parameter space \(\mathbb{R}^N\), associated with \(A\), defined by:

\[ \Theta = \{ a : a \in \mathbb{R}^N, \ \text{rank}(A-sl) < n-1 \text{ for some } s \in \mathbb{C} \} \]

Note \(\Theta \neq \emptyset\) since \(A=0\) is contained in \(\Theta\).
Lemma 3.2.1

Assume \( \text{rank}(A) = n \) (g). Then there exists a proper variety in the parameter space \( \mathbb{R}^N \) which contains \( \Theta \).

Proof:
Define \( b \) as a \((n \times 1)\) vector containing no fixed zeros and consider the single input system \((A, b)\) with associated parameter space \( \mathbb{R}^{N+n} \). Assume \((A, b)\) is structurally uncontrollable. Then since \( b \) has no fixed zeros, Lemma 3.1.2 implies the matrix \([A \ b]\) must have Form (n). But by Theorem 2.4.1, \( \text{rank}[A \ b] < n \) for every data point in \( \mathbb{R}^{N+n} \). This contradicts the assumption that \( \text{rank}(A) = n \) (g). Therefore the structured pair \((A, b)\) must be structurally controllable. Define the polynomial \( \Psi \in \mathbb{R}[\lambda_1, \ldots, \lambda_{n+n}] \) as the sum of the squares of all maximal order minors of the matrix

\[
[b \ Ab \ldots \ A^{n-1} b].
\]

\( \Psi \) defines a variety in \( \mathbb{R}^{N+n} \) which is proper since the pair \((A, b)\) is structurally controllable. Fix \( b \) so that there exist \( a \in \mathbb{R}^N \) for which \( \Psi \neq 0 \) as a polynomial in the nonzero entries of \( A \). Now for every \( a \in \Theta \) the corresponding pair \((A, b)\) is uncontrollable. Thus \( a \in \Theta \) implies \( \Psi = 0 \). Therefore \( \Theta \) is contained in the proper variety \( V \subset \mathbb{R}^N \) defined by

\[
V = \{ a : \Psi = 0 \}
\]
This lemma states that under the assumption \( \text{rank}(A) = n \) (g), all the eigenspaces associated with the eigenvalues of \( A \) are generically of dimension one. This follows since all data points \( a \in V^c \) correspond to matrices having only eigenspaces of dimension one.

Define \( \mathcal{Y} \) as the set of points in \( R^N \), associated with \( A \), defined by:

\[
\mathcal{Y} = \left\{ a : a \in R^N, \text{ there exists at least one left eigenvector of } A \text{ having a zero component} \right\}
\]

Note \( \mathcal{Y} \neq 0 \) since \( A=0 \) is contained in \( \mathcal{Y} \).

Definition 3.2.2 The matrix \( A \) is of Form 1 if there exists a permutation matrix \( P \) satisfying

\[
P^T A P = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\]

with \( A_{11} \) of order \( (t,t) \), \( 1 \leq t \leq n \).

Note that if \([A B]\) is of Form 1, Definition 3.1.2, then \( A \) is of Form 1.

Lemma 3.2.2

Assume \( \text{rank}(A) = n \) (g) and \( A \) does not have Form 1. Then there exists a proper variety in the parameter space \( R^N \) which contains \( \mathcal{Y} \).

Proof:
Consider the set of all structured (nx1) vectors containing at least one zero and one nonzero entry. This set is composed of $2^n-2$ vectors. Let $b_i$ for $i=1,\ldots,2^n-2$ represent these structured vectors. Consider the system pair $(A,b_i)$ with associated parameter space $R^{n\times n_i}$ where $n_i$ is the number of nonzero entries in $b_i$. Assume $(A,b_i)$ is structurally uncontrollable. Since $A$ is not of Form I the matrix $[A b_i]$ is not of Form I. Thus $[A b_i]$ must be of Form (n) from Lemma 3.1.2. From Definition 2.4.2, for some $k$ in the range $1\leq k \leq n+1$, $[A b_i]$ contains a zero submatrix of order $(n-k+2)\times k$. Then $A$ must contain a zero submatrix of order $(n-k+2)\times k-1$. Thus with $s=k-1$, $A$ contains a zero submatrix of order $(n-s+1)\times s$. Then from Theorem 2.4.1, $\text{rank}(A) \leq n$ for every $a \in R^n$ contradicting the assumption that $\text{rank}(A) = n$ (g). Therefore $(A,b_i)$ must be structurally controllable.

Define the polynomial $\Psi_i$ in the nonzero entries of $A$ and $b_i$, as the sum of the squares of all $n\times n_i$ order minors of the matrix

$$[b_i \ Ab_i \ \ldots \ A^{n_i-1} b_i]$$

$\Psi_i$ defines a variety in $R^{n\times n_i}$ which is proper since the pair $(A,b_i)$ is structurally controllable. For each $i \in \{1,\ldots,2^n-2\}$ fix $b_i$ so that there exist $a \in R^n$ for which $\Psi_i \neq 0$ as a polynomial in the nonzero entries of $A$. 
Consider the variety \( V \subset \mathbb{R}^N \) defined by
\[
V = \bigcup_{i=1}^{2^m} V_i
\]
where
\[
V_i = \{ a : \Psi_i = 0 \}
\]
Now \( V \) is proper since each \( V_i \) is proper. Also \( a \in V \) implies \( a \in V \). To see this let \( A' \) be the matrix associated with the data point \( a \). Then there exists a left eigenvector \( x \) of \( A' \) containing a zero component. Therefore there is a \( b_i \) such that the pair \( (A', b_i) \) is uncontrollable, where the nonzero entries of \( b_i \) correspond to zero rows of \( x \). Thus \( \Psi_i = 0 \) which implies \( a \in V \). Therefore \( V \subset V \) where \( V \) is a proper variety in \( \mathbb{R}^N \).

\[\Box\]

This lemma states that under the assumptions, \( \text{rank}(A) = n \) (g) and \( A \) does not have Form 1, all the components of each eigenvector are nonzero generically.

Definition 3.2.3 Consider the matrices \( G \) (m×n) and \( H \) (n×q) having entries \( (g_{ij}) \) and \( (h_{ij}) \). The \( (i, j) \) entry in their product is given by
\[
\sum_{t=1}^{n} g_{it} h_{tj}
\]
The product of \( G \) and \( H \) is identically zero, denoted \( GH = 0 \), if
\[ g_{i,t} h_{t,j} = 0 \]

for every \( i, j, t \).

For example the product of

\[
\begin{bmatrix}
a_1 & 0 & a_2 \\
a_3 \\
a_4 \\
\end{bmatrix}
\begin{bmatrix}
a_2 \\
a_3 \\
a_4 \\
\end{bmatrix}
\]

is not identically zero even though the result vanishes while the product of

\[
\begin{bmatrix}
a_1 & a_2 & 0 \\
0 \\
0 \\
a_3 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

is identically zero.

Lemma 3.2.3

Assume \( \text{rank} \begin{bmatrix} A & B \end{bmatrix} = n \) (g) and \( \text{rank}(A) = q \) (g), \( n-m \leq q < n \).

Then there exists a permutation operation \( P \) such that

\[
P^T A P = \begin{bmatrix}
A_{11} & A_{12} & \ldots & A_{1m+1} \\
A_{21} & A_{22} & \ldots & A_{2m+1} \\
\ldots & \ldots & \ldots & \ldots \\
A_{n1} & A_{n2} & \ldots & A_{nm+1}
\end{bmatrix}
\]

\[
P^T B = \begin{bmatrix}
B_1 \\
B_2 \\
\ldots \\
B_{m+1}
\end{bmatrix}
\]

(3.2.1)

where \( A_{ii} \) is of order \((t_i x t_i)\), \( 0 \leq t_i < n \), and either

\( A_{ii} = 0 \) in which case \( t_i \leq n-q \) or
\[ A_{11} = \begin{bmatrix}
A_{ii} & 0 & \ldots & 0 \\
A_{ii} & A_{ii} & \ddots & \vdots \\
& & \ddots & 0 \\
A_{vi} & A_{vv} & & 
\end{bmatrix} \quad (3,2,2) \]

with \( \text{rank}(A_{11}) = t_1 \) (g), \( 1 \leq v < t_1 \), \( A_{ii} \) of order \((v, v')\), and \( A_{ii} \) not of Form I,

\( 1 \leq k < q-t_1+1 \) and finally \( B_{k,v} \) has no zero rows.

Also the pair

\[ \bar{A} = \begin{bmatrix}
\bar{A}_{11} & A_{12} & \ldots & A_{1,k+1} \\
0 & 0 & A_{23} & \ldots \cdots A_{2,k+1} \\
& \ddots & \ddots & \ddots \\
& & A_{k,v} & A_{k-1,v+1} \\
& & & 0 & A_{k,v+1} \\
0 & \ldots & \ldots & 0 
\end{bmatrix} \quad \bar{B} = P^T B \]

with

\[ \bar{A}_{11} = \begin{bmatrix}
A_{ii} & 0 & \ldots & 0 \\
0 & A_{ii} & \ddots & 0 \\
& & \ddots & \vdots \\
0 & 0 & A_{vv} & 
\end{bmatrix} \]

satisfies
\[
\text{rank}(\tilde{A}) = q \quad (g) \quad (3.2.4)
\]

and

\[
\text{rank}[\tilde{A} \; \tilde{B}] = n \quad (g). \quad (3.2.5)
\]

Proof:

Since \( \text{rank}(A) = q \quad (g) \) and \( \text{rank}[A \; B] = n \quad (g) \) there are \( n-q \) vectors \( x_i \) which satisfy

\[
x_i^T A = 0
\]

and for which no vector \( x \in \{x_1, \ldots, x_{n-q}\} \) satisfies

\[
x^T B \neq 0.
\]

Define the \((nxn-q)\) matrix \( X = (x_1, \ldots, x_{n-q}) \) and the index set \( \theta \in \{1, \ldots, n\} \) designating all nonzero rows of \( B \). Then the rows of \( X \) designated by \( \theta \) contain \( n-q \) nonzero independent entries. For if not there exists a \((n-qx1)\) vector \( y \) satisfying

\[
(Xy)^T B \neq 0
\]

Let \( \theta' \subseteq \theta \) designate any set of \( n-q \) rows of \( X \) containing \( n-q \) nonzero independent entries. Define the permutation operation \( P \) such that the last \( n-q \) rows of \( P^T X \) contain the \( n-q \) nonzero independent entries of the rows designated by \( \theta' \). Then
\[ P^T \mathbf{A} P = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad P^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \] (3.2.6)

where \( \text{rank}(A_1, A_2) = q \) (g), for if not, \( \text{rank}(A) \leq q \) for every data point in \( R^n \). Note that \( B_2 \) has no zeros since \( \Phi^T \subset \Phi \). Set \( A_3, A_4 = 0 \). Then the vectors \( e_j \) satisfy

\[
\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ e_j^T \end{bmatrix} = 0 \text{ and } \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \neq 0
\]

where the \( j^{th} \) entry is the only nonzero entry of \( e_j \), \( 1 \leq j \leq n-q \). Because of this, it is possible to choose values for the nonzero entries of \( A_1, A_2, B_1 \) and \( B_2 \) such that

\[
\text{rank} \begin{bmatrix} A_1 & A_2 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n
\]

and therefore

\[
\text{rank} \begin{bmatrix} A_1 & A_2 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n(q)
\]

The permutation operation required to yield the forms of (3.2.1) and (3.2.3) depend on the properties of \( A_i \).

Three cases are possible.

First if \( \text{rank}(A_1) = q \) (g) and \( A_i \) is not of Form I, (3.2.6) is of the form (3.2.1) with \( A_i = A_{ii} \) and \( k = v = 1 \). Also with
\[ \bar{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

\( \text{rank}(\bar{A}) = q(g) \) and \( \text{rank}(\bar{A} \bar{B}) = n(g) \) as shown above.

In the second case where \( \text{rank}(A_1) = q(g) \) and \( A_1 \) is of Form 1 there exists a permutation operation of the form

\[ P \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \]

such that

\[ P^T A_1 P = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} & 0 \\ & & \ddots \\ & & & A_{vv} \end{bmatrix} \]

where \( 1 \leq v \leq t \), and \( A_{ii} \) is not of Form 1. This follows from repeated application of the permutation operation of Definition 3.2.2 to each resulting block diagonal matrix until (3.2.2) is obtained. That is, since \( A_1 \) is of Form 1 there exists a permutation operation which puts \( A_1 \) in lower block triangular form. If either of the resulting block diagonal matrices is of Form 1 they can be reduced to lower block triangular form. Continue this process until all block diagonal matrices are not of Form 1. Thus (3.2.1) results with \( A_{ii} = P^T A_1 P \), \( k = 1 \) and \( 1 \leq v \leq t \). Also since \( \text{rank}(A_1) = q(g) \), no \( A_{ii} \) has any
fixed zero eigenvalue. This implies \( \text{rank}(A_{\alpha_i}) = v_i \ (g) \), 
\( 1 \leq i \leq v \). Thus with \( A_3 P = 0 \), \( A_4 = 0 \) and

\[
\tilde{A}_n = \begin{bmatrix}
A'_{1} & 0 \\
0 & \cdots \\
0 & A'_{v}
\end{bmatrix}
\]

\( \text{rank}(\tilde{A}) = q \ (g) \) and \( \text{rank} [\tilde{A} \ B] = n \ (g) \) follow as above where

\[
\tilde{A} = \begin{bmatrix}
P^T A_1 P & P^T A_2 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix}
P^T B_1 \\
B_2
\end{bmatrix}
\]

Consider finally the case where \( \text{rank}(A_i) = q' \ (g) \) where 
\( q' < q \). There exist permutation operations such that

\[
\begin{bmatrix}
P^T & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A_1 & A_2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
A'^{i}_1 & A'^{i}_2 & A'^{i}_n \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
P^T & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
= \begin{bmatrix}
B'^{i}_1 \\
B'^{i}_2
\end{bmatrix}
\]

with \( \text{rank}(A'^{i}_1, A'^{i}_2) = q' \ (g) \). Set \( A'^{i}_3, A'^{i}_4 = 0 \). Then

\[
\text{rank}
\begin{bmatrix}
A'^{i}_1 & A'^{i}_2 & A'^{i}_n \\
0 & 0 & A_{zz} \\
0 & 0 & 0
\end{bmatrix}
= q \ (g)
\]

and
\[
\begin{bmatrix}
A'_1 & A'_2 & A'_3 & B'_1 \\
0 & 0 & A'_{22} & B'_2 \\
0 & 0 & 0 & B'_3 \\
\end{bmatrix}
\]

\[
\text{rank}
\begin{bmatrix}
A'_1 & A'_2 & A'_3 & B'_1 \\
0 & 0 & A'_{22} & B'_2 \\
0 & 0 & 0 & B'_3 \\
\end{bmatrix}
= n \quad (g)
\]

by the same argument as before.

Continue this reduction by considering next the generic rank of \( A'_1 \). Eventually a matrix \( \hat{A} \) is considered where either \( \hat{A}=0 \) or \( \text{rank}(\hat{A})=t_1 \) (g) where \( \hat{A} \) is of order \( (t_1 \times t_1) \), \( 1 \leq t_1 < n \). If \( \hat{A}=0 \) the form (3.2.3) is obtained with \( \hat{A}_{tt}=0 \). From the form of the reduction \( \text{rank}(\hat{A})=c \) (g). Thus there exist \( q \) nonzero independent entries in \( \hat{A} \). Therefore \( t_1 \times \text{basis} \). Replacing the zero entries of \( \hat{A} \) with the original indeterminates yields (3.2.1) with \( A_{tt}=0 \).

If \( \hat{A} \neq 0 \), but not of Form 1, then (3.2.3) results with \( \hat{A}_{tt}=\hat{A} \), and \( v=1 \). Again (3.2.4) and (3.2.5) follow as above.

If \( \hat{A} \neq 0 \) and \( \hat{A} \) is of Form 1, the permutation operation of (3.2.7) yields (3.2.2). Since \( \text{rank}(\hat{A})=t_1 \) (g) the form \( \hat{A}_{tt} \) of (3.2.3) is obtained as before. Again (3.2.4) and (3.2.5) follow since \( \text{rank}(\hat{A}_{tt})=t_1 \) (g).

\[\Box\]

Lemma 3.2.4

Assume \( \text{rank} [ A \ B ] = n \) (g) and the system pair \( (A,B) \) is structurally uncontrollable. Then the matrix\([ A \ B ]\) is of Form 1.
Proof:
There are three cases to consider depending on the properties of A.

First assume rank(A) = n (g) and A is not of Form I. Then (A,b) is structurally controllable, where b is any column of B. This result is developed in the proof of Lemma 3.2.2. But this contradicts the assumption of structural uncontrollability of (A,B) and therefore cannot occur.

The second case is when rank(A) = n (g) and A is of Form I. Then there exist permutation operations P which put A in the same form as the matrix of equation (3.2.2). Setting all entries not on the block diagonal to zero yields

\[
\tilde{A} = \begin{bmatrix}
A_{ii} & 0 & \ldots & 0 \\
0 & A_{zz} & 0 \\
& \ddots & \ddots & \ddots \\
& & & A_{ww}
\end{bmatrix}
\]

where 1 \leq v \leq n, A_{ii} (v_i x v_i) is not of Form I. Now since rank(A) = n (g) it follows that rank(A_{ii}) = v_i (g). Let \( \tilde{B} = P^T B \). Since rank(A_{ii}) = v_i (g) and A_{ii} is not of Form I, it follows from Lemma 3.2.1 and Lemma 3.2.2 that parameter values can be chosen such that rank(\( \tilde{A}' \)) = n, the left eigenvectors of \( \tilde{A}' \) are unique up to
multiplicity and of the form

\[(0 \ldots 0 \ x_i \ 0 \ldots 0)^T \quad (3.2.8)\]

where \(x_i\) is a left eigenvector of \(A_i\) with no zero components and \(\tilde{A}'\) denotes the parameter point chosen.

Consider the polynomial in the nonzero entries of \(\tilde{B}\) defined by

\[\Psi = \prod_{i=1}^{j} \sum_{k=1}^{r} (x_i^T b_k)^2\]

where \(x_i\) is a left eigenvector of \(\tilde{A}'\), \(1 \leq j \leq n\) and \(b_k\) is the \(k^{th}\) column of \(\tilde{B}\). Assume no \(x_i\) satisfies \(x_i^T \tilde{B} = 0\). Then \(\Psi\) is a nonconstant polynomial in the nonzero entries of \(\tilde{B}\). Therefore the nonzero entries of \(\tilde{B}\) can be chosen, yielding \(\tilde{B}'\), such that \(\Psi \neq 0\), [Theorem 3, p.18, 35].

Thus \((\tilde{A}', \tilde{B}')\) is a controllable pair contradicting the assumption that \((A, B)\) is structurally uncontrollable. It follows that for some \(i \in \{1, \ldots, j\}\)

\[x_i^T \tilde{B} \neq 0.\]

Therefore from the form of \(x_i\), the permutation operation which exchanges \(A_i\) and \(A_{i'}\) in \(\tilde{A}\) yields a system in Form 1.

The third case to be considered is when \(\text{rank}(A) = q \cdot n\) with \(q < n\). Then by Lemma 3.2.3 there exists a permutation operation \(P\) such that for a suitable choice of certain parameter values,

\[P^T A P = \tilde{A} \quad \text{and} \quad P^T B = \tilde{B}\]
where $\bar{A}$ and $\bar{B}$ are defined in (3.2.3). If $A_{ii} = 0$, all the eigenvalues of $\bar{A}$ are zero. Then since

$$\text{rank}[\bar{A} \; \bar{B}] = n \quad (g)$$

it is possible to choose parameter values $\bar{A}'$ and $\bar{B}'$ such that $(\bar{A}', \bar{B}')$ is a controllable system. This contradicts the assumption that $(A, B)$ is structurally uncontrollable. Therefore assume $A_{ii} \neq 0$. Since

$$\text{rank}(A_{ii}) = t_i \quad (g)$$

and $A_{ii}$ has the form (3.2.3) it follows from Lemma 3.2.1 and Lemma 3.2.2 that values of the nonzero entries of $A_{ii}$ can be chosen such that

$$\text{rank}(A'_{ii}) = t_i$$

the left eigenvectors of $A'_{ii}$ are unique up to multiplicity and of the form (3.2.8), where $A'_{ii}$ denotes the parameter point chosen. Fix the remaining entries of $A'$, yielding $A'$, satisfying $\text{rank}(A') = q$. Then for any $s \in \mathcal{D}(A'_{ii})$,

$$\text{rank}(A'-sI) = n-1.$$ 

Therefore the left eigenvectors of $A'$ associated with eigenvalues of $A'_{ii}$ are unique up to multiplicity. Consider the polynomial in the nonzero entries of $\bar{B}$ defined by

$$\Psi = \prod_{i=1}^{j} \sum_{\tau=1}^{r} (x_i^T b_{\tau})^2$$

where $x_i$ is a left eigenvector of $A'$, $j$ is the number
of eigenvectors of $\bar{A}'$ and $b_n$ is the $k^{th}$ column of $\bar{B}$. Assume no $x_i$, $1 \leq i \leq j$ satisfies

$$x_i^T \bar{B} \equiv 0.$$ 

Then since the $x_i$ associated with $s \in \gamma'(\bar{A}'_{n1})$ are unique up to multiplicity and $B_{n1}$ has no zero rows, $\psi$ is a nonconstant polynomial in the nonzero entries of $\bar{B}$. Therefore the nonzero entries of $\bar{B}$ can be chosen, yielding $\bar{B}'$, such that $\psi \neq 0$. This implies the pair $(\bar{A}', \bar{B}')$ is a controllable system. Since this contradicts the assumption that $(A, B)$ is structurally uncontrollable it follows that some $x=x_i, 1 \leq i \leq j$ must satisfy

$$x^T \bar{B} \equiv 0.$$ 

Note since $B_{n1}$ has no zero rows the last $n-q$ components of $x$ must be zero. The following $(q \times q)$ submatrix of $(\bar{A}' - s1)$, see (3.2.3), satisfies

$$\begin{bmatrix}
\bar{A}'_{n1} - s1 & A_{12} & \ldots & A_{1k} \\
0 & -s1 & A_{23} \\
& \ddots & \ddots & A_{k+n-1,k} \\
& & 0 & -s1 \\
& & & -s1 \\
\end{bmatrix}
\text{rank} = q-1 \quad (3.2.9)$$

for $s \in \gamma'(\bar{A}'_{n1})$. Therefore $x$ is independent of the entries in
\[
\begin{bmatrix}
A_{1,k,n} \\
\vdots \\
\vdots \\
A_{k,k,n}
\end{bmatrix}
\] 

(3.2.10)

in the following sense. Fix \( \bar{A} \) as above except for the entries of (3.2.10). Choose these entries, yielding a \( \bar{A}'' \), such that \( \text{rank}(\bar{A}'') = q \) and \( \bar{A}' \neq \bar{A}'' \). From the above analysis there exists a left eigenvector of \( \bar{A}'' \) satisfying

\[
\bar{x}^T \bar{A}'' \equiv 0.
\]

But since the last \( n-q \) components of \( \bar{x} \) must be zero and since (3.2.9) is true it follows that

\[
\bar{x} = x.
\] 

(3.2.11)

Therefore \( x \) is an eigenvector for any choice of (3.2.10) subject to the resulting matrix having rank \( q \). Consider (3.2.10) as a submatrix of zeros and indeterminates in the matrix \( \bar{A}' \). Define the polynomial \( \Psi \) in the nonzero entries of (3.2.10) as

\[
\Psi = \left[ \sum_{i=1}^{n-q} (x^T a_i)^2 \right] \cdot \delta
\]

where \( a_i \) is the \( i^{th} \) column of (3.2.10) and \( \delta \) is the sum of the squares of all \( q \times q \) order minors of \( \bar{A}' \). Assume
there exist nonzero rows of (3.2.10) corresponding to nonzero components of $x$. This means that

$$\sum_{i=1}^{n-q} (x^T a_i)^2$$

is a nonconstant polynomial in the nonzero entries of (3.2.10). Also since $\text{rank}(\bar{A}^l) = q$, $\delta$ is a nonconstant polynomial in these entries and it follows therefore that $\psi$ is too. So numerical values of the nonzero entries of (3.2.10) can be chosen, yielding a $\bar{A}''$, such that $\psi \neq 0$. Now $\psi \neq 0$ implies $\delta \neq 0$ and therefore $\text{rank}(\bar{A}''') = q$. From the above analysis this implies $x$, of (3.2.11), is a left eigenvector of $\bar{A}''$ associated with $s \in \mathcal{O}(\bar{A}''')$. But $\psi \neq 0$ also implies

$$x^T (\bar{A}'' - sI) \neq 0$$

which means $x$ is not a left eigenvector of $\bar{A}''$. This contradiction implies the rows of (3.2.10) corresponding to nonzero entries of $x$ must be composed entirely of fixed zero entries. Then since (3.2.10) is chosen such that $\delta \neq 0$,

$$\text{rank}(A_{\kappa, \kappa+1}) = t_{\kappa}$$

and therefore the last $t_{\kappa} + (n-q)$ components of $x$ must be zero. But this implies $x$ is independent of the nonzero entries of
\[(A_{1k} \ldots A_{k-1k})^T\]

since

\[
\begin{bmatrix}
\tilde{A}'_{11} - sl & A_{12} & \ldots & A_{1k-1} \\
0 & -sl & \ddots & \\
\vdots & & \ddots & -sl \\
& & & -sl & A_{k-2k-1} \\
& & & & -sl \\
\end{bmatrix} = a_k - t_k - 1
\]

Continuing the above analysis, \(x\) is eventually obtained as a vector of the form \((x' \ 0 \ \ldots \ 0)^T\) where \(x'\) has the structure of (3.2.8) and every row of the matrix

\[
(A_{12} \ A_{13} \ \ldots \ A_{k-1k})
\]

corresponding to nonzero entries of \(x'\) is composed entirely of fixed zero entries. Now because of the form of \(x', (3.2.8), \) and \(\tilde{A}'_{11}, (3.2.3),\) the permutation operation which exchanges \(A'_{12}\) and \(A'_{11}\) of (3.2.3) yields a system of Form 1. Therefore \([A \ B]\) must be of Form 1.

\[\Delta\]

Proof of Theorem 3.2.1:

(Sufficiency)

If the pair \((A, B)\) has Form 1 it is obvious that every data point \(p \in R^{N+M}\) is uncontrollable in the usual sense. If \((A, B)\) is of Form (n) then by Theorem 2.4.1
\[ \text{rank } [A \ B] < n \]

for every \( p \in \mathbb{R}^{N+M} \). Therefore \((A, B)\) is uncontrollable for every data point.

