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A HYBRID COLLOCATION–GALERKIN METHOD FOR THE TWO POINT BOUNDARY VALUE PROBLEM USING CONTINUOUS PIECEWISE POLYNOMIAL SPACES

by

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Chapter I

Introduction

Consider the two point boundary value problem given by

$$(1.1)\qquad \begin{aligned} &Lu \equiv u'' + a(x)u' + b(x)u = f, \qquad x \in I = [0,1] \\ &u(0) = u(1) = 0\,. \end{aligned}$$

Assume that (1.1) has a unique solution for every $f \in C(I)$ and assume a, a′, b, b′ to be in $L^\infty(I)$.

We shell be concerned with the numerical solution of (1.1) by a method of collocation for the particular case in which the approximate solution is a piecewise continuous r^{th} – degree polynomial. More precisely, let $\Delta = \{x_i\}_{i=0}^{n}$ be a partition of I where

$$0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1, \quad I = [0,1]$$

$$h_i = x_{i+1} - x_i, \quad I_i = [x_i, x_{i+1}], \quad i = 0, \cdots, n.$$

Throughout this work we assume that the partition Δ is quasiuniform. i.e., there exists a constant $\sigma \geqslant 1$ such that

$$(1.2)\qquad \max_{0 \leqslant i \leqslant n} h\, h_i^{-1} \leqslant \sigma \;, \text{ where } h = \max_{0 \leqslant i \leqslant n} h_i$$

Let the "hat" function V_i at x_i be the continuous piecewise-linear function that satisfies

$$V_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j\ . \end{cases}$$

The linear span of these functions $Sp\{V_i\}$ is denoted by $M_0^1(\Delta)$. In general, let $M_k^r(\Delta) = \{V \in C^k(I) \mid V|_{I_i} \in P_r(I_i)\;;\; i = 0, \cdots r\;;\; V(0) = V(1) = 0\}$ where $0 \leqslant k < r$ and $P_r(E)$ denotes the class of all polynomials of degree at most r on the set E. Also we let $Z_0^r = Z_0^r(\Delta) = \{V \in M_0^r(\Delta) \mid V(x_i) = 0\;;\; i = 1, \cdots, n\}$. This subspace can be characterized by the following property

$$M_0^r \equiv M_0^r(\Delta) = M_0^1(\Delta) \oplus Z_0^r(\Delta)\,,$$

which can be easily seen by a dimensional argument, and the fact that the M_0^1–interpolant of a function in Z_0^r is identically zero, where the nodes $x = x_i$ are taken as interpolation points.

We shall employ as collocation points the r – 1 points $\{x_{ij}\}_{j=1}^{r-1}$ on each subinterval I_i that are the affine images of the roots of a Jacobi polynomial of degree r – 1.

Although, the collocation conditions at r – 1 points are not enough to completely define an element of M_0^r one cannot impose more than r – 1 collocation-like conditions per subinterval because it would result in an overdetermined system of equations. However, dimensionality requires that we impose only one and exactly one condition per interior node, when imposing r – 1 collocation-like conditions per subinterval.

The condition we choose to impose here is a condition on the jump of the first derivative of the function. It resembles the relation satisfied by the Galerkin procedure.

We define $U \in M_0^r$ the hybrid collocation-Galerkin approximation to (1.1) as a solution to the following equations.

(1.3) $$LU(x_{ij}) = f(x_{ij}), \quad i = 0, \cdots, n\ ;\ j = 1, \cdots, r-1,$$

and

(1.4) $$(LU, V_i) = (f, V_i)\ ;\quad i = 1, \cdots, n\ .$$

Where V_i is the hat function at x_i and $(\cdot\ ,\ \cdot)$ denotes the scalar product between $H^{-1}(I) = H_0^1(I)'$ and $H_0^1(I)$.

Equations (1.3) are the collocation equations and (1.4) are the Galerkin equations. Equations (1.4) are equivalent to impose a condition on the jumps of the first derivative of the approximation as we will show in Chapter II.

This procedure has the advantages that it requires much less quadrature than its counterpart the C^0–Galerkin procedure, replacing most of the integrals by evaluation at points. The quadratures that are left to be evaluated are the simplest. That is, equations (1.4).

In Chapter II, a summary of the results for the C^0–Galerkin procedure is presented for reference. Also presented is a description of the jumps, at the nodes, of the first derivative of the Galerkin approximation.

In Chapter III, we introduce a semi-discrete innerproduct and quadrature method based on the Jacobi polynomials, similar to the quadrature method introduced by Douglas and

and Dupont [3] for the Legendre polynomials. In this chapter, we also prove several technical lemmas that will be used later.

In Chapter IV, using the semi-discrete innerproduct a related variational formulation of the hybrid procedure is given similar to that introduced by Douglas and Dupont [3] for parabolic equations. An Existence and Uniqueness theorem is derived. Optimal rates of convergence are obtained by showing that the hybrid approximation is close to the C^0–Galerkin approximation.

In Chapter V, $O(h^{2r})$ order of superconvergence at the nodes for the function are established. The proof involves similar techniques to those used by de Boor and Swartz [1] to prove $O(h^{2r-2})$ order of superconvergence at the nodes for the C^1–collocation procedure.

Chapter II

The Galerkin Procedure

§2.1 Some notation and preliminaries

For an interval J we let $H^s(J)$ and $W^s(J)$ denote the closure of $C^\infty(J)$ in the norms

$$\|f\|_{H^s(J)} = \Big(\sum_{i=0}^{s} \|f^{(i)}\|^2_{L^2(J)}\Big)^{1/2}$$

and

$$\|f\|_{W^s(J)} = \Big(\sum_{i=0}^{s} \|f^{(i)}\|_{L^\infty(J)}\Big)$$

respectively, where $\|f\|_{L^2(J)} = \int_J f^2(x)dx$ and $\|f\|_{L^\infty(J)} = \sup_{x \in J} |f(x)|$, and s is a positive integer. For simplicity we shall suppress the dependence on J whenever J = I.

The following lemma summarizes some well known properties of the spaces M_0^r.

Lemma 2.1.

