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REINHARDT DOMAINS AND MEROMORPHIC FUNCTIONS

by

Neela Mayur

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's signature:

Reese Harvey

Houston, Texas

May, 1974
To Dnyan
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Introduction

This thesis is divided into two parts dealing with two different topic in several complex variables. The first part contains some results about Reinhardt domains and the second contains some results about meromorphic functions.

A set $\Omega$ contained in $\mathbb{C}^n$ is said to be Reinhardt if a point $(z_1, \ldots, z_n)$ belonging to $\Omega$ implies that the point $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$ belongs to $\Omega$, for arbitrary $(\theta_1, \ldots, \theta_n)$ in $\mathbb{R}^n$.

Cartan and Hartogs have proved the following theorem about such domains:

**Theorem:** Let $\Omega$ be an open connected Reinhardt set containing the origin. The following are equivalent:

i) $\Omega$ is a domain of holomorphy

ii) $\Omega$ is a domain of convergence of a power series.

They have also provided a third equivalent condition, which gives a geometrical characterization of such domains.

iii) $\Omega$ is logarithmically convex and complete, i.e., the associate space $
\hat{\Omega}^* = \{ \xi = (\xi_1, \ldots, \xi_n) \} \text{ where }$
$\xi_j = \log|z_j| \text{ for } z \text{ belonging to } \hat{\Omega} \text{ and each } z_j \neq 0 \}$ is convex and
if a point \((z_1, \ldots, z_n)\) belongs to \(\Omega\) then the polydisc about the origin with radius \(|z_1|, \ldots, |z_n|\) also belongs to \(\Omega\).

In the first chapter, we extend the above theorem to any open connected Reinhardt set which may not contain the origin. Robert Carmignani has provided a geometrical characterization of open Reinhardt sets which are domains of holomorphy. But he has not proved the analogue of (i) and (ii) of Cartan and Hartogs theorem. So we prove the following result: for any open connected Reinhardt set domain of holomorphy is equivalent to domain of convergence of a Laurent series. Also, we provide a geometrical characterization in a more direct way than Carmignani but which of course is equivalent to his.

Now we turn to Chapter II. Let \(f\) be holomorphic in the unit polydisc \(\Delta^n\) around the origin. For a point \(z = (z', z_n)\) in \(\Delta^n\), let \(R(z')\) denote the supremum of the set of \(r\) in \(\mathbb{R}\) such that \(f(z', z_n)\) can be holomorphically continued to the disc of radius \(r\), and \(r(z')\) denote the supremum of the set of \(r\) in \(\mathbb{R}\) such that \(f(z', z_n)\) can be holomorphically continued to an open set containing \(z' \times \Delta(0, r)\). The functions \(R(z')\), \(r(z')\) will be referred to as the separate radius of holomorphy and joint radius of holomorphy respectively.
Hartogs theorem states that if \( f \) is separately holomorphic then it is jointly holomorphic. If the proof of this theorem is closely analyzed it implicitly contains the following properties of \( R(z') \) and \( r(z') \).

i) \( - \log R(z') \) is (nonclassically) plurisubharmonic

ii) \( - \log r(z') \) is (classically) plurisubharmonic

iii) \( - \log r(z') \) is the upper envelope of \( - \log R(z') \)

iv) \( - \log r(z') = - \log R(z') \) except on a set of capacity zero.

The proof of these properties, given in Chapter II.1, depends heavily on the use of Hadamard's formula for separate radius of holomorphy. We want to extend the above theorem of Hartogs to meromorphic functions. Let \( M(z') \), \( m(z') \) be the separate and joint radii of meromorphy analogously defined. Hadamard's formula for the separate radius of meromorphy \( M(z') \), is so much more complicated than the formula for the radius of holomorphy that similar techniques could not be made to work. Analogues to property (iv) above, Rothstein has proved the following theorem.

If \( f \) is holomorphic in a neighborhood of the origin then the joint radius of meromorphy is equal to the separate radius of meromorphy except on a set of capacity zero.

Our proof of this theorem given in II.2 is similar to one suggested by Rothstein but it eliminates the use of
the solution to Levi's problem.

As an application of this theorem, we prove that if \( f \) is separately meromorphic on \( \bigcap \) a subset of \( \mathbb{C}^n \), then it is jointly meromorphic. This proof was suggested by Dr. Bernard Shiffman.
1.1 Completely logarithmically convex open Reinhardt sets

**Definition 1.1.1:** An open set $\Omega \subset \mathbb{C}^n$ is called a Reinhardt set if $(z_1, \ldots, z_n) \in \Omega$ implies 

$$(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \Omega$$

for arbitrary $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$.

**Definition 1.1.2:** An open set $\Omega \subset \mathbb{C}^n$ is called a complete Reinhardt set if for $(z_1, \ldots, z_n) \in \Omega$ and $(t_1, \ldots, t_n) \in \mathbb{C}^n$ with $|t_j| \leq 1$, $j = 1, \ldots, n$ implies 

$$(t_1 z_1, \ldots, t_n z_n) \in \Omega.$$  

**Definition 1.1.3:** For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. A Laurent series $\Sigma \alpha a_\alpha z^\alpha$ is said to be normally convergent in $\Omega$ if it is absolutely uniformly convergent in a neighborhood of every point in $\Omega$.

**Definition 1.1.4:** The domain of convergence of a Laurent series $\Sigma \alpha a_\alpha z^\alpha$ is defined by $\Omega = \{ z \in \mathbb{C}^n : \Sigma \alpha a_\alpha z^\alpha$ converges normally in some neighborhood of $z \}$.

Since a power series is a particular case of a Laurent series where summation is restricted to only positive multi-indices $\alpha$, the domain of convergence of a power series can be defined as in the above definitions.
Note that if $\Omega$ is a domain of convergence of a power series or a Laurent series it is an open Reinhardt set.

**Definition 1.1.5:** An open Reinhardt set $\Omega \subset \mathbb{C}^n$ is called a **logarithmically convex** set if the image $\Omega^*$ of this set in the space of logarithms of the absolute values is a convex set in $\mathbb{R}^n$, i.e.,

$$\Omega^* = \{\xi = (\xi_1, \ldots, \xi_n): \xi_j = \log|z_j|, (j = 1, \ldots, n); \text{ for some } z \in \Omega \text{ and } z_1, \ldots, z_n \neq 0\}$$

is a convex set.

Note that this is a condition on points in $\Omega - \{z: z_1, \ldots, z_n = 0\}$ only.

Cartan [2] and Hartogs [3] have proved the following theorem.

**Theorem:** Let $\Omega$ be an open connected Reinhardt set containing the origin then following are equivalent.

1) $\Omega$ is a domain of holomorphy
2) $\Omega$ is a domain of convergence of a power series
3) $\Omega$ is logarithmically convex and complete.

Now we want to extend this theorem to an open Reinhardt set which may not contain the origin. The third condition, which gives a geometrical characterization of the set, has been extended to a set which may not contain
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Let \( w_\varepsilon^* = (\log|w_1|, \log \varepsilon), z^* = (\log|z_1|, \log|z_2|) \).

The geodesic between \( z \) and \( w_\varepsilon \) in \(|\Omega|\) is the inverse image of the straight line joining \( z^* \) and \( w_\varepsilon^* \) in \( \Omega^* \).

As \( \varepsilon \to 0 \), the straight line joining \( z^* \) and \( w_\varepsilon^* \) tends to a line parallel to \( \rho_2\)-axis. But lines parallel to \( \rho_1\)-axis or \( \rho_2\)-axis get mapped to lines parallel to \( x\)-axis or \( y\)-axis so as \( \varepsilon \to 0 \) the geodesic joining \( z \) and \( w_\varepsilon \) in \(|\Omega|\) tends to the union of the perpendicular from \( z \) to \( x\)-axis and the line joining \( w \) and the foot of the perpendicular.

Now we shall define the geodesics joining any two points in an open Reinhardt set \( \Omega \subset \mathbb{C}^n \).

**Definition 1.1.6:** Let \( \Omega \) be an open connected Reinhardt set.

Let \( z = (|z_1|, \ldots, |z_n|) \)
\[ z^* = (\log|z_1|, \ldots, \log|z_n|) \text{ if } z_i \neq 0 \text{ for all } i \]
Let \(|\Omega| = \{z: z = (z_1, \ldots, z_n) \in \Omega\}\)
\(\Omega^* = \{z^*: z = (z_1, \ldots, z_n) \in \Omega; z_i \neq 0 \text{ for all } i\}\).

For \(v, w\) in \(|\Omega|\), define the geodesics as follows,

**Case (i)** If \(v, w\) both are away from the faces of \(\mathbb{R}^n\)

i.e., \(v_i \neq 0, w_i \neq 0\) for all \(i\), then the geodesic joining \(v\) and \(w\) is the inverse image of the straight line in
\(\Omega^*\) joining \(v^*\) and \(w^*\).

**Case (ii)** If \(v, w\) both belong to the same face, i.e.,

there exists \(j\) such that \(v_j = 0 = w_j\) but there does not exist \(i\) such that \(v_i = 0\) and \(w_i \neq 0\) or \(w_i = 0\) and \(v_i \neq 0\) then the geodesic joining \(v, w\) is the one considering these two points to belong to a space of one (or more) less dimensions.

Let \(P_i\) be the map: \(\mathbb{C}^n \rightarrow \mathbb{C}^n\) defined by

\(P_i(z_1, \ldots, z_i, \ldots, z_n) = (z_1, \ldots, 0, \ldots, z_n)\) for \(i = 1, \ldots, n\).

**Case (iii)** If \(v\) is on one of the faces but not on any of the axes of \(\mathbb{R}^n\) i.e., there exists only one \(j\) such that \(v_j = 0\) and \(w\) is not on any of the faces then the geodesic joining \(v\) and \(w\) consists of two pieces, first the perpendicular from \(w\) onto the face to which \(v\) belongs, i.e., onto \(\{z: z_j = 0\}\) and second the geodesic joining \(v\) and \(P_j(w)\), the foot of the perpendicular from \(w\) onto
\[ \{z: z_j = 0\} \], considering them as points in \( \mathbb{R}^{n-1} \) as defined in case (ii).

**Case (iv)** If \( v, w \) belong to two different faces but not on the axes, i.e., there exist \( i, j \) such that \( i \neq j \) and \( v_i = 0 = w_j \) but \( v_k \neq 0 \) for \( k \neq i \) and \( w_k \neq 0 \) for \( k \neq j \). Then the geodesic joining \( v, w \) consists of three faces: first, the perpendicular from \( w \) onto the face \( \{z: z_i = 0\} \); second, perpendicular from \( v \) onto the face \( \{z: z_j = 0\} \); third, the geodesic joining \( P_i(w) , P_j(v) \); where \( P_i(w) \) is the foot of the perpendicular from \( w \) onto \( \{z: z_i = 0\} \) and \( P_j(v) \) is the foot of the perpendicular from \( v \) onto \( \{z: z_j = 0\} \). Now \( P_i(w) , P_j(v) \) have both \( i^{th} \) and \( j^{th} \) co-ordinate zero so the geodesic between them is defined as in case (ii).

**Case (v)** If one of the points \( v \) or \( w \), say \( v \), belong to an axis, i.e., there exists \( i \) and \( j \) such that \( v_i = 0 = v_j \). Then the point \( v \) belongs to more than one face and so the geodesic between \( v \) and \( w \) can be defined as in case (iii) either by considering \( v \) to belong to the face \( \{z: z_i = 0\} \) or to the face \( \{z: z_j = 0\} \). So for such points geodesic is not defined uniquely.

