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GALERKIN APPROXIMATIONS ON LINEAR ELASTOSTATICS, ELASTODYNAMICS, AND THERMOELASTODYNAMICS

by

Shih-I Chou

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

Thesis Director's Signature:

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1. INTRODUCTION

Rapid strides have been made recently in the development of continuum field theories. It is well recognized, however, that mathematical techniques to solve problems in these theories lag far behind our ability to formulate them. As more and more physical phenomena are incorporated into the theories, it becomes clear that numerical methods offer the most promising tool to yield solutions. Even in linear elasticity, which is one of the oldest subjects in mathematical physics, problems which admit closed-form analytical solutions are extremely few. As a general rule, analytical solutions can be found only when the geometry of the physical system and the loading conditions are relatively simple. Furthermore, expressions of the analytical solutions sometimes do not lead to simple numerical evaluations. Pressed by the need to solve more and more complex problems, structural engineers conceived first the Finite Element Method as a convenient ad hoc numerical scheme. The idea of this scheme turned out to be a very general one, and it developed to be a powerful numerical procedure. Linear elasticity was among the first few fields where the
finite element method was successfully tested.

Recognized as an application of the Ritz-Galerkin technique, the finite element method has been one of the focal points of numerical analysts in recent years. The basic idea of the method is now well-established in general. A consolidation of the fundamentals has also been achieved. For simple systems where only scalar differential equations are involved, the numerical analysis of the method is rather satisfactory. For systems involving vector partial differential equations, however, the numerical analysis of the method is still somewhat underdeveloped. This situation is rather surprising, since usually whenever the analytical theory is well-developed, so are the corresponding numerical methods.

The present work is an attempt to apply the Galerkin formulations to problems in continuum mechanics involving vector partial differential equations. Our primary objective is to solve linear problems in elastostatics, elastodynamics, and coupled thermoelasto-dynamics by numerical procedures and to find their error bounds. It should be noted that the mathematical system governing thermoelasto-dynamics can also be applied to problems in the theories of consolidation of porous medium filled with pore fluid, seepage in elastic material, and some commercial drying
processes, etc. The main results of this work are the formulation of the Galerkin procedures and the derivation of \textit{a priori} error estimates for the aforementioned problems.

Assuming that the exact solutions are sufficiently smooth and that the approximating subspaces used in the Galerkin procedures possess certain prescribed properties, we shall prove that, for the elastostatic problems, the mean square errors of the stress, strain, rotation, and displacement are all of the order $O(h^{k-1})$, where $h$ is the mesh size and $k$ is a smoothness parameter associated with the approximating subspaces. Under certain additional conditions, the displacement errors can be improved to the order $O(h^k)$.

For the elastodynamic problems, the errors of the various quantities and their time derivatives have the same order as those in the elastostatic cases when the continuous time Galerkin procedure is used. In particular, velocity has the same convergence rate as displacement. On the other hand, when the discrete time procedure is used with $\Delta t$ as the step size, then we need to add $O(\Delta t)^2$ to the error bounds. For the coupled dynamic thermoelastic problems, the order of convergence of temperature and heat flux are the same as those of displacement and stress, respectively.
We now summarize briefly the status of our results relative to those reported in both the mechanics and the numerical analysis literature. First, we mention that no mathematical analysis whatsoever about the convergence of the finite element methods in elastostatics can be found in the recent books by Zienkiewicz [55] and Desai and Abel [9]. These books cited only the works of Tong and Pian [50], Key [30], Johnson and McLay [28], and Oliveira [43]. We shall not comment on [43], since convergence rate is not considered there at all. In [50] and [30], the mean square convergence rate of stress, strain, and displacement are all established to be of the order $O(h^{1/2})$ when tetrahedron elements with linear displacement approximation are used. In [28], a two-dimensional rectangular region is considered, and tensor product of piecewise linear functions are employed as shape functions. The mean square convergence rate of stress and strain is shown to be of the order $O(h)$, and that of displacement, also $O(h^1)$, which is not optimal. Even there the authors imposed very restrictive boundary conditions and geometry to obtain the last result.

The only work in numerical analysis literature known to the author where convergence rate of finite element method in elastostatics is considered is that of
Carlson and Hall [5]. Using approximation theorems about some shape functions, they modernized and simplified the method used in [50]. However, they obtained only $O(h)$ for the mean square error of the displacement. Furthermore, their proofs depend essentially on the same restrictions as in [28].

None of these works gave any error bounds for the rotation, i.e., the skew symmetric part of the displacement gradient, nor did they report any result on the traction boundary value problem. A common feature of these works is that static problems are treated by a variational formulation, cf. also Felippa and Clough [17], and Gurtin [22, Sec. 39]. This approach makes the error analysis for the corresponding dynamic problems difficult. Perhaps this is the reason why there exists no reported result on the error analysis of the finite element procedure for solving elastodynamic problems.

Recently, Nickell [38] investigated the three commonly used direct integration schemes for solving the matrix equations of motion in structural dynamics. The idea of this paper is based on the Kreiss-von Neumann-Lax-Richtmyer stability theory, but the scope is limited to problems having one-degree-of-freedom only. We refer the reader to a remark by Strang [49, p. 581] on that theory. In [39]
Nickell and Sackman formulated some variational principles for coupled linear thermoelasticity, and in [40] applied their results to some one-dimensional problems involving thermal shocks. Their algebraic equations turn out to be those of a discrete-time Galerkin approximation, which is locally correct to the first order in time step.

In the present work, most of the deficiencies mentioned above in [50], [30], [28], [5], [38], and [40] have been improved if not completely overcome. As in [50], [30], [28], and [5], our basic assumption is that the strain energy function is positive-definite with respect to the symmetric strain tensor. In problems with displacement boundary conditions, we can even relax this assumption somewhat both in the static case and in the dynamic case.

Primary references of this work are the papers by Douglas and Dupont [10], [11], [12], Dupont [13], Dupont, Fairweather and Johnson [14], Fairweather [15], Friedrich and Keller [21], and Wheeler [54]. Readers not familiar with Galerkin approximations and their error analysis may consult Fairweather's notes [16] for some general background.

In using Galerkin methods one often needs to compute a large number of integrals involving the coefficients of the differential operator to form the coefficients of the approximating algebraic equations. Following [12], we
provide the option to interpolate the coefficients first and then carry out the integrations in most of our Galerkin formulations.

Galerkin approximations for Poisson's equation with Neumann boundary conditions (subject to certain compatibility conditions as usual) have been considered in [21]. Both the exact solution and the approximate solution are unique to within an additive constant only, but they can be chosen uniquely when a normalization condition is imposed. Using the same technique, we consider the elastostatic traction boundary value problem. Of course, our solution is unique to within a rigid translation and rotation, so that our normalization condition is somewhat more complex. Naturally, we consider also the elastostatic displacement boundary value problem in detail.

Although we have not done so, some of our results can be extended to elastostatic problems with mixed boundary conditions upon using the technique of Schultz [46]. Using results of [3], [44], and [45] on the Rayleigh-Ritz procedure for eigenvalue problems, we can also obtain the optimal error estimates for mode shapes of the elastic eigenvalue problems solved by the finite element method using consistent masses.

For dynamic problems, we first approximate the
elastodynamic or coupled thermoelastic dynamic initial-boundary value problems by certain systems of ordinary differential equations. This approximation is accomplished, as usual, by the finite element spatial discretization procedure. We call the solutions to those systems of ordinary differential equations the continuous time Galerkin approximate solutions. Next we approximate the differential equations by finite differences in time and thereby define the discrete time Galerkin approximation solutions. We consider only those discrete time procedures which are correct to within second order in time step, however,

For elastodynamic problems with displacement boundary conditions, we can use the idea of [11] and formulate the Laplace-Modified Galerkin procedure. In this procedure the computational effort of solving the (coupled) momentum equation at each time step is reduced to that of solving n independent scalar n-dimensional wave-equations. Further, if the elastic body is shaped like a rectangle or rectangular parallelepiped, then we can use tensor-product shape functions, and formulate, as in [11], the Alternating Direction Galerkin procedures. In this case the effort of solving each scalar n-dimensional wave-equation is further reduced to that of solving n one-dimensional wave equations, i.e., vibrating string problems. In [11] and [15] only constant
density cases are considered, but their analysis can be extended easily to the non-uniform density cases. The nature of our problem, however, makes a similar extension to the non-uniform density case somewhat more difficult, but we have overcome this difficulty.

For thermoelastic problems, when either the temperature or the displacement satisfies the Dirichlet boundary condition, we formulate a procedure in which the momentum equation and the energy equation are computationally decoupled at each time step. This decoupling procedure is desirable since we can then use the iterative schemes separately for the elastostatic problem and for the heat conduction problem. Furthermore, this decoupling enables us to use the Laplace-Modified (and Alternating Direction) Galerkin Methods for the mechanical subproblems when displacement boundary conditions are used (and when the elastic domain is rectangular).
By \( n \) we denote the dimension of the Euclidean space \( \mathbb{R}^n \) in which the reference configuration of a given elastic or thermoelastic body is embedded. In order to be physically meaningful, \( n \) will take the values 1, 2, or 3 only. By \( \mathbf{x} \), we denote a typical point in \( \mathbb{R}^n \) and by \( (x_1, \ldots, x_n) \) the coordinates of \( \mathbf{x} \) with respect to a fixed Cartesian coordinate system. We denote the reference configuration of the body by \( \Omega \), a bounded domain in \( \mathbb{R}^n \).

For static problems, we describe a deformation of the body by a mapping which carries a point \( \mathbf{x} \in \Omega \) to a new point \( \mathbf{x} + \mathbf{u}(\mathbf{x}) \in \mathbb{R}^n \), and then we require that the new configuration be one of static equilibrium. For dynamic problems, we describe the motion of any point \( \mathbf{x} \in \Omega \) by \( \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \), where \( (\mathbf{x}, t) \in \Omega \times \mathbb{R}^1 \). We call \( \mathbf{x} \) the space variable and \( t \) the time variable.

We assume that \( \Omega \) has the restricted cone property (cf. Agmon [1, p. 11]), i.e. there exist a finite open covering \( \{O_i\}_{i=1}^m \) of the boundary \( \partial \Omega \) of \( \Omega \), and associated open truncated cones \( \{C_i\}_{i=1}^m \) with vertices at the origin, such that for any \( i \) and any \( \mathbf{x} \in O_i \cap \Omega \), the cone \( \mathbf{x} + C_i \equiv \{\mathbf{w}: \mathbf{w} = \mathbf{x} + \gamma \mathbf{y}, \gamma \in C_i\} \) lies in \( \Omega \).
By $C^m(\Omega)$ we denote the class of scalar fields possessing derivatives up to the $m$-th order in $\Omega$ which are continuous on $\Omega$. By $C^m(\partial\Omega)$ we denote the class of vector fields with components in $C^m(\partial\Omega)$, i.e. $\gamma(\xi) \in C^m(\partial\Omega)$ if and only if $\gamma_i(\xi) \in C^m(\partial\Omega)$, $i = 1, \ldots, n$. The space $C^\infty(\partial\Omega)$ is the collection of all infinitely differentiable functions on $\Omega$ which vanish identically outside some compact set contained in $\Omega$. Similarly we define $C^\infty(\partial\Omega)$.

By $W^m(\Omega)$ $(W^m(\Omega))$ we denote the Hilbert space obtained as the completion of $C^\infty(\Omega) (C^\infty(\partial\Omega))$ with respect to the norm $\| \cdot \|_{W^m(\Omega)}$ induced by the inner product

$$< \phi, \psi >_{W^m(\Omega)} = \sum_{k=0}^{m} \sum_{i_1, \ldots, i_k} \frac{\partial^k \phi}{\partial x_{i_1} \ldots \partial x_{i_k}} \frac{\partial^k \psi}{\partial x_{i_1} \ldots \partial x_{i_k}} ,$$

where $< , >$ denotes the usual $L^2(\Omega)$ inner product

$$i.e., < \phi, \psi > = \int_{\Omega} \phi \psi \, dx_1 \ldots dx_n .$$

Since $\Omega$ has the restricted cone property, $W^m(\Omega)$ $(W^m(\partial\Omega))$ coincides with $H^m(\Omega)$ $(H^m(\partial\Omega))$, the set of functions possessing $L^2$-strong generalized derivatives up to the order $m$ in $\Omega$ (Sobolev
space) cf. [1. p. 14]. Generalized derivatives are denoted by the same symbols as the corresponding ordinary derivatives. By \( W^m(\Omega) \) (or \( \partial^m(\Omega) \)) we denote the completion of \( C^m(\Omega) (C^\infty(\Omega)) \) by means of the norm \( \| \cdot \|_{W^m(\Omega)} \) induced by the inner product
\[
< v, w >_{W^m(\Omega)} \equiv \sum_{i=1}^{n} < v_i, w_i >_{\Omega}.
\]
For vector fields we use \( < v, w > \equiv \sum_{i=1}^{n} < v_i, w_i > \).

Let \( X \) be a normed linear space and \([0,T]\) a time interval. By \( C^m(0,T;X) \) we denote the set of functions from \([0,T]\) to \( X \), which possess on \((0,T)\) time derivatives in \( X \) up to the \( m \)-th order, continuous on \([0,T]\). In an analogous way we define the symbols \( L_2(0,T;X) \), \( L_\infty(0,T;X) \). For any \( \phi:[0,T] \to X \), belonging to these spaces, we define
\[
\| \phi \|_{L^2(0,T;X)}^2 = \int_0^T \| \phi(t) \|_X^2 \, dt,
\]
\[
\| \phi \|_{L_\infty(0,T;X)}^2 = \sup_{0 \leq t \leq T} \| \phi(t) \|_X^2.
\]
In computations later, we shall shorten
\[
\| \cdot \|_{L^2(\Omega)} \to \| \cdot \|, \quad L_p(0,T;L_q(\Omega)) \to L_p \times L_q,
\]
and

\[ L_p(O,T;W^1) \to L_p \times W^1, \text{ etc.} \]

We assume also that \( \Omega \) is properly regular in the sense of Fichera [18, p. 21], i.e.,

i) \( \partial \Omega = \partial \Omega, \) and \( \Omega \) is regular (having piecewise continuous unit normal on \( \partial \Omega \));

ii) for any \( x_0 \in \partial \Omega \) there exists a neighbourhood \( I \)
    of \( x_0 \), such that the set \( J = I \cap \Omega \) is homeomorphic to
    the closed half-ball \( B^+: y_n \geq 0 \),

\[ \sum_{i=1}^{n} y_i^2 \leq 1 \]

of the Cartesian space \( \mathbb{R}^n \). In this homeomorphism the set \( \partial \Omega \cap I \) is mapped onto the set \( y_n = 0 \),

\[ \sum_{i=1}^{n-1} y_i^2 \leq 1; \]

iii) The vector-valued function \( \gamma = \gamma(x) \) which maps \( J \) homeomorphically onto \( B^+ \) has piecewise continuous first derivatives and the Jacobian matrix \( \frac{\partial y_i}{\partial x_j} \) has a positive determinant which is bounded away from zero in the whole \( J \).
Under this assumption, integration on $\partial \Omega$ is meaningful, and the divergence theorem is applicable.

The following inequalities will be helpful later:

**Lemma (Poincaré Inequality) (e.g., Fichera [18, p. 19])**

For any $v \in \overset{0}{W}^m(\Omega)$, we have

$$
||v||^2_{\overset{0}{W}^m(\Omega)} \leq C \sum_{i_1, \ldots, i_m} || \frac{\partial^m v}{\partial x_{i_1} \ldots \partial x_{i_m}} ||^2_{L_2(\Omega)} \quad (2.1)
$$

where $C$ is a constant which depends only on $\Omega$. Hence it is useful to define the norm on $\overset{0}{W}^1(\Omega)$

$$
||v||_{\overset{0}{W}^1(\Omega)} = \left( \sum_{i=1}^{n} || \frac{\partial v}{\partial x_i} ||^2_{L_2(\Omega)} \right)^{1/2}.
$$

Note that in $W^1(\Omega)$, $|| \cdot ||_{\overset{0}{W}^1(\Omega)}$ is only a semi-norm.

**Generalized Poincaré Inequality (e.g., Nečas [37, p. 18])**

If $\Omega$ is properly regular, then for any $v \in \overset{0}{W}^m(\Omega)$, we have

$$
||v||_{\overset{0}{W}^m(\Omega)} \leq C \left( \sum_{i_1, \ldots, i_m} || \frac{\partial^m v}{\partial x_{i_1} \ldots \partial x_{i_m}} ||^2_{L_2(\Omega)} \right.

$$

$$
+ \sum_{k<m} \sum_{i_1, \ldots, i_k} \left| \int_\Omega \frac{\partial^k v}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^2 dx \right)^{1/2} \quad (2.2)
$$
The constant $C$ depends only on $\Omega$.

Following Fichera [18, p. 102], we say that the square matrix differential operator $L \equiv (L_{i\ell})_{i,\ell = 1, \ldots, n}$, where

$$L_{i\ell} = - \sum_{j,\ell} \frac{\partial}{\partial x_j} (A_{ij\ell}(x) \frac{\partial}{\partial x_\ell})$$

is uniformly strongly elliptic on $\Omega$ if there exists a positive constant $\eta_s'$ such that

$$\eta_s' \left( \sum_{i=1}^n \xi_i^2 \right) \left( \sum_{j=1}^n \omega_j^2 \right) \leq \sum_{i,j,k,\ell} A_{ij\ell}(x) (\xi_i \omega_j) (\xi_k \omega_\ell),$$

$$\forall \xi \in \mathbb{R}^n, \forall \omega \in \mathbb{R}^n, \forall x \in \Omega. \quad (2.3)$$

Later we need the following theorems:

**Gårding's Inequality.** (as applied to our case)[18, p. 102] If $L$ is uniformly strongly elliptic on $\Omega$, then there exist positive constants $\gamma_0$ and $\lambda_0$ such that for any $v \in \mathcal{W}^1(\Omega)$ the following inequality holds:

$$\gamma_0 \left\| v \right\|_{\mathcal{W}^1(\Omega)}^2 \leq \sum_{i,j,k,\ell} \left< A_{ij\ell}(x) \frac{\partial v_k}{\partial x_\ell}, \frac{\partial v_i}{\partial x_j} \right> + \lambda_0 \left\| v \right\|_{L^2(\Omega)}^2. \quad (2.4)$$
The coefficients $A_{ijkl}(\chi)$ are assumed to be continuous in $\Omega$.

**Remark:** Since there is no lower order terms in the strong-elliptic operator $L$ under consideration, the appearance of $\lambda_0 ||y||^2_{L^2(\Omega)}$ is entirely due to the variation of the coefficients $A_{ijkl}(\cdot)$ over $\Omega$, cf. the proof of Gårding's inequality, e.g. [18, pp. 103-104 or 31, p. 37]. Hence if $A_{ijkl}$ are constant on $\Omega$, then (2.4) can be replaced by

$$\gamma_0 ||y||^2_{W^1(\Omega)} \leq \sum_{i,j,k,l} A_{ijkl} \left< \frac{\partial v_k}{\partial x_i}, \frac{\partial v_l}{\partial x_j} \right>, $$

$$\forall y \in W^1(\Omega), \quad (2.5)$$

where $\gamma_0$ is a positive constant.

We shall also require the following forms of Korn's Inequalities, which are the most important theorems in the mathematical theory of linear elasticity, cf. Friedrichs [20], Fichera [18, pp. 91-92], Hlaváček and Nečas [25], [26], Mikhlin [25, p. 135-136].

**First Korn's Inequality**

For any $y \in W^1(\Omega)$, we have

$$C ||y||^2_{W^1(\Omega)} \leq \sum_{i,j} \left| \left| \frac{\partial v_i}{\partial x_j} \right| \right|^2_{L^2(\Omega)} + \left| \left| \frac{\partial v_i}{\partial x_i} \right| \right|^2_{L^2(\Omega)} \quad (2.6)$$
where the constant $C$ is positive and depends only on $\Omega$.

**Second Korn's Inequality (Classical Form)**

If $\Omega$ satisfies the restricted cone hypothesis, then for any $v \in W^1(\Omega)$ such that

$$
\int_{\Omega} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \, dx = 0,
$$

we have

$$
C ||v||^2_{W^1(\Omega)} \leq \sum_{i,j} || \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} ||^2_{L^2(\Omega)}, \quad (2.7)
$$

where $C$ is a positive constant and depending only on $\Omega$.

**Second Korn's Inequality (Equivalent Form)**

If $\Omega$ satisfies the restricted cone hypothesis, then for any $v \in W^1(\Omega)$ we have

$$
C ||v||^2_{W^1(\Omega)} \leq \sum_{i,j} || \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} ||^2_{L^2(\Omega)}
$$

$$
+ ||v||^2_{L^2(\Omega)}, \quad (2.8)
$$

where the constant $C$ is positive and depends only on $\Omega$. 
In proving the order of convergence for dynamic problems, we need frequently the following two inequalities:

**Gronwall's Inequality** (cf. Bellman [2])

Let \( \phi, \psi, \) and \( \zeta \) be piecewise continuous non-negative functions defined on an interval \( a \leq t \leq b \), and assume that for each \( t \in [a,b] \)

\[
\phi(t) + \zeta(t) \leq \psi(t) + \int_a^t \phi(s) \, ds.
\]

Then we have

\[
\phi(t) + \zeta(t) \leq \int_a^t \exp(t-s)\psi(s)ds + \psi(t),
\]

and if \( \psi \) is nondecreasing, we have

\[
\phi(t) + \zeta(t) \leq \exp(t-a)\psi(t).
\]

**Discrete Analogue of Gronwall's Inequality** [32]

Let \( \phi, \psi, \) and \( \zeta \) be nonnegative functions defined on

\[
\Gamma_\Delta = \{ t \in [a,b] : t = j\Delta t, j = M_1, \ldots, M_2; M_1\Delta t = a, M_2\Delta t = b \},
\]

with \( \psi \) being non-decreasing. If
\[ \phi(t) + \xi(t) \leq \psi(t) + C \Delta t \sum_{j=M_1}^{t/\Delta t - 1} \phi(j \Delta t), \forall \ t \in \Gamma_\Delta, \]

where \( C \) is a positive constant, then

\[ \phi(t) + \xi(t) \leq \psi(t) \exp(Ct), \forall \ t \in \Gamma_\Delta. \]

We shall use \( C \) as a generic constant which is not necessarily the same at each occurrence. In the same spirit we shall use \( \eta \) as a generic symbol for small positive constants. However, these conventions do not apply to those \( C \)'s and \( \eta \)'s with subscripts attached to them.
3. THE CLASSICAL FORMULATIONS OF THE PROBLEMS

In this section we formulate some boundary and initial-boundary value problems in linear elastostatics, elastodynamics, and coupled dynamical thermoelastic problems, all in the classical sense. For the sake of completeness, we prefix a brief summary of the principles of linear elasticity and linear thermoelasticity theory.