There exist constants C_1 and C_2, independent of h, such that for any $\beta \in M_0^r$ and $i = 0, \cdots, n$

(2.1) $$\|\beta'\|_{L^2(I_i)} \leqslant C_1 h_i^{-1} \|\beta\|_{L^2(I_i)} \quad \text{(Inverse property)}$$

(2.2) $$\|\beta^{(j)}\|_{L^\infty(I_i)} \leqslant C_2 h_i^{-1/2} \|\beta^{(j)}\|_{L^2(I_i)}, \quad j = 0, 1 .$$

For a given partition Δ, the L^2-innerproduct on the subinterval I_i is denoted by

$$(f,g)_i = \int_{I_i} f(x)\, g(x)\, dx$$

where both f and g belong to $L^2(I_i)$. If f and g belong to $L^2(I_i)$ for each $i = 0, \cdots, n$, we denote

$$\langle f, g \rangle = \sum_{i=0}^{n} (f,g)_i .$$

Observe that if $\psi \in H^{-1}(I)$ and $\psi \in \bigcup_{i=0}^{n} L^2(I_i)$; then,

$$(\psi,\phi) = \langle \psi,\phi \rangle + \sum_{i=0}^{n} \psi(\phi)(x_i),\ \phi \in L^2(I)$$

where ϕ is continuous at each point in Δ. Furthermore, if ψ and ϕ belong to $L^2(I)$, then $(\psi,\phi) = \langle \psi,\phi \rangle$.

With this notation we obtain the following Green's function representation formulas. Let ξ be a continuous function on I that is C^2 except perhaps at a finite number of points where jump discontinuities can occur on ξ'. i.e., for some partition Δ, $\xi \in C^2(I_i)$ for each $i = 0, \cdots, n$. Let G be the Green's function for the problem (1.1). For $x \in I$ we obtain

$$\xi(x) = (L\xi, G(x,\cdot)) = \langle L\xi, G(x,\cdot)\rangle + \sum_{i=1}^{n} J_1(\xi,x_i)\, G(x,x_i) \tag{2.3}$$

where $J_j(y,p) = y^{(j)}(p+) - y^{(j)}(p-)$ denotes the jump of the j^{th} derivative of the function y.

For the derivative ξ', we obtain

$$\xi(x) = (L\xi, G_x(x,\cdot)) = \langle L\xi, G_x(x,\cdot)\rangle + \sum_{i=1}^{n} J_1(\xi,x_i)\, G_x(x,x_i) \tag{2.4}$$

as long as $x \notin \Delta$.

§2.2 The C^0—Galerkin procedure

Most of the material of this paragraph was presented in Douglas and Dupont [2] and Wheeler [5]. It is included here for later reference.

From the definition of the Galerkin procedure we describe the jumps on the derivative. Let $\chi \in M_0^r$ be the Galerkin approximation to the solution u of (1.1), determined by the relation

$$-(\chi', V') + (a\chi' + b\chi, V) = (f,V),\ \ \forall\, V \in M_0^r\ . \tag{2.5}$$

Integrating (2.5) by parts we obtain for $V \in M_0^r$

$$(L\chi,V) = (\chi'' + a\chi' + b\chi,V) = \langle L\chi,V\rangle + \sum_{i=1}^{n} J_1(\chi,x_i)\, V(x_i)$$

$$= \langle f,V\rangle = \langle Lu,V\rangle\, .$$

Thus,

$$\sum_{i=1}^{n} J_1(\chi,V_i)V(x_i) = \langle Lu - L\chi, V\rangle \ \forall\ V \in M_0^r.$$

Recall now that $M_0^r = M_0^1 \oplus Z_0^r$, and that the hat functions $\{V_i\}_{i=1}^n$ form a basis for M_0^1. Then, we obtain

$$J_1(\chi,x_i) = \langle Lu - L\chi,\ V_i\rangle,\quad i = 1,\cdots,n,$$

where V_i is the hat function at x_i. This expression gives the jump on the first derivative of the Galerkin approximation at the point x_i. Similarly we can find the jump for the hybrid approximation by looking at equations (1.4) as follows, $i = 1,\cdots,n$

$$(LU,V_i) = \langle LU,\ V_i\rangle + \sum_{j=1}^{n} J_1(U,x_j)\ V_i(x_j)$$

$$= \langle LU,V_i\rangle + J_1(U,x_i) = \langle Lu,V_i\rangle\ ;$$

therefore,

$$J_1(U,x_i) = \langle Lu - LU,\ V_i\rangle,\quad i = 1,\cdots,n, \tag{2.6}$$

which says that equations (1.4) are conditions on the jump of the first derivative of the hybrid approximation at the nodes $x = x_i$, as we have already claimed.

Going back to the Galerkin procedure, it is known, [2], that for h sufficiently small (2.5) has a unique solution χ so that

$$\| u - \chi \|_{L^2} + h \| u-\chi \|_{H^1} \leq C\, h^k \|u\|_{H^k(I)},\quad 1 \leq k \leq r+1\ ; \tag{2.7}$$

furthermore,

$$\| u-\chi \|_{H^1} \leq C \inf_{\Gamma \in M_0^r} \| u-\Gamma \|_{H^1}. \tag{2.8}$$

Wheeler [5] obtained the following L^∞ – estimate

$$\| u-\chi \|_{L^\infty} \leq C\, h^{r+1} \|u\|_{W^{r+1}}. \tag{2.9}$$

By means of a slight modification of her proof, we can show that the Galerkin approximation χ satisfies the following estimate

(2.10) $$\| u-\chi \|_{L^\infty} \leqslant C\, h^{r+\frac{1}{2}} \| u \|_{H^{r+1}},$$

result that will prove useful later to obtain the right L^2–estimate for the simple problem $u'' = f$.

In order to prove (2.10) it is enough to prove the following lemma which proof follows almost verbatim that of Wheeler [5] but with slight modification.

Lemma 2.2.

Let $J = (a,b)$ be an interval of length $\mu = b - a$, if $y \in H^{r+1}(J)$ and $Y \in P_r(J)$ satisfy

(2.11)
$$y(a) = Y(a),\ y(b) = Y(b)$$
$$\int_J (y - Y)'(x)\, p'(x)dx = 0, \qquad \forall p \in P_r(J),\ p(a) = p(b) = 0.$$

Then,there exist a constant C independent of μ such that

$$\| y-Y \|_{L^\infty(J)} \leqslant C\, \mu^{r+\frac{1}{2}} \| y^{(r+1)} \|_{L^2(J)}.$$

Proof.