Note that if \( v \) and \( w \) are on different axes then the origin will be on the geodesics joining them.
To determine whether for any two points belonging to an open Reinhardt set $\Omega$, the geodesics joining them also belong to $\Omega$, we would like to have an easier test than the definition 1.1.7. This can be done by using ideas from R. Carmignani [1].

**Lemma 1.1.1:** Suppose $\Omega$ is an open connected Reinhardt set. Then the following are equivalent.

i) $v, w \in \lvert \Omega \rvert$ implies the geodesics joining them belong to $\lvert \Omega \rvert$.

ii) $\Omega - \{z: z_1, \ldots, z_n = 0\}$ is logarithmically convex and $P_i(\Omega) \subset \Omega$ if $\Omega \cap \{z: z_i = 0\} \neq \emptyset$, i.e., there exists $z \in \Omega$ such that $P_i(z) = z$.

**Proof:**

If $\Omega$ satisfies (i) then by the definition of the geodesics, $\Omega - \{z: z_1, \ldots, z_n = 0\}$ is logarithmically convex. To show $P_i(\Omega) \subset \Omega$ if $\Omega \cap \{z: z_i = 0\}$ is trivial.

Now if $\Omega$ satisfies (ii) to show that $v, w \in \lvert \Omega \rvert$ implies the geodesics joining them belong to $\lvert \Omega \rvert$.

Let $v, w \in \lvert \Omega \rvert$ such that $v_i \neq 0$, $w_i \neq 0$ for all $i$, then since $\Omega - \{z: z_1, \ldots, z_n = 0\}$ is logarithmically convex, the geodesic joining $v$ and $w$ as defined in 1.1.6 belongs to $\lvert \Omega \rvert$.

Let $v = (|v_1|, \ldots, |v_n|) \in \lvert \Omega \rvert$ with $z_i \neq 0$ for all $i$ and $w = (|w_1|, \ldots, |w_n|) \in \lvert \Omega \rvert$ such that $w$ is only on one of the faces, say $|w_1| = 0$. 

\[ w_1 = 0 \text{ implies } P_1(w) = w \text{ in } \Omega. \] So by the hypothesis \( P_1(\Omega) \subset \Omega \).

Now we need to show that the geodesic joining \( v \) and \( w \) belongs to \( |\Omega| \).

First we shall show that the perpendicular from \( v \) onto the face \( \{z: z_1 = 0\} \) belongs to \( |\Omega| \).

\( v \in |\Omega| \), and \( P_1(\Omega) \subset \Omega \) implies \( (0,|v_2|,\ldots,|v_n|) \), the foot of the perpendicular from \( v \), belongs to \( |\Omega| \).

\( \Omega \) open implies there exists \( N_1 \times \ldots \times N_n \) open \( \subset \mathbb{C}^n \) such that

\[ (0,v_2,\ldots,v_n) \in N_1 \times \ldots \times N_n \subset \Omega. \]

So there exists \( z_1 \neq 0 \in N_1 \) such that \( (z_1,v_2,\ldots,v_n) \in \Omega \).

Now \( v \) and \( (|z_1|,|v_2|,\ldots,|v_n|) \) belong to \( |\Omega| - \{z: z_1,\ldots,z_n = 0\} \) which is logarithmically convex so geodesic joining \( v \) and \( (|z_1|,\ldots,|v_n|) \) is in \( |\Omega| \) but geodesic joining these two points is the straight line joining them. So the perpendicular from \( v \) to the face \( \{z: z_1 = 0\} \) belongs to \( |\Omega| \).

Now we need to show that geodesic joining \( w \) and \( P_1(v) \) belongs to \( |\Omega| \).

First note that if \( (|u_1|,\ldots,|u_n|) \) and \( (|v_1|,\ldots,|v_n|) \) are any two points in \( \Omega - \{z: z_1,\ldots,z_n = 0\} \). Then the geodesic joining them is the inverse image of the straight line in \( \Omega^* \) joining \( (\log|u_1|,\ldots,\log|u_n|) \),
\((\log|v_1|, \ldots, \log|v_n|)\). So a point on the geodesic is of the form \((e^{\lambda|u_1|+\mu|v_1|}, \ldots, e^{\lambda|u_n|+\mu|v_n|})\) where \(\lambda, \mu \in \mathbb{R}\) is such that \(\lambda + \mu = 1\).

The projections of \((|u_1|, \ldots, |u_n|), (|v_1|, \ldots, |v_n|),
\lambda|u_1|+\mu|v_1|, \ldots, \lambda|u_n|+\mu|v_n|\)
onto the face \(z_1 = 0\)
are \((0, |u_2|, \ldots, |u_n|), (0, |v_1|, \ldots, |v_n|),
\lambda|u_2|+\mu|v_2|, \ldots, \lambda|u_n|+\mu|v_n|\)
respectively and the point \((0, |u_2|, \ldots, |u_n|), (0, |v_2|, \ldots, |v_n|)\)
by definition, is a point on the geodesic joining \((0, |u_2|, \ldots, |u_n|)\) and
\((0, |v_2|, \ldots, |v_n|)\).

So if \(P_1(\Omega) \subset \Omega\) and \(\Omega - \{z: z_1, \ldots, z_n = 0\}\) is logarithmically convex then \((u_1, \ldots, u_n), (v_1, \ldots, v_n) \in \Omega - \{z: z_1, \ldots, z_n = 0\}\) implies that the geodesic joining \((u_1, \ldots, 0, \ldots, u_n), (v_1, \ldots, 0, \ldots, v_n)\) in \(P_1(\Omega)\) is contained in \(\Omega\). In other words, to show that the geodesic joining \((u_1, \ldots, 0, \ldots, u_n)\) and \((v_1, \ldots, 0, \ldots, v_n) \in P_1(\Omega)\) is contained in \(\Omega\) enough to know \(\Omega\) is open, \(P_1(\Omega) \subset \Omega\) and \(\Omega - \{z: z_1, \ldots, z_n = 0\}\) is logarithmically convex.

By the above comments, it is now obvious that the geodesic joining \(w\) and \(P_1(v)\) belong to \(|\Omega|\).

By following the same line of argument, it can be shown that for any two points \(v, w \in |\Omega|\), the geodesics joining them belong to \(|\Omega|\), which completes the proof.
Definition 1.1.7: Suppose $\Omega$ is an open Reinhardt set. $\Omega$ is said to be completely logarithmically convex if any one of the following equivalent conditions are satisfied.

i) $v, w$ in $|\Omega|$ implies the geodesics joining them belong to $|\Omega|$.

ii) $\Omega - \{z: z_1, \ldots, z_n = 0\}$ is logarithmically convex and $P_i(\Omega) \subset \Omega$ if $\Omega \cap \{z: z_1 = 0\} \neq \emptyset$.

Note that if origin belongs to $\Omega$, then $\Omega$ is completely logarithmically convex as in definition 1.1.7 if and only if $\Omega$ is logarithmically convex and complete as defined in definitions 1.1.5 and 1.1.2.

1.2 Extension of Cartan and Hartogs theorem

Definition 1.2.1: An open set $\Omega \subset \mathbb{C}^n$ is called a domain of holomorphy if there are no open sets $\Omega_1$ and $\Omega_2$ in $\mathbb{C}^n$ with the following properties:

(a) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$

(b) $\Omega_2$ is connected and not contained in $\Omega$

(c) for every $f \in \mathcal{O}(\Omega)$ there exists a function $f_2 \in \mathcal{O}(\Omega_2)$ such that $f = f_2$ in $\Omega_1$.

In this section we shall prove the following theorem which is an extension of Cartan and Hartogs to the case of any open connected Reinhardt set.
Theorem 1.2.1: Suppose $\Omega$ is an open connected Reinhardt set. The following are equivalent.

i) $\Omega$ is a domain of holomorphy

ii) $\Omega$ is a domain of convergence for some Laurent series

iii) $\Omega$ is completely logarithmically convex.

Proof (ii) implies (iii):

If $\Omega$ is a domain of convergence of a Laurent series $f = \sum a_\alpha z^\alpha$, then it is completely logarithmically convex.

The series $f = \sum a_\alpha z^\alpha$ splits naturally into $2^n$ parts $f_1, \ldots, f_{2^n}$ where

\[ f_1 = \sum_{\alpha_1 \geq 0} a_\alpha \]

\[ f_2 = \sum_{\alpha_1 < 0} a_\alpha \]

\[ \vdots \]

\[ f_{2^n} = \sum_{\alpha_1 \leq 0} a_\alpha \]

For $i = 1, \ldots, 2^n$ let $\Omega_i$ be the domain of convergence of $f_i$, then clearly $\Omega = \bigcap_{i=1}^{2^n} \Omega_i$. It is then sufficient to show that each $\Omega_i$ is completely logarithmically convex.

For $i = 1, \ldots, 2^n$ let $f_i' = \sum a_\beta w^\beta$ where $w = (w_1, \ldots, w_n)$ with $w_i = z_i$ if $\alpha_i \geq 0$

\[ w_i = \frac{1}{z_i} \text{ if } \alpha_i < 0 \]
and $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_i = \alpha_i$ if $\alpha_i \geq 0$
\[ \beta_i = -\alpha_i \text{ if } \alpha_i < 0. \]

Let $\Omega_i'$ be the domain of convergence of $f_i'$, then there is a relation between $\Omega_i$ and $\Omega_i'$ as follows:

$(z_1', \ldots, z_n') \in \Omega_i$ if and only if $(w_1, \ldots, w_n) \in \Omega_i'$ where $w_i = z_i$ if $\alpha_i \geq 0$, $w_i = \frac{1}{z_i}$ if $\alpha_i < 0$. (Note that since $f_i$ has terms corresponding to negative powers of $z_j$ if $\alpha_i < 0$, so if $(z_1', \ldots, z_n') \in \Omega_i$ then $z_j \neq 0$ if $\alpha_j < 0$.)

Also there is a map between $\Omega_i^*$ and $(\Omega_i')^*$ as follows: if $(\xi_1, \ldots, \xi_n) \in \Omega_i^*$ i.e.,

$(\xi_1', \ldots, \xi_n') \in \Omega_i'$ then $(\eta_1', \ldots, \eta_n') \in (\Omega_i')^*$ where $\eta_i = \xi_i$ if $\alpha_i \geq 0$ and $\eta_i = -\xi_i$ if $\alpha_i < 0$.

Clearly the map between $(\Omega_i')^*$ and $\Omega_i^*$ is linear, so if $(\Omega_i')^*$ is convex then so is $\Omega_i^*$ and if geodesics joining two points are in $|\Omega_i'|$ then so is the case for points in $|\Omega_i|$. $\Omega_i$ is completely logarithmically convex if and only if $\Omega_i'$ is completely logarithmically convex. But $\Omega_i'$ is a domain of convergence of a power series. Hence it is sufficient to show that $\Omega_1$ is completely logarithmically convex. To prove this we shall make use of the following lemma by Abel.