LINEAR ELASTICITY IN THE SENSE OF GREEN

For a material point with configuration $\chi$ at time $t$, let $u(\chi, t)$ be displacement from a stress-free natural state which is selected as the reference configuration. By $\rho(\chi)$ we denote the mass density at the point $\chi$ of the natural state.

We postulate that a stored energy $X(\chi, t)$, quadratic in the deformation gradient $\frac{\partial u_i}{\partial x_j}$, is given by

$$\rho X = \frac{1}{2} \sum_{i,j,k,\ell} C_{ijk\ell} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_\ell},$$

where $[C_{ijk\ell}(\chi)]$ are tensor fields assigned on the reference configuration. We assume that $[C_{ijk\ell}]$ satisfy the symmetry
conditions:

\[ C_{ijkl} = C_{jikl} = C_{klij}, \]  

(3.1)

which ensure, among others, the fact that \( X \) is invariant under infinitesimal rigid rotations. Material symmetry implies additional restrictions on \( C_{ijkl} \); e.g., for isotropic materials we may write (cf. Jeffreys [27, p. 70])

\[ C_{ijkl}(x) = \lambda(x) \delta_{ij} \delta_{kl} + \nu(x) \delta_{ik} \delta_{jl}, \]

where \( \delta_{ij} \) are the usual Kronecker delta symbols, and \( \lambda(x) \) and \( \nu(x) \) are called Lamé coefficients.

The constitutive equation for the stress tensor \([\tau_{ij}]\) is given in general by

\[ \tau_{ij} = \rho \frac{\partial X}{\partial u_i} = \sum_{k,l} C_{ijkl} \frac{\partial u_k}{\partial x_l}. \]  

(3.2)

From (3.1) and (3.2) it follows that \( \tau_{ij} \) is symmetric; this fact implies that the conservation law of moment of momentum is automatically satisfied. The conservation law of linear momentum takes the form

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i, \]  

(3.3)
where \( f(x, t) \) denotes the body force.

Introducing \([t_{ij}]\) from (3.2) into (3.3) and assuming that the coefficients \( C_{ijkl}(x) \) are sufficiently smooth, we obtain the governing equations of linear elastodynamics:

\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j,k,l} \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + \rho f_i. \tag{3.4}
\]

If \( u \) and \( f \) are independent of time, then the body is in static equilibrium, and (3.4) reduces to the governing equations of linear elastostatics:

\[
\sum_{j,k,l} \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + \rho f_i = 0. \tag{3.5}
\]

**LINEAR THERMOELASTICITY**

For a material point with configuration \( x \) at time \( t \), let \( u(x, t) \), \( \theta(x, t) \) be the displacement and the temperature deviation, respectively from a natural state (i.e., stress = 0, temperature = \( \theta_C = \) constant > 0), which is selected as the reference configuration. By \( \rho(x) \) we denote again the mass density at the point \( x \) of the natural state.

We postulate that a free energy function \( X(x, t) \), quadratic in the deformation gradient \( [\partial u_i / \partial x_j] \) and the temperature deviation \( \theta \), is given by
\[
\rho X = \frac{1}{2} \sum_{i,j,k,l} C_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} - \sum_{i,j} m_{ij} \frac{\partial u_i}{\partial x_j} \theta
\]

\[- \frac{1}{2} \rho c_D \frac{1}{\theta_C} \theta^2 ,\]

(3.6)

where \([C_{ijkl}(x)], [m_{ij}(x)], c_D(x)\) are tensor fields assigned on the reference configuration [8]. We assume also that \([C_{ijkl}], [m_{ij}]\) satisfy the symmetry conditions

\[
C_{ijkl} = C_{jikl} = C_{klij} , \quad m_{ij} = m_{ji} ,
\]

(3.7)

which imply that \(X\) is invariant under infinitesimal rigid rotations.

The constitutive equations for the stress tensor \([t_{ij}]\) and the deviation of specific entropy \(s\) are given by

\[
t_{ij} = \rho \frac{\partial X}{\partial (\frac{\partial u_i}{\partial x_j})} = \sum_{k,l} C_{ijkl} \frac{\partial u_k}{\partial x_l} - m_{ij} \theta ,
\]

(3.8)

and

\[
\rho s = \rho \frac{\partial X}{\partial \theta} = \frac{1}{\theta_C} \rho c_D \theta + \sum_{i,j} m_{ij} \frac{\partial u_i}{\partial x_j} .
\]

(3.9)

From (3.9) we see that \(c_D(x)\) may be interpreted as the specific heat at the point \(x\) when \(\partial u_i/\partial x_j + \partial u_j/\partial x_i = 0\), and
\( m_{ij}(x) \) characterizes the thermo-elasto-interaction of the material.

To the list of constitutive equations we add Fourier's law, which asserts that the heat flux vector \( q \) is a linear function of the temperature gradient \( \partial \theta / \partial x_i \), viz,

\[
q_i = \sum_j K_{ij} \frac{\partial \theta}{\partial x_j} ,
\tag{3.10}
\]

where \( [K_{ij}(x)] \) is a tensor field assigned on the reference configuration, called the heat conduction tensor.

From (3.7) and (3.8) it follows that \([t_{ij}]\) is symmetric, as a result the conservation law of moment of momentum is automatically satisfied. The remaining conservation laws of linear momentum and energy read

\[
\rho \frac{\partial ^2 u_i}{\partial t^2} = \sum_j \frac{\partial t_{ij}}{\partial x_j} + \rho f_i ,
\]

\[
\rho \theta C \frac{\partial s}{\partial t} = \sum_i \frac{\partial q_i}{\partial x_i} + \rho C_D r ,
\tag{3.11}
\]

where \( f(x,t) \) is the body force, and \( r(x,t) \) denotes the heat source.

Introducing \([t_{ij}], s, [q_i] \) from (3.8), (3.9), (3.10) into (3.11) and assuming that the coefficients \( C_{ijkl}(x), m_{ij}(x) \) etc. are sufficiently smooth, we obtain the system of
linear thermoelastic equations:

\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j,k,\ell} \frac{\partial}{\partial x_j} (C_{i j k \ell} \frac{\partial u_k}{\partial x_\ell}) - \sum_j \frac{\partial}{\partial x_j} (m_{ij} \theta) + \rho f_i,
\]

\[
\rho c_D \frac{\partial \theta}{\partial t} + \sum_{i,j} m_{ij} \theta C \frac{\partial^2 u_i}{\partial x_j \partial t} = \sum_{i,j} \frac{\partial}{\partial x_i} (K_{ij} \theta \frac{\partial \theta}{\partial x_j}) + \rho c_D r.
\]

(3.12)

We now list the classical formulations of the boundary value problems and the initial-boundary value problems which we shall investigate in this paper. In these formulations, \( \Omega \subseteq \mathbb{R}^n \) is assumed to be a bounded domain, either a polygon or a domain with smooth boundary, and \( \mathbf{n}(\mathbf{x}), \mathbf{x} \in \partial \Omega \), denotes the unit outward normal on \( \partial \Omega \) at \( \mathbf{x} \). Clearly \( \mathbf{n}(\mathbf{x}) \) is defined almost everywhere on \( \partial \Omega \).

ELASTOSTATIC BVP

By a classical solution of the homogeneous displacement elastostatic BVP in \( \Omega \), we mean a function \( u(\mathbf{x}) \) satisfying (3.5) for all \( \mathbf{x} \in \Omega \) together with the boundary condition

\[
u = 0 \quad \text{on} \quad \partial \Omega.
\]

(3.13)

Here we would like to note the reason why we avoid the more general and useful case of inhomogeneous Dirichlet boundary
conditions. The difficulty lies in the construction of a
certain finite-dimensional subspace in a given Sobolev space
which contains the solution of the boundary value problem.
Much effort has been given to bypassing this difficulty, and
many methods have been developed and can be used to extend
the homogeneous problem to the inhomogeneous one.

By a classical solution of the traction elastostatic
BVP in \( \Omega \) we mean a function \( u(x) \) satisfying (3.5) for all
\( x \in \Omega \) together with the boundary condition

\[
\sum_j t_{ij} n_j = \sum_{j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j = g_i \text{ on } \partial \Omega, \quad (3.14)
\]

where \( g = g(x), x \in \partial \Omega \), is the prescribed traction vector
on the boundary. For simplicity, we shall assume that \( g = 0 \).
In order that a solution exists it is necessary ([18, p. 92],
[25], [26], [35]) that the body force \( f \) satisfy the compati-
bility conditions

\[
\int_\Omega \rho f_i \, dx = 0,
\]

\[
\int_\Omega \rho (f_i x_j - f_j x_i) \, dx = 0. \quad (3.15)
\]

Clearly, a solution to the traction BVP can be unique to
within a rigid motion only. Indeed, a solution of the trac-
tion BVP subject to zero body force, i.e.,
\[
\sum_{j,k,\ell} \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial v_k}{\partial x_\ell}) = 0 \quad \text{in } \Omega ,
\]

\[
t_i = \sum_j t_{ij} n_j = \sum_{j,k,\ell} C_{ijkl} \frac{\partial v_k}{\partial x_\ell} n_j = 0 \quad \text{on } \partial \Omega ,
\]
is given by

\[
v_i = a_i + \sum_j W_{ij} x_j,
\]

where \(a_i, W_{ij}\) are arbitrary constants such that \(W_{ij} = -W_{ji}\).

We can force the solution of (3.5), (3.14), (3.15) to be unique by requiring

\[
\int_{\Omega} u_i \, dx = 0
\]

\[
\int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) dx = 0, \quad \text{or } \int_{\Omega} \text{curl } u \, dx = 0 . \quad (3.16)
\]

Of course, there are other alternatives to the above conditions, cf. Mikhlin [35, pp. 135-136], Hlaváček and Nečas [25, 26].

We shall also consider the extra BVP subject to homogeneous displacement (traction-free) boundary condition.

\[- \sum_{j,k,\ell} \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial u_k}{\partial x_\ell}) + \lambda_0 u_i = \rho f_i, \quad x \in \Omega \]
\[- \sum_{j,k,l} \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial u_k}{\partial x_j}) + u_i = \rho f_i, \quad x \in \Omega, \]

\[u_i = 0, \quad x \in \partial \Omega \]

\[(\sum_j t_{ij} n_j = \sum_{j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_j} n_j = 0, \quad x \in \partial \Omega) \quad (3.17)\]

where \(\lambda_0\) is the constant in the Gårding inequality. It should be noted, however, that these problems correspond to no meaningful physical situations. The approximation results derived from these problems will be used solely to obtain approximation results for transient problems.

**ELASTODYNAMIC INITIAL-BOUNDARY VALUE PROBLEM**

Let \(T > 0\). By a classical solution of the elastodynamic homogeneous displacement boundary value problem in the cylinder \(Q_T = \Omega \times (0,T)\), we mean a function \(u(x,t)\) satisfying (3.4) for all \((x,t) \in Q_T\), together with boundary conditions

\[u = 0 \quad \text{on } \partial \Omega \times (0,T) \quad (3.18)\]

and initial conditions

\[(u(x,0), \frac{\partial u}{\partial t}(x,0)) = (u_0(x), \dot{u}_0(x)) \quad \text{on } \Omega, \quad (3.19)\]
where $u_0(x)$, $\dot{u}_0(x)$ are given functions on $\Omega$.

Similarly for the elastodynamic traction boundary value problem we simply replace the boundary conditions (3.18) by

$$\sum_j t_{ij} n_j \equiv \sum_{j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j = g_i \text{ on } \partial \Omega \times (0,T),$$

(3.20)

where $g = g(x,t)$, $(x,t) \in \partial \Omega \times (0,T)$, is the prescribed traction vector on the boundary. For simplicity, we assume that $g = 0$.

**THERMOELASTIC INITIAL-Boundary VALUE PROBLEM**

Let $T > 0$. By a classical solution of the initial-boundary value problem in thermoelasticity in the cylinder $Q_T$, we mean a pair $(u, \theta)$ satisfying (3.12) for all $(x,t) \in Q_T$, together with one of the four possible combinations of boundary conditions:

$$u = 0 \text{ or } \sum_j t_{ij} n_j \equiv \sum_{j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j$$

$$- \left( \sum_j m_{ij} n_j \right) \theta = g_i, \quad \text{on } \partial \Omega \times (0,T)$$

$$\theta = 0 \text{ or } \sum_i q_i n_i \equiv \sum_{i,j} K_{ij} \frac{\partial \theta}{\partial x_j} n_i = q,$$

(3.21)
and initial conditions:

\[
(u(x,0) \frac{\partial u}{\partial t}(x,0), \theta(x,0)) = (u_0(x), \dot{u}_0(x), \theta_0(x,0)) \text{ on } \Omega
\]  \quad (3.22)

where \( g = g(x,t), q = q(x,t), (x,t) \in \partial \Omega \times (0,T), \)

are the prescribed traction vector and heat supply on the boundary, and \( u_0(x), \dot{u}_0(x), \theta_0(x) \) are given functions on \( \Omega \).

For simplicity, we assume that \( g = 0 \) and \( q = 0. \)

In this paper we shall not consider boundary conditions of the rigid contact type, nor those of the mixed type in general, because of the complications arising from these as reported in Mikhlin's text [35, pp. 122-124], and because of the lack of any sufficiently general knowledge about the global regularities of the solutions of the (associated) boundary value problems, \textit{cf.} Fichera [18].
4. THE WEAK FORMULATIONS OF THE PROBLEMS

WEAK FORMULATIONS

I. Elastostatics (Homogeneous Displacement BVP)

A function \( u \in W^1(\Omega) \) is said to be a weak (or generalized) solution of the problem (3.5) and (3.13) if for every \( v \in W^1(\Omega) \)

\[
- \sum_{i,j,k,l} < C_{ijkl} \frac{\partial u_k}{\partial x_l}, \frac{\partial v_i}{\partial x_j} > + \sum_i < \rho f_i, v_i > = 0.
\]  

(4.1)

Elastostatics (Traction BVP)

A function \( u \in \tilde{W}^1(\Omega) \) is said to be a weak solution of the problem (3.5), (3.14), and (3.15) if for every \( v \in \tilde{W}^1(\Omega) \)

\[
- \sum_{i,j,k,l} < C_{ijkl} \frac{\partial u_k}{\partial x_l}, \frac{\partial v_i}{\partial x_j} > + \sum_i < \rho f_i, v_i > = 0.
\]  

(4.2)

A weak solution is called a \textit{normalized} weak solution if it satisfies also (3.16). Since \( \tilde{f} \) verifies (3.15), it
can be shown easily that \( u \) is a normalized weak solution if and only if \( u \in W^1(\Omega)^* \), and

\[
- \sum_{i,j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} + \sum_{i} \rho f_i, v_i > = 0, \quad \forall \ v \in W^1(\Omega)^*,
\]

where

\[
W^1(\Omega)^* \equiv \{ v \in W^1(\Omega) : \int_{\Omega} v dx = 0, \int_{\Omega} \text{curl} \ v dx = 0 \}.
\]

The Extra Elastostatic BVP

A function \( u \in W^1(\Omega) (W^1(\Omega)) \) is said to be a weak solution of the Extra BVP (3.17) if for every \( v \in W^1(\Omega) (W^1(\Omega)) \)

\[
\sum_{i,j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} + \lambda_0 \sum_{i} u_i, v_i > = \sum_{i} \rho f_i, v_i >,
\]

\[
( \sum_{i,j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} + \sum_{i} u_i, v_i > = \sum_{i} \rho f_i, v_i > )
\]

(4.4)

II. Elastodynamics (Homogeneous Displacement BC)

A function \( u \) such that \( u \in L_2(-\infty, T; W^1(\Omega)) \), \( u \) is zero for \( t < 0 \), and the distributional derivative \( \frac{\partial}{\partial t} u \in \)
$L_2(-\infty, T; L_2(\Omega))$ is said to be a weak solution of the problem (3.4), (3.18), and (3.19) if

$$
\frac{d}{dt} \sum_i < \frac{\partial u_i}{\partial t}, v_i > + \sum_{i,j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} - \sum_i < \rho f_i, v_i > \\
- < \rho \ddot{u}_0, \gamma > \delta - < \rho u_0, \gamma > \frac{d\delta}{dt} = 0, \forall \gamma \in W^1_0(\Omega).
$$

(4.5)

Here we assume that $\rho f \in L_2(-\infty, T; L_2(\Omega))$, $\dot{u}_0 \in L_2(\Omega)$, $u_0 \in W^1_0(\Omega)$ and $f$ is zero for $t < 0$. In (4.5), $\delta$ designates the Dirac function at the origin. Equation (4.5) is interpreted in the sense of distribution in time, cf. Lions [33, p. 150, and p. 155].

**Elastodynamics (Traction-Free BC)**

A function $u$ such that $u \in L_2(-\infty, T; W^1_0(\Omega))$, $u$ is zero for $t < 0$, and the distributional derivative $\frac{\partial u}{\partial t} \in L_2(-\infty, T; L_2(\Omega))$ is said to be a weak solution of the problem (3.4), (3.20) and (3.19) if

$$
\frac{d}{dt} \sum_i < \frac{\partial u_i}{\partial t}, v_i > + \sum_{i,j,k,l} C_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} \\
- \sum_i < \rho f_i, v_i > - < \rho \ddot{u}_0, \gamma > \delta - < \rho u_0, \gamma > \frac{d\delta}{dt} = 0, \\
\forall \gamma \in W^1_0(\Omega).
$$

(4.6)
As in (4.5) it is assumed that \( \varphi \in L_2(-\infty, \infty; L_2(\Omega)), \dot{u}_0 \in L_2(\Omega), \dot{u}_0 \in W^1(\Omega), \) and \( \varphi \) is zero for \( t < 0 \).

III. Thermoelastic Initial-Boundary Value Problem

A pair of functions \((u, \theta)\) such that \( u \in L_2(-\infty, \infty; \dot{W}^1(\Omega)) \) \((L_2(-\infty, \infty; W^1(\Omega)))\), \( \theta \in L_2(-\infty, \infty; \dot{W}^1(\Omega)) \) \((L_2(-\infty, \infty; W^1(\Omega)))\), \((u, \theta)\) is zero for \( t < 0 \), and the distributional derivative \( \partial u/\partial t \in L_2(-\infty, \infty; L_2(\Omega)) \) is said to be a weak solution of the problem (3.12), (3.21), and (3.22) if

\[
\frac{d}{dt} \sum_i \varrho \frac{\partial u_i}{\partial t} \cdot v_i + \sum_{i,j,k} C_{ijkl} \frac{\partial u_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} \\
- \sum_{i,j} m_{ij} \vartheta \frac{\partial v_i}{\partial x_j} - \sum_i \rho f_i \cdot v_i - \rho \dot{u}_0 \cdot v > \delta \\
- \rho u_0 \cdot v > \frac{d\delta}{dt} = 0, \forall v \in \dot{W}^1(\Omega)(W^1(\Omega)),
\]

\[
\frac{d}{dt} \left< \rho c_D \theta, \phi \right> + c_d \frac{d}{dt} \sum_{i,j} \frac{\partial u_i}{\partial x_j} \phi \\
+ \sum_{i,j} K_{ij} \frac{\partial \theta}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \rho c_D \theta_0 \cdot \phi > \delta \\
- \theta c_i \sum_{i,j} \frac{\partial u_{0i}}{\partial x_j} \phi > \delta = 0, \forall \phi \in \dot{W}^1(\Omega)(W^1(\Omega)).
\]

(4.7)

Here we assume as before that \( \rho f_i, \rho c_D \varphi \in L_2(-\infty, \infty; L_2(\Omega)), \)
\( \hat{u}_0 \in L^2(\Omega), u_0 \in W^1_0(\Omega), (W^1(\Omega)), \theta_0 \in L^2(\Omega), \) and \((f,r)\) is zero for \(t < 0\).

Clearly, the classical solutions, if they exist, are necessarily weak solutions. Conversely, however, when the weak solutions exist, the classical solutions generally need not exist. In this paper we are mainly interested in problems for which the coefficients of the differential operator, the given data, and the domain \(\Omega\) are all sufficiently smooth. Under these smoothness assumptions it can be shown that the weak solutions are actually smooth and, thus, they are, in fact, classical solutions, cf. Agmon [1], Fichera [18], Lions [33], Nečas [37]. Hence to calculate the classical solutions it suffices to develop some approximation schemes for the weak solutions.

**PHYSICAL ASSUMPTIONS**

In order to establish the existence of solutions, we make the following basic assumptions: We assume that there exist positive constants \(\eta_\rho, C_\rho, \eta_C, C_C, \eta_\theta, C_\theta, \eta_s, \eta_s', C_u\) such that

\[
0 < \eta_\rho \leq \rho(x) \leq C_\rho, \quad \forall x \in \Omega; \quad A(i)
\]

\[
0 < \eta_C \leq C_D(x) \leq C_C, \quad \forall x \in \Omega; \quad A(ii)
\]
\[ \eta_i \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} k_{ij}(\xi) \xi_i \xi_j \leq C_{\theta} \sum_{i=1}^{n} \xi_i^2, \]

\[ \forall \, \xi \in \mathbb{R}^n, \forall \, x \in \bar{\Omega} ; \quad A(iii) \]

\[ \eta_i \sum_{i,j=1}^{n} S_{ij}^2 \leq \sum_{i,j,k,l} C_{ijkl}(\xi) S_{ij} S_{kl}, \]

\[ \forall \, S = S^T \in \mathbb{R}^{n \times n}, \forall \, x \in \bar{\Omega} ; \quad A(iv) \]

\[ \sum_{i,j,k,l} C_{ijkl}(\xi) S_{ij} S_{kl} \leq C_u \sum_{i,j} S_{ij}^2, \]

\[ \forall \, S = S^T \in \mathbb{R}^{n \times n}, \forall \, x \in \bar{\Omega} . \quad A(v) \]

When displacement boundary conditions are considered, \( A(iv) \) may be replaced by the weaker assumption of uniformly strong ellipticity (SE), i.e.

\[ \eta_i \left( \sum_{i=1}^{n} \xi_i^2 \right) \left( \sum_{j=1}^{n} \omega_j^2 \right) \leq \sum_{i,j,k,l} C_{ijkl}(\xi) (\xi_i \omega_j)(\xi_k \omega_l), \]

\[ \forall \, \xi \in \mathbb{R}^n, \forall \, \omega \in \mathbb{R}^n, \forall \, x \in \bar{\Omega} . \quad A(iv)' \]

Remark: For a matrix in general, assumption \( A(iv)' \) does not have any obvious implication, but assumption \( A(iv) \) implies
\[ n_s \sum_{i,j} \left( \frac{A_{ij} + A_{ji}}{2} \right)^2 \leq \sum_{i,j,k,l} C_{ijkl} A_{ij} A_{kl}, \]

\[ \forall A \in \mathbb{R}^{n \times n}, \quad (4.8) \]

The last result is due to (3.1) or (3.7). From (4.8) it can be shown that A(iv) implies A(iv)' with \( n'_s = n_s / 2 \) upon specializing \( A \) in (4.8) to matrices of the tensor product type, cf. the argument in [18, p. 91]. Finally we note that assumption A(v) implies

\[ \sum_{i,j,k,l} C_{ijkl} A_{ij} A_{kl} \leq C_u \sum_{i,j} A_{ij}^2, \forall A \in \mathbb{R}^{n \times n}. \]

This result is due to the simple fact that

\[ \sum_{i,j} \left( \frac{A_{ij} + A_{ji}}{2} \right)^2 \]

is less than \( \sum_{i,j} A_{ij}^2 \), since symmetric matrices and skew-symmetric matrices are orthogonal in the space of arbitrary matrices of the same type.

