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THE COMPUTATION OF OPTIMAL CONTROLS
IN THE PRESENCE OF NONDIFFERENTIAL CONSTRAINTS
BY A SEQUENTIAL CONJUGATE GRADIENT-RESTORATION TECHNIQUE

by

JAMES R. CLOUTIER

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
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1. Introduction

Approximately seven or eight years ago, conjugate gradient techniques began appearing on the optimal control scene. Ladson et al. (Ref. 1) extended the conjugate gradient minimization method of Fletcher and Reeves to optimal control problems in 1967. About the same time, Horwitz and Sarachik (Ref. 2) extended Davidon's method to a real Hilbert space and applied the extension to a quadratic cost-linear dynamics control problem. Shortly thereafter, Ladson (Ref. 3) and Tripathi and Narendra (Ref. 4) also derived extensions of Davidon's method.

One common limitation of the above algorithms is that they are directly applicable only to unconstrained control problems. While they do have the ability to handle problems with terminal conditions and inequality constraints, they can do so only by first converting such problems to an unconstrained form. This conversion is usually achieved by the use of penalty functions.

Conjugate gradient extensions which can solve certain types of constrained problems directly were constructed in Refs. 5-7. Sinnott and Luenberger presented an algorithm for problems with linear terminal constraints in Ref. 5, Heideman and Levy presented an algorithm for problems with arbitrary terminal constraints in Ref. 6, and Pagurek and Woodside presented an algorithm for problems with bounded controls in Ref. 7.

In the area of ordinary gradient methods for constrained problems, Miele et al. (Ref. 8) developed a sequential gradient-restoration algorithm for optimal control problems where the state \( x(t) \), the control \( u(t) \), and the parameter \( \pi \) have to satisfy not only differential constraints and terminal constraints, but also nondifferential constraints everywhere.
along the interval of integration. The importance of that work lies in the fact that (i) many optimization problems arise directly in the form considered there, (ii) problems involving state equality constraints can be reduced to that form through suitable transformations, and (iii) problems involving inequality constraints can be reduced to that form through suitable transformations. Thus, an extremely large class of problems can be handled. Some of the often-encountered problems included in this class are problems with bounded control, problems with bounded state, problems with bounded time rate of change of the state, and problems with a bound on a function of the parameter, the control, the state, and the time rate of change of the state.

This thesis combines the ideas of Ref. 6 and Ref. 8. The result is a conjugate gradient algorithm which can handle arbitrary constraints without resorting to penalty functions. The algorithm defined herein consists of a sequence of conjugate gradient-restoration cycles.

The conjugate gradient portion of each cycle is based upon a conjugate gradient algorithm that is derived for the special case of a quadratic functional subject to linear constraints. This conjugate gradient portion involves a single iteration and is designed to decrease the value of the functional while satisfying the constraints to first order. During this iteration, the first variation of the functional I is minimized subject to the linearized constraints and linearized boundary conditions. This minimization is performed over the class of variations of the control and the parameter which are the same distance from some constant multiple of the corresponding variations of the previous conjugate gradient phase.

The restoration portion of each cycle involves one or more iterations and is designed to restore the constraints to a predetermined accuracy.
while minimizing the norm of the variations of the control and the parameter. To achieve this constraint satisfaction, quasilinearization (Newton's method) is employed.

The algorithm is characterized by two main properties. First, at the end of each conjugate gradient–restoration cycle, the trajectories satisfy the constraints to a given accuracy. Thus, a sequence of feasible suboptimal solutions is produced. Second, due to the fact that the conjugate gradient corrections are of $O(\alpha_g)$ while the restoration corrections are of $O(\alpha_g^2)$, the conjugate gradient stepsize $\alpha_g$ can be chosen sufficiently small so that the restoration phase preserves the descent property of the conjugate gradient phase. Thus, the value of the functional $\hat{I}$ at the end of any cycle is smaller than the value of the functional $I$ at the beginning of the cycle.
2. **Formulation of the Problem**

The problem that we wish to solve numerically is the following.

Minimize the functional

\[
I = \int_0^\tau f^*_\alpha(x, u, \bar{\nu}, \theta) d\theta + \left[ g^*_\alpha(x, \bar{\nu}, \theta) \right]_\tau
\]

with respect to the state \(x(\theta)\), the control \(u(\theta)\), and the parameters \(\bar{\nu}\) and \(\tau\), subject to the differential constraint

\[
dx/d\theta = \phi^*_\alpha(x, u, \bar{\nu}, \theta), \quad 0 \leq \theta \leq \tau
\]

the nondifferential constraint

\[
E^*_\alpha(x, u, \bar{\nu}, \theta) = 0, \quad 0 \leq \theta \leq \tau
\]

and the boundary conditions

\[
(x)_0 = \text{given}, \quad [\psi^*_\alpha(x, \bar{\nu}, \theta)]_\tau = 0
\]

In the above equations, the functions \(f^*_\alpha\) and \(g^*_\alpha\) are scalar, the function \(\phi^*_\alpha\) is an \(n\)-vector, the function \(E^*_\alpha\) is a \(k\)-vector, and the function \(\psi^*_\alpha\) is a \(q\)-vector. The independent variable is the actual time \(\theta\), while the dependent variables are the \(n\)-vector state \(x\), the \(m\)-vector control \(u\), the \(\bar{p}\)-vector parameter \(\bar{\nu}\), and the scalar parameter \(\tau\). At the initial time \(\theta = 0\), \(n\) scalar relations are specified. At the final time \(\theta = \tau\), \(q\) scalar relations are specified, where \(q \leq n + \bar{p}\) if \(\tau\) is fixed and \(q \leq n + \bar{p} + 1\) if \(\tau\) is free.

To facilitate the implementation of the algorithm on a digital computer, we replace the actual time \(\theta\) with the normalized time \(t\). The latter is defined in such a way that the interval of integration is of
unit length. Thus, in normalized form, \( t = 0 \) denotes the time at which the initial boundary (4-1) is left and \( t = 1 \) denotes the time at which the terminal boundary (4-2) is reached. The following linear relation allows passage from the normalized time \( t \) to the actual time \( \theta \):

\[
\theta = \tau t, \quad 0 \leq t \leq 1
\]

(5)

The fact that the normalized final time \((t=1)\) is fixed does not cause any loss of generality in the problem. If the actual final time is free, \( \tau \) simply becomes a parameter to be optimized in the transformed problem. In view of this, we define the \( p \)-vector parameter

\[
\pi = \frac{\pi}{\tau} \quad \text{or} \quad \pi = \begin{bmatrix} \frac{\pi}{\tau} \end{bmatrix}
\]

(6)

where (6-1) holds if \( \tau \) is fixed and (6-2) holds if \( \tau \) is free.

In addition to the normalized time \( t \) and the parameter \( \pi \), we define the following functions:

\[
f(x,u,\pi,t) = \tau f_*(x,u,\bar{\pi},\theta), \quad 0 \leq t \leq 1
\]

(7)

\[
\phi(x,u,\pi,t) = \tau \phi_*(x,u,\bar{\pi},\theta), \quad 0 \leq t \leq 1
\]

(8)

\[
E(x,u,\pi,t) = E_*(x,u,\bar{\pi},\theta), \quad 0 \leq t \leq 1
\]

(9)

\[
g(x,\pi,t) = g_*(x,\bar{\pi},\theta)
\]

(10)

\[
\psi(x,\pi,t) = \psi_*(x,\bar{\pi},\theta)
\]

(11)

Under the above transformation and definitions, problem (1)-(4) can be reformulated as follows: Minimize the functional

\[
I = \int_0^1 f(x,u,\pi,t) dt + [g(x,\pi,t)]_1
\]

(12)
with respect to the state $x(t)$, the control $u(t)$, and the parameter $\pi$
subject to the differential constraint
\[ \dot{x} = \phi(x, u, \pi, t), \quad 0 \leq t \leq 1 \]  \hspace{1cm} (13)

the nondifferential constraint
\[ E(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1 \]  \hspace{1cm} (14)

and the boundary conditions
\[ (x)_0 = \text{given}, \quad [\psi(x, \pi, t)]_1 = 0 \]  \hspace{1cm} (15)

From the calculus of variations, we know that problem (12)-(15) is
one of the Bolza type; it can be recast as that of minimizing the
Lagrangian\(^1\)
\[ J = \int_0^1 F \, dt + (G)_1 \]  \hspace{1cm} (16)

with respect to the state $x(t)$, the control $u(t)$, and the parameter $\pi$
subject to (13)-(15), where the functions $F$ and $G$ are given by
\[ F = f + \lambda^T(x - \phi) + \rho^T E, \quad G = g + \mu^T \psi \]  \hspace{1cm} (17)

and where $\lambda(t)$ is a variable Lagrange multiplier (an $n$-vector), $\rho(t)$ is a
variable Lagrange multiplier (a $k$-vector), and $\mu$ is a constant Lagrange
multiplier (a $q$-vector).