By Peano kernel theorem,

$$(y-Y)(x) = \int_J \mathcal{D}(x,t)\, y^{(s+1)}(t)dt,$$

where for each t, $\mathcal{D}(x,t)$ is the difference between

$$K_t(x) = \begin{cases} \frac{1}{r}(x-t)^r, & x \geqslant t \\ 0\ , & x < t, \end{cases}$$

and its projection into $P_r(J)$ given by (2.11). Hence,

$$\| y-Y \|_{L^\infty(J)} \leqslant \sup_{x \in J} \left(\int_J | \mathcal{D}(x,t) |^2 dt\right)^{\frac{1}{2}} \| y^{(r+1)} \|_{L^2(J)}.$$

Since $\mathcal{D}(0,t) = 0$ for all t,

$$| \mathcal{D}(x,t) | = | \int_a^x \frac{\partial}{\partial x} \mathcal{D}(x,t)dx | \leqslant \int_a^x | \frac{\partial}{\partial x} \mathcal{D}(x,t) | \, dx$$

$$\leqslant \int_J | \frac{\partial}{\partial x} \mathcal{D}(x,t) | \, dx \leqslant \mu^{1/2} (\int_J | \frac{\partial}{\partial x} \mathcal{D}(x,t) |^2 \, dt)^{1/2}$$

$$\leqslant \mu^{1/2} (\int_J | \frac{\partial}{\partial x} K_t(x) |^2 dx)^{1/2} \leqslant \frac{\mu^{1/2} \mu^{r-1} \mu^{1/2}}{(r-1)!^2 (r-2)!} = C \mu^r .$$

Therefore

$$\| y - Y \|_{L^\infty(J)} \leqslant C \mu^{r+1/2} \| y^{(r+1)} \|_{L^2(J)} .$$

Chapter III

Semi–discrete Quadrature and Innerproduct

§3.1 Jacobi polynomials

In this Chapter a semi-discrete quadrature method and innerproduct are defined. They are based on the roots of Jacobi polynomials. Douglas and Dupont [3] defined a similar quadrature and innerproduct for M_1^r using the Gaussian points and they used it to study the C^1 - collocation method for certain parabolic equations. Here we prove similar theorems to those proved in [3].

The Jacobi polynomials are the polynomials on [0,1] given by Rodrigues' formula, for $r = 2, 3, \cdots$

$$(3.1) \qquad J_r(x) \equiv J_{r,\alpha,\beta}(x) = \frac{1}{c\omega(x)} \left(\frac{d}{dx}\right)^{r-1} (\omega(x)\,(x(1-x))^{r-1})$$

where $\omega(x) = x^{\alpha}(1-x)^{\beta}$; $\alpha, \beta > -1$, and c is a constant chosen so the coefficient of x^{r-1} in (3.1) is 1. See [4]. For each pair (α,β), the Jacobi polynomials form a set that satisfies

$$(3.2) \qquad \int_0^1 \omega(x)\, J_r(x)\, x^k dx = 0, \quad 0 \leqslant k \leqslant r-3.$$

We shall consider henceforth the choices $\alpha = \beta = 1$. In this case the Jacobi polynomial J_r can equivalently be interpreted as follows, see [4].

Let $0 < \rho_1 < \cdots < \rho_{r-1} < 1$ and $\omega_k > 0$, $k = 1, \cdots, r-1$ be the unique selection such that

$$(3.3) \qquad \int_0^1 \omega(x)p(x)dx = \sum_{k=1}^{r-1} \omega_j\, p(\rho_k), \qquad p \in P_{2r-3}(I)$$

and $J_r(x) = \prod_{k=1}^{r-1} (x - \rho_k)$, $x \in I$.

§3.2 Semi–discrete innerproduct

Based on (3.3) we will develop an innerproduct as follows. Let v and z be two functions on I with z(0) = z(1) = 0. We define

$$\bar{D}(v,z) = \sum_{j=1}^{r-1} \omega_j \frac{v(\rho_j)\, z(\rho_j)}{\rho_j(1-\rho_j)} \tag{3.4}$$

Observe that if $v \cdot z \in P_{r-1}(I)$; then, $\bar{D}(v,z) = (v,z)$. Since $z(0) = z(1) = 0$, ω divides z; therefore, $v \cdot (z/\omega) \in P_{2r-3}(I)$ and by (3.3)

$$\bar{D}(v,z) = \int_0^1 \omega(x)\, (v(x) \cdot z(x)/\omega(x))\, dx = \int_0^1 v(x)\, z(x) dx \tag{3.5}$$

Given a partition Δ of the interval I, let the collocation points $\{x_{ij}\}_{j=1}^{r-1}$ be the $r-1$ points on each subinterval I_i that are the affine images of the roots $\{\rho_j\}_{j=1}^{r-1}$ of the Jacobi polynomial J_r. We now give the corresponding formula for $\bar{D}$ on each I_i.

For ϕ a function on I with $\phi(0) = \phi(1) = 0$, let $\phi_2(x) = \sum_{i=1}^{n} \phi(x_i)\, V_i(x)$, $\phi_2 \in \mathcal{M}_0^1$, also let $\phi_1 = \phi - \phi_2$. Observe that $\phi_1(x_i) = 0$ for each $i = 0, \cdots, n+1$. We define

$$D_i(\psi, \phi) = \sum_{j=1}^{r-1} \omega_j\, h_i^3 \frac{\psi(x_{ij})\, \phi_1(x_{ij})}{(x_{ij}-x_i)(x_{i+1}-x_i)}, \quad i = 0, \cdots, n,$$

where ψ is a function defined on each I_i. Let $D(\psi, \phi) = \sum_{i=0}^{n} D_i(\psi, \phi)$.

Now we define the semi-discrete innerproduct $\langle\langle\, ,\rangle\rangle$ as follows

$$\langle\langle \psi, \phi \rangle\rangle = D(\psi, \phi_1) + (\psi, \phi_2)\,. \tag{3.6}$$

Observe that (3.6) is defined for $\psi \in H^{-1}(I) \cap (\bigcup_{i=0}^{n} L^2(I_i))$. This innerproduct is called semi-discrete because it has two parts one discrete and one continuous. In Chapter IV we will use the innerproduct (3.6) to provide us with a variational formulation for (1.3)–(1.4) which itself has two parts one discrete and one continuous. In the next paragraph several quadrature relationships are proved.