(Necessity)

Consider the case where \( \text{rank } [A \ B] < n \) for every \( p \in \mathbb{R}^{N+M} \). Then by Theorem 2.4.1 the matrix \([A \ B]\) must have Form (n) and the theorem is valid for this case.

Therefore assume \( \text{rank } [A \ B] \neq n \) for some \( p \in \mathbb{R}^{N+M} \). Then \( \text{rank } [A \ B] = n \) (g) and from Lemma 3.2.4 \([A \ B]\) must be of Form I.

\[ \triangle \]

Theorem 3.2.1 represents the complete solution to the structural controllability (observability) problem. It only remains to consider the computational methods of checking the conditions of the theorem for a given system structure.
3.3 Computational Considerations

A computer algorithm to determine structural controllability using the results of Theorem 3.2.1 must be capable of recognizing Form I and Form (n). However, by reformulating the problem it is possible to determine structural controllability using only FZRF.

Definition 3.3.1 The extended controllability matrix of a system pair is the \( n^2 \times (n+r-1) \) matrix \( \bar{R} \) defined by

\[
\begin{bmatrix}
B & I & 0 & \cdots & \cdots \\
0-A & B & I & 0 & \cdots \\
0 & 0 & 0-A & B & I & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
0 & \cdots & 0-A & B & I & 0 \\
0 & \cdots & 0 & \cdots & 0-A & B
\end{bmatrix}
\tag{3.3.1}
\]

Lemma 3.3.1

Consider \( p \in \mathbb{R}^{N+M} \). For this data point the associated system pair \((A,B)\) is controllable if and only if

\[ \text{rank} (\bar{R}) = n^2. \]

Proof:

see Rosenbrock [33, pp 72-73].

\[ \uparrow \]
Theorem 3.3.1

The following conditions on the system pair \((A, B)\) are equivalent:

1) \((A, B)\) is structurally uncontrollable.

2) The matrix \([A \ B]\) is of Form 1 or Form \((n)\).

3) The matrix \(\bar{R}\) is of Form \((n^2)\).

Proof:

The equivalence of Statements 1) and 2) has already been proved.

Assume Statement 2) is true. If the matrix \([A \ B]\) has Form 1 then there exists a permutation operation such that this matrix has the form

\[
\begin{bmatrix}
A_1 & 0 & 0 \\
A_2 & A_3 & B_1
\end{bmatrix}
\]  \( (3.3.2) \)

where \(A_1\) is of order \((m \times m)\), \(1 \leq m \leq n\). \((3.3.2)\) implies \(\bar{R}\) contains \(nm\) rows and \(n(n+r-1)-n(n-1)\) columns defining a zero submatrix. However \(\bar{R}\) requires only \(n(n-1)+1\) such rows to be of Form \((n^2)\). Since \(v > 1\), \(nw > n(n-1)+1\) and it follows that \(\bar{R}\) is of Form \((n^2)\).

If \([A \ B]\) is of Form \((n)\) then the last \(n\) rows of \(\bar{R}\) are dependent for every \(p \in \mathbb{R}^{N+M}\). Therefore \(\text{rank}(\bar{R}) < n^2\) for every \(p \in \mathbb{R}^{N+M}\) which implies from Theorem 2.4.1 that \(\bar{R}\) is of Form \((n^2)\).

Assume Statement 3) is true. Then \(\text{rank}(\bar{R}) < n^2\) for every \(p \in \mathbb{R}^{N+M}\) from Theorem 2.4.1. Thus from Lemma 3.3.1
the pair \((A, B)\) is uncontrollable for every \(p \in P^M\).

Theorem 3.3.1 allows for the use of \(FZPF\) of Appendix I on the extended controllability matrix to determine the structural controllability of a system pair.
IV. STRUCTURAL SOLVABILITY OF THE SERVOMECHANISM PROBLEM

4.1 Introduction

The analysis of the servomechanism problem has yielded various rank conditions for solvability depending on the version of the problem considered,[3],[5],[11],[12]. The sequence of analytical development has been toward the inclusion of more general forms of parameter uncertainty into the problem formulation. It is the purpose of this chapter to continue this trend toward a more realistic formulation by including structural constraints. The problem is analyzed using the generic formulation of Chapter II.

Consider the linear time invariant model;

\[
\begin{align*}
\dot{x}(t) &= A \ x(t) + B \ u(t) \\
y(t) &= C \ x(t) \\
z(t) &= D \ x(t) \\
x(0) &= x_0
\end{align*}
\] (4.1.1)

Here \( y(\cdot) \in \mathcal{Y} \) is the \( h \) dimensional vector of measurable outputs, \( u(\cdot) \in \mathcal{U} \) is the \( m \) dimensional vector of inputs, \( z(\cdot) \in \mathcal{Z} \) is the \( p \) dimensional output vector to be regulated and \( x(\cdot) \in \mathcal{X} \) is the \( n \) dimensional state vector of the system. Decompose the state space \( \mathcal{X} \) of (4.1.1) as

\[
\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2
\]

where \( \mathcal{X}_1 \) is the state space of the "plant", and \( \mathcal{X}_2 \) is the
state space of the exogenous signals. Let \( d(X) = n \), and \( d(X) = n_2 \). For any \( x = x_1 \oplus x_2 \) define the maps

\[
Ax = (A, x_1 + A_2 x_2) \oplus A_1 x_2 \\
Bu = B u \\
Dx = D_1 x_1 + D_2 x_2 \\
Cx = C, x_1 + C_2 x_2 \\
(A+B) x = (A, +B, F_1) x + (A_3 + B, F_2) x_2 \oplus A_2 x_2
\]

These maps have the matrix representations

\[
A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\
C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}
\]

(4.1.3)

Associated with the system (4.1.1) is the unobservable space \( \eta \) defined as

\[
\eta = \bigcap_{i=1}^{n} \ker CA_{\infty}^i
\]

The original formulation and solution of the regulator problem with internal stability (RPIS) appeared in the comprehensive paper of Wonham and Pearson[3]. RPIS may be stated as determining under what conditions there exists a feedback matrix \( F: X \to U \) such that for every initial state \( x_0 \),

i) \( \lim_{t \to \infty} z(t) = 0 \)

ii) the controllable observable modes of the triple \( (C, A, B) \)
are stabilized.

In terms of the maps of (4.1.1), RPIS is stated as follows.

RPIS

Given the maps \( A: \mathcal{X} \to \mathcal{Y}, B: \mathcal{U} \to \mathcal{X} \), \( D: \mathcal{X} \to \mathcal{Z} \) and a subspace \( \mathcal{Y} \) with \( A \mathcal{Y} \subset \mathcal{Y} \), find \( F: \mathcal{X} \to \mathcal{U} \) such that

\[
\ker F = \mathcal{Y} \quad (4.1.4)
\]

\[
\mathcal{Y}^\times (A+BF) \cap (\langle A|B \rangle + \mathcal{Y}) \subset \mathcal{Y} \quad (4.1.5)
\]

\[
\mathcal{Y}^\times (A+BF) \subset \ker D \quad (4.1.6)
\]

Equation (4.1.4) is the observability constraint, (4.1.5) the internal stability constraint and (4.1.6) the regulation constraint.

The next stage in the development of the servomechanism problem involved the inclusion of robustness into the basic formulation. Although there are several distinct investigations along these lines, Davison [9], Francis, Sebakhy and Wonham [10], Francis and Wonham [43], [44] and Staats and Pearson [12]; it is the techniques used in [12] which are expolited in this thesis.

The result of interest in [12] is the formulation and solution of the robust state feedback problem (RSFP). Without loss of generality assume the map \( B \) is monic and the map \( D \) is epic. In addition it is assumed that
\[ \mathcal{X}(A) \cap \mathcal{X} \subseteq \langle A\vert B \rangle \quad (4.1.7) \]
\[ \mathcal{X}(A_2) \subseteq \mathcal{C}^+ \quad (4.1.8) \]
\[ \eta \cap \mathcal{X}(A) = 0 \quad (4.1.9) \]

Equation (4.1.7) states that the plant of (4.1.1) is stabilizable. Equation (4.1.8) is a natural assumption since no control is required to achieve regulation for modes associated with stable eigenvalues of \( A_2 \). System detectability, (4.1.9) involves no loss of generality since by Theorem 4 of [3] if \( \eta \cap \mathcal{X}(A) \neq 0 \), the problem is reformulated in the factor space \( \mathcal{X}/\eta \cap \mathcal{X}(A) \). For the purposes of this thesis however, (4.1.7) and (4.1.9) are replaced with the assumptions of plant controllability and observability:

\[ \mathcal{X}_i = \langle A\vert B \rangle \quad (4.1.10) \]
\[ \eta \cap \mathcal{X}_i = 0 \quad (4.1.11) \]

This is necessary since by definition a structural property is independent of parameter values. However since stabilizability and detectability depend on eigenvalue locations, these properties are not amenable to structural analysis. In the following discussion the relevant results of [12] are restated in terms of assumptions (4.1.10) and (4.1.11). For this purpose let \((\bar{A}, \bar{B}, \bar{D})\) represent maps induced in the factor space \( \bar{\mathcal{X}} = \mathcal{X}/\eta \cap \mathcal{X}(A) \), i.e., \( PA = \bar{A}P \), \( PB = \bar{B} \), \( \bar{D}P = \bar{D} \) where \( P: \mathcal{X} \rightarrow \bar{\mathcal{X}} \) is the canonical projection.
Let $F: \mathcal{X} \to \mathcal{U}$ be subject to perturbations of the form

$$\tilde{F} = F + \varepsilon \frac{\hat{F}}{\|\hat{F}\|}$$

(4.1.12)

where $\varepsilon$ is a scalar and $\hat{F} \neq 0$ is arbitrary.

**Definition 4.1.1** Given $\varepsilon_0 > 0$ and $F$ such that $\ker F \supset \gamma$ then $\tilde{F}$ is an admissible perturbation if $\ker \hat{F} \supset \gamma$ and $0 < \varepsilon < \varepsilon_0$.

Then RSFP is stated as:

**RSFP**

Given the maps $A: \mathcal{X} \to \mathcal{X}$, $B: \mathcal{U} \to \mathcal{X}$, $D: \mathcal{X} \to \mathcal{Z}$, and a subspace $\gamma$ such that $A\gamma \subset \gamma$. Under what conditions do there exist $F: \mathcal{X} \to \mathcal{U}$ and $\varepsilon_0 > 0$ such that with $\ker F \supset \gamma$

$$\mathcal{X}^*(A + BF) \subset \ker D$$

(4.1.13)

$$\mathcal{X}^*(A + BF) \cap (\langle A | B \rangle + \gamma) \subset \gamma$$

(4.1.14)

for every admissible $\tilde{F}$.

Clearly RPIS must be solvable in order for a solution to exist. A necessary condition is therefore [3],

$$\gamma \cap \mathcal{X}^*(A) \subset \ker D$$

(4.1.15)

Assuming (4.1.15) is true, the solvability of RSFP can be divided into two cases, [11]. First it is possible that in the factor space $\tilde{X}$.

* The matrix norm is defined as $\| \hat{F} \| = m \cdot \max |f_{ij}|$ where $\hat{F}$ is of order $(m \times n)$ with entries $f_{ij}$.
\[ \bar{\mathcal{Y}}^*(\bar{A}) \subset \langle \bar{A} | \bar{B} \rangle. \]  
(4.1.16)

In this case any \( F: \bar{X} \to \mathcal{U} \) satisfying \( F^\theta = 0, \sigma'(\bar{A} + \bar{BF}) \subset \mathcal{C} \) is a solution to RPIS.

**Theorem 4.1.1[11]**

Assume (4.1.15) and (4.1.16) are true. Any solution to RPIS is a solution to RSFP.

The second case

\[ \bar{\mathcal{Y}}^*(\bar{A}) \neq \langle \bar{A} | \bar{B} \rangle \]  
(4.1.17)

is the most likely to occur. The solution to RSFP in terms of the induced maps in the factor space \( \bar{X} \) is given in the following theorem.

**Theorem 4.1.2[12]**

Assume (4.1.8), (4.1.10), (4.1.11), (4.1.15) and (4.1.17) are true for (4.1.1) with the structure of (4.1.3). RSFP is solvable if and only if

i) the reduced RPIS is solvable, i.e., there exists \( F: \bar{X} \to \mathcal{U} \) such that

\[ \bar{\mathcal{Y}}^*(\bar{A} + \bar{BF}) \subset \ker \bar{D} \]

\[ \bar{\mathcal{Y}}^*(\bar{A} + \bar{BF}) \cap \langle \bar{A} | \bar{B} \rangle = 0 \]

ii) for any solution \( F: \bar{X} \to \mathcal{U} \) to the reduced RPIS,

\[ \bar{D} \subset \bigcap_{\epsilon = 1}^{K_\epsilon} (\bar{A} + \bar{BF} - s_\epsilon i)^\epsilon \ker \bar{D} \]  
(4.1.18)

for each \( s_\epsilon \in \sigma'(\bar{A}_\epsilon) \), where \( k_\epsilon \) is the multiplicity of
s_i as a root of the minimal polynomial of \( \tilde{A}_2 \).

The formulation of RSFP and the results of Theorem 4.1.2 are used to introduce structural constraints into the analysis of the servomechanism problem. In Section 4.2 an equivalent representation of the conditions of Theorem 4.1.2, especially suited to structural analysis, is developed. In particular new necessary and sufficient matrix rank conditions for solvability of RPIS are derived and extended to include RSFP.

A restricted class of exogenous signals considered in the remainder of the chapter is defined in Section 4.3. This restricted class is composed of multiple step and ramp functions. The implications of generic solvability of RSFP are then considered. The concept of well-posed RPIS and RSFP is introduced and shown to guarantee the robust properties desired. Any resulting state variable feedback solution is realized using output feedback compensation. The means of dealing with this compensation is then considered.

In Section 4.4 the computational aspects involved in the structural analysis of RSFP are considered. A detailed investigation is made into the practical methods necessary to check the genericity of the basic problem assumptions and the well-posedness of a given system structure.

Section 4.5 deals with the situation where RPIS is well-posed but RSFP is not. An algorithm is developed to
determine if a given system can be extended in such a way as to obtain a well-posed RSFP solution for the extended system.

The last section of this chapter contains examples demonstrating many of the structural effects considered above.
4.2 Solvability of RSFP

Consider the system (4.1.1), (4.1.2) and (4.1.3). Assume the map \( B_1 \) is monic and the map \( D_1 \) is epic. To facilitate the development, the form of the matrix representations of \( A_2, A_3, D_2 \) and \( C_2 \) are examined in more detail. Assume \( A_2 \) is in Jordan form:

\[
A_2 \sim \begin{bmatrix}
J_{i_1} & & \\
& J_{i_2} & \\
& & \ddots \\
& & & J_{i_r}
\end{bmatrix} \quad (4.2.1)
\]

where \( J_{i_j} \) is a \( k_{i_j} \times k_{i_j} \) Jordan matrix for \( s_{i_j} \in d(A_2) \), and the blocks have been arranged so that \( k_{i_1} \geq k_{i_2} \geq \cdots \geq k_{i_r} \) for \( i \in \{1, \ldots, r\} \). Partitioning the matrices \( A_3, D_2 \) and \( C_2 \) consistently with (4.2.1) yields

\[
A_3 = \begin{bmatrix}
A_{i_1} & A_{i_2} & \cdots & A_{i_r}
\end{bmatrix}
\]

\[
C_2 = \begin{bmatrix}
C_{i_1} & C_{i_2} & \cdots & C_{i_r}
\end{bmatrix} \quad (4.2.2)
\]

and

\[
D_2 = \begin{bmatrix}
D_{i_1} & D_{i_2} & \cdots & D_{i_r}
\end{bmatrix}
\]

where \( A_{i_j}, D_{i_j} \) and \( C_{i_j} \) are \( (n_i, xk_{i_j}), (pxk_{i_j}) \) and \( (hxk_{i_j}) \) matrices respectively. It is assumed that the partitioned blocks of \( A_3, D_2 \) and \( C_2 \) have the structure

\[
A_{i_j} = \begin{bmatrix}
a_{i_j} & 0 & \cdots & 0
\end{bmatrix}
\]
\[ C_{ij} = \begin{bmatrix} c_{ij} & 0 & \ldots & 0 \end{bmatrix} \] (4.2.3)

and

\[ D_{ij} = \begin{bmatrix} d_{ij} & 0 & \ldots & 0 \end{bmatrix} \]

As pointed out in [12], these matrix structural assumptions on \( A_2, A_3, C_2 \) and \( D_2 \) are consistent with the formulation of the servomechanism problem. That is, due to the form of (4.1.1) and (4.1.2), \( A_2 \) can be placed in Jordan form by inspection and the structure of (4.2.3) results naturally. The results of this section are presented under the assumption that \( A_2 \) has only real roots of its minimal polynomial. This is done to simplify the presentation. The modifications necessary for complex roots are exactly the same as those used in [40; Appendix A]. That is, the only change required in the proofs to follow is the use of complexified spaces and maps [45], as presented in [40]. The matrix representation of the system is therefore:

\[
A = \begin{bmatrix}
A_i & A_ii \\
0 & J_{ii0} & \ddots & 0 \\
& \ddots & \ddots & \ddots \\
& & 0 & \ddots & 0 \\
& & & \ddots & J_{ii0} \\
& & & & \ddots & \ddots \\
& & & & & \ddots & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
B_i \\
0 \\
\ddots \\
0 \\
\ddots \\
0 \\
B_i \\
0
\end{bmatrix}
\]

(4.2.4)
\[ D = \begin{bmatrix} D_1 & D_{12} & \ldots & D_{1r} \\ D_{21} & D_2 & \ldots & D_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ D_{r1} & D_{r2} & \ldots & D_r \end{bmatrix} \]

\[ C = \begin{bmatrix} C_1 & C_{12} & \ldots & C_{1r} \\ C_{21} & C_2 & \ldots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \ldots & C_r \end{bmatrix} \]

where \( A_3 \), \( C_2 \), and \( D_2 \) have the structure of (4.2.3) for each \( i, j \) and \( J_{ij} \) is a \( k_{ij} \times k_{ij} \) Jordan matrix for \( s_i \in \sigma(A_2) \).

Since the plant of (4.2.4) is assumed controllable and observable, the plant matrices induced in the factor space \( \widetilde{X} \) are unaltered. The system maps induced in \( \widetilde{X} \) are of the general form

\[
\begin{bmatrix}
A_i & A_n & A_F & A_{F\tilde{F}} \\
0 & J_0 & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \ddots & \ddots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{B}_i \\
\end{bmatrix}
\]

(4.2.5)

\[
\tilde{D} = \begin{bmatrix} D_1 & D_{12} & \ldots & D_{1r} \\ D_{21} & D_2 & \ldots & D_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ D_{r1} & D_{r2} & \ldots & D_r \end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix} C_1 & C_{12} & \ldots & C_{1r} \\ C_{21} & C_2 & \ldots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \ldots & C_r \end{bmatrix}
\]

where

\[
\tilde{A}_{ij} = \begin{bmatrix} \tilde{a}_{ij} & 0 & \ldots & 0 \\ \end{bmatrix}
\]

\[
\tilde{B}_{ij} = \begin{bmatrix} \tilde{b}_{ij} & 0 & \ldots & 0 \\ \end{bmatrix}
\]

\[
\tilde{C}_{ij} = \begin{bmatrix} \tilde{c}_{ij} & 0 & \ldots & 0 \\ \end{bmatrix}
\]

for each \( i, j \), \( \tilde{J}_{ij} \) is a \( (k_{ij}, k_{ij}) \) Jordan matrix for \( s_i \in \sigma(A_2) \).
and \( \bar{r} < r \), \( \bar{q}_i < q_i \). The following matrix rank conditions given in terms of the maps of (4.2.5) are obtained for solvability of RPIS.

**Theorem 4.2.1**

Assume (4.1.8), (4.1.10), (4.1.11) and (4.1.15) are true. RPIS is solvable if and only if for all \( i \in \{1, \ldots, \bar{r}\} \) and \( j \in \{1, \ldots, \bar{q}_i\} \)

\[
\begin{bmatrix}
B_i & A_i & -s_i & 1 \\
D_i & & & \\
-1 & B_i & A_i & -s_i & 1 \\
& \ddots & \ddots & \ddots & \ddots \\
& & & D_i & -1 & a_{ij} & -s_i & 1 \\
& & & & D_i & -1 & a_{ij} & 0 \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots & \ddots \\
& & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & B_i & A_i & -s_i & 1 \\
& & & & & & & & D_i \\
\end{bmatrix}
\]

(4.2.6)
where the matrices of (4.2.6) are of order \( \mathbf{F}_{ij} (n, m) \) and \( \mathbf{k}_{ij} (n, m) \), respectively.

