INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again -- beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms
300 North Zeab Road
Ann Arbor, Michigan 48106
WONG, Hin-Wai, 1944-
CONVECTIVE INSTABILITIES OF FLUIDIZED PARTICLES.

Rice University, Ph.D., 1973
Engineering, chemical

University Microfilms, A XEROX Company, Ann Arbor, Michigan
RICE UNIVERSITY

CONVECTIVE INSTABILITIES OF FLUIDIZED PARTICLES

by

Wong, Hin-Wai

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's Signature:

Houston, Texas

May 1973
ABSTRACT

Title: Convective Instabilities of Fluidized Particles

Author: WONG Hin-Wai

A linearized hydrodynamic stability analysis is applied to two-dimensional systems of fluidized particles based on the hydrodynamic model proposed by Anderson and Jackson.

It is shown that such systems are inherently unstable against small perturbations and the instabilities appear in the form of convective cells, the width of which is determined by the wave length of the dominant mode of disturbance.

The analysis also shows that the distributor pressure drop plays an important role in suppressing the convective instabilities of the fluidized particle assemblies and that the instability is several times faster with the gas-fluidized beds than with the liquid-fluidized beds.
ACKNOWLEDGEMENT

My years at Rice University as a graduate student have been an intellectually satisfying and rewarding experience. To all my teachers at Rice I express my thanks.

I consider myself very fortunate to have had the opportunity of studying under Professor Roy Jackson. I have benefited greatly from his excellent teaching as well as his patient guidance in my research work. To Professor Jackson, I express my deepest gratitude.

Many of my fellow students at Rice have provided me with useful and stimulating discussions on academic matters. Of these, I particularly appreciate the many fruitful discussions I have had with James Medlin who is currently conducting research in a closely related area.

Finally, I wish to thank Dr. L. McIntire and Dr. R. Bowen for serving in my thesis committee and I gratefully acknowledge the financial support from Rice University in the form of Rice fellowship.
TABLE OF CONTENTS

Abstract ................................................................. ii
Acknowledgement ......................................................... iii
Table of Contents ......................................................... iv
List of Figures ........................................................... vi
List of Symbols ........................................................... ix

1. INTRODUCTION ......................................................... 1
   1.1 The Hydrodynamic Stability Theory ......................... 1
   1.2 The Stability of Fluidized Beds .......................... 2
   1.3 The Convective Instabilities of Fluidized Beds ... 3
   1.4 The Notation Conventions ................................. 4

2. DERIVATION OF THE PERTURBATION EQUATIONS ......... 5
   2.1 The Hydrodynamic Model of Fluidized Beds .......... 5
   2.2 Linearized Perturbation Equations .................... 7
   2.3 The Dimensionless Form of the Perturbation Equations .... 9

3. THE CASE OF ZERO FLUID DENSITY AND ZERO DISTRIBUTOR PRESSURE DROP .................. 13
   3.1 The Perturbation Equations ............................... 13
   3.2 The Modal Analysis ....................................... 16
   3.3 The Boundary Conditions ................................. 20
   3.4 The Characteristic Equations ........................... 22
   3.5 Computations .............................................. 23
   3.6 Results and Discussion .................................. 27
4. THE ZERO DENSITY CASE WITH FINITE DISTRIBUTOR PRESSURE DROP .................................. 35
   4.1 Introduction ................................................. 35
   4.2 The Boundary Conditions ................................. 36
   4.3 Results and Discussion .................................... 42

5. THE CASE OF FINITE FLUID DENSITY ..................... 45
   5.1 Introduction ................................................ 45
   5.2 Modal Analysis of the Perturbation Equations ........ 45
   5.3 The Boundary Conditions ................................. 59
   5.4 The Characteristic Equations ........................... 67
   5.5 Results and Discussions .................................. 69

6. CONCLUSIONS .................................................. 75

REFERENCES ..................................................... 77
LIST OF FIGURES

(3.1) Determinant vs. $s$ for a given $k$.

Parameters specified in equation (3.45) ; $k = 1$

(3.2) Particle velocity field at the onset of instability.

$\mu^s = \lambda^s = 0$ ; other parameters as specified in equation (3.45)

(3.3) $s^*$ vs. $k^*$ for the zero fluid density case.

Parameters specified in equation (3.45)

(3.4) Effect of particle phase viscosities.

$\mu^s = \lambda^s = 0.1 ; 1.0 ; 10 ; 20$ poise

(3.5) Effect of varying particle phase shear viscosity.

$\mu^s = 5 ; 10 ; 20$ poise

(3.6) Effect of varying particle phase bulk viscosity.

$\lambda^s = 5 ; 10 ; 20$ poise

(3.7) Effect of varying inter-particle pressure.

$\frac{dP^*}{d\varepsilon^s_o} = 0 ; 200$ dynes/cm$^2$

(3.8) Effect of solid density.

$\rho^s = 1.43 ; 2.86 ; 5.72$ g/cm$^3$

(3.9) Effect of particle diameter.

$d^*_p = 0.043 ; 0.086 ; 0.172$ cm
(3.10) Effect of varying bed voidage.
\[ \varepsilon_o = 0.42; 0.46; 0.50 \]

(4.1) Effect of increasing distributor pressure drop.
\[ \frac{\Delta p^{*}_{\text{distributor}}}{\Delta p^{*}_{\text{bed}}} = 0; 0.004; 0.005; 0.01; 0.03; 0.05; 0.10 \]

(5.1) Schematic sketch of the fluidized bed.

(5.2) Determinant vs. s for the genuine root.
\[ k = 86 \text{ parameters specified in equation (5.62)} \]

(5.3) s vs. k plot for the genuine and the spurious roots.
Parameters specified in equation (5.62)

(5.4) Typical s* vs. k* curve for the standard set of parameters.
Parameters specified in equation (5.62)

(5.5) Effect of varying the particle phase shear viscosity.
\[ \mu_o^{S*} = 5; 10; 20 \text{ poise} \]

(5.6) Effect of varying the particle phase bulk viscosity.
\[ \lambda_o^{S*} = 5; 10; 20 \text{ poise} \]

(5.7) Effect of varying \[ \frac{dp^{*}}{d\varepsilon_o} \]
\[ \frac{dp^{*}}{d\varepsilon_o} = 0; 20 \text{ dynes/cm}^2 \]

(5.8) Effect of solid density.
\[ \rho_s^{*} = 2.5; 2.86; 3.2 \text{ g/cm}^3 \]
(5.9) Effect of varying the bed voidage.
\[ \varepsilon_o = 0.42; 0.46; 0.50 \]

(5.10) Effect of particle diameter.
\[ d_p^* = 0.080; 0.096 \text{ cm} \]

(5.11) Effect of neglecting the fluid density.
\[ \rho_i^* = 0; 0.0012 \text{ g/cm}^3 \]

(5.12) Effect of varying the fluid density.
\[ \rho_f^* = 0.25; 0.5; 1.0; 1.5; 2.0 \text{ g/cm}^3 \]

(5.13) Effect of varying the fluid viscosity.
\[ \mu_f^* = 0.008; 0.010; 0.012 \text{ poise} \]

(5.14) Effect of boundary condition distances.
\[ m = n = 0.005; 0.0075; 0.01; \infty \]
### LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>d*</td>
<td>Bed height</td>
<td>[cm]</td>
</tr>
<tr>
<td>d_p</td>
<td>Particle diameter</td>
<td>[cm]</td>
</tr>
<tr>
<td>E(x)</td>
<td>Pre-exponential part of $\varepsilon$, defined in Eq. (3.10)</td>
<td></td>
</tr>
<tr>
<td>$E^*$</td>
<td>Fluid stress tensor</td>
<td>[dynes/cm²]</td>
</tr>
<tr>
<td>$E^{*}$</td>
<td>Particle phase stress tensor</td>
<td>[dynes/cm²]</td>
</tr>
<tr>
<td>f*</td>
<td>Interaction force between fluid and particles</td>
<td>[dynes]</td>
</tr>
<tr>
<td>Fr</td>
<td>Froude number $= \frac{u_o}{g^* d^*}$</td>
<td></td>
</tr>
<tr>
<td>g*</td>
<td>Gravitational acceleration</td>
<td>[cm/sec²]</td>
</tr>
<tr>
<td>i</td>
<td>Complex number $(0, 1)$</td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>Unit vector in the positive x direction</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>Unit tensor</td>
<td></td>
</tr>
<tr>
<td>k*</td>
<td>Wave number of disturbance $= \frac{(cm^{-1})}{(cm^{-1})}$</td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>Dimensionless wave number $= \frac{k^<em>}{d^</em>}$</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>(Pressure drop across distributor)/(Pressure drop across bed)</td>
<td></td>
</tr>
<tr>
<td>n*</td>
<td>Number of particles per unit bed volume, $= \frac{(cm^3)^{-1}}{(cm^3)^{-1}}$</td>
<td></td>
</tr>
<tr>
<td>n_o</td>
<td>Number of particles per unit bed volume $= \frac{(cm^3)^{-1}}{(cm^3)^{-1}}$</td>
<td></td>
</tr>
<tr>
<td>p*</td>
<td>Fluid pressure $= \frac{p^<em>}{\rho_s g^</em> d^*}$</td>
<td>[dynes/cm²]</td>
</tr>
<tr>
<td>p</td>
<td>Dimensionless fluid pressure $= \frac{p^<em>}{\rho_s g^</em> d^*}$</td>
<td></td>
</tr>
</tbody>
</table>
\( p_o^* \quad \text{Fluid pressure at lower bed surfaces at steady state, [dynes/cm}^2\text{]} \)

\( p_b^* \quad \text{Fluid pressure just below bed support, [dynes/cm}^2\text{]} \)

\( p_s^* \quad \text{Particle phase pressure, [dynes/cm}^2\text{]} \)

\( p_s^* \quad \text{Dimensionless particle pressure}, \quad = \frac{p_s^*}{p_s^* g d^*} \)

\( p_o^* \quad \text{Particle pressure at steady state of uniform suspension, [dynes/cm}^2\text{]} \)

\( p_o^{s^*} \quad = \left( \frac{dp^s}{dn} \right)_{n=n_o^*}, \quad \text{[dynes cm]} \)

\( p_o^{s^*} \quad \text{Dimensionless } p_o^{s^*}, \quad = \frac{n_o^* s^{s^*}}{p_o^* g d^*} \)

\( R^* \quad \text{Bed support pressure drop coefficient, [dynes sec/cm}^3\text{]} \)

\( R \quad \text{Dimensionless } R^*, \quad = M/\epsilon_o \)

\( s^* \quad \text{Disturbance growth rate, [(sec.)}^{-1}\text{]} \)

\( s \quad \text{Dimensionless } S^*, \quad = \frac{s^* d^*}{u_o^*} \)

\( t^* \quad \text{Time, [sec.]} \)

\( t \quad \text{Dimensionless time}, \quad = \frac{t u_o^*}{d^*} \)

\( u^* \quad \text{Fluid velocity vector, [cm/sec]} \)
\[\begin{align*}
\overline{u} & \quad \text{Dimensionless } u^*, \quad = \frac{\overline{u}}{u_o} \\
\overline{u}_o & \quad \text{Magnitude of fluid velocity at steady state of uniform suspension, [cm/sec]} \\
\overline{u}_x & \quad \text{x-component of } \overline{u} \\
\overline{u}_y & \quad \text{y-component of } \overline{u} \\
\overline{U}_x (x) & \quad \text{Pre-exponential part of } u_x, \text{ defined in eqn. (3.10)} \\
\overline{U}_y (x) & \quad \text{Pre-exponential part of } u_y, \text{ defined in eqn. (3.10)} \\
\overline{U}_{mf} & \quad \text{Minimum fluidizing velocity, [cm/sec]} \\
\overline{v} & \quad \text{Particle velocity vector, [cm/sec]} \\
\overline{v} & \quad \text{Dimensionless } v^*, \quad = \frac{\overline{v}}{u_o} \\
\overline{v}_x & \quad \text{x-component of } \overline{v} \\
\overline{v}_y & \quad \text{y-component of } \overline{v} \\
\overline{V}_x (x) & \quad \text{Pre-exponential part of } v_x, \text{ as defined in eqn. (3.10)} \\
\overline{V}_y (x) & \quad \text{Pre-exponential part of } v_y, \text{ as defined in eqn. (3.10)} \\
x^* & \quad \text{Vertical displacement, [cm]} \\
x & \quad \text{Dimensionless } x^*, \quad = \frac{x}{d^*} \\
y^* & \quad \text{Horizontal displacement, [cm]} \\
y & \quad \text{Dimensionless } y^*, \quad = \frac{y}{d^*}
\end{align*}\]
GREEK SYMBOLS

\( \epsilon \)  Bed voidage

\( \epsilon_o \)  \( \epsilon \) at steady state of uniform suspension

\( \beta^* (\epsilon) \)  Drag coefficient between fluid and particles, [dynes sec/cm]

\( \beta_o^* \)  = \( \beta^* (\epsilon_o) \)

\( \beta_o^* (\epsilon) = \left( \frac{d\beta^*}{dn^*} \right)_{n=n_o^*} \)

\( \beta_o^* \) = \( \frac{n_o^* \beta_o^* u_o^*}{\rho_s^* g} \)

\( C(\epsilon) \)  Virtual mass coefficient

\( C_o \)  = \( C(\epsilon_o) \)

\( \lambda^* \)  Coefficient of bulk viscosity of fluid, [g/cm sec]

\( \lambda^* \)  Wave length of disturbance, \( = \frac{2\pi}{\lambda^*} \)

\( \lambda_o^{s*} \)  Particle phase bulk viscosity coefficient, [g/cm sec]

\( \lambda_o^{s*} \)  Dimensionless \( = \frac{\lambda_o^{s*} u_o^{s*}}{\rho_s^* g (d^*)^2} \)

\( \mu^* \)  Coefficient of shear viscosity of fluid, [g/cm sec]

\( \mu_s^* \)  Particle phase shear viscosity coefficient, [g/cm sec]
\[ \mu_0^s \quad \text{Dimensionless, } \mu_0^{s*} = \frac{\mu_0 u_o}{\rho_s^* g (d)^2} \]

\[ \rho_f^* \quad \text{Fluid density, } [\text{g/cm}^3] \]

\[ \rho_s^* \quad \text{Solid density of the particles, } [\text{g/cm}^3] \]

\[ \rho^* \quad \text{Dimensionless fluid density, } = \frac{\rho_f^*}{\rho_s^*} \]

\[ \nu^* \quad \text{Particle volume, } [\text{cm}^3] \]

\[ \nabla^* \quad \text{Spatial gradient operator, } [(\text{cm})^{-1}] \]

\[ \nabla^2* \quad \text{Laplacian operator, } [(\text{cm})^{-2}] \]

\[ \nabla \quad \text{Dimensionless } \nabla^*, \quad = \nabla^* d^* \]

\[ \nabla^2 \quad \text{Dimensionless } \nabla^2*, \quad = \nabla^2*(d^*)^2 \]

\[ E_{xy}^s \quad \text{Shear stress tensor component} \]

**Subscripts**

\[ o \quad \text{Quantities at the steady state of uniform suspension} \]

\[ 1 \quad \text{Small perturbation quantities} \]

**Superscripts**

\[ ^* \quad \text{Dimensional quantities} \]

\[ ^- \quad \text{Quantities in the clear fluid regions above and below the bed} \]
1. INTRODUCTION

1.1 THE HYDRODYNAMIC STABILITY THEORY

The basic principles of linear hydrodynamic stability theory are clearly presented in the treatises by Lin\(^\text{(1)}\) and by Chandrasekhar\(^\text{(2)}\). Stated briefly, the technique used in the stability analysis is to introduce infinitesimal perturbations into a system at steady state. The growth or decay of such perturbations is then followed with time as a consequence of the mechanical behavior of the system. If a particular mode of disturbance grows with time, then the system is said to be unstable to that mode of disturbance. A system is stable if it is stable against all modes of disturbances. On the other hand, a system is unstable even if there is only one mode of disturbance to which the system is unstable.

A hydrodynamic system is usually described by a set of non-linear partial differential equations which may have simple, steady-state solutions. Upon introducing infinitesimal perturbations into the system about the steady state, the equations are linearized. The linear set of perturbation equations is then solved under a given set of boundary conditions which are usually homogeneous. Upon seeking a solution
separable in space and time, one obtains a characteristic value problem for a set of homogeneous linear ordinary differential equations and corresponding boundary conditions, with the separation constant appearing as the parameter, which takes the characteristic values. One particular problem that has been treated successfully by linearized hydrodynamic stability analysis is that of convective instabilities in a layer of fluid with an adverse temperature gradient. This is the so-called Bénard problem\(^{(2)}\). It will be seen later in this work that the analysis of the convective instabilities of fluidized beds is analogous to that of the Bénard problem.

1.2 THE STABILITY OF FLUIDIZED BEDS

The existence of two types of fluidized beds is well known\(^{(10)}\). Particulate fluidized beds, which are almost exclusively liquid-fluidized beds, have a smooth appearance with the particles more or less evenly spaced from one another. Almost all gas-fluidized beds, on the other hand, are aggregative fluidized beds, which are characterized by the occurrence of gas bubbles in the bed and the apparent turbulence caused by the passage of these bubbles.