§3.3 Quadrature relationships

There are several relationships between the semi-discrete innerproduct in (3.6) and the usual L^2–innerproduct. They are contained in the Lemmas to follow. We begin by making the observation that if $\alpha \in \mathcal{M}_0^r$ and $\beta \in Z_0^r$, by (3.5) we obtain

$$D(\alpha'', \beta) = -(\alpha', \beta') \tag{3.7}$$

and

(3.8) $$D(\alpha', \beta) = (\alpha', \beta).$$

Lemma 3.1

If $u \in H^1(I)$ and $\beta \in M_0^r$, then

(3.9) $$|\langle\langle u, \beta \rangle\rangle - (u, \beta)| \leqslant ch \| u' \|_{L^2} \|\beta\|_{L^2},$$

where the constant c is independent of h.

Proof.

It is enough to prove it for $\beta \in Z_0^r$. Then

$$|D(u,\beta) - (u,\beta)| \leqslant |\sum_{i=0}^{n} D_i(u,\beta) - (u,\beta)_i|$$

$$\leqslant \sum_{i=0}^{n} |D_i(u,\beta) - (u,\beta)_i|.$$

Now, by (3.5) $D_i(u,\beta) = (u, \beta)_i$ whenever u is a constant on I_i. For $u \in H^1$, the fundamental theorem of calculus implies that for $x \in I_i$

$$u(x) = u(x_i) + \int_{x_i}^{x} u'(x)dx.$$

Therefore,

$$|D_i(u,\beta) - (u,\beta)_i| = |\sum_{j=1}^{r-1} \omega_j h_i^3 \frac{\beta(x_{ij}) \int_{x_i}^{x_{ij}} u'(t)dt}{(x_{ij}-x_i)(x_{i+1}-x_i)} - \int_{x_i}^{x_{i+1}} \beta(x) \int_{x_i}^{x} u'(t)dt\, dx|$$

$$\leqslant Ch^{3/2} \|\beta\|_{L^\infty(I_i)} \|u'\|_{L^2(I)} + Ch \|\beta\|_{L^2(I')} \|u'\|_{L^2(I_i)}$$

$$\leqslant Ch \|\beta\|_{L^2(I_i)} \|u'\|_{L^2(I_i)},$$

where we have used (2.2). The constant C clearly can be chosen independent of i; therefore, Cauchy's inequality implies

$$\sum_{i=0}^{n} |D_i(u, \beta) - (u, \beta)_i| \leqslant C \|u'\|_{L^2} \|\beta\|_{L^2},$$

which concludes the proof. We have the following two corollaries.

Corollary 3.1.

Let a be a continuous function on I and let $u \in H^1(I)$. If $a' \in L^\infty$, then

(3.10) $$|\langle\langle au, \beta \rangle\rangle| \leqslant C \|u\|_{H^1} \|\beta\|_{L^2}, \quad \forall \beta \in M_0^r.$$

Corollary 3.2

Let a be as in corollary 3.1. Let $\alpha \in M_0^r$, then

(3.11) $$|\langle\langle a\alpha, \beta \rangle\rangle - (a\alpha, \beta)| \leqslant Ch \|\alpha\|_{H^1} \|\beta\|_{L^2}, \quad \forall \beta \in M_0^r.$$

Lemma 3.2

Let a be a continuous function on I. Let $\alpha, \beta \in M_0^r$. If $a' \in L^\infty$, then

(3.12) $$|\langle\langle a\alpha', \beta \rangle\rangle - (a\alpha', \beta)| \leqslant Ch \|\alpha\|_{H^1} \|\beta\|_{L^2}.$$

Proof.

This proof is due to Douglas and Dupont [3] for the case of Legrendre polynomials. However, their idea generalizes here and the details are included for completeness.

As in Lemma 3.1 we can assume that $\beta \in Z_0^r$. Let $a_i = h_i^{-1}(a,1)_i$, then

$$\begin{aligned} D_i(a\alpha', \beta) &= a_i D_i(\alpha', \beta) + D_i((a - a_i)\alpha', \beta) \\ &= a_i(\alpha', \beta)_i + D_i((a - a_i)\alpha', \beta) \\ &= (a\alpha', \beta)_i + \{D_i((a - a_i)\alpha', \beta) - ((a - a_i)\alpha', \beta)_i\}. \end{aligned}$$

By definition

$$\begin{aligned} D_i((a - a_i)\alpha', \beta) &= \sum_{j=1}^{r-1} h_i^3 \omega_j \frac{(a(x_{ij}) - a_i)\, \alpha'(x_{ij})\, \beta(x_{ij})}{(x_{ij} - x_i)(x_{i+1} - x_{ij})} \\ &\leqslant Ch_i \|a - a_i\|_{L^\infty(I_i)} \|\alpha'\|_{L^\infty(I_i)} \|\beta\|_{L^\infty(I_i)}, \end{aligned}$$

where the constant C is independent of i. Using (2.2) and the definition of a_i, we obtain

$$|D(a\alpha', \beta) - (a\alpha', \beta)| \leqslant Ch\Big(\sum_{i=0}^{n} \|\alpha'\|_{L^2(I_i)} \|\beta\|_{L^2(I_i)}\Big).$$

Cauchy's inequality completes the proof. We obtain the following important result.

Corollary 3.3

For $\alpha, \beta \in M_0^r$, we have

$$|\langle\langle L\alpha,\beta\rangle\rangle - (L\alpha,\beta)| \leqslant Ch\,\|\alpha\|_{H^1}\,\|\beta\|_{L^2} \tag{3.13}$$

Proof.

It follows by (3.7), Corollary 3.2 and Lemma 3.2. The following lemma is a technical result we need to use later.

Lemma 3.3

Let a be a continuous function on I, $u \in H^1$ and $\beta \in M_0^r$. Then, there is a constant C independent of h such that

$$|\langle\langle au',\beta\rangle\rangle| \leqslant C\,\|u'\|_{L^\infty}\|\beta\|_{L^2}\,. \tag{3.14}$$

Proof.

As in Lemma 3.1 it is enough to prove it for $\beta \in Z_0^r$. For $i = 0, \cdots, n$

$$\begin{aligned} D_i(au',\beta) &= \sum_{j=1}^{r-1} \omega_j\, h_i^3\, \frac{a(x_{ij})\,u'(x_{ij})\,\beta(x_{ij})}{(x_{ij}-x_i)\,(x_{i+1}-x_{ij})} \\ &\leqslant C\,h_i\,\|u'\|_{L^\infty(I_i)}\,\|\beta\|_{L^\infty(I_i)} \\ &\leqslant C\,h_i^{1/2}\,\|u'\|_{L^\infty(I_i)}\,\|\beta\|_{L^2(I_i)}, \end{aligned}$$

where we use (2.2). The constant C can be chosen independent of i and h. We can bound $\|u'\|_{L^\infty(I_i)}$ by $\|u'\|_{L^\infty(I)}$. Therefore, summing over i and using Cauchy's inequality, we obtain (3.14).