**Abel's lemma:** The power series $\Sigma a_\alpha z^\alpha$ converges normally in the polydisc $|z_j| < |w_j|$, $j = 1, \ldots, n$ if there exists $c$ such that $|a_\alpha w^\alpha| \leq c$ for all $\alpha$. 
Proof:

By hypothesis there is a constant \( c \) such that \( |a_\alpha w^\alpha| \leq c \). If \( |z_j| \leq k_j |w_j| \) and \( k_j < 1 \) for every \( j \), we obtain \( |a_\alpha z^\alpha| \leq ck^\alpha \) and since \( \sum_{j=1}^{n} k^\alpha = \prod_{j=1}^{n}(1-k_j)^{-1} < \infty \), the lemma is proved.

Now let \( D_1 = \{ z : |a_\alpha z^\alpha| < c \text{ for some } c \text{ and for all } \alpha \text{ such that } \alpha_i \geq 0 \} \). Then by Abel's lemma, it is clear that \( \Omega_1 = \text{interior of } D_1 = D_1^0 \). Hence it is enough to show \( D_1 \) is completely logarithmically convex.

Now by definition 1.1.7, it is necessary to show that \( D_1 - \{ z : z_1, \ldots, z_n = 0 \} \) is logarithmically convex and if there exists \( z \in D \), with \( P_1(z) = z \) for some \( i \) then \( P_1(D_1) \subset D_1 \). Let \( (\varepsilon_1, \ldots, \varepsilon_n), (\eta_1, \ldots, \eta_n) \in D_1^* \), i.e., \((\varepsilon_1, \ldots, \varepsilon_n), (\eta_1, \ldots, \eta_n) \in |D_1| \).

So for all \( \alpha \geq 0 \) there exists \( c \) such that

\[
|a_\alpha| \exp \left( \sum_{j=1}^{n} \alpha_j \varepsilon_j \right) \leq c
\]

\[
|a_\alpha| \exp \left( \sum_{j=1}^{n} \alpha_j \eta_j \right) \leq c
\]

If \( \lambda, \mu \geq 0 \) such that \( \lambda + \mu = 1 \) then

\[
|a_\alpha| \exp \left( \sum_{j=1}^{n} \alpha_j (\lambda \varepsilon_j + \mu \eta_j) \right)
\]

\[
= |a_\alpha| \exp \left( \sum_{j=1}^{n} \lambda \alpha_j \varepsilon_j \right) \exp \left( \sum_{j=1}^{n} \mu \alpha_j \eta_j \right)
\]
\[
|a_\alpha| \exp \left( \sum_{j=1}^{n} \alpha_j \xi_j \right) |a_\alpha| \exp \left( \sum_{j=1}^{n} \alpha_j \eta_j \right) \\
\leq c^\lambda \cdot c^{1-\lambda} = c.
\]

This implies \((\lambda \xi_1 + \mu \eta_1, \ldots, \lambda \xi_n + \mu \eta_n) \in D^*_1\), i.e., \(D^*_1\) is a convex set, i.e., \(D_1 = \{z : z_1, \ldots, z_n = 0\}\) is logarithmically convex.

Now \(0 \in D_1\) implies for all \(i = 1, \ldots, n\), \(P_i(0) = 0\) and so we need to show \(P_i(D_1) \subseteq D_1\) for all \(i\).

Let \((z_1, \ldots, z_n) \in D_1\), i.e., there exists \(c\) such that for all \(\alpha = (\alpha_1, \ldots, \alpha_n)\), \(|a_\alpha| |z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}| \leq c\).

Now \(P_i(z_1, \ldots, z_n) = (z_1, \ldots, 0, \ldots, z_n)\) if \(\alpha_i > 0\) then \([P_i(z_1, \ldots, z_n)]^{\alpha} = 0\) so trivially \(|a_\alpha| |[P_i(z_1, \ldots, z_n)]^{\alpha}| \leq c\). If \(\alpha_i = 0\) then

\[
|a_\alpha| |[P_i(z_1, \ldots, z_n)]^{\alpha}| \\
= |a_\alpha| |z_1^{\alpha_1}, \ldots, z_i^{\alpha_i-1}, \ldots, z_{i+1}^{\alpha_{i+1}}, \ldots, z_n^{\alpha_n}| \leq c.
\]

Since \((z_1, \ldots, z_n) \in D_1\), so \(P_i(z_1, \ldots, z_n) \in D_1\).

This completes the proof that \(D_1\) is completely logarithmically convex and hence (ii) implies (iii).

Now we shall prove a theorem that will enable us to prove (i) implies (ii).

**Theorem 1.2.2:** Let \(\Omega\) be an open connected Reinhardt set, then \(f \in \mathcal{O}(\Omega) = \text{holomorphic functions on } \Omega\), can be expanded in an absolutely and uniformly convergent Laurent series.
Proof:

Let \( f \in \mathcal{S}(\Omega) \). Choose \( r = (r_1^-, \ldots, r_n^-, r_1^+, \ldots, r_n^+) \in \mathbb{R}^{2n} \) such that \( r_i^- \geq 0, r_i^+ \geq 0 \) for all \( i = 1, \ldots, n \) and \( \mathcal{B}_r \), \( \sigma = (\sigma_1, \ldots, \sigma_n) \) where \( \sigma_i \) takes the value + or - . Define \( \Gamma(r, \sigma) = \{ \zeta : |\zeta_i| = r_i^\sigma_i, i = 1, \ldots, n \} \) and

\[
\mathcal{L}_\sigma^r(f)(z) = \frac{1}{(2\pi i)^n} \oint_{\Gamma(r, \sigma)} \frac{f(\zeta)}{(\zeta - z)} d\zeta
\]

where \( (\zeta - z)^I = (\zeta_1 - z_1), \ldots, (\zeta_n - z_n) \) and the integral in \( \mathcal{L}_\sigma^r(f) \) around \( |\zeta_i| = r_i^- \) is taken clockwise and around \( |\zeta_i| = r_i^+ \) is taken counterclockwise.

Define \( f_r(z) = \sum \mathcal{L}_\sigma^r(f)(z) \) where summation is carried over all possible choices of \( \sigma = (\sigma_1, \ldots, \sigma_n) \) with \( \sigma_i = + \) or - for \( i = 1, \ldots, n \).

Now we shall prove that for each choice of \( \sigma \), \( \mathcal{L}_\sigma^r \) is a Laurent series in \( z = (z_1, \ldots, z_n) \) which converges normally and whose coefficients do not depend on \( r \).

In particular, consider \( \sigma = (\sigma_1, \ldots, \sigma_n) \) with \( \sigma_i = - \) for all \( i = 1, \ldots, n \). Let \( \mathcal{L}_\sigma^r(f) = \mathcal{L}_1^r \) for this choice of \( \sigma \). Then

\[
\mathcal{L}_1^r(f) = \frac{1}{(2\pi i)^n} \oint_{\Gamma(r, \sigma)} \frac{f(\zeta)}{(\zeta - z)} d\zeta
\]

\[
= \frac{1}{\Gamma(r, \sigma)} \frac{f(\zeta_1)}{z_1} \cdots \frac{f(\zeta_n)}{z_n} \prod_{i=1}^n \frac{\zeta_i}{z_i - 1}, \ldots, \frac{\zeta_n}{z_n - 1}\)
Now $\zeta_i \in \Gamma(r, \sigma)$ implies $|\zeta_i| = r_i^-$ and $r_i^- < |z_i^-| < r_i^+$

implies $\left| \frac{\zeta_i}{z_i} \right| < 1$ for all $i$ so $\frac{1}{(z-1)^{\Gamma}} = \sum_{a_i \geq 0} \left( \frac{\zeta_i}{z_i} \right)^{a_i} \cdots,$

$\left( \frac{\zeta_n}{z_n} \right)^{a_n}$, with convergence being normal for $z \in B^o_r$ and $\zeta$ in the distinguished boundary of $B_r$, i.e., $\Gamma(r, \sigma)$.

Now $f(\zeta)$ is bounded on $\Gamma(r, \sigma)$ so the integration and summation in $\mathcal{L}_1^r(f)$ can be interchanged.

$$\mathcal{L}_1^r(f) = \sum_{a_i \geq 0} \left\{ \frac{1}{z^{a_i+1}} \int_{\Gamma(r, \sigma)} \frac{f(\zeta)}{\zeta^{-a}} \, d\zeta \right\}$$

where $\zeta^{-a} = \zeta_1^{-a_1}, \ldots, \zeta_n^{-a_n}, z^{a+1} = z_1^{a_1+1}, \ldots, z_n^{a_n+1}.$

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ define $a^\beta_{\sigma}(r) = \int_{\Gamma(r, \sigma)} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta,$

$\zeta^{\beta+1} = \zeta_1^{\beta_1+1}, \ldots, \zeta_n^{\beta_n+1}.$ For the choice of $\sigma = (-, \ldots, -)$, let $a^\beta_{\sigma}(r) = a^\beta_{1}(r) = \int_{r_1} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta$. Then we get

$$\mathcal{L}_1^r(f) = \sum_{a_i \geq 0} a_i^{a}(r) z^{-a}$$

where $a = (a_1, \ldots, a_n)$ and $z^{-a} = z_1^{-a_1}, \ldots, z_n^{-a_n}.$

Now it can easily be seen that by using similar techniques each $\mathcal{L}_1^r(f)$ can be expanded in a Laurent series with positive or negative powers $z_j$ depending on whether
\( \sigma_j \) is + or -, with coefficients \( a_\sigma^\beta(r) \). In each case the convergence is normal.

It remains to show that \( a_\sigma^\beta(r) \) is in fact independent of \( r \). It is sufficient to show that any small change in \( r \) leaves \( a_\sigma^\beta(r) \) constant. Since the interchange in order of integration leaves the integral in \( a_\sigma^\beta(r) \) unchanged, it is enough to show that if there is a small change in \( r_n^- \), \( a_\sigma^\beta(r) \) is constant.

Let \( r = (r_1^-, \ldots, r_n^-, r_1^+, \ldots, r_n^+) \) and \( r' = (r_1^-, \ldots, r_n^-, r_1^+, \ldots, r_n^+) \) where \( \varepsilon > 0 \) is such that \( B_{r'} \subset \Omega \).

Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be such that \( \sigma_n = - \). Then

\[
\begin{align*}
\frac{a_\sigma^\beta(r) - a_\sigma^\beta(r')}{\varepsilon} &= \int_{\Gamma(r, \sigma)} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta - \int_{\Gamma(r', \sigma)} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta \\
&= \int_{|\zeta_1| = r_1}^{\sigma_1} \ldots \int_{|\zeta_n| = r_{n-1}}^{\sigma_n} \left[ \int_{|\zeta_n| = r_n}^{\sigma_n} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta - \int_{|\zeta_n| = r_n + \varepsilon} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta \right].
\end{align*}
\]

But \( \frac{f(\zeta)}{\zeta^{\beta+1}} \) is holomorphic in the annulus \( r_1^- < |\zeta_1| < r_1^- + \varepsilon \)

as a function of \( \zeta_n \). So

\[
\begin{align*}
\frac{a_\sigma^\beta(r) - a_\sigma^\beta(r')}{\varepsilon} &= \int_{|\zeta_n| = r_n}^{\sigma_n} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta - \int_{|\zeta_n| = r_n + \varepsilon} \frac{f(\zeta)}{\zeta^{\beta+1}} \, d\zeta \\
&= 0. \quad \text{This implies} \quad a_\sigma^\beta(r) = a_\sigma^\beta(r').
\end{align*}
\]

Hence \( f_r = \sum_{\sigma} \xi_\sigma^r(f) \) can be expanded in a Laurent series whose coefficients do not depend on \( r \) and which
converges normally in $B_r$. But by Cauchy's formula, there exists $q_i^-, a_i^+, i=1,\ldots,n$ such that $f_r(z) = f(z)$ in $r_i^- < q_i^- \leq |z_i| \leq q_i^+ < r_i^+$. But $\Omega$ is connected so $f_r(z) = f(z)$ in $\Omega$, i.e., $f(z) = \sum_\sigma e_\sigma(f)$ in $\Omega$. This completes the proof of the theorem.