The optimal solution $x(t), u(t), \pi$ must satisfy (13)-(15), the
Euler equations

\(^1\)In Eq. (16), it is tacitly assumed that the initial condition (15-1) is satisfied.
\[ F_\vec{x} - (d/dt) F_\vec{x} = 0, \quad F_u = 0, \quad \int_0^1 F_\pi dt + (G_\pi)_L = 0 \quad (18) \]

and the following natural condition arising from the transversality condition:

\[ (F_\vec{x} + G_\vec{x})_L = 0 \quad (19) \]

Thus, we seek functions \( x(t), u(t), \pi \) and multipliers \( \lambda(t), \rho(t), \mu \) which satisfy the constraints (13)-(15) and the following optimality conditions:

\[ \dot{\lambda} - f_\vec{x} \lambda - E_\vec{x} \rho = 0, \quad 0 \leq t \leq 1 \quad (20) \]

\[ f_u - \phi_u \lambda + E_u \rho = 0, \quad 0 \leq t \leq 1 \quad (21) \]

\[ \int_0^1 (f_\pi - \phi_\pi \lambda + E_\pi \rho) dt + (g_\pi + \psi \mu)_L = 0 \quad (22) \]

\[ (\lambda + g_\vec{x} + \psi \mu)_L = 0 \quad (23) \]

**Approximate Methods**

In general, the differential system (13)-(15) and (20)-(23) is nonlinear. Therefore, some type of iterative approximation must be employed in its solution. For this purpose, let us define the scalar functionals \( P \) and \( Q \) which denote the error in the constraints and the error in the optimality conditions, respectively. We have

\[ P = \int_0^1 N(\dot{x} - \phi) dt + \int_0^1 N(E) dt + N(\psi)_L \quad (24) \]

and
\[ Q = \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda - E_x \rho) dt + \int_0^1 N(f_u - \phi_u \lambda + E_u \rho) dt \]
\[ + N \left[ \int_0^1 (f_\pi - \phi_\pi \lambda + E_\pi \rho) dt + (g_\pi + \psi_\pi \mu)_1 \right] + N(\lambda + g_x + \psi_x \mu)_1 \tag{25} \]

where \(N(.)\) denotes the square of the vector norm, i.e.,

\[ N(a) = a^T a \tag{26} \]

for given vector \(a\).

For the optimal solution, \(P = 0\) and \(Q = 0\). For an approximation to the optimal solution,

\[ P \leq \varepsilon_1, \quad Q \leq \varepsilon_2 \tag{27} \]

where \(\varepsilon_1\) and \(\varepsilon_2\) are small, preselected numbers.

**Notation**

For any iteration of the algorithms which will be constructed in the next two sections, let us adopt the following terminology: Let \(x(t), u(t), \pi\) denote the nominal functions and let \(\bar{x}(t), \bar{u}(t), \bar{\pi}\) denote the varied functions. Furthermore, let \(\alpha\) denote the stepsize and let \(A(t), B(t), C\) denote displacements per unit stepsize. Then,

\[ \bar{x}(t) = x(t) + \Delta x(t), \quad \bar{u}(t) = u(t) + \Delta u(t), \quad \bar{\pi} = \pi + \Delta \pi \tag{28} \]

and

\[ \Delta x(t) = \alpha A(t), \quad \Delta u(t) = \alpha B(t), \quad \Delta \pi = \alpha C \tag{29} \]

which imply

\[ \bar{x}(t) = x(t) + \alpha A(t), \quad \bar{u}(t) = u(t) + \alpha B(t), \quad \bar{\pi} = \pi + \alpha C \tag{30} \]
3. **Conjugate Gradient Algorithm for Quadratic Functional and Linear Constraints**

In this section we construct an algorithm for solving problem (12)-(15) for the special case where the functions \( f, g \) are quadratic and the functions \( \phi, \psi, E \) are linear. In each iteration of this algorithm, our objective is to reduce the functional \( I \) or the augmented functional \( J \) while remaining on the constraints.

Assuming that nominal functions \( x(t), u(t), \pi \) satisfying (13)-(15) are available, there are several ways to compute the variations \( \Delta x(t), \Delta u(t), \Delta \pi \) through which we can achieve our objective. One way, for example, can be found in Ref. 8. There, the gradient of \( J \) was projected onto the tangential hyperplane of the constraints, and the variations were selected by searching along the negative direction of the projection. It was shown there that equivalent variations could be obtained by solving the following problem: From the class of variations of the control and the parameter having the same norm, find the variations \( \Delta u(t), \Delta \pi \), along with the state variation \( \Delta x(t) \), which minimize the first variation of \( I \) \( \delta I \) subject to the first order change of (13)-(15).

For our algorithm we will use a slightly different approach from that of Ref. 8. Here, we will use information from the previous iteration as well as from the present iteration. In this case, we will restrict our attention to the class of variations of the control and the parameter which are the same distance from some constant multiple of their predecessors. Thus, we seek variations \( \Delta x(t), \Delta u(t), \Delta \pi \) which minimize

\[
\delta I = \int_0^1 (f_x^T \Delta x + f_u^T \Delta u + f_\pi^T \Delta \pi) dt + (g_x^T \Delta x + g_\pi^T \Delta \pi)_1
\]

(31)
subject to the linearized constraint equations

\[
\Delta \dot{x} - \phi_x^T \Delta x - \phi_u^T \Delta u - \phi_{\pi}^T \Delta \pi = 0, \quad 0 \leq t \leq 1
\]  
(32)

\[
E_x^T \Delta x + E_u^T \Delta u + E_{\pi}^T \Delta \pi = 0, \quad 0 \leq t \leq 1
\]  
(33)

the linearized boundary conditions

\[
(\Delta x)_0 = 0, \quad (\psi_x^T \Delta x + \psi_{\pi}^T \Delta \pi)_1 = 0
\]  
(34)

and the isoperimetric constraint

\[
K = \int_0^1 (\Delta u - \varepsilon \Delta \hat{u})^T (\Delta u - \varepsilon \Delta \hat{u}) dt + (\Delta \pi - \varepsilon \Delta \hat{\pi})^T (\Delta \pi - \varepsilon \Delta \hat{\pi})
\]  
(35)

where \(K, \varepsilon\) are undetermined constants. The variations \(\Delta \dot{x}(t), \Delta \dot{u}(t), \Delta \hat{\pi}\) are the variations leading from the previous solution \(\dot{x}(t), \dot{u}(t), \hat{\pi}\) to the present solution \(x(t), u(t), \pi\), that is,

\[
\Delta \dot{x}(t) = x(t) - \dot{x}(t), \quad \Delta \dot{u}(t) = u(t) - \dot{u}(t), \quad \Delta \pi = \pi - \hat{\pi}
\]  
(36)

For the first iteration of the algorithm we take

\[
\Delta \dot{x}(t) = 0, \quad \Delta \dot{u}(t) = 0, \quad \Delta \hat{\pi} = 0
\]  
(37)

From the calculus of variations, we know that problem (31)-(35) is one of the Bolza type with an added isoperimetric constraint on the variations of the control and the parameter; it can be recast as that of minimizing the Lagrangian\(^2\)

\[
J' = \int_0^1 F' dt + (G')_1
\]  
(38)

\(^2\)In Eq. (38), it is tacitly assumed that the initial condition (34-1) is satisfied.
with respect $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ subject to (32)-(35), where the functions $F'$ and $G'$ are given by

$$F' = f_x^T \Delta x + f_u^T \Delta u + f_{\pi}^T \Delta \pi + \lambda^T (\Delta \dot{x} - \phi_x^T \Delta x - \phi_u^T \Delta u - \phi_{\pi}^T \Delta \pi)$$
$$+ \rho^T (E_x^T \Delta x + E_u^T \Delta u + E_{\pi}^T \Delta \pi) + (1/2\alpha) (\Delta u - \varepsilon \Delta \hat{u})^T (\Delta u - \varepsilon \Delta \hat{u})$$

(39)

$$G' = g_x^T \Delta x + g_{\pi}^T \Delta \pi + \mu^T (\psi_x^T \Delta x + \psi_{\pi}^T \Delta \pi) + (1/2\alpha) (\Delta \pi - \varepsilon \Delta \hat{\pi})^T (\Delta \pi - \varepsilon \Delta \hat{\pi})$$

(40)

and where $\lambda(t)$ is a variable Lagrange multiplier (an n-vector), $\rho(t)$ is a variable Lagrange multiplier (a k-vector), $\mu$ is a constant Lagrange multiplier (a q-vector), and $1/2\alpha$ is a constant Lagrange multiplier (a scalar). Later, it will be seen that $\alpha$ represents the conjugate gradient stepsize.

The optimal variations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ must satisfy the constraints (32)-(35), the Euler equations

$$F'_{\Delta x} - (d/dt)F'_{\Delta x} = 0, \quad F'_{\Delta u} = 0, \quad \int_0^1 F'_{\Delta \pi} dt + (G'_{\Delta \pi})_l = 0$$

(41)

and the following natural condition arising from the transversality condition:

$$(F'_{\Delta x} + G'_{\Delta x})_l = 0$$

(42)

Hence, we seek variations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ and multipliers $\lambda(t)$, $\rho(t)$, $\mu$, $1/2\alpha$ which satisfy the constraints (32)-(35) and the following optimality conditions:

$$\dot{\lambda} = -f_x^T \lambda - E_x^T \rho = 0, \quad 0 \leq t \leq 1$$

(43)

$$(\Delta u - \varepsilon \Delta \hat{u})/\alpha + f_u^T \lambda + E_u^T \rho = 0, \quad 0 \leq t \leq 1$$

(44)
\[
\frac{(\Delta \pi - \varepsilon \Delta \hat{\pi})}{\alpha} + \int_0^1 (f_{\pi} - \phi_\pi \lambda + E_{\pi} \rho)dt + (g_{\pi} + \psi_\pi \mu)_1 = 0
\]  
(45)

\[
(\lambda + g_{x} + \psi_x \mu)_1 = 0
\]  
(46)