Proof:

(Necessity)

Assume PPIS is solvable. This implies from Theorem 4 of [3] and (4.1.4), (4.1.5) and (4.1.6) that there exists a map \( F: x \rightarrow U \) satisfying

\[
\mathcal{X}^+ (\mathbf{A} + \mathbf{B} \mathbf{F}) \cap \langle \mathbf{A} \mathbf{F} \rangle = 0
\]

\[
\mathcal{X}^+ (\mathbf{A} + \mathbf{B} \mathbf{F}) \subset \ker \mathbf{F}
\]

Partition \( \mathbf{F} \) consistently with (4.2.5) as:

\[
\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 \\ \mathbf{F}_3 & \mathbf{F}_4 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} & \mathbf{F}_{13} & \cdots & \mathbf{F}_{1n} \\ \mathbf{F}_{21} & \mathbf{F}_{22} & \mathbf{F}_{23} & \cdots & \mathbf{F}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{m1} & \mathbf{F}_{m2} & \mathbf{F}_{m3} & \cdots & \mathbf{F}_{mn} \\ \mathbf{F}_{n1} & \mathbf{F}_{n2} & \mathbf{F}_{n3} & \cdots & \mathbf{F}_{nn} \end{bmatrix}
\]

where \( \mathbf{F}_{ij} \) is a \((m \times k_{ij})\) matrix. Since \( \mathbf{F} \) solves PPIS

\( \sigma (\mathbf{A}, + \mathbf{B}, \mathbf{F}) \subset \mathcal{C}^- \). Thus there exist \((n, m \times k_{ij})\) matrices \( \mathbf{V}_{ij} \)

such that

\[
\mathcal{X}^+ (\mathbf{A} + \mathbf{B} \mathbf{F}) = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} & \cdots & \mathbf{V}_{1n} \\ \mathbf{V}_{21} & \mathbf{V}_{22} & \cdots & \mathbf{V}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{V}_{n1} & \mathbf{V}_{n2} & \cdots & \mathbf{V}_{nn} \end{bmatrix}
\]

Now \( \mathcal{X}^+ (\mathbf{A} + \mathbf{B} \mathbf{F}) \) is an \( \mathbf{A} + \mathbf{B} \mathbf{F} \) invariant subspace and \( \mathcal{X}^+ (\mathbf{A} + \mathbf{B} \mathbf{F}) \)
\[ \text{ker } \overline{D}. \text{ So for each } i,j \text{ such that } i \in \{1, \ldots, r\} \text{ and } j \in \{1, \ldots, q_i \} \text{ the following equations are true:} \]

\[ (A_i + B_i \overline{F}_i - s_i \mathbb{1}) v_{ij} + (A_i + B_i \overline{F}_i - s_i \mathbb{1}) = V_{ij} \overline{J}_{ij} \]

\[ D_i v_{ij} + \overline{D}_{ij} = 0 \]  \( (4.2.7) \)

Write the matrices \( V_{ij} \) and \( \overline{F}_{ij} \) as

\[ V_{ij} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{q_i} \end{bmatrix} \text{ and } \overline{F}_{ij} = \begin{bmatrix} f_1 & f_2 & \cdots & f_{q_i} \end{bmatrix} \]

Then by \( (4.2.7) \)

\[ (A_i + B_i \overline{F}_i - s_i \mathbb{1}) v_1 + B_i f_1 + \overline{a}_{ij} = 0 \]

\[ D_i v_1 + \overline{d}_{ij} = 0 \]

\[ (A_i + B_i \overline{F}_i - s_i \mathbb{1}) v_2 + B_i f_2 - v_1 = 0 \]

\[ D_i v_2 = 0 \]  \( (4.2.8) \)

\[ \cdots \]

\[ (A_i + B_i \overline{F}_i - s_i \mathbb{1}) v_{q_i} + B_i f_{q_i} - v_{q_i-1} = 0 \]

\[ D_i v_{q_i} = 0 \]

The solvability of these equations imply with

\[ A_{F_i} = A_i + B_i \overline{F}_i; \]

\[ \begin{bmatrix}
B_i & A_{F_i} - s_i \mathbb{1} \\
D_i & 0 \\
-1 & B_i \\
\vdots & \ddots \\
B_i & A_{F_i} - s_i \mathbb{1} \\
D_i & 0
\end{bmatrix} \]

\[ \text{rank } \begin{bmatrix}
B_i & A_{F_i} - s_i \mathbb{1} \\
D_i & 0 \\
-1 & B_i \\
\vdots & \ddots \\
B_i & A_{F_i} - s_i \mathbb{1} \\
D_i & 0
\end{bmatrix} \]  \( (4.2.9) \)
\[
\begin{bmatrix}
B_i & A_{F_i} & -s_i & 1 \\
& D_i & \bar{a}_{ij} \\
& -1 & B_i \\
& & \bar{d}_{ij} \\
& & & \ddots
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
& & & & B_i & A_{F_i} & -s_i & 1 \\
& & & & & D_i \end{bmatrix}
\]

The relations of (4.2.6) follow immediately for all \(i \in \{1, \ldots, \bar{r}\}\), and \(j \in \{1, \ldots, \bar{n}_i\}\).

(Sufficiency)

Assume equations (4.2.6) are true. Then by (4.1.10) there exists an \(\bar{F}_i\) such that \(d(A_i + B_i \bar{F}_i) \in \mathbb{C}^{\bar{r}}\). Consider any \(i \in \{1, \ldots, \bar{r}\}\) and \(j \in \{1, \ldots, \bar{n}_i\}\). Using this \(\bar{F}_i\) in (4.2.6), equation (4.2.9) results. This implies there exists a vector \([f_{i_1}, v, \ldots, f_{\bar{x}_{ij}}, \bar{v}_{\bar{x}_{ij}}, f_{\bar{x}_{ij}}]^T\) such that (4.2.8) is true. Define \(V_{ij} = [v, v_2, \ldots, v_{\bar{x}_{ij}}]\) and \(\bar{F}_{ij} = [f_{i_1}, f_2, \ldots, f_{\bar{x}_{ij}}]\). Then with these definitions

\[
(A_i + B_i \bar{F}_i) V_{ij} + (\bar{A}_{i,j} + B_i \bar{F}_{ij}) = V_{ij} \bar{J}_{ij}
\]

\[
D_i V_{ij} + \bar{D}_{ij} = 0
\]

(4.2.10)

Repeating this development for each \(i, j\) defines matrices \(V_{ij}, V_{ij}, \ldots, V_{\bar{x}_{ij}}\), and \(\bar{F}_i, \bar{F}_{ij}, \ldots, \bar{F}_{\bar{x}_{ij}}\). Define \(\bar{U}\) as
\[
\tilde{V} = \begin{pmatrix}
V_{11} & V_{12} \\
1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & \ddots & \ddots \\
0 & 0 & 1
\end{pmatrix}
\]

and \( \bar{F} = [\bar{F}_1, \bar{F}_2] = [\bar{F}_1, \bar{F}_{12}, \ldots, \bar{F}_{12^n}]. \) Now from (4.2.10), \( \tilde{V} \) is \((\bar{A} + \bar{B} \bar{F})\) invariant and \( \tilde{V} \subset \text{ker} \bar{D}. \) Also by construction \( \tilde{V} \subset \bar{X}^+ (\bar{A} + \bar{B} \bar{F}) \) and \( d(\tilde{V}) = d(\bar{X}^+(\bar{A} + \bar{B} \bar{F})). \) Therefore \( \tilde{V} = \bar{X}^+(\bar{A} + \bar{B} \bar{F}) \) and so \( \bar{X}^+(\bar{A} + \bar{B} \bar{F}) \cap \bar{A} \bar{F} = 0. \) Then by assumption (4.1.15) and Theorem 4 of [3], RPIIS is solvable. △

This result is extended to include matrix rank conditions for the solvability of PSFP in the following theorem.

**Theorem 4.2.2**

Subject to the assumptions of Theorem 4.2.1, PSFP is solvable if and only if for every \( i \in \{1, \ldots, \bar{r}\} \) and \( j \in \{1, \ldots, \bar{r}_i\} \), (4.2.6) is satisfied and is of rank \( \bar{k}_{ij} n_i \).

**Proof:**

In the following define \( A_{\bar{r}} = (A_{\bar{r}} + B_{\bar{r}} \bar{F}_{\bar{r}}). \) Consider the \( t(n_i + p) \times t(n_i + m), 1 \leq t \leq \bar{k}_{ij}, \) matrix
Using elementary column operations on (4.2.11) the following matrix of equal rank is obtained:
The rank of (4.2.12) is equal to
\[
\begin{bmatrix}
B_1, [A_{F_1} - s_{i_1}] B_1 & \cdots & [A_{F_1} - s_{i_1}] B_1 & [A_{F_1} - s_{i_1}] B_1 \\
0 & D_1, [A_{F_1} - s_{i_1}] B_1 & D_1 [A_{F_1} - s_{i_1}] B_1 \\
0 & 0 & \cdots & D_1 [A_{F_1} - s_{i_1}] B_1 & D_1 [A_{F_1} - s_{i_1}] B_1 \\
\end{bmatrix}
\]

\[
\text{rank} \begin{bmatrix}
B_1, [A_{F_1} - s_{i_1}] & \cdots & [A_{F_1} - s_{i_1}] t^{t-1} B_1 & [A_{F_1} - s_{i_1}] t^t \\
0 & D_1, [A_{F_1} - s_{i_1}] t^{t-1} B_1 & D_1 [A_{F_1} - s_{i_1}] t^t \\
0 & 0 & \cdots & D_1 [A_{F_1} - s_{i_1}] t^t \\
\end{bmatrix} = (t-1)n, \quad (4.2.13)
\]

(Necessity)

Assume RSFP is solvable. Then by Theorem 4.1.2, reduced RPIS is solvable and for any solution \( F: \bar{A} \to U \), (4.1.18) is satisfied where \( k_i = \bar{k}_i \). If \( \bar{F} \) is partitioned as \([\bar{F}_i, \bar{F}_\bar{i}]\), then by Lemma 2.2 of [12],

\[
B_i \subseteq \bigcap_{i=1}^{\bar{k}_i} [A_{F_i} - s_{i_1}] t^{t} \ker D_i, \quad (4.2.14)
\]

and from (4.2.14) and the fact that \( A_{F_i} - s_{i_1} \) is invertible for every \( s_{i_1} \in \sigma(\bar{A}_i) \)

\[
\text{rank} \begin{bmatrix}
[A_{F_i} - s_{i_1}] t^t \\
D_i \\
\end{bmatrix} = \text{rank} \begin{bmatrix}
[A_{F_i} - s_{i_1}] t^t B_i \\
D_i \quad 0 \\
\end{bmatrix} = n, \quad (4.2.15)
\]

for all \( t \in \{1, \ldots, \bar{k}_i\} \). From (4.2.5) there exist matrices \( H_{e_1} \) of order \((n, xm)\) satisfying

\[
\begin{bmatrix}
[A_{F_i} - s_{i_1}] t^t B_i & H_{e_1} \\
D_i & 0 \\
\end{bmatrix} \begin{bmatrix}
B_i \\
H_{e_1} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}, \quad (4.2.16)
\]

for \( t=1, \ldots, \bar{k}_i \), where \( H_{e_i} = (A_{F_i} - s_{i_1}) H_{e_{i-1}} \). Therefore
\[
\begin{bmatrix}
[A_{hi} - s_i I]^{\theta_i} B_i & [A_{hi} - s_i I]^{\theta_i} D_i [A_{hi} - s_i I]^{\theta_i - 1} B_i & [A_{hi} - s_i I]^{\theta_i} D_i [A_{hi} - s_i I]^{\theta_i - 1} D_i [A_{hi} - s_i I]^{\theta_i - 1} \\
\cdot & \cdot & \cdot \\
D_i B_i & \cdot & \cdot \\
0 & \cdot & \cdot \\
\cdot & D_i & D_i \\
\end{bmatrix}
\]

for \( g \in \{0, \ldots, t-1\} \). From (4.2.17) and (4.2.16) the rank of the matrix of (4.2.13) is equal to the dimension of the plant, \( n_i \). Thus the rank of (4.2.11) is \( t \cdot n \), for any \( t \in \{1, \ldots, \bar{k}_{i\bar{t}}\} \). Since RSFP has been assumed solvable, it follows that RPI is solvable and from Theorem 4.2.1 that equation (4.2.6) is true. Then since the rank of (4.2.11) is \( t \cdot n \), for every \( t \in \{1, \ldots, \bar{k}_{i\bar{t}}\} \), where \( \bar{k}_{i\bar{t}} \gg \bar{k}_{i\bar{t}_2} \gg \ldots \gg \bar{k}_{i\bar{t}_m} \), the proof of necessity is complete.

(Sufficiency)
Assume (4.2.6) is satisfied and is of rank \( \bar{k}_{i\bar{j}} \cdot n \), for every \( i \in \{1, \ldots, \bar{r}\} \), and \( j \in \{1, \ldots, \bar{q}_{i\bar{t}}\} \). From Theorem (4.2.1) this implies RPI is solvable. Thus there exists an \( \bar{F} : \mathcal{I} \rightarrow \mathcal{U} \) such that with \( \bar{F} = [\bar{F}_1 \bar{F}_2] \),
where the matrix is of type \((\bar{k}_{\epsilon}, (n+p) \times \bar{k}_{\epsilon}, (n+m))\) and \(A_F - s_\epsilon I\) is invertible for every \(s_\epsilon \in \mathcal{A}(\bar{A})\). From (4.2.18) and (4.2.13), with \(t = \bar{k}_{\epsilon_1}\),

\[
\begin{bmatrix}
    B_i & [A_F - s_{\epsilon_1} I] B_i & [A_F - s_{\epsilon_1} I] & [A_F - s_{\epsilon_1} I] \\
    0 & D_i & B_i & D_i \bar{k}_{\epsilon_1} \\
    & \ddots & \ddots & \ddots \\
    & & B_i & D_i [A_F - s_{\epsilon_1} I] \\
\end{bmatrix} = \begin{bmatrix}
    \bar{k}_{\epsilon_1} n_i \ \\
    \end{bmatrix}
\]

(4.2.18)

Equation (4.2.19) implies

\[
\text{rank} \begin{bmatrix}
    B_i & [A_F - s_{\epsilon_1} I] \\
    0 & D_i \\
\end{bmatrix} = \text{rank} \begin{bmatrix}
    [A_F - s_{\epsilon_1} I] \\
    D_i \\
\end{bmatrix} = n_i
\]

Postmultiplying the last block of \(n_i\) columns of (4.2.19) by \([A_F - s_{\epsilon_1} I]^{-1}\) yields
rank \[
\begin{bmatrix}
B_i & [A_{Fi} - s_i \cdot \bar{I}] \\
0 & D_i
\end{bmatrix}
\] = \rank \[
\begin{bmatrix}
[A_{Fi} - s_i \cdot \bar{I}] \\
D_i
\end{bmatrix}
\] = n_i.

Continuing in this manner yields

\[
\mathcal{B}_i \subset \bigcap_{t=1}^{K_i} [A_{Fi} - s_i \cdot \bar{I}]^t \ker D_i
\]

Then from Lemma 2.2 of [12]

\[
\mathcal{B} \subset \bigcap_{t=1}^{K_i} (\bar{A} + \bar{B} \bar{E} - s_i \cdot \bar{I})^t \ker \bar{D}
\]

for each \( s_i \in \mathcal{J}(\bar{A}_2) \), where \( k_i = k_{i'} \). Therefore RSFP is solvable from Theorem 4.1.2.

Also from [12] Corollary 2.2, the following lemma is true.

Lemma 4.2.1 [12]

Subject to the assumptions of Theorem 4.2.2, if RSFP is solvable then any \( \bar{F} : {\bar{A}U} \) satisfying

\[
\bar{Y}^+(\bar{A} + \bar{B} \bar{E}) \cap \langle \bar{A} \bar{B} \rangle = 0
\]

is a solution.
4.3 Structural Solvability of RSFP

RSFP discussed in the previous section provides conditions under which a solution \( F : \mathcal{X} \to \mathcal{U} \) to RPIS may perturb and yet remain a solution to the problem. A more realistic situation of considerable interest is when perturbations are also allowed in the system parameters.

Consider a structured system of the form (4.1.1) with the detailed representation of (4.1.3) and (4.2.4). Let \( \mathbb{R}^l \) be the parameter space associated with (4.2.4) of dimension equal to the number of nonzero entries in \( A_1, A_2, B, C \) and \( D \). Consider a data point \( p \in \mathbb{R}^l \) and define perturbations about \( p \) as perturbations in the nonzero entries of \( A_1, A_2, B, C \) and \( D \) of the form;

\[
\begin{align*}
\tilde{A}_1 &= A_1 + \varepsilon \hat{A}_1 / \| \hat{A}_1 \| \\
\tilde{A}_2 &= A_2 + \varepsilon \hat{A}_2 / \| \hat{A}_2 \| \\
\tilde{B} &= B + \varepsilon \hat{B} / \| \hat{B} \| \\
\tilde{C} &= C + \varepsilon \hat{C} / \| \hat{C} \| \\
\tilde{D} &= D + \varepsilon \hat{D} / \| \hat{D} \| 
\end{align*}
\]

(4.3.1)

where \( \varepsilon > 0 \) is a scalar and \( \hat{A}_1, \hat{A}_2, \hat{B}, \hat{C} \) and \( \hat{D} \) are nonzero arbitrary matrices structurally equivalent to \( A_1, A_2, B, C \) and \( D \). Then the system matrix perturbations are of the form

\[
\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & A_2 \end{bmatrix}
\]

The following definition formalizes what is meant in this
thesis by a robust solution to the servomechanism problem.

Definition 4.3.1 RSFP is solvable for perturbations in
(A, B, C, D) of the form (4.3.1) if there
exists a \( \varepsilon_0 > 0 \) such that RSFP is solvable for
\((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) with \( 0 < \varepsilon < \varepsilon_0 \).

Consider a special form of the system (4.2.4) in which
the exogenous signals consist of step and ramp functions.
With \( k_1 \) step and \( k_2 \) ramp signals the system matrix
description of (4.2.4) becomes

\[
A = \begin{bmatrix}
A_1 & a_{11} & \cdots & a_{1k} & a_2 & 0 & \cdots & a_{2k} & 0 \\
0 & 0 & 0 & & & & & & \\
\vdots & & & & & & & & \\
0 & 0 & & & & & & & \\
\end{bmatrix}
B = \begin{bmatrix}
B_1 \\
0 \\
\vdots \\
0 \\
0 \\
\end{bmatrix}
\]

(4.3.2)
\[
D = \begin{bmatrix}
D_1 & d_{11} & \ldots & d_{1K_1} & d_{12} & 0 & \ldots & d_{1K_2} & 0
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
C_1 & c_{11} & \ldots & c_{1K_1} & c_{12} & 0 & \ldots & c_{1K_2} & 0
\end{bmatrix}
\]

As before, \(R\) is the parameter space associated with (4.3.2). From (4.2.5) the form of the structured system maps of (4.3.2) in the factor space \(\tilde{X} = \eta / \eta \wedge \tilde{X}^*(A)\) are,

\[
\tilde{A} = \begin{bmatrix}
A_1 & \tilde{a}_{11} & \ldots & \tilde{a}_{1K_1} & \tilde{a}_{12} & 0 & \ldots & \tilde{a}_{1K_2} & 0
0 & 0 & 0 & & & & & & 1
0 & 0 & & & & & & & 0
& & & & & & & & 0
\end{bmatrix}
\]

\[
\tilde{B} = \begin{bmatrix}
B_1
\end{bmatrix}
\]

\[
\tilde{D} = \begin{bmatrix}
D_1 & \tilde{d}_{11} & \ldots & \tilde{d}_{1K_1} & \tilde{d}_{12} & 0 & \ldots & \tilde{d}_{1K_2} & 0
\end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix}
C_1 & \tilde{c}_{11} & \ldots & \tilde{c}_{1K_1} & \tilde{c}_{12} & 0 & \ldots & \tilde{c}_{1K_2} & 0
\end{bmatrix}
\]

where \(K_1, (K_2)\) is the number of steps (ramps) induced in \(\tilde{X}\).

Assume for (4.3.2),

\[
\chi_1 = \langle A | B \rangle \quad (g) \quad (4.3.4)
\]

\[
\eta \wedge \chi_1 = 0 \quad (g) \quad (4.3.5)
\]

\[
\eta \wedge \chi^*(A) \subseteq \ker D \quad (g) \quad (4.3.6)
\]

As discussed previously, two cases can arise in the solution.
of RSFP; either $\tilde{\chi}(\bar{A}) \subseteq \langle \bar{A} | \bar{B} \rangle$ or $\tilde{\chi}(\bar{A}) \not\subseteq \langle \bar{A} | \bar{B} \rangle$.

Lemma 4.3.1

Assume (4.3.4), (4.3.5) and (4.3.6) are true for (4.3.2) and let $p \in R^t$ be a typical data point with respect to these assumptions*. Then

$$\tilde{\chi}^+(\bar{A}) \subseteq \langle \bar{A} | \bar{B} \rangle$$  \hspace{1cm} (4.3.7)

if and only if $\bar{k}_1 = \bar{k}_2 = 0$.

Proof:

(Sufficiency)

From (4.3.3) if $\bar{k}_1 = \bar{k}_2 = 0$, then $\langle \bar{A} | \bar{B} \rangle = \bar{\chi}$ and so (4.3.7) is true.

(Necessity)

Assume $\bar{k}_1 \neq 0$ and/or $\bar{k}_2 \neq 0$. Decompose $\bar{\chi}$ as $\bar{\chi} = \bar{\chi}_1 \oplus \bar{\chi}_2$ where $\bar{\chi}_1 = \chi_i$. Since $\tilde{\chi}^+(\bar{A}) \subseteq \bar{\chi}_1$,

$$\tilde{\chi}^+(\bar{A}) = \left\{ \begin{array}{c} x_1 \ \ x_2 \\ 0 \ \ x_3 \end{array} \right\}$$

where $x_3 (n_2 \times \bar{k}_1 + 2\bar{k}_2)$ has full column rank.

For if $\text{rank}(x_3) = n_1$ then there exists $y (n_2 \times 1)$ satisfying $x_3 y = 0$ and $(x_2 y 0)^T \in \tilde{\chi}^+(\bar{A})$. But from the form of $\bar{A}$ this implies $x_2 y \in \{x_i\}$

* Recall a property $\pi_i$ is generic if there exists a proper variety $V_i$ such that $\ker \pi_i \subseteq V_i$. Then a data point $p$ is typical, with respect to $\pi_i, i=1, \ldots, t$ if $p \in (V \cup \ldots \cup V)^c$. 
Then since \( \langle A | B \rangle \subset \bar{\mathcal{A}}_1 \),
\[
\langle \bar{\mathcal{A}} | \bar{\mathcal{B}} \rangle = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}
\]

Therefore
\[
\begin{pmatrix} x_1, x_2 \\ 0, x_3 \end{pmatrix} \cap \begin{pmatrix} R_1 \\ 0 \end{pmatrix} * \begin{pmatrix} R_1 \\ 0 \end{pmatrix}
\]

Thus the only circumstance for which \( \bar{\mathcal{X}}(\mathcal{A}) \subset \langle \bar{\mathcal{A}} | \bar{\mathcal{B}} \rangle \) is when \( \bar{\mathcal{A}} \approx \mathcal{A}, \bar{\mathcal{B}} \approx \mathcal{B}, \bar{\mathcal{C}} \approx \mathcal{C}, \) and \( \bar{\mathcal{D}} \approx \mathcal{D}. \)

Definition 4.3.2 RSFP is structurally solvable if it is solvable at all data points \( p \) in the complement of a proper variety in \( R' \).