A theoretical basis for the existence of the two types of fluidized beds was first proposed by Jackson (1963)\(^{(3)}\) who successfully distinguished them on the basis of a linearized hydrodynamic stability analysis.
applied to a system of fluidized particles. In the subsequent years, several similar attempts have been made in the analysis of fluidized bed stabilities\(^{(4-8)}\) and these are reviewed in a recently published article (Jackson, 1971).\(^{(9)}\)

In 1967, Anderson and Jackson\(^{(6)}\) proposed a hydrodynamic model of fluidized beds. In essence, the model views the fluidized assemblage of particles as a system of two inter-penetrating continua and derives the local averages through a formal mathematical manipulation of the point variables. This model has been applied to the analysis of the stability of the state of uniform fluidization\(^{(7)}\) and it will be used in this work as the basis for the analysis of the convective instabilities of the fluidized particle assembly.

1.3 THE CONVECTIVE INSTABILITIES OF FLUIDIZED BEDS

Consider a two-dimensional system of fluidized particles which extends a finite height in the vertical (z) direction and extends to infinity in the horizontal (y) direction. Suppose initially the system is at a steady state in which all particles are uniformly suspended and stationary while the fluidizing fluid percolates through the particle assembly at a uniform velocity. If at this state, a set of small perturbations is introduced into the system, how will the system react to these perturbations? In particular, will such a system sustain a convective type of
instability similar to that of the Bénard problem? These are the questions which the present work seeks to answer.

Three distinct cases are studied:

(i) The zero fluid density case with zero distributor pressure drop.

(ii) The zero fluid density case with finite distributor pressure drop.

(iii) The finite fluid density case with zero distributor pressure drop.

These cases are treated in Chapters (3), (4), and (5) respectively.

In Chapter (2) the derivation of the perturbation equations in their dimensionless form is given, which is the starting point for the stability analysis. The conclusions of this work are presented in Chapter (6).

1.4 THE NOTATION CONVENTIONS

Throughout this work, the superscript * is used to denote dimensional variables. The same symbols without the superscript * will denote the corresponding dimensionless variables. The subscript 1 is used to denote small perturbation quantities and the subscript o is used to denote the steady-state conditions. The subscripts s and f denote solid and fluid respectively. The super-bar — is used to denote variables in the clear fluid regions above and below the bed surfaces.
2. DERIVATION OF THE PERTURBATION EQUATIONS

2.1 THE HYDRODYNAMIC MODEL OF FLUIDIZED BEDS

The Anderson-Jackson Model consists of four equations: two scalar equations of continuity and two vector equations of motion.

Equation of continuity for fluid

\[ \frac{\partial \varepsilon}{\partial t} + \nabla^* \cdot (\varepsilon \mathbf{u}^*) = 0 \]  \hspace{1cm} (2.1)

Equation of continuity for particles

\[ \frac{\partial (1-\varepsilon)}{\partial t} + \nabla^* \cdot [(1-\varepsilon) \mathbf{v}^*] = 0 \]  \hspace{1cm} (2.2)

Equation of motion for fluid

\[ \rho_f^* \left[ \frac{\partial \mathbf{u}^*}{\partial t} + \left( \mathbf{u}^* \cdot \nabla^* \right) \mathbf{u}^* \right] = \nabla^* \cdot \mathbf{F}^* - \frac{\gamma_f^*}{\varepsilon} + \rho_f^* \mathbf{g}^* \]  \hspace{1cm} (2.3)

Equation of motion for particles

\[ \rho_s^* (1-\varepsilon) \left[ \frac{\partial \mathbf{v}^*}{\partial t} + \left( \mathbf{v}^* \cdot \nabla^* \right) \mathbf{v}^* \right] - \rho_f^* (1-\varepsilon) \left[ \frac{\partial \mathbf{u}^*}{\partial t} + \left( \mathbf{u}^* \cdot \nabla^* \right) \mathbf{u}^* \right] \]

\[ = \frac{\gamma_f^*}{\varepsilon} + (1-\varepsilon) \left( \rho_s^* - \rho_f^* \right) \mathbf{g}^* + \nabla^* \cdot \mathbf{e}_{sp}^* \]  \hspace{1cm} (2.4)

Where \( \varepsilon, \mathbf{u}^*, \mathbf{v}^*, \rho_s^*, \rho_f^*, \mathbf{g}^* \) are respectively the local mean voidage, fluid velocity, particle velocity, solid density, fluid density and gravitational
acceleration. \( \mathbf{n}^* \) is the interaction force per unit bed volume between the fluid and the particles and \( \mathbf{E}^*, \mathbf{E}^{*s} \) are the stress tensors of the fluid and particle phases:

\[
\mathbf{n}^* = \varepsilon \beta^*(\varepsilon) (\mathbf{u}^* - \mathbf{v}^*) + (1 - \varepsilon) C(\varepsilon) \rho_f^* \frac{d}{dt}^* (\mathbf{u}^* - \mathbf{v}^*) \tag{2.5}
\]

\[
\mathbf{E}^* = -p^* \mathbb{I} + \lambda^*(\varepsilon)(\nabla^* \cdot \mathbf{u}^*) \mathbb{I} + \mu^*(\varepsilon) \left[ \nabla^* \mathbf{u}^* + (\nabla^* \mathbf{u}^*)^T - \frac{2}{3} (\nabla^* \cdot \mathbf{u}^*) \mathbb{I} \right] \tag{2.6}
\]

\[
\mathbf{E}^{*s} = -p^{*s} \mathbb{I} + \lambda^{*s}(\varepsilon)(\nabla^* \cdot \mathbf{v}^*) \mathbb{I} + \mu^{*s} \left[ \nabla^* \mathbf{v}^* + (\nabla^* \mathbf{v}^*)^T - \frac{2}{3} (\nabla^* \cdot \mathbf{v}^*) \mathbb{I} \right] \tag{2.7}
\]

where \( \mathbb{I} \) is the unit tensor; \( p^*, p^{*s} \) are the fluid and particle phase effective pressures respectively; \( \lambda^*, \mu^* \) and \( \lambda^{*s}, \mu^{*s} \) are the effective bulk and shear viscosities for the two phases. \( \beta^*(\varepsilon) \) is the drag coefficient between the fluid and the particles; \( C(\varepsilon) \) is the virtual mass coefficient.

Equations (2.4) to (2.4) have a steady-state solution which corresponds to the state of uniform suspension:

\[
\mathbf{u}^* = u_o^* \mathbb{I} \quad \mathbf{v}^* = 0 \quad \varepsilon = \varepsilon_o \tag{2.8}
\]
In this state, the equations of motion become:

\[ \nabla p_o^* + i \beta (\varepsilon_o) u_o^* + i \rho_i^* g^* = 0 \]  
\[ (1 - \varepsilon_o) (\rho_s^* - \rho_i^*) g^* = 0 \]

where \( \hat{i} \) is the unit vector pointing vertically upwards.

2.2 LINEARIZED PERTURBATION EQUATIONS

Introduce small perturbations into the steady-state solution above:

\[ u^* = u_o^* + u_1^* \]
\[ v^* = v_1^* \]
\[ \varepsilon = \varepsilon_o + \varepsilon_1 \]
\[ p^* = p_o^* + p_1^* \]

where the subscript 1 denotes small perturbation quantities. Substituting (2.11) into equations (2.4) to (2.5) and neglecting all terms of degree greater than one in the small perturbation quantities leads to a set of linear partial differential equations.

\[ \frac{\partial \varepsilon_1}{\partial t} + u_o^* \frac{\partial \varepsilon_1}{\partial x} + \varepsilon_o \nabla^* \cdot u_1^* = 0 \]  

(2.12)
\[-\frac{\partial \varepsilon_1^*}{\partial t^*} + (1 - \varepsilon_0^*) \nabla \cdot \mathbf{v}_1^* = 0 \quad (2.13)\]

\[\rho_f^* \left(1 + \frac{1 - \varepsilon_0^*}{\varepsilon_0^*} C_o^* \right) \left( \frac{\partial u_1^*}{\partial t^*} + u_1^* \frac{\partial u_1^*}{\partial x^*} \right) - \rho_f^* \left(1 - \varepsilon_0^* \right) C_o^* \frac{\partial \varepsilon_1^*}{\partial t^*} \]

\[= -\nabla \cdot \mathbf{p}_1^* + \left(\lambda_0^* + \frac{\mu_0^*}{3}\right) \nabla (\nabla \cdot \mathbf{u}_1^*) + \mu_0^* \nabla^2 \mathbf{u}_1^* \]

\[-\beta_o^* (u_1^* - \mathbf{v}_1^*) + \frac{iu_o^*}{\nabla \cdot \mathbf{v}_1^*} \varepsilon_1^* \quad (2.14)\]

\[\left[\rho_s^* (1 - \varepsilon_0^*) + \rho_f^* \left(1 + \frac{1 - \varepsilon_0^*}{\varepsilon_0^*} C_o^* \right) \frac{\partial \varepsilon_1^*}{\partial t^*} - \rho_f^* (1 - \varepsilon_0^*) \left(1 + \frac{C_o^*}{\varepsilon_0^*} \right) \left( \frac{\partial u_1^*}{\partial t^*} + u_1^* \frac{\partial u_1^*}{\partial x^*} \right) \right] \]

\[= \imath (\rho_s^* - \rho_f^*) \varepsilon_1^* + \beta_o^* (u_1^* - \mathbf{v}_1^*) - \frac{iu_o^*}{\nabla \cdot \mathbf{v}_1^*} \varepsilon_1^* \]

\[+ \frac{p_s^*}{\nabla \cdot \mathbf{v}_1^*} \varepsilon_1^* + (\lambda_0^* + \frac{4}{3} \mu_0^*) \nabla (\nabla \cdot \mathbf{v}_1^*) + \mu_0^* \nabla^2 \mathbf{v}_1^* . \quad (2.15)\]

Where

\[\beta_o^* = \beta^* (\varepsilon_0^*), \quad C_o^* = C(\varepsilon_0^*)\]

\[\beta_o^* = \left( \frac{\partial \beta^*}{\partial n} \right)_{n=n_0^*} = -\nabla \cdot \left( \frac{\partial \beta^*}{\partial \varepsilon} \right)_{\varepsilon=\varepsilon_0^*} \]

\[\beta_o^* = \left( \frac{\partial \beta^*}{\partial n} \right)_{n=n_0^*} = -\nabla \cdot \left( \frac{\partial \beta^*}{\partial \varepsilon} \right)_{\varepsilon=\varepsilon_0^*} \]

\[p_o^* = \left( \frac{\partial \beta^*}{\partial n} \right)_{n=n_0^*} = -\nabla \cdot \left( \frac{\partial \beta^*}{\partial \varepsilon} \right)_{\varepsilon=\varepsilon_0^*}\]
where

\( x^* \) is the vertical axis;

\( v^* \) is the volume of one particle;

\( n \) is the number of particles per unit volume.

Subscript \( \text{o} \) denotes the steady state of uniform fluidization.

In equation (2.14) the viscous terms containing \( \lambda^*_o \) and \( \mu^*_o \)
are in general much smaller than the inertial terms, so they can be
neglected. Equation (2.14) reduces to:

\[
\rho_f^* \left( 1 - \left( \frac{1 - \varepsilon_o^*}{\varepsilon_o^*} \right) C_o \right) \left( \frac{\partial u^*_4}{\partial t^*} + u^*_o \frac{\partial u^*_4}{\partial x^*} \right) - \rho_f^* \left( \frac{1 - \varepsilon_o^*}{\varepsilon_o^*} \right) C_o \frac{\partial v^*_4}{\partial t^*} = - \nu^*_p - \beta_o^* (u^*_4 - v^*_4) + \frac{u_o^* \beta_o^*}{\nu^*_o} \varepsilon_4
\]

Equations (2.12), (2.13), (2.15), (2.16) are then the linearized per-
turbation equations which are the basis of the linearized hydrodynamic
stability analysis.

### 2.3 THE DIMENSIONLESS FORM OF THE PERTURBATION EQUATIONS

By choosing proper reference quantities, the set of linearized
collection equations can be transformed into dimensionless form.
The following reference quantities are chosen:

Length: \( d^* \)

Velocity: \( u_o^* \)

Density: \( \rho_s^* \)

Time: \( \frac{d^*}{u_o^*} \)

and dimensionless variables are then defined in the following way:

\[
\begin{align*}
    x &= \frac{x^*}{d^*} \\
    y &= \frac{y^*}{d^*} \\
    t &= \frac{u_t^*}{u_o^*} \\
    \underline{u} &= \frac{u^*}{u_o^*} \\
    v &= \frac{v^*}{u_o^*} \\
    \rho &= \frac{\rho_f^*}{\rho_s^*}
\end{align*}
\]
\[ p = \frac{p^*}{\rho_s g d^*} \]

\[ p^* = \frac{p}{\rho_s g d} \]

\[ \mu_o^* = \frac{\mu_o u_o^*}{\rho_s g (d^*)^2} \]

\[ \lambda_o^* = \frac{\lambda_o u_o^*}{\rho_s g (d^*)^2} \]

\[ \rho_o^* = \frac{(p_o)^* n_o^*}{\rho_s g d^*} \]

\[ \beta_o^* = \frac{n_o^* u_o^*}{\rho_s g} \]

\[ \text{Fr} = \frac{(u_o^*)^2}{g d^*} = \text{Froude number}. \]  

(2.17)

Recognizing \( \beta_o^* = \frac{(1-c_o^*)(\rho_s^* - \rho_f^*) g d^*}{u_o^*} \),

Equations (2.12), (2.13), (2.15), (2.16) are now written in terms of these dimensionless variables. Also the subscript 1 will be dropped for convenience from here on.
\frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial x} + \varepsilon_o \nabla \cdot \mathbf{u} = 0 \quad (2.18)

- \frac{\partial \varepsilon}{\partial t} + (1 - \varepsilon_o) \nabla \cdot \mathbf{v} = 0 \quad (2.19)

\rho Fr \left ( 1 + \left ( \frac{1 - \varepsilon_o}{\varepsilon_o} \right ) C_o \right ) \left ( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right ) - \rho Fr \left ( \frac{1 - \varepsilon_o}{\varepsilon_o} \right ) C_o \frac{\partial v}{\partial t}

= - \nabla p - (1 - \varepsilon_o)(1 - \rho)(u - v) + \frac{\beta'_o \varepsilon}{1 - \varepsilon_o} \quad (2.20)

Fr \left ( (1 - \varepsilon_o) + \rho \left ( \frac{1 - \varepsilon_o}{\varepsilon_o} \right ) C_o \right ) \frac{\partial \gamma}{\partial t} - \rho Fr(1 - \varepsilon_o) \left ( 1 + \frac{C_o}{\varepsilon_o} \right ) \left ( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right )

= \frac{\beta'_o \varepsilon}{1 - \varepsilon_o} + (1 - \rho)(u - v) - \frac{\beta'_o \varepsilon}{1 - \varepsilon_o}

+ \frac{p_s'}{(1 - \varepsilon_o)} \nabla \varepsilon + \left ( \lambda_o + \frac{\mu_s}{3} \right ) \nabla(\nabla \cdot \mathbf{v}) + \mu_o \nabla^2 \mathbf{v} \quad (2.21)

Equations (2.18), (2.19), (2.20), (2.21) form the basis of the stability analysis developed in Chapters (3), (4), and (5).
3. THE CASE OF ZERO FLUID DENSITY AND ZERO DISTRIBUTOR PRESSURE DROP

3.1 THE PERTURBATION EQUATIONS

When the fluidizing fluid is a gas, the ratio $\rho_f^*/\rho_g^*$ is usually negligibly small. The perturbation equations (2.18) to (2.21) can then be simplified. Thus, setting $\rho = 0$, equations (2.18) to (2.21) are reduced to the following:

\[
\begin{align*}
\frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial x} + \varepsilon_g \nabla \cdot \mathbf{u} &= 0 \quad (3.1) \\
\frac{\partial \varepsilon}{\partial t} + (1-\varepsilon_g) \nabla \cdot \mathbf{v} &= 0 \quad (3.2) \\
-\nabla p - (1-\varepsilon_g)(\mathbf{u} \cdot \mathbf{v}) + i \frac{\beta_g^* \varepsilon}{(1-\varepsilon_g)} &= 0 \quad (3.3)
\end{align*}
\]

\[
Fr(1-\varepsilon_g) \frac{\partial \mathbf{v}}{\partial t} = \mathbf{i} \cdot \varepsilon + (1-\varepsilon_g)(\mathbf{u} \cdot \mathbf{v}) - i \frac{\beta_g^* \varepsilon}{(1-\varepsilon_g)} + \frac{p_g^*}{(1-\varepsilon_g)} \varepsilon + \left( \lambda_g^* + \frac{\mu_g^*}{3} \right) \nabla (\nabla \cdot \mathbf{v}) + \mu_g^* \nabla^2 \mathbf{v} \quad (3.4)
\]

(3.3) and (3.4) may be combined to give:

\[
Fr(1-\varepsilon_g) \frac{\partial \mathbf{v}}{\partial t} = \mathbf{i} - \nabla p + \frac{p_g^*}{(1-\varepsilon_g)} \varepsilon + \left( \lambda_g^* + \frac{\mu_g^*}{3} \right) \nabla (\nabla \cdot \mathbf{v}) + \mu_g^* \nabla^2 \mathbf{v} \quad (3.5)
\]
Take the curl of (3.5), using the identity curl grad = 0:

\[ \text{Fr}(1-\varepsilon) \frac{\partial}{\partial t} (\nu \times \nu) = \nu \times (\varepsilon) + \mu_o \nu^2 (\nu \times \nu) \]  
(3.6)