Due to Eqs. (29), Eqs. (32)-(34) and (43)-(46) can be rewritten as

\[
\dot{A} - \phi^T_A - \phi^T_{u} - \phi^T_C = 0, \quad 0 \leq t \leq 1
\]  
(47)

\[
E^T_{\pi} + E^T_{u} + E^T_{\pi} C = 0, \quad 0 \leq t \leq 1
\]  
(48)

\[
\dot{\lambda} - f_{x} + \phi_\lambda - E_{x} \rho = 0, \quad 0 \leq t \leq 1
\]  
(49)

\[
B - \gamma \hat{B} + f_{u} - \phi_\lambda + E_{u} \rho = 0, \quad 0 \leq t \leq 1
\]  
(50)

\[
c - \gamma \hat{C} + \int_0^1 (f_{\pi} - \phi_\pi \lambda + E_{\pi} \rho)dt + (g_{\pi} + \psi_\pi \mu)_1 = 0
\]  
(51)

\[
(A)_o = 0
\]  
(52)

\[
(\psi^T_{x} + \psi^T_{\pi} C)_1 = 0
\]  
(53)

\[
(\lambda + g_{x} + \psi_x \mu)_1 = 0
\]  
(54)

where the directional coefficient \(\gamma\) is given by \(\gamma = \varepsilon \lambda / \alpha\), \(\alpha\) being the previous conjugate gradient stepsize. In addition, the isoperimetric constraint (35) can be rewritten as

\[
K = \alpha^2 \left[ \int_0^1 (B - \gamma \hat{B})^T (B - \gamma \hat{B})dt + (C - \gamma \hat{C})^T (C - \gamma \hat{C}) \right]
\]  
(55)

The solution of the differential system (47)-(54) is given by

\[
A(t) = A_*(t) + \gamma \hat{A}(t), \quad B(t) = B_*(t) + \gamma \hat{B}(t), \quad C = C_* + \gamma \hat{C}
\]  
(56)

\[
\lambda(t) = \lambda_*(t), \quad \rho(t) = \rho_*(t), \quad \mu = \mu_*
\]  
(57)
where the functions $A_*(t)$, $B_*(t)$, $C_*$ and the multipliers $\lambda_*(t)$, $\rho_*(t)$, $\mu_*$ represent the particular solution of Eqs. (47)-(54) for the case of $\gamma = 0$:

$$\dot{A}_* - \phi^T_x A_* - \phi^T_u B_* - \phi^T_\pi C_* = 0, \quad 0 \leq t \leq 1 \quad (58)$$

$$E^T_x A_* + E^T_u B_* + E^T_\pi C_* = 0, \quad 0 \leq t \leq 1 \quad (59)$$

$$\dot{\lambda}_* - \phi^T_x \lambda_* - E^T_x \rho_* = 0, \quad 0 \leq t \leq 1 \quad (60)$$

$$\dot{B}_* + f_u - \phi^T_u \lambda_* + E^T_u \rho_* = 0, \quad 0 \leq t \leq 1 \quad (61)$$

$$C_* + \int_0^1 (f_\pi - \phi^T_\pi \lambda_* + E_\pi \rho_*) dt + (g_\pi + \psi_\pi \mu_*)_1 = 0 \quad (62)$$

$$(A_*)_0 = 0 \quad (63)$$

$$(\psi^T_x A_* + \psi^T_\pi C_*)_1 = 0 \quad (64)$$

$$(\lambda_* + g_x + \psi_x \mu_*)_1 = 0 \quad (65)$$

Equations (58)-(65) form a linear, two-point boundary-value problem, which can be solved in several ways. One, the method of particular solutions, is given in Section 4.4.

Once the functions $A_*(t)$, $B_*(t)$, $C_*$ and the multipliers $\lambda_*(t)$, $\rho_*(t)$, $\mu_*$ are known, the functions $A(t)$, $B(t)$, $C$ can be computed from (56) provided $\gamma$ is specified, while the variations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ can be computed from (29) provided $\alpha$ is specified. Since the varied value of the Lagrangian

$$\tilde{J} = \int_0^1 \left[ \tilde{f} + \lambda^T (\tilde{x} - \tilde{\phi}) + \rho^T \tilde{E} \right] dt + (\tilde{g} + \mu^T \tilde{\psi})_1 \quad (66)$$
can be written as a function of \( \alpha \) and \( \gamma \),

\[
\tilde{J} = \tilde{J}(\alpha, \gamma)
\]  

(67)

it is desirable to find the \((\alpha, \gamma)\) pair that minimizes (67). This pair must satisfy

\[
\tilde{J}_\alpha(\alpha, \gamma) = 0, \quad \tilde{J}_\gamma(\alpha, \gamma) = 0
\]

(68)

or, explicitly,

\[
\int_0^1 \left[ (\tilde{f}_x - \tilde{\phi}_x \lambda + \tilde{E}_x \rho)^T A + \lambda^T A + (\tilde{f}_u - \tilde{\phi}_u \lambda + \tilde{E}_u \rho)^T B \\
+ (\tilde{f}_\pi - \tilde{\phi}_\pi \lambda + \tilde{E}_\pi \rho)^T C \right] dt + \left[ (\tilde{g}_x + \tilde{\psi}_x \mu)^T A + (\tilde{g}_\pi + \tilde{\psi}_\pi \mu)^T C \right]_1 = 0
\]

(69)

\[
\int_0^1 \left[ (\tilde{f}_x - \tilde{\phi}_x \lambda + \tilde{E}_x \rho)^T A + \lambda^T A + (\tilde{f}_u - \tilde{\phi}_u \lambda + \tilde{E}_u \rho)^T B \\
+ (\tilde{f}_\pi - \tilde{\phi}_\pi \lambda + \tilde{E}_\pi \rho)^T C \right] dt + \left[ (\tilde{g}_x + \tilde{\psi}_x \mu)^T A + (\tilde{g}_\pi + \tilde{\psi}_\pi \mu)^T C \right]_1 = 0
\]

(70)

Multiplying Eq. (70) by \( \gamma \), subtracting the product from Eq. (69), and combining the result with Eq. (56) yields

\[
\int_0^1 \left[ (\tilde{f}_x - \tilde{\phi}_x \lambda + \tilde{E}_x \rho)^T A_* + \lambda^T A_* + (\tilde{f}_u - \tilde{\phi}_u \lambda + \tilde{E}_u \rho)^T B_* \\
+ (\tilde{f}_\pi - \tilde{\phi}_\pi \lambda + \tilde{E}_\pi \rho)^T C_* \right] dt + \left[ (\tilde{g}_x + \tilde{\psi}_x \mu)^T A_* + (\tilde{g}_\pi + \tilde{\psi}_\pi \mu)^T C_* \right]_1 = 0
\]

(71)

From Eqs. (47), (48), (53) and Eqs. (56), (57), (62), it can be seen that Eqs. (69)-(71) hold for any \( \lambda, \rho, \mu \). Accounting for this, and further accounting for the linearity of \( \phi, \psi, \) and \( E \), we can rewrite Eqs. (69)-(71) as
\begin{align}
\int_0^1 \left[ (\tilde{f}_x - \phi_x \tilde{\lambda} + E_x \tilde{\phi}) T_A + \tilde{\lambda} T_A + (\tilde{f}_u - \phi_u \tilde{\lambda} + E_u \tilde{\phi}) T_B \right. \\
+ (\tilde{f}_\pi - \phi_\pi \tilde{\lambda} + E_\pi \tilde{\phi}) T_C \bigg] dt + \left[ (\tilde{g}_x + \psi_x \tilde{\mu}) T_A + (\tilde{g}_u + \psi_u \tilde{\mu}) T_B \right. \\
+ (\tilde{g}_\pi + \psi_\pi \tilde{\mu}) T_C \bigg]_1 = 0 \quad (72) \\
\int_0^1 \left[ (\tilde{f}_x - \phi_x \tilde{\lambda} + E_x \tilde{\phi}) \tilde{T}_A + \tilde{\lambda} \tilde{T}_A + (\tilde{f}_u - \phi_u \tilde{\lambda} + E_u \tilde{\phi}) \tilde{T}_B \right. \\
+ (\tilde{f}_\pi - \phi_\pi \tilde{\lambda} + E_\pi \tilde{\phi}) \tilde{T}_C \bigg] dt + \left[ (\tilde{g}_x + \psi_x \tilde{\mu}) \tilde{T}_A + (\tilde{g}_u + \psi_u \tilde{\mu}) \tilde{T}_B \right. \\
+ (\tilde{g}_\pi + \psi_\pi \tilde{\mu}) \tilde{T}_C \bigg]_1 = 0 \quad (73) \\
\int_0^1 \left[ (\tilde{f}_x - \phi_x \tilde{\lambda} + E_x \tilde{\phi}) T_{A*} + \tilde{\lambda} T_{A*} + (\tilde{f}_u - \phi_u \tilde{\lambda} + E_u \tilde{\phi}) T_{B*} \right. \\
+ (\tilde{f}_\pi - \phi_\pi \tilde{\lambda} + E_\pi \tilde{\phi}) T_{C*} \bigg] dt + \left[ (\tilde{g}_x + \psi_x \tilde{\mu}) T_{A*} + (\tilde{g}_u + \psi_u \tilde{\mu}) T_{B*} \right. \\
+ (\tilde{g}_\pi + \psi_\pi \tilde{\mu}) T_{C*} \bigg]_1 = 0 \quad (74)
\end{align}

After integrating by parts and using Eqs. (49), (52), (54) for the next iteration, we obtain the following orthogonality conditions:

\begin{align}
\int_0^1 \tilde{b}_{x}^{T}B_{x} \ dt + \tilde{c}_{x}^{T}C_{x} = 0 \quad (75) \\
\int_0^1 \tilde{b}_{x}^{T}\tilde{B}_{x} \ dt + \tilde{c}_{x}^{T}\tilde{C}_{x} = 0 \quad (76) \\
\int_0^1 \tilde{b}_{x}^{T}B_{x*} \ dt + \tilde{c}_{x}^{T}C_{x*} = 0 \quad (77)
\end{align}