Note: If $0 \in \Omega$ then $r_i^-$ can be taken arbitrarily small then consider a $\sigma$ for which $\sigma_i = -$ for some $i$ and $\beta_i + 1 < 0$ in the integral

$$a_\sigma^\beta(r) = \frac{f(z)}{\Gamma(r,\sigma) z^{\beta+1}} d\zeta.$$

By Cauchy's inequalities, $|a_\sigma^\beta(r)| \leq \frac{M}{(r_i^-)^{\beta_i+1}}$ where $M$ depends on $r_j^\sigma$ $j \neq i$ and the supremum of $f$ in $\Gamma(r,\sigma)$.

Since $\beta_i + 1 < 0$, $|a_\sigma^\beta(r)| \to 0$ as $r_i^- \to 0$. This shows that the coefficients of negative powers of $z_i$ ($i=1,\ldots,n$) are zero if the origin is in $\Omega$, implying that the Laurent series reduces to a power series, i.e., if $0 \in \Omega$ then every holomorphic function $f$ on $\Omega$ can be expanded in a power series with normal convergence which is the result of Cartan [2] and Hartogs [3]. Now we complete the proof of theorem 1.2.1.

**Proof of (i) implies (ii):**

In view of the theorem 1.2.2, this can easily be proved by using the following characterization of domains of holomorphy by Cartan and Thullen.
**Theorem A:** Let $\Omega$ be an open set in $\mathbb{C}^n$. $\Omega$ is a domain of holomorphy if and only if for every $z \in \partial \Omega$ there exists $f_z \in \Theta(\Omega)$ such that $f_z$ cannot be continued analytically to $z$, i.e., for every $z \in \partial \Omega$ and for every ball $B$ containing $z$ there does not exist $\tilde{f} \in \Theta(B)$ such that $f_z = \tilde{f}$ in $B \cap \Omega$.

Since $\Omega$ is a domain of holomorphy by the above theorem there exists $f \in \Theta(\Omega)$ which cannot be continued analytically beyond $\Omega$. But by theorem 1.2.2, this function $f \in \Theta(\Omega)$ can be expanded in a Laurent series with normal convergence in $\Omega$ and so this Laurent series cannot be continued beyond $\Omega$, i.e., $\Omega$ is a domain of convergence for some Laurent series.

**Proof of (iii) implies (i):**

To prove this, we shall use the above theorem of Cartan and Thullen and some ideas from R. Carmignani [1].

For every point $p \in \partial \Omega$ we shall construct a function $f_p \in \Theta(\Omega)$ such that $f_p$ cannot be continued to $p$.

Let $p = (p_1, \ldots, p_n) \in \partial \Omega$. We need to consider the following three cases.

**Case (i)** $p \in \partial \Omega - \{z: z_1, \ldots, z_n = 0\}$. $\Omega$ is completely logarithmically convex so $\Omega - \{z: z_1, \ldots, z_n = 0\}$ is
logarithmically convex. By the geometric form of Hahn Banach theorem and the convexity of $\Omega^*$ there exists a hyperbolic half space $H^\alpha_H^c>$ or $H^\alpha_H^c<$ with the property

that $\Omega \subset H^\alpha_H^c>$ or $\Omega \subset H^\alpha_H^c<$ and $p \in \partial H^\alpha_H^c>$ or $p \in \partial H^\alpha_H^c<$

where $H^\alpha_H^c>$ or $H^\alpha_H^c<$ is defined as follows: for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ let $\alpha_{ik}$ be $\geq 0$ $(0 \leq k \leq n)$ and $\alpha_{ij+1}$ be $< 0$ $(k \leq j \leq n)$ such that for each $k$, $\alpha_{ik}$ is one and only one component of $\alpha$.

$H^\alpha_H^c> = \{ z : |z_{i_1}|^{\alpha_{i_1}} \ldots |z_{i_k}|^{\alpha_{i_k}} |z_{i_{j+1}}^{\alpha_{i_{j+1}}} \ldots |z_{i_{n-k+j}}^{\alpha_{i_{n-k+j}}})

H^\alpha_H^c< = \{ z : |z_{i_1}|^{\alpha_{i_1}} \ldots |z_{i_k}|^{\alpha_{i_k}} |z_{i_{j+1}}^{\alpha_{i_{j+1}}} \ldots |z_{i_{n-k+j}}^{\alpha_{i_{n-k+j}}}) \}.$

If we show that $H^\alpha_H^c<(H^\alpha_H^c>)$ is a domain of holomorphy then by Cartan Thullen theorem, there exists $f \in \mathfrak{d}(H^\alpha_H^c)$ that cannot be continued to $p$. Let $f = F|_{\Omega}$ then $f \in \mathfrak{d}(\Omega)$ and cannot be continued to $p$.

**Lemma 1.2.1:** $H^\alpha_H^c>(H^\alpha_H^c)<$ are domains of holomorphy.

**Proof:** We shall prove the lemma for $H^\alpha_H^c>$, the proof for $H^\alpha_H^c<$ is similar.

Also, we shall indicate the proof for $n = 2$, the case for arbitrary $n$ can be proved similarly.
First let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 > 0$, $\alpha_2 > 0$ and

$$\alpha_1 = \frac{p_1}{q_1}, \quad \alpha_2 = \frac{p_2}{q_2} \quad (q_1 \neq 0, \ q_2 \neq 0) \ i.e., \ \alpha_1, \ \alpha_2 \ are \ positive \ rational \ numbers. \ Then \ let$$

$$f(z_1, z_2) = \frac{1}{z_1^{p_1 q_2} z_2^{p_2 q_1} - c q_1 q_2}.$$ 

$f \in \wp(H_c^\alpha)$ since the denominator vanishes only if

$$|z_1|^{p_1 q_2} |z_2|^{p_2 q_1} = c q_1 q_2 \ i.e., \ |z_1|^{p_1 / q_1} |z_2|^{p_2 / q_2} = c$$

and so, $f$ cannot be continued beyond $H_c^\alpha$ analytically.

If $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = \frac{p_1}{q_1} > 0$ and $\alpha_2 = \frac{p_2}{q_2} < 0$, 

i.e., $H_c^{\alpha} = \{z: |z_1|^\alpha_1 < c |z_2|^{-\alpha_2}\}$ then let

$$f(z_1, z_2) = \frac{1}{z_1^{p_1 q_2} - c q_1 q_2 z_2^{p_2 q_1}}.$$ 

$f \in \wp(H_c^{\alpha})$ since the denominator vanishes only on $\wp(H_c^{\alpha})$ so $f$ cannot be continued analytically.

Hence in either case $H_c^{\alpha}$ is a domain of holomorphy if $\alpha$ is rational.

If $\alpha_1, \alpha_2$ are irrational then $H_c^{\alpha}$ can be written as an increasing union of domains of holomorphy and then by using the following theorem of Behnke and Stein, $H_c^{\alpha}$ will be a domain of holomorphy.
**Theorem B**: The union of an increasing sequence of domains of holomorphy is a domain of holomorphy.

Let $K_i$ be an open set in $|z|^2$ bounded by

$|z_1| = A_1 = \text{constant}, \quad |z_2| = A_2 = \text{constant},$

$|z_1|^{p_1/q_1}|z_2|^{p_2/q_2} = A_3 = \text{constant and } (p_1/q_1, p_2/q_2)$

are positive rationals.

Now $K_i$ is clearly a domain of holomorphy since it is the intersection of $\{z: (z_1) < A_1\}$, $\{z: |z_2| < A_2\}$, $\{z: |z_1|^{p_1/q_1}|z_2|^{p_2/q_2} < A_3\}$ each of which are domains of holomorphy. If $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 0$ and irrational, it is clear that $I = (A_1, A_2, A_3, p_1/q_1, p_2/q_2)$ can be so chosen that $K_i \subset H_{c^<}^\alpha$. Also by suitably choosing the constants in $I$, we can get an increasing sequence of $K_i$ such that $H_{c^<}^\alpha = \bigcup K_i$. Hence $H_{c^<}^\alpha$ is a domain of holomorphy.
Case (ii) \( p \in \mathfrak{a}\Omega \) with \( p_j = 0 \) for some \( j \) and 
\( \Omega \cap P_j(\Omega) = \emptyset \) then let 
\[
    f_p(z_1, \ldots, z_n) = \frac{1}{z_j}.
\]

\( f_p \in \mathfrak{A}(\Omega) \) and cannot be analytically continued to \( p \).

Case (iii) \( p \in \mathfrak{a}\Omega \) with \( p_j = 0 \) for some \( j \) and 
\( \Omega \cap P_j(\Omega) \neq \emptyset \).

The proof of this case will be based on induction on \( n \) - the dimension of the space.

Let \( Q_n \) be the proposition that for \( \Omega = \) completely logarithmically convex open Reinhardt set in \( \mathbb{C}^n \), and for 
\( p \in \mathfrak{a}\Omega \) with \( p_j = 0 \) for some \( j \) with \( \Omega \cap P_j(\Omega) \neq \emptyset \) 
there exists a function \( f_p \) in \( \mathfrak{A}(\Omega) \) such that \( f_p \) cannot be analytically continued to the point \( p \).

For \( n = 1 \) this proposition is true since every annulus in \( \mathbb{C} \) is a domain of holomorphy. Now assume that 
\( Q_{n-1} \) is true and we shall prove that \( Q_n \) is true.

Let \( p \in \mathfrak{a}\Omega \) with \( p_1 = 0 \) and \( \Omega \cap P_1(\Omega) \neq \emptyset \). We shall now show that \( P_1(\Omega) \) as a subset of \( \mathbb{C}^{n-1} \) satisfies the hypothesis of the proposition with \( p \in \mathfrak{a}[P_1(\Omega)] \).

\( P_1(\Omega) \) is clearly an open subset of \( \mathbb{C}^{n-1} \) since the hypothesis on \( \Omega \), \( P_1(p) = p \) and \( P_1(\Omega) \subset \Omega \) implies 
\[
    P_1(\Omega) = ([0] \times \mathbb{C}^{n-1}) \cap \Omega.
\]

Also, \( \Omega \) is a Reinhardt set and \( P_1(\Omega) \subset \Omega \) implies \( P_1(\Omega) \)
is a Reinhardt set. Since \( p \notin \Omega \) and \( P_1(\Omega) \subset \Omega \), \( p \notin P_1(\Omega) \) and \( p \in \partial \Omega \) implies \( p \in \partial [P_1(\Omega)] \).

Now to apply induction hypothesis to \( P_1(\Omega) \) it only remains to show that \( P_1(\Omega) \) is completely logarithmically convex set.

Since \( \Omega \) is a completely logarithmically convex, by definition 1.1.7 it is enough to show that if \((0,z_2',...,z_n') \in P_1(\Omega) \) and for some \( j > 1 \), \( z_j' = 0 \) then \( P_j(P_1(\Omega)) \) is contained in \( P_1(\Omega) \).