Written in the expanded form equation (3.6) becomes:

\[ \text{Fr}(1-\varepsilon) \frac{\partial}{\partial t} \left( \frac{\partial \nu}{\partial x} - \frac{\partial \nu}{\partial y} \right) = \frac{\partial \varepsilon}{\partial y} + \mu_o \left( \frac{\partial^3 \nu}{\partial x^3} + \frac{\partial^3 \nu}{\partial x \partial y^2} - \frac{\partial^3 \nu}{\partial y \partial x^2} - \frac{\partial^3 \nu}{\partial y^3} \right) \]  
(3.6')

From equation (3.2):

\[ \nu \cdot \nu = \left( \frac{1}{1-\varepsilon} \right) \frac{\partial \varepsilon}{\partial t} \]  
(3.2')

or

\[ \frac{\partial \nu_y}{\partial y} = \left( \frac{1}{1-\varepsilon} \right) \frac{\partial \varepsilon}{\partial t} - \frac{\partial \nu_x}{\partial x} \]  
(3.2'')

Combining (3.2'') and (3.6') by eliminating \( \nu_y \):

\[ \text{Fr}(1-\varepsilon) \left[ \left( \frac{1}{1-\varepsilon} \right) \frac{\partial \varepsilon}{\partial t} \left( \frac{\partial^3 \nu_x}{\partial t^2 \partial x} - \frac{\partial^3 \nu_x}{\partial t \partial x^2} \right) \right] \]

\[ = - \frac{\partial^2 \varepsilon}{\partial y^2} + \mu_o \left[ \left( \frac{1}{1-\varepsilon} \right) \left( \frac{\partial^4 \varepsilon}{\partial t^3 \partial x} + \frac{\partial^4 \varepsilon}{\partial t \partial x \partial y^2} \right) \right. \]

\[ - \left. \left( \frac{\partial^4 \nu_x}{\partial x^4} + 2 \frac{\partial^4 \nu_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \nu_x}{\partial y^4} \right) \right] \]
(3.7)
Take the divergence of equation (3.4):

$$\nabla \cdot \left[ (1 - \varepsilon) \frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}) \right] = \frac{\partial \varepsilon}{\partial x} + (1 - \varepsilon) (\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{v})$$

$$- \left( \frac{\beta^i_o}{1 - \varepsilon} \right) \frac{\partial \varepsilon}{\partial x} + \left( \frac{p^s_o}{1 - \varepsilon} \right) \nabla^2 \varepsilon$$

$$+ \left( \lambda^s_o + \frac{4\mu^s_o}{3} \right) \nabla^2 (\nabla \cdot \mathbf{v})$$

(3.8)

From equation (3.1')

$$\nabla \cdot \mathbf{u} = \left( \frac{1}{\varepsilon^o} \right) \left[ - \frac{\partial \varepsilon}{\partial t} - \frac{\partial \varepsilon}{\partial x} \right]$$

(3.1')

Eliminate \( \mathbf{u} \) and \( \mathbf{v} \) from equation (3.8) by substituting (3.1') and (3.2') into it:

$$\nabla \cdot \left[ \frac{\partial^2 \varepsilon}{\partial t^2} \right] = \frac{\partial \varepsilon}{\partial x} - \left( \frac{1 - \varepsilon}{\varepsilon^o} \right) \left( \frac{\partial \varepsilon}{\partial t} + \frac{\partial \varepsilon}{\partial x} \right) - \frac{\partial \varepsilon}{\partial t} - \left( \frac{\beta^i_o}{1 - \varepsilon} \right) \frac{\partial \varepsilon}{\partial x}$$

$$+ \left( \frac{p^s_o}{1 - \varepsilon} \right) \left( \frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial y^2} \right) + \left( \lambda^s_o + \frac{4\mu^s_o}{3} \right) \left( \frac{1}{1 - \varepsilon} \right)$$

$$+ \left( \frac{\partial^3 \varepsilon}{\partial t \partial x^2} + \frac{3 \partial \varepsilon}{\partial t \partial y^2} \right)$$

(3.9)

Equations (3.7) and (3.9) contain \( \varepsilon \) and \( v_x \) as the dependent variables.

The remaining dependent variables \( v_y, u_x, u_y, p \) can be expressed in terms of \( \varepsilon, v_x \) and their derivatives.
Thus equation (3.2') relates \( v_y \) to \( v_x \) and \( \varepsilon \). Equation (3.4)
can be rearranged to express \( u \) in terms of \( v \) and \( \varepsilon \).

\[
u = v + Fr \frac{\partial v}{\partial t} - i \left( \frac{\varepsilon}{1-\varepsilon_0} \right) + i \frac{\beta^t \varepsilon}{(1-\varepsilon_0)^2} - \frac{p_o s'}{(1-\varepsilon_0)^2} v \varepsilon
\]

\[
\left( \frac{\lambda^s o + \frac{\mu^s o}{3}}{(1-\varepsilon_0)} \right) \nabla (v \cdot v) - \frac{\mu^s o}{(1-\varepsilon_0)} v^2 v
\]

(3.4')

and equation (3.3) expresses \( p \) in terms of \( u, v \) and \( \varepsilon \).

\[
\nabla p = i \frac{\beta^t \varepsilon}{(1-\varepsilon_0)} - \left( 1 - \varepsilon_0 \right) (u-v)
\]

(3.3')

3.2 THE MODAL ANALYSIS

Suppose \( u, v, \varepsilon, p \) admit solutions which are the real parts of
the following complex forms

\[
\varepsilon = E(x)e^{iky}e^{st}
\]

\[
v_x = V_x(x)e^{iky}e^{st}
\]

\[
v_y = V_y(x)e^{iky}e^{st}
\]

\[
u_x = U_x(x)e^{iky}e^{st}
\]

\[
u_y = U_y(x)e^{iky}e^{st}
\]

\[
p = P(x)e^{iky}e^{st}
\]

(3.10)

These represent disturbances which are periodic in \( y \) and \( k \) is the
wave number of the disturbance, which is related to the wave length \( \lambda \) by
\[ k = \frac{2\pi}{\lambda} \]  

(3.11)

\( s \) is generally a complex number. The real part of \( s \) indicates the growth rate of the disturbance and the ratio of the imaginary part and \(|k|\) measures the velocity of the travelling wave. In this work, only real \( s \) are investigated. Therefore, only the convective type of instability is studied and the compression wave type instability is not considered. This entails the supposition that the upper surface of the fluidized bed is maintained stationary so that compression wave modes are suppressed. Physically this is realized by creating an interface which keeps the upper surface of the bed from free movement and at the same time is perfectly slippery to horizontal movement of the particles.

Substituting (3.10) into (3.9) gives:

\[
\frac{d^2E}{dx^2} \left[ \left( \lambda_o + \frac{4}{3} \mu_o \right) s + p_o^s \right] + \frac{dE}{dx} \left[ 2(1-\epsilon_o) - \rho_o' - \frac{(1-\epsilon_o)}{\epsilon_o} \right] \\
- E \left[ Fr(1-\epsilon_o) s^2 + \frac{(1-\epsilon_o)}{\epsilon_o} s + sk^2 \left( \lambda_o + \frac{4}{3} \mu_o \right) + p_o^s k^2 \right] = 0 \tag{3.12}
\]

Substituting (3.10) into (3.7) gives:

\[
\frac{d^4V}{dx^4} - \left[ 2k^2 + \frac{(1-\epsilon_o)Fr_s}{\mu_o^s} \right] \frac{d^2V}{dx^2} + \left[ \frac{k^2(1-\epsilon_o)Fr_s}{\mu_o^s} + k^4 \right] V_x \\
= \left( \frac{s}{1-\epsilon_o} \right) \frac{d^3E}{dx^3} - \left( \frac{k^2s}{(1-\epsilon_o)} + \frac{Fr_s^2}{\mu_o^s} \right) \frac{dE}{dx} + \left( \frac{k^2}{\mu_o^s} \right) E \tag{3.13}
\]
Then from (3.24)
\[ V_y = \left( \frac{1}{ik} \right) \left( \frac{1}{1-\varepsilon_0} \right) sE - \frac{dV_x}{dx} \]  \hspace{1cm} (3.14)

and from (3.41)
\[ U_x = \left( -\frac{\mu_0}{(1-\varepsilon_0)} \right) \frac{d^2V_x}{dx^2} + \left[ \frac{\mu_0}{(1-\varepsilon_0)} \frac{k_s^2}{1 + Frs} \right] V_x \]
\[ - \left[ \frac{p_o^{s'}}{(1-\varepsilon_0)^2} + \left( \frac{\lambda_0 k_o s}{(1-\varepsilon_0)^2} \right) \frac{dE}{dx} + \left( \frac{\beta_o^{s'}}{(1-\varepsilon_0)^2} - \frac{1}{(1-\varepsilon_0)} \right) E \right] \]  \hspace{1cm} (3.15)

\[ U_y = \left( \frac{-\mu_0}{1-\varepsilon_0} \right) \frac{d^2V_y}{dx^2} + \left[ \frac{k_s^2}{(1-\varepsilon_0)^2} \frac{\mu_0}{1 + Frs} \right] V_y \]
\[ - \frac{ik}{(1-\varepsilon_0)^2} \left[ p_o^{s'} + \left( \frac{\lambda_0 k_o s}{(1-\varepsilon_0)^2} \right) \right] E \]  \hspace{1cm} (3.16)

while from (3.34)
\[ P = \left( \frac{-1}{ik} \right) (1-\varepsilon_0) (U_y - V_y) \]  \hspace{1cm} (3.17)

Equation (3.12) is a homogeneous second order ordinary differential equation which has the following solution:
\[ E(x) = A_1 e^{a_1 x} + A_2 e^{a_2 x} \]  \hspace{1cm} (3.18)
where

\[ a_1, a_2 = \frac{-\alpha_1 \pm \sqrt{\left(\alpha_1^2 - 4\alpha_2\alpha_0\right)}}{2\alpha_2} \]  

(3.19)

\[ \alpha_2 = \left(\lambda_0^s + \frac{4}{3}\mu_0^s\right)s + p_0^s' \]

\[ \alpha_1 = 2(1-\varepsilon_0) - \beta_0' - \frac{(1-\varepsilon_0)}{\varepsilon_0} \]

\[ \alpha_o = -Fr(1-\varepsilon_o) s^2 - \frac{(1-\varepsilon_o) s}{\varepsilon_o} - sk^2 \left(\lambda_0^s + \frac{4}{3}\mu_0^s\right) - \frac{b}{p}k^2 \]  

(3.20)

A_1, A_2 are arbitrary constants.

Substituting (3.18) into (3.13) and solving the 4th order inhomogeneous ordinary differential equation leads to:

\[ V(x) = B_1 e^{b_1 x} + B_2 e^{b_2 x} + B_3 e^{b_3 x} + B_4 e^{b_4 x} \]

\[ + \phi(a_1)A_1 e^{a_1 x} + \phi(a_2)A_2 e^{a_2 x} \]  

(3.21)

where

\[ b_1 = k \]

\[ b_2 = -k \]

\[ b_3 = \sqrt{k^2 + (1-\varepsilon_0)Fr\sigma/\mu_o^s} \]

\[ b_4 = -\sqrt{k^2 + (1-\varepsilon_0)Fr\sigma/\mu_o^s} \]  

(3.22)
\[
\phi(\gamma) = \frac{\left(\frac{s}{1-\epsilon_0}\right)^3 - \left(\frac{k_s^2}{1-\epsilon_0} + \frac{\text{Fr}_s^2}{\mu_o}\right)\gamma + \frac{k_s^2}{\mu_o}}{\left[\gamma^4 - \left(\frac{ik_s^2}{\mu_o} + \frac{(1-\epsilon_0)\text{Fr}_s}{\mu_o}\right)\gamma^2 + \frac{k_s^2(1-\epsilon_0)\text{Fr}_s}{\mu_o} + k_s^2\right]} \tag{3.23}
\]

\(A_1, A_2, B_1, B_2, B_3, B_4\) are six arbitrary constants which have to be determined by the boundary conditions.

3.3 THE BOUNDARY CONDITIONS

At the upper and lower surfaces of the fluidized bed, the perturbations must satisfy certain imposed boundary conditions.

Consider the lower bed surface, \(x = 0\). Assuming the no-slip condition applies to the particle movement,

\[
V_x(0) = 0 \tag{3.24}
\]

\[
V_y(0) = 0 \tag{3.25}
\]

and by maintaining constant pressure in the fluid below the bed,

\[
P(0) = 0 \tag{3.26}
\]

Consider the upper surface which is maintained constant at \(x = 1\) as discussed in section (3.2). A steady upper surface free from any movement entails that the vertical component of this particle velocity must vanish, thus:

\[
V_x(1) = 0 \tag{3.27}
\]
Also it has been assumed that the interface at the upper surface is perfectly slippery to the horizontal movement of the particles, therefore:

\[ E_{xy}^s(1) = 0 \]  
(3.28)

If, in addition, the upper surface is exposed to a constant fluid pressure, then:

\[ P(1) = 0 \]  
(3.29)

This simple pressure boundary condition (3.29) is no longer attainable in the case of finite fluid density because of the requirement of the Bernoulli's principle.

Equations (3.24) to (3.29) can be rewritten into the following form:

\[ V_x(o) = 0 \]  
(3.30)

\[ \frac{dV_x}{dx}(o) - \left( \frac{s}{1-\varepsilon_o} \right) E(o) = 0 \]  
(3.31)

\[ \left( \frac{k^2}{1-\varepsilon_o} \right)^2 \left[ \left( \lambda_o^s + \frac{\mu_o^s}{3} \right) s + p_o^s \right] E(o) - \left( \frac{\mu_o^s}{1-\varepsilon_o} \right) \frac{d^2E}{dx^2}(o) + \left( \frac{\mu_o^s}{1-\varepsilon_o} \right) \frac{d^3V}{dx^3}(o) = 0 \]  
(3.32)

\[ V_x(1) = 0 \]  
(3.33)

\[ -\mu_o^s \frac{d^3V}{dx^3}(1) + \left[ F_r(1-\varepsilon_o)s + k \mu_o^s \right] \frac{dV_x}{dx}(1) + \left( \frac{\mu_o^s}{1-\varepsilon_o} \right) \frac{d^2E}{dx^2}(1) \]

\[ - \left[ F_r s^2 + \frac{k \left( \lambda_o^s + \frac{4}{3} \mu_o^s \right)}{(1-\varepsilon_o)} + \frac{\mu_o^s k^2}{(1-\varepsilon_o)} \right] E(1) = 0 \]  
(3.34)
\[
\left(\frac{s}{1-e_0}\right) \frac{dE}{dx}(1) - \frac{d^2V}{dx^2}(1) = 0
\]  
(3.35)

3.4 THE CHARACTERISTIC EQUATIONS

Substituting equations (3.18) and (3.21) into equations (3.30) to (3.35) results in a set of six homogeneous linear algebraic equations with six unknowns \( A_1, A_2, B_1, B_2, B_3, B_4 \):

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \phi(a_1) & \phi(a_2) \\
b_1 & b_2 & b_3 & b_4 & \psi_2(a_1) & \psi_2(a_2) \\
-\mu_s b_1^3 & -\mu_s b_2^3 & -\mu_s b_3^3 & -\mu_s b_4^3 & \psi_3(a_1) & \psi_3(a_2) \\
e^{b_1} & e^{b_2} & e^{b_3} & e^{b_4} & \phi(a_1)e^{a_1} & \phi(a_2)e^{a_2} \\
\delta(b_1)e & \delta(b_2)e & \delta(b_3)e & \delta(b_4)e & \psi_5(a_1)e^{a_1} & \psi_5(a_2)e^{a_2} \\
b_1^2 e^{b_1} & b_2^2 e^{b_2} & b_3^2 e^{b_3} & b_4^2 e^{b_4} & \psi_6(a_1)e^{a_1} & \psi_6(a_2)e^{a_2}
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
A_1 \\
A_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]  
(3.36)

where \( b_1, b_2, b_3, b_4 \) are defined in equation (3.22)
\[ \psi_2(\gamma) = \phi(\gamma) \gamma - \left( \frac{s}{1 - e_0} \right) \]

\[ \psi_3(\gamma) = -\mu_o s \phi(\gamma) \gamma^3 + \frac{\mu_o s^2}{(1 - e_0)^2} - \frac{k^2}{(1 - e_0)^2} \left[ \left( \frac{s}{\lambda_o + \frac{4 \mu_o}{3}} \right) s + p_o^1 \right] \]

\[ \psi_5(\gamma) = -Fr s^2 - \frac{p_o^1 k^2}{(1 - e_0)^2} - \frac{\lambda_o + \frac{4 \mu_o}{3}}{(1 - e_0)^2} + \mu_o^s \left( \frac{s}{1 - e_0} \right)^2 \gamma^2 \]

\[ + \left\{ Fr(1 - e_0) s + k^2 \mu_o^s \right\} \gamma - \mu_o^s \gamma^3 \left\{ \phi(\gamma) \right\} \]

\[ \psi_6(\gamma) = \phi(\gamma) \gamma^2 - \frac{s^2}{(1 - e_0)} \]

\[ \delta(\gamma) = \left[ Fr(1 - e_0) s + k^2 \mu_o^s \right] \gamma - \mu_o^s \gamma^3 \]

(3.37)

3.5 COMPUTATIONS

In order for equation (3.36) to have non-trivial solutions, the determinant of the coefficient matrix must vanish. This requirement leads to a relationship between the wave number, \( k \), and the growth factor, \( s \), for any given set of parameters of the system:

\[ f(k, s, Fr, \mu_o^s, \lambda_o^s, p_o^1, e_o^1, \beta_o^1) = 0 \]

(3.38)

For each set of parametric values, the determinant of the coefficient matrix is computed, for a given \( k \), over a range of \( s \) on the
real axis. Thus for each $k$, repeated computations yield $s$ which is a real root of equation (3.8). Care has to be taken to distinguish the "spurious" roots from the true ones. From inspection of (3.36) it is clear that the determinant of the coefficient matrix will vanish if one of the following situations arise.

$$
\begin{align*}
  b_1 &= b_2 \rightarrow k = -k = 0 \\
  b_3 &= b_4 - k^2 + \frac{(1 - \epsilon_0)F}{\mu_0} = 0 \\
  a_1 &= a_2 - \frac{\alpha_1^2}{\alpha_0} - 4\alpha_1 \alpha_2 = 0
\end{align*}
$$

(3.39)

Where $\alpha_1, \alpha_0, \alpha_2$ are defined in (3.20).