Continuing our manipulations in an effort to find explicit expressions for \( \alpha \) and \( \gamma \), we expand the gradients of \( f \) and \( g \) in Eqs. (69)-(70). Since the functions \( f \) and \( g \) are quadratic and the functions \( \phi \), \( \psi \), and \( E \) are linear, we have
\[ \int_0^1 \left[ (f_\lambda - \phi_\lambda + E_\lambda\rho)^T A + \lambda^T A + (f_u - \phi_u + E_u\rho)^T B \right. \\
+ (f_\pi - \phi_\pi + E_\pi\rho)^T C \bigg] dt + \left[ (g_x + \psi_\mu)^T A + (g_\pi + \psi_\mu)^T C \bigg] \bigg]_1 \\
+ \alpha \left[ \int_0^1 Y^T_{fyy} Y dt + (Z^T_{gzz} Z)_1 \right] = 0 \quad (78) \]

\[ \int_0^1 \left[ (f_\lambda - \phi_\lambda + E_\lambda\rho)^T A + \lambda^T A + (f_u - \phi_u + E_u\rho)^T B \right. \\
+ (f_\pi - \phi_\pi + E_\pi\rho)^T C \bigg] dt + \left[ (g_x + \psi_\mu)^T A + (g_\pi + \psi_\mu)^T C \bigg] \bigg]_1 \\
+ \alpha \left[ \int_0^1 \hat{Y}^T_{fyy} \hat{Y} dt + (\hat{Z}^T_{gzz} \hat{Z})_1 \right] = 0 \quad (79) \]

where

\[ y = \begin{bmatrix} x \\ u \\ \pi \end{bmatrix}, \quad z = \begin{bmatrix} x \\ \pi \end{bmatrix}, \quad Y = \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{bmatrix}, \quad Z = \begin{bmatrix} A \\ C \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} \hat{C} \end{bmatrix} \quad (80) \]

Integration by parts and use of Eqs. (49), (52), (54) produces

\[ \left[ \int_0^1 B_x^T B dt + C_x^T C \right] - \alpha \left[ \int_0^1 Y^T_{fyy} Y dt + (Z^T_{gzz} Z)_1 \right] = 0 \quad (81) \]

\[ \left[ \int_0^1 B_x^T \hat{B} dt + C_x^T \hat{C} \right] - \alpha \left[ \int_0^1 \hat{Y}^T_{fyy} \hat{Y} dt + (\hat{Z}^T_{gzz} \hat{Z})_1 \right] = 0 \quad (82) \]

Due to Eqs. (56) and (75), Eqs. (81)-(82) reduce to

\[ \alpha = \left[ \int_0^1 B_x^T B_x dt + C_x^T C_x \right] / \left[ \int_0^1 Y^T_{fyy} Y dt + (Z^T_{gzz} Z)_1 \right] \quad (83) \]
and

$$\int_0^1 Y_{yy}^T \dot{y} \, dt + (Z_{zz}^T \dot{z})_1 = 0$$

Equation (83) gives the optimum conjugate gradient stepsize\(^3\)

while Eq. (84) shows the conjugacy between the present and previous variations per unit stepsize. By substituting Eq. (56) into Eq. (84) and using Eq. (83) written for the previous iteration, we obtain

$$\hat{a} \left[ \int_0^1 Y_{yy}^T \dot{y} \, dt + (Z_{zz}^T \dot{z})_1 \right] + \gamma \left[ \int_0^1 \hat{E}_{B}^T \dot{e} \, dt + \hat{C}_{C}^T \right] = 0 \tag{85}$$

In order to further simplify Eq. (85), we note the following:

$$\left[ \int_0^1 Y_{yy}^T \dot{y} \, dt + (Z_{zz}^T \dot{z})_1 \right] - \left[ \int_0^1 y^T (f_y - \dot{f}_y) \, dt + \left[ Z_{zz}^T (g_z - \dot{g}_z) \right]_1 \right] = 0 \tag{86}$$

$$\int_0^1 \left[ \lambda^T (A - \phi_x A - \phi_u B - \phi C) + \rho^T (E^T A + E^T B + E^T C) \right] dt$$

$$+ \left[ \mu^T (\psi A + \psi C) \right]_1 = 0 \tag{87}$$

$$\int_0^1 \left[ \lambda^T (A - \phi_x A - \phi_u B - \phi C) + \hat{\rho}^T (\hat{E}^T A + \hat{E}^T B + \hat{E}^T C) \right] dt$$

$$+ \left[ \hat{\mu}^T (\hat{\psi} A + \hat{\psi} C) \right]_1 = 0 \tag{88}$$

\(^3\)Since Eq. (83) involves second derivatives, computation of \(a\) via Eq. (83) will be bypassed in favor of a one-dimensional search utilizing cubic interpolation.
Equation (86) is based on the first order Taylor expansions of the gradients of \( f \) and \( g \), which, in this instance, are exact. Equations (87)-(88) are based on Eqs. (58), (59), (64) and the linearity of \( \phi, \psi, \psi_{E} \). By subtracting (86) and (88) from (85), adding (87) to the result, integrating by parts, and calling upon Eqs. (60), (63), (65), (77), we find that

\[
\gamma = \left[ \int_{0}^{1} B_{\alpha}^{T} B_{\alpha} \, dt + C_{\alpha}^{T} C_{\alpha} \right] \left/ \int_{0}^{1} \hat{B}_{\alpha}^{T} B_{\alpha} \, dt + \hat{C}_{\alpha}^{T} \hat{C}_{\alpha} \right. \tag{89}
\]

Since Eqs. (60) and (65) are satisfied at every iteration, the error in the optimality conditions given by Eq. (25) reduces to

\[
Q = \int_{0}^{1} B_{\alpha}^{T} B_{\alpha} \, dt + C_{\alpha}^{T} C_{\alpha} \tag{90}
\]

Thus, Eq. (89) can be rewritten as

\[
\gamma = Q/\hat{Q} \tag{91}
\]

By the use of Eqs. (29), (49), (54), (56), (61), (62), it can be shown that the first variation of \( J \) is given by

\[
\delta J = - \alpha Q - \alpha \gamma \left[ \int_{0}^{1} B_{\alpha}^{T} B_{\alpha} \, dt + C_{\alpha}^{T} C_{\alpha} \right] \tag{92}
\]

with the implication that

\[
\hat{J}_{\alpha}(0) = - Q - \gamma \left[ \int_{0}^{1} B_{\alpha}^{T} B_{\alpha} \, dt + C_{\alpha}^{T} C_{\alpha} \right] \tag{93}
\]
Equation (75) implies that

$$\delta J = - \alpha Q$$  \quad (94)$$

while Eqs. (32)-(34) imply that

$$\delta I = \delta J$$  \quad (95)$$

By noting that $Q > 0$, we can see from Eqs. (94)-(95) that a descent property on both the augmented functional $J$ and the functional $I$ is attained for $\alpha$ sufficiently small.

**General Orthogonality and Conjugacy Conditions**

One can generalize the orthogonality conditions (75)-(77) and the conjugacy condition (84) by induction. It can be show that

$$\int_{0}^{1} B^T_{*} B^o \ dt + C^T_{*} C^o = 0$$  \quad (96)$$

$$\int_{0}^{1} B^T_{*} B^o \ dt + C^T_{*} C^o = 0$$  \quad (97)$$

$$\int_{0}^{1} Y^T f_{yy} Y^o \ dt + (Z^T g_{zz} Z^o) = 0$$  \quad (98)$$

where the superscript $^o$ denotes any iteration preceding the present iteration.
4. **Sequential Conjugate Gradient-Restoration Algorithm**

The algorithm which will be constructed to solve problem (12)-(15) for the general case where \( f, g \) are nonquadratic and/or \( \phi, \psi, E \) are non-linear is an extension of the sequential conjugate gradient-restoration algorithm of Ref. 6. The algorithm consists of a sequence of two-phase processes or cycles, each composed of a conjugate gradient phase and a restoration phase.

4.1. **Conjugate Gradient Phase.** The conjugate gradient phase is started only when Ineq. (27-1) is satisfied and involves a single iteration. In this iteration, the objective is to reduce the functional \( I \) or the augmented functional \( J \) while satisfying the constraints to first order. To attain this objective, we use the algorithm developed in Section 3 for a quadratic functional subject to linear constraints.

Due to the fact that the orthogonality conditions (96)-(97) do not hold in the general case, the first variation of \( J \) given by Eq. (94) is valid only for the first iteration of the conjugate gradient phase. During this initial iteration, the inequality

\[
\tilde{J}_\alpha(0) < 0
\]  
(99)

holds, with the consequence that \( \delta J < 0 \). Therefore, for \( \alpha \) sufficiently small, a descent property is achieved:

\[
\tilde{J}(\alpha) < \tilde{J}(0)
\]  
(100)

For subsequent iterations of the conjugate gradient phase, \( \delta J \) and \( \tilde{J}_\alpha(0) \) are given by Eqs. (92) and (93), respectively, and Ineqs. (99)-(100) may or may not hold. In order to retain a descent property, whenever Ineq. (99) is violated, the conjugate gradient phase must be interrupted,
and the algorithm must be restarted with an ordinary gradient phase, characterized by $\gamma = 0$.

Equations (58)-(65), in combination with Eqs. (56)-(57) and (91), uniquely define the variations per unit stepsize of the conjugate gradient phase.

At the end of the conjugate gradient phase, the varied functions $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ are known and the varied constraint error $\tilde{P}$ can be computed with Eq. (24). If $\tilde{P}$ satisfies Ineq. (27-1), the conjugate gradient phase is repeated using the varied functions $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ of the previous conjugate gradient phase for nominal functions. If Ineq. (27-1) is violated, a restoration phase must be employed prior to repeating the conjugate gradient phase.

4.2. Restoration Phase. As mentioned above, the restoration phase is started only when Ineq. (27-1) is violated. During this restoration phase, the objective is to reduce the constraint error $P$ to a level compatible with Ineq. (27-1) while preserving the descent property of the conjugate gradient phase. To achieve this constraint satisfaction, quasilinearization (Newton's method) is employed.