Suppose for \( j = 2 \), \( z_j' = z_2' = 0 \), i.e., \((0,0,z_3',...,z_n') \in P_1(\Omega) \). Then we want to show that if \((0,z_2,...,z_n) \in P_1(\Omega) \) then \((0,0,z_3,...,z_n) \in P_1(\Omega) \).

Since \( P_1(\Omega) \subset \Omega \), \((0,0,z_3',...,z_n') \in \Omega \) such that \( P_2(0,0,z_3',...,z_n') = (0,0,...,z_n') \in \Omega \), so by hypothesis on \( \Omega \), \( P_2(\Omega) \subset \Omega \), i.e., \((0,z_2',...,z_n) \in P_1(\Omega) \subset \Omega \) implies \((0,0,z_3,...,z_n) \in \Omega \), i.e., \((0,0,z_3,...,z_n) \in P_1(\Omega) \) by definition of \( P_1(\Omega) \). Hence \( P_2(P_1(\Omega)) \subset P_1(\Omega) \).

Applying inductive hypothesis to \( P_1(\Omega) \), we get that there exists a function \( \tilde{f}_p \) in \( \Theta(P(\Omega)) \) such that \( \tilde{f}_p \) cannot be continued to \( p \). Let \( f_p(z_1',...,z_n) = \tilde{f}_p(0,z_2',...,z_n) \) for \((z_1',...,z_n) \in \Omega \). Then clearly \( f_p \in \Theta(\Omega) \) and cannot be continued to \( p \).

Hence \( \Omega \) is a domain of holomorphy. This completes the proof of theorem 1.2.1.
1.3 Envelope of Holomorphy

With the results of Hartogs [4] and Cartan [5] as mentioned in section 1.1, it can be seen that for an open connected Reinhardt set \( \Omega \) containing the origin, every function \( f \in \Theta(\Omega) \) can be extended to the logarithmically convex envelope (i.e., the interior of the intersection of all the logarithmically convex and complete Reinhardt sets containing \( \Omega \)). Also, this intersection is a domain of holomorphy. In view of results in section 1.2 now we can prove a similar result for \( \Omega \) which may not contain the origin.

**Definition 1.3.1:** Let \( \Omega \) be an open connected Reinhardt set, then the interior of the intersection of all completely logarithmically convex Reinhardt domains containing \( \Omega \) is called the completely logarithmically convex envelope of \( \Omega \) and will be denoted by \( \hat{\Omega} \).

**Theorem 1.3.1:** Suppose \( \Omega \) is an open connected Reinhardt set, then for every \( f \in \Theta(\Omega) \) there exists \( \hat{f} \in \Theta(\hat{\Omega}) \) such that \( f = \hat{f} \) on \( \Omega \). Furthermore \( \hat{\Omega} \) is a domain of holomorphy.

**Proof:**

First we shall show that in fact, \( \hat{\Omega} = \text{interior of} \ (\cap D_f) \) where \( D_f = \text{domain of convergence of the Laurent} \ f \in \Theta(\Omega) \)
series which converges to \( f \) normally on \( \Omega \). (This
Laurent series exists by theorem 1.2.2.) Let \( \mathcal{S} = \text{set} \)
of completely logarithmically convex Reinhardt sets con-
taining \( \Omega \). By theorem 1.2.1 \( \{D_f: f \in \mathcal{O}(\Omega)\} \subset \mathcal{S} \). Now
let \( D \in \mathcal{S} \), then by theorem 1.2.1, \( D \) is a domain of
convergence for some Laurent series on \( D \), i.e., \( D = D_g \)
for some \( g \in \mathcal{O}(D) \). Let \( g_1 = g|_{\Omega} \) then clearly,
\( D_{g_1} = D \) and \( g_1 \in \mathcal{O}(\Omega) \). So \( D \in \{D_f: f \in \mathcal{O}(\Omega)\} \). This
implies \( \mathcal{S} = \{D_f: f \in \mathcal{O}(\Omega)\} \). So \( \hat{\Omega} = (\cap D_f)^0 \).

Now given \( f \in \mathcal{O}(\Omega) \), \( \hat{\Omega} \subset D_f \), i.e., the Laurent
series corresponding to \( f \) also converges normally on \( \hat{\Omega} \)
which implies that there exists \( \hat{f} \in \mathcal{O}(\hat{\Omega}) \) with \( \hat{f} = f \) on \( \Omega \).

Intersection of completely logarithmically convex
Reinhardt set is clearly a completely logarithmically convex
set. Now \( \hat{\Omega} \) is the interior of completely logarithmically
convex set and hence by the following lemma and theorem 1.2.1,
\( \hat{\Omega} \) is a domain of holomorphy.

**Lemma 3.1.1:** Suppose \( \Omega \) is a completely logarithmically
convex Reinhardt set contained in \( C^n \), then the interior
of \( \Omega \) is also a completely logarithmically convex Reinhardt
set.

**Proof:**

If a set is convex then its interior is also convex
\( \circ \) implies \( \Omega - \{z: z_1, \ldots, z_n \neq 0\} \) is logarithmically convex,
and if there exists a point \( z \in \overset{\circ}{\Omega} \cap \{ z : z_i = 0 \} \) then by the assumptions on \( \overset{\circ}{\Omega} \), \( P_i(\overset{\circ}{\Omega}) \subset \overset{\circ}{\Omega} \), but \( P_i(\overset{\circ}{\Omega}) \) is open implying \( P_i(\overset{\circ}{\Omega}) \subset \overset{\circ}{\Omega} \) which proves the lemma.

Now we wish to define the concept of envelope of holomorphy. In this section we will restrict our attention to subsets of \( \mathbb{C}^n \) only. For a detailed discussion of this concept, please see Malgrange [6].

**Definition 1.3.2:** Let \( \Omega \subset \mathbb{C}^n \), a set \( \overset{\sim}{\Omega} \subset \mathbb{C}^n \) containing \( \Omega \) is called an (schlicht) envelope of holomorphy if every \( f \in \mathfrak{C}(\Omega) \) extends to \( \overset{\sim}{\Omega} \), i.e., there exists \( \overset{\sim}{f} \in \mathfrak{C}(\overset{\sim}{\Omega}) \) such that \( f = \overset{\sim}{f} \) on \( \Omega \) and if there exists a function \( f \in \mathfrak{C}(\overset{\sim}{\Omega}) \) which cannot be continued past \( \overset{\sim}{\Omega} \) holomorphically.

Theorem 1.3.1 immediately shows that \( \overset{\sim}{\Omega} = \text{interior of the intersection of all completely logarithmically convex Reinhardt sets containing } \Omega \), is in fact the (schlicht) envelope of holomorphy of \( \Omega \).
Chapter II

Hartogs' Theorem: If \( f \) is a complex valued function defined in the open set \( \Omega \subset \mathbb{C}^n \) and \( f \) is analytic in each variable \( z_j \) when the other variables are given fixed values, then \( f \) is analytic in \( \Omega \).

In this chapter, we shall prove the analogue of the above theorem of Hartogs for the case of meromorphic functions, using a theorem of Rothstein.

2.1 Radii of holomorphy

In the following three definitions \( f \) will denote a function defined on an open set \( \Omega \subset \mathbb{R}^n \) with values in \([-\infty, \infty)\).

**Definition 2.1.1:** \( \lim_{y \to x} f(y) = \inf_{\varepsilon > 0} \sup_{|x-y| < \varepsilon} f(y) \).

**Definition 2.1.2:** \( f \) is said to be upper semi continuous on \( \Omega \) if for all \( x \in \Omega \), \( \lim_{y \to x} f(y) = f(x) \).

**Definition 2.1.3:** \( \lim_{y \to x} f(y) \) is defined to be the upper envelope of \( f \).

Note that an upper envelope of a function is always upper semi continuous.
Definition 2.1.4: $f$ defined on $\Omega \subset \mathbb{C}$ with values in $[-\infty, \infty)$ is said to be (classically) subharmonic if

i) $f$ is upper semi continuous on $\Omega$ as a subset of $\mathbb{R}^2$

ii) $f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})d\theta$ for all $z \in \Omega$ and for sufficiently small $r$ such that $\{z + re^{i\theta}: 0 \leq \theta \leq 2\pi\} \subset \Omega$.

Definition 2.1.5: Let $f \in L^1_{\text{loc}}(\Omega)$, $f$ is said to be (non-classically) subharmonic if $\Delta f \geq 0$ in the sense of distribution theory, i.e., for all $\phi \in C_0^\infty(\Omega)$ and $\phi \geq 0$, $\int f\Delta \phi \, d\lambda \geq 0$ where $\lambda$ is Lebesgue measure.

Proposition 2.1.1

i) If $f$ satisfies the definition 2.1.4 then it also satisfies the definition 2.1.5.

ii) If $f$ satisfies the definition 2.1.5 then there is a unique (classically) subharmonic function $g$ in $\Omega$ which is equal to $f$ almost everywhere and which is given by

$$g(x) = \lim_{\varepsilon \to 0} \text{ess sup} \ f(y) \text{ where ess sup } f(y) = \inf_{|x-y|<\varepsilon} \sup_{|x-y|<\varepsilon} h(y).$$

Proof:


Definition 2.1.6: If $f$ satisfies the definition 2.1.5 then $g$ as obtained in proposition 2.1.1 will be called the canonical representative of $f$. 
Definition 2.1.7: \( f \) defined on \( \Omega \subset \mathbb{C}^n \) with values in \([-\infty, \infty)\) is called (classically) plurisubharmonic if

i) \( f \) is upper semi-continuous on \( \Omega \) as a subset of \( \mathbb{R}^{2n} \)

ii) for arbitrary \( z, w \in \mathbb{C}^n \) the function \( \tau \to f(z + \tau w) \) is (classically) subharmonic in the part of \( \mathbb{C} \) where it is defined.

Definition 2.1.8: \( f \in L'_{\text{loc}}(\Omega) \) is said to be (non-classically) plurisubharmonic if the Hermitian form \( (H(z,f)a,a) = \sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} a_j \bar{a}_k \) is a positive measure, i.e., for \( \phi \in C_0^\infty(\Omega) \)

and \( \phi \geq 0 \), \( \sum_j \int f(z) \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \, dx dy a_j \bar{a}_k \) is positive.

Proposition 2.1.2: If \( f \in D'(\Omega) \) is such that the Hermitian form \( (H(z,f),a,a) = \sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} a_j \bar{a}_k \) is a positive measure then there exists a unique (classically) plurisubharmonic function on \( \Omega \) which coincides with \( f \) almost everywhere and this unique function is given by \( \text{ess } \lim_{w \to z} f(w) \).

Proof:

Let \( \phi \geq 0 \) and in \( C_0^\infty(\mathbb{C}^n) \) such that \( \phi \) is radial, \( \phi(z) = 0 \), \( |z| > 1 \), and \( \int \phi(z) d\lambda(z) = 1 \) where \( d\lambda \) is Lebesgue measure.

Let \( f_\epsilon(z) = \int f(z - \epsilon \zeta) \varphi(\zeta) d\lambda(\zeta) \) then it is a standard result that \( f(z) = \lim_{\epsilon \to 0} f_\epsilon(z) \) is (classically)
plurisubharmonic function and \( \hat{F}(z) = f(z) \) almost everywhere.