If $a_1 = a_2$, then equation (3.18) will no longer be the valid solution for $E(x)$. Instead,

$$
E(x) = (A_1 + A_2 x)^{a_2 x}
$$

(3.40)

must be used.

If either $b_1 = b_2$, or $b_3 = b_4$

then equation (3.21) will no longer be the valid solution for $V_x(x)$. Instead either

$$
\begin{align*}
  V_x(x) &= (B_1 + B_2 x)^{a_1 x} + B_3 e^{b_3 x} + B_4 e^{b_4 x} \\
          &+ \phi(a_1) A_1 e^{a_1 x} + \phi(a_2) A_2 e^{a_2 x}
\end{align*}
$$

(3.41)

or

$$
V_x(x) = B_1 e^{b_1 x} + B_2 e^{b_2 x} + (B_3 + B_4 x)e^{b_3 x} + \phi(a_1) A_1 e^{a_1 x} + \phi(a_2) A_2 e^{a_2 x}
$$

(3.42)

24
must be used.

Consequently, whenever any of the equations in (3.39) is satisfied, equation (3.36) will no longer be valid. Therefore, the "roots" determined under these conditions have no physical significance.

The computation of the determinant is done by the method of pivotal condensation. The matrix is transformed into the triangular form first and then the diagonal elements are multiplied together to give the determinant.

However, since the coefficient matrix contains elements which have very large numerical values, the computation of the determinant is performed by representing the original matrix by two different matrices in the computer program. One contains the pre-exponential part of the original elements and the other contains the power indices of the exponential terms in the original matrix. So that the coefficient matrix in equation (3.36) is split into the following matrices A and B

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & \phi(a_1) & \phi(a_2) \\
\mu_o b_1 & b_2 & b_3 & b_4 & \psi_2(a_1) & \psi_2(a_2) \\
-s_1 b_3 & -s_2 b_3 & -s_3 b_3 & -s_4 b_3 & \psi_3(a_1) & \psi_3(a_2) \\
1 & 1 & 1 & 1 & \phi(a_1) & \phi(a_2) \\
\delta(b_1) & \delta(b_2) & \delta(b_3) & \delta(b_4) & \psi_5(a_1) & \psi_5(a_2) \\
\mu o b_2 & b_2 & b_3 & b_4 & \psi_6(a_1) & \psi_6(a_2)
\end{bmatrix}
\] (3.43)
where $A$ is the pre-exponential matrix and $B$ is the power matrix.

Each elementary operation of the original matrix is performed by operating on both matrices in the program. For example, a multiplication of two elements in the original matrix is computed by two steps:

(i) the multiplication of the two corresponding elements in the pre-exponential matrix,

(ii) the addition of the two corresponding elements in the power index matrix.

Denote the elements in the original coefficient matrix by $C_{ij}$ and elements in matrices $A$ and $B$ by $A_{ij}$ and $B_{ij}$. The operation

$$C_{ij} = C_{ij} \times C_{ji}$$

is computed in two operations:

$$A_{ij} = A_{ij} \times A_{ji}$$

$$B_{ij} = B_{ij} + B_{ji}$$
Where $C_{ij}$ is related to $A_{ij}$ and $B_{ij}$ by:

$$C_{ij} = A_{ij} e^{B_{ij}}.$$ 

Since the method of pivotal condensation involves the triangularization of the matrix, the six columns in the original matrix should be arranged in ascending numerical order. This is necessary in order to avoid small numbers being subtracted by very large numbers.

In order to locate the root at which the determinant of the coefficient matrix vanishes, a marching technique is used to search for the interval in which the root lies. Once the interval is found, the root is determined to a prescribed degree of accuracy through a bisection process. The criterion used for stopping the algorithm is

$$\frac{(s_1 - s_2)}{(s_1 + s_2)} < 0.001$$

where $s_1$, $s_2$ prescribes the interval in which the root lies.

Figure (3.1) shows a typical plot of determinant vs. $s$ for a given $k$, the intersection of the curve with the horizontal axis gives the root.

3.6 RESULTS AND DISCUSSION

The system of glass beads fluidized by air is chosen for computations. Unlike the Benard problem or the Hagen-Poiseuille flow problem, the number of parameters in this problem is very large whether dimensional or in dimensionless groups. So a complete investigation of dependence of stability on parameters is not possible.
The following set of parameters will be taken as standard: (in cgs units)

\[ \rho^*_g = 2.86 \]

\[ d^*_p = 0.086 \]

\[ \lambda^*_o = 10 \]

\[ \mu^*_o = 10 \]

\[ \left| \frac{d p^*_s}{d \varepsilon} \right|_{\varepsilon_0} = 0 \]

\[ \varepsilon_0 = 0.42 \]

\[ u^*_o = u^*_m f \]

\[ d^* = 86.0 \] \hfill (3.45)

A parametric study is performed to test the sensitivity of the instabilities to variation of the individual parameters from these values.

The results are presented in graphical form. Figure (3.3) shows a typical stability curve with the standard set of parameters described by (3.45). \( s^* \) rises with increasing \( k^* \) initially until \( k^* \) reaches a value of approximately 1.20 where \( s^* \) reaches its maximum and then descends gently with further increase in \( k^* \).
Since

\[ \lambda^* = \frac{2\pi}{k^*}, \text{ therefore at maximum } s^* \]

\[ \lambda^* \approx \frac{2\pi}{1.2} \approx 5.2 \text{ cm} \]

The superscript (*), as before, denotes dimensional quantities. Using the reference quantities defined in section (2.3), the dimensional variables are related to the corresponding dimensionless variables by the following equations:

\[ k^* = \frac{k}{d^*} \]

\[ s^* = \frac{s u_o}{d^*} \]

\[ \lambda^* = \lambda d^* \]  \hspace{1cm} (3.46)

Physically \( \lambda^* \) corresponds to the width of the convection cell. That the instability manifests itself in the form of convection cells is shown in Fig. (3.2) which displays the particle velocity field at the onset of instability. The particle velocity field is computed according to equation (3.10) where \( V_x(x) \) and \( V_y(x) \) are provided by equations (3.21) and (3.14) respectively. In a bed of infinite width, \( \lambda^* \) will be the dominant mode of disturbance. In a bed of finite width, assuming the vertical boundaries of the bed are perfectly slippery to particle motion, \( \lambda^* \) will be the dominant
mode of disturbance only if the width of the bed is an integral multiple of \( \lambda^* \).

The computation is done to about \( k^* = 6 \text{ cm}^{-1} \) which corresponds to \( \lambda^* = 1 \text{ cm} \). Disturbances of wavelengths comparable with the particle size are not meaningful in a fluidized bed, therefore the computation is terminated at a point which corresponds to \( \lambda^* = 10 \text{ d}^* \).

Figure (3.4) shows a family of curves which differ from one another only in values of \( \lambda_o^{s*} \) and \( \mu_o^{s*} \), the particle phase viscosity coefficient. Starting from the standard case of \( \lambda_o^{s*} = \mu_o^{s*} = 10 \text{ poise increasing } \lambda_o^{s*} \) and \( \mu_o^{s*} \) to 20 poise suppresses the instability and shifts the dominant \( k^* \) towards the origin. This means the width of the convection cell is increased. On the other hand, decreasing \( \lambda_o^{s*} \) and \( \mu_o^{s*} \) to 1.0 poise greatly enhances the instability of the system and shifts the dominant \( k^* \) away from the origin. This implies a reduced width of the convection cell. Further decrease of \( \lambda_o^{s*} \) and \( \mu_o^{s*} \) to 0.1 poise increases the instability of the system further. The dominant \( k^* \) is not seen when the computation terminates at \( k^* = 6 \text{ cm}^{-1} \). The convection cell width is so narrow that it no longer has physical meaning.

The trend seems to be clear: the instability of the system is strongly dependent upon the viscosities of the particle assemblage, the instability is suppressed by increases in the particle phase viscosities. Figure (3.5)
is a study of the effect of varying the shear viscosity coefficient, $\mu_o^{s*}$, alone. Results similar to those of Fig. (3.4) are obtained. The instability is decreased when $\mu_o^{s*}$ is increased from 10 to 20 poise and the instability is enhanced when $\mu_o^{s*}$ is decreased from 10 to 5 poise. Similarly the dominant $k^*$ is increased upon decrease in $\mu_o^{s*}$ and it is decreased upon increase in $\mu_o^{s*}$.

The effects of variations of the bulk viscosity, $\lambda_o^{s*}$, are shown in Fig. (3.6). The effect is qualitatively the same as in the case of $\mu_o^{s*}$ except the variations of $\lambda_o^{s*}$ do not affect the instability of the system to as large an extent as do the variations of $\mu_o^{s*}$.

Increasing the quantity $|\frac{dp^{s*}}{dc}|_o^{s*}$ from 0 to 200 dynes/cm$^2$ decreases the instability of the system as shown in Fig. (3.7). However, the extent of the decrease in instability due to increase in $|\frac{dp^{s*}}{dc}|_o^{s*}$ is rather small. Similarly the dominant $k^*$ is not significantly changed, hence the width of the convection cell remains relatively unchanged. The instability of the system is significantly affected by variations in solid density, $\rho_s^{*}$. This is shown in Fig. (3.8). Starting with the standard density of 2.86 g/cm$^3$, doubling $\rho_s^{*}$ to 5.72 g/cm$^3$, significantly enhances the instability of the system, at the same time the dominant $k^*$ is shifted away from the origin implying a decrease in the width of the convection cell. On the other hand, a decrease of $\rho_s^{*}$ to 1.43 g/cm$^3$
significantly lowers the instability of the system and shifts the dominant 
$k^*$ towards the origin and, therefore, an increase in the width of the 
convection cell. It should be noted here that in reality, the change of 
$\rho_s^*$ could also have produced changes in $\epsilon_o$, $\mu_o^*$, $\lambda_o^*$, $\frac{d\rho_s}{d\epsilon_o}\bigg|_{\epsilon_o}$ and $U_{mf}^*$. Here, owing to the lack of clear understanding on the effects of 
variations of $\rho_s^*$ to bed structure, all of these variables except $U_{mf}^*$ are 
held constant as $\rho_s^*$ is changed. $U_{mf}^*$ is computed using Rowe's 
correlation (10)

$$U_{mf}^* = \frac{0.00081 (\rho_s^* - \rho_f^*) g \frac{d^*}{p} \mu^*}{\mu^*}$$

(3.47)

Similarly, in the following discussion of the effects of variations 
of $d_p^*$ all parameters except $U_{mf}^*$ have been held constant, although 
physically it may not be realizable.

Variations of particle diameter, $d_p^*$, however, do not affect the 
instability of the system to the same significant extent as do the variations 
of $\rho_s^*$. As shown in Fig. (3.9), a slight increase in instability accompanies 
the doubling of $d_p^*$ to 0.17. A similarly small decrease in instability is 
observed as $d_p^*$ is halved to 0.043 cm. It is interesting to note that here 
unlike the preceding graphs, the dominant $k$ is not changed. So that 
although the growth rates of the disturbance are slightly altered the size 
of the convection cell remains unchanged.
Variations of the initial voidage, $\varepsilon_o$, also induce small changes in the stability of the system. Here again, it has been assumed that $\varepsilon_o$ could be varied independent of the other parameters of the bed such as $u_0^*, \lambda_0^*, \mu_0^*, \beta_0^*$, $\left|\frac{dp}{d\varepsilon}\right|^*_0$, and $Fr$. In reality, clearly $\varepsilon_o$ could only be changed if $u_0^*$ is changed for a given bed of particles. The simultaneous changes in $u_0^*$ and $\varepsilon_o$ will inevitably cause changes in $\lambda_0^*, \mu_0^*, \beta_0^*$, $\left|\frac{dp}{d\varepsilon}\right|^*_0$ and $Fr$. Thus the effect of expanding the bed really entails the complicated interactions of many parameters. The physical relations among these parameters are, therefore, needed before a realistic study of the effect of bed expansion could be made. However, as shown in Fig. (3.10), unlike the variations of $d_p^*$ the dominant $k_p^*$ is also slightly changed here.

A decrease in $\varepsilon_o$ favors instability and makes the width of the convection cell slightly narrower. On the other hand, an increase in $\varepsilon_o$ suppresses the instability slightly and increases the width of the convection cell.

Assuming bed height $d_p^*$ could be varied independently, the results of computations for the cases:

- $d_p^* = 43 \text{ cm}$
- $d_p^* = 86 \text{ cm}$
- $d_p^* = 172 \text{ cm}$
show that to the accuracy of computation, the bed instability appears to be independent of bed height.

Summarizing the parametric study then, the most favorable situation for instability to develop is in a bed which consists of large, dense particles rather closely spaced and with low coefficients of bulk and shear viscosities as well as a low value of \( \left| \frac{dp}{d\varepsilon} \right|^{*}_{\varepsilon_0} \).

The most stable case, on the other hand, is a bed which consists of small, light particles loosely spaced with high viscosities and a high value of \( \left| \frac{dp}{d\varepsilon} \right|^{*}_{\varepsilon_0} \).
4. THE ZERO DENSITY CASE WITH FINITE DISTRIBUTOR PRESSURE DROP

4.1 INTRODUCTION

It is a well-known fact that a much more smoothly fluidized particle bed is obtained when a high pressure drop fluid distributor is used in place of one which has negligible pressure drop.\(^{(14)}\)\(^{(15)}\) The objective of this section is to determine the effect of the distributor pressure drop upon the instability of the fluidized bed.

The same set of perturbation equations and their solutions derived in sections (3.1) and (3.2) are valid here. The only change required from the analysis in Chapter 3 is one of the six boundary conditions.

Equation (3.26) states \(P(o) = 0\). In the present situation this is no longer valid since variations in flow through the distributor induce variations in pressure drop, and hence variations in pressure at the lower surface of the bed, even when the pressure below the distributor is constant and uniform. A development of an appropriate boundary condition will be given in (4.2). The effects of introducing a finite pressure drop distributor are then discussed in (4.3).
4.2 THE BOUNDARY CONDITIONS

In the uniformly fluidized state,
\[ \overline{u}^* = \overline{i u_o} \]
\[ \overline{v}^* = 0 \]
\[ \varepsilon = \varepsilon_o \] (4.1)

Let \( p_b^* \) be the pressure immediately below the distributor plate where \( p_b^* \) is kept constant at all times.

Let \( p^* \) be the pressure immediately above the distributor plate and \( p^*_b = p^*_o \) in the uniformly fluidized state.

Suppose: \( (p_b^* - p_o^*) = R^* \left( u_o^* \varepsilon_o \right) \) (4.2)

where \( R^* \) is the pressure drop coefficient of the distributor

\[ R^* = \frac{p_b^* - p_o^*}{(u_o^* \varepsilon_o)} \]

\[ = \frac{p_b^* \left[ \frac{p_a^* + d^* (1 - \varepsilon_o) \left( \rho_s^* - \rho_f^* \right) g^*}{(u_o^* \varepsilon_o)} \right]}{(u_o^* \varepsilon_o)} \] (4.2')

\[ = \frac{(p_b^* - p_a^*) - \Delta p_o^*}{(u_o^* \varepsilon_o)} \]

where \( p_a^* \) is the constant pressure above the bed; \( \Delta p_o^* \) is the pressure drop across the uniformly fluidized bed.
Introduce the following perturbations at $x = 0$:

$$
\begin{align*}
\vec{p}^* &= \vec{p}_o^* + \vec{p}_1^* \\
\vec{c} &= \vec{c}_o + \vec{c}_1 \\
\vec{u}_x &= \vec{u}_o^* + \vec{u}_x^*
\end{align*}
$$

(4.3)

In the perturbed state, equation (4.2) becomes:

$$
(p_b^* - p^*) = R^* (u_x^* c^*)
$$

(4.4)

Substitute (4.3) into (4.4) and rearrange:

$$
\begin{align*}
\vec{p}_b^* &= \vec{p}_o + R^* \vec{u}_o c_o + \vec{p}_1^* + R^* (c_o \vec{u}_1^* + u_o \vec{c}_1^*) \\
\vec{p}_1^* + R^* (u_o c_1^* + u_1 c_o^*) &= 0
\end{align*}
$$

(4.5)

Subtracting (4.2) from (4.5) gives:

$$
\begin{align*}
\frac{\partial p_1^*}{\partial y} + R^* \frac{\partial u_1}{\partial y} = 0
\end{align*}
$$

(4.6)

Take $\left( \frac{\partial}{\partial y} \right)^*$ of (4.6), which holds for all $y$ at $x = 0$.

$$
\begin{align*}
\frac{\partial p_1^*}{\partial y} + R^* \frac{\partial u_1^*}{\partial y} + R^* \frac{\partial c_1^*}{\partial y} = 0
\end{align*}
$$

(4.7)

Define the dimensionless quantity $R$:

$$
R = \frac{R^* u_o^*}{\rho_s (1-c_o) g d^*}
$$

(4.8)
Combining equations (4.2) and (4.8), it is seen that:

\[ R = \left( \frac{1}{\varepsilon_o} \right) \left( \frac{p_b^* - p_o^*}{p_b^* (1-\varepsilon_o) g d^*} \right) \]  

(4.8')

That is:

\[ R = \left( \frac{1}{\varepsilon_o} \right) \left( \frac{\Delta p_{\text{dist.}}}{\Delta p_{\text{bed.}}} \right) \]  

(4.8'')

All the other dimensionless quantities are defined by equation (2.17).