**INTERPRETATIONS OF THE PHYSICAL ASSUMPTIONS**

Assumption A(i), A(ii), A(iii), A(v) above are obviously in agreement with physical experience. Assumption A(iii) may be interpreted with the help of the
Clausius-Duhem inequality

\[ \sum_{i} q_{i} \frac{\partial \theta}{\partial x_{i}} = \sum_{i,j} K_{ij} \frac{\partial \theta}{\partial x_{i}} \frac{\partial \theta}{\partial x_{j}} > 0 \]

whenever

\[ \sum_{i} ( \frac{\partial \theta}{\partial x_{i}} )^2 \neq 0. \]

The mechanical interpretations for assumptions A(iv) and A(iv)' are less trivial and will be explained in somewhat detail here, as they are probably responsible for the present status of art on numerical analysis of BVP arising from elasticity theory.

Assumption A(iv) states that the fourth order tensor \( C_{ijkl}(x) \) is uniformly (with respect to \( x \in \Omega \)) positive definite over the space of second order symmetric tensors, the dimension of that space being \( n(n+1)/2 \), i.e. 3 when \( n = 2 \) and 6 when \( n = 3 \). Physically assumption A(iv) requires that the stress work in any non-rigid deformation be positive. For isotropic materials, assumption A(iv) reduces to the familiar ones

\[ \mu > 0 \text{ and } 3\lambda + 2\mu > 0. \]

In fact we may take \( \eta_{s}(C_{ijkl}) \) to be the min (max) of \( \{ 3\lambda(x) + \frac{\chi_{i}}{\chi_{j}} \chi_{i}, 2\mu(x), +2\mu(x) \} \), see e.g. Gurtin [22, Secs. 22, 24].
In this case, another characterization of A(iv) is that the shear stress in simple shear shall point in the direction of the shear effected, and that pressure in mean is required to decrease the volume, tension in mean to increase it. For these and other interpretations see Truesdell [51, p. 153] or Wang and Truesdell [53].

Assumption A(iv)' states that the fourth order tensor \( C_{ijkl}(x) \) is uniformly positive definite over the set of matrices of the tensor product type. This set is a manifold in \( \mathbb{R}^n \), its dimension is 3 when \( n = 2 \) and 5 when \( n = 3 \). Assumption A(iv)' is necessary and sufficient in order that the squared speeds of waves corresponding to all amplitudes satisfying the propagation condition be positive. For isotropic materials, assumption A(iv)' reduces to \( \mu > 0 \) and \( \lambda + 2\mu > 0 \). In fact, we may take \( n' \) to be the \( \min_{x \in \mathbb{R}} \{ \lambda(x) + 2\mu(x), \mu(x) \} \), see e.g. Gurtin [22, Secs. 25, 70]. The reduced inequalities mean that in simple shear, the shearing stresses be directed in the same sense as the shear effected, and that when a cube of isotropic material is lengthened along any particular direction the tensile force in the same direction must be increased, but to shorten it, the tensile force must be reduced. For these and other interpretations see Hayes [23], Truesdell [51, p. 168] or Wang and Truesdell [53]. Note for isotropic materials, assumption A(iv)
obviously implies assumption A(iv)' but not vice versa, a fact we have observed in general before.

Douglas and Dupont ([10], [11, p. 178]), in their analysis of Galerkin methods for parabolic systems of the form

$$\frac{\partial v_i}{\partial t} = \sum_{k=1}^{m} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (A_{ijkl}(v, x) \frac{\partial v_k}{\partial x_l}), \quad i=1, \ldots, m,$$

assumed for the differential operator on the right-hand side the following pointwise condition

$$n_D \sum_{i,j,k,l} \xi_{ij}^2 \leq \sum_{i,j,k,l} A_{ijkl} \xi_{ij} \xi_{kl}, \quad \forall \xi \in \Omega, \forall \xi \in \mathbb{R}^{m \times n},$$

(4.9)

where $\xi$, as noted, need not be square. In their derivation of the error estimates for alternating-direction Galerkin methods on rectangles for this system, this pointwise assumption was used in a crucial way [11, pp. 178-179]. It should be noted that the analogue of this assumption is not acceptable in elasticity theory, since we know that

$$\sum_{i,j,k,l} C_{ijkl} w_{ij} w_{kl} = 0, \forall w = -W^T \in \mathbb{R}^{n \times n}, \forall x \in \Omega, n \geq 2,$$

because of the symmetry relations of $C_{ijkl}$ (cf. equations (3.1) or (3.7)1). Therefore, any positive definiteness assumptions on $C_{ijkl}$ should be made on a proper subset of
which can intersect the space of skew-symmetric n×n matrices only at 0. It is clear that the assumptions A(iv) and A(iv)' obey this rule.

EXISTENCE AND UNIQUEENESS OF WEAK SOLUTIONS

We first write down some global coercivity conditions implied by the assumptions A(iii), A(iv), A(iv)', and A(v). Assumption A(iii) implies that

\[ \eta \| \nabla \phi \|^2_{L^2(\Omega)} \leq \langle K \nabla \phi, \nabla \phi \rangle \leq C_\eta \| \nabla \phi \|^2_{L^2(\Omega)} , \]

\[ \forall \phi \in W^1(\Omega) . \] (4.10)

Assumptions A(iv) and A(v), together with the First Korn's Inequality and (4.8), imply that there is a positive constant \( \eta_u^{(1)} \) such that

\[ \eta_u^{(1)} \| y \|^2_{W^1(\Omega)} \leq \sum_{i,j,k,l} C_{ijkl} \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \]

\[ \leq C_u \| y \|^2_{W^1(\Omega)} , \forall y \in W^1(\Omega) . \] (4.11)

Assumptions A(iv), and A(v), together with the Classical Form of the Second Korn's Inequality and (4.8), imply that there is a positive constant \( \eta_u^{(2)} \) such that
\[ \eta_u^{(2)} \| v \|_{W^1_0(\Omega)}^2 \leq \sum_{i,j,k,l} < C_{ijkl} \frac{\partial v_k}{\partial x_i} \frac{\partial v_l}{\partial x_j} > \]
\[ \leq C_u \| v \|_{W^1_0(\Omega)}^2 , \]
\[ \forall v \in W^1_0(\Omega) \text{ s.t. } \int_{\Omega} (\frac{\partial v}{\partial x_j} - \frac{\partial v}{\partial x_i}) dx = 0. \quad (4.12) \]

Assumptions A(iv) and A(v), together with the Equivalent Form of the Second Korn's Inequality and (4.8), yield that there is a positive constant \( \eta_u^{(3)} \) such that

\[ \eta_u^{(3)} \| v \|_{W^1_0(\Omega)}^2 \leq \sum_{i,j,k,l} < C_{ijkl} \frac{\partial v_k}{\partial x_i} \frac{\partial v_l}{\partial x_j} > + \| v \|_{L^2(\Omega)}^2 , \]
\[ \forall v \in W^1_0(\Omega). \quad (4.13) \]

Assumption A(iv)' and A(v), together with the Garding's Inequality, imply that there is a positive constant \( \eta_u^{(4)} \) such that

\[ \eta_u^{(4)} \| v \|_{W^1_0(\Omega)}^2 \leq \sum_{i,j,k,l} < C_{ijkl} \frac{\partial v_k}{\partial x_i} \frac{\partial v_l}{\partial x_j} > \]
\[ + \lambda_0 \| v \|_{L^2(\Omega)}^2 , \]
\[
\sum_{i,j,k,l} C_{ijkl} \frac{\partial v_k}{\partial x_k} \frac{\partial v_l}{\partial x_l} \leq C_u \|v\|_{W^2(\Omega)}^2
\]

\forall v \in W^2(\Omega). \quad (4.14)

Note that the inequalities (4.14), a consequence of A(iv)', does not imply global uniqueness for the solution of the Displacement Elastostatic BVP. Indeed, Edelstein and Fosdick (see [31, p. 39]) have constructed a counter example to show that non-uniqueness for the displacement BVP of bounded regions is possible under the assumption of strong ellipticity. It is known that in sufficiently small subsets of \( \Omega \), strongly elliptic systems with Dirichlet boundary conditions are always uniquely solvable however, cf. [1, p. 102]. So as to consider well posed problems only, we shall use the assumption A(iv)' only when the elasticities \( C_{ijkl} \) are uniform. Then from (2.5) of the remark after Gårding's inequality, we have the coercivity condition, i.e.

\[
n_{(4)} u \|v\|_{W^1(\Omega)}^2 \leq \sum_{i,j,k,l} C_{ijkl} \frac{\partial v_k}{\partial x_k} \frac{\partial v_l}{\partial x_l} \leq C_u \|v\|_{W^2(\Omega)}^2
\]

\forall v \in W^2(\Omega). \quad (4.15)

Hence under assumptions A(v), A(iv) or A(iv)' with uniform elasticity, the Displacement BVP of Elastostatics is well-posed.
by the Lax-Milgram Theorem [1, p. 99]. Similarly under assumptions A(v) and A(iv), the Traction BVP and the Extra BVP of Elastostatics with Traction-Free BC are well-posed. Further, under assumptions A(v) and A(iv)', the Extra BVP of Elastostatics with Homogeneous Displacement BC is also well-posed.

The elastodynamic I-BVP with displacement BC is well-posed under assumptions A(i), A(v) and A(iv)', even if the elasticities are not necessarily uniform, cf. Lions [33, pp. 150-158]. Note that the foregoing statement remains true if the SE assumption is replaced by the positive-definiteness assumption A(iv). Similarly the elastodynamic I-BVP with traction BC is well-posed under assumptions A(i), A(v), and A(iv).

For the Homogeneous Displacement Elastostatic BVP, we say that the elasticity operator is \textit{o-regular} if the elasticities $C_{ijkl}(x)$ satisfy (4.11) or (4.15) and are sufficiently smooth such that the elastostatic problem defined by (4.1) verifies the following $W^2(\Omega)$-estimate:

$$
||u||_{W^2(\Omega)} \leq C||f||_{L^2(\Omega)}.
$$

Similarly, for the Traction Free Elastostatic BVP, we say that the elasticity operator is \textit{o-regular} if the elasticities $C_{ijkl}(x)$ satisfy (4.12) and are sufficiently smooth
such that the elastostatic problem defined by (4.3) verifies the \( \underline{W}^2(\Omega) \)-estimate (4.16). Finally, for the Extra Elastostatic BVP we say the elasticity operator is \( \alpha \)-regular if the elasticities \( C_{ijkl}(x) \) satisfy (4.14)((4.13)) and are sufficiently smooth such that the elastostatic problem defined by (4.4) verifies the \( \underline{W}^2(\Omega) \)-estimate (4.16).
5. FORMULATION OF GALERKIN APPROXIMATION

We shall approximate the weak solution \( y \) of the *elasticity problems* by a function \( \bar{y} \) belonging to a certain finite-dimensional subspace of \( W^1_0(\Omega) \) or \( W^1(\Omega) \) (for each \( t \) in the case of dynamic problems). Similarly, we shall approximate the weak solution \((u, \theta)\) of the *thermoelasticity problems* by a pair \((\bar{u}, \bar{\theta})\) belonging to a certain finite dimensional subspace of the corresponding Sobolev product spaces for each \( t \).

In using Galerkin methods, one needs to compute a large number of integrals involving the coefficients of the differential operator to form the coefficients of the approximating algebraic equations. Certain simplifications are often used to evaluate these integrals, e.g., interpolation of the coefficients in the differential operator. Experience indicates that these simplifications usually do not affect the quality of the approximate solution. McIay [36], using a variational principle with Lagrange multipliers, considered the effect of this kind of simplification in a two-point boundary value problem, namely, a rod with varying sectional properties and loaded with varying axial body forces. Douglas and Dupont [12] propose interpolating the coefficients in nonlinear parabolic Galerkin procedures and showed
that for a rather general collection of approximation schemes the resulting approximate solution is essentially as good as if the integrals are evaluated exactly. We shall follow Douglas and Dupont's approach to formulate the approximating procedures and later to derive the error estimates in our cases.

For elasticity problems, we shall denote by \( M_N \) a certain \( N \)-dimensional subspace of \( \bar{W}^1(\Omega) \) or \( W^1(\Omega) \) having the following form:

\[
M_N = M_{1_{N_1}} \times \ldots \times M_{n_{N_n}} \text{ with } N = N_1 + \ldots + N_n,
\]

where each \( M_{i_{N_i}} \) is a finite dimensional subspace of \( \bar{W}^1(\Omega) \) or \( W^1(\Omega) \), with dimension \( N_i \). Note that we do not require each component of the approximating solution \( \bar{v} \) to be contained in a common finite-dimensional space. Suppose that \( \{ \bar{v}_{i,1}, \ldots, \bar{v}_{i,N_i} \} \) is a basis for \( M_{i_{N_i}} \), and let

\[
a = \sum_{j=1}^{i-1} N_j + \lambda.
\]

Then we define \( \bar{v}^\alpha \) to be the \( R^n \) vector-valued function having its \( i \)-th component non-zero only and that component is equal to the function \( \bar{v}_{i,\lambda} \). We can regard \( \bar{v}^\alpha \) as an element in \( M_N \) by the identification
\[ \mathbf{v}^a(x) = (0, \ldots, 0, \mathbf{V}_{i, z}^a(x), 0, \ldots, 0). \]

For thermoelasticity problems, we shall denote by \( \mathcal{M}'_N \times \mathcal{M}'_{N_0} \) a certain \((N+N_0)\)-dimensional subspace of \( \tilde{\mathbf{w}}^1(\Omega) \)
\( (\mathbf{w}^1(\Omega)) \times \mathbf{w}^1(\Omega) \) having a similar form.

(I) **Elastostatics (Homogeneous Displacement BVP)**

Let \( \mathcal{M}'_N \) be the \( N \)-dimensional subspace of \( \tilde{\mathbf{w}}^1(\Omega) \) just introduced, viz.,

\[ \mathcal{M}'_N = \text{Sp}(\mathbf{v}^1, \ldots, \mathbf{v}^N). \]

We shall approximate \( u \) of (4.1) by a function \( \mathbf{y} \) in \( \mathcal{M}'_N \),

\[ u = \sum_{\beta=1}^{N} \sigma_\beta \mathbf{y}_\beta \quad \text{or} \quad U_k = \sum_{\beta=1}^{N} \sigma_\beta \mathbf{v}_k^\beta, \quad k=1, \ldots, n, \]

(5.1)

which satisfies

\[ - \sum_{i,j,k,l} C_{ijkl} \frac{\partial U_k}{\partial x_i} \frac{\partial V_l}{\partial x_j} + \sum_i \langle \rho f_i \rangle, \mathbf{V}_i \rangle = 0, \]

\[ \mathbf{V} \in \mathcal{M}'_N. \]

(5.2)

Here, when the elasticities are not uniform, we use the
physical assumption A(iv), then $\tilde{C}_{ijkl}$ and $(\rho f_i)^\sim$ are some approximations of $C_{ijkl}$ and $\rho f_i$ belonging to $\mathcal{V}$, a finite dimensional subspace of $L_\infty(\Omega)$. Of course, we require that

$$\tilde{C}_{ijkl} = \tilde{C}_{jilk} = \tilde{C}_{klij};$$

(5.3)

further,

$$\max_{i,j,k,\ell} \{ ||\tilde{C}_{ijkl}(\cdot) - C_{ijkl}(\cdot)||_{L_2(\Omega)} \},$$

$$|| (\rho f_i)^\sim(\cdot) - \rho f_i(\cdot) ||_{L_2(\Omega)} \} \leq \chi,$$

(5.4)

where $\chi$ is of the same order of magnitude as

$$\inf\{ ||u - \gamma||_{L_2(\Omega)} : \nu_i \in \mathcal{V} \}.$$

In order that the Galerkin approximation solution be well-defined, we assume further that (4.11) remain valid with $n_u^{(1)}$ replaced by $\frac{n_u}{2}$ and with $C_u$ replaced by $\frac{3}{2} C_u$, i.e.

$$\frac{n_u}{2} ||\gamma||_{W^1(\Omega)}^2 \leq \sum_{i,j,k,\ell} \tilde{C}_{ijkl} \frac{\partial v_k}{\partial x_i} \frac{\partial v_i}{\partial x_j},$$

$$\leq \frac{3C_u}{2} ||\gamma||_{W^1(\Omega)}^2,$$

(5.5)
where for simplicity we put \( \eta_{u} = \min (\eta_{u}^{(1)}, \eta_{u}^{(2)}, \eta_{u}^{(3)}, \eta_{u}^{(4)}) \). This assumption is automatically satisfied when \( \hat{C}_{ijkl} \) is sufficiently close to \( C_{ijkl} \). Otherwise some procedures must be devised to insure the algebraic problem be well-posed. (Douglas and Dupont [12] devised such a scheme for the case of a one-dimensional nonlinear parabolic equation using the Galerkin method with Hermite cubic basis functions and with the (nonlinear) coefficients interpolated in a subspace of \( L_{\infty}(\Omega) \).) For isotropic materials satisfying assumptions A(iv) and A(v), i.e.,

\[
0 < \eta_{s} = \min_{\chi \in \bar{\Omega}} \{3\lambda(\chi) + 2\mu(\chi), 2\mu(\chi)\} \leq \max_{\chi \in \bar{\Omega}} \{3\lambda(\chi) + 2\mu(\chi), 2\mu(\chi)\} = C_{u} < \infty ,
\]

it suffices to require that

\[
0 < \frac{\eta_{s}}{2} = \min_{\chi \in \bar{\Omega}} \{3\lambda(\chi) + 2\mu(\chi), 2\mu(\chi)\} \leq \frac{3C_{u}}{2} < \infty ,
\]

In particular, the perturbed bulk modules and the perturbed shear modules both must remain positive and finite.

Substituting (5.1) and \( \bar{Y} = Y^{B}, \beta = 1, \ldots, N \), into (5.2), we obtain the linear algebraic system
\( \tilde{\zeta} \mathbf{q} = \tilde{\Phi} \),

\[ (5.6) \]

where \( \zeta \equiv [\zeta^{\alpha\beta}], \mathbf{q} \equiv [\mathbf{q}^\beta], \tilde{\zeta} \equiv [\tilde{\zeta}^\alpha], \tilde{\Phi} \equiv [\tilde{\Phi}^\alpha], \) with

\[ \zeta^{\alpha\beta} = \sum_{i,j,k,l} \langle \tilde{C}_{ijkl} \frac{\partial V_k}{\partial x_k}, \frac{\partial V_l}{\partial x_l} \rangle, \]

\[ \tilde{\Phi}^\alpha = \sum_{\mathbf{i}} \langle \rho f_i, \mathbf{v}_i^\alpha \rangle. \]

From (5.5), we know that the symmetric matrix \( \tilde{\zeta} \) is positive definite. Thus the Galerkin solution \( \mathbf{U} \) is defined. In engineering literature, \( \tilde{\zeta} \) is called the stiffness matrix, \( \tilde{\Phi} \) is called the load vector.

When the elasticities are uniform, we do not need to interpolate the coefficients and we have a choice in making assumptions A(iv) or A(iv)''. In this case the linear algebraic system takes the form

\( \tilde{\zeta} \mathbf{q} = \Phi \),

\[ (5.7) \]

where \( \zeta \equiv [\zeta^{\alpha\beta}], \mathbf{q} \equiv [\mathbf{q}^\beta], \Phi \equiv [\Phi^\alpha] \) with

\[ \zeta^{\alpha\beta} = \sum_{i,j,k,l} \langle C_{ijkl} \frac{\partial V_k}{\partial x_k}, \frac{\partial V_l}{\partial x_l} \rangle, \]

\[ \Phi^\alpha = \sum_{\mathbf{i}} \langle \rho f_i, \mathbf{v}_i^\alpha \rangle. \]
The symmetric matrix $\zeta$ is positive definite because of (4.11) or (4.15).

**Elastostatics (Traction BVP)**

Let $\mathcal{M}_N$ be an $N$-dimensional subspace of $W^1(\Omega)$, such that

$$\mathcal{M}_N = \text{Sp}\{y^1, \ldots, y^N\}.$$ 

We define a Galerkin approximation solution $U \in \mathcal{M}_N$ of the problem (3.15) and (4.2) by

$$- \sum_{i,j,k,l} \zeta_{ijkl} \left( \frac{\partial U_k}{\partial x_i} \frac{\partial V_l}{\partial x_j} \right) + \sum_{i=1}^m (\rho f_i)^* V_i = 0,$$

$$\forall v \in \mathcal{M}_N. \quad (5.8)$$

Here $\zeta_{ijkl}$, $(\rho f_i)^*$ are defined as before, with precautions taken on $\zeta_{ijkl}$ as in the case of homogeneous displacement BVP, namely

$$\frac{n u}{2} \| \delta^2 v \|_{W^1(\Omega)}^2 \leq \sum_{i,j,k,l} \zeta_{ijkl} \left( \frac{\partial v_k}{\partial x_i} \frac{\partial v_l}{\partial x_j} \right)$$

$$\leq \frac{3C_u}{2} \| v \|_{W^1(\Omega)}^2, \forall v \in W^1(\Omega), \text{ such that}$$
\[
\int_{\Omega} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) \, dx = 0. \tag{5.9}
\]

In deriving (5.9), we have used the assumptions A(iv) and A(v), the Classical Form of the Second Korn's Inequality, and the precautions taken in calculating \( \zeta \). We impose also additional restrictions on \( (\rho f_i)^- \) analogous to the compatibility conditions of the original problem, namely,

\[
\int_{\Omega} (\rho f_i)^- \, dx = 0,
\]

\[
\int_{\Omega} ((\rho f_i)^- x_j - (\rho f_j)^- x_i) \, dx = 0. \tag{5.10}
\]

If the integrals are evaluated exactly, then (5.10) is nothing but (3.15).