Lemma 4.3.2
Assume (4.3.4), (4.3.5) and (4.3.6) are true for (4.3.2). Then RSFP is structurally solvable if and only if

\[
\text{rank} \begin{bmatrix} B, A \\ 0, D \end{bmatrix} = \text{rank} \begin{bmatrix} B, A, \bar{\mathcal{A}}_{31} \\ \cap \ D, \bar{\mathcal{D}}_{21} \end{bmatrix} = n, \quad (g) \quad (4.3.8)
\]

where \( \bar{\mathcal{A}}_{31} = [\bar{a}_{11}, \ldots, \bar{a}_{1r}] \) and \( \bar{\mathcal{D}}_{21} = [\bar{d}_{11}, \ldots, \bar{d}_{1r}] \), and

\[
\text{rank} \begin{bmatrix} B, A \\ D \end{bmatrix} = \text{rank} \begin{bmatrix} B, A, \bar{\mathcal{A}}_{32} \\ -1 B, A \end{bmatrix} = 2n, \quad (g) \quad (4.3.9)
\]

where \( \bar{\mathcal{A}}_{32} = [\bar{a}_{21}, \ldots, \bar{a}_{2r}] \) and \( \bar{\mathcal{D}}_{32} = [\bar{d}_{21}, \ldots, \bar{d}_{2r}] \).
Proof:

(Necessity)
From assumptions (4.3.4), (4.3.5), (4.3.6) and Definition 4.3.2 there exists a proper variety \( V \subset \mathbb{R}^r \) such that all \( p \in V^c \) are typical with respect to these assumptions, and \( \text{RSFP} \) is solvable. Then from Theorem 4.2.2 all \( p \in V^c \) satisfy

\[
\text{rank } \begin{bmatrix} B_i & A_i \\ 0 & D_i \end{bmatrix} = \text{rank } \begin{bmatrix} B_i & A_i & \bar{a}_{1i} \\ 0 & D_i & \bar{d}_{1i} \end{bmatrix} = n_i
\]
for \( i = 1, \ldots, \bar{k}_1 \), and

\[
\text{rank } \begin{bmatrix} B_i & A_i \\ -1 & B_i & A_i \\ 0 & D_i \\ -1 & D_i \end{bmatrix} = \text{rank } \begin{bmatrix} B_i & A_i & \bar{a}_{2i} \\ -1 & B_i & \bar{a}_{2i} \\ 0 & D_i & \bar{d}_{2i} \end{bmatrix} = 2n_i
\]
for \( i = 1, \ldots, \bar{k}_2 \), \( 0 \leq \bar{k}_i \), and \( 0 \leq \bar{k}_2 \leq k_2 \). Then defining \( \bar{A}_{3i} \), \( \bar{A}_{32} \), \( \bar{D}_{2i} \), and \( \bar{D}_{22} \) as above, (4.3.8) and (4.3.9) follow.

(Sufficiency)
From the genericity of equations (4.3.4), (4.3.5), (4.3.6), (4.3.8) and (4.3.9) it follows that there exists a proper variety \( V \) such that all \( p \in V^c \) are typical with respect to (4.3.4), (4.3.5) and (4.3.6) and

\[
\text{rank } \begin{bmatrix} B_i & A_i \\ 0 & D_i \end{bmatrix} = \text{rank } \begin{bmatrix} B_i & A_i & \bar{A}_{3i} \\ 0 & D_i & \bar{D}_{3i} \end{bmatrix} = n_i
\]
\[
\begin{bmatrix}
B_1 & A_1 \\
0 & D_1 \\
-1 & B_1 & A_1 \\
0 & D_1 & C_1 & D_1
\end{bmatrix}
= \begin{bmatrix}
B_1 & A_1 & \tilde{A}_{21} \\
0 & D_1 & \tilde{D}_{21} \\
-1 & B_1 & A_1 \\
0 & D_1 & C_1 & D_1
\end{bmatrix} = 2n,
\]

Then from the definitions of \(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{D}_{21}, \tilde{D}_{22}\) and Theorem 4.2.2, RSFP is solvable at all points \(p \in V^{c}.\)

Note that if the rank equalities of (4.3.8) and (4.3.9) are true generically but not equal to \(n_1\) and \(2n_1\), then RPIS is structurally solvable but RSFP is not.

Definition 4.3.3 The open neighborhood, \(N(p, \epsilon_0)\), of the data point \(p \in R^1\) with radius \(\epsilon_0\) consists of all \(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\) such that

\[
N(p, \epsilon_0) = \left\{ \begin{array}{l}
\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} : \|\tilde{A} - A\| < \epsilon_0, \|\tilde{B} - B\| < \epsilon_0, \|\tilde{C} - C\| < \epsilon_0, \\
\|\tilde{D} - D\| < \epsilon_0 \end{array} \right\}
\]

Definition 4.3.4 RSFP is well-posed at \(p \in R^1\) if it is solvable at all points in some open neighborhood of \(p\).

Lemma 4.3.3

Assume (4.3.4), (4.3.5) and (4.3.6) are true for (4.3.2). There exist well-posed data points with respect to RSFP if and only if RSFP is structurally solvable.
Proof:
(Sufficiency)
Assume RSFP is solvable at all data points $p \in V^c$, where $V$ is a proper variety in $\mathbb{R}^i$. Then from Lemma 2.2.1 every $p \in V^c$ is a well-posed data point with respect to RSFP.

(Necessity)
Assume RSFP is well-posed at $p$ and $p$ is typical with respect to $(4.3.4)$, $(4.3.5)$ and $(4.3.6)$. If $p$ is not a typical data point with respect to these assumptions then there always exists a $p'$ in a neighborhood of $p$ which is typical and for which RSFP is well-posed. This follows as a result of Lemma 2.2.1 and the genericity of assumptions $(4.3.4)$, $(4.3.5)$ and $(4.3.6)$.

There exists a neighborhood $N(p, \epsilon_*)$ of $p$ for which the assumptions and results of Lemma 4.3.2 are true. Consider the following matrices from $(4.3.8)$ and $(4.3.9)$.

\[
\begin{bmatrix}
B_i & A_i \\
0 & D_i
\end{bmatrix}
\quad (4.3.10)
\]

\[
\begin{bmatrix}
B_i & A_i & \tilde{A}_g \\
0 & D_i & \tilde{D}_{2i}
\end{bmatrix}
\quad (4.3.11)
\]
\[
\begin{bmatrix}
B, A,
0 & D,
-1 & B, A,
\end{bmatrix}
\]
\(\text{(4.3.12)}\)

\[
\begin{bmatrix}
B, A, & \bar{A}_{32}
0 & D, & \bar{D}_{32}
-1 & B, A, &
\end{bmatrix}
\]
\(\text{(4.3.13)}\)

Define the polynomials \(\Psi_i, i=1,2,3,4\) as the sum of the squares of all minors having order greater than \(n, n, 2n, 2n\) of \((4.3.10),(4.3.11),(4.3.12)\) and \((4.3.13)\) respectively. Since RSFP is solvable at all points \(N(p, e_0)\), the polynomial

\[
\Theta = \sum_{i=1}^{4} \Psi_i
\]

is zero in this neighborhood. But this implies \(\Theta\) vanishes identically in \(R^1\) since it vanishes in a neighborhood of \(p, [35]\). Thus for any \(p \in R^1\) the ranks of \((4.3.10),(4.3.11),(4.3.12)\) and \((4.3.13)\) are less than or equal to \(n, n, 2n, 2n\) and \(2n\). Consider the variety defined by

\[
\prod_{i=1}^{4} Y_i = 0
\]

where \(Y_i, i=1,2,3,4\) is the sum of the squares of all minors of order \(n, n, 2n, 2n\) of \((4.3.10),(4.3.11)\),
(4.3.12) and (4.3.13) respectively. Since there exist well-posed data points this variety is proper. Therefore RSFP is solvable at all data points, typical with respect to (4.3.4), (4.3.5), (4.3.6), in the complement of this proper variety. ▲

Thus if RSFP is structurally solvable, practically every data point in $R^l$ is well-posed with respect to RSFP.

The main result of this section concerns the obvious equivalence of well-posed data points and the definition of robustness of Definition 4.3.1.

Theorem 4.3.1

Assume (4.3.4), (4.3.5) and (4.3.6) are true for (4.3.2). Let $p \in R^l$ be a data point typical with respect to these assumptions and for which RSFP is solvable. The RSFP is solvable for $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with $0 \leq \varepsilon \leq \varepsilon_0$ if and only if RSFP is well-posed at $p$.

Proof:
(Necessity)
Define the open neighborhood $N(p, \varepsilon_0)$ of $p$. Then by assumption every $p' \in N(p, \varepsilon_0)$ admits a solution to RSFP. Therefore by Definition 4.3.1, RSFP is well-posed at $p$.

(Sufficiency)
If RSFP is well-posed at $p$ there exists a neighborhood $N(p, \varepsilon_0) \subset R^l$ such that RSFP is solvable at all
\( p' \in N(p, \varepsilon_o) \). Then for \( 0 < \varepsilon < \varepsilon_o \), \( (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \) admits a solution to RSFP.

Remarks:
1) Let \( F : \mathcal{X} \rightarrow \mathcal{U} \) be a solution to RSFP at a well-posed data point. Then there exists an open neighborhood \( N(p, \varepsilon_o) \), of \( p \) such that for any \( p' \in N(p, \varepsilon_o) \) the resulting closed loop system \( (\tilde{A} + \tilde{B}F, \tilde{D}) \) solves RPIES for any admissible \( \tilde{F} \). Hence the nomenclature "robust solution".

2) From Lemma 4.3.3, if RSFP is structurally solvable then practically any data point is well-posed. Thus to determine if a system admits a robust solution in the sense of Definition 4.3.1 requires the verification of equations (4.3.8) and (4.3.9).

The realization of such a solution using output feedback is accomplished in this thesis using a Brasch type compensator, [23].

This form of compensation involves extending the state space of (4.3.2) with integrators. In general the system is augmented with \( t \) integrators such that the matrix representation of the system relative to the extended state space \( \mathcal{X}^e = \mathcal{X} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \) is;
\[
A^e = \begin{bmatrix}
A_1 & 0 & a_{1n} & \cdots & a_{1k_1} & a_{21} & 0 & \cdots & a_{2k_2} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

\[
B^e = \begin{bmatrix}
B_1 & 0 \\
0 & 1 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
\end{bmatrix}
\]

\[
C^e = \begin{bmatrix}
C_1 & 0 & c_{1n} & \cdots & c_{1k_1} & c_{21} & 0 & \cdots & c_{2k_2} & 0 \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\end{bmatrix}
\]

(4.3.14)

where \(u^e = u \otimes u_e\) is the input space and \(y^e = y \otimes y_e\) is the output space. Note that since the extended state variables are both controllable and observable, the basic assumptions of plant structural controllability and observability remain valid for the extended system. Also if (4.3.8) and (4.3.9) are
true for the original system then they remain true for the extended system from the form of $B^e$. Applying output feedback to (4.3.14) of the form

$$K: y^e \rightarrow u^e$$

(4.3.15)

yields the closed loop plant matrix;

$$(A_i^e + B_i^e K C_i^e) = \begin{bmatrix}
A_i + B_i K_{ii} C_i & B_i K_{ij} \\
K_{ij} C_i & K_{jj}
\end{bmatrix}$$

where

$$K = \begin{bmatrix}
K_{ii} & K_{ij} \\
K_{ij} & K_{jj}
\end{bmatrix}$$

Definition 4.3.5 The controllability index $\gamma$ of the system $(A,B)$ is the smallest non-negative integer such that

$$\text{rank}[B \ AB \ \ldots \ A^i B] = \text{rank}[B \ AR \ \ldots \ A^{i+1} B]$$

Definition 4.3.6 The observability index $\delta$ of the system $(A,C)$ is the smallest non-negative integer such that

$$\text{rank} \begin{bmatrix}
C \\
CA \\
CA^i
\end{bmatrix} = \text{rank} \begin{bmatrix}
C \\
CA \\
CA^{i+1}
\end{bmatrix}$$

Note that output feedback of the form of (4.3.15) guarantees
equivalent admissible state variable feedback laws with respect to compensator perturbations. That is; \( K^{e} : x^{e} \to u^{e} \) satisfies \( \gamma^{e} = \gamma \subset \ker \tilde{K}C^{e} \) for perturbations in \( K \) defined as \( \tilde{K} = K + \epsilon \hat{K} / \| \hat{K} \| \) where \( \hat{K} \) is arbitrary.

Theorem 4.3.2[23]

If \((A, B, C)\) is a controllable, observable system and \( t = \min(\nu_{c} - 1, \nu_{o} - 1) \) then for every choice of a self-conjugate set \( \sqcup \) of \( n + t \) complex numbers, there exists a

\[ K : y^{e} \to u^{e} \]

such that

\[ \gamma'(A^{e} + B^{e}K^{e}) = \sqcup \]

Let \( p \) be a well-posed data point with respect to RSFP. As discussed above either (4.1.16) or (4.1.17) are true. From Theorem 4.1.1 or Lemma 4.2.1, for either of these cases any admissible \( F : x \to u \) satisfying \( \gamma'(A_{i} + B_{i}F_{i}) \subset C^{-} \) is a solution. But by Theorem 4.3.2, \( \gamma \) can always be chosen such that

\[ \gamma' \left[ \begin{array}{c} A_{i} + B_{i}, \tilde{K}_{i}, \tilde{C}_{i}, \tilde{B}_{i}, \tilde{K}_{i} \end{array} \right] \subset C^{-} \quad (4.3.16) \]

and \( \tilde{K}C^{e} \) is admissible. Therefore \( K : y^{e} \to u^{e} \) yields a robust closed loop solution.

Employing a Brasch compensator to stabilize the plant
of a system for which RSFP is structurally solvable yields a closed loop system insensitive to very general system and compensator parameter perturbations. Specifically any perturbations such that (4.3.16) is true are admissible.

The computational problems involved in determining the structural solvability of RSFP for a given system structure are considered in the next section.
4.4 Computational Considerations

Well-posed data points with respect to RSFP admit robust solutions, as in Definition 4.3.1. Since practically every data point is well-posed if RSFP is structurally solvable, Lemma 4.3.3, it is desirable to be able to verify structural solvability of RSFP for a given system.

Assume the given system has form (4.3.2) relative to the state space \( \mathbf{X} \) and (4.3.6) relative to the factor space \( \tilde{\mathbf{X}} \). Let \( \mathbf{R}' \) be the parameter space associated with (4.3.2) as defined previously. Assume for (4.3.2) that (4.3.4), (4.3.5) and (4.3.6) are true. Then Lemma 4.3.2 gives necessary and sufficient conditions for RSFP to be structurally solvable; (4.3.8) and (4.3.9). Therefore to practically check structural solvability of RSFP for a given system structure requires the verification of assumptions (4.3.4), (4.3.5) and (4.3.6), and the checking of (4.3.8) and (4.3.9).

An important special case of (4.3.2) occurs frequently in practice. This special case is when the outputs to be regulated are directly measurable. The output matrix of (4.3.2) is then assumed of the form

\[
\begin{bmatrix}
C_1 \ c_1 \ \ldots \ c_{ik} \ 0 \ \ldots \ c_{jk} \ 0 \\
D_1 \ d_u \ \ldots \ d_{iu} \ d_{it} \ 0 \ \ldots \ d_{jk} \ 0
\end{bmatrix}
\]

(4.4.1)

Measurable outputs of this form lead to certain simplifications in the computational procedures developed below.

In the remainder of this section, algorithmic
procedures are considered which computationally verify assumptions (4.3.4), (4.3.5) and (4.3.6), and the rank conditions of (4.3.8) and (4.3.9). The modifications of these algorithms for outputs of the form (4.4.1) are also presented.

Plant Structural Controllability (Observability)

Structural controllability and observability of the triple $(C, A, B_i)$ are checked using the results of Theorem 3.3.1. The following algorithm is presented in terms of plant controllability. In the case of plant observability replace $A_i \sim A_i^T$ and $B_i \sim C_i^T$. The algorithm is equally valid under the assumption of (4.4.1).

Step_1: Using the matrices $A_i$ and $B_i$, form the extended controllability matrix $\tilde{R}$; (3.3.1).

Step_2: Determine if $\tilde{R}$ has Form $(n^2)$ using FZRF.

Step_3: If $\tilde{R}$ does not have Form $(n^2)$, go to Step_4.

Otherwise $(A_i, B_i)$ is structurally uncontrollable; end of algorithm.

Step_4: Since $\tilde{R}$ is not of Form $(n^2)$, $(A_i, B_i)$ is structurally controllable; end of algorithm.

$\gamma \cap x^+(A) \subset \ker D (g)$

The assumption of plant structural observability permits certain simplifications in the computational verification of (4.3.6).
Lemma 4.4.1

Consider the system (4.3.2) with the assumption of (4.3.5). Let \( p \) be a typical data point with respect to (4.3.5). Then

\[
\eta = \eta \cap X^+(A)
\]

Proof:

Since all the exogenous signals of (4.3.2) are unstable, \( X^-(A) \subset X_1 \). Also since \( p \) represents a system having an observable plant, \( \eta \cap X_1 = 0 \). Now \( \eta, X^-(A) \) and \( X^+(A) \) are all \( A \) invariant subspaces, so;

\[
\eta = \eta \cap X^+(A) \oplus \eta \cap X^-(A) = \eta \cap X^+(A)
\]

Thus the verification of (4.3.6) is equivalent to the verification of

\[
\eta \subseteq \ker D (p)
\]

(4.4.2)

The vector space condition of (4.4.2) is converted into matrix rank conditions in the next two lemmas.

Lemma 4.4.2

Consider the system (4.3.2) and assume (4.3.5) is true. Let \( p \) be a typical data point with respect to (4.3.5). Then
\[ \eta = \ker \begin{bmatrix} C \\ A \end{bmatrix} \]

when \( k_1 \neq 0, k_2 = 0 \) and

\[ \eta = \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix} \]

when \( k_2 \neq 0 \).

Proof:
Define the subspace \( m\mathcal{X} \) as,

\[ \mathcal{X}^*(A) = \mathcal{X}^*(A) \cap \chi_1 + \mathcal{X}^*(A) \cap \eta + m \]

From Lemma 4.4.1, \( \eta = \eta \cap \mathcal{X}^*(A) \) yielding

\[ \mathcal{X}^*(A) = \mathcal{X}^*(A) \cap \chi_1 \oplus \eta \oplus m \]  \hspace{1cm} (4.4.3)

Write \( \chi \) as

\[ \chi = \mathcal{X}^*(A) \oplus \mathcal{X}^-(A) \]  \hspace{1cm} (4.4.4)

Combining (4.4.3) and (4.4.4)

\[ \chi = \mathcal{X}^*(A) \cap \chi_1 \oplus \eta \oplus \mathcal{X}^-(A) \oplus m \]

\[ = \chi_1 \oplus m \oplus \eta \]

since \( \mathcal{X}^*(A) \subset \chi_1 \). Now \( A\chi_1 \subset \chi_1 \), \( A\eta \subset \eta \), \( Am \subset \chi_1 \oplus m \oplus \eta \). Let \( T \) be a coordinate transformation formed from basis vectors of \( \chi_1, m \) and \( \eta \). Then relative to this new basis,
A has the form

$$
\begin{bmatrix}
A_1 & A_{12} & 0 \\
0 & A_{22} & 0 \\
0 & A_{32} & A_{33}
\end{bmatrix}
$$

where the minimal polynomial of

$$
\begin{bmatrix}
A_{22} & 0 \\
A_{32} & A_{33}
\end{bmatrix}
$$

is either $s$ when $k_1 \neq 0$, $k_2 = 0$ or $s^2$ when $k_2 \neq 0$. Thus if $k_2 = 0$ it follows that $A_{22} = 0$, $A_{22} = 0$, $A_{33} = 0$ and if $k_2 \neq 0$ the square of (4.4.5) is

$$
\begin{bmatrix}
A_1^2 & V & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

where $V = A_1 A_{12} + A_{12} A_{22}$. Since relative to this basis the unobservable space has the form

$$
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
$$

It follows that

$$(T^* A T)T^{-1} \eta = 0$$
or

\[(T^\tau AT)^2 T^\tau \eta = 0 \quad (4.4.7)\]

Therefore if \( k_2 = 0 \), \( A\eta = 0 \) and if \( k_2 \neq 0 \), \( A^2 \eta = 0 \).

(Minimality)

Assume \( k_1 = 0 \), \( k_2 \neq 0 \) and let \( x \in \eta \). Then since \( x \in \ker A \) it follows that

\[
\begin{bmatrix}
C \\
CA \\
CA^{n-1} \\
A
\end{bmatrix}
\in \ker
\begin{bmatrix}
C \\
A
\end{bmatrix}
\]

On the other hand if

\[
\begin{bmatrix}
C \\
A
\end{bmatrix}
\begin{bmatrix}
x \\
\end{bmatrix}
\in \ker
\begin{bmatrix}
C \\
A
\end{bmatrix}
\]

then

\[
\begin{bmatrix}
C \\
CA \\
CA^{n-1}
\end{bmatrix}
\in \eta
\]

Now assume \( k_2 \neq 0 \) and let \( x \in \eta \). Then
\[
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1} \\
A^2
\end{bmatrix}
\]

\[ x \in \ker \]

But from (4.4.7), \( x \in \ker A^2 \). Therefore

\[
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1} \\
A^2
\end{bmatrix}
\]

\[ = \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix} \]

(Sufficiency)

Let

\[
\begin{bmatrix}
C \\
CA \\
A^2
\end{bmatrix}
\]

Then
Lemma 4.4.3

Consider the situation where (4.3.5) is true. Let \( \eta \) be a tuple with respect to (4.3.5). Then

\[
x \in \ker \begin{bmatrix} C & CA & \cdots & CA^{n-1} \end{bmatrix} = \eta
\]

if and only if

\[
\ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix} = \ker \begin{bmatrix} C \\ CA \\ A^2 \\ \vdots \end{bmatrix}
\]

(4.4.8)

Proof:
(Necessity)
Assume \( \eta \in \ker D \) and let
Lemma 4.4.3

Consider the system (4.3.2) and assume (4.3.5) is true. Let $p$ be a typical data point with respect to (4.3.5). Then

$$\eta \in \ker D$$

If and only if

$$\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-3}(A^2)
\end{bmatrix} = \begin{bmatrix}
C \\
CA \\
A^2 \\
D
\end{bmatrix}$$

(4.4.8)

Proof:

(Necessity)

Assume $\eta \in \ker D$ and let
$x \in \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix}$

Then from Lemma 4.4.2, $x \in \eta$ and so $x \in \ker D$. Therefore

$\begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix}$

On the other hand

$x \in \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix}$ implies $x \in \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix}$

So (4.4.8) is true.

(Sufficiency)

Assume (4.4.8) is true. Let $x \in \eta$. Then by Lemma 4.4.2

$x \in \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix} = \ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix}$
Therefore \( x \in \ker D \) and so

\[ \gamma \subseteq \ker D \]

Therefore (4.4.2) is equivalent to the condition

\[
\ker \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix} = \ker \begin{bmatrix} C \\ CA \\ A^2 \\ D \end{bmatrix} \quad (\gamma) \quad (4.4.9)
\]

It follows that

\[
\text{rank} \begin{bmatrix} C \\ CA \\ A^2 \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \\ A^2 \\ D \end{bmatrix} \quad (\gamma) \quad (4.4.10)
\]

if and only if (4.4.9) is true. Therefore verification of (4.4.9), and thus (4.3.6) involves the calculation of the generic ranks of (4.4.10). Note that if (4.4.1) is true, \( \ker D \subseteq \ker C \) (\( \gamma \)) and (4.4.10) is always satisfied. Thus it is not necessary to verify (4.3.6) for this case.