In terms of dimensionless variables, equation (4.7) may be written as:

\[ \frac{\partial p_1}{\partial y} + R(1-\varepsilon_o) \varepsilon_o \frac{\partial u_1}{\partial y} + R(1-\varepsilon_o) \frac{\partial \varepsilon_1}{\partial y} = 0 \]  

(4.9)

Rewrite (4.9), dropping the subscript 1 for convenience:

\[ \frac{\partial p}{\partial y} + R(1-\varepsilon_o) \varepsilon_o \frac{\partial u}{\partial y} + R(1-\varepsilon_o) \frac{\partial \varepsilon}{\partial y} = 0 \]  

(4.9')

but from equation (3.3'),

\[ \frac{\partial \varepsilon}{\partial y} = -(1-\varepsilon_o) (u_y - v_y) \]  

(4.10)

and from equation (3.4'),

\[ u_x = v_x + Fr \frac{\partial v_x}{\partial t} - \left( \frac{\varepsilon}{1-\varepsilon_o} \right) + \frac{\beta_o^1 \varepsilon_o}{(1-\varepsilon_o)^2} - \frac{p_o^s}{(1-\varepsilon_o)^2} \frac{\partial \varepsilon}{\partial x} \]

\[ - \left( \frac{\lambda_o^s + \mu_o^s}{3} \right) \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon}{\partial t} \right) - \frac{\mu_o^s}{(1-\varepsilon_o)^2} \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \]  

(4.11)
\[ u_y = v_y + Fr \frac{\partial v}{\partial t} - \frac{p_o s_i}{(1-\varepsilon_o)^2} \frac{\partial \varepsilon}{\partial y} - \left( \frac{\mu_o}{3} \right) \frac{\partial}{\partial y} \left( \frac{\partial \varepsilon}{\partial t} \right) \]

\[ - \left( \frac{\mu_o}{1-\varepsilon_o} \right) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \]

(4.12)

Substitute equations (4.10), (4.11), (4.12) into equation (4.9'),

\[ \left[ -(1-\varepsilon) Fr \frac{\partial v}{\partial t} + \left( \frac{p_o s_i}{(1-\varepsilon_o)^2} \right) \frac{\partial \varepsilon}{\partial y} + \left( \frac{\lambda_o + \mu_o}{3} \right) \frac{\partial}{\partial y} \left( \frac{\partial \varepsilon}{\partial t} \right) + \mu_o \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \right) \right] \]

\[ + R(1-\varepsilon_o)(\varepsilon_o) \left[ \frac{\partial v}{\partial x} + Fr \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial y} \right) - \frac{1}{(1-\varepsilon_o)^2} \frac{\partial \varepsilon}{\partial y} + \frac{\beta_o}{(1-\varepsilon_o)^2} \frac{\partial \varepsilon}{\partial x} - \frac{p_o s_i}{(1-\varepsilon_o)^2} \frac{\partial^2 \varepsilon}{\partial x \partial y} \right] \]

\[ - \left( \frac{\lambda_o + \mu_o}{3} \right) \frac{\partial}{\partial x \partial y} \left( \frac{\partial \varepsilon}{\partial t} \right) - \frac{\mu_o}{(1-\varepsilon_o)^2} \frac{\partial}{\partial y} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right] + R(1-\varepsilon_o) \frac{\partial \varepsilon}{\partial y} = 0 \]

(4.13)

Introducing the modal analysis by substituting equation (3.10) into equation (4.13):

\[ \left( \frac{1}{1-\varepsilon_o} \right) \left[ R(1-\varepsilon_o)^2 + p_o s_i + \left( \frac{\lambda_o + \mu_o}{3} \right) \varepsilon_o - R(1-\varepsilon_o) \varepsilon_o + R \left( \varepsilon_o \right) \right] E \]

\[ - \frac{R \varepsilon_o}{(1-\varepsilon_o)} \left[ p_o s_i + \left( \frac{\lambda_o + \mu_o}{3} \right) \varepsilon_o \right] \frac{dE}{dx} + R(1-\varepsilon_o) \varepsilon_o \left[ 1 + Frs + \frac{k^2 \mu_o}{(1-\varepsilon_o)} \right] V_x \]

\[ - R \left( \varepsilon_o \mu_o \right) \frac{d^2 V}{dx^2} + \left( \frac{\mu_o}{ik} \right) \frac{d^2 V}{dx^2} - \left( \frac{1}{ik} \right) \left[ (1-\varepsilon_o) Frs + k^2 \mu_o \right] V_y = 0 \]

(4.14)
But from equation (3.21):

\[(1k)V_y = \left(\frac{s}{1-\varepsilon_o}\right)E - \frac{dV}{dx}\]  \hspace{1cm} (4.15)

and substituting equation (4.15) into equation (4.14):

\[
\left(\frac{1}{1-\varepsilon_o}\right)\left[R(1-\varepsilon_o)^2 + \rho s^t + \left(\lambda_o + \frac{\mu_o}{3}\right) s - R(1-\varepsilon_o)\varepsilon_o + R \varepsilon_o s^t + \left(\frac{1-\varepsilon_o}{k^2}\right) Fr_o^2 + \mu_s^o \right]E
\]

\[-\frac{R \varepsilon_o}{(1-\varepsilon_o)} \left[p_s^t + \left(\lambda_o + \frac{\mu_o}{3}\right) s\right] \frac{dE}{dx} - \frac{\mu_s^o}{(1-\varepsilon_o)k^2} \frac{d^2E}{dx^2}
\]

\[+ R(1-\varepsilon_o)\varepsilon_o \left[1 + Fr_o + \frac{k^2 s^o}{(1-\varepsilon_o)}\right] V_x - \frac{1}{k^2}(1-\varepsilon_o) Fr_o + k \frac{s^o}{\mu_o} \frac{dV}{dx}
\]

\[- R \varepsilon_o \mu_s^o \frac{d^2V}{dx^2} + \left(\frac{\mu_o}{k^2}\right) \frac{d^3V}{dx^3} = 0 \] \hspace{1cm} (4.16)

Equation (4.16) valid only at \(x = o\), of course, is the required new boundary condition which replaces equation (3.32) in the zero pressure drop case.

Equation (3.32) can be recovered by setting \(R = o\) in equation (4.16).

Substituting equations (3.18) and (3.21) into (4.16) gives:

\[\theta(b_1)B_1 + \theta(b_2)B_2 + \theta(b_3)B_3 + \theta(b_4)B_4 + \Omega(a_1)A_1 + \Omega(a_2)A_2 = 0 \] \hspace{1cm} (4.17)
where

\[ \theta(\gamma) = R(1 - \epsilon_0)\epsilon_o \left[ 1 + Frs + \frac{k^2 \mu_s}{(1 - \epsilon_0)} \right] + \left( \frac{\mu_o}{k^2} \right) \gamma^3 - (R \epsilon_o \mu_s^o) \gamma^2 \]

\[ \Omega(\gamma) = \left( \frac{1}{1 - \epsilon_o} \right) \left[ R(1 - \epsilon_o)^2 + p_s^i + \left( \lambda_o + \frac{\mu_o}{3} \right) s - R(1 - \epsilon_o)\epsilon_o + R \epsilon_0 \beta_s^i \right] \]

\[ - \frac{R \epsilon_o}{(1 - \epsilon_o)} \left[ p_s^i + \left( \lambda_o + \frac{\mu_o}{3} \right) s \right] \gamma - \left( \frac{\mu_o}{(1 - \epsilon_o)k^2} \right) \gamma^2 \]

\[ + R(1 - \epsilon_o)\epsilon_o \left[ 1 + Frs + \frac{k^2 \mu_s}{(1 - \epsilon_0)} \right] \phi(\gamma) \]

\[ + \left( \frac{\mu_o}{k^2} \right) \phi(\gamma) \gamma^3 - (R \epsilon_o \mu_s^o) \phi(\gamma) \gamma^2 \]

(4.18)

So the new set of characteristic equations becomes:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \phi(a_1) & \phi(a_2) \\
\theta(b_1) & \theta(b_2) & \theta(b_3) & \theta(b_4) & \Omega(a_1) & \Omega(a_2) \\
b_1 & b_2 & b_3 & b_4 & \psi_2(a_1) & \psi_2(a_2) \\
\phi(a_1) & \phi(a_2) & \phi(a_1) & \phi(a_2) & \phi(a_1) & \phi(a_2) \\
\delta(b_1)e & \delta(b_2)e & \delta(b_3)e & \delta(b_4)e & \psi_5(a_1)e & \psi_5(a_2)e \\
2b_1e & 2b_2e & 2b_3e & 2b_4e & \psi_6(a_1)e & \psi_6(a_2)e \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
A_1 \\
A_2 \\
\end{bmatrix}
\]

\[ = \]

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(4.19)
which replaces equations (3.36).

Once again, in order to have non-trivial solutions for equations (4.19), the determinant of the coefficient matrix must vanish.

Following similar computational procedures as before, a set of $s^* \text{ vs. } k^*$ curves are computed for various values of the physical parameters.

4.3 RESULTS AND DISCUSSION

Figure (4.1) shows a family of curves obtained with the standard set of parameters, given in equation (3.45), with the curves differing only in the pressure drop across the distributor.

The pressure drop across the distributor is defined in terms of multiples of the pressure drop across the bed under uniform fluidization conditions.

Let

$$(M + 1) \Delta p_{\text{bed}} = \Delta p_t \quad (4.20)$$

where

$\Delta p_t$ = the total pressure drop across the bed and the distributor

$\Delta p_{\text{bed}}$ = the pressure drop across the bed alone

and

$$M = \frac{\Delta p_t - \Delta p_{\text{bed}}}{\Delta p_{\text{bed}}} = \frac{\Delta p_{\text{dist}}}{\Delta p_{\text{bed}}} \quad (4.20')$$
Comparing equation (4.8′) and (4.20′), it is clear that:

\[ M = \epsilon R^o \]  

Figure (4.1), then, is a family of curves of increasing \( M \). It is seen that as \( M \) is increased, that is as the distributor pressure drop is increased, the instability is appreciably reduced. In addition, the dominant \( k^* \) is shifted towards the origin indicating an increasing width of the circulation cells.

Starting from the case of \( M = 0 \), which is the zero distributor pressure drop case described in Chapter (3), increasing \( M \) is accompanied by successive reductions in instability and when the distributor pressure drop reaches 10% of the bed pressure drop the growth rate of the dominant mode of disturbance becomes only 1/50 of that of the zero distributor pressure drop case. At the same time, the convection cell width is increased from approximately 6 cm to approximately 600 cm a 100 fold increase.

The results then seem to bear out the general observation, that is: increasing the distributor pressure drop does seem to reduce the level of instability and in physical terms this presumably means smoother fluidized conditions.

Within the range of computation, none of these curves is found to intercept the \( k \) axis. The tail portion of the curves simply drops to smaller and smaller values of \( s \) while remaining positive. This implies that the
beds are never absolutely stable, only the growth rate of the instability changes with changing $k$. 
5. THE CASE OF FINITE FLUID DENSITY

5.1 INTRODUCTION

Of the two types of fluidization, particulate fluidization is usually associated with liquid fluidized beds while almost all gas fluidized beds fall within the category of aggregative fluidization which is characterized by its turbulent appearance caused by the rising gas bubbles. Jackson's earlier analysis successfully distinguished these two types of fluidizations in terms of their vastly different disturbance growth rates, subsequent works confirm this finding. It is the objective of this section to study the convective instabilities of fluidized systems in which the fluid densities are not negligible compared with the solid densities of the particles.

The treatment of the finite density case follows closely that of the zero density case in chapter. The starting point here is the set of dimensionless perturbation equations derived in section (2,3).

5.2 MODAL ANALYSIS OF THE PERTURBATION EQUATIONS

Equations (2.18), (2.19), (2.20), (2.21) are the four perturbation equations into which the following modal expressions for the small disturbances are substituted:
\[ \varepsilon = E(x) e^{-iky e^i \omega t} \]
\[ u_x = U_x(x) e^{-iky e^i \omega t} \]
\[ u_y = U_y(x) e^{-iky e^i \omega t} \]
\[ v_x = V_x(x) e^{-iky e^i \omega t} \]
\[ v_y = V_y(x) e^{-iky e^i \omega t} \]
\[ p = P(x) e^{-iky e^i \omega t} \]  

These substitutions lead to the following set of equations,

\[ sE + \frac{dE}{dx} + \varepsilon \left( \frac{dU_x}{dx} + ikU_y \right) = 0 \]  

\[ -sE + \varepsilon_o \left( \frac{dV_x}{dx} + ikV_y \right) = 0 \]

\[ \rho Fr C_o \left( sU_x + \frac{dU_x}{dx} \right) - \rho Fr \left( \varepsilon_o \frac{\varepsilon_o}{\varepsilon_o} \right) C_o sV_x = \frac{-dF}{dx} - \varepsilon_o \rho (U_x-V_x) + \frac{\beta^i \varepsilon}{\varepsilon_o} \]

\[ \rho Fr C_o \left( sU_y + \frac{dU_y}{dx} \right) - \rho Fr \left( \varepsilon_o \frac{\varepsilon_o}{\varepsilon_o} \right) C_o sV_y = -ikF - \varepsilon_o \rho (U_y-V_y) \]

\[ Fr \left( \varepsilon_o + \rho \frac{\varepsilon_o}{\varepsilon_o} C_o \right) sV_x - \rho Fr \varepsilon_o \left( 1 + \frac{C_o}{\varepsilon_o} \right) \left( sU_x + \frac{dU_x}{dx} \right) \]

\[ = \frac{p_o}{\varepsilon_o} \frac{dE}{dx} + \left( \lambda_o + \frac{\mu_o}{3} \right) \frac{s}{\varepsilon_o} \frac{dE}{dx} + \rho \frac{E}{\varepsilon_o} (U_x-V_x) - \frac{\beta^i E}{\varepsilon_o} + \mu_o \left( \frac{d^2V_x}{dx^2} - k^2V_x \right) \]

(5.6)
\[
Fr\left(\varepsilon_o + \rho\frac{\varepsilon_o}{\varepsilon_o} C_o\right) sV_y - \rho Fr\varepsilon_o \left(1 + \frac{C_o}{\varepsilon_o}\right) \left(sU_x + \frac{dU_y}{dx}\right) \\
= -\rho\varepsilon_o (U_y - V_y) + \frac{\rho_s}{\varepsilon_o} (ik) E + \left(\frac{s}{\varepsilon_o} + \frac{\mu_o}{3}\right) (ik) \left(\frac{dV_x}{dx} + ikV_y\right) + \mu_o \left(\frac{d^2V_y}{dx^2} - k^2 V_y\right) 
\]

(5.7)

where

\[
\varepsilon_o = (1 - \varepsilon_o) \\
\rho = (1 - \rho) \\
C_o' = \left[1 + \left(\frac{1 - \varepsilon_o}{\varepsilon_o}\right) C_o\right] 
\]

(5.8)

Equations (5.2) to (5.7) form a set of six simultaneous linear ordinary differential equations containing six dependent variables:

\[E(x), P(x), U_x(x), U_y(x), V_x(x), V_y(x)\].

These equations will now be solved.

From (5.3)

\[
V_y = \left(\frac{1}{ik}\right) \left(\frac{sE}{\varepsilon_o} - \frac{dV_x}{dx}\right) 
\]

(5.3')

From (5.2)

\[
U_y = \left(\frac{1}{ik}\right) \left(-\frac{sE}{\varepsilon_o} - \frac{1}{\varepsilon_o} \frac{dE}{dx} - \frac{dU_x}{dx}\right) 
\]

(5.2')
From (5.5)

\[ P = \left(\frac{4}{1k}\right) \left\{ -\frac{\varepsilon_0}{\rho_0} (U_y - V_y) + \rho Fr \left(\frac{\varepsilon_0}{\varepsilon_o} C_o s V_y\right) \\ -\rho Fr C_o' \left( sU_y + \frac{dU_y}{dx} \right) \right\} \]  

(5.5')

Substitute (5.2'), (5.3') into (5.5'), giving

\[ P = \left(\frac{-4}{k^2}\right) \left[ \frac{\varepsilon_0}{\varepsilon_o} \left( \frac{\varepsilon_0}{\varepsilon_o} \rho s + \frac{\rho Fr C_o s^2}{\varepsilon_o} + \frac{\rho Fr C_o' s^2}{\varepsilon_o} \right) \right. \\
+ \frac{dE}{dx} \left( \frac{\varepsilon_0}{\varepsilon_o} + \frac{2 \rho Fr s C_o'}{\varepsilon_o} \right) + \frac{d^2 E}{dx^2} \left( \frac{\rho Fr C_o'}{\varepsilon_o} \right) \\
+ \frac{dV_x}{dx} \left( -\frac{\varepsilon_0}{\varepsilon_o} \rho - \rho Fr \left(\frac{\varepsilon_0}{\varepsilon_o} C_o s \right) \right) + \frac{dU_x}{dx} \left( \frac{\varepsilon_0}{\varepsilon_o} \rho + \rho Fr C_o' s \right) \\
+ \frac{d^2 U_x}{dx^2} (\rho Fr C_o') \right] \]  

(5.9)

From equations (5.2'), (5.3') and (5.9), it is seen that the dependent variables \( U_y, V_y \), and \( P \) can be expressed in terms of the remaining dependent variables \( U_x, V_x, E \) and their derivatives.