Let \( \psi \) take the form of (5.1). Then the resulting linear algebraic system is the same as that of (5.6). However in this case the symmetric matrix \( \zeta \) is only positive semi-definite, and \( \psi \) is determined uniquely only to within a rigid displacement. As pointed out by Friedrich and Keller [21], to solve the system numerically, one may follow the "advice to programmer" given by Forsythe and Wasow [19, Section 25.9, p. 376]. More general iterative methods for solving this and other singular systems have been given by Keller [29].
In order to derive specific error estimates of the Galerkin approximation of the traction BVP, we force the Galerkin solution to be unique by imposing conditions analogous to (3.16). We define the normalized Galerkin approximation solution to be the function $\Psi \in \mathcal{M}_N^{**}$ such that

$$\sum_{i,j,k,l} \epsilon^{ijkl} \frac{\partial U_k}{\partial x_l} \frac{\partial V_i}{\partial x_j} + \sum_i \langle (\rho f_i)^-, V_i \rangle = 0,$$

$$\forall \Psi \in \mathcal{M}_N^{**},$$

(5.11)

where

$$\mathcal{M}_N^{**} = \{ \Psi \in \mathcal{M}_N : \int_{\Omega} \Psi dx = 0, \int_{\Omega} \text{curl} \, \Psi dx = 0 \}.$$ 

That $\Psi$ is well-defined is a consequence of (5.9) and the Generalized Poincaré Inequality. Since we shall not implement this formulation on computer, no display of the algebraic system is given.

ELASTOSTATICS (THE EXTRA BVP)

Let $\mathcal{M}_N = \text{Sp}\{\mathcal{V}_1, \ldots, \mathcal{V}_N\}$ be an $N$-dimensional subspace of $\mathcal{W}_1^1(\Omega)(\mathcal{W}_1^1(\Omega))$ as before. We define the Galerkin approximation $\Psi \in \mathcal{M}_N$ to the Extra Elastostatic BVP (4.4) by
\[
\sum_{i,j,k,\ell} C_{i j k \ell} \frac{\partial U_k}{\partial x_i} \frac{\partial V_{\ell}}{\partial x_j} + \lambda_0 \sum_i \langle U_i, V_i \rangle = \sum_i \langle \rho f_i, V_i \rangle
\]

\[
(\sum_{i,j,k,\ell} C_{i j k \ell} \frac{\partial U_k}{\partial x_i} \frac{\partial V_{\ell}}{\partial x_j} + \sum_i \langle U_i, V_i \rangle = \sum_i \langle \rho f_i, V_i \rangle),
\]

\[\psi \in \mathcal{M}_N.\]  \hspace{1cm} (5.12)

From (4.14)((4.13)), it is clear that the Galerkin solution \( \mathcal{U} \) is well-defined. Since we shall study this problem only for deriving the \( L_2(\Omega) \)-error estimates for dynamical problems with homogeneous displacement or traction-free boundary conditions, we do not exhibit the linear algebraic system explicitly, nor consider interpolation of coefficients.

II. **Elastodynamics (Continuous Time Galerkin Approximation)**

Let \( \mathcal{M}_N = \text{Sp}\{v^1, \ldots, v^N\} \) be an \( N \)-dimensional subspace of \( \mathcal{W}^1(\Omega) \) or \( \mathcal{W}^1(\bar{\Omega}) \), when the problem is a homogeneous displacement elastodynamic problem (4.5) or a traction free elastodynamic problem (4.6) respectively. We define the continuous time Galerkin approximation \( \mathcal{U}(\cdot, t) \in \mathcal{M}_N \) to the elastodynamic problem (4.5) or (4.6) by
\[ \sum_i \rho \frac{\partial U_i}{\partial t^2} V_i + \sum_{i,j,k,l} \tilde{C}_{ijkl} \frac{\partial U_k}{\partial x_i} \frac{\partial V_i}{\partial x_j} \]

\[ - \sum_i \langle (\rho f_i) - , V_i \rangle = 0, \quad t > 0, \quad (5.13) \]

where the initial conditions for \( U \) are to be specified and where \( \tilde{\rho}, \tilde{C}_{ijkl}, \) and \( (\rho f_i)^- \) are defined as before, except that we require also \( \tilde{\rho}(x) \in \mathcal{A} \), and

\[ 0 < \frac{\eta \tilde{\rho}}{2} \leq \tilde{\rho}(x) \leq \frac{3}{2} C_{\rho}, \quad \forall x \in \Omega. \quad (5.14) \]

For elastodynamic problems with homogeneous displacement boundary conditions, we shall use assumptions A(i), A(v), and A(iv) or A(iv)', and for elastodynamic problems with traction free boundary conditions, assumptions A(i), A(v), and A(iv). In the former problem, when assumption A(iv)' is used, we assume that (4.14) remains valid after the interpolation procedure, i.e.

\[ \frac{\eta u}{2} \| y \|^2_{H^1(\Omega)} \leq \sum_{i,j,k,l} \tilde{C}_{ijkl} \frac{\partial v_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} \]

\[ + \lambda_0 \| y \|^2_{L^2(\Omega)}, \]

\[ \sum_{i,j,k,l} \tilde{C}_{ijkl} \frac{\partial v_k}{\partial x_i} \frac{\partial v_i}{\partial x_j} \leq \frac{3}{2} C_u \| v \|^2_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega), \quad (5.15) \]
while when assumption A(iv) is used we simply set \( \lambda_0 = 0 \) in the above condition. In the latter problem, we assume that (4.13) remains valid after the interpolation procedure, i.e.,

\[
\frac{\eta u}{2} \|y\|_{W^1(\Omega)}^2 \leq \sum_{i,j,k,l} \tilde{C}_{ijkl} \frac{\partial v_k}{\partial x_k} \frac{\partial v_l}{\partial x_l} + \frac{3}{2} \|y\|_{L^2(\Omega)}^2,
\]

\( \forall y \in W^1(\Omega) \). (5.16)

For isotropic materials, we have the obvious analogue of the physical interpretations for (5.5).

We shall now specify the choices of the initial conditions for the Galerkin approximation \( U \). First, we may choose \( U(\cdot,0) \) and \( \partial U/\partial t(\cdot,0) \) to be the \( \mathcal{M}_N \) interpolates of \( u_0 \) and \( \dot{u}_0 \), respectively. Second, we may choose \( U(\cdot,0) \) to be the weighted \( W^1(\Omega) \) projection of \( u_0 \) associated with the extra Elastostatic BVP, i.e., we define \( U(\cdot,0) \) by

\[
\sum_{i,j,k,l} \tilde{C}_{ijkl} \frac{\partial}{\partial x_k}(u_0 - u_0k) \frac{\partial v_i}{\partial x_l} + \sum_i \frac{\partial v_i}{\partial x_i} \dot{u}_0 = 0, \forall u \in \mathcal{M}_N,
\]

(5.17)
where \( \Lambda \) has the value \( \lambda_0 \) for displacement BC and the value 1 for traction BC, and we choose \( \partial U/\partial \tau(\cdot,0) \) to be the \( L_2(\Omega) \) projection of \( \hat{u}_0 \), i.e., we have

\[
\sum_i \left< \frac{\partial U_i}{\partial \tau}(\cdot,0), V_i \right> = \sum_i \left< \hat{u}_0i, V_i \right>, \forall V \in \mathcal{M}_N.
\]

These two possible choices will be used separately in the \( W^1(\Omega) \) and the \( L_2(\Omega) \) error estimates, respectively in Section 8.

Let

\[
U(x,t) = \sum_{\beta=1}^{N} \tilde{\sigma}_\beta(t)Y\beta(x).
\]

By substituting \( V = Y^\beta, \beta = 1, \ldots, N \) into (5.13), we obtain the following system of linear ordinary differential equations

\[
\begin{align*}
\mathcal{M} \frac{d^2 \tilde{\gamma}}{dt^2}(t) + \mathcal{C} \tilde{\gamma}(t) &= \tilde{F}(t), \quad t > 0,
\end{align*}
\]

(5.18)

where

\[
\mathcal{M} \equiv [\mathcal{M}^{\alpha\beta}], \quad \mathcal{C} \equiv [\mathcal{C}^{\alpha\beta}], \quad \tilde{F}(t) \equiv [\tilde{F}^\alpha(t)],
\]

with

\[
\mathcal{M}^{\alpha\beta} \equiv \left< \rho Y^\alpha, Y^\beta \right>, \quad \tilde{F}^\alpha(t) \equiv \sum_i \left< (\rho f_i(t)), V_i^\alpha \right>,
\]
\[
\zeta^{\alpha\beta} = \sum_{i,j,k,l} \left< \tilde{C}_{ijkl} \frac{\partial}{\partial x_k} v^\alpha_i, \frac{\partial}{\partial x_j} v^\beta_i \right>.
\]

The initial conditions for \( \tilde{\varphi} \) are chosen appropriately in accordance with the initial conditions for \( \Theta \) specified above.

**Elastodynamics (Discrete Time Galerkin Approximation)**

Let \( M \) be a positive integer and let \( \Delta t = T/M \). Then in general for any function \( \phi \) of time, \( \phi^m \) denotes the value of \( \phi \) at \( t = t^m = m\Delta t \), where \( m \) is an integer between 0 and \( M \). We use also the following notation:

\[
\phi^{m+1/2} = (\phi^{m+1} + \phi^m)/2,
\]

\[
\phi^{m,b} = b \phi^{m+1} + (1-2b) \phi^m + b \phi^{m-1},
\]

\[
\Delta_t \phi^{m+1/2} = (\phi^{m+1} - \phi^m)/\Delta t,
\]

\[
\Delta_t^2 \phi^m = (\phi^{m+1} - 2\phi^m + \phi^{m-1})/(\Delta t)^2,
\]

where \( b \) is a parameter whose value will be suitably restricted to insure stability of the numerical scheme.

We define the discrete time Galerkin approximation by discretization in time on the systems of ordinary differential equations (5.18). Specifically we replace (5.18) by the difference equations
\[ \tilde{\mathcal{M}} \Delta_t^2 \tilde{\zeta}_m + \tilde{\zeta}_m^\prime, \tilde{\zeta}_m^\prime, b = \tilde{r}_m^b, \quad 1 \leq m \leq M-1, \]  

(5.19)

and we choose \( b \) to be no less than 1/4 for stability purpose. We assume that \( \tilde{\zeta}_0 \) and \( \tilde{\zeta}_1 \) are such that the corresponding \( \tilde{y}_0 \) and \( \tilde{y}_1 \) belong to \( \mathcal{Y}_N \); the choice of \( \tilde{y}_0 \) and \( \tilde{y}_1 \) will be discussed in detail later. In inner product form (5.19) becomes

\[ \sum_i \langle \rho \Delta_t^2 U_i^m, V_i \rangle + \sum_{i,j,k,\lambda} \langle \tilde{C}_{ijk\lambda} \frac{\partial U_k^m}{\partial x_k}, \frac{\partial V_i}{\partial x_j} \rangle \]

\[ - \sum_i \langle (\rho f_i^m, b)^\prime, V_i \rangle = 0 \]

\[ 1 \leq m \leq M-1, \quad V \in \mathcal{Y}_N. \]  

(5.20)

Now we show that the discrete time Galerkin approximation \( \{U_i^m\}_{i=2}^M \) is well-defined by (5.19). First, consider the case with homogeneous displacement BC. If we use assumptions A(i), A(v), and A(iv)', then from (5.15) we know that \( \{U_i^m\}_{i=2}^M \) is well-defined by (5.19) if \( \Delta t \) is sufficiently small. Indeed, the matrix, \( b(\tilde{\zeta} + \tilde{\zeta}/(\Delta t)^2 \) is positive definite for small \( \Delta t \), e.g., when \( \Lambda = 2\lambda_0/\eta_\rho \leq b^{-1}(\Delta t)^{-2} \). If we use assumptions A(i), A(v) and A(iv) (or A(iv)' with uniform elasticity), then \( \Lambda_0 = 0 \) and \( \{U_i^m\}_{i=2}^M \) is well-defined for any positive \( \Delta t \). Next we consider the case with traction-free BC. Again we use
assumptions A(i), A(v) and A(iv). Then from (5.16) and the fact that the positive coefficient of the term $||\mathbf{y}||_{L_2(\Omega)}^2$ there is actually arbitrary (that coefficient was chosen earlier as 1 for convenience), we see that $\{\mathbf{u}_m\}_{m=2}^M$ is well-defined for any positive $\Delta t$.

We now mention the starting procedure used by Dupont in [13] to choose $\mathbf{u}_0^0$ and $\mathbf{u}_1^1$. In this procedure solutions of two Galerkin elastostatic problems are required. In order to obtain optimal $L_2(\Omega)$ error estimates in our procedures, we shall calculate the integrals in the stiffness matrices and the load vectors exactly. As before in the starting procedures, the symbol $\Lambda$ has the value $\lambda_0$ for problems with homogeneous displacement BC and it has the value 1 for problems with traction-free BC. First we project $\mathbf{u}_0$ into $\mathcal{N}$ to obtain $\mathbf{u}_0^0$, i.e., $\mathbf{u}_0^0$ is defined by (5.17). Then we define $\mathbf{u}^*$, an approximation to $\mathbf{u}(\mathbf{x}, \Delta t)$, by

$$
\mathbf{u}^* = \mathbf{u}_0 + \Delta t \mathbf{u}_0 + \left(\frac{\Delta t}{2}\right)^2 \frac{\partial^2 \mathbf{u}}{\partial t^2}(\mathbf{x}, 0),
$$

where $\frac{\partial^2 \mathbf{u}}{\partial t^2}(\mathbf{x}, 0)$ is obtained from (3.4), that is

$$
\frac{\partial^2 \mathbf{u}_i}{\partial t^2}(\mathbf{x}, 0) = \frac{1}{\rho(\mathbf{x})} \sum_{j,k,l} \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial \mathbf{u}_0}{\partial x_k}) + f_i(\mathbf{x}, 0).
$$
Next we define $\tilde{U}^1$ by

$$\sum_{i,j,k,\xi} \tilde{c}_{ijk\xi} \frac{\partial}{\partial x_\xi} (U_k^1-u_k^*), \frac{\partial V_i}{\partial x_j} >$$

$$+ \Lambda \sum_i < U_i^1-u_i^*,V_i > = 0,$$  \hspace{1cm} (5.21)

We now show that $\tilde{U}^0$ and $\tilde{U}^1$ are well-defined by (5.17) and (5.21).

We consider the case with displacement BC first.
If assumptions A(v) and A(iv) or A(iv)' (with uniform elasticity) are used, then $\Lambda = 0$ and $\tilde{U}^0$ and $\tilde{U}^1$ are well-defined. On the other hand when the body has non-uniform elasticity and satisfy the assumptions A(v) and A(iv)' only, we simply use a large value for $\Lambda$. Then $\tilde{U}^0$ and $\tilde{U}^1$ are also well-defined as guaranteed by Garding's inequality. The optimal value for $\Lambda$ is usually not readily calculable, however.
For problems with traction BC, $\tilde{U}^0$ and $\tilde{U}^1$ are well-defined as a result of the Second Korn's inequality.

There is a second starting procedure which is less cumbersome than (5.17) and (5.21). Instead of solving two Galerkin elastostatic problems, we obtain $\tilde{U}^0$ and $\tilde{U}^1$ by solving $2n$ individual scalar Laplacian problems. Define $U_i^0$ for the homogeneous displacement BC (traction-free BC) by
- \sum_{j} \left< \frac{3}{\partial x_j^J} (U_i^0 - u_0^i), \frac{3}{\partial x_j^J} \right> = 0,
\forall \, V \in \mathcal{M}_{1N_1} \subset W^1(\Omega)

(-\sum_{j} \left< \frac{3}{\partial x_j^J} (U_i^0 - u_0^i), \frac{3}{\partial x_j^J} \right> + \left< U_i^0 - u_0^i, V \right> = 0,
\forall \, V \in \mathcal{M}_{1N_1} \subset W^1(\Omega), \, i=1, \ldots, n, \quad (5.22)

and \, \, U_i^1 \, by

- \sum_{j} \left< \frac{3}{\partial x_j^J} (U_i^1 - u^*_i), \frac{3}{\partial x_j^J} \right> = 0, \forall \, V \in \mathcal{M}_{1N_1} \subset W^1(\Omega)

(-\sum_{j} \left< \frac{3}{\partial x_j^J} (U_i^1 - u^*_i), \frac{3}{\partial x_j^J} \right> + \left< U_i^1 - u^*_i, V \right> = 0,
\forall \, V \in \mathcal{M}_{1N_1} \subset W^1(\Omega), \, i=1, \ldots, n, \quad (5.23)

when \, u^* \, is \, the \, same \, as \, in \, (5.21). \, Since \, only \, the \, Laplace \, operator \, appears \, in \, the \, principal \, part \, of \, (5.22)-(5.23), \, \Psi^0 \, and \, \Psi^2 \, are \, clearly \, well-defined.

There is no proof that this second procedure will yield an optimal \, L^2(\Omega) \, error estimate for the Galerkin
approximation, however.

Laplace-Modified Galerkin Approximation
for Elastodynamic I-BVP with Homogeneous
Displacement BC

For problems with homogeneous displacement BC, we can follow Douglas and Dupont [11] to formulate a more efficient discrete Galerkin method. Replace the continuous time Galerkin approximation system (5.18) by the following difference equations:

$$\tilde{M} \Delta_t^2 \tilde{\sigma}^m + b(\Delta t) \frac{2}{\beta} \tilde{\sigma}^m + \tilde{C} \tilde{\sigma}^m = \tilde{F}^m,$$

$$1 \leq m \leq M-1,$$  \hspace{1cm} (5.24)

where $\beta = [\beta_{\alpha \beta}]$ is an $n$-block diagonal matrix with

$$\beta_{\alpha \beta} \equiv < \nabla \gamma^\alpha, \nabla \gamma^\beta >, \hspace{1cm} \alpha, \beta = 1, \ldots, N.$$

In (5.24) $b$ is a parameter whose value will be restricted to be greater than $C_u/4$ (3$C_u/8$) to guarantee stability if integrals are calculated exactly (approximately). The algebraic system to be used is

$$(\tilde{M} + b(\Delta t) \frac{2}{\beta} \Delta_t^2) \tilde{\sigma}^m = \tilde{F}^m - \tilde{C} \tilde{\sigma}^m, \hspace{0.5cm} 1 \leq m \leq M-1.$$  \hspace{1cm} (5.25)
Since this system has an \( n \)-block diagonal coefficient matrix, it is, in fact, a system of \( n \) independent subsystems. In inner product form (5.24) becomes

\[
\sum_i < \tilde{\rho} \tilde{\Delta}_t^2 U_i^m, V_i > + \sum_{i,j} \frac{\partial}{\partial x_j} \left[ (U_i^{m+1} - 2U_i^m + U_i^{m-1}) \frac{\partial V_i}{\partial x_j} \right] \\
+ \sum_{i,j,k,l} C_{ijkl} \frac{\partial U_k^m}{\partial x_l} \frac{\partial V_l^i}{\partial x_j} - \sum_i < \rho f_i^m, V_i > = 0,
\]

\[\forall \; V \in \mathcal{M}_N < V^1 \left( \Omega \right), \; 1 \leq m \leq M-1. \quad (5.26)\]

The second term on the left-hand side distinguishes this system from the previous system (5.20). Since only Laplace operator appears in the principal part of (5.26), \( \{ U^m \}_{m=2}^M \) are clearly well-defined. In order to start the process (5.26), it is necessary to specify \( U^0 \) as well as \( U^1 \). We can use the same starting procedures as in the discrete time Galerkin method.

Alternating Direction Galerkin Methods on Rectangles for Elastodynamic I-BVP with Homogeneous Displacement BC

In [11], Douglas and Dupont formulated an alternating direction Galerkin procedure for the numerical solution
of second order parabolic and (scalar) hyperbolic problems. In [15], Fairweather formulated an alternating direction procedure for plate problems. Here we shall formulate a similar procedure for the elastodynamic I-BVP. As in [11] and [15], our results are restricted to cases with Dirichlet BC for rectangular domains. In the Laplace-Modified Galerkin procedure we have reduced the coupled dynamic problem at each time step to n independent n-dimensional scalar problems. Now for rectangular regions, we can reduce further each n-dimensional scalar problem to n one-dimensional scalar problems which are simply problems of vibrating strings.

For definiteness we assume that \( \Omega \) is a unit hypercube in \( \mathbb{R}^n \), i.e., \( \Omega = [0,1]^n \). In order to use Alternating Direction Galerkin procedure, it is necessary to choose a subspace \( \mathcal{M}_{N_1} \) having a tensor product basis for each displacement component in the Galerkin process. Specifically for each \( i,j=1,\ldots,n \), let

\[
\{ V^{(ij)}_{\alpha} | \alpha = 1, \ldots, N_{ij} \} \subset W^1([0,1])
\]

be linearly independent, and put

\[
\{ (x_j^i)_{iN_1} \} \equiv \text{Sp}\{ V^{(ij)}_1, \ldots, V^{(ij)}_{N_{ij}} \} \subset W^1([0,1]).
\]
Then we define

\[ M_{iN_1} \equiv M_{iN_1}^{(x_1)} \otimes \ldots \otimes M_{iN_1}^{(x_n)} \]

\[ = \text{Sp}(V^{(il)}_1 \ldots V^{(in)}_1, \ldots, V^{(il)}_{N_1} \ldots V^{(in)}_{N_{N_1}}), \]

so that

\[ N_i = \sum_{j=1}^{N_i} N_{ij} \quad \text{(Recall that } N = \sum_{i=1}^{n} N_i). \]

(Note: \( V^{(ij)}_\alpha \) corresponds to the \( \alpha \)-th basis element of a vibrating string in the \( j \)-th direction associated with the one-dimensional scalar problem of the \( \alpha \)-th component of displacement.) Let

\[ \langle \phi, \psi \rangle_{x_j} \equiv \int_0^1 \phi \psi dx_j, \]

and for each \( i,j=1, \ldots, n, \alpha, \beta=1, \ldots, N_{ij} \), let

\[ \lambda^{(ij)} \equiv [\lambda^{(ij)}_{\alpha\beta}], \beta^{(ij)} \equiv [\beta^{(ij)}_{\alpha\beta}] \]

where

\[ \lambda^{(ij)}_{\alpha\beta} \equiv \langle V^{(ij)}_\alpha, V^{(ij)}_\beta \rangle_{x_j}, \beta^{(ij)}_{\alpha\beta} \equiv \frac{d}{dx_j} V^{(ij)}_\alpha, \]

\[ \frac{d}{dx_j} V^{(ij)}_\beta \rangle_{x_j}. \]
These matrices are clearly symmetric and positive definite.