The matrices of (4.4.10) can be rewritten such that
\[
\begin{align*}
\text{rank } & \begin{bmatrix} C & \end{bmatrix} \quad = \quad \text{rank } & \begin{bmatrix} C & 0 \\ A - I & \end{bmatrix} - n. \quad (g) \quad (4.4.11) \\
& \begin{bmatrix} A^2 \\ 0 & C \\ 0 & A \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\text{rank } & \begin{bmatrix} C \\ CA \\ D \end{bmatrix} \quad = \quad \text{rank } & \begin{bmatrix} C & 0 \\ A - I & \end{bmatrix} - n \quad (g) \quad (4.4.12) \\
& \begin{bmatrix} 0 & C \\ 0 & A \\ D & 0 \end{bmatrix}
\end{align*}
\]

The algorithmic procedure used to determine the generic ranks of the expanded matrices of \((4.4.11)\) and \((4.4.12)\) is used in several subsequent rank checks. For this reason the detailed discussion of this procedure is contained in Appendix II for easy reference.

Consider the following algorithm to verify \((4.3.6)\).

**Step 1:** Form the matrices

\[
\begin{align*}
\begin{bmatrix} C & 0 \\ A - I & \end{bmatrix} \quad & \quad \begin{bmatrix} C & 0 \\ A - I & \end{bmatrix} \\
0 & C \quad & \quad 0 & C \\
0 & A \quad & \quad 0 & A \\
D & 0 \quad & \quad D & 0
\end{align*}
\]
Step_2: Using the procedure of Appendix II determine the
generic ranks of the matrices of Step_1.
Step_3: If these ranks are equal, go to Step_4.
Otherwise (4.3.6) is not true; end of algorithm.
Step_4: (4.3.6) is true; end of algorithm.

Equations (4.3.8) and (4.3.9)
The matrix rank conditions of (4.3.8) and (4.3.9) are
in terms of the maps induced in the factor space
\( \bar{\mathcal{X}} = \mathcal{X}/\mathcal{H} \cap \mathcal{X}^*(A) \). By Lemma 4.4.2 the structure of the
vectors in

\[
\begin{bmatrix}
  \mathcal{C} \\
  \text{ker} \mathcal{C}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \mathcal{C} \\
  \mathcal{C} \mathcal{A} \\
  \mathcal{A}^2
\end{bmatrix}
\]

determine the form of these induced maps. Consider a (2nx1) vector \((w, w_2)^T\) contained in the kernel of the matrix

\[
\begin{bmatrix}
  \mathcal{C} & 0 \\
  \mathcal{A} & -1 \\
  \mathcal{C} \\
  0 & \mathcal{A}
\end{bmatrix}
\]

(4.4.13)

Lemma 4.4.4
If \((w, w_2)^T\) is a vector contained within the kernel of
(4.4.13), then \( w_i \in \mathcal{N} \).

Proof:
Since \((w_i, w_2)^T\) is in the kernel of (4.4.13),

\[
\begin{bmatrix}
C & 0 \\
A - I & -A \\
0 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
w_i \\
w_2
\end{bmatrix} = 0
\]  
(4.4.14)

Then from (4.4.14)

\[
\begin{bmatrix}
C & 0 \\
0 & -1 \\
CA & C \\
A^2 & A
\end{bmatrix}
\begin{bmatrix}
w_i \\
-Aw_i + w_2
\end{bmatrix} = 0
\]  
(4.4.15)

Implying \(-Aw_i + w_2 = 0\). Therefore

\[
w_i \in \ker \begin{bmatrix}
C \\
CA \\
A^2
\end{bmatrix} = \mathcal{N}
\]

by Lemma 4.4.2.

Consider a decomposition of the state space \( \mathcal{X} \) of (4.3.2) defined by,
\[
\chi = \chi_i \otimes \chi_i' \otimes \ldots \otimes \chi_i^k \otimes \chi_i^2 \otimes \ldots \otimes \chi_i^z \quad (4.4.1f)
\]

where \(x_i'\) spans \(\chi_i'\), \(i=1, \ldots, k\), and \(x_i^2, \ldots x_i^z\) spans \(\chi_i^z\) with \(Ax_i^z = x_i^z\), \(i=1, \ldots k\). Lemma 4.4.4 provides a means of determining the structure of a basis for \(\chi\) relative to the decomposition (4.4.16). The procedure used to determine this basis is contained in Appendix III. Let the sets \(W_1\) of order \(r\), \(W_2\) of order \(r\) and \(W_3\) of order \(s\) be a basis of \(\chi\) obtained from Appendix III for (4.3.2).

Lemma 4.4.5

Consider the system (4.3.2) under the assumptions (4.3.4), (4.3.5) and (4.3.6). Given the sets of structured vectors \(W_1\), \(W_2\) and \(W_3\), the form of the maps induced in the factor space \(\bar{\chi}\) are

\[
\begin{bmatrix}
A_1, a_{1j}, \ldots a_{1j_k}, -w_1, \ldots -w_g, a_{2i}, 0, \ldots a_{2i_k}, 0 \\
0, 0, 0, 0, 0, 0, 0, 0, 0, 0
\end{bmatrix}
\begin{bmatrix}
R_1 \\
0
\end{bmatrix}
\]

\[
\bar{A} =
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\vdots \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]

\[
\bar{\theta} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\bar{\gamma} =
\begin{bmatrix}
D_1, d_{1j}, \ldots d_{1j_k}, 0, \ldots, 0, d_{2i}, 0, \ldots d_{2i_k}, 0 \\
\end{bmatrix}
\quad (4.4.17)
\]

where \(k_i' = k_i - t - k, \quad g = r - s - k, \quad k_i^2 = k_i - s\) for some \(k\) in the
range 0 \leq k \leq r-s, and \( w_i \in X_i \cap \{W_i \}, i \in \{1, \ldots, r\} \).

Proof:
Consider \( w \in W_i \). Since \( \gamma \) is \( A \) invariant, (4.3.5) is true and \( x_{2j}^2 = 0 \) for all \( j \in \{1, \ldots, k_i\} \), \( Aw = 0 \). Define the coordinate transformation

\[
T = \begin{bmatrix}
    1_{x_i} \\
    1_{z-i} \\
    0 & \ldots & w \\
    1_{2k_i + k_i - 1}
\end{bmatrix}
\]  \hspace{1cm} (4.4.18)

Then in the factor space \( \bar{X}/w \) the induced system maps have the form

\[
\begin{bmatrix}
    A_i & a_{i1} & \ldots & a_{i,i-1} & a_{i,i+1} & \ldots & a_{i,2k_i} & 0 \\
    0 & 0 \\
    0 & 1 \\
    0 & 0 \\
    \vdots \\
    0 & 1 \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    R_i \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    D_i & d_{i1} & \ldots & d_{i,i-1} & d_{i,i+1} & \ldots & d_{i,2k_i} & 0
\end{bmatrix}
\]

Thus each \( w \in W_i \) results in the removal of the associated step exogenous signal in the factor space \( \bar{X} \).
Since the order of \( W_i \) is \( t \), \( k_i \geq k_i - t \) in (4.4.17).
Consider \( w \in W_3 \). Since \( \mathcal{H} \) is a \( \text{A} \) invariant, \( A w = w' \) where \( w' \in W_2 \) by Lemma III.2. Define the coordinate transformation

\[
T = \begin{bmatrix}
I_{n \times K_1} & & & \\
& I_{2(i-1)} & & \\
& 0 & \cdots & 0\ w'\ w \\
& 0 & & 0 \\
& & & I_{2(k_i-\iota)}
\end{bmatrix}
\]

The induced maps in the factor space \( \mathcal{H}/w_3w' \) are

\[
\begin{bmatrix}
A_1, a_1, \ldots, a_{k_1}, a_1, \ldots, a_{k_1-1}, 0, a_{2i+1}, 0, \ldots, a_{k_2}, 0 \\
0 & 0 & & & \\
0 & 1 & & & \\
0 & 0 & & & \\
\vdots & & & \ddots & \\
0 & 1 & & & \\
0 & 0 & & & \\
D_1, d_1, \ldots, d_{k_1}, d_{2i}, 0, \ldots, d_{2i-1}, 0, d_{2i+1}, 0, \ldots, d_{k_2}, 0
\end{bmatrix}
\begin{bmatrix}
B_1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Thus each vector \( w \in W_3 \) results in the removal of the associated ramp signal in the factor space \( \mathcal{H} \). Since the order of \( W_3 \) is \( s, k_i' = k_i - s \) in (4.4.17).

Consider \( w \in \bar{W}_2 \) where \( \bar{W}_2 \) is the set of vectors left in \( W_2 \) after the \( w' \) vectors used above have been removed,
The order of $\bar{W}_2$ is $r-s$. Each vector $w_i \in \bar{W}_2$ can be written as

$$w = w_i + \alpha_1 x_1 + \ldots + \alpha_{n_i} x_{n_i} + \alpha_{n_i+1} x_{n_i+1} + \ldots + \alpha_{n+k} x_{n+k}$$

where $w_i \in X_i$. Let $\bar{W}_2^i$ be a set of vectors of $\bar{W}_2$ containing the largest number of nonzero independent entries possible in rows $n_i+1$ through $n_i+k_i$. Thus the vectors of $\bar{W}_2^i$ form a basis for $\{\bar{W}_2^i \cap \{X_1 \cup \ldots \cup X_i\}\}$. Assume the order of $\bar{W}_2^i$ is $k$. Using the transformation of (4.4.18), $k$ additional exogenous steps can be eliminated from the induced maps in $\bar{X}$. Thus the number of remaining steps in $\bar{X}$ is $k_i = k - t - k_i$.

Consider $w_i \in \bar{W}_2$, $w_i \in \bar{W}_2^i$. Denote this set of vectors as $\bar{W}_2^i$. There exist coefficients $\alpha_j^i$, $i=1,\ldots,k_i$ such that $w_i$ defined by

$$w_i' = w_i + \alpha_1 w_i + \ldots + \alpha_{k_i} w_i$$

with $w_i' \in \bar{W}_2^i$ satisfies $\alpha_j^i = 0$ for all $j \in \{1,\ldots,k_i\}$.

Partition $w_i'$ as

$$w_i' = \begin{bmatrix} w_i \\ w_2 \end{bmatrix}$$

where $w_i \in X_i$. Since $\alpha_j^i = 0$ for all $j \in \{1,\ldots,k_i\}$ and $x_{ij}^i = 0$ for all $j \in \{1,\ldots,k_i\}$, a vector $v$ can always be defined
such that $A\nu = w_1 + w$. That is $\nu$ is defined as

$$
\nu = \sum_{z=1}^{K_z} \rho_z x_{2z}^2
$$

$\rho_z = 0$ whenever $x_{2z}^2 \neq 0$ in the representation of $w^t$, in the basis $\langle 4, 4, 16 \rangle$. Define the following coordinate transformation

$$
T = \begin{bmatrix}
I_{n_1} & w_1 \\
0 & I_{K_v + 2(z_{-1})} & \nu & w_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{2(K_z - 1)}
\end{bmatrix}
$$

The induced maps in the factor space $\chi / w^t$ are of the form

$$
\begin{bmatrix}
A_1 & a_{1v} & \cdots & a_{1K_v} & a_{2v} & 0 & \cdots & a_{2K_v} & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
R_v \\
0
\end{bmatrix}
$$

Thus for each $w^t$, a ramp disturbance is reduced to a
step disturbance in $\tilde{\mathbf{x}}$. Since there are $r-s-k$ such vectors, $\varepsilon=r-s-k$ in (4.4.17).

The following procedure determines the validity of (4.3.8) and (4.3.9). No modifications are required under the assumption of (4.4.1).

Step_1: Using the procedure of Appendix III determine the sets $W_1, W_2$ and $W_3$.

Step_2: For every $w \in W_1$, eliminate the appropriate exogenous step signal of (4.3.2).

Step_3: For every $w \in W_2$, isolate the corresponding $w' \in W_2$ according to Lemma 4.4.5 and eliminate the appropriate exogenous ramp signal from (4.3.2).

Step_4: With the information concerning the $w'$ of Step_3, obtain $\tilde{W}_2$.

Step_5: By a repeated application of Lemma III.3 determine the structure of each $w \in \tilde{W}_2$; i.e., determine the nonzero $\xi'_i$'s in

$$
W = W_1 + \sum_{i=1}^{k_1} \xi'_i x'_i + \sum_{j=1}^{2} \sum_{i=1}^{k_2} \xi'^2_{ij} x'^2_{ij}
$$

where $w, \varepsilon \xi'_i$.

Step_6: Determine the vectors in $\tilde{W}_2$ by locating the largest set of nonzero independent entries in rows $n+i$ through $n+i+k$, of the vectors of $\tilde{W}_2$.

Step_7: For every $w \in \tilde{W}_2$ remove the appropriate exogenous step from (4.3.2).
Step 8: Apply Appendix III to the resulting reduced system to determine \( \tilde{w}_j^i \); and therefore \( w_1, \ldots, w_g \) of (4.4.17).

Step 9: Using FZRF check (4.3.8) for \( i=j, \ldots, j_m \) of (4.4.17). If (4.3.8) fails, RSFP is not structurally solvable; end of algorithm.

Step 10: For the exogenous step signals \( w_1, \ldots, w_g \) a straightforward application of FZRF is not possible since the entries of \( w_i^j, i \in \{1, \ldots, g\} \), \( A, \) and \( C \), are not independent.

Let \( w = (w', w'')^T \in \tilde{w}_j^i \) with \( w' \in \{w_1, \ldots, w_g\} \). Then from Appendix III for some \( i \in \{1, \ldots, k_x\} \)

\[
\begin{bmatrix}
  C, F, C_{x_i}
  [A, F_2 a_{x_i}]
\end{bmatrix}
\begin{bmatrix}
w' \\
w''
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

(4.4.19)

where

\[
\begin{bmatrix}
  C, F_i \\
  A, F_2
\end{bmatrix}
\]

has full generic column rank and \( (F, F_2)^T \) is of order \( (n_1 + h x f) \) and composed of vectors of the form \( (c_{x_j} a_{x_j})^T, j < i \).

Lemma 4.4.6

If \( p \) is a typical data point with respect to (4.3.4) and (4.4.19) is true then
\[
\begin{align*}
\text{rank } \begin{bmatrix} B, & A, & \end{bmatrix} & = \text{rank } \begin{bmatrix} B, & A, & -w' \end{bmatrix} = n_1 \quad (4.4.20) \\
\begin{bmatrix} 0 & D, & \end{bmatrix} & \begin{bmatrix} 0 & D, & 0 \end{bmatrix}
\end{align*}
\]

if and only if

\[
\begin{align*}
\text{rank } \begin{bmatrix} B, & A, & -I \end{bmatrix} & = \text{rank } \begin{bmatrix} B, & A, & -I \end{bmatrix} = 2n_1 + f \\
\begin{bmatrix} 0 & D, & 0 \end{bmatrix} & \begin{bmatrix} C, & F, & c_{z_i} \\ A, & F_2 & a_{z_i} \end{bmatrix}
\end{align*}
\]

(4.4.21)

Proof:

(Necessity)

Assume (4.4.20) is true. Then there exists a vector \((x, x_0)^T\) such that

\[
\begin{align*}
\begin{bmatrix} B, & A, & \end{bmatrix} & \begin{bmatrix} x, \end{bmatrix} = \begin{bmatrix} w', \end{bmatrix} \\
\begin{bmatrix} 0 & D, & \end{bmatrix} & \begin{bmatrix} x_0, \end{bmatrix} = \begin{bmatrix} 0, \end{bmatrix}
\end{align*}
\]

(4.4.22)

Now by assumption

\[
\begin{align*}
\begin{bmatrix} C, & F, & c_{z_i} \end{bmatrix} & \begin{bmatrix} w', \end{bmatrix} = \begin{bmatrix} 0, \end{bmatrix} \\
\begin{bmatrix} A, & F_2 & a_{z_i} \end{bmatrix} & \begin{bmatrix} w'' \end{bmatrix} = \begin{bmatrix} 0, \end{bmatrix}
\end{align*}
\]

(4.4.23)

Combining (4.4.22) and (4.4.23) implies that
\[
\begin{bmatrix}
B, A, -1 \\
0, D, \\
C, F, c_2
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_j \\
w^t
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

admits a solution. Then since \((A_i, B_i)\) is controllable and

\[
\operatorname{rank}
\begin{bmatrix}
C, F_i \\
A_i, F_2
\end{bmatrix}
= n_i + f
\]

(4.4.21) follows.

(Sufficiency)

Assume (4.4.21) is true. Then there exists a vector

\((x, x_2, x_3, x_4)^T\) satisfying

\[
\begin{bmatrix}
B, A_i \\
0, D_i
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_j
\end{bmatrix}
= \begin{bmatrix}
x_3 \\
0
\end{bmatrix}
\]

(4.4.24)

and

\[
\begin{bmatrix}
C, F_i, c_2 \\
A_i, F_2, a_2
\end{bmatrix}
\begin{bmatrix}
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(4.4.25)

By (4.4.25), \((x_3, x_4)^T \in \mathcal{N}\). But since

\[
\begin{bmatrix}
C, F_i \\
A_i, F_2
\end{bmatrix}
\]

has full column rank, \((x_3, x_4)^T\) is unique up to multiplicity. Thus from (4.4.15), \(x_3 = w^t\). Then since \((A_i, B_i)\) is controllable and
\[
\text{rank } \begin{bmatrix} C_i & F_i \\ A_i & F_2 \end{bmatrix} = n_i + f
\]

(4.4.20) follows.

Thus the validity of (4.3.8) can be checked using the equivalent condition

\[
\begin{bmatrix} B_i & A_i & -I \\ 0 & D_i \end{bmatrix} \quad \text{rank } \begin{bmatrix} B_i & A_i & -I \\ 0 & D_i \end{bmatrix} = \text{rank } \begin{bmatrix} B_i & A_i & -I \\ 0 & D_i \end{bmatrix} = 2n_i + f \quad (\gamma)
\]

for

\[
\begin{bmatrix} F_i & c_{2i} \\ F_2 & a_{2i} \end{bmatrix}
\]

defined by \( w \in \tilde{W}^U \) as in (4.4.19).

Using the procedure of Appendix II check (4.4.21) for each \( w_i \), \( i \in \{1, \ldots, K\} \) of (4.4.17). If (4.4.21) fails for any \( i \in \{1, \ldots, K\} \), RSFP is not structurally solvable; end of algorithm.

Step 11: Using the procedure of Appendix II check (4.4.9) for \( i \in \{1, \ldots, l_{K_2}\} \). If (4.4.9) fails, RSFP is not structurally solvable; end of algorithm.

Step 14: RSFP is structurally solvable.
4.5 Structural Extension

Consider the system described by (4.3.2) and the results of Lemma 4.3.2. Structural solvability of RSFP can fail for two reasons. In particular, the first rank equality of either (4.3.8) or (4.3.9) can fail or either of the second equalities can fail. In the first case RPIS is not structurally solvable, by Theorem 4.2.1 and no solution is possible via state space extension. This follows from Theorem 2 of [3] which states that extended RPIS is not solvable if RPIS is not solvable. However if the first rank equality of (4.3.8) and (4.3.9) are true but either of the second fails, it may be possible to extend in such a way that the extended system satisfies the assumptions and conditions of Lemma 4.3.2.

Assume (4.3.4), (4.3.5) and (4.3.6) are true and the system maps induced in the factor space $\bar{X}=X/η∩X(A)$ have the form

$$
A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}
$$

(4.5.1)

where $A, B, D$ and $C$ have the detailed form of (4.3.2). Further assume

RPIS is structurally solvable

(4.5.2)
RSFP is not structurally solvable  \[(4.5.3)\]

Extend the system \[(4.5.1)\] with \(t\) new state variables as

\[
\dot{x}_c = A_c x_c + B_c C x
\]

\[(4.5.4)\]

where \(X = \mathcal{X} \otimes \mathcal{X}_c\), \(A_c : \mathcal{X}_c \rightarrow \mathcal{X}_c\) and \(B_c : \mathcal{Y} \rightarrow \mathcal{X}_c\). The extended system has the form

\[
A_{i}^e = \begin{bmatrix}
A_i & 0 \\
B_c C_i & A_c
\end{bmatrix},
B_{i}^e = \begin{bmatrix}
R_i \\
0
\end{bmatrix},
A_{3}^e = \begin{bmatrix}
A_i \\
B_c C_i
\end{bmatrix},
\]

\[
D_{i}^e = \begin{bmatrix}
D_i & 0
\end{bmatrix},
D_{3}^e = D_2
\]

\[
C_{i}^e = \begin{bmatrix}
C_i & 0 \\
0 & 1
\end{bmatrix},
C_{3}^e = \begin{bmatrix}
C_2
\end{bmatrix}
\]

\[(4.5.5)\]

Let \(R^n\) be the parameter space associated with the system \[(4.5.5)\]. \(d(R^n)\) includes the number of nonzero entries in \((A_c, B_c)\).

**Extended Structural RSFP (ESRSFP)**

Given \((A, B, C, D)\); under what conditions does there exist a state space extension \((A_c, B_c)\) such that for \((A^e, B^e, C^e, D^e)\) RSFP is structurally solvable with respect to the parameter space \(R^n\).

For the specific class of systems considered in this section, the solution to the extended structural RSFP is contained in;
Theorem 4.5.1

Assume (4.3.4), (4.3.5) and (4.3.6) are true for (4.5.1). ESRSFP has a solution if and only if there exists an extension $(A_c, B_c)$ such that relative to $R^n$

$(A_i^e, B_i^e)$ is structurally controllable (4.5.6)

$(A_i^e, C_i^e)$ is structurally observable (4.5.7)

$$\text{rank} \begin{bmatrix} A_i^e & B_i^e \\ D_i^e & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} A_i^e & B_i^e & a_{i\iota}^e \\ D_i^e & 0 & d_{i\iota}^e \end{bmatrix} = n_i + t \quad (\text{r})$$

for $l=1, \ldots, k_i$, and

$$\text{rank} \begin{bmatrix} B_i^e & A_i^e \\ D_i^e \\ -1 & B_i^e & A_i^e \\ D_i^e \end{bmatrix} = \text{rank} \begin{bmatrix} B_i^e & A_i^e & a_{i\iota}^e \\ D_i^e \\ -1 & B_i^e & A_i^e \\ D_i^e \end{bmatrix} = 2(n_i + t) \quad (\text{s})$$

for $l=1, \ldots, k_i$, where $d_{i\iota}^e = d_{i\iota}^c$, $d_{2i}^e = d_{2i}^c$ and

$$a_{i\iota}^e = \begin{bmatrix} a_{i\iota}^c \\ B_c C_i^c \end{bmatrix} \quad \text{and} \quad a_{2i}^e = \begin{bmatrix} a_{2i}^c \\ B_c C_{2i}^c \end{bmatrix}$$

Proof:

This result follows directly from the form of (4.5.1) and Lemma 4.3.2 under the assumption that $\eta = 0$.