Let \( \hat{f}(z) = \text{ess } \lim_{w \to z} f(w) \)

\[
= \inf_{g=f \text{ a.e.}} \inf_{\varepsilon > 0} \sup_{|w-z| < \varepsilon} g(w).
\]

Since \( \hat{F}(z) = f(z) \) almost everywhere, \( \hat{f}(z) \leq \hat{F}(z) \). But

\[
f_{\varepsilon}(z) = \int f(z - \varepsilon \zeta) \varphi(\zeta) d\lambda(\zeta)
\]

\[
= \int \frac{f(w) \varphi(z-w)}{|z-w| < \varepsilon} d\lambda(w)
\]

\[
\leq \text{ess sup } f(w) \cdot \int \frac{\varphi(z-w)}{|z-w| < \varepsilon} d\lambda(w)
\]

\[
f_{\varepsilon}(z) \leq \text{ess sup } f(w).
\]

So taking limits as \( \varepsilon \to 0 \) we get

\[
\hat{F}(z) \leq \hat{f}(z).
\]

This completes the proof of proposition 2.1.2.

**Definition 2.1.9**: If \( f \) satisfies the definition 2.1.8 then the unique (classically) plurisubharmonic function which coincides with \( f \) almost everywhere obtained in proposition 2.1.2 will be called the canonical representative of \( f \).

Here are some properties of plurisubharmonic (classically or non-classically) functions which we will use in the next sections, whose proofs can be found in Hörmander [5] or Vladimirov [9].
i) If \( f \in \mathcal{O}(\Omega) \) then \( \log|f| \) is (classically) plurisubharmonic on \( \Omega \).

ii) If \( f_n \) is a sequence of (classically) plurisubharmonic functions on \( \Omega \) which are locally uniformly bounded then \( \sup_n f_n \) is (non-classically) plurisubharmonic and \( \lim_{z \to w} \sup_n f_n(w) \), \( \lim_{z \to w} \lim_{n \to \infty} f_n(w) \) are both (classically) plurisubharmonic.

iii) If \( f_n \) is a decreasing sequence of (non-classically) plurisubharmonic function then the decreasing limit is also (non-classically) plurisubharmonic.

Now we shall state and prove a theorem about radii of holomorphy which will be used to prove Rothstein's theorem in the next section.

But first, following Helms [4], we shall define the notion of capacity.

**Definition 2.1.10:** Let \( \Omega \subset \mathbb{R}^n \). A Green's function for \( \Omega \) is an extended real valued function \( G \) defined on \( \Omega \times \Omega \) such that

\[
G(x,y) = -\text{constant } \log|x-y| + h_x(y) \quad \text{for } n = 2
\]

\[
G(x,y) = \frac{\text{constant}}{|x-y|^{n-2}} + h_x(y) \quad \text{for } n > 2
\]

where for each \( x \in \Omega \), \( h_x(y) \) is harmonic such that for all \( y \in \partial \Omega \), \( G(x,y) = 0 \).
Definition 2.1.11: Let Ω be an open subset of \( \mathbb{R}^n \) with Green's function \( G \). For every compact set \( K \subset \Omega \) define the capacity of \( K \) by \( C(K) = \sup\{ \mu(K): G\mu(x) \leq 1, \mu \text{ is a measure with support in } K \text{ and } G\mu(x) = \int G(x,y) d\mu(y) \} \).

Note that for \( n > 2 \) \( \Omega \) can be taken to be all of \( \mathbb{R}^n \) for

\[
G(x,y) = \frac{\text{const.}}{|x-y|^{n-2}}
\]

is Green's function for \( \mathbb{R}^n \) since

\[
\lim_{y \to \infty} G(x,y) = 0.
\]

But in case of \( n = 2 \) \( \Omega \) can be taken as any bounded set.

Definition 2.1.12: For any \( \Omega \subset \mathbb{R}^n \) with Green's function \( G \), given \( E \subset \Omega \) define

\[
C(E) = \text{capacity of } E = \sup\{ C(K): K \lll E \}.
\]

Sets are capacity zero are called polar sets and are equivalent to the removable sets for (classically) subharmonic functions as seen in the following theorem.

Theorem: Let \( \Omega \) be an open set. \( A \) is relatively closed polar subset of \( \Omega \) if and only if \( f \) is classically subharmonic on \( \Omega - A \) and locally bounded above on \( \Omega \) implies \( f \) has a unique subharmonic extension to \( \Omega \).

Proof: See Helms [4].
Theorem 2.1.1: Suppose $f$ is holomorphic on some open set in $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ containing a disc $\Delta'(0,r') \subset \mathbb{C}^{n-1}$. For $z' \in \Delta'(0,r') \subset \mathbb{C}^{n-1}$, let $r(z') = \sup\{r: \text{germ of } f \text{ at } (z',0) \text{ has a holomorphic extension to an open set containing } (z') \times \Delta(0,r)\}$ and $R(z') = \sup\{r: f(z',z_n) \text{ can be holomorphically continued to } \Delta(0,r) \text{ as a function of } z_n \in \mathbb{C}\}$. We shall refer to $r$ as joint radius of meromorphy and to $R$ as separate radius of holomorphy. Then the following are true.

i) $-\log R(z')$ is (non-classically) plurisubharmonic

ii) $-\log r(z')$ is (classically) plurisubharmonic

iii) $-\log r(z')$ is the upper envelope of $-\log R(z')$

iv) $-\log R(z') = -\log r(z')$ on $\Delta'(0,r')$ except on a polar set, i.e., a set of capacity zero. In particular $-\log r(z')$ is the canonical representative of $-\log R(z')$.

Proof:

(i): Since $f$ is holomorphic on some open set in $\mathbb{C}^n$ containing a disc $\Delta'(0,r') \subset \mathbb{C}^{n-1}$ for $z' \in \Delta(0,r')$ we can
expand \( f \) as a power series in \( z_n \), i.e.,

\[
f(z', z_n) = \sum_{j=1}^{\infty} a_j(z') z_j^n \quad \text{with} \quad a_j(z') \in \mathcal{G}(\Delta').
\]

Hadamard's theorem implies \( 1/R(z') = \lim_{n \to \infty} |a_n(z')|^{1/n} \), so

\[
-\log R(z') = \lim_{n \to \infty} \frac{1}{n} \log |a_n(z')|.
\]

Since \( a_n(z') \in \mathcal{G}(\Delta') \), by property (i) page 32, \( \log |a_n(z')| \) is (classically) plurisubharmonic. By property (ii) on page 32, \( \sup_{j \geq n} \frac{1}{j} \log |a_j(z')| \) will be non-classically plurisubharmonic if \( \log |a_j(z')|^{1/j} \) is locally uniformly bounded above. By applying Cauchy's inequalities it can immediately be seen that \( \log |a_j(z')|^{1/j} \) are locally uniformly bounded above. Now by applying property (iii) on page 32, to \( \sup_{j \geq n} \frac{1}{j} \log |a_j(z')| \),

\[
\log R(z') = \text{decreasing limit of} \sup_{j \geq n} \frac{1}{j} \log |a_j(z')| \quad \text{as} \quad n \to \infty
\]
is non-classically plurisubharmonic.

(ii) Before proving (ii), we shall derive the following formula for \(-\log r(z')\) that will enable us to prove the claim:

\[
-\log r(z') = \lim_{n \to \infty} \sup_{\zeta' \to z'} \frac{1}{j} \log |a_j(\zeta')|.
\]

Since \( \lim_{n \to \infty} \sup_{\zeta' \to z'} \frac{1}{j} \log |a_j(\zeta')| \)

\[
= \lim_{n \to \infty} \lim_{\epsilon \to 0^+} \sup_{|z' - \zeta'| < \epsilon} \sup_{j \geq n} \frac{1}{j} \log |a_j(\zeta')|
\]
\[ \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \sup_{j \geq n} \sup_{|z' - \zeta'| < \varepsilon} \frac{1}{j} \log |a_j(z')| = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log |a_n|_\Delta(z', \varepsilon) \]

where \[ |a_n|_\Delta(z', \varepsilon) = \sup_{|z' - \zeta'| < \varepsilon} |a_n(\zeta')|. \] It is enough to show that \[-\log r(z') = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log |a_n|_\Delta(z', \varepsilon) \] or equivalently \[ \frac{1}{r}(z') = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} |a_n|_{\Delta(z', \varepsilon)}^{1/n}. \] Let \[ r(z') > r, \] this implies that there exists \( \varepsilon_0 \) such that \[ f \in \Theta(\Delta(z', \varepsilon_0) \times \Delta(0, r)) \] and hence bounded say by \( M \), on \( \Delta(z', \varepsilon_0) \times \Delta(0, r) \). Using Cauchy's inequalities, we get

\[ |a_n|_{\Delta(z', \varepsilon_0)} \leq \frac{M}{r^n}, \text{ i.e., } |a_n|_{\Delta(z', \varepsilon_0)}^{1/n} \leq \frac{M^{1/n}}{r}. \] So

\[ \lim_{n \to \infty} |a_n|_{\Delta(z', \varepsilon_0)}^{1/n} \leq \frac{1}{r}. \] So

\[ \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} |a_n|_{\Delta(z', \varepsilon)}^{1/n} \leq \frac{1}{r}. \]

This implies that \[ \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} |a_n|_{\Delta(z', \varepsilon)}^{1/n} \leq \frac{1}{r(z')} . \] Now let \( r \) be such that \[ \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} |a_n|_{\Delta(z', \varepsilon)}^{1/n} \leq \frac{1}{r}. \] Then there exists \( \varepsilon_0 > 0 \) and \( n_0 \), a positive integer such that

\[ |a_n|_{\Delta(z', \varepsilon_0)} r^n \leq 1 \text{ for all } n \geq n_0. \] But this implies that \[ \sum a_j(z')z_n^j \] converges uniformly on \( \Delta(z', \varepsilon_0) \times \Delta(0, r) \). By definition of \( r(z') \), \( r \leq r(z') \), which proves the formula for \(-\log r(z')\), i.e., \(-\log r(z') = \lim_{n \to \infty} \sup_{\zeta' \to z'} \frac{1}{j} \log |a_j(\zeta')| \).
But since \( \log |a_j(z')|^{1/j} \) are locally uniformly bounded above by property (ii) on page 32, \( \lim_{\zeta' \to z'} \sup_{j \geq n} \frac{1}{j} \log |a_j(\zeta')| \) is classically plurisubharmonic and \( -\log r(z') \) is a decreasing limit of classically plurisubharmonic, hence itself a classically plurisubharmonic.