In [11] and [15], only cases with uniform density are considered, but their analysis can be extended easily to cases with non-uniform density. For elastodynamic problems, a similar extension is somewhat more complex. We write (3.4) and (4.5) in the form

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho} \sum_{j,k,\lambda} \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + f_i, \quad (5.27)$$

and

$$\langle \frac{\partial^2 u_i}{\partial t^2}, v \rangle + \sum_{j,k,\lambda} \langle C_{ijkl} \frac{\partial u_k}{\partial x_l}, \frac{\partial v}{\partial x_j} \rangle - \langle f_i, v \rangle = 0,$$

$$v \in W^1(\Omega). \quad (5.28)$$

Then a $(\Delta t)^2$-correct perturbation of (5.28) similar to (5.26) is given by

$$\langle \Delta_t^2 u_i^m, v \rangle + b(\Delta t)^2 \sum_j \langle \frac{\partial}{\partial x_j} (\Delta_t^2 u_i^m), \frac{\partial v}{\partial x_j} \rangle$$

$$= - \sum_{j,k,\lambda} \langle C_{ijkl} \frac{\partial u_k^m}{\partial x_l}, \frac{\partial}{\partial x_j} (\frac{v}{\rho}) \rangle$$

$$+ \langle f_i^m, v \rangle + O(\Delta t)^2, v \in W^1(\Omega).$$
We shall now develop the Alternating Direction Galerkin procedure. First, we replace $u_i^m$ in (5.29) by

$$u_i^m(x) = \sum_{a_1=1}^{N_{il}} \cdots \sum_{a_n=1}^{N_{in}} c^{(i)m}_{a_1 \cdots a_n} V^{(il)\alpha_1}(x_1) \cdots V^{(in)\alpha_n}(x_n),$$

and $v$ similarly by a basis element among

$$V^{(il)\beta_1} \cdots V^{(in)\beta_n} \text{ with } 1 \leq \beta_j \leq N_{ij}, j=1, \ldots, n.$$ 

Then (5.29) yields a linear algebraic system for the vector

$$\sigma^{(i)m} = (\sigma_{a_1 \cdots a_n}^{(i)m})^T.$$ 

The left-hand side of the linear system has the form

$$[\mathcal{A}^{(il)} \otimes \cdots \otimes \mathcal{A}^{(in)} + b(\Delta t)^2 \otimes (\mathcal{A}^{(il)} \otimes \cdots \otimes \mathcal{A}^{(i,n-1)} \otimes \mathcal{B}^{(in)})] \cdot (\Delta t^2 \sigma^{(i)m}).$$

Here we have used the fact

$$< \frac{\partial}{\partial x_j} (V^{(il)\alpha_1} \cdots V^{(in)\alpha_n}), \frac{\partial}{\partial x_j} (V^{(il)\beta_1} \cdots V^{(in)\beta_n}) >$$
\[ \begin{align*}
&= \left< V^{(il)a_1}, V^{(il)b_1} \right> x_1 \cdots \left< \frac{d}{dx_j} V^{(ij)a_j}, \frac{d}{dx_j} V^{(ij)b_j} \right> x_j \\
&\cdots \left< V^{(in)a_n}, V^{(in)b_n} \right> x_n \\
&= \mathcal{A}^{(il)}a_1 b_1 \cdots \mathcal{B}^{(ij)} a_j b_j \cdots \mathcal{A}^{(in)}a_n b_n, \text{ etc.}
\end{align*} \]

In order to factorize the coefficient matrix of \( \Delta t^2 g_{(i)m} \), we add some higher order terms in \( b(\Delta t)^2 \) in such a way that the new left-hand side of the linear system becomes

\[ \left[ (\mathcal{A}^{(il)} + b(\Delta t)^2 \mathcal{B}^{(il)}) \otimes \cdots \otimes \mathcal{A}^{(in)} \right. \\
\left. + b(\Delta t)^2 \mathcal{B}^{(in)}) \right] (\Delta t^2 g_{(i)m}). \]

This perturbation does not increase the order of the original local discretization error, but the new coefficient matrix is now a product of \( n \) positive definite and symmetric matrices.

The right-hand side of the linear system has the form

\[ F^{(i)m} = (F_{1}^{(i)m}, \ldots, F_{a_1 a_n}^{(i)m}, \ldots, F_{N_{i1} \cdots N_{in}}^{(i)m})^T. \]

We define \( F_{z}^{(i)m} \) by
\[
\sum_{a_1 \ldots a_n} F_{(i)m} a_{\alpha_1 \ldots a_n} = - \sum_{j,k,\xi} \sum_{\beta_1 = 1}^{N_i} \sum_{\beta_n = 1}^{N_i} c_{\beta_1 \ldots \beta_n} (k \beta_1 \ldots \beta_n) V_{(k\xi) \beta_\xi \ldots \beta_n},
\]
\[
> C_{ijk\xi} V_{ij} \ldots \frac{d}{dx_\xi} V_{kn} \beta_n,
\]
\[
\frac{\partial}{\partial x_j} (\frac{1}{\rho} V_{il} a_{\alpha_1 \ldots \alpha_n}) >
\]
\[
+ \sum_{i} f_{i}^m, V_{i} a_{\alpha_1 \ldots \alpha_n} >.
\]

In inner product forms, \( \{ y^m \}_{m=2}^M \) are then defined when \( n=2 \) by

\[
\sum_{i} < \Delta_t^2 U_i^m, V_i > + b (\Delta t)^2 \sum_{i,j} \frac{\partial}{\partial x_j} (\Delta_t^2 U_i^m), \frac{\partial V_i}{\partial x_j} >
\]
\[
+ b^2 (\Delta t)^4 \frac{\partial}{\partial x_1 \partial x_2} (\Delta_t^2 U_i^m), \frac{\partial}{\partial x_1 \partial x_2} V_i >
\]
\[
+ \sum_{i,j,k,\xi} C_{ijk\xi} \frac{\partial U_i^m}{\partial x_\xi}, \frac{\partial}{\partial x_j} \left( \frac{V_i}{\rho} \right) >
\]
\[
- \sum_{i} f_{i}^m, V_i > = 0, \forall V \in \mathcal{M}, 1 < m < M - 1, (5.30)
\]

and when \( n = 3 \) by
\[ \sum_i \langle \Delta t U_i^m, V_i \rangle + b(\Delta t)^2 \sum_{i,j} \frac{\partial}{\partial x_j}(\Delta t U_i^m), \frac{\partial V_i}{\partial x_j} \]

\[ + b^2(\Delta t)^4 \sum_i \{ \frac{\partial^2}{\partial x_1^2 \partial x_2^2}(\Delta t U_i^m), \frac{\partial^2}{\partial x_1^2 \partial x_2^2} V_i \} \]

\[ + \frac{\partial^2}{\partial x_2^2 \partial x_3^2}(\Delta t U_i^m), \frac{\partial^2}{\partial x_2^2 \partial x_3^2} V_i \} \]

\[ + \frac{\partial^2}{\partial x_3^2 \partial x_1^2}(\Delta t U_i^m), \frac{\partial^2}{\partial x_3^2 \partial x_1^2} V_i \} \}

\[ + b^3(\Delta t)^6 \sum_i \left\{ \frac{\partial^3}{\partial x_1^3 \partial x_2 \partial x_3}(\Delta t U_i^m), \frac{\partial^3}{\partial x_1^3 \partial x_2 \partial x_3} V_i \right\} \]

\[ + \sum_{i,j,k,l} C_{ijkl} \frac{\partial U_k^m}{\partial x_j}, \frac{\partial (\frac{V_i}{\rho})}{\partial x_j} \right\} - \sum_i \langle f_i, V_i \rangle = 0 \]

\[ \forall \forall \in \mathcal{M}_N, \quad 1 \leq m \leq M-1. \] (5.31)

In (5.30) or (5.31), the value of \( b \) is restricted to be greater than \( C_u/(4\eta_p) \) so as to guarantee stability.

Like all discrete Galerkin procedures, this process needs to be properly initiated. We define \( \Psi^0 \) and \( \Psi^1 \) in the following way: we choose \( \Psi^0 \) to be the Hermite interpolate of \( u^0 \), and we choose \( \Psi^1 \), when \( n = 2 \), by
\[
\langle y^1 - y^0, v \rangle + (\Delta t) \langle v(y^1 - y^0), \nabla v \rangle \\
+ (\Delta t)^4 \left\langle \frac{\partial^2}{\partial x_1 \partial x_2} (y^1 - y^0), \frac{\partial^2}{\partial x_1 \partial x_2} v \right\rangle \\
= \langle u^*, v \rangle + (\Delta t) \langle \nabla u^*, \nabla v \rangle \\
+ (\Delta t)^4 \left\langle \frac{\partial^2}{\partial x_1 \partial x_2} u^*, \frac{\partial^2}{\partial x_1 \partial x_2} v \right\rangle , \quad (5.32)
\]

and when \( n = 3 \) by

\[
\langle y^1 - y^0, v \rangle + (\Delta t) \langle v(y^1 - y^0), \nabla v \rangle \\
+ (\Delta t)^4 \left\langle \frac{\partial^2}{\partial x_1 \partial x_2} (y^1 - y^0), \frac{\partial^2}{\partial x_1 \partial x_2} v \right\rangle \\
+ \left\langle \frac{\partial^2}{\partial x_2 \partial x_3} (y^1 - y^0), \frac{\partial^2}{\partial x_2 \partial x_3} v \right\rangle \\
+ \left\langle \frac{\partial^2}{\partial x_3 \partial x_1} (y^1 - y^0), \frac{\partial^2}{\partial x_3 \partial x_1} v \right\rangle \\
+ (\Delta t)^8 \left\langle \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} (y^1 - y^0), \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} v \right\rangle \\
= \langle u^*, v \rangle + (\Delta t) \langle \nabla u^*, \nabla v \rangle 
\]
\[ + (\Delta t)^4 \left\{ \frac{a^2}{\partial x_1 \partial x_2} u^* - \frac{a^2}{\partial x_1 \partial x_2} V \right\} \]

\[ + \frac{a^2}{\partial x_2 \partial x_3} u^*, \frac{a^2}{\partial x_2 \partial x_3} V > \]

\[ + \left\{ \frac{a^2}{\partial x_1 \partial x_1} u^*, \frac{a^2}{\partial x_1 \partial x_1} V \right\} \]

\[ + (\Delta t)^8 \left\{ \frac{a^3}{\partial x_1 \partial x_1 \partial x_3} \frac{a^3}{\partial x_1 \partial x_2 \partial x_3} > , \right\} \]

(5.33)

where

\[ u^* = (\Delta t) \dot{u}_{0i} + \frac{(\Delta t)^2}{2 \rho} \left( \sum_{j,k,l} \frac{\partial}{\partial x_j} \left( C_{ijk\ell} \frac{\partial u_{0k}}{\partial x_\ell} + \rho f_i \right) \right) \]

Since (5.32) and (5.33) generally cannot be factorized, each \( U_i^1 \) must be solved by iteration as in [11].

III. Coupled Dynamic Thermoelastic Problems

(Continuous Time Galerkin Approximation)

For simplicity we shall consider cases using assumptions A(i), A(ii), A(iii), A(iv), and A(v). For displacement BC problems, assumption A(iv) may be relaxed to A(iv)!

Let \( \mathcal{M}_N = \text{Sp}\{V^1, \ldots, V^N\} \) be an N-dimensional subspace of \( W^1(\Omega) \) or \( W^1(\Omega) \) as before. Let \( \mathcal{M}_N^\theta = \text{Sp}\{\phi^1, \ldots, \phi^N\} \) be
an \( N_e \)-dimensional subspace of \( W^1_0(\Omega) \) or \( W^1(\Omega) \). We define the continuous time Galerkin approximation \((U, \Theta) \in M_N \times M_{N_\Theta} \) for the thermoelastic problem (4.7) by

\[
\sum_i \langle \tilde{\rho} \frac{\partial^2 U_i}{\partial t^2}, V_i \rangle + \sum_{i,j,k,l} \langle \tilde{C}_{ijkl} \frac{\partial U_k}{\partial x_j}, \frac{\partial V_i}{\partial x_j} \rangle \\
- \sum_{i,j} \langle \tilde{m}_{ij} \Theta, \frac{\partial}{\partial x_j} V_i \rangle - \sum_i \langle (\rho f_i)^\sim, V_i \rangle = 0, \quad t > 0,
\]

\[
\langle (\rho c_D)^\sim \frac{\partial \Theta}{\partial t}, \phi \rangle + c_C \sum_{i,j} \langle \tilde{m}_{ij} \Theta, \frac{\partial^2 U_i}{\partial x_j \partial t}, \phi \rangle \\
+ \sum_{i,j} \langle \tilde{K}_{ij} \frac{\partial \Theta}{\partial x_j}, \frac{\partial \phi}{\partial x_i} \rangle - \langle (\rho c_D r)^\sim, \phi \rangle = 0, \quad t > 0,
\]

\[
\sum_{i,j,k,l} \langle \tilde{C}_{ijkl} \frac{\partial^2 U_i}{\partial x_j}, \frac{\partial^2 U_k}{\partial x_l}, V_i \rangle \\
+ \Lambda \sum_i \langle U_i(\cdot,0)-u_{0i}, V_i \rangle = 0,
\]

\[
\sum_i \langle \frac{\partial U_i}{\partial \tilde{t}}(\cdot,0), V_i \rangle = \sum_i \langle \tilde{u}_0, V_i \rangle,
\]

\[
\langle \Theta(\cdot,0), \phi \rangle = \langle \Theta_0, \phi \rangle,
\]

\[
(5.34)
\]

for all \((V, \phi) \in M_N \times M_{N_\Theta} \). Here \( \tilde{\rho}, \tilde{C}_{ijkl}, (\rho f_i)^\sim, \tilde{m}_{ij}, \)
\((\rho c_D)^{-}, \tilde{K}_{ij}, (\rho c_D^{-})^{-}\) are defined as before except that we also require

\[
\frac{\eta^g}{2} \| \nabla \phi \|_{L^2(\Omega)}^2 \leq \langle \tilde{K} \nabla \phi, \nabla \phi \rangle \leq \frac{3}{2} C_\Theta \| \nabla \phi \|_{L^2(\Omega)}^2,
\]

\(\forall \phi \in W^1(\Omega),\) \hspace{1cm} (5.35)

and 1 has the value \(\lambda_0\) for displacement BC and the value 1 for traction BC.

Let

\[
\bar{U}(\bar{x},t) = \sum_{\beta=1}^{N} \tilde{c}_{\beta}(t) \tilde{V}^\beta(\bar{x}),
\]

and

\[
\bar{\Theta}(\bar{x},t) = \sum_{a=1}^{N_\Theta} \tilde{\omega}_{a}(t) \tilde{\gamma}^a(\bar{x}).
\]

Substituting \(\bar{V} = \bar{V}^\gamma, \gamma = 1, \ldots, N, \tilde{\gamma} = \tilde{\gamma}^a, a = 1, \ldots, N_\Theta\), into (5.34), we obtain

\[
\bar{\gamma} \frac{d^2 \bar{\sigma}}{dt^2}(t) + \bar{\sigma} \frac{d\bar{\omega}}{dt}(t) - \bar{\gamma} T \bar{\omega}(t) = \bar{F}(t), \ t > 0,
\]

\[
\bar{\omega} \frac{d\bar{\omega}}{dt}(t) + \Theta \bar{\gamma} \frac{d\bar{\omega}}{dt}(t) + \bar{K} \bar{\omega}(t) = \bar{J}(t), \ t > 0,
\]
\[ (\mathcal{C} + \Lambda \mathcal{A}) \tilde{\omega}(0) = \omega_0, \]
\[ \dot{\mathcal{A}} \frac{\partial \sigma}{\partial t}(0) = \dot{\omega}_0, \]
\[ \mathcal{D} \tilde{\omega}(0) = \omega_0, \]

(5.36)

where

\[ \mathcal{H} = [\mathcal{H}^{\alpha\gamma}], \Xi = [\Xi^{\alpha\beta}], \mathcal{K} = [\mathcal{K}^{\alpha\beta}], \]
\[ \mathcal{J} = [\mathcal{J}^{\alpha}], \mathcal{A} = [\mathcal{A}^{\alpha\gamma}], \sigma_0 = [\sigma_0], \]
\[ \dot{\omega}_0 = [\dot{\omega}_0], \mathcal{E} = [\mathcal{E}^{\alpha\beta}], \omega_0 = [\omega_0^\alpha], \]

with

\[ \mathcal{H}^{\alpha\gamma} = \sum_{i,j} <\tilde{m}_{ij} \phi^\alpha, \frac{3}{\partial x_j} V_i^\gamma >, \Xi^{\alpha\beta} = \langle \rho c_D \phi^\alpha \phi^\beta \rangle, \]
\[ \mathcal{K}^{\alpha\beta} = \langle \tilde{\kappa} \phi^\alpha \phi^\beta \rangle, \mathcal{J}^{\alpha} = \langle \rho c_D \phi^\alpha \phi^\alpha \rangle, \]
\[ \mathcal{A}^{\alpha\gamma} = \langle \Xi^{\alpha\gamma} \Xi^{\alpha\gamma} \rangle, \mathcal{K}^{\alpha\beta} = \langle \phi^\alpha \phi^\beta \rangle, \]
\[ \sigma_0^\gamma = \sum_{i,j,k,\lambda} C_{ijk\lambda} \frac{3}{\partial x_k} u_0 k \frac{3}{\partial x_j} V_i^\gamma >, \]
\[ + \Lambda \sum_i <u_0, V_i^\gamma >, \]
\[ \ddot{\gamma}_0 = \langle \dot{u}_0, \dot{v}_0 \rangle, \quad \omega_0 = \langle \dot{\theta}_0, \dot{\phi}_0 \rangle, \]

\[ \alpha, \beta = 1, \ldots, N_\theta, \quad \gamma, \gamma' = 1, \ldots, N. \]

Note that \( \tilde{\mathcal{K}} \) is positive definite.

That \( (\tilde{\gamma}, \tilde{\omega}) \) is well-defined by (5.36) can be seen readily by rewriting (5.36) as a first order system of ordinary differential equations.

**Thermoelastic I-BVP (Discrete Time Galerkin Approximation)**

Since the energy equation is first order in time, we have a choice between using the Crank-Nicolson [10] and the three level [14] difference equations. Here we can use the three-level equations.

In the discrete-time Galerkin Method for the elastodynamic I-BVP, we introduced a parameter \( b \). Here we need a parameter \( b_u \) for the displacement and another parameter \( b_\theta \) for the temperature. For \( b_u \geq 1/4, b_\theta \geq 1/4 \) it can be shown that the discrete Galerkin scheme is stable independent of \( \Delta t \) and \( \mathcal{M}_u x \mathcal{M}_\theta \). In practice one always chooses \( b_u = b_\theta = 1/4 \) to minimize the truncation error. Hence there is no great loss of generality by simply taking \( b_u \) and \( b_\theta \) a common value \( b \). This choice will simplify somewhat
our analysis later.

As we shall see in Section 9, the $W^1$-norms of the temperature errors are not $\|v(\theta - e)^m\|$ but $\|v(\theta - e)^m,^b\|$. Hence we approximate the term

$$< \dot{m}_{ij}\theta^m, \frac{\partial}{\partial x_j} V_i > \text{ in (5.34)}_1 \text{ by}$$

$$< \dot{m}_{ij}\theta^m,^b, \frac{\partial}{\partial x_j} V_i > .$$

Now we are ready to define the discrete time Galerkin approximate solution $(\mathbf{y}_m, \theta^m)_m^M$. Suppose that $(\mathbf{y}_m, \theta^m)_m^1$ are given. Then we calculate $(\mathbf{y}_m, \theta^m)_m^2$ by

$$\sum_i < \dot{\Delta}^2 t^m \mathbf{v}_i , V_i > + \sum_{i,j,k,l} < \widehat{C}_{ijkl} \frac{\partial \mathbf{y}_k^m,^b}{\partial x_k}, \frac{\partial V_i}{\partial x_j} >$$

$$- \sum_{i,j} < \dot{m}_{ij}\theta^m,^b, \frac{\partial}{\partial x_j} V_i >$$

$$- \sum_i < (\rho f_i^m,^b), V_i > = 0 ,$$

$$< (\rho c_D) \frac{\theta^{m+1} - \theta^{m-1}}{2\Delta t}, \phi > + \sum_{i,j} < \tilde{K}_{ij} \frac{\partial}{\partial x_j} \theta^m,^b, \frac{\partial}{\partial x_i} \phi >$$
\[ + \frac{\theta C}{2} \sum_{i,j} <m_{ij} \frac{\partial}{\partial x_j} (\Delta_t u_i^{m+1/2} + \Delta_t u_i^{m-1/2}), \phi > \]

\[ - <(\rho c_D r)^{-m}, \phi > = 0, \]

\[ \forall \psi \in \mathcal{M}_N, \phi \in \mathcal{M}_N, 1 \leq m \leq M-1. \quad (5.37) \]

To prove that \( \{u^m, \phi^m\}_{m=2}^M \) are well-defined by (5.37), it suffices to show that the corresponding algebraic system has the trivial solution \( u_i^{m+1} = \phi^{m+1} = 0 \) only when \( u_i^m = u_i^{m-1} = \phi^m = \phi^{m-1} = f^m, b = x^m = 0 \). Let \( \phi = \phi^m, b \) in (5.37)_1, and choose \( \psi = 1/2(\Delta_t u_i^{m+1/2} + \Delta_t u_i^{m-1/2}) \) in (5.37)_2. We multiply (5.37)_1 by \( \theta C \) and add the result to (5.37)_2 obtaining

\[ \frac{\theta C}{2} \left| \frac{\partial}{\partial x_j} (\Delta_t u_i^{m+1/2}) \right|^2 + \frac{\theta C b}{2 \Delta_t} \sum_{i,j,k,l} \]

\[ < \tilde{C}_{ijkl} \frac{\partial u_k^{m+1}}{\partial x_l}, \frac{\partial u_i^{m+1}}{\partial x_j} > \]

\[ + \frac{b}{2 \Delta t} \left| (\rho c_D)^{-1/2} \phi^{m+1} \right|^2 \]

\[ + b^2 < \tilde{K}_{ij} \frac{\partial}{\partial x_j} \phi^{m+1}, \frac{\partial}{\partial x_i} \phi^{m+1} > = 0. \]

Since \( \tilde{C}_{ijkl} \) and \( \tilde{K}_{ij} \) are positive definite over symmetric...
matrices and vectors respectively, it follows that $U_1^{m+1} = \varepsilon_1^{m+1} = 0$.