Structural Extension Algorithm

Define the following matrices

$$B_n = B_i, \quad A_n = A_i, \quad D_n = D_i, \quad \text{and} \quad D_{2i} = D_{2i}^c$$

Step 1: Using FZRF determine the generic rank of
\[
\begin{bmatrix}
B, A_i \\
0, D_i
\end{bmatrix} \quad (4.5.10)
\]

From (4.3.4) this rank is \( n_i + t_i \), where \( t_i \geq 0 \). If \( t_i = 0 \) define the index set \( J'' = \{1\} \) and go to Step 9. Otherwise continue.

Step_2: Using FZRF isolate the \( t_i \) generically independent rows of \( D_i \) in (4.5.10). Denote these rows as

\[
\tilde{D} = \begin{bmatrix}
\tilde{D}_1 \tilde{D}_2
\end{bmatrix}. \quad (4.5.11)
\]

Step_3: Consider all groups of rows of \( C_i \), which are generically independent with respect to the rows of \((B_i, A_i)\). In particular using FZRF determine all subsets of the \( h \) rows of \( C \) such that

\[
\text{rank} \begin{bmatrix}
B_i, A_i \\
0, S_{ij}
\end{bmatrix} = n_i + r_j \quad (g) \quad (4.5.12)
\]

for \( j = 1, \ldots, q \), where \((S_{ij} , S_{2j})\) is a \( r_j \) rowed subset of the rows of \( C \) and

\[
q = \sum_{k=0}^{h-1} \frac{h!}{((h-k)!k!)}.\]

In actual practice this number of rank checks is generally not required since any row of \( C \) generically dependent on the rows of \((B_i, A_i)\) results in \( 2^{h-1} \) sets falling (4.5.12).
Lemma 4.5.1

If no $S_{ij}$, $1 \leq j \leq q$ exists such that (4.5.12) is true, then ESRSFP is not solvable.

Proof:

Assume no $S_{ij}$, $1 \leq j \leq q$ exists such that (4.5.12) is true.

Then

$$\begin{bmatrix} B_i & A_i \\ 0 & B_c S_{ij} & A_c \end{bmatrix} = \begin{bmatrix} B_i & A_i \\ 0 & 0 & A_c \end{bmatrix} = n_i + t_i + \text{rank}(A_c) \quad (4,5,13)$$

for any $j \in \{1, \ldots, q\}$, and

$$\text{rank} \begin{bmatrix} B_i & A_i \\ B_c S_{ij} & A_c \end{bmatrix} = \text{rank} \begin{bmatrix} B_i & A_i \\ 0 & A_c \end{bmatrix} = n_i + \text{rank}(A_c) \quad (4,5,14)$$

for any $j \in \{1, \ldots, q\}$. Now (4.5.6) implies from (4.5.14) that $\text{rank}(A_c) = t$. But (4.5.8) implies from (4.5.13) that $\text{rank}(A_c) < t$ since $t_i > 0$. From this contradiction, ESRSFP is not solvable.

\[ \triangle \]

If no $S_{ij}$, $1 \leq j \leq q$ exists such that (4.5.12) is true, ESRSFP is not solvable; end of algorithm. Otherwise assume there exists an index set $J \subseteq \{1, \ldots, q\}$ such that for all $j \in J$, (4.5.12) is true.

Step 4: Using FZRF determine the generic rank of
\[
\begin{bmatrix}
B_i \ A_i \\
0 \ S_{ij} \\
0 \ D_i
\end{bmatrix}
\quad (4.5.15)
\]

for all \( j \in J \).

Lemma 4.5.2

If no \( S_{ij}, j \in J \) exists such that the generic rank of (4.5.15) is \( n_i + r_j \), then ESRSFP is not solvable.

Proof:

By assumption

\[
\text{rank}
\begin{bmatrix}
B_i \ A_i \\
S_{ij} \\
D_i
\end{bmatrix}
> n_i + r_j \quad (g) 
\quad (4.5.16)
\]

for every \( j \in J \). Assume ESRSFP has a solution. Then from (4.5.6) and (4.5.8),

\[
n_i + t = \text{rank}
\begin{bmatrix}
B_i \ A_i \\
B_c S_{ij} \ A_c
\end{bmatrix} = \text{rank}
\begin{bmatrix}
B_i \ A_i \\
B_c S_{ij} \ A_c \\
n_i
\end{bmatrix} \quad (g)
\]

for some \( j \in \{1, \ldots, q\} \). But by Lemma 4.5.1 no \( j \in J^c \) solves ESRSFP. Thus for some \( j \in J \) there exists a matrix such that
\[
\begin{bmatrix}
B_i & A_i \\
S_{ij} & D_i
\end{bmatrix}
= 0
\]

or
\[
\begin{bmatrix}
B_i & A_i \\
S_{ij} & D_i
\end{bmatrix}
= 0
\]

This contradicts (4.5.16), and therefore ESRSFP is not solvable.

\[\square\]

If no \( S_{ij}, j \in J \) exists such that the generic rank of (4.5.15) is \( n_i + r_j \), ESRSFP is not solvable; end of algorithm. Otherwise assume for all \( j \in J' \subseteq J \), (4.5.15) has generic rank \( n_i + r_j \).

**Step 5:** If \( k_i = 0 \), let \( J'' = J' \) and go to **Step 8**.

**Step 6:** Using FZRF determine for each \( j \in J' \) the generic rank of

\[
\begin{bmatrix}
B_i & A_i & a_{ij} \\
S_{ij} & S_{ij}' & S_{ij}'' \\
\bar{D}_i & \bar{D}_{ij} & \bar{D}_{ij}'
\end{bmatrix}
\]  
(4.5.17)

for \( i = 1, \ldots, k_i \), where \( S_{ij} = [s_{ij}, s_{ij}', s_{ij}'' \ldots s_{ij}'', 0, \ldots] \).

Lemma 4.5.3

If no \( j \in J' \) exists such that the generic rank of (4.5.17) is \( n_i + r_j \), then ESRSFP is not solvable.
Proof:

By assumption for every $j \in J'$

$$\begin{bmatrix} B_i & A_i & a_{i\lambda} \\ S_{ij} & s_{i\lambda} \\ \bar{D}_i & \bar{d}_{i\lambda} \end{bmatrix} \quad \text{rank} \quad n_i + r_j \quad (g) \quad (4.5.18)$$

for some $i \in \{1, \ldots, k_i\}$. Assume ERSFSP is solvable. Then from (4.5.6) and (4.5.8)

$$n_i + t = \text{rank} \begin{bmatrix} B_i & A_i \\ B_c S_{ij} & A_c \end{bmatrix} = \text{rank} \begin{bmatrix} B_i & A_i \\ B_c S_{ij} & A_c \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} B_i & A_i & a_{i\lambda} \\ B_c S_{ij} & A_c & B_c s_{i\lambda} \\ \bar{D}_i & \bar{d}_{i\lambda} \end{bmatrix} \quad (p) \quad (4.5.19)$$

for all $i \in \{1, \ldots, k_i\}$, for some $j \in \{1, \ldots, n_j\}$. But from Lemma 4.5.2, no $j \in J'^c$ solves ERSFSP. Therefore (4.5.19) is true for some $j \in J'$. Then from (4.5.19) there exists a matrix of order $(t, x_n, t + t_i)$ such that

$$(G_i, G_2, 1) \begin{bmatrix} B_i & A_i & a_{i\lambda} \\ B_c S_{ij} & B_c s_{i\lambda} \\ \bar{D}_i & \bar{d}_{i\lambda} \end{bmatrix} = 0$$

for all $i \in \{1, \ldots, k_i\}$ for some $j \in J'$. Therefore
which contradicts (4.5.18).

\[ \text{If no } j \in J' \text{ exists such that the generic rank of } (4.5.17) \text{ is } n_i + r_j, \text{ ERSSEP is not solvable; end of algorithm. Otherwise assume all } j \in J'' \subseteq J', \text{ (4.5.17) has generic rank } n_i + r_j. \]

**Step 7:** If \( k_2 \neq 0 \) go to **Step 8**. Otherwise define the extended system

\[
\begin{align*}
A_{ij} &= \begin{bmatrix} A_1 & 0 \\ S_{ij} & 0 \end{bmatrix}, \\
B_{ij} &= \begin{bmatrix} R_i \\ 0 \end{bmatrix}, \\
A_{3j} &= \begin{bmatrix} A_3 \\ S_{3j} \end{bmatrix}, \\
D_{ij} &= \begin{bmatrix} D_i & 0 \end{bmatrix}, \\
D_{3j} &= D_2, \\
C_{ij} &= \begin{bmatrix} C_i & 0 \\ 0 & I \end{bmatrix}, \\
C_{3j} &= \begin{bmatrix} C_2 \end{bmatrix}
\end{align*}
\]

(4.5.20)

with \( n_i^j = n_i + r_j \). The structural controllability and observability of the extended plant of (4.5.20) follows from Lemma 3.1.3. For controllability consider \( s \in \mathcal{G}, s \neq 0 \). Then

\[
\text{rank } (A_{ij} - sI \ B_{ij}) = \text{rank } \begin{bmatrix} A_1 - sI & 0 & B_1 \\ S_{ij} & -sI \end{bmatrix} = r_j + \text{rank } (A_1 - sI \ B_1) \quad (g)
\]

With \( s = 0 \)
\[ \text{rank}(A_{ij} R_{ij}) = \text{rank} \begin{bmatrix} A_i & R_i \\ S_{ij} & 0 \end{bmatrix} = n_i + r_j \] (g)

Therefore \((A_{ij}, R_{ij})\) is structurally controllable.

For observability consider any \(s \in \mathcal{C}\). Then

\[ \text{rank} \begin{bmatrix} A_{ij} - sI \\ C_{ij} \end{bmatrix} = \text{rank} \begin{bmatrix} A_i - sI & 0 \\ S_{ij} & -sI \\ C & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} A_i - sI \\ C_i \\ 0 \end{bmatrix} + r_d \] (g)

Therefore \((A_{ij}, C_{ij})\) is structurally observable. Also

\[ \text{rank} \begin{bmatrix} B_{ij} A_{ij} \\ 0 & D_{ij} \end{bmatrix} = \text{rank} \begin{bmatrix} B_i & A_i & 0 \\ 0 & S_{ij} & 0 \\ 0 & D_i & 0 \end{bmatrix} \]

\[ = \text{rank} \begin{bmatrix} B_i & A_i & 0 & a_{ij} \\ 0 & S_{ij} & 0 & s^j_i \end{bmatrix} + r_d \] (g)

for all \(i \in \{1, \ldots, k_j\}\), and so RSFP is structurally solvable for (4.5.20). Choose the extended system of least dimension; end of algorithm.

Step 8: For each \(j \in J\) define the extended system of order \((n, +2r_j + n_z)\),

\[ A_{ij} = \begin{bmatrix} A_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{ij} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad C_{ij} = \begin{bmatrix} A_i \\ 0 \end{bmatrix}, \quad S_{ij} = \begin{bmatrix} S_{ij} \\ 0 \end{bmatrix}, \quad D_{ij} = \begin{bmatrix} D_i & 0 & 0 \end{bmatrix} \]

\[ D_{ij} = \begin{bmatrix} D_j \\ 0 \end{bmatrix}, \quad D_{ij} = D_2 \] (4.5.21)
\[
\begin{bmatrix}
C_i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \quad \quad
\begin{bmatrix}
C_i \\
0 \\
0
\end{bmatrix}
\]

where \( A_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3, 0, \ldots, a_{ij}^{k_j}) \). The structural controllability and observability of the plant of (4.5.21) is shown below. For controllability consider \( s \in \mathbb{C}, s \neq 0 \). Then

\[
\text{rank}(A_{ij}, -sl, B_{ij}) = \text{rank}
\begin{bmatrix}
A_i & -sl & 0 & 0 & B_i \\
0 & -sl & 1 & 0 \\
S_{ij} & 0 & -sl & 0
\end{bmatrix}
= n_i + 2r_j
\]

For \( s=0, \) \( \text{rank}(A_{ij}, B_{ij}) = n_i + 2r_j \) (as by the way the \( S_{ij} \) sets were chosen. Therefore \( (A_{ij}, B_{ij}) \) is structurally controllable since \( (A_i, B_i) \) is structurally controllable. For observability consider any \( s \in \mathbb{C} \). Then

\[
\text{rank}
\begin{bmatrix}
A_{ij}^* & -sl \\
C_{ij}
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
A_i & -sl & 0 & 0 \\
0 & -sl & 1 \\
S_{ij} & 0 & -sl \\
C_i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= n_i + 2r_j
\]

Therefore \( (A_{ij}, C_{ij}) \) is structurally observable since \( (A_i, C_i) \) is structurally observable. Note also that (4.5.21) satisfies (4.5.8); i.e.,
\[
\text{rank}\left[ \begin{array}{cc}
A_{ij} & B_{ij} \\
D_{ij} & 0 \\
\end{array} \right] = \text{rank}\left[ \begin{array}{cccc}
A_i & 0 & 0 & B_i \\
0 & 0 & 1 & 0 \\
S_{ij} & 0 & 0 & 0 \\
D_i & 0 & 0 & 0 \\
\end{array} \right] = n_i + 2r_{ij} \quad (g)
\]

from Step_4 since \(j \in J'\) and
\[
\text{rank}\left[ \begin{array}{cc}
A_{ij} & B_{ij} \quad a_{ij} \ \\
D_{ij} & 0 \quad d_{ij} \\
\end{array} \right] = \text{rank}\left[ \begin{array}{cccc}
A_i & 0 & 0 & R_i \quad a_{ij} \\
0 & 0 & 1 & 0 \\
S_{ij} & 0 & 0 & s_{ij} \\
D_i & 0 & 0 & d_{ij} \\
\end{array} \right] = n_i + 2r_{ij} \quad (g)
\]

from Step_6 since \(j \in J''\).

Step_9: Use the procedure of Appendix II to determine, for every \(j \in J''\), the generic rank of the matrix
\[
\left[ \begin{array}{cc}
B_{ij} & A_{ij} \\
D_{ij} & -1 & B_{ij} & A_{ij} \\
\end{array} \right] \quad (4.5.22)
\]

The generic rank of (4.5.22) is \(2n_i + t_{2j}\), where \(t_{2j} > 0\). For all \(j \in J''\) for which \(t_{2j} = 0\), ERSFEP is solved. Choose the extension of (4.5.21) of least dimension; end of algorithm. If \(t_{2j} > 0\) for every \(j \in J''\), continue.

Step_10: Let \(j \in J''\). From (4.5.22) determine the \(t_{2j}\) generically independent rows of \(D_{ij}\), using Appendix II, such that
\[
\text{rank} \begin{bmatrix}
B_{ij} & A_{ij} \\
-1 & B_{ij} & A_{ij} \\
\]

with
\[
\hat{n}_j = (\hat{n}_j, \hat{S}_{ij}) = (\hat{d}_{ij}, \hat{d}_{ik}, \hat{d}_{ik}, \hat{d}_{ik}, 0, \ldots, \hat{d}_{ik}, 0)
\]

**Step 11:** Using the procedure of Appendix II determine all subsets \( S_{ik} \) of the rows of \( C_{ij} \) disjoint from \( (S_{ij} 0 0) \) such that

\[
\text{rank} \begin{bmatrix}
B_{ij} & A_{ij} \\
& S_{ik} & A_{ij} \\
\]

Let \( \bar{K} \) denote the index set of all rows of \( C_{ij} \) disjoint from \( (S_{ij} 0 0) \).

**Lemma 4.5.4**

If no \( S_{ik}, k \in \bar{K} \) exists such that (4.5.23) is true, then ERSFSP is not solvable.

**Proof:**

Assume no \( S_{ik}, k \in \bar{K} \) exists such that (4.5.23) is true.

Then

\[
\text{rank} \begin{bmatrix}
B_{ij} & A_{ij} \\
0 & S_{ik} \\
\end{bmatrix} = n^j_i (g) \quad (4.5.24)
\]

for otherwise there would exist a \( k \in \bar{K} \) satisfying (4.5.23).
Assume ESRSFP is solvable. Then from (4.5.6) there exists a $S_{ik}, k \in \{1, \ldots, n\}$ such that

$$\text{rank} \begin{bmatrix} B_{ij} & A_{ij} & 0 \\ 0 & B_c S_{ik} & A_c \end{bmatrix} = n_i^j + t (g)$$

This implies that for $k \in \tilde{K}$ rank $A_c = t (g)$ from (4.5.24). From (4.5.9),

$$2(n_i^j + t) = \text{rank} \begin{bmatrix} B_{ij} & A_{ij} \\ \\ B_c S_{ik} & A_c \\ \hat{D}_{ij} \\ -1 & B_{ij} & A_{ij} & 0 \\ -1 & 0 & B_c S_{ik} & A_c \\ \hat{D}_{ij} \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} B_{ij} & A_{ij} \\ 0 & A_c \\ -1 & 0 & B_{ij} & A_{ij} & 0 \\ -1 & B_c S_{ik} & A_c \\ \hat{D}_{ij} \end{bmatrix} = 2n_i^j + t_{ij} + 2t (g)$$

This is a contradiction since $t_{ij} > 0$. Therefore ESRSFP is not solvable.

\[ \blacksquare \]

If no $S_{ik}, k \in \tilde{K}$ exists such that (4.5.63) is true, then repeat Step_10 with an unconsidered $j \in J''$. Otherwise let $K \subseteq \tilde{K}$ specify those sets of rows of $C_{ij}$
for which (4.5.23) is true and continue.

Step 12: Using the procedure of Appendix II determine the generic rank of

\[
\begin{bmatrix}
B_{ij} & A_{ij} \\
S_{ik} & \hat{D}_{ij} \\
-1 & B_{ij} & A_{ij} \\
S_{ik} & \hat{D}_{ij}
\end{bmatrix}
\] (4.5.25)

for all \( k \in K \).

Lemma 4.5.5

If no \( S_{ik}, k \in K \) exists such that (4.5.25) has generic rank \( 2(n^i_j + r_k) \), then ESRSFP is not solvable.

Proof:

Assume no \( S_{ik}, k \in K \) exists such that (4.5.25) has generic rank \( 2(n^i_j + r_k) \). Then

\[
\begin{bmatrix}
B_{ij} & A_{ij} \\
S_{ik} & \hat{D}_{ij} \\
-1 & B_{ij} & A_{ij} \\
S_{ik} & \hat{D}_{ij}
\end{bmatrix}
\]

rank \( >2(n^i_j + r_k) \) (g) (4.5.26)

for every \( k \in K \). Assume ESRSFP is solvable. Then from (4.5.6), (4.5.5) and Lemma 4.5.4
\[
2(n_i^d + t) = \text{rank}\left[
\begin{array}{cccc}
B_{ij} & A_{ij} & & \\
& & B_c S_{ik} & A_c \\
& -1 & B_{ij} & A_{ij} \\
-1 & & B_c S_{ik} & A_c \\
\end{array}
\right]
\]

\[
= \text{rank}\left[
\begin{array}{cccc}
B_{ij} & A_{ij} & & \\
& & B_c S_{ik} & A_c \\
& -1 & B_{ij} & A_{ij} \\
-1 & & B_c S_{ik} & A_c \\
\hat{D}_{ij} & & &
\end{array}
\right] \tag{g} \tag{4.5.27}
\]

for some \( k \in K \). Therefore there exists a matrix such that

\[
(G, G_2 \mid G, G_4 \mid) \left[
\begin{array}{cccc}
B_{ij} & A_{ij} & & \\
& S_{ik} & & \\
& & B_{ij} & A_{ij} \\
& & S_{ik} & \\
\hat{D}_{ij} & & &
\end{array}
\right] = 0
\]

This contradicts \((4.5.26)\) and so ESPSFP is not solvable.

▲

If no \( S_{ik} \), \( k \in K \) exists such that \((4.5.25)\) has generic rank \(2(n_i^d + r_k)\), then repeat Step 10 with an unconsidered \( j \in J'' \). Otherwise assume for all \( k \in K' \subset K \), the generic rank of \((4.5.25)\) is \(2(n_i^d + r_k)\) and continue.
Step_13: Using the procedure of Appendix II determine for $k \in K'$ the generic rank of
\[
\begin{bmatrix}
B_{ij} & A_{ij} & a_{ij}^j & a_{ij}^k \\
0 & S_{ik} & s_{ik}^j & s_{ik}^k \\
\hat{D}_{ij} & \hat{D}_{ij} & \hat{D}_{ij}^j & \hat{D}_{ij}^k \\
-1 & B_{ij} & A_{ij} \\
\end{bmatrix}
\quad (4.5.28)
\]
for $i=1,\ldots,k_2$.

Lemma 4.5.6

If no $k \in K'$ exists such that the generic rank of
(4.5.28) is $2(n_i^j+r_k)$, for all $i \in \{1,\ldots,k_2\}$, then ESRSFP is not solvable.

Proof:
Assume no $(S_{ik}, S_{2k})$, $k \in K'$ exists such that (4.5.28) has generic rank $2(n_i^j+r_k)$, for every $i \in \{1,\ldots,k_2\}$. Then for every $k \in K'$
\[
\text{rank}
\begin{bmatrix}
B_{ij} & A_{ij} & a_{ij}^j & a_{ij}^k \\
0 & S_{ik} & s_{ik}^j & s_{ik}^k \\
\hat{D}_{ij} & \hat{D}_{ij} & \hat{D}_{ij}^j & \hat{D}_{ij}^k \\
-1 & B_{ij} & A_{ij} \\
\end{bmatrix}
> 2(n_i^j+r_k) \quad (g) (4.5.29)
\]
for some \( i \in \{1, \ldots, k_2\} \).

Assume ESRSFP is solvable. Then for some \( k \in \{1, \ldots, q\} \)

\[
2(n_i^j + t) = \text{rank} \begin{bmatrix}
B_{ij} & A_{ij} \\
 & & B_c S_{ik} & A_c \\
 & -1 & B_{ij} & A_{ij} \\
& -1 & 0 & B_c S_{ik} & A_c \\
\end{bmatrix}
\]

\[
= \text{rank} \begin{bmatrix}
B_{ij} & A_{ij} \\
 & & B_c S_{ik} & A_c & \hat{d}_{ij} \\
 & -1 & B_{ij} & A_{ij} \\
& -1 & 0 & B_c S_{ik} & A_c \\
\end{bmatrix}
\]

\[
= \text{rank} \begin{bmatrix}
B_{ij} & A_{ij} & a_{ij}^j \\
 & & B_c S_{ik} & A_c & \hat{d}_{ij} \\
 & -1 & B_{ij} & A_{ij} & \hat{d}_{ij}^j \\
& -1 & 0 & B_c S_{ik} & A_c \\
\end{bmatrix}
\]  \( (p) \)  \( (4.5.30) \)

for all \( i \in \{1, \ldots, k_2\} \). Since \( (4.5.30) \) is not true for any \( k \in K'^c \) from Lemma 4.5.5, assume \( k \in K' \). Equation \( (4.5.30) \) implies there exists a matrix of order \( (2t_i^j \times 2(n_i^j + t + t_{ij})) \) such that
\[ (G_1, G_2, 1, G_3, G_{k_i}) \begin{bmatrix} B_{i,j} & A_{i,j} & a_{i,j}^k \\ S_{1,k} & S_{k_i}^j & s_{k_i}^j \\ \delta_{i,j} & \delta_{k_i}^j & \delta_{k_i}^j \\ -1 & B_{i,j} & A_{i,j} \\ S_{1,k} & S_{k_i}^j & s_{k_i}^j \\ \delta_{i,j} & \delta_{k_i}^j & \delta_{k_i}^j \end{bmatrix} = 0 \]

for all \( i \in \{1, \ldots, k_i\} \). But this implies

\[ \begin{bmatrix} B_{i,j} & A_{i,j} & a_{i,j}^k \\ S_{1,k} & S_{k_i}^j & s_{k_i}^j \\ \delta_{i,j} & \delta_{k_i}^j & \delta_{k_i}^j \\ -1 & B_{i,j} & A_{i,j} \\ S_{1,k} & S_{k_i}^j & s_{k_i}^j \\ \delta_{i,j} & \delta_{k_i}^j & \delta_{k_i}^j \end{bmatrix} \preceq 2(n_i^j + r_k) \]

This contradicts (4.5.29) and so ESPSFP is not solvable. \[ \triangle \]

Therefore if no \( k \in K' \) exists such that the generic rank of (4.5.28) is \( 2(n_i^j + r_k) \), then repeat Step 10 with an unconsidered \( j \in J'' \). Otherwise assume for all \( k \in K'' \subseteq K' \), (4.5.28) has generic rank \( 2(n_i^j + r_k) \), and continue.