The following manipulations are performed on equations (5.2) to (5.7) to obtain expressions for \( U_x, V_x, \) and \( E \).
Rearrange equation (5.6) to give:

\[
\left[ \frac{\rho \varepsilon_o}{\varepsilon_o} + \rho Fr \frac{\varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) \left( s + \frac{d}{dx} \right) \right] U_x
\]

= \left[ \left( \frac{\beta}{\varepsilon_o} \right) - \beta \right] E + \left[ - \frac{p_s}{\varepsilon_o} \left( \frac{s}{\varepsilon_o} + \frac{\mu_s}{3} \right) \right] \frac{dE}{dx}

+ \left[ Fr \left( \frac{\varepsilon_o}{\varepsilon_o} + p \left( \frac{\varepsilon_o}{\varepsilon_o} \right) C_o \right) s + \rho \varepsilon_o + \mu_s k^2 \right] V_x + \left( \frac{\mu_s}{2} \right) \frac{d^2 V_x}{dx^2} \quad \text{(5.6')}

Rearrange equation (5.7) to give,

\[
\left[ \frac{\rho \varepsilon_o}{\varepsilon_o} + \rho Fr \frac{\varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) s + \rho Fr \frac{\varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) \frac{d}{dx} \right] U_y
\]

= \left[ Fr \left( \frac{\varepsilon_o}{\varepsilon_o} + p \left( \frac{\varepsilon_o}{\varepsilon_o} \right) C_o \right) s + \rho \varepsilon_o + \left( \frac{\mu_s}{3} \right) \right] V_y

- \mu_s \frac{d^2 V_y}{dx^2} - \left( \frac{\mu_s}{3} \right) \frac{dV_y}{dx} - \frac{p_s}{\varepsilon_o} \left( ik \right) E \quad \text{(5.7')}

Substitute (5.2') and (5.3') into (5.7'), giving
- \left[ \left( \frac{s}{\varepsilon_o} + \rho \, \frac{Fr}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) s \right) \frac{d}{dx} + \rho \, Fr \, \frac{\varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) \frac{d^2}{dx^2} \right] U_x \\
= \left[ Fr \left( 1 + \rho \, \frac{C_o}{\varepsilon_o} \right) s^2 + p_o \rho s + \left( \frac{s}{\varepsilon_o} + \frac{4 \mu_o}{3} \lambda_o \right) k^2 \left( \frac{s}{\varepsilon_o} \right) + \rho \frac{p_{s}}{\varepsilon_o} \right] E \\
+ \left[ \frac{1}{\varepsilon_o} \left( \frac{s}{\varepsilon_o} + \rho \, Fr \, \frac{\varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) s \right) \right] \frac{dE}{dx} \\
+ \left[ - \frac{\mu_s \varepsilon_o}{\varepsilon_o} + \frac{1}{\varepsilon_o} Fr \, \rho \, \frac{s}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) \right] \frac{d^2E}{dx^2} \\
+ \left[ - Fr \left( \frac{s}{\varepsilon_o} + \frac{\rho \, \varepsilon_o \, C_o}{\varepsilon_o} \right) s - \frac{s}{\varepsilon_o} - \mu_o k^2 \right] \frac{dV}{dx} + \mu_o \frac{d^3V}{dx^3} \right] (5.10)

Take the derivative of both sides of (5.6') with respect to \(x\):

\left[ \left( \frac{s}{\varepsilon_o} + \rho \, \frac{Fr}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) s \right) \frac{d}{dx} + \rho \, Fr \, \frac{\varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) \frac{d^2}{dx^2} \right] U_x \\
= \left( \frac{\beta_o}{\varepsilon_o} - \rho \right) \frac{dE}{dx} - \left[ \frac{p_{s}}{\varepsilon_o} + \left( \frac{s}{\varepsilon_o} + \frac{\mu_s}{3} \lambda_o \right) \frac{s}{\varepsilon_o} \right] \frac{d^2E}{dx^2} \\
+ \left[ Fr \, s \left( \frac{s}{\varepsilon_o} + \frac{\rho \, \varepsilon_o \, C_o}{\varepsilon_o} \right) + \rho \, \frac{s}{\varepsilon_o} + \mu_o k^2 \right] \frac{dV}{dx} + \mu_o \frac{d^3V}{dx^3} \right] (5.11)
Adding equations (5.10) and (5.11) then gives:

\[
\left[ \frac{\rho Fr \varepsilon_o}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) - \frac{p_s}{\varepsilon_o} \left( \frac{s}{\varepsilon_o} + \frac{4}{3} \frac{\mu_o}{\varepsilon_o} \right) \left( \frac{s}{\varepsilon_o} \right) \right] \frac{d^2E}{dx^2} \\
+ \left[ \frac{\rho \varepsilon_o}{\varepsilon_o} + 2\rho Fr \varepsilon_o \left( 1 + \frac{C_o}{\varepsilon_o} \right) \left( \frac{s}{\varepsilon_o} \right) - \frac{\beta'_o}{\varepsilon_o} \right] \frac{dE}{dx} \\
+ \left[ \frac{p_s}{\varepsilon_o} + \frac{p_o s^2}{\varepsilon_o} + \left( \frac{\lambda_o}{\varepsilon_o} + \frac{4}{3} \frac{\mu_o}{\varepsilon_o} \right) \frac{k^2 s}{\varepsilon_o} + Fr s^2 \left( 1 + \frac{C_o}{\varepsilon_o} \right) \right] \\
\left( \frac{\varepsilon_o}{\varepsilon_o} \right) \left( 1 + \frac{C_o}{\varepsilon_o} \right) E = 0
\]

Equation (5.12) is a homogeneous second order ordinary differential equation in \( E \) only.

It will be shown in the following development that similar manipulations lead to a fifth order equation in \( V_x \) and \( E \). Take the derivative of (5.5) with respect to \( x \) and rearrange, giving

\[
\frac{dP}{dx} = \left( -\frac{1}{ik} \right) \left\{ \left( \frac{\rho Fr C_o s + \varepsilon_o}{\varepsilon_o} \right) \frac{d}{dx} + \rho Fr C_o \frac{d^2}{dx^2} \right\} U_y \\
- \left( \frac{\rho Fr \varepsilon_o C_o s}{\varepsilon_o} - \frac{\varepsilon_o}{\varepsilon_o} \frac{\partial}{\partial x} \right) \frac{dV_y}{dx}
\]

Substitute (5.13) into (5.4), and eliminate \( U_y, V_y \) in the resultant equation using equations (5.2') and (5.3')

51
\[
\left[ \frac{d^3}{dx^3} + \left( s + \frac{\varepsilon \rho}{\rho Fr C'_o} \right) \frac{d^2}{dx^2} + (-k^2) \frac{d}{dx} + \left( -k^2 s - \frac{k^2 \varepsilon \rho}{\rho Fr C'_o} \right) \right] U_x \\
= \left( \frac{\varepsilon \rho}{Fr \rho C'_o} + \frac{\varepsilon C_o}{\varepsilon C'_o} \right) \frac{d^2 V}{dx^2} + \left( \frac{k^2 s \varepsilon C_o}{\varepsilon C'_o} + \frac{k^2 \varepsilon \rho}{Fr \rho C'_o} \right) V_x + \left( -\frac{4}{\varepsilon_o} \right) \frac{d^3 E}{dx^3} \\
+ \left( -\frac{2s}{\varepsilon_o} - \frac{\varepsilon \rho}{\varepsilon_o \rho Fr C'_o} \right) \frac{d^2 E}{dx^2} + \left( -\frac{s^2}{\varepsilon_o} - \frac{\varepsilon_o s^2}{\varepsilon C'_o} - \frac{\varepsilon \rho s}{\varepsilon_o \rho Fr C'_o} + \frac{\rho_s}{\varepsilon_o \rho Fr C'_o} \right) \frac{dE}{dx} \\
+ \left( -\frac{k^2 \beta}{\varepsilon_o \rho Fr C'_o} \right) E
\] \\
(5.14)

Equation (5.14) can be combined with equation (5.6') to eliminate terms containing \( U_x \) and its derivatives. Thus multiplying (5.14) by

\[
\left[ \frac{\bar{\rho} \varepsilon}{\varepsilon_o} + \rho Fr \frac{\varepsilon}{\varepsilon_o} \left( 1 + \frac{C_o}{\varepsilon_o} \right) \left( s + \frac{d}{dx} \right) \right]
\]

and (5.6') by

\[
\left[ \frac{d^3}{dx^3} + \left( s + \frac{\varepsilon \rho}{\rho Fr C'_o} \right) \frac{d^2}{dx^2} - k^2 \frac{d}{dx} - \left( k^2 s + \frac{k^2 \varepsilon \rho}{\rho Fr C'_o} \right) \right],
\]

then adding the two resultant equations:

\[
h_5 \frac{d^5 V}{dx^5} + h_4 \frac{d^4 V}{dx^4} + h_3 \frac{d^3 V}{dx^3} + h_2 \frac{d^2 V}{dx^2} + h_1 \frac{dV}{dx} + h_0 V = d_4 \frac{d^4 E}{dx^4} \\
+ d_3 \frac{d^3 E}{dx^3} + d_2 \frac{d^2 E}{dx^2} + d_1 \frac{dE}{dx} + d_0 E
\]
(5.15)

52
where

\[ h_5 = \frac{F_\nu \rho C^2 \mu^s_o \mu^s_s}{\rho \epsilon \epsilon_o} \]

\[ h_4 = \frac{F_\nu \rho C^2 \mu^s_s \mu^s_o}{\rho \epsilon \epsilon_o} + \frac{\mu^s_s}{\epsilon_o} \]

\[ h_3 = -\frac{2F_\nu \rho C^2 \mu^s_s k^2}{\rho \epsilon \epsilon_o} - \frac{2F_\nu \rho s}{\rho \epsilon} (C^2 + \rho C_o) - \rho F \]

\[ h_2 = -\frac{2F_\nu \rho C^2 \mu^s_s k^2}{\rho \epsilon \epsilon_o} - \frac{F_\nu \rho s 2}{\rho \epsilon} (C^2 + \rho C_o) - \frac{2k^2 \mu^s_o}{\epsilon} - \rho F s - \frac{\epsilon}{\epsilon_o} \frac{F s}{\epsilon_o} \]

\[ h_1 = \frac{F_\nu \rho C^2 \mu^s_s k^4}{\rho \epsilon \epsilon_o} + \frac{F_\nu \rho s k^2}{\rho \epsilon} (C^2 + \rho C_o) + \rho F k^2 \]

\[ h_o = \frac{F_\nu \rho C^2 \mu^s_s k^4}{\rho \epsilon \epsilon_o} + \frac{F_\nu \rho s k^2}{\rho \epsilon} (C^2 + \rho C_o) + \frac{\mu^s_s k^4}{\epsilon_o} \]

\[ + \frac{k^2 \epsilon F s}{\epsilon_o} \]

\[ d_4 = \frac{F_\nu \rho C^2 \mu^s_s}{\rho \epsilon \epsilon_o} \]

(5.16)
\[ d_3 = \frac{Fr \rho C'_o \mu^s_s}{\rho \varepsilon_o \varepsilon'_o} + \frac{\mu^s_s}{\varepsilon_o (1 - \varepsilon'_o)} \]

\[ d_2 = -\frac{Fr \rho C'_o \mu^s_s k^2}{\rho \varepsilon_o \varepsilon'_o} - \frac{2 k^2 Fr^2}{\varepsilon_o \varepsilon'_o} (C'_o + \rho C'_o) - \frac{s Fr \rho}{\varepsilon'_o} \]

\[ d_1 = \frac{-Fr \rho C'_o \mu^s_s k^2}{\rho \varepsilon_o \varepsilon'_o} + \frac{k^2 Fr C'_o}{\varepsilon_o \varepsilon'_o} - \frac{s^3 Fr^2 \rho}{\rho \varepsilon'_o} (C'_o + \rho C'_o) - \frac{k^2 Fr \beta'_o \rho}{\varepsilon'_o} \]

\[ \frac{k^2 \mu^s_s}{\varepsilon'_o} \frac{s Fr}{\rho \varepsilon'_o} - \frac{Fr s^2}{\varepsilon'_o} - \frac{2 \rho Fr}{\varepsilon'_o} \frac{k^2}{\varepsilon'_o} \]

\[ d_0 = \frac{sk^2 \rho Fr C'_o}{\varepsilon_o \varepsilon'_o} - \frac{sk^2 Fr \beta'_o \rho}{\varepsilon'_o} + \frac{sk^2}{\rho \varepsilon'_o} + \frac{sk^2}{\varepsilon'_o} \]

(5.16) (cont)

In the following, it will be shown that \( U_x \) can be expressed in terms of \( E, \ V_x \) and their derivatives.

From Eq. (5.6'):

\[ \frac{dU_x}{dx} = \frac{1}{\rho Fr (C'_o - \varepsilon'_o)} \left\{ \left[ Fr \varepsilon'_o s \left( 1 + \frac{\rho C'_o}{\varepsilon'_o} \right) + \frac{\mu^s_s k^2}{\varepsilon'_o} + \frac{\rho \varepsilon'_o}{\varepsilon'_o} \right] V_x - \left( \mu^s_s \frac{d^2 V_x}{dx^2} \right) \right\} \]

\[ + \left( \frac{\beta'_o}{\varepsilon'_o} - \frac{\rho}{\varepsilon'_o} \right) E + \left[ -\frac{\rho s^s}{\varepsilon'_o} - \left( \frac{\mu^s_s}{\varepsilon'_o} + \frac{s}{\varepsilon'_o} \right) \right] \frac{dE}{dx} \left( \frac{-\rho \varepsilon'_o}{\rho Fr (C'_o - \varepsilon'_o) - s} \right) U_x \]

(5.6'')
From (5.6) \( \frac{d^2 U_x}{dx^2} \) \( \frac{d^3 U_x}{dx^3} \) can be expressed in terms of \( U_x \) together with \( E, V_x \) and their derivatives.

Substituting these into equation (5.14) gives:

\[
U_x = \frac{1}{\xi} \left[ e_3 \frac{d^3 E}{dx^3} + e_2 \frac{d^2 E}{dx^2} + e_1 \frac{dE}{dx} + e \frac{dV}{dx} + W \frac{d^4 V}{dx^4} + W_3 \frac{d^3 V}{dx^3} + W_2 \frac{d^2 V}{dx^2} + W_1 \frac{dV}{dx} + W_0 V_x \right]
\]

(5.17)

where

\[
\xi = \left\{ - \left( \frac{-\rho \epsilon_o}{\rho Fr(C'_o - \epsilon_o)} + s \right) \left( \frac{-\rho \epsilon_o}{\rho Fr(C'_o - \epsilon_o)} + s \right)^2 \left( \frac{-\rho \epsilon_o}{\rho Fr(C'_o - \epsilon_o)} + s \right) \right\}
\]

\[
e_3 = \frac{\rho FrC'_o \left[ \frac{p_s}{\epsilon_o} + \left( \frac{s}{\epsilon_o} \right) \right]}{\rho Fr(C'_o - \epsilon_o)} - \frac{\rho FrC'_o}{\epsilon_o}
\]

\[
e_2 = \frac{\rho FrC'_o \left( \frac{\beta}{\epsilon_o} - s \beta \right)}{\rho Fr(C'_o - \epsilon_o)} - \frac{\rho FrC'_o \left[ \frac{s}{\epsilon_o} + \left( \frac{\mu_s}{\epsilon_o} \right) \right]}{\rho Fr(C'_o - \epsilon_o)}
\]

\[
\left[ \left( \frac{-\rho \epsilon_o}{\rho Fr(C'_o - \epsilon_o)} \right) \left( Frp C'_o \right) - Frp C'_o - \epsilon_o \beta \right]
\]

\[
- \frac{2 \rho FrC'_o}{\epsilon_o} \frac{-\epsilon_o}{\epsilon_o}
\]