Now it remains to define $\{U_0^m, \varepsilon_0^m\}_{m=0}^1$. First, we use (5.17) and (5.21) to calculate $U_0^0$ and $U_1^1$. (In calculating $U_1^1$, we use (3.12) to evaluate $\partial^2 y/\partial t^2(\cdot,0)$.) Next we define $\varepsilon_0$ and $\varepsilon_1$ as the Hermite interpolates of $\varepsilon^0$ and $\varepsilon^*$ respectively, where

$$\varepsilon^* = \varepsilon(x,0) + \Delta t \frac{\partial \varepsilon}{\partial t}(x,0)$$

with $\partial \varepsilon/\partial t(x,0)$ evaluated from the differential equation (3.12)2.

Thermoelastic I-BVP (Discrete Time Galerkin Approximation with the Thermal and Mechanical Subproblems Computationally Decoupled)

The discrete Galerkin method presented so far is applicable to any of the four possible combinations of boundary conditions. If either the temperature or the displacement vector satisfies homogeneous Dirichlet BC, then there are other alternatives to (5.37). For example the Crank-Nicolson formulation can be used for the energy equation, but we shall not consider this alternative here. Rather we shall modify the discretization of the thermal-mechanical coupling terms in (5.37) in such a way that
the energy equation and the momentum equation become computationally decoupled at every time step. Such a decoupling is desirable, since the problem is reduced to two smaller subproblems, and we can use the iterative schemes for elasticity and heat conduction to solve them. In particular, Laplace-Modified (and Alternating Direction) Galerkin methods for the mechanical subproblem with displacement BC (and with rectangular domains) can be used. Of course, we must decide which one of $Y^{m+1}$, $\theta^{m+1}$ is calculated first. If $Y^{m+1}$ is calculated first, then we need to extrapolate $\theta^{m-1,b}$, $\theta^{m-2,b}$ and $\theta^{m-3,b}$ to get a $(\Delta t)^2$-accurate approximation for $\theta^{m,b}$; the implementation of this scheme is very messy. However, if $\theta^{m+1}$ is calculated first, we need only to replace $\partial U_i / \partial t(t^m)$ in the coupling term of the energy equation in (5.34) by the $(\Delta t)^2$-accurate extrapolate $2\Delta t U_i^{m-1/2} - \Delta t U_i^{m-3/2}$. Hence we calculate $\theta^{m+1}$ first at each time step.

We define the discrete Galerkin approximate solution $\{U^m, \theta^m\}_{m=0}^M$ in this order $U^2$, $\theta^2$, $U^3$, $\theta^3$, ..., $U^M$, $\theta^M$ by

$$\sum_i \left< \Delta_t^2 U_i^m, V_i \right> + \sum_{i,j,k,l} \left< \Delta_t \frac{\partial U_i^m}{\partial x_k}, \frac{\partial V_j}{\partial x_l} \right>$$
\[- \sum_{i,j} \langle \tilde{m}_{ij} \phi^{m,b}, \frac{\partial}{\partial x_j} V_i \rangle\]

\[- \sum_i \langle (\rho f_i^{m,b})^{-m}, V_i \rangle = 0, \ 1 \leq m \leq M-1, \ \forall \in M_N,\]

\[\langle \rho c_D \frac{\phi^{m+1} - \phi^{m-1}}{2\Delta t}, \phi \rangle + \sum_{i,j} \langle \tilde{k}_{ij} \frac{\partial}{\partial x_j} \phi^{m,b}, \frac{\partial}{\partial x_i} \phi \rangle\]

\[+ \frac{\theta_C}{2} \sum_{i,j} \langle \tilde{m}_{ij} [3\Delta t (\frac{\partial}{\partial x_j} U_i)]^{m-1/2} \]

\[- \Delta t (\frac{\partial}{\partial x_j} U_i)^{m-3/2}, \phi \rangle\]

\[- \langle (\rho c_D r)^{-m}, \phi \rangle = 0, \ 2 \leq m \leq M-1, \ \in M_N.\]

We define \([U^m, \phi^m]_{m=0}^1\) as before. \(\phi^2\) is defined as the Hermite interpolate of \(\phi^{**}\) where

\[\phi^{**} = \phi(x,0) + 2\Delta t \frac{\partial \phi}{\partial t}(x,0).\]

When displacement BC (and rectangular domain) is used, the formulations of the Laplace-Modified (and Alternating Direction) Galerkin Methods for the mechanical subproblem are similar to those of the elastodynamic I-BVP.
6. APPROXIMATING SUBSPACE

Most of our a priori estimates which we shall derive apply to any finite dimensional approximating subspaces. In order to make the corresponding algebraic problems computationally simple, we shall, however, use only subspaces which are local, i.e. subspaces whose basic functions are nonzero only over a small portion of $\Omega$. Readers who are interested in knowing possible interpretations of our results within the context of certain non-local approximating subspaces such as global multivariate trigonometric or polynomial approximations should consult Schultz's paper [47].

We shall subdivide $\Omega$ into some finite elements say triangles, for a planar domain. Engineers have been remarkably ingenious in using approximating subspaces which are formed by piecewise polynomials of degree $j-1$ and which possess convenient local bases. It is usually left to the numerical analysts to prove that these finite-element spaces of degree $j-1$ actually achieve approximations for an arbitrary smooth function $\phi$ to within error of order $h^j$ and for its derivatives of order $m$ to within error of order $h^{j-m}$, where $h$ is the mesh size, i.e., the maximum diameter of the element used.

6-1
Whenever possible, we shall exhibit in our a priori estimates the asymptotic rate of convergence of the approximating scheme as the mesh size and the time increment (if applicable) tend to zero under the assumption that the approximating subspaces have good approximating properties. Specifically we require that \( \mathcal{N}_N \) be an \( S_{1,k}^0(\Omega) \)-space or an \( S_{1,k}^h(\Omega) \)-space. These spaces are defined as follows [3]:

Let \( 0 < h < 1 \), and suppose that \( k \) is an integer greater than 1. Then \( S_{1,k}^h,0(\Omega) \) denotes a finite dimensional subspace of \( \tilde{W}^1(\Omega) \) which satisfies the following condition: There exists a constant \( C \), such that for any \( \phi \in \tilde{W}^j(\Omega) \cap \tilde{W}^1(\Omega) \) we can find a certain \( \tilde{\phi} \in S_{1,k}^h(\Omega) \) satisfying the inequality

\[
||\phi - \tilde{\phi}||_{\tilde{W}^\varepsilon(\Omega)} \leq C h^{j-\varepsilon}||\phi||_{\tilde{W}^j(\Omega)} \tag{6.1}
\]

for all integers \( j \) and \( \varepsilon \) with \( \varepsilon = 0, 1 \) and \( \varepsilon \leq j \leq k \). Likewise \( S_{1,k}^h(\Omega) \) denotes a finite dimensional subspace of \( \tilde{W}^1(\Omega) \) such that for any \( \phi \in \tilde{W}^j(\Omega) \) there exists a certain \( \tilde{\phi} \in S_{1,k}^h(\Omega) \) satisfying (6.1) for all integers \( j \) and \( \varepsilon \) with \( \varepsilon = 0, 1 \) and \( \varepsilon \leq j \leq k \).

Three main categories of finite elements are used in Galerkin approximation of linear elasticity or linear thermoelasticity problems: \( n \)-simplex elements, curved elements,
and n-dimensional rectangular elements. We shall now consider these in detail.

A. n-Simplex Elements

For problems with Dirichlet BC, we cover \( \Omega \) by a finite number of non-degenerate n-simplices, i.e. intervals \( (n=1) \), triangles \( (n=2) \), and tetrahedrons \( (n=3) \). Each pair of simplices share a vertex, an edge, a face, or are disjoint. For problems with natural BC, n-Simplex elements are often used for polyhedral domains; we simply subdivide the domain into a finite number of non-overlapping n-simplices.

For more detail about the use of n-simplex elements, we refer the reader to the works of Ciarlet and Raviart \[6\], and Strang \[48\]. \[6\] adopts a powerful coordinate-free approach, so that its results are not restricted to Cartesian coordinates only. In this thesis, however, we shall restrict ourselves to using Cartesian coordinates. \[6\] also contains a big bibliography on this subject.

We refer the reader to \[6\] for the general definition and discussion of the Lagrangian and Hermite interpolation problems. Here we shall present from \[6\] some examples which are useful for our purpose. For some of our Galerkin procedures, we need to compute explicitly interpolates of
material coefficients, loading functions, initial data, etc. These interpolates pose no difficulty as long as the function to be interpolated are defined, such as in the case when no derivative of order $k-(n/2)$ or larger is involved for functions belonging to $W^k(\Omega)$. The error estimates for these cases are obtained from [6].

Since an arbitrary function $\phi \in W^k(\Omega)$, such as the solution of the exact problems, need not possess derivatives of order greater than $k-n/2$, sometimes it is necessary to smooth or mollify $\phi$ to obtain smoothened function for which interpolation is meaningful. Fortunately, in our analysis such smoothened functions are needed only as upper bounds of the majorizing side of our error estimate, so that they need not be computed explicitly. The error estimates for these cases are obtained from [6] and [48].

The error estimate (6.1) can be established if we can obtain some local error estimates for the interpolation problems of each element, and then make use of some uniformity assumption about the shapes and the relative sizes of the elements. We give some examples here, using their designations in [6]. First we define some notation. Let $K$ be a non-degenerate $n$-simplex of $R^n$ with vertices $a_i$, $1 \leq i \leq n + 1$. Then we denote by $a_{ij} = a_{ji}$ the mid-point of the edge joining the vertices $a_i$ and $a_j$, $i \neq j$. We denote by $a_{ijk} = a_{jik} = \ldots = a_{kji}$ the barycentre of each
triangle with vertices $a_i$, $a_j$, and $a_k$, with $i \neq j$, $j \neq k$, $k \neq i$, and we put

$$h = \text{diameter of } K,$$

$$h' = \sup\{\text{diameter of the sphere contained in } \Omega\}.$$

**Type I (Type II): Lagrangian Interpolation**

Given a function $\phi$ defined on $\{a_i\}_{i=1}^{n+1}$ (and $\{a_{ij}\}_{1 \leq i < j \leq n+1}$), we say that $\tilde{\phi}$ is its interpolating polynomial of type I (resp. of type II), if it is the polynomial of degree $\leq 1$ (degree $\leq 2$) verifying

$$\tilde{\phi}(a_i) = \phi(a_i), \quad 1 \leq i \leq n+1$$

(and $\tilde{\phi}(a_{ij}) = \phi(a_{ij}), \quad 1 \leq i < j \leq n+1$).

We construct such polynomials for each simplex of the given decomposition of $\Omega$ and let $M_N$, our approximating subspace, be the space of functions whose restrictions on each simplex are those polynomials. If $\phi \in W^2(K)$ ($\phi \in W^3(K)$), then $\phi \in C(K)$ and the interpolate $\tilde{\phi}$ is defined. Further, we have

$$||\phi - \tilde{\phi}||_{W^m(K)} \leq C ||\phi||_{W^2(K)} \frac{h^2}{h^m}, \quad 0 \leq m \leq 2 \quad (6.2)$$
\[ \left\| \phi - \tilde{\phi} \right\|_{W^m(K)} \leq C \left\| \phi \right\|_{W^3(K)} \frac{h^3}{h^m}, \quad 0 \leq m \leq 3, \quad (6.3) \]

where \( C \) depends on \( n \) and on the type of the interpolations used.

**Type III: Hermite Interpolation**

Given a function \( \phi \) defined on \( \{a_i\}_{i=1}^{n+1} \) and \( \{a_{ijk}\}_{i,j,k=1}^{n+1} \), \( i,j,k \) distinct, we say that \( \tilde{\phi} \) is its interpolating polynomial of type III if it is the polynomial of degree \( \leq 3 \) verifying

\[ \tilde{\phi}(a_i) = \phi(a_i), \quad 1 \leq i \leq n+1, \]

\[ \tilde{\phi}(a_{ijk}) = \phi(a_{ijk}), \text{ for all distinct} \]

\[ i,j,k \text{ with } 1 \leq i,j,k \leq n+1, \]

\[ \frac{\partial \tilde{\phi}}{\partial x_j}(a_i) = \frac{\partial \phi}{\partial x_j}(a_i), \quad 1 \leq i \leq n+1, \quad 1 \leq j \leq n. \]

The approximating subspace \( M_N \) is constructed in the same way as before. If \( \phi \in W^4(K) \), then \( \phi \in C^2(K) \) and the interpolate \( \tilde{\phi} \) of \( \phi \) is defined. Further, we have

\[ \left\| \phi - \tilde{\phi} \right\|_{W^m(K)} \leq C \left\| \phi \right\|_{W^4(K)} \frac{h^4}{h^m}, \quad 0 \leq m \leq 4, \quad (6.4) \]
where \( C \) depends on \( n \).

**Type III': Hermite Interpolation**

Type III' differs from Type III in that the value of the interpolating polynomial \( \hat{\phi} \) at \( a_{ijk} \) will no longer be considered as a free parameter; instead we require \( \hat{\phi}(a_{ijk}) \) to have the value

\[
\frac{1}{3}(\phi(a_i) + \phi(a_j) + \phi(a_k))
\]

\[
- \frac{1}{6}(\nabla \phi(a_i) \cdot (a_i - a_{ijk}) + \nabla \phi(a_j) \cdot (a_j - a_{ijk}) + \nabla \phi(a_k) \cdot (a_k - a_{ijk}))
\]

The approximating subspace \( M_N \) is then constructed in the same way as before. If \( \phi \in W^3(K) \), then \( \phi \in C(K) \) when \( n = 2 \) or \( 3 \) and \( \phi \in C^2(K) \) when \( n = 1 \). Hence when \( n = 1 \), the usual interpolate \( \hat{\phi} \) is defined. But when \( n = 2 \) or \( 3 \), we need to smoothen \( \phi \) first and then get the interpolate \( \tilde{\phi} \) of the smoothened function. (Here smoothing of \( \phi \) is possible, since we have assumed that \( \Omega \) satisfies the restricted cone condition. For more detail, see [48, p. 95]). Whatever \( n \) may be, if \( \phi \in W^3(K) \), then

\[
||\phi - \tilde{\phi}||_{W^m(K)} \leq C ||\phi||_{W^3(K)} \frac{h^3}{m}, \quad 0 \leq m \leq 4,
\]  

(6.5)
where $C$ depends on $n$.

Having considered local estimates for interpolation problems of each element, we now decompose $\tilde{\Omega}$ into a finite number of $n$-simplices $K^\alpha$. To each such decomposition of $\tilde{\Omega}$, we define two parameters

$$h = \max_{\alpha} \{\text{diameter of } K^\alpha\}$$

$$h' = \min_{\alpha} \{\text{diameter of the inscribed sphere of } K^\alpha\}.$$

Then we say that a family of such decompositions is regular if there exists a constant $C > 0$ such that $h_i \leq C h_1$ for all $i$, where $i$ denotes the index of the family. For such a family, we can rewrite (6.2)-(6.5) in the following form:

$$||\phi - \tilde{\phi}||_{W^m(\tilde{\Omega})} \leq C ||\phi||_{W^{k+1}(\tilde{\Omega})} h^{k+1-m}$$

$$0 \leq m \leq k+1,$$

where $k = 1, 2, 3, 2$ for type I, II, III, III', respectively. This inequality means that approximating subspaces of regular family of each of the four types form $S^h_{1,k}(\Omega)$-spaces. Approximating subspaces of regular family of types I and II form $S^h_{1,k}(\Omega)$-spaces if we impose further the condition that each basis function of the subspace vanishes on $\partial\Omega$. 
B. Curved Elements

We refer the reader to the paper of Ciarlet and Raviart for the theory of interpolation over curved elements [7]. There they obtained some abstract theoretical results, and applied these to Lagrange and Hermite interpolations. They obtained also asymptotic estimates of the form (6.1) for regular families of some curved simplicial elements and curved quadrilateral elements.

C. n-Dimensional Rectangular Elements
Using Tensor Product of One-Dimensional Hermite Interpolation

For this type of elements, we describe briefly the fundamental results of Bramble and Hilbert [4] and Hilbert [24].

We describe first the case with natural boundary conditions, and then the case with essential boundary conditions.

Consider any closed hypercube \( \mathcal{K} \) in \( \mathbb{R}^n \), vertices denoted by \( a_\xi, 1 \leq \xi \leq 2^n \) as before. For \( \phi \in C^{2m-1} (\mathcal{K}) \), the \( m \)-th Hermite interpolation of \( \phi \) in \( \mathcal{K} \) is defined to be a polynomial \( \phi_m \) having degree \( 2m-1 \) in each of its \( n \) variables and satisfying at each vertex \( a_\xi \) the condition
for any $i_1, \ldots, i_n$ with $0 \leq i_j \leq m-1$ for all $1 \leq j \leq n$. Next we partition $\mathbb{R}^n$ into non-overlapping hypercubes with sides of length $h$, $\mathbb{R}^n = U_\alpha K^\alpha$. Then given any $\phi \in C^{2m-1}(\mathbb{R}^n)$, we can find its interpolation $\phi_m$ by piecing together its interpolates on each $K^\alpha$. For a function $\phi \in W^{2m}(\mathbb{R}^n)$, we may have to smoothen $\phi$ first and then obtain the interpolate $\phi_m$ of the smoothened function, see [24]. We define $H(m)$ to be the subspace of $C^{m-1}(\mathbb{R}^n)$ consisting in functions $\phi$ which are polynomials of degree $\leq 2m-1$ in each variable on each $K^\alpha$ and are globally of class $C^{m-1}$. $H(m)(\Omega)$ is the set of restrictions to $\Omega$ of the elements of $H(m)$. The following result is due to Hilbert (Thm. 4.3 of [24]), and Bramble and Hilbert [4].

**Theorem:** Let $\phi$ belong to $W^{2m}(\Omega)$. Then there exists an element $\tilde{\phi}$ in $H(m)(\Omega)$ such that
\[
\left| \frac{\partial^{i_1 + \ldots + i_n}}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} (\phi - \phi) \right| \leq C h^{2m-(i_1 + \ldots + i_n)}
\]

\[
\sum_{(j_1, \ldots, j_n) \in J} \left| \frac{\partial^{j_1 + \ldots + j_n}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}} \phi \right|
\]

where \(C\) is independent of \(h\) and \(\phi\) for any index \((i_1, \ldots, i_n)\) with \(i_1 + \ldots + i_n \leq 2m\), and \(J\) is the set of all indices \((j_1, \ldots, j_n)\) with \(j_1 + \ldots + j_n = 2m\) such that the polynomial \(x_1^{j_1} \ldots x_n^{j_n}\) does not coincide with its \(m\)-th Hermite interpolation.

Clearly \(H^{(m)}(\Omega)\) is an \(S^h_{1,k}(\Omega)\)-space. One can also partition \(\mathbb{R}^n\) into rectangular parallelepipeds instead of hypercubes. The same type of analysis then yields local error bounds in terms of the longest side and the shortest side of each rectangular parallelepiped. In particular, if the ratio of the longest side to the shortest side of each parallelepiped is bounded above and below over all parallelepipeds, and if the longest sides of the parallelepipeds are bounded, then we can obtain error bounds by simply replacing \(h\) by the upper bound of the longest sides.

For problems with homogeneous Dirichlet boundary conditions, we shall use tensor-product type basis elements when the region is a union of rectangular parallelepipeds.
Clearly the resulting approximation spaces are \( S_{I,k}^h,0(\Omega) \)-spaces, and the basis functions vanish on \( \partial \Omega \).

We remark that Hilbert [24] also obtained similar results for splines.

We shall need the following lemmas of Douglas and Dupont [10], Dupont [13] to convert the error bounds for transient problems into rates of convergence under the assumption that the approximating subspace \( \mathcal{M}_h \) is an \( S_{I,k}^{h,0}(\Omega) \)-space or an \( S_{I,k}^h(\Omega) \)-space.

**Lemma 6.1.** Let \( m \) be an integer such that \( 0 \leq m \leq 2 \). Let \( \phi: [0,T] \rightarrow W^k(\Omega) \) be a function of \( t \) such that \( \partial^m \phi / \partial t^m \in L_2(0,T; W^k(\Omega)) \). Let \( \tilde{\phi}(\cdot; t) \) be the interpolate of \( \phi(\cdot, t) \) in \( \mathcal{M}_h \). Then we have

\[
\| \frac{\partial^m \phi}{\partial t^m} (\cdot, t) \|_{L_2(0,T; W^k(\Omega))} \leq C h^{k-\ell} \| \phi \|_{L_2(0,T; W^k(\Omega))}.
\]

This result was originally stated [10, p. 624] only for those functions \( \phi \) having the property:

\[
\frac{\partial^m \phi}{\partial t^m} (\cdot, t) \in C^{k-1}(\overline{\Omega}), \forall \ t \in [0,T].
\]

This condition can be removed, however, since the time derivatives of the interpolate of \( \phi \) are just the interpolates
of the time derivatives of $\phi$. From the ways Hilbert [24] and Strang [48] constructed the mollified function, it is clear that the mollified function retains the same time differentiability of the original function.
7. A PRIORI ESTIMATE FOR GALERKIN APPROXIMATION FOR ELASTOSTATIC BOUNDARY VALUE PROBLEMS

In this section we present bounds for the error induced by using the solutions of (5.2), (5.11), and (5.12) to approximate solutions of (4.1), (4.3), (4.4) respectively. The error bounds are expressed in terms of the norm of \( u - \hat{u} \), where \( u \) is the exact solution, and \( \hat{u} \) is the "best possible" approximation of \( u \) which lies in the approximating subspace \( M_N \). Thus in each case we reduce the question of bounding the error in the approximate solution to a question in approximation theory.