Chapter IV

Variational Procedure, L^2–estimates

§4.1 The Variational Procedure.

Douglas and Dupont [3] succeeded in defining a variational formulation for the C^1–collocation using a discrete innerproduct based on the Gaussian points and use it to study collocation procedures for certain parabolic equations. Here by giving a similar formulation we are able to prove existence and uniqueness as well as obtain optimal rates of convergence of the hybrid collocation–Galerkin procedure defined by (1.3)–(1.4).

The variational formulation should not be used to compute. (1.3)–(1.4) is a more simple system and it should be used for practical computations.

The variational procedure can be described as follows, let $W \in M_0^r$ be the solution to

(4.1) $$\langle\langle LW, V \rangle\rangle = \langle\langle f, V \rangle\rangle, \quad \forall V \in M_0^r .$$

It is clear by the definition of the semi–discrete innerproduct of §3.2 that any solution of (1.3)–(1.4) is a solution of (4.1). Therefore, to show existence of the hybrid collocation–Galerkin approximation it suffices to show it for (4.1), and this is guaranteed by the following

Theorem 4.1

For h small enough, there exist a unique $U \in M_0^r$ satisfying (4.1).

Proof.

This proof is basicly an adaptation of the proof of existence and uniqueness of the Galerkin solution. It is due to Schatz and reported in [2].

Assume $f \equiv 0$. Then, the problem becomes that of showing $Z \equiv 0$ where $Z \in M_0^r$ satisfies the following equation

(4.2) $$\langle\langle LZ, V \rangle\rangle = 0 , \quad \forall V \in M_0^r .$$

Let $v \in H_0^1(I) \cap H^2(I)$ be such that

$$(4.3) \qquad \begin{aligned} L^* v &= Z \quad \text{on } I \\ v(0) &= v(1) = 0, \end{aligned}$$

then, elliptic regularity for (4.3) which follows from our hypotheses on L, implies that

$$\|v\|_{H^2} \leq C\|Z\|_{L^2}.$$

Thus,

$$\begin{aligned} \|Z\|_{L^2}^2 &= (Z,Z) = (L^*v,Z) \\ &= -(v',Z') + (v,aZ' + bZ) \\ &= -(Z',v') + (aZ' + bZ,v) = (LZ,v). \\ &= (LZ,v) - \langle\!\langle LZ, \psi \rangle\!\rangle \end{aligned}$$

for $\psi \in M_0^1$, thus

$$\begin{aligned} \|Z\|_{L^2}^2 &= (LZ,v) - (LZ,\psi) = (LZ,v-\psi) \\ &\leq C\|Z\|_{H^1} \inf_{v \in M_0^1} \|v-\psi\|_{H^1} \end{aligned}$$

Then

$$\|Z\|_{L^2}^2 \leq C\|Z\|_{H^1}\, h\|v\|_{H^2},$$

by approximation theory. Hence

$$\|Z\|_{L^2}^2 \leq Ch\|Z\|_{H^1}\|Z\|_{L^2}.$$

Therefore,

$$(4.4) \qquad \|Z\|_{L^2} \leq Ch\|Z\|_{H^1}.$$

Now

$$(LZ,Z) = -(Z',Z') + (aZ' + bZ,Z),$$

hence

$$\|Z'\|^2_{L^2} \leqslant |(LZ,Z)| + |(aZ'+bZ,Z)|$$
$$\leqslant |(LZ,Z)| + C\|Z\|_{H^1}\|Z\|_{L^2}$$
$$\leqslant |\langle\langle LZ,Z\rangle\rangle| + Ch\|Z\|_{H^1}\|Z\|_{L^2} + C\|Z\|_{H^1}\|Z\|_{L^2},$$

where we have use Corollary 2.3. Using (4.2) and (4.4), we conclude that for h small enough

$$\|Z\|^2_{H^1} \leqslant C\,h\,\|Z\|^2_{H^1},$$

which concludes the proof.

Following the same outline as in the proof of (4.4) in the proof of Theorem 4.1, we can conclude the following

Lemma 4.1

Let $e = u - U$, where u is the solution to (1.1), if $u \in H^1(I)$. Then, for h small enough, we have

$$\|e\|_{L^2} \leqslant Ch\|e\|_{H^1}.$$

§4.2 Global Convergence

In this paragraph we dealt with the question of L^2–convergence of the hybrid procedure defined by (1.3)–(1.4), (or equivalently by (4.1)). To obtain estimates we use the known properties of the Galerkin procedure presented in Chapter II. The main result of this paragraph is the following

Theorem 4.2

Let u be the solution to (1.1), let $U \in \mathcal{M}^r_0$ be the solution to (4.1). If $u \in W^{r+1}(I)$, then

$$(4.5) \qquad \|u-U\|_{L^2} + h\|u-U\|_{H^1} \leqslant Ch^{r+1}\|u\|_{W^{r+1}}.$$

Before we go into the proof of the theorem we shall examine in detail the case $u'' = f$. Let χ be the Galerkin solution and U the hybrid solution. That is χ and U satisfy

$$-(\chi',V') = (f,V), \quad \forall V \in \mathcal{M}^r_0$$

and

$$\langle\langle U'',V \rangle\rangle = \langle\langle f,V \rangle\rangle, \quad \forall V \in M_0^r.$$

Recall that $J_1(W,x_i) = \langle L(u-W), V_i \rangle$, $i = 1, \cdots, n$, for both $W = \chi$ and $W = U$; then,

$$\sum_{j=1}^{n} J_1(W,x_j)\, G(x,x_j) = \langle L(u-W), \sum_{j=1}^{n} V_j\, G(x_i,x_j)\rangle$$

where G is the Greens function for $u'' = f$. Since $G(x,z)$ as a function of z, for fixed $x = x_i$, is in M_0^1, we obtain

$$\sum_{j=1}^{n} J_1(W,x_j)\, G(x_i,x_j) = \langle L(u-W), G(x_i,\cdot)\rangle.$$

Then by the Green's function representation formula (2.3)

$$(u-W)(x_i) = \langle L(u-W), G(x_i,\cdot)\rangle + \sum_{j=1}^{n} J_1((u-W),x_j)\, G(x_i,x_j)$$

$$= \langle L(u-W), G(x_i,\cdot)\rangle - \sum_{j=1}^{n} J_1(W,x_j)\, G(x_i,x_j)$$

$$= \langle L(u-W), G(x_i,\cdot)\rangle - \langle L(u-W), G(x_i,\cdot)\rangle = 0.$$

therefore, $u(x_i) = \chi(x_i) = U(x_i)$.