While the conjugate gradient phase involves a single iteration, the restoration phase may involve several iterations. This is due to the fact that the constraint equations (13)-(15) are considered only in linearized form during the restoration phase. In each restorative iteration, the objective is to reduce the functional $P$ while satisfying the constraints to first order and while minimizing the norm of the variations of the control and the parameter. The restoration phase is terminated whenever Ineq. (27-1) is satisfied.

The functions $\hat{A}(t), \hat{B}(t), \hat{C}$ are known from the previous conjugate gradient phase.
For the first restorative iteration, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ of the previous conjugate gradient phase are used for nominal functions. For any subsequent restorative iteration, the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ of the previous restorative iteration are used for nominal functions.

Let $x(t)$, $u(t)$, $\pi$ denote nominal functions satisfying condition (15-1), and let $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ denote varied functions satisfying (13)-(15). To first order, the perturbations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ must satisfy the
linearized constraint equations

$$
\dot{\Delta x} - \phi_x^T \Delta x - \phi_u^T \Delta u - \phi_\pi^T \Delta \pi + \alpha(\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1 \quad (101)
$$

$$
E_x^T \Delta x + E_u^T \Delta u + E_\pi^T \Delta \pi + \alpha E = 0, \quad 0 \leq t \leq 1 \quad (102)
$$

and the linearized boundary conditions

$$
(\Delta x)_0 = 0, \quad (\psi_x^T \Delta x + \psi_\pi^T \Delta \pi + \alpha \psi)_1 = 0 \quad (103)
$$

where $\alpha$ denotes a scaling factor (restoration stepsize) in the range $0 \leq \alpha \leq 1$.

In order to ensure a decrease in the functional $P$, we must have $\Delta P < 0$. This inequality can be enforced at every iteration providing the stepsize $\alpha$ is sufficiently small and $\Delta P < 0$. The first variation of $P$ is given by

$$
\delta P = 2 \int_0^1 (\dot{x} - \phi)^T (\Delta \dot{x} - \phi_x^T \Delta x - \phi_u^T \Delta u - \phi_\pi^T \Delta \pi) dt
$$

$$
+ 2 \int_0^1 E (E_x^T \Delta x + E_u^T \Delta u + E_\pi^T \Delta \pi) dt + 2 [\psi_x^T (\psi_x^T \Delta x + \psi_\pi^T \Delta \pi)]_1 \quad (104)
$$
When the variations defined by (101)-(103) are employed, the first variation of the constraint error (24) becomes

$$\delta P = -2\alpha P$$  \hspace{1cm} (105)$$

Since $P > 0$, Eq. (105) shows that $\delta P < 0$. Hence, for $\alpha$ sufficiently small, a decrease in the constraint error $P$ is attained.

In order to uniquely define\(^5\) the restorative iteration, an additional requirement must be introduced. We require that the restoration be accomplished while minimizing the norm of the variations of the control and the parameter. Thus, we seek the minimum of the quadratic functional

$$K = \int_0^1 \Delta u^T \Delta u dt + \Delta \pi^T \Delta \pi$$  \hspace{1cm} (106)$$

with respect to the perturbations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ subject to the linearized constraints (101)-(102) and the linearized boundary conditions (103). From calculus of variations, we know that the solution minimizing (106) subject to (101)-(103) must satisfy the following optimality conditions:

$$\Delta u = \alpha (\phi_u \lambda - E_u \rho), \quad 0 \leq t \leq 1$$  \hspace{1cm} (107)$$

$$\Delta \pi = \alpha \left[ \int_0^1 (\phi_\pi \lambda - E_\pi \rho) dt - (\psi_\mu)_1 \right]$$  \hspace{1cm} (108)$$

$$\dot{\lambda} + \phi_x \lambda - E_x \rho = 0, \quad 0 \leq t \leq 1$$  \hspace{1cm} (109)$$

$$(\lambda + \psi_x \mu)_1 = 0$$  \hspace{1cm} (110)$$

Due to Eqs. (29), Eqs. (101)-(103) and (107)-(110) can be rewritten as

\(^5\)Eqs. (101)-(103) admit an infinite number of solutions.
\[
\dot{x} - \phi^T_A x - \phi^T_B \dot{u} - \phi^T_C \pi + (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1
\]

(111)

\[
E^T_A x + E^T_B \dot{u} + E^T_C \pi + \phi = 0, \quad 0 \leq t \leq 1
\]

(112)

\[
\dot{x} + \phi^T_A \pi - E^T_C \phi = 0, \quad 0 \leq t \leq 1
\]

(113)

\[
\dot{u} - \phi^T_B \pi + E^T_C \phi = 0, \quad 0 \leq t \leq 1
\]

(114)

\[
C + \int_0^1 (- \phi^T_A \pi + E^T_C \phi) dt + \left(\psi^T_{\pi} \phi\right)_1 = 0
\]

(115)

\[
(A)_C = 0
\]

(116)

\[
(\psi^T_A + \psi^T_C + \psi)_1 = 0
\]

(117)

\[
(\dot{x}^T + \psi^T_{\pi} \phi)_1 = 0
\]

(118)

Equations (111)-(118) uniquely define the variations per unit stepsize of the restoration phase.

4.3. General Form of the Variations Per Unit Stepsize of a Cycle.

Upon comparing the particular variations per unit stepsize of the conjugate gradient phase [Eqs. (58)-(65)] with the variations per unit stepsize of the restoration phase [Eqs. (111)-(118)] it can be seen that the two sets of variations are quite similar. The sequential conjugate gradient-restoration algorithm can be made computationally efficient by taking advantage of this similarity. These two sets of variations can be embedded in a one-parameter family of variations per unit stepsize in which the parameter \( \beta \) has the following values:

\[
\beta = 1 \quad \text{during the conjugate gradient phase}
\]

\[
\beta = 0 \quad \text{during the restoration phase}
\]

(119)

The variations per unit stepsize of a cycle can then be represented in the following general form:
\[ A^c - \phi^T_{x}A^c - \phi^T_{u}B^c - \phi^T_{\pi}C^c + (1 - \beta) (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1 \]  
(120)

\[ E^T_{x}A^c + E^T_{u}B^c + E^T_{\pi}C^c + (1 - \beta)E = 0 \quad 0 \leq t \leq 1 \]  
(121)

\[ (A^c)_{0} = 0, \quad [(\psi^T_{x}A^c + \psi^T_{\pi}C^c + (1 - \beta)\psi)l] = 0 \]  
(122)

\[ B^c = -(\beta f_{u} - \phi_{u} \lambda + E_{u} \rho), \quad 0 \leq t \leq 1 \]  
(123)

\[ C^c = -\left[ \int_{0}^{1} (\beta f_{\pi} - \phi_{\pi} \lambda + E_{\pi} \rho)dt + (\beta g_{\pi} + \psi_{\pi} \mu)l \right] \]  
(124)

\[ \dot{x} - \beta f_{x} + \phi_{x} \lambda - E_{x} \rho = 0, \quad 0 \leq t \leq 1 \]  
(125)

\[ (\lambda + \beta g_{x} + \psi_{x} \mu)l = 0 \]  
(126)

where

\[
\begin{bmatrix}
A^c \\
B^c \\
C^c
\end{bmatrix} =
\begin{bmatrix}
A^* \\
B^* \\
C^*
\end{bmatrix}
\]

during the conjugate gradient phase  
(127)

and

\[
\begin{bmatrix}
A^c \\
B^c \\
C^c
\end{bmatrix} =
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
\]

during the restoration phase  
(128)

4.4. Linear, Two-Point Boundary-Value Problem. Equations (120)-(126) form a two-member family of linear, two-point boundary-value problems. The linear system obtained by setting \(\beta = 1\) defines the particular variations per unit stepsize of the conjugate gradient phase. The linear system obtained by setting \(\beta = 0\) defines the variations per unit stepsize of the restoration phase. These linear, two-point boundary-value problems can be solved in the same way.
We integrate the differential system under consideration \( n + p + 1 \) times using a forward integration scheme in combination with the method of particular solutions (Refs. 9-12). In each integration (subscript \( i \)), we assign a different set of values to the \( n \)-vector \( \lambda(0) \) and the \( p \)-vector \( C^c \), specifically,

\[
\lambda_i(0) = [\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}]^T
\]

\[
C_i^c = [\delta_{i(n+1)}, \delta_{i(n+2)}, \ldots, \delta_{i(n+p)}]^T
\]

(129)

where \( i = 1, 2, \ldots, n + p + 1 \) and where the Kronecker delta \( \delta_{ij} \) is such that

\[
\delta_{ij} = 1 \text{ if } i = j, \quad \delta_{ij} = 0 \text{ if } i \neq j
\]

(130)

With the above vectors specified, the differential system is integrated forward employing (a) Eqs. (120), (121), (122-1), (123), (125) and by-passing (b) Eqs. (122-2), (124), (126). In this way, one obtains the particular solutions

\[
A_i^c(t), B_i^c(t), C_i^c, \quad i = 1, 2, \ldots, n + p + 1
\]

(131)

\[
\lambda_i(t), \rho_i(t), \quad i = 1, 2, \ldots, n + p + 1
\]

(132)

Now, consider the linear combinations

\[
A_i^c(t) = \Sigma k_i A_i^c(t), \quad B_i^c(t) = \Sigma k_i B_i^c(t), \quad C_i^c = \Sigma k_i C_i^c
\]

(133)

\[
\lambda(t) = \Sigma k_i \lambda_i(t), \quad \rho(t) = \Sigma k_i \rho_i(t)
\]

(134)

where the summation is taken over the index \( i \). Here, the symbols \( k_i \)
denote undetermined, scalar constants. These linear combinations satisfy conditions (a) providing

$$\Sigma k_i = 1 \quad (135)$$

and satisfy conditions (b) providing

$$\Sigma k_i \left( \psi^T A^c_i + \psi^T C^c_i \right) + [ (1 - \beta) \psi ]_1 = 0, \quad \Sigma k_i (\lambda_i)_1 + (\beta g^x)_1 + (\psi)_1 \mu = 0 \quad (136)$$

$$\Sigma k_i \left[ \int_0^1 (\phi^1 - E^0 - \beta f_i) dt - C^c_i \right] - (\psi^1) \mu - (\beta g^x)_1 = 0 \quad (137)$$

The linear system (135)-(137) contains n + p + q + 1 equations in which the unknowns are the n + p + 1 constants $k_i$ and the q components of the multiplier $\mu$.