**Step 14:** For each \( k \in K'' \) form the extended system

\[
A_i^e = \begin{bmatrix} A_{i,j} & 0 \end{bmatrix}, \quad B_i^e = \begin{bmatrix} B_{i,j} \\ 0 \end{bmatrix},
\]

\[
D_i^e = \begin{bmatrix} D_{i,j} & 0 \end{bmatrix}, \quad D_2^e = D_{2,j}
\]

(4.5.31)
The system of (4.5.31) is structurally controllable and observable. Following Step_7, consider \( s \in \mathcal{C}, s \neq 0 \).

Then

\[
\text{rank}(A_i^e-sI \ B_i^e) = \text{rank} \begin{bmatrix}
A_{ij} - sI & 0 & B_{ij} \\
S_{ik} & -sI & 0
\end{bmatrix} = n_i^j + r_k
\]

For \( s=n_i^j \),

\[
\text{rank}(A_i^e B_i^e) = \text{rank} \begin{bmatrix}
A_{ij} & 0 & B_{ij} \\
S_{ik} & 0 & 0
\end{bmatrix} = n_i^j + r_k
\]

from the way \( S_{ik} \) was chosen. Therefore \((A_i^e, B_i^e)\) is structurally controllable for any \( k \in \mathbb{K}^\prime \) since \((A_{ij}, B_{ij})\) is structurally controllable. For observability consider any \( s \in \mathcal{C} \). Then

\[
\text{rank} \begin{bmatrix}
A_i^e - sI \\
C_i^e
\end{bmatrix} = \text{rank} \begin{bmatrix}
A_{ij} - sI & 0 \\
S_{ik} & -sI \\
C & 0 \\
0 & 1
\end{bmatrix} = n_i^j + r_k
\]

Therefore \((C_i^e, A_i^e)\) is structurally observable since \((A_{ij}, C_{ij})\) is structurally observable. Similarly (4.5.9) is true by the way \( S_{ik} \) was chosen in Step_12 and Step_13.

Repeat Step_10 for any unconsidered \( j \in \mathbb{J}^\prime \).

Step_15 If after all \( j \in \mathbb{J}^\prime \) have been considered no extension of the form (4.5.31) exists, then ESRSFP is not
solvable; end of algorithm. Otherwise go to Step_16.

Step_16: Choose the extended system having the least dimension.

Under the assumption of (4.1.1) certain simplifications of the above algorithm are possible. In particular delete Steps 3, 4, 5 and 6, and replace Steps 7 and 8 with:

Step_7': If $k_2 \neq 0$ go to Step_8'. Otherwise extend with $t$, new state variables as

$$
A_i^e = \begin{bmatrix} A_i & 0 \\ \bar{D}_i & 0 \end{bmatrix}, \quad B_i^e = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad C_i^e = \begin{bmatrix} C_i & 0 \end{bmatrix}, \quad C_2^e = \begin{bmatrix} C_2 \\ 0 \end{bmatrix}, \quad A_3^e = \begin{bmatrix} A_3 \\ \bar{D}_2 \end{bmatrix}
$$

$$
D_i^e = \begin{bmatrix} D_i & 0 \end{bmatrix}, \quad n_2^e = n_2
$$

with $n_i^e = n_i + t_i$. This extension simply attaches integrators to the $t_i$ measurable outputs $\bar{y}$. Thus the description of the new state variables in the extended plant and the $\bar{y}$ outputs perturb together. The extended plant of (4.5.32) is structurally controllable. That is

$$
\text{rank}(A_i^e - sI, B_i^e) = \text{rank} \begin{bmatrix} A_i - sI & 0 & B_i \\ \bar{D}_i & -sI & 0 \end{bmatrix} = n_i + t_i, \quad (g)
$$

for any $s \in \mathbb{C}$ since $(A_i, B_i)$ is structurally controllable. Structural observability follows since
\[
\begin{bmatrix}
A_i^e & -sI & 0 \\
C_i^e & & \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
A_i & -sI & 0 \\
0 & -sI & \\
0 & 0 & 1
\end{bmatrix}
\text{ for any } s \in \mathbb{C} \text{ since } (C_i, A_i) \text{ is structurally observable.}
\]

Also
\[
\begin{bmatrix}
A_i^e & B_i^e \\
D_i^e & 0
\end{bmatrix}
= \begin{bmatrix}
A_i & 0 & B_i \\
D_i & 0 & 0
\end{bmatrix}
\text{ from Step_2 and the assumption that the extended plant matrix and the } D_i \text{ outputs perturb together.}
\]

Similarly it follows that (4.5.9) is satisfied for (4.5.32).

**Step_8**: Extend with 2t, new state variables as

\[
A_i^e = \begin{bmatrix}
A_i & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
B_i^e = \begin{bmatrix}
B_i \\
0 \\
0
\end{bmatrix}
C_i^e = \begin{bmatrix}
C_i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
C_2^e = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
A_3^e = \begin{bmatrix}
a_1 \cdots a_{1k} & a_2 & 0 \cdots a_{2k} & 0 \\
0 & \cdots & 0 & 0 & 0 \\
\bar{d}_1 & \cdots & \bar{d}_{1k} & \bar{d}_2 & 0 \cdots \bar{d}_{2k} & 0
\end{bmatrix}
D_1^e = \begin{bmatrix}
D_1 & 0 & 0 \\
D_2 & 0 & 0
\end{bmatrix}
D_2^e = D_2
\]

where \( \bar{D}_i \) is of order \((t, x_n)\) and with \( n_i^e = n_i + 2t \). This extension is obtained simply by attaching integrators to the \( t \), measurable outputs \( \bar{D} \). As in Step_7 the description of these new state variables perturb with
the output matrix. Therefore (4.5.8) becomes

\[
\begin{bmatrix}
R, A, 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & D, 0 & 0 \\
0 & D, 0 & 0 \\
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
R, A, 0 & 0 & a_i^e \\
0 & 0 & 0 & 1 & 0 \\
0 & D, 0 & 0 & d_i \tilde{D} & 0 \\
0 & D, 0 & 0 & d_i^e & 0 \\
\end{bmatrix}
= n, +2t, (g)
\]

for \( i=1, \ldots, k \). Similarly it is easy to show that under this assumption of simultaneous perturbations (4.5.6) and (4.5.7) are true. Also (4.5.9) is true for the outputs \( \tilde{D} \). Therefore if any further extension is required it is the result of other rows of \( D \).

Define the regulated outputs of the extended system (4.5.33) as \( \hat{D} \); i.e., \( D^e \) with the \( \tilde{D} \) rows removed.

Continue the previous algorithm with

\[
B_{10} = B_i^e \quad A_{10} = A_i^e \quad C_{10} = C_i^e \quad C_{20} = C_2^e \quad A_{30} = A_3^e
\]

\[
D_{10} = (\hat{D}, 0) \quad D_{20} = \hat{D}_2 \quad S_{10} = \tilde{D}_i
\]

and with \( J^u = \{ n \} \).

As an example of this procedure consider the system described by the matrices

\[
A_1 = \begin{bmatrix}
0 & 0 \\
a, & 0 \\
0 & 0
\end{bmatrix} \quad B_1 = \begin{bmatrix} b_i \\
0 \\
0
\end{bmatrix} \quad A_2 = \begin{bmatrix} a_2 & 0 \\
0 & 0 \\
0 & 0 
\end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
D_1 = \begin{bmatrix} d_1, n \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \quad C = I_w
\]

Note that the outputs to be regulated are directly
measurable.

**Step_1:**

\[
\begin{bmatrix} B, A \end{bmatrix} \begin{bmatrix} 0, & d \end{bmatrix} = \begin{bmatrix} b, & 0, & 0 \end{bmatrix} \begin{bmatrix} 0, & a, & 0 \end{bmatrix} = 2 \quad (g)
\]

Thus \( t_1 = 0 \); go to **Step_9**.

**Step_9:** Since \( k_2 = 1 \neq 0 \), (4.4.15) becomes

\[
\begin{bmatrix} b, & 0, & 0 \\
0, & a, & 0 \\
d, & 0 \\
-1, & 0, & b, & 0, & 0 \\
-1, & 0, & a, & 0 \\
& & & d, & 0 \\
\end{bmatrix}
\]

Therefore \( t_2 = t_2 = 1 \).

**Step_10:** Obviously \( \hat{D} = 0 \).

**Step_11:** Now \( C = I_2 \), so

\[
S_{11} = (1, 0), \quad S_{12} = (0, 1) \quad \text{and} \quad S_{13} = I_2.
\]

Then (4.4.16) yields for \( S_{11} \),

\[
\begin{bmatrix} b, & 0, & 0 \\
0, & a, & 0 \\
1, & 0 \\
-1, & 0, & b, & 0, & 0 \\
-1, & 0, & a, & 0 \\
& & & 1, & 0 \\
\end{bmatrix}
\]

\[\text{rank} \quad = 5\]
Therefore $1 < K$, and $3 < K$. For $S_{12}$

$$\begin{bmatrix}
    b, & 0 & 0 \\
    0 & a, & 0 \\
    1 & 0 \\
    -1 & 0 & b, & 0 & 0 \\
    -1 & 0 & a, & 0 \\
    1 & 0
\end{bmatrix} \text{ rank } = 6 \ (g)$$

so $K=\{2\}$.

Step 12: For $K=\{2\}$, $(4,4,19)$ becomes

$$\begin{bmatrix}
    b, & 0 & 0 \\
    0 & a, & 0 \\
    0 & 1 \\
    d, & 0 \\
    -1 & 0 & b, & 0 & 0 \\
    -1 & 0 & a, & 0 \\
    0 & 1 \\
    d, & 0
\end{bmatrix} \text{ rank } = 6 \ (g)$$

Step 13: Since $D_2=(0 \ 0)$, $(4,4,70)$ has generic rank 6.

Step 14: The solution to ESRSFP is

$$A_i^e = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad B_i^e = \begin{bmatrix} b_i \end{bmatrix}, \quad C_i^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_i^e = \begin{bmatrix} d_i & 0 & 0 \end{bmatrix}, \quad D_2^e = \begin{bmatrix} 0 & 0 \end{bmatrix} \text{ and } C^e = 15.$$
4.6 Examples

Example 1

Consider the structural controllability of the following system:

\[
\begin{bmatrix}
  a_1 & 0 & 0 & 0 \\
  0 & a_2 & a_3 & a_4 \\
  0 & a_5 & a_6 & a_7 \\
  0 & 0 & 0 & a_8 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  b_1 & 0 \\
  0 & b_2 \\
  0 & 0 \\
  0 & 0 \\
\end{bmatrix}
\]

The extended controllability matrix \( \tilde{R} \) of Definition 3.3.1 is

\[
\begin{bmatrix}
  b, 0 & 1 & 0 & 0 & 0 \\
  0 & b_2 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  -a, 0 & 0 & 0 & b, 0 & 1 & 0 & 0 & 0 \\
  0-a_5-a_6 & 0 & b_2 & 0 & 1 & 0 & 0 \\
  0-a_5-a_7 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 & 1 \\
  -a, 0 & 0 & 0 & b, 0 & 1 & 0 & 0 & 0 \\
  0-a_5-a_6 & 0 & b_2 & 0 & 1 & 0 & 0 \\
  0-a_5-a_7 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 & 1 \\
  -a, 0 & 0 & 0 & b, 0 \\
  0-a_5-a_6 & 0 & b_2 \\
  0-a_5-a_7 & 0 & 0 \\
  0 & 0 & 0 & -a_8 & 0 & 0 \\
\end{bmatrix}
\]
FZRF gives the structural rank of this (16x18) matrix as 15. This dependency can be seen by noting that rows 4, 8, 12 and 16 contain a block of zeros of order (4x17). From Theorem 2.4.1 these four rows must be generically dependent. Therefore the system is structurally uncontrollable.
Example 2

Consider the system

\[
\begin{align*}
\dot{x}_1 &= a_1 x_1 + a_2 x_2 \\
\dot{x}_2 &= a_3 x_1 + b_1 u + a_w w \\
z_i &= x_2 \\
y_i &= x_1 \quad y_2 = x_2
\end{align*}
\]

where \( w \) is a step disturbance. Then \( n_i = 2, \ r = 1, \ c_i = 1, \ k_w = 1 \) and

\[
A_i = \begin{bmatrix} a_1 & a_2 \\ a_3 & 0 \end{bmatrix}, \quad a_{ii} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix}
\]

\[
D_i = \begin{bmatrix} 0 & d_i \end{bmatrix}, \quad D_2 = 0
\]

Note that this system comes under the assumption of (4.4.1). From the structure of the matrix \((A, B, C)\), the plant is structurally controllable. From Lemma 4.3.3, since \( \gamma = 0 \), structural solvability of RSFP requires,

\[
\begin{align*}
\text{rank} \begin{bmatrix} B_i & A_i \\ 0 & D_i \end{bmatrix} &= \text{rank} \begin{bmatrix} B_i & a_{ii} \\ 0 & D_i \end{bmatrix} = 2 \quad (\alpha)
\end{align*}
\]

However from an application of FZRF,

\[
\begin{align*}
\text{rank} \begin{bmatrix} 0 & a_1 & a_2 \\ b_i & a_3 & 0 \\ 0 & 0 & d_i \end{bmatrix} &= \text{rank} \begin{bmatrix} 0 & a_1 & a_2 & 0 \\ b_i & a_3 & 0 & a_w \\ 0 & 0 & d_i & 0 \end{bmatrix} = 3 \quad (\beta)
\end{align*}
\]

Therefore RPIS is structurally solvable but RSFP is not.
Now applying the algorithm of Section 4.5 to determine solvability of ESRSFP yields;

**Step_1:**

\[
\begin{bmatrix}
  n & a_1 & a_2 \\
  b_1 & a_3 & 0 \\
  0 & 0 & d_1
\end{bmatrix}
\]

\[\text{rank} = 3\ (g)\]

**Step_2:**

\[\overline{D}_1 = 0_1\]

Since (4.4.1) is applicable and \(k_2 = 0\), extend as in **Step_7** yielding;

\[
\begin{align*}
A^e &= \begin{bmatrix}
  a_1 & a_2 & 0 \\
  a_3 & 0 & 0 \\
  0 & d_1 & 0
\end{bmatrix} \\
B^e &= \begin{bmatrix}
  0 \\
  b_1 \\
  0
\end{bmatrix} \\
a^e &= \begin{bmatrix}
  0 \\
  a_4 \\
  0
\end{bmatrix} \\
D^e &= \begin{bmatrix}
  0 & d_1 & 0
\end{bmatrix}
\end{align*}
\]

Then

\[
\text{rank} \begin{bmatrix}
  B^e \\
  0 & D^e
\end{bmatrix} = \text{rank} \begin{bmatrix}
  0 & a_1 & a_2 & 0 \\
  b_1 & a_3 & 0 & 0 \\
  0 & 0 & d_1 & 0 \\
  0 & 0 & d_1 & 0
\end{bmatrix} = 3\ (g)\]
\[ \text{rank} \begin{bmatrix} B^e & A^e & a^e_n \\ 0 & D^e & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & a & 0 & 0 \\ b & a_z & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & d & 0 \end{bmatrix} = 3 \quad (g) \]

and a robust solution exists for the extended system.
Example 3

Consider the system described by

\[
\begin{bmatrix}
0 & a_1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad a_{2t} = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \quad R_t = \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix}
\]

\[
D_t = \begin{bmatrix} d_1 & 0 & d_2 \\
0 & d_3 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad d_{2t} = \begin{bmatrix} 0 \\ 0 \\ d_w \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
C_t = \begin{bmatrix}
c_1 & 0 & 0 \\
0 & c_2 & 0 \\
c_3 & 0 & c_4 \\
0 & c_5 & 0
\end{bmatrix} \quad c_{2t} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_6 \end{bmatrix}
\]

The plant is structurally controllable and observable; in fact \( \eta = 0 \). Then checking structural solvability of RSFP yields

\[
\begin{bmatrix}
B_t & A_t \\
0 & 0 \\
-1 & B_t & A_t \\
0 & 0 & 0 \\
0 & 0 & D_t
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 0 & 0 & a, 0 \\
b, 0 & 0 & 0 & 0 \\
0 & b_2 & 0 & 0 & 0 \\
d, 0 & d_2 & 0 & 0 \\
0 & d_3 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & a, 0 \\
0 & -1 & 0 & b_2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & b_2 & 0 & 0 & 0 \\
d, 0 & d_2 & 0 & 0 \\
0 & d_3 & 0 & 0
\end{bmatrix}
\]

\[
\text{rank} = 9 (\xi)
\]

and

\[
\begin{bmatrix}
P, A, & a_2 \\
0 & D, & d_2 \\
-1 & P, A, \\
0 & D, 
\end{bmatrix}
\]

\[
\text{rank} = 10 (\rho)
\]

\[
\begin{bmatrix}
0 & 0 & 0 & a, 0 & a_2 \\
b, 0 & 0 & 0 & 0 & 0 \\
0 & b_2 & 0 & 0 & 0 \\
d, 0 & d_2 & 0 & 0 \\
0 & d_3 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & a, 0 \\
0 & -1 & 0 & b_2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & b_2 & 0 & 0 & 0 \\
d, 0 & d_2 & 0 & 0 \\
0 & d_3 & 0 & 0
\end{bmatrix}
\]

Therefore RPIS is not structurally solvable.
Example 4

Consider the system

\[
A_1 = \begin{bmatrix}
0 & a_1 & a_2 \\
a_3 & 0 & a_4 \\
a_5 & 0 & a_6 \\
\end{bmatrix},
A_2 = \begin{bmatrix}
a_7 \\
0 \\
0 \\
\end{bmatrix},
A_3 = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}, \quad R = \begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
\end{bmatrix}
\]

\[
C_1 = D_1 = \begin{bmatrix} d_1 & 0 & d_2 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & c_2 \end{bmatrix}
\]

\[
C_2 = D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

The plant is both structurally controllable and observable.

Since \( C = D \), assumption (4.4.1) is valid and \( \eta \eta^T(A) \subseteq \ker D \).

Structural solvability of R-PIS requires (4.3.8) and (4.3.9) to be true in the factor space \( \tilde{X} = \chi / \eta \eta^T(A) \). A structured basis for \( \eta \) is obtained using the procedure of Appendix III.

Since there are no step disturbances, Step 5 is performed first. Now

\[
\text{rank } \begin{bmatrix} C_1 & c_2 \end{bmatrix} = \text{rank } \begin{bmatrix}
c_1 & 0 & c_2 & 0 \\
0 & a_1 & a_2 & a_7 \\
a_3 & 0 & a_4 & 0 \\
a_5 & 0 & a_6 & 0 \\
\end{bmatrix} = 3 \ (\pi)
\]

and
\[
\begin{bmatrix}
  c, 0 & c_2 & 0 \\
  0 & a_1 & a_2 & 0 \\
  a_3 & 0 & a_4 & a_5 \\
  a_6 & 0 & a_7 & 0 \\
\end{bmatrix}
\]

\[
\text{rank} = 4 \quad (g)
\]

From Lemma 11.1 the second ramp is thus observable. The next rank check, from Step_10 is

\[
\begin{bmatrix}
  c, 0 & c_2 & 0 & 0 & 0 & 1 & 0 & 0 \\
  0 & a_1 & a_2 & 0 & 1 & 0 & 0 & 0 \\
  a_3 & 0 & a_4 & a_5 & 0 & 1 & 0 & 0 \\
  a_6 & 0 & a_7 & 1 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{rank} = 10 \quad (g)
\]

Since there are 10 rows and 11 columns the generic rank of this matrix must be 10, Therefore the first ramp is completely unobservable. The maps induced in the factor space \( \tilde{\mathcal{F}} \) are

\[
\tilde{A} = \begin{bmatrix} A_1 & a_{12} & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} B_1 \end{bmatrix}, \quad \tilde{b} = [n, 0, 0, 0]
\]

Checking (4.3.9) with these maps yields
\[
\begin{bmatrix}
  b, & 0 & a, & a_2 \\
 0 & a_3 & 0 & a_4 \\
0 & a_5 & 0 & a_6 \\
0 & 0 & -1 & 0 \\
 d, & 0 & d_2 \\
\end{bmatrix}
\]

\[\text{rank} \begin{bmatrix}
  -1 & 0 & 0 & b, & 0 & a, & a_2 \\
 0 & -1 & 0 & 0 & a_3 & 0 & a_4 \\
0 & 0 & -1 & 0 & a_5 & 0 & a_6 \\
 d, & 0 & d_2 \\
\end{bmatrix} = 7 \quad (g)\]

and

\[
\begin{bmatrix}
  b, & 0 & a, & a_2 & 0 \\
 0 & a_3 & 0 & a_4 & a_8 \\
0 & a_5 & 0 & a_6 & 0 \\
0 & 0 & -1 & 0 & 0 \\
 d, & 0 & d_2 & 0 \\
\end{bmatrix}
\]

\[\text{rank} \begin{bmatrix}
  -1 & 0 & 0 & b, & 0 & a, & a_2 & 0 \\
 0 & -1 & 0 & 0 & a_3 & 0 & a_4 & a_8 \\
0 & 0 & -1 & 0 & a_5 & 0 & a_6 & 0 \\
 d, & 0 & d_2 & 0 & 0 \\
\end{bmatrix} = 8 \quad (g)\]

Therefore structural RPIS is not solvable.
V. SUMMARY AND CONCLUSIONS

5.1 Summary

In this thesis it has been assumed that an important characteristic of practical systems is the structure often fixed by physical considerations. Also since the nonzero parameter values of a linear system model are seldom known precisely, these parameter values are assumed to be subject to arbitrary perturbations. Problems involving structure and imprecise data are analyzed in a parameter space formulation yielding results readily programmable on a digital computer. In fact, because the above two basic assumptions describe the operating characteristics of a digital computer, exact zero representation but imprecise data due to rounding errors; etc., computer verification of structural results is an exact process.

The first structural property investigated is the rank of a structured matrix. The structural rank of a matrix is defined intuitively as that rank obtained by a given structured matrix for practically any set of parameter values. Necessary and sufficient conditions are derived for the determination of structural rank. These conditions are shown to involve the locations of the fixed zeros in the matrix. Because of the importance of the rank concept in the analysis of linear time invariant system problems, the ability to determine the rank of a structured matrix permits
the analysis of various system structural properties.

Structural controllability of multi-input systems is defined and related to the standard concept of controllability. The necessary and sufficient conditions derived for the structural controllability of multi-input systems are shown to be the same as those obtained by Lin for single input systems. By suitably reformulating the problem, the conditions for structural controllability are shown to be equivalent to a certain structured matrix having full rank, Theorem 3.3.1.

Structural solvability of the servomechanism problem is also investigated. New necessary and sufficient matrix rank conditions for solvability of RSFP are derived. These conditions are translated into a set of necessary and sufficient conditions for the structural solvability of RSFP for the step and ramp exogenous signal case. It is shown that structural solvability of RSFP implies that under very general restrictions on plant and feedback compensator parameter perturbations, RPIS is solved. These restrictions involve only the retention of plant stability.