Now by Lemma 2.2 we know that the Galerkin solution χ satisfies

$$\| (u-\chi)'' \|_{L^\infty(I_i)} \leq C h_i^{r-3/2} \|u\|_{H^{r+1}(I_i)};$$

therefore,

$$| (\chi - U)''(x_{ij}) | \leq C h_i^{r-3/2} \|u\|_{H^{r+1}(I_i)}$$

Let $\theta = \chi - U$, then $\theta'' \in P_{r-2}(I_i)$ which implies by (2.2)

(4.6) $$\| \theta'' \|_{L^2(I_i)} \leq C h_i^{r-1} \|u\|_{H^{r+1}(I_i)}.$$

For $x \in I_i$

$$\theta(x) = \theta(x_i) + \int_{x_i}^{x} \theta'(x)dx = \int_{x_i}^{x} \theta'(x)dx,$$

since $\theta(x_i) = 0$; therefore,

$$\theta(x_{i+1}) = \int_{x_i}^{x_{i+1}} \theta'(x)dx = 0,$$

but

$$\int_{x_i}^{x_{i+1}} \theta'(x)dx = \int_{x_i}^{x_{i+1}} \theta'(x_i)dx + \int_{x_i}^{x_{i+1}} \int_{x_i}^{x} \theta''(y)dy\,dx.$$

Thus,

$$h_i \,|\, \theta'(x_i) \,| \leqslant h_i^{3/2} \|\theta''\|_{L^2(I_i)} \,;$$

however,

$$\begin{aligned} |\,\theta(x)\,| &= \left|\int_{x_i}^{x} \theta'(x_i)dx + \int_{x_i}^{x} \int_{x_i}^{y} \theta''(z)dz\,dy\right| \\ &\leqslant h_i \,|\, \theta'(x_i)\,| + h_i^{3/2} \|\theta''\|_{L^2(I_i)} \\ &\leqslant 2h_i^{3/2} \|\theta''\|_{L^2(I_i)} . \end{aligned}$$

Hence, by (2.2)

$$C\,\|\theta\|_{L^2(I_i)} \leqslant h_i^{1/2} \|\theta\|_{L^\infty(I_i)} \leqslant Ch^2 \|\theta''\|_{L^2(I_i)} .$$

Which together with (4.6) implies

$$\|\theta\|_{L^2(I_i)} \leqslant Ch_i^{r+1} \|u\|_{H^{r+1}(I_i)} .$$

Thus

$$\begin{aligned} \|\theta\|_{L^2} = \left(\sum_{i=1}^{n} \|\theta\|^2_{L^2(I_i)}\right)^{1/2} &\leqslant Ch^{r+1}\left(\sum_{i=1}^{n} \|u\|^2_{H^{r+1}(I_i)}\right)^{1/2} \\ &= Ch^{r+1} \|u\|_{H^{r+1}} . \end{aligned}$$

We have just given a proof of the following

Lemma 4.2

If U is the hybrid approximation, and χ the Galerkin approximation to the problem

$$u'' = f, \quad \text{on } I$$

$$u(0) = u(1) = 0$$

Then

$$\| \chi - U \|_{L^2} \leqslant Ch^{r+1} \|u\|_{H^{r+1}} .$$

Furthermore

$$\| u - U \|_{L^2} \leqslant Ch^{r+1} \|u\|_{H^{r+1}} .$$

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2

The idea employed in this proof of comparing the approximate solution of (1.1) with that of the more simple problem $u'' = f$, was originated by Wheeler. See [5].

Because of Lemma 4.1 all we have to estimate is $\| u - U \|_{H^1}$. Let $e = u - U$, $\check{e} = u - \chi$ and $\bar{e} = \chi - U$, where $U \in M_0^r$ is the solution to (4.1) and $\chi \in M_0^r$ is the solution to (2.5). Note that $e = \check{e} + \bar{e}$, $\bar{e} \in M_0^r$ and $(L\check{e},\bar{e}) = 0$. Thus

$$(Le,e) = -(e',e') + (ae' + be,e),$$

then

$$\|e'\|_{L^2}^2 \leqslant C \|e\|_{H^1} \|e\|_{L^2} + |(Le,e)| .$$

Therefore, by Lemma 4.1 for h small enough

$$\begin{aligned}
(4.7) \qquad \|e\|_{H^1}^2 &\leqslant C |(Le,e)| = C|(Le,\check{e}) + (L\check{e},\bar{e}) + (L\bar{e},\bar{e})| \\
&\leqslant C \|e\|_{H^1} \|\check{e}\|_{H^1} + C|(L\bar{e},\bar{e})| \\
&\leqslant C\|e\|_{H^1} \|\check{e}\|_{H^1} + C|\langle\langle L\bar{e},\bar{e}\rangle\rangle| + Ch\|\bar{e}\|_{H^1}\|\bar{e}\|_{L^2} .
\end{aligned}$$

where we have use Corollary 3.3. However,

$$\begin{aligned}
(4.8) \qquad -\langle\langle L\bar{e},\bar{e}\rangle\rangle &= \langle\langle L\check{e},\bar{e}\rangle\rangle \\
&= \langle\langle \check{e}'',\bar{e}\rangle\rangle + \langle\langle a\check{e}',\bar{e}\rangle\rangle + \langle\langle b\check{e},\bar{e}\rangle\rangle \\
&\leqslant |\langle\langle \check{e}'',\bar{e}\rangle\rangle| + C\{\|\check{e}'\|_{L^\infty}\|\bar{e}\|_{L^2} + C\|\check{e}\|_{H^1}\|\bar{e}\|_{L^2}\},
\end{aligned}$$

by Lemma 3.3 and Corollary 3.1.