Upon determining the constants $k_i$, two methods for constructing the solution of the linear, two-point boundary-value problem are possible.

The first method requires the saving of the n + p + 1 particular solutions (131)-(132). In this case, the composite solution can be obtained directly via Eqs. (133)-(134). The second method requires the saving of the initial conditions (129) employed to generate the particular solutions. In this case, the composite solution is obtained by first using the constants $k_i$ to define $\lambda(0)$ and $C^c$ and then integrating the linear differential system forward once more. The latter technique requires less computer storage and is used in the examples of this thesis.

4.5. Step size Determination. From the solution of the linear, two-point boundary-value problem, the functions $A^c(t)$, $B^c(t)$, $C^c$ and the multipliers $\lambda(t)$, $\rho(t)$, $\mu$ are known, and the variations per unit stepsize $A(t)$,
B(t), C can be computed. With these variations, one forms the one-parameter family of solutions (30) for which the augmented functional \( J \) and the constraint error \( P \) take the following form:

\[
\tilde{J} = \tilde{J}(\alpha), \quad \tilde{P} = \tilde{P}(\alpha)
\]  

(138)

Then, a one-dimensional search scheme is employed. The type of search scheme used depends upon the phase.

In the conjugate gradient phase, \( \alpha \) must be selected in such a way that the inequality

\[
\tilde{J}(\alpha) < \tilde{J}(0)
\]  

(139)

is satisfied while keeping

\[
\tilde{P}(\alpha) \leq \varepsilon_3 \quad \text{and} \quad \tilde{\tau}(\alpha) \geq 0
\]  

(140)

Here, \( \varepsilon_3 \) is a small, preselected number. Satisfaction of (139) is guaranteed by the descent property of the conjugate gradient phase. Satisfaction of (140-1) is desirable in order to limit the constraint violation which is due to the use of the linearized constraint equations (32)-(34). Satisfaction of (140-2) is automatic in problems where the actual final time \( \tau \) is fixed and is required in problems where the actual final time is free.

Any violation of the above inequalities necessitates a reduction in the size of \( \alpha \). Such a reduction can be obtained by employing a bisection process, starting from a suitably chosen reference stepsize \( \alpha = \alpha_0 \). In practice, an upper limit must be imposed on the number of stepsize bisections which are allowable. During the conjugate gradient phase, if this upper limit is reached when \( \gamma = 0 \), the algorithm must be stopped.
If this upper limit is reached when $\gamma \neq 0$, the algorithm must be restarted with an ordinary gradient phase, characterized by $\gamma = 0$.

The conjugate gradient reference stepsize is determined by finding the minimizer of $\tilde{J}(\alpha)$. This minimizer is found by a numerical search technique, which is conducted only when Ineq. (99) is satisfied. During this search, $\tilde{J}_\alpha(\alpha) \ast$ is computed at each point in the sequence $\{0, 1, 2, 4, 8, 16, 32, \ldots\}$ until two consecutive values, denoted by $\alpha_1$ and $\alpha_2$, satisfy $\tilde{J}_\alpha(\alpha_1) < 0$ and $\tilde{J}_\alpha(\alpha_2) > 0$.

In order to compute $\alpha_0$, $J(\alpha)$ is approximated by a cubic,

$$
\tilde{J}(\alpha) = K_0 + K_0 \alpha + K_2 \alpha^2 + K_3 \alpha^3 \quad (141)
$$

whose coefficients are determined from the values of the ordinate and the slope at $\alpha = \alpha_1$ and $\alpha = \alpha_2$. This yields the linear system

$$
\tilde{J}(\alpha_1) = K_0 + K_1 \alpha_1 + K_2 \alpha_1^2 + K_3 \alpha_1^3 
$$

(142)

$$
\tilde{J}(\alpha_2) = K_0 + K_1 \alpha_2 + K_2 \alpha_2^2 + K_3 \alpha_2^3 
$$

(143)

$$
\tilde{J}_\alpha(\alpha_1) = K_1 + 2K_2 \alpha_1 + 3K_3 \alpha_1^2 
$$

(144)

$$
\tilde{J}_\alpha(\alpha_2) = K_1 + 2K_2 \alpha_2 + 3K_3 \alpha_2^2 
$$

(145)

After solving Eqs. (142)-(145) for the coefficients, the stepsize $\alpha_0$ is obtained from the relations

$$
\tilde{J}_\alpha(\alpha_0) = K_1 + 2K_2 \alpha_0 + 3K_3 \alpha_0^2 = 0 
$$

(146)

$$
\tilde{J}_\alpha(\alpha_0) = 2K_2 + 6K_3 \alpha_0 > 0 
$$

(147)

which imply that
\[ a_0 = \left[ -K_2 + \sqrt{(K_2^2 - 3K_1K_3)} \right] / (3K_3) \]  \hfill (148)

Once \( a_0 \) is known, two possibilities exist: either (i) \( \bar{J}_a(a_0) > 0 \) or (ii) \( \bar{J}_a(a_0) < 0 \). In case (i), the cubic interpolation is repeated between \( a_1 \) and \( a_0 \). In case (ii), the cubic interpolation is repeated between \( a_0 \) and \( a_2 \). This process is continued until

\[ \bar{J}_a^2(a_0) / \bar{J}_a^2(0) \leq \varepsilon_4 \]  \hfill (149)

where \( \varepsilon_4 \) is a small, preselected number.

When \( K_3 \) approaches zero, the numerator and denominator of Eq. (148) tend to zero simultaneously, leaving \( a_0 \) indeterminable. In this situation, L'Hospital's rule implies that

\[ a_0 = -K_1 / 2K_2 \]  \hfill (150)

which is the same equation that is obtained if \( \bar{J}(a) \) is approximated by a quadratic whose coefficients are determined from the values of the ordinate and the slope at \( a = a_1 \) and the value of the ordinate at \( a = a_2 \).

The conjugate gradient reference stepsize is obtained from Eq. (148) whenever \( K_3^2 > \varepsilon_5 \) and from Eq. (150) whenever \( K_3^2 \leq \varepsilon_5 \), where \( \varepsilon_5 \) is a small, preselected number.

In the restoration phase, \( \alpha \) must be selected in such a way that the inequality

\[ \bar{P}(\alpha) < \bar{P}(0) \]  \hfill (151)

is satisfied while keeping
Satisfaction of Ineqs. (151) and (152) is guaranteed for \( \alpha \) sufficiently small. Any violation of the above inequalities necessitates a reduction in the size of \( \alpha \). Again, such a reduction can be obtained by employing a bisection process, starting from a suitably chosen reference stepsize \( \alpha = \alpha_0 \).

The restoration reference stepsize is obtained by setting \( \alpha_0 = 1 \).

4.6. Descent Property. After completing a conjugate gradient-restoration cycle, we must check to see if the restoration phase has preserved the descent property of the conjugate gradient phase. Let \( I \) denote the value of the functional (12) at the beginning of the conjugate gradient phase and let \( \hat{I} \) denote the value of (12) at the end of the restoration phase. We would like to have

\[
\hat{I} < I
\]  

(153)

If Ineq. (153) is satisfied at the end of the restoration phase, the next conjugate gradient phase is started. If Ineq. (153) is violated, one returns to the previous conjugate gradient phase and reduces the conjugate gradient stepsize\(^6\) until, after restoration, Ineq. (153) is satisfied.

That the above procedure leads to satisfaction of Ineq. (153) is guaranteed by the fact that the conjugate gradient corrections are of \( O(\alpha) \) while the restoration corrections are \( O(\alpha^2) \). Hence, if the conjugate gradient stepsize \( \alpha \) is sufficiently small, the restoration phase preserves the descent property of the conjugate gradient phase.

4.7. Restarting Conditions. The conjugate gradient reference stepsize \( \alpha_0 \), i.e., the optimum stepsize given either by Eq. (148) or Eq. (150)

---

\(^6\)For example, one can use a bisection process.
and satisfying Ineq. (149), is very important. In the case of a quadratic functional subject to linear constraints, the general orthogonality and conjugacy conditions (96)–(98) depend upon its use. If \( \alpha_0 \) must be bisected due to violation of either Ineq. (139), (140-1), (140-2), or (153), the next cycle of the algorithm must be started with an ordinary gradient phase, characterized by \( \gamma = 0 \).

Also, an upper limit must be imposed on the number of search iterations required to attain Ineq. (149). If this upper limit is reached with Ineq. (149) still violated, the resulting suboptimal \( \alpha_0 \) is used. However, the next cycle of the algorithm must be started with an ordinary gradient phase, again characterized by \( \gamma = 0 \).
5. **Numerical Computations**

In order to illustrate the theory, thirteen numerical examples from Rice University, Aero-astronautics Report\(^7\) No. 111 (AAR-111) were considered. Two of the examples are presented in detail below, while an overview of all of the examples is given in Tables 13-14. Every example was first run with the sequential conjugate gradient-restoration algorithm (SCGRA). Each example was then rerun with \(\gamma = 0\) in every conjugate gradient phase; this yields the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 8.