(iii) Now the upper envelope of \( -\log R(z') \) is

\[
\lim_{\epsilon \to 0^+} \sup_{|z' - \zeta'| < \epsilon} -\log R(\zeta') = \lim_{\epsilon \to 0^+} \sup_{|z' - \zeta'| < \epsilon} \lim_{n \to \infty} \frac{1}{n} \log |a_n(\zeta')| .
\]

So to show that \( -\log r(z') \) is the upper envelope of \( -\log R(z') \), it is enough to show \( \lim_{\epsilon \to 0^+} \sup_{|z' - \zeta'| < \epsilon} \lim_{n \to \infty} \frac{1}{n} |a_n(\zeta')|^{1/n} \)

\[
= \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \sup_{|z' - \zeta'| < \epsilon} \log |a_n(\zeta')|^{1/n} .
\]

For \( \zeta' \) such that \( |z' - \zeta'| < \epsilon \), \( \lim_{n \to \infty} \log |a_n(\zeta')|^{1/n} \leq \lim_{n \to \infty} \sup_{|z' - \zeta'| < \epsilon} \log |a_n(\zeta')|^{1/n} \)

so taking limits as \( \epsilon \to 0^+ \), we get

\[
\lim_{\epsilon \to 0^+} \lim_{n \to \infty} \sup_{|z' - \zeta'| < \epsilon} \frac{1}{n} \log |a_n(\zeta')| \leq 
\]

\[
\lim_{\epsilon \to 0^+} \lim_{n \to \infty} \sup_{|z' - \zeta'| < \epsilon} \frac{1}{n} \log |a_n(\zeta')| .
\]

Now to show the reverse inequality, let

\( \lambda \geq \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \sup_{|z' - \zeta'| < \epsilon} \frac{1}{n} \log |a_n(\zeta')| \), then there exists \( \epsilon_0 \) such that for all \( \epsilon < \epsilon_0 \)

\( \lambda \geq \lim_{n \to \infty} \frac{1}{n} \log |a_n(\zeta')| \) for
all $\zeta'$ such that $|\zeta'-z'|<\varepsilon$. Since the functions 
$\frac{1}{n} \log |a_n(\zeta')|$ are (classically) plurisubharmonic and
locally uniformly bounded above on \{ $\zeta': |z'-\zeta'|<\varepsilon$ \} we
can apply the following theorem of Hartogs on the sequences
of (classically) plurisubharmonic functions to the sequence
$\{ \frac{1}{n} \log |a_n(\zeta')| \}$.

**Hartogs' theorem:** Let $v_n$ be a sequence of (classically)
plurisubharmonic functions on $\Omega$ which
are uniformly bounded above on every compact set contained in $\Omega$ and assume
that $\lim_{n \to \infty} v_n(z) \leq C$ for every $z \in \Omega$.

For every $\eta > 0$, and every compact set
$K \subset \Omega$, one can find $N$ such that
$v_n(z) \leq C + \eta$, $z \in K$, $n \geq N$.

**Proof:** See Hörmander [5], theorem 1.6.13.

Now applying the above theorem, we get for arbitrary
$\eta > 0$, $0 < \varepsilon' < \varepsilon$ there exists $N$ such that
$\frac{1}{n} \log |a_n(\zeta')| \leq \lambda + \eta$ for $\zeta'$ such that $|\zeta'-z'|<\varepsilon'$,

$n \geq N$, i.e., \sup_{|\zeta'-z'|<\varepsilon'} \frac{1}{n} \log |a_n(\zeta')| \leq \lambda + \eta$ for $n \geq N$.

This implies that $\lim_{n \to \infty} \sup_{|\zeta'-z'|<\varepsilon'} \frac{1}{n} \log |a_n(\zeta')| \leq \lambda + \eta$,

since $\eta$ is arbitrary, $\lim_{n \to \infty} \sup_{|\zeta'-z'|<\varepsilon'} \frac{1}{n} \log |a_n(\zeta')| \leq \lambda$. 

Hence \( \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{|z' - \zeta'| < \epsilon} \frac{1}{n} \log |a_n(\zeta')| \leq \lambda \). This proves that the upper envelope of \(-\log R(z')\) is \(-\log r(z')\).

(iv) To prove (iv) we shall make use of the following theorem of Cartan.

**Cartan's theorem:** If \( \mathfrak{F} \) is the family of subharmonic functions which are locally bounded above on an open set \( \Omega \) then \( v = \sup\{u: u \in \mathfrak{F}\} \) differs from the upper envelope \( \tilde{v} \) of \( v \) at most on a set of capacity zero.

**Proof:** See Helms [4], page 155.

Now let \( \mathfrak{F}_n = \{\frac{1}{j} \log |a_j(\zeta')| ; j \geq n\} \) and

\[
u_n = \sup\{u: u \in \mathfrak{F}_n\} = \sup_{j \geq n} \frac{1}{j} \log |a_j(\zeta')|
\]

let \( \hat{\nu}_n = \) upper envelope of \( \nu_n \)

\[
\hat{\nu}_n = \lim_{\zeta' \to z'} \sup_{j \geq n} \frac{1}{j} \log |a_j(\zeta')|
\]

\( \mathfrak{F}_n \) is a family of plurisubharmonic functions of \( n \) variable so as functions of \( 2n \) variables \( \mathfrak{F}_n \) is a family of subharmonic functions. Applying Cartan's theorem, we get that there exists a set \( B_n \) of capacity zero such that
\[ u_n = \hat{u}_n \] except on \( B_n \).

Let \( B = \bigcup_{n=1}^{\infty} B_n \) then since for each \( n \), capacity of \( B_n \)
is zero, capacity of \( B \) is also zero and for each \( n \),
\[ u_n = \hat{u}_n \] except on \( B \). Hence \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} \hat{u}_n \) except on \( B \). But \( -\log R(z') = \lim_{n \to \infty} u_n \) and \( -\log r(z') = \lim_{n \to \infty} \hat{u}_n \).

So \( -\log R(z') = -\log r(z') \) except on a set of capacity zero.

Now \( -\log r(z') \) is classically plurisubharmonic and agrees with \( -\log R(z') \) almost everywhere, hence by uniqueness \( -\log r(z') \) must be the canonical representative of \( -\log R(z') \).

This completes the proof of theorem 2.1.1.

2.2 Radii of Meromorphy and Rothstein's theorem

In this section, we shall define two radii of meromorphy and prove the Rothstein's theorem about the radii of meromorphy which is analogous to part (iv) of the theorem 2.1.1 in the previous section. We shall prove that the two radii of meromorphy are equal except on a set of capacity zero.

Definition 2.2.1: Let \( f(z) \) be a complex valued function defined in \( \Omega \), open connected, non-empty set in \( \mathbb{C}^n \), except on variety \( E \), where \( E \) is closed nowhere dense in \( \Omega \) and for every neighborhood \( N \) of an arbitrary point in \( E \), \( N - E \) is connected. Suppose (i) \( f \) is holomorphic on \( \Omega - E \),
(ii) at every point \( z \in \Omega \), there exists neighborhood \( N_z \) and two functions \( g_z, f_z \) holomorphic in \( N_z \) such that \( g_z = h_z f \) in \( N - E \). \( f \) is then said to be meromorphic in \( \Omega \). If \( f \) satisfies definition 2.2.1 on \( \Omega \), we will say \( f \) is jointly meromorphic.

Definition 2.2.2: Let \( f(z) \) be defined in the unit polydisc \( \Delta^n(0,1) = \{ z \in \mathbb{C}^n : |z_j| < 1 ; j = 1, \ldots, n \} \) except on \( E \), where \( E \) is closed, nowhere dense in \( \Delta(0,1) \) and for every neighborhood \( N \) of an arbitrary point in \( E \), \( N - E \) is connected.

Let \( \hat{\Delta}_j = \{ \hat{z}_j = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbb{C}^{n-1} \) such that \( |z_i| < 1 \), \( i \neq j \} \). And let \( S_j = \{ \hat{z}_j \in \hat{\Delta}_j \) such that there exists an open set \( N_j \) in \( \mathbb{C}^{n-1} \) such that \( (z_1, \ldots, z_j, \ldots, z_n) \in E \) for all \( z_j \in N_j \} \). For each \( j \), and for every point \( (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) \in \hat{\Delta}_j - S_j \) each of the functions \( f(a_1, \ldots, a_{j-1}, z_j, \ldots, a_n) \) as a function of one variable \( z_j \) is meromorphic in \( |z_j| < 1 \) then we say \( f \) is separately meromorphic in each variable in \( \Delta^n(0,1) \).

Now we state Rothstein's theorem. The proof of this theorem given below is based essentially on the same techniques suggested by him through his personal correspondence with Dr. Shiffman. But a few changes in his proof has enabled us to eliminate the use of the Levi-Oka theorem on domains of holomorphy and meromorphy.
Rothstein's theorem: Let $f \in \mathcal{O}(\Delta^{n+1})$ where $\Delta^{n+1} = \Delta^n \times \Delta^1 = \{(z,w) \in \mathbb{C}^{n+1}, |z_j| < 1 \text{ for } j = 1, \ldots, n, |w| < 1\}$. For $z \in \Delta^n$ let $M(z) = \sup\{r: f_z(w) = f(z,w) \text{ as a function of } w \text{ is meromorphic in } \Delta'(0,r)\}$ and $m(z) = \sup\{r: f(z,w) \text{ is meromorphic jointly in neighborhood of } z \times \Delta^1(0,r)\}$. We shall refer to $M(z)$ as separate radius of meromorphy and to $m(z)$ as joint radius of meromorphy. Then $m(z) = M(z)$ for all $z \in \Delta^n(0,1)$ except on a set of capacity zero.

Proof:

It is obvious that $m(z) \leq M(z)$ for all $z \in \Delta^n(0,1)$. So it is enough to show that the capacity of $Q = \{z: m(z) < M(z)\}$ is zero. The proof will be based on reductio-ad-absurdum.

Suppose capacity of $Q$ is positive. Let $c \in Q$, i.e., $m(c) < M(c)$. We can choose $n(c)$, a positive integer with $m(c) + \frac{1}{n-1} < M(c)$. Now $f_c(w)$ is meromorphic in $\Delta'(m(c) + \frac{1}{n})$. Let $\{a_1, \ldots, a_n\}$ denote the poles of $f_c$ in $\Delta'(m(c) + \frac{1}{n})$.

Since $\{a_j\}$ are poles of a meromorphic function of one variable, the set $\{a_j\}$ is discrete as a subset of $\mathbb{C}$.
So we can choose disjoint open discs $K_1, \ldots, K_N$ such that for each $j = 1, \ldots, N$

I.  
   i) $a_j \in K_j$  
   ii) $K_j$ has rational radius and center  
   iii) $\frac{1}{K_j} < \Delta'(m(c) + \frac{1}{n-1})$  
   iv) $K_j \cap \Delta'(0,1) = \emptyset$  
   v) $\frac{1}{f_c}$ is holomorphic on $K = \bigcup_{j=1}^{N} K_j$.

Now it is clear that we can find $K_1, \ldots, K_N$ such that properties (i)-(iv) are satisfied. For them to satisfy (v), it is enough to note that $f_c(w) \to \infty$ as $w \to a_j$ in $K_j$ so we can choose $K_j$ small enough that $f_c(w)$ does not vanish on $K$.

Let $R = \Delta'(m(c) + \frac{1}{n}) - K$. $R$ is compact and so can be covered by discs $\{H_j\}_{j=1}^{M}$ such that for each $j = 1, \ldots, M$

II.  
   i) $a_j \notin H_j$  
   ii) $H_j$ has rational radius and center  
   iii) $H_j \subseteq \Delta'(m(c) + \frac{1}{n-1})$  
   iv) Let $H = \bigcup_{j=1}^{M} H_j$ then $f_c(w)$ is holomorphic on $H$.

For each $c \in Q$, there corresponds a tuple of rational numbers $L(c) = \{n(c), K_1(c), \ldots, K_N(c), H_1(c), \ldots, H_M(c), M(c)\}$. The set $\{L(c): c \in Q\}$ is a countable set and $\{c \in Q: L(c) = L^* = \text{constant}\}$ forms a disjoint countable
cover of \( Q \). But since capacity of \( Q \) is positive there exists \( Q^* \subset Q \) such that capacity of \( Q^* > 0 \) and for all \( c \in Q^* \), \( L(c) = \text{constant} = L^* \).