Let $U^* \in M_0^r$ be the solution to

$$\langle\langle U^{*''},V\rangle\rangle = \langle\langle (f - au' - bu), V\rangle\rangle, \quad \forall\, V \in M_0^r .$$

Let $\eta = u - U^*$, by Lemma 4.2 $\|\eta\|_{H^1} \leq Ch^r \|u\|_{H^{r+1}}$. Thus

$$\langle\langle \check{e}'', \bar{e} \rangle\rangle = \langle\langle \eta'', \bar{e} \rangle\rangle - \langle\langle (\chi - U^*)'', \bar{e} \rangle\rangle \tag{4.9}$$
$$= ((\chi - U^*)', \bar{e}')$$
$$= (\check{e}', e') + (\eta', \bar{e}')$$
$$\leq \|\check{e}\|_{H^1} \|\bar{e}\|_{H^1} + \|\eta\|_{H^1} \|\bar{e}\|_{H^1}$$
$$= \|\bar{e}\|_{H^1} (\|\check{e}\|_{H^1} + \|\eta\|_{H^1}).$$

Because of (2.8)

$$\|\check{e}\|_{H^1} \leq C\|e\|_{H^1},$$

therefore, by the triangle inequality

$$\|\bar{e}\|_{H^1} \leq C\|e\|_{H^1}.$$

By inequalities (4.8) and (4.9), we obtain from (4.7) that for h small enough

$$\|e\|^2_{H^1} \leq C\|e\|_{H^1} \{\|\check{e}\|_{H^1} + \|\check{e}'\|_{L^\infty} + \|\eta\|_{H^1}\},$$

thus

$$\|e\|_{H^1} \leq C \{\|\check{e}\|_{H^1} + \|\check{e}'\|_{L^\infty} + \|\eta\|_{H^1}\}.$$

Estimates (2.7) and (2.9) and lemma 4.2 complete the proof of the theorem.

We have obtained optimal rates of convergence for the hybrid procedure defined by (1.3) – (1.4). The next task is to observe that at some special points this rate of convergence is actually improved. For this we shall need the following

Lemma 4.3

There exist a constant C that depends only on $\max_{0 \leq i \leq n} h h_i^{-1}$ and $\|u\|_{W^{r+1}}$ such that for h small enough

$$\|U^{(\alpha)}\|_{L^\infty(I_i)} \leq C \qquad \text{for all } \alpha > 0.$$

Proof.

Equation (4.4), quasiuniformity and the triangular inequality implies, for $\alpha = 0, \cdots, r$

$$\| U^{(\alpha)} \|_{L^2(I_i)} \leq Ch^{r+1} h_i^{-\alpha} \| u \|_{W^{r+1}} + \| u^{(\alpha)} \|_{L^2(I_i)}$$
$$\leq Ch^{1+r-\alpha} \| u \|_{W^{r+1}} + h^{1/2} \| u^{(\alpha)} \|_{L^\infty(I_i)}$$
$$\leq Ch^{1/2} \| u \|_{W^{r+1}} \cdot$$

therefore, by (2.2)

$$\| U^{(\alpha)} \|_{L^\infty(I_i)} \leq Ch_i^{-1/2} \| U^{(\alpha)} \|_{L^2(I_i)} \leq C \| u \|_{W^{r+1}} \cdot$$

For $\alpha > r$, $U^{(\alpha)} \equiv 0$ which concludes the proof.

Chapter V

Superconvergence Estimates at the Nodes

§5.1 Estimates at the nodes.

The global convergence rate of Theorem 4.2 is of optimal order. We cannot expect faster rates of global convergence to hold. However, de Boor and Swartz [1] have shown that the error at the nodes $x = x_i$ for C^1–collocation method for the problem (1.1), using the Gaussian points as collocation points, yields a $O(h^{2r-2})$ rate of convergence in the uniform interval case. Furthermore, Douglas and Dupont [2] have shown that the Galerkin procedure for the problem (1.1) using M_0^r yields a $O(h^{2r})$ rate of convergence at the nodes. The object of this chapter is to obtain $O(h^{2r})$ rate of convergence at the nodes for the hybrid collocation–Galerkin procedure.

Let u be the solution of (1.1), U be the collocation solution of (4.1) and G be the Green's function for (1.1). Also let $e = u - U$. We write $e^{(j)} = (d/dx)^j e$ and $G_j(x,t) = (\partial/\partial x)^j G(x,t)$ for $j = 0,1$. By the Green's function representation formula (2.3) – (2.4) we obtain that for $k = 1, \cdots, n$, $j = 0,1$.

$$\begin{aligned} e^{(j)}(x_k+) &= (Le, G_j(x_k+,\cdot)) \\ &= \langle Le, G_j(x_k+,\cdot)\rangle + \sum_{i=1}^{n} J_1(e,x_i)\, G_j(x_k+,x_i) \\ &= \langle Le, G_j(x_k+,\cdot)\rangle - \sum_{i=1}^{n} J_1(U,x_i)\, G_j(x_k+,x_i) \\ &= \langle Le, G_j(x_k+,\cdot)\rangle - \sum_{i=1}^{n} \langle Le, V_i\rangle G_j(x_k+,x_i) \end{aligned}$$

because of (2.6). Thus

$$\sum_{i=1}^{n} \langle Le, V_i\rangle G_j(x_k+,x_i) = \langle Le, \sum_{i=1}^{n} G_j(x_k+,x_i) V_i\rangle, \quad k = 1, \cdots, n, \; j = 0,1.$$

Therefore, we obtain

$$e^{(j)}(x_k+) = \langle Le, g_k^j\rangle, \quad k = 1, \cdots, n; \; j = 0,1. \tag{5.1}$$

where $g_k^j(\xi) = G_j(x_k+, \xi) - \sum_{i=1}^{n} G_j(x_k+, x_i)V_i(\xi)$, $\xi \in \mathring{I}_i$, $k = 1, \cdots, j = 0,1$. Now we are ready to estimate the rate of convergence at the nodes $x = x_i$ for the function and its derivate.