The algorithm was programmed in FORTRAN IV. Double-precision arithmetic was used. The problems were solved on a CDC 6700 digital computer. In each case, the interval of integration was divided into 100 steps. The differential equations were integrated by Hamming's modified predictor-corrector method using a special Runge-Kutta starting procedure. The definite integrals \(I, J, P, Q\) were computed by a modified Simpson's rule.

The determination of the conjugate gradient stepsize or the restoration stepsize was performed in accordance with Section 4.5. For the conjugate gradient phase, the stepsize was subject to the inequalities

\[
\tilde{J}(\alpha) < \tilde{J}(0), \quad \tilde{P}(\alpha) \leq 10, \quad \tilde{r}(\alpha) \geq 0 \tag{154}
\]

The iterative search was limited to 20 iterations and was terminated whenever

\[
\tilde{J}^2(\alpha) / \tilde{J}^2(0) \leq 10^{-6} \tag{155}
\]

For the restoration phase, the stepsize was subject to the inequalities

\[
\tilde{P}(\alpha) < \tilde{P}(0), \quad \tilde{r}(\alpha) \geq 0 \tag{156}
\]

\(^7\)See Ref. 13.
The restoration phase was terminated whenever

$$\tilde{P}(\alpha) \leq 10^{-8}$$  \hfill (157)

Convergence of the algorithm was defined by

$$P \leq 10^{-8}, \quad Q \leq 10^{-6}$$  \hfill (158)

Nonconvergence of the algorithm was defined by

(a) $N > 100$
(b) $N_g > 10$ when $\gamma = 0$
(c) $N_r > 10$
(d) $\tilde{J}_\alpha(0) > 0$ when $\gamma = 0$
(e) Exponential overflow

where $N$ is the number of iterations and $N_g$ and $N_r$ are the number of bisections of the stepsize $\alpha$ required to satisfy Ineqs. (154) and (156), respectively.

Example 5.1 (Ex. 4.3 of AAR-111). Consider the problem of minimizing the functional

$$I = \int_0^1 (x^2 + u^2) \, dt$$  \hfill (159)

with respect to the state variable $x(t)$ and the control variable $u(t)$ which satisfy the differential constraint

$$\dot{x} = x^2 - u$$  \hfill (160)

the nondifferential constraint

$$x - 0.8 - t + t^2 \geq 0$$  \hfill (161)
and the boundary conditions

\[ x(0) = 1, \quad x(1) = 1 \]  \hspace{1cm} (162)

Introduce the auxiliary state variable \( y \) defined by

\[ x - 0.8 - t + t^2 = y^2 \]  \hspace{1cm} (163)

and replace Ineq. (161) with Eq. (163). Next, compute the time derivative of Eq. (163), discard Eq. (163), and replace it with the nondifferential constraint

\[ x^2 - u - 1 + 2t - 2yv = 0 \]  \hspace{1cm} (164)

the differential constraint

\[ \dot{y} = v \]  \hspace{1cm} (165)

and the initial condition

\[ y(0) = \sqrt{0.2} \]  \hspace{1cm} (166)

Then, problem (159)-(162) becomes that of minimizing the functional

\[ I = \int_0^1 (x^2 + u^2) \, dt \]  \hspace{1cm} (167)

with respect to the state variables \( x(t), y(t) \) and the control variables \( u(t), v(t) \) subject to

\[ \dot{x} = x^2 - u, \quad \dot{y} = v \]  \hspace{1cm} (168)

\[ x^2 - u - 1 + 2t - 2yv = 0 \]  \hspace{1cm} (169)
\[ x(0) = 1, \quad y(0) = \sqrt{0.2} \quad (170) \]
\[ x(1) = 1 \quad (171) \]

In this problem,
\[ n = 2, \quad m = 2, \quad p = 0 \quad (172) \]
\[ k = 1, \quad q = 1 \quad (173) \]

Since \( n + p + 1 = 3 \), three particular solutions are needed for each conjugate gradient or restorative iteration.

Assume the nominal functions
\[ x = 1, \quad y = \sqrt{0.2} \quad (174) \]
\[ u = 1, \quad v = 1 \quad (175) \]

which are consistent with the boundary conditions (170)-(171) but violate the constraints (168)-(169). Since these nominal functions do not constitute a feasible solution, both the SOGRA and the SCGRA start with a restoration phase. The SOGRA has converged to the required accuracy, after a total of \( N = 62 \) iterations; the SCGRA has converged to the required accuracy after a total of \( N = 44 \) iterations. Tables 1-3 show the convergence history, the optimal control variables, and the optimal state variables, respectively, for the SOGRA. Tables 4-6 show the convergence history, the optimal control variables, and the optimal state variables, respectively, for the SCGRA.

In Table 4, \( N_c \) denotes the number of cycles without restarting the algorithm. In particular, \( N_c = 0 \) denotes a cycle in which a suboptimal conjugate gradient stepsize is used. Note that the minimum values produced by the two algorithms agree to five significant figures. As expected, these values are higher than the minimum value of the functional \( (I = 1.5371) \) in the absence of Ineq. (161).

**Example 5.2 (Ex. 5.1 of AAR-Jll).** Consider the problem of minimizing the functional
\[ I = \tau \quad (176) \]

with respect to the state variables \( x(\theta), y(\theta), z(\theta) \), the control variable \( u(\theta) \), and the parameter \( \tau \) which satisfy the differential constraints

\[ \frac{dx}{d\theta} = z \cos u, \quad \frac{dy}{d\theta} = z \sin u, \quad \frac{dz}{d\theta} = \sin u \quad (177) \]

the inequality constraint

\[ \frac{dy}{d\theta} \leq \frac{1}{3} \quad (178) \]

and the boundary conditions

\[ x(0) = 0, \quad y(0) = 0, \quad z(0) = 0 \quad (179) \]

\[ x(\tau) = 1 \quad (180) \]

Substitute Eq. (177-2) into Ineq. (178) to obtain

\[ \frac{1}{3} - z \sin u \geq 0 \quad (181) \]

Next, introduce the auxiliary control variable \( v \) defined by

\[ \frac{1}{3} - z \sin u = v^2 \quad (182) \]

and replace Ineq. (181) with Eq. (182). Finally, introduce the normalized time \( t = \theta/\tau \) so that problem (176)-(180) becomes that of minimizing the functional

\[ I = \tau \quad (183) \]

with respect to the state variables \( x(t), y(t), z(t) \), the control variables \( u(t), v(t) \), and the parameter \( \tau \) subject to

\[ \dot{x} = \tau z \cos u, \quad \dot{y} = \tau z \sin u, \quad \dot{z} = \tau \sin u \quad (184) \]
\[ \frac{1}{3} - z \sin u - v^2 = 0 \]  \hspace{1cm} (185)
\[ x(0) = 0, \quad y(0) = 0, \quad z(0) = 0 \]  \hspace{1cm} (186)
\[ x(1) = 1 \]  \hspace{1cm} (187)

In this problem,

\[ n = 3, \quad m = 2, \quad p = 1 \]  \hspace{1cm} (188)
\[ k = 1, \quad q = 1 \]  \hspace{1cm} (189)

Since \( n + p + 1 = 5 \), five particular solutions are needed for each conjugate gradient or restorative iteration.

Assume the nominal functions

\[ x = t, \quad y = 0, \quad z = t \]  \hspace{1cm} (190)
\[ u = 1, \quad v = 1, \]  \hspace{1cm} (191)
\[ \tau = 1 \]  \hspace{1cm} (192)

which are consistent with the boundary conditions (186)-(187) but violate the constraints (184)-(185). Since these nominal functions do not constitute a feasible solution, both the SOGRA and the SCGRA start with a restoration phase. The SOGRA has converged to the required accuracy after a total of \( N = 30 \) iterations; the SCGRA has converged to the required accuracy after a total of \( N = 27 \) iterations. Tables 7-9 show the convergence history, the optimal control variables, and the optimal state variables, respectively, for the SOGRA. Tables 10-12 show the convergence history, the optimal control variables, and the optimal state variables, respectively, for the SCGRA. In Table 10, \( N_c \) denotes the number of cycles without restarting the algorithm. In particular, \( N_c = 0 \) denotes a cycle in which a suboptimal conjugate gradient stepsize is used. Note that the minimum values produced by the two algorithms
agree to six significant figures. As expected, these values are higher than the minimum value of the functional \( I = 1.7724 \) in the absence of Ineq. (178).

**Overview of Thirteen Examples.** As mentioned above, a total of thirteen examples were examined. Tables 13-14 show the number of iterations taken by the SOGRA and the SCGRA, respectively, to converge to various levels of \( Q \). For those problems where the algorithms were not able to converge to the required accuracy, the corresponding table entries are blank.

In achieving the levels of \( Q \leq 10^{-2}, Q \leq 10^{-3}, Q \leq 10^{-4}, Q \leq 10^{-5}, Q \leq 10^{-6} \), the SCGRA took approximately 98\%, 87\%, 72\%, 86\%, 87\% fewer iterations, respectively, than the SOGRA.
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<th>I</th>
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Table 1. Convergence History of the SOGRA for Example 5.1 (Cont'd).

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<tr>
<td>0.6</td>
<td>0.4401</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3997</td>
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</tr>
</tbody>
</table>

\[ \tau = 1.82022 \]

Table 9. Optimal State Variables of the SOGRA for Example 5.2.

<table>
<thead>
<tr>
<th>t</th>
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<th>z</th>
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<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0020</td>
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<td>0.1810</td>
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<tr>
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<td>0.3559</td>
</tr>
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<td>0.1240</td>
<td>0.4980</td>
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<td>0.1847</td>
<td>0.6077</td>
</tr>
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<td>0.3684</td>
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<td>0.7823</td>
</tr>
<tr>
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<td>0.3667</td>
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Table 10. Convergence History of the SCGRA for Example 5.2.