55
\[ e_1 = \left( \frac{\bar{\beta}_o - \bar{\rho}}{\bar{\epsilon}_o} \right) \left[ \left( \frac{\bar{\rho} \bar{\epsilon}_o}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} + s \right) \left( \rho \text{Fr}C'_o - \rho \text{Fr}C'_o s - \bar{\rho} \bar{\epsilon}_o \right) \right] \]

\[ \left[ \left( \frac{\bar{\rho} s}{\bar{\epsilon}_o} + \left( \lambda_o + \frac{\mu_o s}{3} \right) \left( \frac{s}{\bar{\epsilon}_o} \right) \right) \right] \left[ \rho \text{Fr}(C'_o - \bar{\epsilon}_o) \right] \]

\[ - \left( \rho \text{Fr}C'_o s + \bar{\epsilon}_o \bar{\rho} \right) \left( \frac{\bar{\rho} \bar{\epsilon}_o}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} + s \right) + k^2 \rho \text{Fr}C'_o \]

\[ - \left( \frac{1}{\bar{\epsilon}_o} \right) \left( \rho \text{Fr}C'_o s^2 + \rho \text{Fr}C'_o s^2 - \bar{\rho} s \right) \]

\[ e_o = \left( \frac{\bar{\beta}_o - \bar{\rho}}{\bar{\epsilon}_o} \right) \left[ - \rho \text{Fr}C'_o \left( \frac{\bar{\rho} \bar{\epsilon}_o}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} + s \right) + \left( k^2 \rho \text{Fr}C'_o + \bar{\epsilon}_o \bar{\rho} \right) \right] \]

\[ \left( \frac{\bar{\rho} \bar{\epsilon}_o}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} + s \right) + k^2 \rho \text{Fr}C'_o \right] - \left( \frac{k^2 \beta'_o}{\bar{\epsilon}_o} \right) \]

\[ W_4 = \frac{\mu_o \rho \text{Fr}C'_o}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} \]

\[ W_3 = \frac{-\mu_o \rho \text{Fr}(C'_o - \bar{\epsilon}_o)}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} \left[ \left( \frac{\bar{\rho} \bar{\epsilon}_o}{\rho \text{Fr}(C'_o - \bar{\epsilon}_o)} + s \right) \rho \text{Fr}C'_o - \rho \text{Fr}C'_o s - \bar{\epsilon}_o \bar{\rho} \right] \]
\[ W_2 = -\frac{\rho F r C^i}{\rho F r (C^i_o - \varepsilon_o)} \left[ \frac{F r \varepsilon_o}{\varepsilon_o} \left( 1 + \frac{\rho \varepsilon_o}{\varepsilon_o} \right) + \mu_o k^2 + \frac{\rho \varepsilon_o}{\varepsilon_o} \right] \]

\[ \quad - \frac{\mu_o}{\rho F r (C^i_o - \varepsilon_o)} \left[ - F r \rho C^i_o \left( \frac{\rho \varepsilon_o}{F r \rho (C^i_o - \varepsilon_o)} + s \right) \right]^2 \left( \frac{\rho \varepsilon_o}{F r \rho (C^i_o - \varepsilon_o)} + s \right) \]

\[ \left( F r \rho C^i_o s + \rho \varepsilon_o \right) + k^2 \rho F r C^i_o \right] + \rho \varepsilon_o + \frac{\rho F r \varepsilon_o C^i_o s}{\varepsilon_o} . \]

\[ W_1 = \left( \frac{\rho F r \varepsilon_o}{\varepsilon_o} \left( 1 + \frac{\rho \varepsilon_o}{\varepsilon_o} \right) + \mu_o k^2 + \rho \varepsilon_o \right) \]

\[ x \left( \frac{\rho \varepsilon_o}{\rho F r (C^i_o - \varepsilon_o)} + s \right) \left( \rho F r C^i_o \right) - \rho F r C^i_o s - \varepsilon_o \rho \right] . \]

\[ W_0 = \left( \frac{\rho F r \varepsilon_o}{\varepsilon_o} \left( 1 + \frac{\rho \varepsilon_o}{\varepsilon_o} \right) + \mu_o k^2 + \frac{\rho \varepsilon_o}{\varepsilon_o} \right) \]

\[ - \rho F r C^i_o \left( \frac{\rho \varepsilon_o}{\rho F r (C^i_o - \varepsilon_o)} + s \right) \]

\[ + \left( \frac{\rho \varepsilon_o}{\rho F r (C^i_o - \varepsilon_o)} + s \right) \left( \rho F r C^i_o s + \varepsilon_o \rho \right) + k^2 \rho F r C^i_o \right] \]

\[ - k^2 \rho \varepsilon_o - \frac{k^2 \rho F r \varepsilon_o C^i_o s}{\varepsilon_o} . \]  

(5.18)

Thus the original set of six equations, (5.2) to (5.7), has now been transformed into equations (5.12), (5.15), (5.17), (5.9), (5.3'), (5.2') which are amenable to solution.
It is seen that by setting \( \rho = 0 \), equations (5.12) and (5.15) reduce to equations (3.12) and (3.13) respectively which are the corresponding equations for the zero density case treated in Chapter (3). Equation (5.12) can be solved to give:

\[
E(x) = R_1 e^{r_1 x} + R_2 e^{r_2 x} \tag{5.19}
\]

where

\[
r_1, r_2 = -\frac{\eta_1 \pm \sqrt{\eta_1^2 - 4 \eta_2 \eta_0}}{2 \eta_2} \tag{5.20}
\]

and

\[
\eta_2 = \left[ \frac{\rho \bar{F}\bar{r} \bar{e}_o}{\bar{e}_o} \left( 1 + \frac{C_o}{\bar{e}_o} \right) - \frac{\lambda_o}{\lambda_o} - \left( \frac{\lambda_s + 4}{3} \frac{\mu_o}{\mu_o} \right) \left( \frac{\bar{e}_o}{\bar{e}_o} \right) \right]
\]

\[
\eta_1 = \left[ \frac{\bar{\rho} \bar{e}_o}{\bar{e}_o} + 2 \rho \bar{F}\bar{r} \bar{e}_o \left( 1 + \frac{\bar{e}_o}{\bar{e}_o} \right) \left( \frac{\bar{e}_o}{\bar{e}_o} \right) - \frac{\beta_o}{\beta_o} \right]
\]

\[
\eta_0 = \left[ \frac{\bar{\rho} \bar{e}_o}{\bar{e}_o} + \frac{\lambda_s + 4}{3} \frac{\mu_o}{\lambda_o} \frac{\beta_o^2}{\beta_o^2} + \left( \frac{\lambda_s + 4}{3} \frac{\mu_s}{\mu_o} \right) \frac{k^2}{k^2} + \bar{F}_s \bar{r}^2 \left( 1 + \frac{\rho}{\rho \bar{C}_o} \right) \right.
\]

\[
+ \rho \bar{F}_s^2 \left( \frac{\bar{e}_o}{\bar{e}_o} \right) \left( 1 + \frac{C_o}{\bar{e}_o} \right) \right] \tag{5.21}
\]

\( R_1, R_2 \) are two arbitrary constants to be determined by boundary conditions.
Equation (5.15) has the following general solution:

\[ V_x(x) = Q_1 e^{q_1 x} + Q_2 e^{q_2 x} + Q_3 e^{q_3 x} + Q_4 e^{q_4 x} + Q_5 e^{q_5 x} + R_1 f(r_1) e^{r_1 x} \]
\[ + R_2 f(r_2) e^{r_2 x} \]  

(5.22)

where \( q_i \) are the roots of the polynomial:

\[ h_5 q_i^5 + h_4 q_i^4 + h_3 q_i^3 + h_2 q_i^2 + h_1 q_i + h_0 = 0 \]  

(5.23)

and

\[ f(\gamma) = \left( \frac{d_4 \gamma^4 + d_3 \gamma^3 + d_2 \gamma^2 + d_1 \gamma + d_0}{h_5 \gamma^5 + h_4 \gamma^4 + h_3 \gamma^3 + h_2 \gamma^2 + h_1 \gamma + h_0} \right). \]  

(5.24)

\( Q_1, Q_2, Q_3, Q_4, Q_5 \) are five arbitrary constants to be determined by boundary conditions.

5.3 THE BOUNDARY CONDITIONS

The sum of the orders of equations (5.12) and (5.16) is 7. So we expect 7 boundary conditions to be needed.

Consider the situation depicted in Fig. (5.1) where the fluidizing fluid is introduced uniformly at \( x = -n \) below the bed and the system is exposed to a constant pressure at \( x = m \), some distance above the bed.

In the clear fluid regions immediately above and below the bed surface, the motion of the inviscid fluid must obey the following set of equations:

\[ \rho Fr \left[ \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} \right] = - \nabla \bar{p} - \rho_i \]  

(5.25)
\[ \nabla \cdot \vec{u} = 0 \quad (5.26) \]

Equations (5.25) and (5.26) are dimensionless equations with the reference quantities given in Section (2.3). The super bar refers to quantities in the clear fluid regions.

Equations (5.25) and (5.26) have the following steady-state solutions:

\[ \bar{u} = \bar{u}_o \bar{i} \]

\[ \nabla p = -\bar{p} \bar{i} \quad (5.27) \]

where \( \bar{u}_o = \varepsilon u_o \).

Introduce small perturbations \( \bar{u}_{x1} \), \( \bar{u}_{y1} \) and \( \bar{p}_1 \) into the steady-state solutions, so that

\[ \bar{u}_x = \bar{u}_o + \bar{u}_{x1} \]

\[ \bar{u}_y = \bar{u}_{y1} \]

\[ \bar{p} = \bar{p}_o + \bar{p}_1 \quad (5.28) \]

This leads to the following set of perturbation equations, dropping the subscripts 1 for convenience

\[ \rho Fr \left( \frac{\partial \bar{u}_x}{\partial t} + \varepsilon \frac{\partial \bar{u}_x}{\partial x} \right) = -\frac{\partial \bar{p}}{\partial x} \]

\[ \rho Fr \left( \frac{\partial \bar{u}_y}{\partial t} + \varepsilon \frac{\partial \bar{u}_y}{\partial x} \right) = -\frac{\partial \bar{p}}{\partial y} \quad (5.29) \]

\[ \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_x}{\partial x} = 0 \]
Let
\[ \overline{u}_x = \overline{U}_x(x) e^{iky} e^{st} \]
\[ \overline{u}_y = \overline{U}_y(x) e^{iky} e^{st} \]
\[ \overline{p} = \overline{P}(x) e^{iky} e^{st} \]  \hspace{1cm} (5.30)

Substituting equation (5.30) into equation (5.29) leads to the following equation
\[ \varepsilon_0 \frac{d^3 \overline{U}_x}{dx^3} + s \frac{d^2 \overline{U}_x}{dx^2} - \varepsilon_0 k^2 \frac{d\overline{U}_x}{dx} - sk^2 \overline{U}_x = 0 \]  \hspace{1cm} (5.31)

Equation (5.31) has the following solution:
\[ \overline{U}_x(x) = A e^{kx} + B e^{-kx} + C e^{-\left(\frac{s}{\varepsilon_0}\right)x} \]  \hspace{1cm} (5.32)

where \( A, \ B, \ C \) are arbitrary constants.

Since
\[ -ik\overline{U}_y = \frac{d\overline{U}_x}{dx} \]  \hspace{1cm} (5.33)

it follows that
\[ \overline{U}_y(x) = \begin{bmatrix} \left( -\frac{1}{ik} \right) A e^{kx} - B k e^{-kx} - C \left( \frac{s}{\varepsilon_0} \right) e^{-\left(\frac{s}{\varepsilon_0}\right)x} \end{bmatrix} \]  \hspace{1cm} (5.34)
Also from (5.29) and (5.30),

$$\bar{P}(x) = -\left(\frac{\rho\bar{R}}{ik}\right) \left(s\bar{U}_y + \epsilon_0 \frac{d\bar{U}}{dx}\right)$$  \hspace{1cm} (5.35')

which leads to:

$$\bar{P}(x) = -\left(\frac{\rho\bar{R}}{k}\right) \left[ A(s + \epsilon_0 k)e^{kx} + B(\epsilon_0 k - s)e^{-kx}\right]$$ \hspace{1cm} (5.35)

Now, at

$$x = -n \quad , \quad \bar{U}_x = \bar{U}_y = 0$$ \hspace{1cm} (5.36)

This leads to:

$$\bar{C} = \begin{bmatrix} 2\epsilon_0 - (k + \frac{s}{\epsilon_0})n \\ -\frac{\epsilon_0 k}{(\epsilon_0 k - s)} e^{2kn} \end{bmatrix} \bar{A}$$ \hspace{1cm} (5.36')

$$\bar{B} = \begin{bmatrix} \left(\frac{\epsilon_0 k + s}{\epsilon_0 k - s}\right) e^{-2kn} \end{bmatrix} \bar{A}$$

at x = 0, assume the following relationships hold:

$$(\bar{u}_o + \bar{u}_1) = (\epsilon_0 + \epsilon_1) (u_o + u_1)$$ \hspace{1cm} (5.37)

$$(p_o + p_1) + \frac{\rho}{2} \left[ (\epsilon_o u_o + \bar{u}_{x1})^2 + (\bar{u}_{y1})^2 \right] = (p_o + p_1) + \frac{\rho}{2} \left[ (u_o + u_{x1})^2 + (u_{y1})^2 \right]$$ \hspace{1cm} (5.38)

Physically, equation (5.37) states that as the advancing fluid flows into the bed, the direction of the flow remains unchanged and the magnitude of the fluid velocity is increased by a factor of $$\frac{1}{\epsilon}$$ due to the constricted flow path.
Equation (5.38) is the Bernoulli equation which states that the sum of the pressure energy and the kinetic energy on both sides of the interface must be equal.

Equations (5.37) and (5.38) provide three relationships between the perturbation quantities as follows (where the subscript 1 has been dropped for convenience).

\[
\begin{align*}
\overline{U}_x &= E + \varepsilon_o U_x \\
\overline{U}_y &= \varepsilon_o U_y \\
\bar{P} &= \overline{P} + \rho Fr (\varepsilon_o \overline{U}_x - U_x)
\end{align*}
\]

Substituting the solutions (5.32), (5.34), (5.35) for \( \overline{U}_x, \overline{U}_y, \overline{P} \) into equations (5.39), (5.40), (5.41) then leads to two boundary conditions at \( x = 0 \),

\[
\left( \frac{g}{k} - \Gamma \right) E(o) + \left( \frac{1}{k} \right) \frac{dE}{dx}(o) - \varepsilon_o \Gamma U_x(o) + \left( \frac{\varepsilon_o}{k} \right) \frac{dU}{dx}(o) = 0
\]

\[
(\Lambda') E(o) - Z_2 \frac{dE}{dx}(o) - Z_3 \frac{d^2E}{dx^2}(o) - V_2 \frac{dV}{dx}(o) + (\Lambda') U_x(o) + u_2 \frac{dU}{dx}(o)
\]

\[
+ u_3 \frac{d^2U}{dx^2}(o) = 0
\]

63
where

\[ \Gamma(k, s, n) = \frac{\left[ 1 - \left( \frac{\varepsilon_o}{\varepsilon_o k - s} \right) e^{-2kn} + \left( \frac{2\varepsilon}{\varepsilon_o k} \right) \left( \frac{s}{\varepsilon_o k - s} \right) e^{-(k + \frac{s}{\varepsilon_o})n} \right]}{\left[ 1 + \left( \frac{\varepsilon_o}{\varepsilon_o k - s} \right) e^{-2kn} - \left( \frac{2\varepsilon}{\varepsilon_o k} \right) \left( \frac{s}{\varepsilon_o k - s} \right) e^{-(k + \frac{s}{\varepsilon_o})n} \right]} \]

\[ \Lambda' = [\rho Frk^2(\varepsilon_o - \Lambda) - Z_1] \]

\[ \Lambda'' = [\rho Frk^2(-\varepsilon_o \Lambda + \varepsilon_o^2 - 1)] \]

\[ \Lambda(k, s, n) = \frac{\left[ \left( \frac{s + \varepsilon}{\varepsilon_o} \right) \right]}{1 + \left( \frac{\varepsilon_o k + s}{\varepsilon_o k - s} \right) e^{-2kn} - \left( \frac{2\varepsilon}{\varepsilon_o k} \right) \left( \frac{s}{\varepsilon_o k - s} \right) e^{-(k + \frac{s}{\varepsilon_o})n}} \] (5.44)

and

\[ Z_1 = \left[ -\rho FrC' \frac{2}{\varepsilon_o} - \rho Fr \left( \frac{s}{\varepsilon_o} \right) C_o - \frac{\varepsilon_o \bar{\rho}}{\varepsilon_o} \left( \frac{s}{\varepsilon_o} \right) - \bar{\rho} s \right] \]

\[ Z_2 = \left[ -\rho FrC' \frac{2s}{\varepsilon_o} - \frac{\varepsilon_o \bar{\rho}}{\varepsilon_o} \right] \]

\[ Z_3 = \rho FrC' \frac{s}{\varepsilon_o} \]

\[ V_2 = [\varepsilon_o \bar{\rho} + \rho Fr \left( \frac{\varepsilon_o}{\varepsilon_o} \right) C_o s] \]

\[ u_2 = [\rho FrC' \frac{s}{\varepsilon_o} + \frac{\varepsilon_o \bar{\rho}}{\varepsilon_o}] \]

\[ u_3 = \rho FrC' \]

(5.45)
In equation (5.44), as \( n \to \infty \)

\[
\Gamma (k, s, \infty) = \begin{cases} 
1 & \text{for } (k + \frac{s}{\varepsilon_o}) > 0 \\
-\left(\frac{s}{\varepsilon_o k}\right) & \text{for } (k + \frac{s}{\varepsilon_o}) < 0
\end{cases}
\]

\[
\Lambda (k, s, \infty) = \begin{cases} 
\left(\frac{s + \varepsilon_0 k}{k}\right) & \text{for } (k + \frac{s}{\varepsilon_o}) > 0 \\
0 & \text{for } (k + \frac{s}{\varepsilon_o}) < 0
\end{cases}
\] (5.44')

Physically, \( n = \infty \) corresponds to the case of maintaining the constant fluid velocity \( \bar{u} = \bar{u}_0 \) a large distance below the fluidized bed.

In addition to equations (5.42) and (5.43), there are two other boundary conditions at \( x = 0 \), namely:

\[ V_x (0) = 0 \] (5.46)

\[ V_y (0) = 0 \]

or, equivalently

\[
\left(\frac{s}{1 - \varepsilon_o}\right) \bar{\xi}(0) + \frac{dV_x}{dx} (0) = 0
\] (5.47)

In the clear fluid region above the bed, equations (5.32) and (5.34) again apply.