HOMOGENEOUS DISPLACEMENT BVP

We first consider the case when (5.2) is used to approximate the solution of (4.1). Let \( \hat{u} \) be any element in the subspace \( M_N \). We put

\[
\bar{e} = \hat{u} - u, \quad \hat{e} = u - \hat{u}, \quad e = u - u = \hat{e} + \bar{e},
\]

where \( U \) is the Galerkin approximate solution. If we have some knowledge about \( \bar{e} \) for a particular choice of \( \hat{u} \) (e.g. the interpolate of \( u \) if applicable), then to estimate \( e \) it suffices to bound \( \hat{e} \).

7-1
Theorem 7.1 \((\mathcal{W}^1(\Omega))\)-Estimate for Homogeneous Displacement BVP with Interpolation of Coefficients

Under assumptions \(A(\text{iv})\), \(A(\text{v})\), (5.5) and \(\|v_u\|_{L_\infty(\Omega)} < \infty\), there exists a constant \(C > 0\), independent of \(\tilde{u}\) and the choice of \(\mathcal{M}_N\), such that

\[
\|U - u\|_{\mathcal{W}^1(\Omega)} \leq C(\inf\{\|U - \tilde{u}\|_{\mathcal{W}^1(\Omega)} : \tilde{u} \in \mathcal{M}_N\} + \chi)
\]

(7.1)

where

\[
x = \|\rho \tilde{f} - \rho f\|_{L_2(\Omega)}
\]

\[
+ \|v_u\|_{L_\infty(\Omega)} \sum_{i,j,k} \|\tilde{C}_{ijkl} - C_{ijkl}\|_{L_2(\Omega)}.
\]

(7.2)

If furthermore \(\mathcal{M}_N\) is an \(S_{1,k}^0(\Omega)\)-space, and \(u \in \mathcal{W}^k(\Omega) \cap \mathcal{W}^1(\Omega)\) with \(k > 1\), then we have

\[
\|U - u\|_{\mathcal{W}^1(\Omega)} \leq C(h^{k-1} \|u\|_{\mathcal{W}^k(\Omega)} + \chi),
\]

(7.3)

where \(C\) is a positive constant independent of \(u\) and \(h\).

Notice that for a given \(u\), \(\chi\) is a measure of the accuracy of the interpolated coefficients.
Proof: Subtracting (4.1) from (5.2), and using $v = Y = \hat{e}$, we obtain

$$- \sum_{i,j,k} < \tilde{C}_{ijkl} \frac{\partial \hat{e}_k}{\partial x_k}, \frac{\partial \hat{e}_i}{\partial x_j} >$$

$$- \sum_{i,j,k} < (\tilde{C}_{ijkl} - C_{ijkl}) \frac{\partial u_k}{\partial x_k}, \frac{\partial \hat{e}_i}{\partial x_j} >$$

$$- \sum_{i,j,k} < \tilde{C}_{ijkl} \frac{\partial \hat{e}_k}{\partial x_k}, \frac{\partial \hat{e}_i}{\partial x_j} >$$

$$+ \sum_{i} < (\rho f_i)^{-1} - \rho f_i, \hat{e}_i > = 0.$$ 

Using the Cauchy-Schwarz inequality, and (5.5), we get

$$\frac{\eta u_2}{2} ||\hat{e}_i||^2_{W^1(\Omega)} \leq \sum_{i,j,k} ||\tilde{C}_{ijkl} - C_{ijkl}||$$

$$\cdot ||v_u|| \cdot ||v\hat{e}_i||$$

$$+ \frac{3C}{2} ||v\hat{e}_i|| ||\hat{e}_i|| + ||(\rho f_i)^{-1} - \rho f_i|| ||\hat{e}_i||$$

$$\leq \eta ||\hat{e}_i||^2_{W^1(\Omega)} + C(||v\hat{e}_i||^2 + ||(\rho f)^{-1} - \rho f||^2 ||\hat{e}_i||^2$$
In deriving the last inequality, we have used the following inequality several times

\[ ab \leq \eta a^2 + b^2/(4\eta). \]

If in (7.4), \( \eta \) is chosen to be sufficiently small, e.g., less than \( \eta u/4 \), then the term \( \eta ||e||^2_{W^1(\Omega)} \) can be absorbed into the left hand side, or equivalently, we have

\[ ||\hat{e}||_{W^1(\Omega)} \leq C( ||\hat{\nu}|| + ||(\rho \hat{f}) - \rho f|| ) \]

\[ + ||\nu u||_{L^\infty(\Omega)} ( \sum_{i,j,k,l} ||\tilde{C}_{ijkl} - \tilde{C}_{ijkl}|| ). \]

Then our result is proved upon using the triangle inequality

\[ ||e||_{W^1(\Omega)} \leq ||\hat{e}||_{W^1(\Omega)} + ||\tilde{e}||_{W^1(\Omega)}. \quad \text{Q.E.D.} \]

If the integrals are evaluated exactly, then \( \chi \) is zero. In particular, we have the following result.
Corollary 7.1 \((W^1(\Omega))-Estimate for Homogeneous
Displacement BVP Without Interpolation
Of Coefficients\)

Under assumptions A(v), A(iv), or A(iv)', with uniform elasticity, there exists a constant \(C\), independent of \(u\) and \(M_N\), such that

\[
||u - \tilde{u}||_{W^1(\Omega)} \leq C \inf \{ ||u - \tilde{u}||_{W^1(\Omega)} : \tilde{u} \in M_N \}. \tag{7.5}
\]

If \(M_N\) is an \(S_{1,k}^0(\Omega)\)-space, and \(u \in W^k(\Omega) \cap W^1(\Omega)\) with \(k \geq 1\), then we have

\[
||u - \tilde{u}||_{W^1(\Omega)} \leq C h^{k-1} ||u||_{W^k(\Omega)}, \tag{7.6}
\]

where \(C\) is a positive constant independent of \(u\) and \(h\).

If the stiffness matrix \(\tilde{G}\) and the load vector \(\tilde{F}\) are both computed exactly, then (7.5) indicates that the accuracy of the Galerkin approximation, measured by the mean-square errors of strain and rotation, hinges on how well we can approximate the exact solution by functions in the approximating subspace. If the stiffness matrix \(\tilde{G}\) and the load vector \(\tilde{F}\) are not computed exactly, then (7.1) tells us that the errors depend also on the mean-square errors in the perturbed coefficients used in
computing \( \tilde{\mathcal{C}} \) and \( \tilde{F} \).

For example, if \( \mathcal{M}_N \) is chosen to be the subspace consisting of linear functions on simplicial elements (piecewise linear globally) and if \( \tilde{C}_{ijkl} \) and \( (\rho f_i) \) are the Lagrange interpolates of \( C_{ijkl} \) and \( \rho f_i \) in \( \mathcal{M}_N \), then whether we use \( \mathcal{C} \) and \( F \) or \( \tilde{\mathcal{C}} \) and \( \tilde{F} \), the error bounds in (7.1) and (7.5) are of the order: \( O(h) \), provided \( u \) belongs to \( W^2(\Omega) \cap \overset{0}{W}^1(\Omega) \). However, the interpolated system is much easier to solve than the exact system.

Error estimates (7.1) and (7.5) trivially give rise to a bound for the mean-square error of displacement of the Galerkin approximation, i.e. \( ||u - \tilde{u}||_{L^2(\Omega)} \). But this bound turns out to be not sharp. If \( \mathcal{M}_N \) is an \( S_{k,m}^0(\Omega) \)-space, then using the now standard technique due to Nitsche [41, 42], we have the following sharper result.

Theorem 7.2 (\( L^2(\Omega) \)-Estimate for Homogeneous Displacement BVP with Interpolation of Coefficients)

Under the assumptions A(iv), A(v), (5.5), and that there is a positive constant \( C_1 \) independent of \( \mathcal{M}_N \) such that \( ||\nabla w||_{L^\infty(\Omega)} \leq C_1 \), if the elasticity operator for the homogeneous displacement elastostatic BVP is \( 0 \)-regular,
and if \( \mathcal{M}_N \) is an \( S^{h,0}_{1,k}(\Omega) \)-space, then there exists a positive constant \( C \), independent of \( u \) and \( h \), such that

\[
||U-u||_{L^2(\Omega)} \leq C(h^k||u||_{W^k(\Omega)} + \chi)
\]

(7.7)

provided \( u \in W^k(\Omega) \cap \tilde{W}^k(\Omega) \) with \( k \geq 1 \). Here

\[
\chi = ||(\rho \tilde{f}) - \rho f||_{L^2(\Omega)}
\]

\[
+ C_1 \sum_{i,j,k,\ell} ||\tilde{C}_{ijkl} - C_{ijkl}||_{L^2(\Omega)},
\]

and \( \tilde{W} \) is the Galerkin solution when integrals are computed exactly, i.e., \( \tilde{W} \in \mathcal{M}_N \) and

\[
\sum_{i,j,k,\ell} <C_{ijkl} \frac{\partial}{\partial x_j} (u_i - \tilde{W}_i), \frac{\partial \tilde{W}_k}{\partial x_\ell}> = 0,
\]

\( \forall \tilde{W} \in \mathcal{M}_N \) .

(7.8)

**Proof:** Let \( \tilde{e} = U - \tilde{W}, \ \hat{e} = U - W, \ e = U - u = \hat{e} + \tilde{e} \). Subtracting (4.1) from (5.2), and using \( \nu = \tilde{y} = \hat{e} \), we obtain

\[
- \sum_{i,j,k,\ell} <\tilde{C}_{ijkl} \frac{\partial}{\partial x_\ell} (U_k - \tilde{W}_k), \frac{\partial \hat{e}_i}{\partial x_j}> \]

...
Using (5.5), the Cauchy-Schwartz inequality, and (7.8), we have

\[ \frac{n}{2} \| \hat{e} \|^2 _{W^1(\Omega)} \leq \| \nabla W \| _{L_\infty(\Omega)} \| \nabla \hat{e} \| 
\]

\[ \quad \times \sum_{i,j,k,\ell} |\tilde{C}_{ijk\ell} - C_{ijk\ell}| + \| (\rho \hat{f}) - \rho f \| \| \hat{e} \| 
\]

\[ \leq n \| \hat{e} \|^2 _{W^1(\Omega)} + C(C^2_i \sum_{i,j,k,\ell} |\tilde{C}_{ijk\ell} - C_{ijk\ell}|^2 + \| (\rho \hat{f}) - \rho f \| ^2). \]

Now if \( n \) is chosen to be sufficiently small, then

\[ \| \hat{e} \| \leq \| \hat{e} \| _{W^1(\Omega)} \]
\[
\leq C(C_{1, i, j, k, l} \|\bar{\epsilon}_{ijkl} - \epsilon_{ijkl}\| + \|\rho \bar{\epsilon} - \rho \bar{\epsilon}\|) \tag{7.9}
\]

Note when \(\bar{\epsilon}_{ijkl} = \epsilon_{ijkl}\), \((\rho \bar{\epsilon})^{\sim} = \rho \bar{\epsilon}\), (7.9) implies \(\bar{\epsilon} = 0\), as it should.

Next we use Nitsche's argument. Let \(\gamma = (\gamma_1, \ldots, \gamma_n)\) be the solution of the following BVP

\[
\sum_{j, k, l} \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial \gamma_k}{\partial x_l} \right) = \bar{\epsilon}_i,
\]

\(\forall i = 1, \ldots, n, \quad \forall x \in \Omega\),

\(\gamma = 0\),  \(\forall x \in \partial \Omega\).

Then

\[
\|\bar{\epsilon}\|^2 = \langle \bar{\epsilon}, \bar{\epsilon} \rangle
\]

\[
= - \sum_{i, j, k, l} \langle C_{ijkl} \frac{\partial \gamma_k}{\partial x_l}, \frac{\partial \bar{\epsilon}_i}{\partial x_j} \rangle
\]

\[
= - \sum_{i, j, k, l} \langle C_{ijkl} \frac{\partial}{\partial x_k} (\gamma_k - \gamma_{\tilde{k}}), \frac{\partial \bar{\epsilon}_i}{\partial x_j} \rangle, \quad \forall \gamma \in \mathcal{M}_N.
\]
The last equality follows from (7.8). Using (5.5), the properties of an \( S^{h,0}_{k,m}(\Omega) \) space, and the global regularity assumption, we have

\[
||\tilde{e}||^2 \leq C_u ||y - \tilde{y}||_{W^1(\Omega)} \inf \{ ||y - \tilde{y}||_{W^1(\Omega)} : \tilde{y} \in \mathcal{M}_N \}
\]

\[
\leq C_u (\inf \{ ||y - \tilde{u}||_{W^1(\Omega)} : \tilde{u} \in \mathcal{M}_N \})
\cdot (Ch ||y||_{W^2(\Omega)})
\]

\[
\leq Ch ||\tilde{e}|| \inf \{ ||y - \tilde{u}||_{W^1(\Omega)} : \tilde{u} \in \mathcal{M}_N \}
\]

or, equivalently,

\[
||\tilde{e}|| \leq Ch \inf \{ ||y - \tilde{u}||_{W^2(\Omega)} : \tilde{u} \in \mathcal{M}_N \}.
\]  \hspace{1cm} (7.10)

Adding (7.9) and (7.10), and using the triangle inequality, we obtain

\[
||y - u|| \leq C(h \inf \{ ||y - \tilde{u}||_{W^1(\Omega)} : \tilde{u} \in \mathcal{M}_N \} + \chi),
\]

which implies (7.7) since \( \mathcal{M}_N \) is an \( S^{h,0}_{1,m}(\Omega) \) space. Q.E.D.

In the course of the proof, we obtained also the
following result when the integrals are evaluated exactly.

Corollary 7.2  \((L_2(\Omega))-\text{Estimate for Homogeneous Displacement BVP Without Interpolation of Coefficients})

Under the assumptions \(A(v), A(iv), \) or \(A(iv)' \) with uniform elasticity, if the elasticity operator for the homogeneous displacement elastic BVP is \(0\)-regular, and if \(w_N^h\) is an \(S_{1,k}^{h,0}(\Omega)\)-space, then there exists a positive constant, independent of \(u\) and \(h\), such that

\[
||\Psi - u||_{L_2(\Omega)} \leq C h^k ||u||_{W^k(\Omega)}
\]  \(7.11\)

provided \(u \in W^k(\Omega) \cap W^1(\Omega)\) with \(k \geq 1\).

**TRACTION FREE BVP**

Now consider the case when \((5.11)\) is used to approximate the solution of the normalized problem \((4.3)\), with the understanding that the compatibility condition \((3.15)\) is satisfied. When the stiffness matrix and the load vector are computed by interpolation, due precautions are supposedly already taken on \(C_{ijkl}^n\) and \((\rho f_i)^n\) as explained before.
The proof of the following theorems are parallel to those for the scalar Neumann BVP in the works of Friedrich and Keller [21], and Schultz [47].

Theorem 7.3 (W¹(Ω)-Estimate for Traction
Free BVP with Interpolation of Coefficients)

Under the assumptions A(iv), A(v), (5.9), and
||v||^\infty \in Ω < ∞, there exists a constant C > 0, independent of v and the choice of MN such that

||Y - v||^1 \in Ω \leq C(\inf(||Y - \tilde{v}||^1 \in Ω : \tilde{v} \in MN} + \chi) \quad (7.12)

where \chi is defined by (7.2). If furthermore MN is an S_{1,k}(Ω)-space, and v \in W^k(Ω) with k ≥ 1, then we have

||Y - v||^1 \in Ω \leq C(h^{k-1}||v||^k \in Ω} + \chi), \quad (7.13)

where C is a positive constant independent of v and h.

Proof: We first recall the definition

MN** = {v \in MN: % integral over Ω of v dx = 0,}
\[ \int_{\Omega} \text{curl} \, \mathbf{v} \, dx = 0 \].

Let \( \tilde{u}^{**} \) be any element in \( M_N^{**} \). Then we define
\[ \tilde{e} = \tilde{u}^{**} - u, \quad \hat{e} = U - \tilde{u}^{**}, \quad \varepsilon = U - u = \hat{e} + \tilde{e}. \]

Following the argument used in the proof of Theorem 7.1 and using (4.12), we have
\[ \| \hat{\varepsilon} \|_{W^1(\Omega)} \leq \eta \| \hat{\varepsilon} \| + C(\| \nabla \tilde{e} \| + \chi). \]

The term \( \| \hat{\varepsilon} \| \) on the right hand side is due to the fact that in the Classical Second Korn's Inequality only the semi-norm term appears on the dominated side. Upon using the Generalized Poincaré inequality, we have
\[ \| \hat{\varepsilon} \|_{W^1(\Omega)} \leq C(\| \nabla \tilde{e} \| + \chi), \]

or after further using the triangle inequality,
\[ \| U - u \|_{W^1(\Omega)} \leq C(\| U - \tilde{u}^{**} \|_{W^1(\Omega)} + \chi), \quad (7.14) \]

\[ \nabla \tilde{u}^{**} \in M_N^{**}. \]

The inequality (7.14) is almost the same as (7.12)
except that in (7.14) \( \tilde{u}^{**} \) is restricted to the subspace \( M_N^{**} \). To remove this restriction, we define for any \( \tilde{u} \in M_N \), two associated elements \( \tilde{u}^* \) and \( \tilde{u}^{**} \) by

\[
\tilde{u}^* \equiv \tilde{u} - \left( \int_\Omega \text{curl} \tilde{u} \, dx \right) \times \chi/(2m(\Omega)),
\]

\[
\tilde{u}^{**} \equiv \tilde{u}^* - \left( \int_\Omega \tilde{u}^* \, dx \right)/m(\Omega),
\]

where \( \chi \) stands for the usual vector cross product and \( m(\Omega) \) denotes the Lebesgue measure of \( \Omega \). Clearly we have

\[
\int_\Omega \tilde{u}^{**} \, dx = 0.
\]

From the vector identity

\[
\text{curl}(\alpha \times \chi) = 2\alpha
\]

for any constant vector \( \alpha \), it follows that

\[
\int_\Omega \text{curl} \tilde{u}^* \, dx = 0.
\]

Hence \( \tilde{u}^{**} \in M_N^{**} \). Then to prove (7.12), it suffices to show that

\[
\| \| u - \tilde{u}^{**} \|_{W^1(\Omega)} \| \leq C \| u - \tilde{u} \|_{W^1(\Omega)}, \; \forall \tilde{u} \in M_N.
\]

(7.15)
Upon using (3.16), the Cauchy-Schwartz inequality, and the fact that \( \Omega \) is bounded, we have

\[
||u - \tilde{u}^*||_{H^1(\Omega)}^2 = ||u - \tilde{u}^*||_{H^1(\Omega)}^2
\]

\[= ||u - \tilde{u} + (\int_\Omega \text{curl}(\tilde{u} - u)dx)||_{H^1(\Omega)}^2 \]

\[ \leq 2||u - \tilde{u}||_{H^1(\Omega)}^2 + 2||\int_\Omega \text{curl}(u - \tilde{u})dx||_{H^1(\Omega)}^2 \]

\[\times \frac{\chi}{2m(\Omega)}||u||_{H^1(\Omega)}^2 \times \frac{\chi}{2m(\Omega)}||u||_{H^1(\Omega)}^2 \cdot \]

\[\leq 2||u - \tilde{u}||_{H^1(\Omega)}^2 + C\int_\Omega ||\text{curl}(u - \tilde{u})dx||^2 \]

\[\leq C||u - \tilde{u}||_{H^1(\Omega)}^2 \]

\[\text{and} \]

\[||u - \tilde{u}^*||^2 \leq 2||u - \tilde{u}^*||^2 \]

\[+ 2||\int_\Omega (u - \tilde{u}^*)dx/m(\Omega)||^2 \]
\[ \leq 2 \| u - \tilde{u}^* \|^2 + C \int_\Omega (u - \tilde{u}^*) dx \leq (2 + Cm(\Omega)) \| u - \tilde{u}^* \|^2 \]
\[ \leq C(\| u - \tilde{u} \|^2 + \| \int_\Omega \text{curl} (u - \tilde{u}) dx \times \chi / (2m(\Omega)) \|^2) \]
\[ \leq C(\| u - \tilde{u} \|^2 + \| \chi \|^2 \int_\Omega \text{curl}(u - \tilde{u}) dx \|^2) \]
\[ \leq C \| u - \tilde{u} \|^2 \| \chi \| _{\mathcal{W}^1(\Omega)}. \]

Thus (7.15) is proved. \[ \text{Q.E.D.} \]

As before if the integrals are evaluated exactly, then we have the following result:

**Corollary 7.3 (\(\mathcal{W}^1(\Omega)\)-Estimate for Traction BVP Without Interpolation of Coefficients)**

Under the assumptions A(iv) and A(v), there exists a constant C, independent of \( \tilde{u} \) and \( \mathcal{M}_N \), such that

\[ \| \Psi - u \| _{\mathcal{W}^1(\Omega)} \leq C \inf \{ \| u - \tilde{u} \| _{\mathcal{W}^1(\Omega)} : \tilde{u} \in \mathcal{M}_N \}. \] (7.16)
If $\mathcal{M}_N$ is an $S_{1,k}^h(\Omega)$-space, and $u \in \mathring{W}^k(\Omega)$ with $k \geq 1$, then we have

$$\| u-u \|_{\mathring{W}^1(\Omega)} \leq C h^{k-1} \| u \|_{\mathring{W}^k(\Omega)}, \quad (7.17)$$

where $C$ is a positive constant independent of $u$ and $h$.