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<tr>
<th>N</th>
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<th>Q</th>
<th>I</th>
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<td>.2203 E-01</td>
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<td></td>
<td>.5453 E-07</td>
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<tr>
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<tr>
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<td>1.82033</td>
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<td>.6766 E-04</td>
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<td>.2439 E-04</td>
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<td>.5771 E-07</td>
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<tr>
<td>24</td>
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<td>.1558 E-10</td>
<td>.3032 E-05</td>
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<tr>
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<tr>
<td>27</td>
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<td>1.82022</td>
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</table>
### Table 11. Optimal Control Variables of the SCGRA for Example 5.2.

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<tr>
<th>t</th>
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<td>0.5773</td>
</tr>
<tr>
<td>0.1</td>
<td>1.3871</td>
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</tr>
<tr>
<td>0.2</td>
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</tr>
<tr>
<td>0.3</td>
<td>0.7333</td>
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</tr>
<tr>
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<tr>
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<tr>
<td>0.6</td>
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<td>0.0000</td>
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<tr>
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</table>

\( \tau = 1.82022 \)

---

### Table 12. Optimal State Variables of the SCGRA for Example 5.2.

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
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<tr>
<td>0.0</td>
<td>0.0000</td>
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<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0020</td>
<td>0.0163</td>
<td>0.1810</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0157</td>
<td>0.0633</td>
<td>0.3559</td>
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<td>0.3</td>
<td>0.0644</td>
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<td>0.4980</td>
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<td>0.4</td>
<td>0.1450</td>
<td>0.1847</td>
<td>0.6077</td>
</tr>
<tr>
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<td>0.7005</td>
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<td>0.3684</td>
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<td>0.7823</td>
</tr>
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<td>0.7</td>
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<td>0.3667</td>
<td>0.8564</td>
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</table>
Table 13. Number of Iterations for Convergence ($Q \leq 10^{-8}$).

<table>
<thead>
<tr>
<th>Example (AAR-111)</th>
<th>$Q \leq 10^{-2}$</th>
<th></th>
<th>$Q \leq 10^{-3}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>SCGRA</td>
<td>SOGRA</td>
<td>SCGRA</td>
</tr>
<tr>
<td>4.1</td>
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<td>9</td>
<td>9</td>
<td>9</td>
</tr>
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<td>14</td>
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<td>12</td>
<td>10</td>
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<tr>
<td>4.6</td>
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<td>20</td>
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<td>4.8</td>
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<td>7</td>
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<td>11</td>
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<td>14</td>
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<td>5.2</td>
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<td>9</td>
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<td>11</td>
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<tr>
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<td>42</td>
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<td>6.3</td>
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<td>14</td>
<td>30</td>
<td>23</td>
</tr>
<tr>
<td>TOTAL</td>
<td>175</td>
<td>171</td>
<td>283</td>
<td>246</td>
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</table>
Table 14. Number of Iterations for Convergence ($P \leq 10^{-8}$).

<table>
<thead>
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<th>Q $\leq 10^{-5}$</th>
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</thead>
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<td>SCGRA</td>
<td>SOGRA</td>
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<td>4.3</td>
<td>25</td>
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<td>30</td>
</tr>
<tr>
<td>4.4</td>
<td>22</td>
<td>16</td>
<td>31</td>
</tr>
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<td>4.5</td>
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<td>16</td>
<td>22</td>
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<td></td>
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<tr>
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<tr>
<td>TOTAL</td>
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<td>224</td>
<td>286</td>
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</table>

*For Ex. 4.8, the SOGRA only achieved a level of $Q = .1224 \times 10^{-5}$.
6. **Summary**

An algorithm has been developed to solve optimal control problems in the presence of nondifferential constraints. In order to facilitate numerical integrations, the algorithm has been constructed based on an interval of integration of unit length. This has been done without loss of generality.

The algorithm is composed of a sequence of cycles, each cycle consisting of two phases, a conjugate gradient phase and a restoration phase. The objective of each cycle is to decrease the functional $I$ while satisfying the constraints to the predetermined accuracy of (27-1).

The conjugate gradient phase involves a single iteration and is designed to decrease the augmented functional $J$ while satisfying the constraints to first order. During this iteration, the first variation of the functional $I$ is minimized subject to the linearized constraints and linearized boundary conditions. This minimization is performed over the class of variations of the control and the parameter which are the same distance from some constant multiple of their predecessors. The conjugate gradient phase is started only when the nominal functions $x(t)$, $u(t)$, $\pi$ satisfy the constraints (13)-(15) within the preselected accuracy of (27-1). When this occurs, the nominal functions are used to compute the vectors $f_x$, $f_u$, $f_\pi$ and the matrices $\phi_x$, $\phi_u$, $\phi_\pi$ and $E_x$, $E_u$, $E_\pi$ along the interval of integration and to evaluate the vectors $g_x$, $g_\pi$ and the matrices $\psi_x$, $\psi_\pi$ at the final time $t = 1$. The linear, two-point boundary-value problem (120)-(126) is then solved by the method of particular solutions with $\beta = 1$, and the directional coefficient $\gamma$ is determined from Eq. (91). With the quantities $A_*(t)$, $B_*(t)$, $C_*(t)$, $\gamma$ known, the functions $A(t)$, $B(t)$, $C(t)$ are obtained from Eq. (56). Using these functions, the conjugate gradient stepsize is
computed in accordance with Section 4.5. The perturbations $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ are then determined from Eq. (29). Finally, the varied functions are calculated from Eq. (28).

If, before computing the conjugate gradient stepsize, it is determined that Ineq. (99) is violated, the conjugate gradient phase is interrupted and the algorithm is restarted with an ordinary gradient phase, characterized by $\gamma = 0$. If, in the process of computing the conjugate gradient stepsize, it is determined that a suboptimal stepsize must be used, the present conjugate gradient phase is completed using that stepsize, and the next cycle of the algorithm is begun with an ordinary gradient phase, again characterized by $\gamma = 0$. Use of a suboptimal stepsize can be necessitated either by nonconvergence of the iterative search or by violation of either Ineq. (139), (140-1), (140-2), or (153).

At the end of the conjugate gradient phase, the constraint error (24) is computed. If Ineq. (27-1) is satisfied, the restoration phase is bypassed and the next cycle of the algorithm is started, using the varied functions of the previous conjugate gradient iteration for nominal functions. Otherwise, the restoration phase is begun.

The restoration phase involves one or more iterations and is designed to restore the constraints to a level compatible with (27-1). Each iteration is designed to decrease the constraint error $P$ while minimizing the norm of the variations of the control and the parameter. In achieving constraint satisfaction, quasilinearization (Newton's method) is employed.

For the first restorative iteration the nominal functions are identical with the varied functions of the previous conjugate gradient iteration. For any subsequent restorative iteration the nominal functions are identical with the varied functions of the previous restorative iteration. In either
case, the nominal functions satisfy condition (15-1) but violate conditions (13)-(14), (15-2).

These nominal functions are used to compute the vector \( \dot{x} - \phi \), the vector \( E \), and the matrices \( \phi_x, \phi_u, \phi_\pi \) and \( E_x, E_u, E_\pi \) along the interval of integration and to evaluate the vector \( \psi \) and the matrices \( \psi_x, \psi_\pi \) at the final time \( t = 1 \). The linear, two-point boundary-value problem (120)-(126) is then solved by the method of particular solutions with \( \beta = 0 \). In this way, the functions \( A(t), B(t), C \) are obtained. Using these functions, the restoration stepsize is computed in accordance with Section 4.5. The perturbations \( \Delta x(t), \Delta u(t), \Delta \pi \) are then determined from Eq. (29). Finally, the varied functions are calculated from Eq. (28).

At the end of each restorative iteration, the constraint error (24) is computed. If Ineq. (27-1) is still violated, one or more restorative iterations are performed until Ineq. (27-1) is satisfied. Once this occurs, the functional (12) is evaluated. If Ineq. (153) is satisfied, the next cycle of the algorithm is started. If not, the previous conjugate gradient phase is returned to and the previous conjugate gradient stepsize \( \alpha_g \) is bisected until, after restoration, Ineq. (153) is satisfied.

The above bisection procedure is guaranteed to lead to satisfaction of Ineq. (153). This is due to the fact that the conjugate gradient corrections are of \( O(\alpha_g) \) while the restoration corrections are of \( O(\alpha_g^2) \). If this bisection takes place, the next cycle of the algorithm is started with an ordinary gradient phase, characterized by \( \gamma = 0 \).

At the end of each cycle, the error in the optimality conditions (25) is computed. The algorithm is terminated whenever Ineqs. (27-1) and (27-2) are satisfied simultaneously.
It should be remarked that the algorithm can be started with nominal functions satisfying condition (15-1) but violating conditions (13), (14), (15-2). If Ineq. (27-1) is satisfied, the algorithm starts with an ordinary gradient phase. If Ineq. (27-1) is violated, the algorithm starts with a restoration phase.

A final remark concerning the solution of the linear, two-point boundary-value problem should be made. For given values $A^c, C^c, \lambda$, relations (121) and (123) constitute a system of $m + k$ equations in the $m + k$ components of the vectors $B^c$ and $\rho$. The system admits a unique solution providing

$$\det \begin{bmatrix} I & E_u \\ E_u^T & 0 \end{bmatrix} = (-1)^k \det [E_u^T E_u] \neq 0$$

where $I$ denotes the $m \times m$ identity matrix and $0$ denotes the $k \times k$ null matrix. This implies that, while the state $x$ and/or the parameter $\pi$ can be absent from Eq. (14), the control $u$ can never be absent from Eq. (14). In fact, $u$ must be present in each of the scalar components of $E$. Therefore, suitable transformations must be introduced to convert problems where the function $E$ does not involve the control into problems where the function $E$ involves the control.
References


Additional Bibliography