Now we will show that for \( c \in Q^* \), \( f \) is jointly meromorphic on a neighborhood of \( c \times \Delta'(m(c) + \frac{1}{n^*}) \) which obviously is a contradiction by the definition of \( m(c) \).

Before completing the proof, we need the following lemma.

**Lemma 2.2.1:** Suppose \( f \in \mathcal{G}(\Delta^{n+1}) \), \( \Delta^{n+1} = \Delta^n \times \Delta^1 \) and \( G \) is an open set containing \( \Delta' \). If \( P = \{ c \in \Delta^n : f_c(w) \text{ extends holomorphically to } G \} \) has positive capacity then there exists \( P_1 \subset P \) with capacity \( P_1 > 0 \) such that for all \( c \) in \( P_1 \) and for arbitrary \( G \subset \subset G \), \( f \) extends holomorphically to a neighborhood of \( \{c\} \times G \). In fact, \( P_1 \) and \( P \) differ by a set of capacity zero.

**Proof:**

We can assume that \( G \) is a disc \( \Delta(0,g) \subset \mathcal{C} \), \( g \geq 1 \). For \( c \in \Delta^n \), let \( R(c) \), \( r(c) \) be the separate and joint radii of holomorphy respectively as defined on page 42. Then \( f_c(w) \in \mathcal{G}(G) \) implies \( R(c) \geq g \) for all \( c \in P \). Let \( P_1 = P \cap \{c : r(c) \geq g - \varepsilon\} \) where \( \varepsilon \) is arbitrarily small positive number. It is obvious that

\[
CP_1 \cap P \subset \{ c \in \Delta^n : r(c) < R(c) \}.
\]
But by theorem 2.1.1 \{c ∈ Δ^n: r(c) < R(c)\} has capacity zero so \(CP_1 \cap P\) has capacity zero. But capacity \(P > 0\), hence the capacity of \(P_1\) is positive, i.e., \(P_1 = \{c ∈ P: f\) extends holomorphically to a neighborhood of \(c × Δ'(0, g-ε)\}\} has positive capacity. This completes the proof of the lemma 2.2.1. Now the remaining proof of Rothstein's theorem is as follows.

First apply lemma 2.2.1 with \(G = \mathbb{H}^\ast\) since \(Q_1^\ast \subset \{c ∈ Δ^n: f_c(w) \) extends holomorphically to \(\mathbb{H}^\ast\}\) and capacity of \(Q_1^\ast > 0\), there exists \(Q_1^\ast\) with capacity of \(Q_1^\ast > 0\) and for all \(c ∈ Q_1^\ast\), \(f\) extends holomorphically to a neighborhood of \(c × \mathbb{H}^\ast\), where \(\mathbb{H}^\ast \subset \mathbb{H}^\ast\). Now \(\mathbb{H}^\ast\) can be so chosen that there exists \(\mathbb{H}^\ast \subset \mathbb{H}\) such that \(\mathbb{H}\) also satisfies the properties II and \(\mathbb{K}_j^\ast \cap \mathbb{H}^\ast \neq \emptyset\), for all \(j\). Let \(F\) denote the extension of \(f\) which is jointly holomorphic on some neighborhood of \(Q_1^\ast × \mathbb{H}^\ast\).

Now for each \(j = 1, \ldots, N\), there exists a point \(w_j^\ast \in \mathbb{K}_j^\ast \cap \mathbb{H}^\ast\) such that \(F\) is jointly holomorphic in a neighborhood of \(Q_1^\ast × w_j^\ast\). So choose \(G_j^\ast\) open in \(\mathbb{C}\), \(G_j^\ast \subset \mathbb{K}_j^\ast\) such that \(F\) is jointly holomorphic in a neighborhood of \(Q_1^\ast × G_j^\ast\). But by I-(v), \(F_c = f_c\) does not vanish on \(G_j^\ast \subset \mathbb{K}_j^\ast\) for \(c ∈ Q_1^\ast\). Therefore \(\frac{1}{F}\) is jointly holomorphic in a neighborhood of \(Q_1^\ast × G_j^\ast\) and \(\frac{1}{F_c}\) has holomorphic extension to \(\mathbb{K}_j^\ast\) for each \(c ∈ Q_1^\ast\). Now apply lemma 2.2.1 with \(G^\ast = \bigcup_{j=1}^{N} G_j^\ast\) to \(\frac{1}{F}\). So there exists a set \(Q_2^\ast \subset Q_1^\ast \subset Q^\ast\).
with capacity $Q_2^* > 0$ such that $\frac{1}{F}$ has a holomorphic extension to a neighborhood of $Q_2^* \times K_j^*$. Consequently $F$ is jointly meromorphic in $Q_2^* \times (H_j^* \cup K_j^*)$. But for $c \in Q_2^* \subset Q^*$, $\Delta'(m(c) + \frac{1}{n}) \subset H_j^* \cup K_j^*$. This proves that there exists $c \in Q^*$ such that $f$ has a jointly meromorphic extension to a neighborhood of $c \times \Delta'(m(c) + \frac{1}{n})$. It is an obvious contradiction. So the capacity of $Q = \{c : m(c) < M(c)\}$ is zero, i.e., $m(c) = M(c)$ except on a set of capacity zero.

2.3 Analogue of Hartogs' theorem

In this section, we shall prove the analogue of the Hartogs' theorem mentioned on page 35 for the case of meromorphic functions. The proof is based on Rothstein's theorem proved in section 2.2 and suggestions by Dr. B. Shiffman in his personal correspondence.

Notation: i) \( \pi_j : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1} \) given by \( \pi_j(z_1, \ldots, z_n) = (z_1, \ldots, z_j-1, z_{j+1}, \ldots, z_n) \), and \( \pi^j : \mathbb{C}^n \longrightarrow \mathbb{C} \) given by \( \pi^j(z_1, \ldots, z_n) = z_j \).

ii) for $E \subset \mathbb{C}^n$, $w \in \mathbb{C}^{n-1}$, $E_{j,w} = E \cap \pi^{-1}_j(w) = \{(z_1, \ldots, z_n) \in E \text{ such that } z_i = w_i, i \neq j\}$.

iii) If $f : E \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ then $f_{j,w} = f|_{E_{j,w}}$. 


so for \( w \in C^{n-1} \) domain of \( f_{j,w} \) is contained in \( \pi_j(E) \subset C \).

**Theorem 2.3.1:** Let \( f: \Delta^n \to C \) where \( E \) is a subset of \( \Delta^n = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_n = \{z: |z_j| < 1, j = 1, \ldots, n\} \) such that for \( 1 \leq j \leq n \), there exists a closed subset of measure zero \( S_j \subset \pi_j(\Delta^n) \subset C^{n-1} \) such that for all \( w \in \pi_j(\Delta^n) - S_j \), \( E_{j,w} \) is discrete and \( f_{j,w} \) extends to a meromorphic function on \( \Delta_j \), then \( f \) is jointly meromorphic on \( \Delta^n \).

**Proof:**

We shall prove the result by induction on \( n \). First, we shall prove it for \( n = 2 \). The proof is based on Rothstein's theorem, so it is necessary to show that \( f \) is holomorphic in some open set \( U \times V \subset \Delta^2 \). This can be done with the use of Baire category theorem.

Let \( \Delta^2 = \Delta_1 \times \Delta_2 \), \((z,w) \in \Delta^2 \) implies \( |z| < 1 \), \( |w| < 1 \). Let \( \{U_n\}^\infty \), be the basis for the topology of \( \Delta_2 - S_2 \). Since \( S_1 \) is closed, \( \Delta_1 - S_1 \) is a Baire space. We shall now construct a countable covering of \( \Delta_1 - S_1 \) by closed sets.

Let \( E_{m,n} = \{z \in \Delta_1 - S_1: \|f_2,z\|_{U_n} \leq m\} \)

\[
= \bigcap_{w \in U_n} \{z: \Delta_1 - S_1: |f_2,z(w)| \leq m\}.
\]
Since \( w \in U_n \subset \Delta_2 - S_2 \) and \( z \in \Delta_1 - S_1 \), \( f_{2, z}(w) = f_{1, w}(z) = f(z, w) \). So \( E_{m, n} = \bigcap_{w \in U_n} \{ z \in \Delta_1 - S_1 : |f_{1, w}(z)| \leq m \} \).

For fixed \( w \in U_n \), \( f_{1, w} \) is meromorphic function of \( z \). So the set \( \{ z \in \Delta_1 - S_1 : |f_{1, w}(z)| \leq m \} \) is closed. Hence \( E_{m, n} \), the intersection of closed sets is closed and clearly \( E_{m, n} \) covers \( \Delta_1 - S_1 \). So by applying the Baire Category theorem to \( \Delta_1 - S_1 \), we get that there exists \( E_{m, n} \) which contains an open set, i.e., there exists an open set \( U \times V \subset \Delta^2 \) such that \( f \) is bounded on \( U \times V \), therefore \( f \) is holomorphic on \( U \times V \).

Now apply Rothstein's theorem to \( f \) which is holomorphic on \( U \times V \) and for fixed \( z \in U \), \( f_2(w) \) extends meromorphically as a function of \( w \) to \( \Delta_2 \). Since \( \{ z : m(z) \neq M(z) \} \) is a set of capacity zero by Rothstein's theorem, we get \( M(z) \geq 1 \) implies \( m(z) \geq 1 \) for all \( z \in U \) except on a set of capacity zero, say \( A \).

Hence \( f \) is jointly meromorphic on \( (U \times V) \cup (U - A) \times \Delta_2 \). But \( A \) is a set of measure zero, so \( f \) can be extended as a jointly meromorphic function on \( U \times \Delta_2 \).

Again applying Rothstein's theorem and above argument to the set \( V \), we get \( f \) is jointly meromorphic on \( \Delta_1 \times V \). Hence \( f \) is jointly meromorphic on \( (U \times \Delta_2) \cup (\Delta_1 \times V) \). Since \( f \) is jointly meromorphic on \( (\Delta_1 - U) \times V \), it is holomorphic on an open dense set of points in \( (\Delta_1 - U) \times V \). By repeating the above argument to those set of points \( f \) can be proved to be jointly meromorphic on \( \Delta_1 \times \Delta_2 \).
The following steps will complete the proof of the theorem for arbitrary \( n \). The proof has not been given in details since the notation involved makes the proof appear much harder and unreadable than it really is.

i) Assume that the result is true for \( n - 1 \).

ii) Show that there exists a set \( A \) of measure zero such that for fixed \( z_n \in \Delta_n - A \) the function \( f \) as a function of \( (z_1, \ldots, z_{n-1}) \) satisfies the hypothesis of the theorem for the case of \( n - 1 \) variables. So by (i) for all \( z_n \in \Delta_n - A \), \( f \) will be jointly meromorphic on \( \Delta^{n-1} \).

iii) Using the fact that \( f \) is jointly meromorphic as a function of \( z_1, \ldots, z_{n-1} \) construct a closed cover of \( \Delta_n - A \) in the same way as was done for \( n = 2 \).

iv) Now apply Baire Category theorem to \( \Delta_n - A \) and show that \( f \) is bounded on some open set \( U_1 \times \ldots \times U_n \subset \Delta^n \) and hence holomorphic on it.

v) As indicated in the proof of \( n = 2 \) apply Rothstein's theorem to complete the proof.
Bibliography