§5.2 Estimates for the function

To obtain estimates at the nodes we will use (5.1). We observe that since G is a continuous function on I x I, the function $g_k = g_k^0$ is a continuous function on I with $g_k(x_i) = 0$, $i = 0, \cdots, n$. On each subinterval I_i we let $\bar{g}_k$ be the function that satisfies the equation

$$g_k(\xi) = (\xi - x_i)(x_{i+1} - \xi)\, \bar{g}_k(\xi), \quad \xi \in I_i .$$

We also observe that $Le(x_{ij}) = 0$, $i = 0, \cdots, n$; $j = 1, \cdots, r-1$. And we let Γ be the function that on each subinterval I_i satisfies the equation

$$Le = \prod_{j=1}^{r-1} (\xi - x_{ij})\, \Gamma(\xi), \quad \xi \in I_i .$$

Thus, multiplying Le by g_k and integrating over I_i, we obtain

$$(5.2) \qquad (Le, g_k)_i = \int_{I_i} (\xi - x_i)(x_{i+1} - \xi) \prod_{j=1}^{r-1} (\xi - x_{ij})\, \bar{g}(\xi)\Gamma(\xi) d\xi.$$

Now we take a Taylor series on the subinterval I_i of the function $\bar{g} \cdot \Gamma$ about the point $x = x_i$ for convenience. We obtain

$$(5.3) \qquad (\bar{g}_k \Gamma)(\xi) = P_s(\xi) + \frac{(\xi - x_i)^{s+1}}{(s+1)!} \left(D^{s+1}(\bar{g}_k \Gamma)(\tilde{\xi})\right)$$

where P_s is a polynomial on I_i of degree at most s, $\tilde{\xi} = \tilde{\xi}(\xi) \in I_i$, and $-1 \leqslant s \leqslant r-2$.

Combining (5.2) and (5.3), we obtain

$$(5.4) \qquad (Le, g_k)_i = \int_{I_i} (\xi - x_i)(x_{i+1} - \xi) \prod_{j=1}^{r-1} (\xi - x_{ij})\, P_s(\xi) d\xi$$
$$+ \int_{I_i} (\xi - x_i)(x_{i+1} - \xi) \prod_{j=1}^{r-1} (\xi - x_{ij}) \frac{(\ \ -x_i)^{s+1}}{(s+1)!} \left(D^{s+1}(\bar{g}_k \Gamma)(\tilde{\xi})\right) d\xi .$$

By the way the collocation points $\{x_{ij}\}$ were selected —(3.2). We find that the first term in the right hand side of (5.4) is Zero, for $-1 \leqslant s \leqslant r-2$. Now note that if the coefficients a and b in (1.1) are elements of $C^{2r-2}(I)$, then

$$G(x_k, \cdot) \in W^{2r-1}[0, x_k] \cap W^{2r-1}[x_k, 1],$$

thus,

(5.5) $$\| D^{s+3-\ell}G(x_k,\cdot) \|_{L^\infty(I_i)} \leq C, \quad \ell = 0,\cdots,s+1, \; -1 \leq s \leq r-2.$$

By lemma 4.3, and the assumptions on a and b,

(5.6) $$\| D^{r+\ell-1} Le \|_{L^\infty(I_i)} \leq C \|u\|_{W^{r+\ell+1}}, \; \ell = 0,\cdots,s+1, \; -1 \leq s \leq r-2.$$

and by the definitions of $\bar{g}$ and Γ, and the mean value theorem,we find

$$D^{s+1}(\bar{g},\Gamma)(\xi) = \frac{1}{2\cdot(r-1)!} \sum_{\ell=0}^{s+1} \binom{s+1}{\ell} (D^{s+2-\ell}G(x_k,\breve{\xi}))\cdot(D^{r+\ell-1}Le(\bar{\xi})),$$

where $\breve{\xi} = \breve{\xi}(\xi)$ and $\bar{\xi} = \bar{\xi}(\xi)$ both belong to I_i. Therefore, combining (5.5) and (5.6) we obtain

$$\| D^{s+1}(\bar{g}\Gamma) \|_{L^\infty(I_i)} \leq C \|u\|_{W^{r+s+2}}, \; -1 \leq s \leq r-2,$$

which in turn implies

$$|(Le,g_k)_i| \leq Ch^{r+s+3} \|u\|_{W^{r+s+2}}, \quad -1 \leq s \leq r-2.$$

Collecting and summing on i, we obtain

$$\max_{1\leq k\leq n} |(u-U)(x_i)| \leq Ch^{r+s+2} \|u\|_{W^{r+s+2}}, \; -1 \leq s \leq r-2$$

§5.3 Estimates for the derivative

The argument we will use here to estimate the rate of convergence for the derivative is similar to that one used for the function in the previous paragraph. However,some difficulties arise mainly because of the jump discontinuity of the derivative of the Green's function G.

We let Γ be as in §5.2 and observe that for $i \neq k$, $g_k^1(x_i+) = g_k^1(x_{i+1}-) = 0$. As in §5.2 we obtain for $i \neq k$

$$|(Le,g_k^1)_i| \leq Ch^{r+s+3} \|u\|_{W^{r+s+2}}, \; -1 \leq s \leq r-2.$$

But for $i = k$, the argument breaks down as $g_k^1(x_{k+1}-) \neq 0$. However, since $g_k^1(x_k+) = 0$ we let $\bar{g}_k^1$ on I_k be the solution to the equation

$$g_k^1(\xi) = (\xi - x_k)\, g_k^1(\xi), \quad \xi \in I_k,$$

therefore

$$|\,(Le, g_k^1)_k\,| \leqslant |\int_{I_k} (\xi - x_k) \prod_{j=1}^{r-1} (\xi - x_{kj})\, \bar{g}_k^1(\xi)\, \Gamma(\xi)\, d\xi\,|,$$

where the term in the right of the inequality can be bounded by $Ch^{r+1} \|u\|_{W^{r+1}}$. Therefore, collecting terms and summing over i, we obtain

$$\max_{1 \leqslant k \leqslant n} |\,(u-U)'(x_k+)\,| \leqslant Ch^{r+1} \|u\|_{W^{r+2}},$$

which is an order of superconvergence more than the global order.

§5.4 Conclusion

We can summarize the result of this chapter in the following theorem.

Theorem 5.1

Let u be a solution of (1.1). Let $a, b \in c^{\ell-2}(I)$ for some $\ell > r+1$. Let $U \in \mathcal{M}_0^r$ be the solution to (1.3) – (1.4). Then, there exists a constant C independent of h such that

$$\max_{1 \leqslant i \leqslant n} |\,(u-U)(x_i)\,| \leqslant Ch^s \|u\|_{W^s}, \quad s = \min(2r, \ell)$$

further, if $\ell \geqslant r+2$,

$$\max_{1 \leqslant i \leqslant n} |\,(u-U)'(x_i+)\,| \leqslant Ch^{r+1} \|u\|_{W^{r+2}},$$

and

$$\max_{1 \leqslant i \leqslant n} |\,(u-U)'(x_i-)\,| \leqslant Ch^r \|u\|_{W^{r+1}}.$$

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