At \( x = m \)

\[ \bar{F}(m) = 0 \] (5.48)

and equation (5.48) together with the matching conditions (5.37) and (5.38)
provide one boundary condition for the system at \( x = 1 \), namely

\[
\sigma_1 E(1) + \sigma_2 \frac{dE}{dx}(1) + Z_3 \frac{d^2E}{dx^2}(1) + V_2 \frac{dV}{dx}(1) + \sigma_3 U_x(1)
\]

\[
+ \sigma_4 \frac{dU}{dx}(1) - u_3 \frac{d^2U}{dx^2}(1) = 0
\]

(5.49)

where

\[
\sigma_1 = [\rho Frk(2s\sigma - \epsilon_o k) + Z_4]
\]

\[
\sigma_2 = [\rho Frk\sigma + Z_2]
\]

\[
\sigma_3 = [\rho Frk(\epsilon_o s\sigma - \epsilon_o^2 k + k)]
\]

\[
\sigma_4 = [\rho Frk \epsilon_o \sigma - u_2]
\]

\[
\sigma(k, m) = \frac{1 - e^{2k(m-1)}}{1 + e^{2k(m-1)}}
\]

(5.50)

Equation (5.49) is derived through the following manipulations. Substituting equation (5.35) into equation (5.48) gives:

\[- \left( \frac{\rho Fr}{k} \right) \left[ \overline{A}(s + \epsilon_o k)e^{km} + \overline{B}(\epsilon_o k - s)e^{-km} \right] = 0
\]

(5.48')

Substituting equations (5.32), (5.34), (5.35) for \( \overline{U}_x', \overline{U}_y, \overline{P} \) and the appropriate expressions for \( U_x', U_y, P \) into equations (5.39), (5.40) and (5.41) results in a set of three simultaneous algebraic equations in terms of \( \overline{A}, \overline{B} \) and \( \overline{C} \). This set of equations together with equation (5.48') lead to the boundary condition (5.49).
From equation (5.50), as \( m \to \infty \)

\[
\sigma(k, \infty) = -1.
\]

(5.50')

Physically, the case of \( m = \infty \) corresponds to exposing the fluidizing fluid to a constant pressure at a large distance above the fluidized bed.

In addition to equation (5.49), there are two other boundary conditions at \( x = 1 \), namely:

\[
V_x(1) = 0 \quad (5.51)
\]

\[
E_{xy}(1) = 0 \quad (5.52')
\]

or, equivalently

\[
\frac{d^2 V}{dx^2}(1) - \left( \frac{s}{1 - \varepsilon_0} \right) \frac{dE}{dx}(1) = 0 \quad (5.52)
\]

Thus equations (5.42), (5.43), (5.46), (5.47), (5.49), (5.51) and (5.52) provide seven homogeneous boundary conditions for the perturbations.

These boundary conditions must be satisfied by the solutions (5.19) and (5.22), and this condition generates a set of 7 homogeneous linear algebraic equations in the 7 arbitrary constants.

5.4 THE CHARACTERISTIC EQUATIONS

Substituting equations (5.19) and (5.22) into the set of boundary conditions derived in Section (5.3) yields the following set of homogeneous linear algebraic equations. For non-trivial solutions to exist it is necessary that the coefficient matrix be singular.
\[
\sum_{i=1}^{5} Q_i + \sum_{i=1}^{2} R_i f(r_i) = 0 \quad (5.53)
\]

\[
\sum_{i=1}^{5} Q_i q_i + \sum_{i=1}^{2} R_i \left[ f(r_i) r_i - \left( \frac{s}{1-e_o} \right) \right] = 0 \quad (5.54)
\]

\[
\sum_{i=1}^{5} Q_i \left[ \frac{G(q_i)}{\zeta} \epsilon_o \left( \frac{q_i}{k} - \Gamma \right) \right]
\]

\[
+ \sum_{i=1}^{2} R_i \left\{ \left[ \frac{E(r_i) + G(r_i) f(r_i)}{\zeta} \right] \epsilon_o \left( \frac{r_i}{k} - \Gamma \right) + \left( \frac{s}{k} - \Gamma \right) + \frac{r_i}{k} \right\} = 0 \quad (5.55)
\]

\[
\sum_{i=1}^{5} Q_i \left[ - V_2 q_i + \frac{G(q_i)}{\zeta} \left( \Lambda'' + u_2 q_i + u_3 q_i^2 \right) \right]
\]

\[
+ \sum_{i=1}^{2} R_i \left\{ \left[ \frac{E(r_i) + G(r_i) f(r_i)}{\zeta} \right] \left( \Lambda'' + u_2 r_i + u_3 r_i^2 \right) - V_2 f(r_i) r_i - Z_3 r_i^2 - Z_2 r_i + \Lambda' \right\} = 0 \quad (5.56)
\]

\[
\sum_{i=1}^{5} Q_i e_i + \sum_{i=1}^{2} R_i f(r_i) r_i = 0 \quad (5.57)
\]

\[
\sum_{i=1}^{5} Q_i^2 q_i + \sum_{i=1}^{2} R_i r_i \left[ r_i^2 f(r_i) - \left( \frac{s}{1-e_o} \right) r_i \right] = 0 \quad (5.58)
\]
\[
\sum_{i=1}^{5} Q_i e_1 \left[ \frac{G(q_i)}{\zeta} (\sigma_3 + \sigma_4 q_i - u_3 q_i^2) + V_2 q_i \right] \\
+ \sum_{i=1}^{2} R_i e_1 \left[ \frac{(E(r_i) + G(r_i)f(r_i))}{\zeta} (\sigma_3 + \sigma_4 r_i - u_3 r_i^2) \\
+ V_2 f(r_i) r_i + Z_3 r_i^2 + \sigma_2 r_i + \sigma_1 \right] = 0
\]  

(5.59)

where

\[
\zeta = - \rho Fr C'_o t^3 + (\rho Fr C'_o + \bar{\rho} \bar{\epsilon}) t^2 + k^2 \rho Fr C'_o t \\
- k^2 \rho Fr C'_o s - k^2 \rho \epsilon_o
\]  

(5.60)

\[
t = \frac{\bar{\rho} \epsilon_o}{\rho Fr (C'_o - \epsilon_o)} + s
\]

\[
E(\gamma) = e_3 \gamma^3 + e_2 \gamma^2 + e_1 \gamma + e_0
\]

\[
G(\gamma) = W_4 \gamma^4 + W_3 \gamma^3 + W_2 \gamma^2 + W_1 \gamma + W_0
\]  

(5.61)

The coefficients in equation (5.61) are defined in equation (5.18).

5.5 RESULTS AND DISCUSSIONS

The determinants of the coefficient matrix of the linear system of equations (5.53) - (5.59) are computed for given sets of parameters. For each mode of disturbance represented by the wave number \( k \), a value of \( s \) is determined which makes the coefficient matrix singular. A series of \( s^* \) vs. \( k^* \)
curves are plotted to reveal the instabilities of the fluidized systems with finite density fluids.

In the computations of determinant vs. $s$ for a given $k$, care must be taken to distinguish the real zero from the spurious zero. For example, for the case $k = 86$, computations show that the determinant changes signs at two locations on the $s$ axis. Labelling these two locations as root (1) and root (2) respectively, close examination shows that root (2) is actually a spurious zero and root (1) is a genuine zero (Fig. 5.2). Root (2) is a spurious zero in the sense that it corresponds to the situation where the 5th order polynomial (Equation 5.23) has repeated roots. This makes the determinant of the coefficient matrix identically zero. Repeating the determinant vs. $s$ computations for different values of $k$ reveal the loci of root (1) and root (2) (Fig. 5.3). The locus of root (1) in the $s$ vs. $k$ plot therefore shows the instability of the system.

Figure (5.3) shows a typical $s^*$ vs. $k^*$ curve for the following set of parameters which is used as the standard in this series of curves.

\[
\begin{align*}
\frac{\rho^*}{s} &= 2.86 \\
\frac{d^*}{p} &= 0.086 \\
\mu_o^* &= 10 \\
\lambda_o^* &= 10 \\
\frac{dp^*_o}{d\varepsilon} &= 20 \\
\varepsilon_o &= 0.46 \\
d^* &= 86
\end{align*}
\]
fluidizing fluid: water
\[ \rho_f^* = 1.0 \]
\[ \mu_o^f = 0.01 \]
\[ C_o = 0.5 \]
\[ u_o^* = U_{mf}^* \]
\[ m = n = \infty \] (in (cgs) units) (5.62) (cont)

Note that \( \mu_o^f \) enters the analysis only through \( U_{mf}^* \) which is defined in Equation (3.47). The general shape of the curve in Fig. (5.4) is similar to that of Fig. (3.3) which is the zero fluid density case. However, the growth rate for the dominant disturbance is only approx. one-third of that of the zero density case. This indicates that water fluidized beds are less unstable than air fluidized beds. The width of the convection cells is slightly increased from that of the zero density case as inciated by the shifting of the dominant wave number.

The case of \( m = n = \infty \) corresponds to the case where the boundary conditions
\[ \overline{p}(m) = 0 \]
\[ \overline{u}(-n) = 0 \] (5.63)
are set at infinite distance away from the bed surfaces. The effects of setting these boundary conditions at finite distances above and below the bed surfaces are also studied.

The effects of varying the shear viscosity coefficient, \( \mu_o^s \), are shown in Fig. (5.5). An increase of \( \mu_o^s \) from 10 to 20 reduces the
instability of the system and the dominant \( k^* \) is shifted towards the origin indicating a widening of the circulation cell. A decrease of \( \mu^*_{\text{o}} \) from 10 to 5 enhances the instability and shifts the dominant \( k^* \) away from the origin indicating a narrowing down of the convection cell.

Figure (5.6) shows the similar effects caused by increase and decrease of the bulk viscosity coefficients, \( \lambda^*_{\text{o}} \). The magnitude of the changes in this case is, however, smaller than that due to the variations in \( \mu^*_{\text{o}} \).

A decrease of \( \frac{dp^*_{\text{o}}}{d\varepsilon} \) from 20 to 0 causes a slight increase in the instability of the system as shown in Fig. (5.7). Similarly an increase in solid density causes a rise in the instability and a decrease in solid density is accompanied by a reduction in the instability. A small shifting of the dominant \( k^* \) is also observed in Fig. (5.8) indicating a wider convection cell for the more stable systems.

Figure (5.9) shows the effects of varying the voidage \( \varepsilon_{\text{o}} \). Instability is greater for systems of smaller \( \varepsilon_{\text{o}} \)'s. In Fig. (5.9), computations are based on the assumption that \( \varepsilon_{\text{o}} \) can be changed independently from other parameters. In reality, for a given type of particles, \( \varepsilon_{\text{o}} \) is changed by changing the fluid velocity \( u^*_{\text{o}} \). Since the exact relationship between \( \varepsilon_{\text{o}} \) and \( u^*_{\text{o}} \) is not clear, the effects of simultaneous changes in \( u^*_{\text{o}} \) and \( \varepsilon_{\text{o}} \) are not studied. The effects of varying particle size, \( d^*_{\text{p}} \), are shown in Fig. (5.10) which indicates higher instability for systems of larger particles.
Since the fluidizing velocity $u^*_0$ is proportional to $d^{*2}_p$ according to Rowe's correlation (equation 3.47), Fig. (5.10), really shows the effects of variations of $u^*_0$ on stability.

So far the effects of varying $\mu^*_o$, $\lambda^*_o$, $\left|\frac{dp^*}{d\varepsilon^*}\right|$, $\rho^*_s$, $d^*_p$, $\varepsilon^*_o$ upon the instability of the system follow closely that of the zero density case discussed in section (3.6). The computations made for the parameters $C^*_o$, $d^*$, the virtual mass coefficient and the bed height, show that these parameters do not significantly affect the instability of the system.

In order to test the significance of neglecting the fluid density in the case of an air fluidized bed, the $s^*-k^*$ curve computed for very small but non-zero $\rho^*_f$ is compared with the one obtained in chapter 3 where the fluid density is strictly zero. As shown in Fig. (5.11), the effect of neglecting air density is to increase slightly the instability of the system. The close agreement of these two curves provides an excellent check on the accuracy of the algebra and the computations for the zero and the non-zero $\rho^*_f$ cases.

Figure (5.12), shows a family of curves obtained for various fluid densities. An orderly increase of instability follows on decreasing the fluid density. The shape of the curves remains much the same for all the fluid densities, also the dominant $k^*$ is not shifted as $\rho^*_f$ is varied. The effects of varying the fluid viscosity, $\mu^*_f$, upon the instability are shown in Fig. (5.13). An increase in $\mu^*_f$ is seen to decrease the instability. Here again the dominant $k^*$ is not changed. As pointed out before, $\mu^*_o$ enters computation only through $U^*_m$. 

73
The effects of setting the boundary conditions on the clear fluid equation (5.63) at finite distances above and below the bed surfaces are indicated in Fig. (5.4). It is found that only when \((m-1) = n \leq 500 \frac{d^*}{D}\), there is any detectable deviation of the curves from the \(m = n = 0\) case. Even by setting \(m\) and \(n\) to a few particle diameters away from the bed surfaces, the effect on the instability of the system is quite small. As \(m\) and \(n\) are decreased, the system is stabilized, as might be expected. In addition the dominant \(k^*\) shifts away from the origin indicating a narrowing of the width of the convection cells.

Summarizing then: the growth rate of the dominant mode of disturbance in water fluidized beds is only about one-third of that in air fluidized beds with the same set of parameters. The effects of varying the parameters of the system on instability are similar in the cases of finite and zero fluid densities. Here again a more unstable system is one that has large, dense particles fluidized by a light fluid of low viscosity and with low particle phase viscosities and low \(\left| \frac{dp_s^*}{d \epsilon} \right| \epsilon_o^*\) value. On the other hand, the more stable system is one consisting of small, light particles fluidized by a dense, highly viscous fluid and has high particle phase viscosities and \(\left| \frac{dp_s^*}{d \epsilon} \right| \epsilon_o^*\).
6. CONCLUSIONS

Convective motions of the particles in a two-dimensional fluidized bed with a low pressure drop distributor is a commonly observed phenomenon.\textsuperscript{(12,13)} In this work, it has been shown that such convective instabilities are in fact predicted by linearized hydrodynamic stability analysis using the Anderson-Jackson model of fluidized beds. The stability analysis also indicates that gas-fluidized systems are several times more unstable than liquid-fluidized systems and that the distributor pressure drop plays an important role in suppressing the convective instabilities of the gas-fluidized systems. All of these are in good qualitative agreement with qualitative observations.

It has been demonstrated here that linearized hydrodynamic stability analysis can be applied successfully to two dimensional fluidized particle assemblies, as it has earlier been applied to the Bénard problem. The three-dimensional case will be a trivial extension of the present analysis.

Parametric studies have been made on a fluidized bed of glass spheres. The effects of individual parameters upon the convective instabilities are presented in plots of $s^*$ vs. $k^*$ where $s^*$ is the growth rate of the disturbance and $k^*$ is the wave number of that disturbance.

In applying the linearized hydrodynamic stability analysis in this work, only the convective type of instabilities are investigated. In the systems
studied, instabilities in the form of travelling waves are not investigated. Therefore, no conclusion concerning the travelling wave type instabilities is drawn here. However, it is conceivable that the imposition of the "flat surfaces" boundary condition at the upper bed surface would probably suppress the compression wave modes.
REFERENCES


FIG(3.1) DETERMINANT VS S FOR A GIVEN X
FIG (3.4) EFFECT OF PARTICLE PHASE VISCOSITIES

1. $\mu_0 \approx \lambda_0 \approx 0.1$
2. $\mu_0 \approx \lambda_0 \approx 1.0$
3. $\mu_0 \approx \lambda_0 \approx 10.0$
4. $\mu_0 \approx \lambda_0 \approx 20.0$
FIG(3.5) EFFECT OF PARTICLE PHASE SHEAR VISCOSITY

- $\mu_0^{s^y} = 20.0$ POISE
- $\mu_0^{s^y} = 10.0$ POISE
- $\mu_0^{s^y} = 5.0$ POISE
FIG.(3.9) EFFECT OF PARTICLE DIAMETER

- $d_\delta = 0.172$ cm
- $d_\delta = 0.086$ cm
- $d_\delta = 0.043$ cm

$\delta (eV)$

$\chi (cm)$

30 20 10 0
FIG(4.1) EFFECT OF DISTRIBUTOR PRESSURE DROP

<table>
<thead>
<tr>
<th>CURVE</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.000</td>
</tr>
<tr>
<td>2</td>
<td>.001</td>
</tr>
<tr>
<td>3</td>
<td>.005</td>
</tr>
<tr>
<td>4</td>
<td>.010</td>
</tr>
<tr>
<td>5</td>
<td>.030</td>
</tr>
<tr>
<td>6</td>
<td>.050</td>
</tr>
<tr>
<td>7</td>
<td>.100</td>
</tr>
</tbody>
</table>

\[ M = \frac{\Delta P_{\text{dist.}}}{\Delta P_{\text{bed}}} \]
FIG(5.1) SCHEMATIC SKETCH OF THE FLUIDIZED BED
FIG.(5.2) DET. VS S FOR THE FINITE FLUID DENSITY CASE
FIG. (5,6) EFFECT OF PARTICLE PHASE VISCOSITY
FIG(5.7) EFFECT OF PARTICLE PHASE PRESSURE
FIG (5.8) EFFECT OF SOLID DENSITY

1  $\tau_s = 3.20$  a/cm$^3$
2  $\tau_s = 2.86$  a/cm$^3$
3  $\tau_s = 2.50$  a/cm$^3$
FIG(5,12) EFFECT OF FLUID DENSITY