Using the results concerning structural controllability (observability) and structural solvability of RSFP, algorithmic procedures are formulated to computationally determine the existence of a solution to the servomechanism problem. Finally the possibility of using state space extension to obtain the desired structural properties is
investigated. An algorithmic procedure is formulated to
determine if a given structured system can be extended using
measurable outputs in such a way that RSFP is structurally
solvable for the extended system.

For systems of the form considered, the procedures of
this thesis permit the computational determination of the
structural solvability of RSFP. If RSFP is structurally
solvable for a given system structure then practically any
set of parameter values yields a system for which RSFP is
solvable. The resulting closed loop solution, for
practically any nominal set of parameter values, is robust
with respect to system and compensator parameter
perturbations.
5.2 Suggestions for Future Work

Many practical systems are of the large sparse type; i.e., containing large numbers of fixed zeros, it is anticipated that the structural analysis techniques of this thesis can be used to advantage in the analysis of this type of system. It would be desirable to verify that in fact structural analysis is of practical importance in the analysis of these or other types of "real world" systems.

There are many additional results of linear system theory which yield conditions which are expressible in terms of matrix rank checks. Two examples are the decoupling and model matching problems. The structural analysis of these problems could yield useful results.

There are several areas of investigation remaining with respect to the servomechanism problem. The limitation encountered thus far is in the ability to determine the rank of a structured matrix having repeated nonzero entries; i.e., entries whose perturbations are related. For the step and ramp exogenous signal case this problem is successfully avoided. Also if more general types of exogenous signals are considered, the ranks of structured matrices containing fixed nonzero entries is involved. However since these fixed entries generally occur only in certain known locations, Theorem 4.2.2, their inclusion into the problem is not anticipated to involve serious complication.
Appendix I

Let \( A \) be a \((n \times m)\) structured matrix with \( n \leq m \). The following algorithm determines the largest integer \( t \) for which there does not exist a \( k, m-t < k \leq m \) such that \( A \) contains a zero submatrix of order \((n+m-t-k+1) \times k\); i.e., \( A \) is of Form(j), of Definition 2.4.2, for \( j > t \), but not of Form(t). Then by Theorem 2.4.2 the generic rank of \( A \) is \( t \).

Step_1: Set the integer variable \( v \) to 1 and \( A(0) = A \).

Step_2: Search \( A(0) \) for any rows composed entirely of zeros. Assume there exists \( i(0) \) such rows. Form the matrix \( A(1) \) by removing these \( i(0) \) zero rows from \( A(0) \).

Step_3: Identify a row in \( A(v) \) having the least number of nonzero entries and all rows structurally equivalent to this row. Let \( j(v) \) be the number of nonzero entries and \( i(v) \) the number of such structurally equivalent rows.

Step_4: Using row and column exchanges put \( A(v) \) into the form

\[
\begin{bmatrix}
  R_k & 0 \\
  A_1 & A_2
\end{bmatrix}
\]

(1.1)

where \( R_k \) is of order \((i(v) \times j(v))\) and contains the nonzero entries of the rows identified in Step_3.

Step_5: Set \( v = v + 1 \).

Step_6: If \( A_2 \) is of a nontrivial order; i.e., having one or
more rows and columns, let $A(v) = A_2$ and go to Step_3. Otherwise go to Step_7.

**Step_7:** The $A$ matrix is now of the form

$$
\begin{bmatrix}
0 & \cdots & 0 \\
R_1 & 0 & \cdots & 0 \\
X & R_2 & 0 & \cdots & 0 \\
X & X & R_3 & \cdots & 0 \\
X & \cdots & \cdots & \cdots & \cdots \\
X & \cdots & \cdots & \cdots & R_w
\end{bmatrix}
$$

where $R_w$ is of order $(i(v) \times j(v))$ with $v = 1, \ldots, w$ and $w < n$. Then the generic rank of $A$ is obtained from

$$
t = n - \max_{\sigma \in \mathbb{S}_w} \left[ \sum_{s=0}^{\delta} \left[ i(s) - j(s) \right], 0 \right]^* 
$$

with $j(0) = 0$. This follows since

$$
\max_{\sigma \in \mathbb{S}_w} \left[ \sum_{s=0}^{\delta} \left[ i(s) - j(s) \right], 0 \right]
$$

is the maximum number of generically dependent rows in $A$. Otherwise, from Theorem 2.4.2, for some $j < t$ there exists a $k, m-j < k < m$ defining a zero submatrix of order $(n+m-j-k+1) \times k$. But from the method of the above algorithm and since $j < t$ this implies there exist zero entries in some $R_\delta$ of (1.2) or there exist additional zero rows of $A$. Since neither of these possibilities occur, $\text{rank}(A) = t \ (\rho)$.

* An equivalent expression is $t = \sum_{q=1}^{\delta} \min(i(q), j(q))$
Step 8: End of algorithm.

Consider the following example of the application of the above algorithm. Assume Step 7 has been reached and A has the form

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X & 0 & 0 & 0 & 0 & 0 & 0 \\
X & X & 0 & 0 & 0 & 0 & 0 \\
X & X & 0 & 0 & 0 & 0 & 0 \\
0 & X & X & 0 & 0 & 0 & 0 \\
X & 0 & X & X & X & X & 0
\end{bmatrix}
\]

where \( n=6, m=8 \), the X's represent nonzero entries and \( R_i = X, i(1)=j(1)=1; \)

\[ R_2 = \begin{bmatrix} X \\ X \end{bmatrix}, i(2)=2, j(2)=1; \]

\[ R_3 = \begin{bmatrix} X \end{bmatrix}, i(3)=j(3)=1; \]

\[ R_4 = X \times X \times X, i(4)=1, j(4)=4, \]

and \( l(0)=1. \)

Then with \( l=4 \) and \( j(0)=0, \)

\[
\max_{0 \leq g \leq 4} \left[ \sum_{s=0}^{g} [i(s) - j(s)], n \right] = \max(1,1,2,2,\ldots) = 2.
\]

Therefore \( \text{rank}(A) = 4 \) (g).
Appendix II

Consider the structured system (4.3.2) and assume (4.3.5) is true. The following algorithmic procedure determines the number of generically independent columns in the matrix

\[
\begin{bmatrix}
  C & 0 \\
  A & -1 \\
  0 & C \\
  0 & A \\
\end{bmatrix}
\]

(11.1)

The detailed matrix representation of (11.1) is
\[
\begin{bmatrix}
C_0 & c_{1i} \cdots c_{ki} & C_2 & 0 & \cdots & c_{2k_i} & 0 & 0 \\
A_0 & a_{1i} \cdots a_{ki} & a_2 & 0 & \cdots & a_{2k_i} & 0 & -1 \\
0 & -1 \\
0 & 1 \\
0 & 0 \\
\vdots & \vdots \\
0 & 1 \\
0 & 0 \\
\vdots & \vdots \\
0 & 1 \\
0 & 0 \\
C_0 & c_{1i} \cdots c_{ki} & C_2 & 0 & \cdots & c_{2k_i} & 0 \\
A_0 & a_{1i} \cdots a_{ki} & a_2 & 0 & \cdots & a_{2k_i} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix}
\]

(11.2)

Step 1: By assumption (4.3.5)

\[
\text{rank } \begin{bmatrix} C_i \\ A_i \end{bmatrix} = n_i \ (g) 
\]  

(11.3)

Using FZR and retaining the columns of (11.3) identify and eliminate the generically dependent columns of the matrix.
\[
\begin{bmatrix}
C_1 & C_2 & \cdots & C_{14} & C_{15} & \cdots & C_{23} \\
A_1 & a_{11} & \cdots & a_{113} & a_{12} & \cdots & a_{23}
\end{bmatrix}
\] (11.4)

Define

\[
\begin{bmatrix}
G_i \\
G_2
\end{bmatrix} =
\begin{bmatrix}
c_{1i} & \cdots & c_{114} \\
a_{1i} & \cdots & a_{113}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
G_3 \\
G_4
\end{bmatrix} =
\begin{bmatrix}
c_{25} & \cdots & c_{223} \\
a_{25} & \cdots & a_{223}
\end{bmatrix}
\]

where \( s \leq k_1 \) and \( r \leq k_2 \), as the generically independent columns retained. Then

\[
\text{rank}
\begin{bmatrix}
C_1 & G_1 & G_3 \\
A_1 & G_2 & G_4
\end{bmatrix}
= n_1 + s + r \quad (g)
\]

Step_2: The generic rank of (11.2) is equal to the generic rank of the matrix

\[
\begin{bmatrix}
C_1 & G_1 & G_3 \\
A_1 & G_2 & G_4 & -I \\
& & & 0 & \quad I_{n_1+2k_2} \\
& & C_2 & \cdots & C_{2k_2} \\
& A_1 & a_{11} & \cdots & a_{12k_2}
\end{bmatrix}
\] (11.5)

Consider

\[
\begin{bmatrix}
C_1 & G_1 & G_3 \\
A_1 & G_2 & G_4 & -I \\
& & & 0 & \quad I_{n_1+2k_2} \\
& & C_2 & \cdots & C_{2k_2} \\
& A_1 & a_{11} & \cdots & a_{12k_2}
\end{bmatrix}
= \begin{bmatrix} I_{n_1} \end{bmatrix}
\begin{bmatrix}
C_1 & G_1 & G_3 \\
A_1 & G_2 & G_4 & -I_{n_1} \\
& & & I_{n_1} \\
& & C_2 & \cdots & C_{2k_2} \\
& A_1 & a_{11} & \cdots & a_{12k_2}
\end{bmatrix}
\] (11.6)

Now by Sylvester's inequality, [36; p. 33], (11.6) implies
\[(h+2n_1+r)+(2n_2+2r+s)-(h+2n_1+r) \leq \text{rank} \begin{bmatrix} C_1 & G_1 & G_3 \\ A_1 & G_2 & G_y - I \\ C_1 & G_3 \\ A_1 & G_y \end{bmatrix} \leq \min((h+2n_1+r),(2n_1+2r+s)) \quad (g)\]

Then

\[2n_1+2r+s \leq \text{rank} \begin{bmatrix} C_1 & G_1 & G_3 \\ A_1 & G_2 & G_y - I \\ C_1 & G_3 \\ A_1 & G_y \end{bmatrix} \leq 2n_1+2r+s \quad (g)\]

since \(n_1+h \geq s+r\). Therefore

\[\text{rank} \begin{bmatrix} C_1 & G_1 & G_3 \\ A_1 & G_2 & G_y - I \\ C_1 & G_3 \\ A_1 & G_y \end{bmatrix} = 2n_1+2r+s \quad (g) \quad (11.7)\]

Thus the matrix of (11.7) generically has full column rank and is therefore not of form \((2n_1+2r+s)\). Rewrite (11.5) in the form

\[
\begin{bmatrix}
C_1 & G_1 & G_3 \\
A_1 & G_2 & G_y - I \\
0 & I_{n_1+2r+s} & \\
C_1 & G_1 & H_1 \\
A_1 & G_y & H_2
\end{bmatrix}
\quad (11.8)
\]
where

\[
\begin{bmatrix}
c_2, \ldots, c_{2k_4} \\
a_2, \ldots, a_{2k_4}
\end{bmatrix}
\sim
\begin{bmatrix}
g_1 & h_1 \\
g_2 & h_2
\end{bmatrix}
\]

after a suitable permutation of columns.

**Step 3:** To determine the generic rank of (11.2) it is only necessary to determine the generic dependance of the last \(k_2 - r\) columns of (11.8). Let \((h_1, h_2)^T\) be a column of \((H, H_2)^T\).

**Lemma 11.1**

\[
\begin{bmatrix}
C_1 & G_1 & G_3 \\
A_1 & G_2 & G_4 - 1
\end{bmatrix}
\]

\[
\text{rank}
\begin{bmatrix}
0 \\
C_1 & G_3 \\
A_1 & G_4
\end{bmatrix}
= 2n_1 + 2r + s + 1 \quad (g) \quad (11.9)
\]

if and only if the matrix of (11.9) does not have

From \((2n_1 + 2r + s + 1)\).

**Proof:**

If \(2h = 2r + s\), the maximal rank of the matrix of (11.9) is \(2n + 2r + s\). Therefore assume \(2h > 2r + s\).

(Necessity)

Assume the matrix of (11.9) has form \((2n_1 + 2r + s + 1)\). Then from Theorem 2.4.1 the matrix can not have full rank for any set of parameter values.
(Sufficiency)
Assume the matrix of \((11, 9)\) does not have Form \((2n, +2r+s+1)\). Fix \(C, A, G, G, G\) and \(G\) such that

\[
\begin{bmatrix}
C, G, G, G, G, G
\end{bmatrix}
\begin{bmatrix}
A, G, G, G, G, G
\end{bmatrix}
= \begin{bmatrix}
0, 0, 0, 0, 0, 0
\end{bmatrix}
= 2n, +2r+s
\]

Then the polynomial \(\psi\) defined as the sum of the squares of all \(2n, +2r+s+1\) order minors of \((11, 9)\) is a nonconstant polynomial in the nonzero entries of \((h, h')^T\).

Therefore there exist values of \((h, h')^T\) such that the rank of the matrix of \((11, 9)\) is \(2n, +2r+s+1\). Then \((11, 9)\) follows from the definition of a generic property.

Using FZRF and the results of Lemma 11.1 locate and eliminate the remaining generically dependent columns of \((11, 9)\).

Step 4: If \(f\) is the number of columns of \((u, u')^T\) not eliminated in Step 3, then

\[
\begin{bmatrix}
C, 0
\end{bmatrix}
= 2n, +2r+s+f \ (g)
\]

\[
\begin{bmatrix}
A, -1
\end{bmatrix}
= \begin{bmatrix}
0, 0, 0, 0, 0, 0
\end{bmatrix}
= 2n, +2r+s, +f \ (g)
\]

\[
\begin{bmatrix}
0, G
\end{bmatrix}
= \begin{bmatrix}
0, 0, 0, 0, 0, 0
\end{bmatrix}
= 2n, +2r+s, +f \ (g)
\]

\[
\begin{bmatrix}
0, A
\end{bmatrix}
= \begin{bmatrix}
0, 0, 0, 0, 0, 0
\end{bmatrix}
= 2n, +2r+s, +f \ (g)
\]

\[
\begin{bmatrix}
0, A
\end{bmatrix}
= \begin{bmatrix}
0, 0, 0, 0, 0, 0
\end{bmatrix}
= 2n, +2r+s, +f \ (g)
\]
Appendix III

Consider the state space decomposition (4.4.16) with respect to the structured system (4.3.2). A basis for $\mathcal{H}$ relative to the decomposition (4.4.16) is determined as follows:

Step 1: By Lemma (4.4.4) any generic dependency among the first $n$ columns of (11.2) specifies the structure of a vector in $\mathcal{H}$. Using FZRF with $i \in \{1, \ldots, k_i\}$ determine the generic rank of

\[
\begin{bmatrix}
C_i & L_{i-1} & c_{i'} \\
A_i & L_{2i-1} & a_{i'}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
C_i & L_{i-1} \\
A_i & L_{2i-1}
\end{bmatrix}
\]

has full generic column rank, $(L_{i-1}, L_{2i-1})^\top$ is composed of vectors $(c_{ij}, a_{ij})^\top$, $j < i$, and $(L_{10}, L_{20})^\top = (0, 0)^\top$.

Step 2: If (11.1) has full generic column rank, define

\[
\begin{bmatrix}
L_{i'} \\
L_{2i'}
\end{bmatrix} = \begin{bmatrix}
L_{i-1} & c_{i'} \\
L_{2i-1} & a_{i'}
\end{bmatrix}
\]

and go to Step 4.

Step 3: If (11.1) does not have full generic column rank,
then there exists a vector \( w \in \mathcal{H} \) of the form \( x_i^j \neq 0, \ x_j^i = 0 \) for \( i < j < k \), and \( x_{ij}^j = x_{ij}^i = 0 \) for all \( j \in \{1, ..., k_2\} \).

**Step 4:** If \( i < k_1 \), repeat Step 1 with \( i = i + 1 \). If \( i = k_1 \), let

\[
\begin{bmatrix}
G_1 \\
G_2
\end{bmatrix} =
\begin{bmatrix}
L_{1k_1} \\
L_{2k_1}
\end{bmatrix}
\]

and let \( W_i \) be the set of \( t \) vectors of \( \mathcal{H} \) found in Step 3; \( t \leq k_1 \).

**Step 5:** Using FZRF with \( i \in \{1, ..., k_2\} \) determine the generic rank of

\[
\begin{bmatrix}
C_i & G_i & L_{i+1} & c_{2i} \\
A_i & G_{2i} & L_{2i+1} & a_{2i}
\end{bmatrix}
\tag{III.2}
\]

where

\[
\begin{bmatrix}
C_i & G_i & L_{i+1} \\
A_i & G_{2i} & L_{2i+1}
\end{bmatrix}
\]

has full generic column rank, \( (L_{i+1} \ L_{2i+1})^T \) is composed of vectors \( (c_{2j} \ a_{2j})^T, j < 1 \), and

\[
(L_{10} \ L_{20})^T = (0 \ 0)^T.
\]

**Step 6:** If (III.2) has full generic column rank, define

\[
\begin{bmatrix}
L_{1i} \\
L_{2i}
\end{bmatrix} =
\begin{bmatrix}
L_{i+1} & c_{2i} \\
L_{2i+1} & a_{2i}
\end{bmatrix}
\]

and go to Step 8.

**Step 7:** If (III.2) does not have full generic column rank,

then there exists a vector \( w \in \mathcal{H} \) of the form \( x_i^j \neq 0, \ x_j^i = 0 \),
for $i < j \leq k_2$, and $x_{ij} = 0$ for all $j \in \{1, \ldots, k_2\}$.

Step_8: If $i < k_2$ repeat Step_5 with $i = i + 1$. If $i = k_2$, let

$$
\begin{bmatrix}
G_3 \\
G_4
\end{bmatrix} =
\begin{bmatrix}
L_{1k_2} \\
L_{2k_2}
\end{bmatrix}
$$

and let $W_2$ be the set of $r$ vectors of $\eta$ found in Step_7; $r \leq k_2$.

Step_9: Rewrite the matrix of (111.2) as

$$
\begin{bmatrix}
C, G_1, G_3 \\
A, G_2, G_4, -I \\
I_{k_2}, E, E_2 \\
C, G_3, H_1 \\
A, G_4, H_2 \\
I_{k_1 + 2k_2}
\end{bmatrix}
$$

(111.3)

where

$$
\begin{bmatrix}
E, E_2 \\
G_3, H_1 \\
G_4, H_2
\end{bmatrix} =
\begin{bmatrix}
I \\
c_{21}, \ldots, c_{2k_1} \\
a_{z}, \ldots, a_{z_{k_2}}
\end{bmatrix}
$$

after a suitable permutation of columns, and

$$
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} =
\begin{bmatrix}
c_{zj_1}, \ldots, c_{zj_r} \\
}\end{bmatrix}
\begin{bmatrix}
a_{zj_1}, \ldots, a_{zj_r}
\end{bmatrix}
$$

Step_10: From the results of Lemma 11.1, FZRF can be used to determine the generic rank of
\[
\begin{bmatrix}
C_1, G_1, G_2 \\
A_1, G_2, G_3 \\
1_k \\
E\_j, E_3, e_i \\
C_1, G_3, L_{i,i-1} \\
A_1, G_4, L_{i,i-1} \\
a_{i,j_i}
\end{bmatrix}
\quad (\text{III},4)
\]

where \( i \in \{j, \ldots, j_r\} \), (III,4) has full generic column rank up to the last column,

\[
\begin{bmatrix}
L_{i,i-1} & C_{2j_i} \\
L_{2i-1} & a_{2j_i}
\end{bmatrix}
\]

is composed of vectors from the \((n, +hr)\) matrix \((H, H_2)^T\)
\((L_{i,i}, L_{2i})^T = (0, 0)^T\), and \(e_i\) is a \((k_2 \times 1)\) vector of zeros with unity in the \(j_i^{\text{th}}\) row.

**Step 11:** If (III,4) has full generic column rank, define

\[
\begin{bmatrix}
L_{i,i} \\
L_{2i}
\end{bmatrix} =
\begin{bmatrix}
L_{i,i-1} & C_{2j_i} \\
L_{2i-1} & a_{2j_i}
\end{bmatrix}
\]

and go to Step 13.

**Step 12:** If (III,4) does not have full generic column rank then there exists a vector \(\omega \in \gamma\) of the form \(x_{\omega_i} = 0\).

**Step 13:** If \( i < r \), repeat Step 10 with \( i = i + 1 \). If \( i = r \), let \( W_3 \) be the set of s vectors of \( \gamma \) found in Step 12; \( s < r \).

end of algorithm.

**Lemma III.1**

\[W_1 \circ W_2 \circ W_3 = \gamma\]
Proof:
From the above algorithm and the form of the vectors in $W_1$, $W_2$, and $W_3$, it is obvious their sum is contained in $\gamma$. However from the manner in which the vectors of $W_1 + W_2 + W_3$ were obtained,

$$t + r + s = d \ker \left[ \begin{array}{c} C \\ CA \\ A_1 \end{array} \right]$$

Then from Lemma 4.4.2

$$W_1 \cap W_2 \cap W_3 = \gamma$$

The two following lemmas demonstrate properties of the above algorithm.

Lemma III.2

Assume $w \in W_3$ is of the form $x_{i_2}^2 \neq 0$, $i \in \{i_1, \ldots, i_p\}$. Then there exists a vector $w' \in W_2$ of the form $x_{i_1}^2 \neq 0$, $x_{i_2}^2 = 0$, for $i < j < k_2$ and $x_{i_j}^2 = 0$ for all $j \in \{1, \ldots, k_2\}$.

Proof:
Assume no $w' \in W_2$ exists of the proper form. Then from Step 8, $(c_{z_1} a_{z_2})^T$ is a column of $(C, G_w)^T$. Therefore $i \notin \{i_1, \ldots, i_p\}$ and $w \notin W_3$.  

\[ \triangle \]
Lemma 111.3

Assume \( \text{rank}(h, \ldots, h_i) = \text{rank}(h, \ldots, h_i; h) = m \) where
\((h, \ldots, h_i)\) is of order \((n \times m)\), with \(m \leq n\). Then

\[
\text{rank}(h, \ldots, h_{j-1}, h_j, \ldots, h_i) = \text{rank}(h, \ldots, h_{j-1}, h_{j+1}, \ldots, h_i; h)(111.5)
\]

for some \(j \in \{1, \ldots, i\}\) if and only if \(x_j \neq 0\) in any vector such that

\[
(h, \ldots, h_i) \begin{bmatrix}
x_i \\
\vdots \\
x_j \\
\vdots \\
x_i 
\end{bmatrix} = h \quad (111.6)
\]

Proof:

(Necessity)

Assume \(x_j = 0\) in some vector satisfying (111.6). Then

\[
h = \sum_{\substack{k = 1 \atop k \neq j}}^{i} x_k h_k
\]

and (111.5) is not true.

(Sufficiency)

Assume

\[
\text{rank}(h, \ldots, h_{j-1}, h_{j+1}, \ldots, h_i) = \text{rank}(h, \ldots, h_{j-1}, h_{j+1}, \ldots, h_i; h)
\]

Then

\[
h = \sum_{\substack{k = 1 \atop k \neq j}}^{i} x_k h_k
\]
and so there exists a vector satisfying (111.6) for which $x_j = 0$. ▲
REFERENCES