Now let $\mathcal{M}_N$ be a $S_{1,k}^h(\Omega)$-space. To obtain the sharp mean-square error estimate of the displacement for the traction BVP, we again define $\mathring{w}$ to be the Galerkin solution with the stiffness matrix and the load vector computed by exact integration. Following a proof strictly parallel to that of Theorem 7.2, we have:

Theorem 7.4 ($L_2(\Omega)$-Estimate for Traction-Free BVP with Interpolation of Coefficients)

Under the assumptions A(iv), A(v), (5.9), and that there is a positive constant $C_1$ independent of $\mathcal{M}_N$ such that $\| \nabla \mathring{w} \|_{L^\infty(\Omega)} \leq C_1$, if $\mathcal{M}_N$ is an $S_{1,k}^h(\Omega)$-space, and if the elasticity operator for the traction-free elastostatic BVP is 0-regular, then there exists a positive constant $C$, independent of $u$ and $h$, such that

$$\| u-u \|_{L_2(\Omega)} \leq C (h^k \| u \|_{\mathring{W}^k(\Omega)} + \gamma), \quad (7.18)$$
provided \( u \in \mathcal{W}^k(\Omega) \), \( k \geq 1 \). Here \( \chi \) is defined in Theorem 7.2, and \( \mathcal{W} \) is the Galerkin approximating solution when integrals are computed exactly.

We have a simpler result when the integrals are computed exactly.

**Corollary 7.4 (\( L_2(\Omega) \)-Estimate for the Traction-Free BVP Without Interpolation of Coefficients)**

Under the assumptions A(iv) and A(v), if \( \mathcal{M}_N \) is an \( S_{1,k}^h(\Omega) \)-space and if the elasticity operator for the traction-free elastostatic BVP is \( O \)-regular, then there exists a positive constant \( C \), independent of \( u \) and \( h \), such that

\[
\| \boldsymbol{\Psi} - \mathcal{U} \|_{L_2(\Omega)} \leq C h^k \| u \|_{\mathcal{W}^k(\Omega)},
\]

provided \( u \in \mathcal{W}^k(\Omega) \), \( k \geq 1 \).

**THE EXTRA BVP**

We consider the case when (5.12) is used to approximate the Extra Problem (4.4). Naturally we have the analogue of Corollaries 7.1 and 7.2 upon using (4.14) ((4.13)).
Theorem 7.5 \( \mathcal{W}^1(\Omega) \) - and \( L^2(\Omega) \) - Estimate for the Extra BVP Without Interpolation of Coefficients

Under the assumptions A(v) and A(iv)'(A(iv)),

(i) There exists a constant \( C \), independent of \( \bar{y} \) and \( \mathcal{M}_N \), such that

\[
||y-y||_{\mathcal{W}^1(\Omega)} \leq C \inf \{ ||y-y||_{\mathcal{W}^1(\Omega)} : \bar{y} \in \mathcal{M}_N \}. \tag{7.20}
\]

If \( \mathcal{M}_N \) is an \( S_{1,k}^0(\Omega) \) \((S_{1,k}^h(\Omega))\)-space, and \( u \in \mathcal{W}^k(\Omega) \) with \( k \geq 1 \), then we have

\[
||y-y||_{\mathcal{W}^1(\Omega)} \leq C h^{k-1} ||y||_{\mathcal{W}^k(\Omega)}, \tag{7.21}
\]

where \( C \) is a positive constant independent of \( \bar{y} \) and \( h \).

(ii) If \( \mathcal{M}_N \) is an \( S_{1,k}^0(\Omega)(S_{1,k}^h(\Omega)) \)-space and if the elasticity operator for the extra elastostatic BVP is \( O \)-regular, then there exists a constant \( C \), independent of \( y \) and \( h \), such that

\[
||y-y||_{L^2(\Omega)} \leq C h^k ||y||_{\mathcal{W}^k(\Omega)}, \tag{7.22}
\]

provided \( y \in \mathcal{W}^k(\Omega) \) with \( k \geq 1 \).
8. A PRIORI ESTIMATE FOR GALERKIN APPROXIMATION
FOR ELASTODYNAMIC INITIAL-BOUNDARY
VALUE PROBLEMS

In this section we present error bounds for the various approximating procedures formulated before to solve the elastodynamic I-BVP. These error bounds are expressed in terms of some norms of \( \tilde{u} - \hat{u} \), where \( \tilde{u} \) is the exact solution and \( \hat{u}(\cdot, t) \) is the "best possible" approximate solution belonging to the subspace \( \mathcal{M}_N \) at each \( t \). As a result we reduce the question of bounding the error to a question in approximation theory.

Continuous Time Galerkin Approximation for
the Elastodynamic I-BVP

We consider first I-BVP with homogeneous displacement BC. Let \( \tilde{u} \) be any element in the subspace \( \mathcal{M}_N \). We define

\[
\hat{e} = \tilde{u} - u, \quad \hat{\epsilon} = \hat{U} - \tilde{u}, \quad \epsilon = U - u = \hat{\epsilon} + \hat{e}.
\]

In order to estimate \( \epsilon \), it suffices to bound \( \hat{\epsilon} \) provided that we can bound \( \hat{e} \) first by choosing a particular \( \tilde{u}(\cdot, t) \). We need to consider only the case when
assumption A(iv)' is used, since the case when assumption A(iv) is used can be treated similarly by setting $\lambda_0 = 0$, where $\lambda_0$ is the coefficient in Gårding's inequality.

To obtain $W^1(\Omega)$ error estimates, let $\tilde{u}$ be an arbitrary element in $\mathcal{M}_N$, e.g. the Hermite interpolate of $u$ (if applicable). To obtain $L_2(\Omega)$ error estimates we choose $\tilde{u}$ to be the element in $\mathcal{M}_N$ such that

$$\sum_{i,j,k,l} C_{ijkl} \frac{\partial}{\partial x_k} (u_k(\cdot,t) - \tilde{u}_k(\cdot,t)), \frac{\partial V_i}{\partial x_j} >$$

$$+ \lambda_0 \sum_i < u_i(\cdot,t) - \tilde{u}_i(\cdot,t), V_i > = 0,$$

$$\forall \, \forall \, \forall \, \forall \in \mathcal{M}_N, \forall \, t \in [0,T]. \quad (8.1)$$

Then we use the optimal $L_2(\Omega)$ error estimates obtained in Theorem 7.5 for the associated Extra Elastostatic BVP to derive the optimal error estimates for the dynamic problem. We need the following lemma of Dupont [13] in our error analysis.

Lemma 8.1. Let $\xi(0) \in L_2(\Omega)$ and $\partial \xi / \partial t \in L_2(0,T;L_2(\Omega))$. Then there exists a positive constant $C$ such that
\[ \|\xi\|^2_{L^2(\Omega)^{(\tau)}} \leq C(\|\xi\|^2_{L^2(\Omega)^{(0)}}) \]
\[ + \int_0^\tau \|\frac{\partial \xi}{\partial t}\|^2_{L^2(\Omega)} \, dt, \quad \forall \tau \in [0, T]. \]

Our first result of error bound for elastodynamic I-BVP is

**Theorem 8.1 (W^1(\Omega)-Estimate for Continuous Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP with Interpolation of Coefficients)**

Under assumptions A(i), A(v), A(iv)', and (5.15), if

\[ \forall u_0 \in L^\infty(\Omega), \quad \forall \frac{\partial u}{\partial t} \in L^\infty(0, T; L^\infty(\Omega)), \]
\[ \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^\infty(\Omega)), \]

then there exists a positive constant C, independent of u and the choice of \( M_N \), such that

\[ \|\|\cdot\|\|_{L^\infty(0, T; W^1(\Omega))} + \|\frac{\partial}{\partial t}(\cdot)\|_{L^\infty(0, T; L^2(\Omega))} \]

[\( \cdot \)]
\[ \leq C \left( \inf \left\{ \left\| \frac{\partial}{\partial t} (u - \tilde{u}) \right\|_{L^2(0, T; L^2(\Omega))} \right\} \right. \\ \left. + \left\| \frac{\partial^2}{\partial t^2} (u - \tilde{u}) \right\|_{L^2(0, T; L^2(\Omega))} \right) \\ + \left\| u - \tilde{u} \right\|_{H^{-1}(\Omega)} (0) + \left\| \frac{\partial}{\partial t} (u - \tilde{u}) \right\|_{L^2(\Omega)} (0) \\ + \left\| u - \tilde{u} \right\|_{H^{-1}(\Omega)} (0) + \left\| \frac{\partial}{\partial t} (u - \tilde{u}) \right\|_{L^2(\Omega)} (0) \right) + \chi, \quad (8.2) \]

where

\[ \chi \equiv \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^\infty(0, T; L^\infty(\Omega))} \left\| \tilde{\rho} - \rho \right\|_{L^2(\Omega)} \left( \left\| (\rho \bar{f}) - \rho \bar{f} \right\|_{L^2(0, T; L^2(\Omega))} \right. \\ \left. + \left( \left\| \nabla u_0 \right\|_{L^2(\Omega)} + \left\| \frac{\partial}{\partial t} \right\|_{L^2(0, T; L^2(\Omega))} \right) \right) \right. \\ \left. \cdot \sum_{i, j, k, \ell} \left\| \tilde{C}_{i j k \ell} - C_{i j k \ell} \right\|_{L^2(\Omega)}. \right) \quad (8.3) \]

If furthermore \( \mathcal{M}_N \) is an \( H^0_{L, k}(\Omega) \)-space, \( u_0, \tilde{u}_0 \in H^k(\Omega) \cap \tilde{W}^{1}(\Omega), (\frac{\partial^2}{\partial t^2}) u \in L^2(0, T; \tilde{W}^k(\Omega)) \) with \( k > 0 \),
and \( Y(0) \) and \( \partial Y / \partial t(0) \) are chosen to be the \( M_N \) interpolates of \( u_0 \) and \( \dot{u}_0 \) respectively, then we have

\[
\| Y - u \|_{L^\infty(0,T; L^2(\Omega))} + \| \frac{\partial}{\partial t}(Y - u) \|_{L^\infty(0,T; L^2(\Omega))} \leq C(h^{k-1} + \epsilon), \quad (8.4)
\]

where \( C \) is a positive constant independent of \( h \).

**Remark:** \( \nabla \partial Y / \partial t \in L^\infty(0,T; L^\infty(\Omega)) \) implies \( \| \nabla \dot{u}_0 \| < \infty \), which in turn implies \( \| \dot{u}_0 \| < \infty \). From Lemma 8.1, we see that \( \nabla u \in L^\infty(0,T; L^\infty(\Omega)) \). Indeed, we have

\[
\| \nabla u \|_{L^\infty(\Omega)}^2(\tau) \leq C(\| \nabla u_0 \|_{L^\infty(\Omega)}^2 + \int_0^T \| \nabla \frac{\partial u}{\partial t} \|_{L^\infty(\Omega)}^2 \, dt), \quad \forall \tau \in [0,T].
\]

Similarly, if \( \partial^2 / \partial t^2 u \in L^2(0,T; W^k(\Omega)) \) and \( \dot{u}_0 \in W^k(\Omega) \), then \( \partial / \partial t \dot{u} \in L^\infty(0,T; W^k(\Omega)) \).

**Proof:** Subtracting \((4.5)_1\) from \((5.13)_1\), and using \( \gamma = \nabla \partial e / \partial t \), we obtain
\[
\sum_i \left< \rho \frac{a^2 e_i}{\partial t^2}, \frac{ae_i}{\partial t} \right> + \sum_i \left< (\rho - \rho) \frac{a^2 u_i}{\partial t^2}, \frac{ae_i}{\partial t} \right> \\
+ \sum_i \left< \rho \frac{a^2 e_i}{\partial t^2}, \frac{ae_i}{\partial t} \right> \\
+ \sum_{i,j,k,\lambda} \left< \tilde{C}_{ijkl} \frac{ae_k}{\partial x_{\lambda}}, \frac{a^2 e_i}{\partial x_j} \right> \\
+ \sum_{i,j,k,\lambda} \left< (\tilde{C}_{ijkl} - C_{ijkl}) \frac{a u_k}{\partial x_{\lambda}}, \frac{a^2 e_i}{\partial x_j} \right> \\
+ \sum_{i,j,k,\lambda} \left< \tilde{C}_{ijkl} \frac{ae_k}{\partial x_{\lambda}}, \frac{a^2 e_i}{\partial x_j} \right> \\
- \sum_i \left< (\rho f_i)^{\sim} - \rho f_i, \frac{ae_i}{\partial t} \right> = 0. \quad (8.5)
\]

Then using the Cauchy-Schwartz inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \sum_{i,j,k,\lambda} \left< \tilde{C}_{ijkl} \frac{ae_k}{\partial x_{\lambda}}, \frac{ae_i}{\partial x_j} \right> \\
+ \frac{1}{2} \frac{d}{dt} \sum_i \left< \rho \frac{ae_i}{\partial t}, \frac{ae_i}{\partial t} \right>
\]
\[
\begin{align*}
&= - \sum_i \left< \frac{a^2 e_i}{\partial t^2}, \frac{a e_i}{\partial t} \right> - \sum_i \left(\rho - \rho_i \right) \frac{a^2 u_i}{\partial t^2}, \frac{a e_i}{\partial t} > \\
&- \sum_{i,j,k,l} \left< \bar{C}_{ijkl} - C_{ijkl} \right> \frac{a u_k}{\partial x_j}, \frac{a^2 e_i}{\partial t} \frac{a e_i}{\partial x_j} > \\
&- \sum_{i,j,k,l} \left< \bar{C}_{ijkl} \frac{a e_k}{\partial x_j}, \frac{a^2 e_i}{\partial t} \right> > \\
&+ \sum_i \left< (\rho f_i) - \rho f_i, \frac{a e_i}{\partial t} \right> \\
&\leq \frac{3}{2} C_\rho \left[ \left\| \frac{a^2 e_i}{\partial t^2} \right\| + \left\| \frac{a e_i}{\partial t} \right\| + \left\| \frac{a^2 u_i}{\partial t^2} \right\| \right] \left\| \bar{\rho} - \rho \right\| \\
&\cdot \left( \left\| \frac{a e_i}{\partial t} \right\| + \left\| (\rho f_i) - \rho f_i \right\| \right) \left\| \frac{a e_i}{\partial t} \right\| \\
&- \sum_{i,j,k,l} \left< \bar{C}_{ijkl} - C_{ijkl} \right> \frac{a u_k}{\partial x_j}, \frac{a^2 e_i}{\partial t} \frac{a e_i}{\partial x_j} > \\
&- \sum_{i,j,k,l} \left< \bar{C}_{ijkl} \frac{a e_k}{\partial x_j}, \frac{a^2 e_i}{\partial t} \right> .
\end{align*}
\]

We can integrate this equation with respect to \( t \) obtaining for each \( t \in (0,T] \), the inequality
\[ \frac{1}{2} \sum_{i,j,k,l} \hat{C}_{ijkl} \frac{\partial e_k}{\partial x_i} \frac{\partial e_i}{\partial x_j} \geq (\tau) \]

\[ + \frac{1}{2} \sum_i \hat{\rho} \frac{\partial e_i}{\partial t} \geq (\tau) \]

\[ \leq \left( \left\| \frac{\partial^2 e}{\partial t^2} \right\|_{L^2(0,T;L^2(\Omega))}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} \right) \left\| \hat{\rho} \right\|^2 \]

\[ + \left\| (\rho f) - \rho \hat{f} \right\|_{L^2(0,T;L^2(\Omega))}^2 + \int_0^T \left\| \frac{\partial e}{\partial t} \right\|^2 dt + \left\| \psi e \right\|^2 (0) + \left\| \frac{\partial e}{\partial t} \right\|^2 (0) \]

\[ - \sum_{i,j,k,l} \int_0^T \left( \hat{C}_{ijkl} - C_{ijkl} \right) \frac{\partial u_k}{\partial x_i} \frac{\partial^2 e_i}{\partial t \partial x_j} dt \]

\[ - \sum_{i,j,k,l} \int_0^T \left( \hat{C}_{ijkl} \frac{\partial e_k}{\partial x_i} \frac{\partial^2 e_i}{\partial t \partial x_j} \right) dt. \quad (8.6) \]

In deriving this result we have used the inequality \( ab \leq na^2 + b^2/(4n) \) as before. Upon using (5.5) or (5.15), we
see that the first term in (8.6) is bounded below by

\[(\eta_0/4) ||\vec{v}\vec{e}||^2(\tau) - (\lambda_0/2) ||\vec{\phi}||^2(\tau),\]

where in (5.15)_1, \(\lambda_0\) is the coefficient in Gårding’s Inequality, and in (5.5)_1, \(\lambda_0 = 0\). Upon using (5.14), we can bound the second term in (8.6) from below by, \((\eta_0/4) ||\partial \vec{e}/\partial t||^2(t)\). The last two terms in (8.6) can be integrated by parts, and their results are bounded from above by

\[\sum_{i,j,k,\ell} \int_0^\tau (\tilde{C}_{ijkl} - C_{ijkl}) \frac{\partial u_k}{\partial x_i} \frac{\partial \vec{e}_i}{\partial \xi_j} \, dt\]

\[= - \sum_{i,j,k,\ell} (\tilde{C}_{ijkl} - C_{ijkl}) \frac{\partial u_k}{\partial x_i} \frac{\partial \vec{e}_i}{\partial x_j} \bigg|_0^\tau\]

\[+ \int_0^\tau (\tilde{C}_{ijkl} - C_{ijkl}) \frac{\partial^2 u_k}{\partial t \partial x_i} \frac{\partial \vec{e}_i}{\partial x_j} \, dt\]

\[\leq C(1 ||v||^2_{L^\infty(0,T;L^\infty(\Omega))} + ||v_{\partial t}||^2_{L^\infty(0,T;L^\infty(\Omega))} + \sum_{i,j,k,\ell} ||\tilde{C}_{ijkl} - C_{ijkl}||^2\]}
\[ + ||\hat{v}||^2(0) + \int_0^T ||\hat{v}||^2 dt \]
\[ + \eta ||v||^2(\tau), \quad (8.7) \]

and

\[ - \sum_{i,j,k,\ell} \int_0^\tau <\hat{C}_{ijk\ell} \frac{\partial \hat{e}}{\partial x_k}, \frac{\partial \hat{e}}{\partial x_j}> dt \]
\[ = - \sum_{i,j,k,\ell} <\hat{C}_{ijk\ell} \frac{\partial \hat{e}}{\partial x_k}, \frac{\partial \hat{e}}{\partial x_j}> |\tau_0 \]
\[ + \sum_{i,j,k,\ell} \int_0^\tau <\hat{C}_{ijk\ell} \frac{\partial^2 \hat{e}}{\partial t \partial x_k}, \frac{\partial \hat{e}}{\partial x_j}> dt \]
\[ \leq C(||\hat{v}||^2_{L_{\infty} x L_2} + ||\hat{v}||^2_{L_2 x L_2} \]
\[ + ||\hat{v}||^2(0) + \int_0^\tau ||\hat{v}||^2 dt \]
\[ + \eta ||v||^2(\tau). \quad (8.8) \]

In deriving these two bounds, we have used, among others, the standard results
\[ \| \cdot \|_{L^p(0, \tau; L^2(\Omega))} \leq \| \cdot \|_{L^p_x L^2_t}, \ p=2 \text{ or } \infty, \]

and

\[ \| \cdot \|_{L^2_x L^2_t} \leq \tau^{1/2} \| \cdot \|_{L^\infty_x L^2_t}. \]

Substituting into (8.6) the various lower and upper bounds just derived, we get

\[
\left( \frac{n_u}{4} \right) \| \varphi \|^2(\tau) + \left( \frac{n_p}{4} \right) \| \frac{\partial \hat{e}}{\partial t} \|^2(\tau)
\]

\[
\leq C \left( \| \varphi \|^2_{L^\infty_x L^2_t} + \| \frac{\partial \hat{e}}{\partial t} \|^2_{L^2_x L^2_t}
\]

\[
+ \| \frac{\partial^2 e}{\partial t^2} \|^2_{L^2_x L^2_t} + \| \varphi \|^2(0) + \| \frac{\partial \hat{e}}{\partial t} \|^2(0) + \chi^2
\]

\[
+ C \left( \int_0^\tau \| \varphi \|^2 dt + \int_0^\tau \| \frac{\partial \hat{e}}{\partial t} \|^2 dt \right)
\]

\[
+ \eta \| \varphi \|^2(\tau) + \left( \frac{\lambda_0}{2} \right) \| \hat{e} \|^2(\tau).
\]

It should be noted that the first two terms on the right hand side of (8.9) can be traced to the term
in (8.5). The last four terms on the right hand side of (8.9) can be disposed of in the following way: The term \((\lambda_0/2)|\hat{e}|^2(\tau)\) can be disposed of in view of Lemma 8.1 and Poincare's Inequality, while the term \(\eta||v\hat{e}||^2(\tau)\) can be absorbed into the left hand side if \(\eta\) is chosen to be sufficiently small, say less than \(\eta_u/8\). The two remaining integrals involving derivatives of \(\hat{e}\) can then be disposed of by using Gronwall's Inequality. Thus (8.9) can be rewritten as

\[
|\|v\hat{e}\||^2(\tau) + |\|\hat{a}\hat{e}\||^2(\tau)
\]

\[
\leq C\left( |\|v\hat{e}\||^2_{L^\infty_x L^2_t} + |\|\hat{a}\hat{e}\||^2_{L^\infty_x L^2_t} + |\|v\hat{e}\||^2_{L^2_x L^2_t}
\right. \\
\left. + |\|\hat{a}\hat{e}\||^2_{L^2_x L^2_t} + |\|\hat{e}\||^2(0)
\right. \\
\left. + |\|\hat{a}\hat{e}\||^2(0) + x^2 \right).
\]

This inequality can be simplified further in the following way: Applying Lemma 8.1 to \(v\hat{e}\) and \(\hat{a}\hat{e}/\hat{a}t\), we obtain
\[ \| e \|^2_{L^2_x L^2_t} \leq C \left( \| e \|^2_0 + \| \frac{\partial e}{\partial t} \|^2_{L^2_x L^2_t} \right). \]

and

\[ \| \frac{\partial e}{\partial t} \|^2_{L^2_x L^2_t} \leq C \left( \| \frac{\partial e}{\partial t} \|^2_0 + \| \frac{\partial^2 e}{\partial t^2} \|^2_{L^2_x L^2_t} \right). \]

Then we have

\[ \| \frac{\partial e}{\partial t} \|_{(\tau)} + \| \frac{\partial e}{\partial t} \|_{(\tau)} \]

\[ \leq C \left( \| \frac{\partial e}{\partial t} \|_{L^2_x L^2_t} + \| \frac{\partial^2 e}{\partial t^2} \|_{L^2_x L^2_t} \right) \]

\[ + \| e \|_{(0)} + \| \frac{\partial e}{\partial t} \|_{(0)} \]

\[ + \| \frac{\partial e}{\partial t} \|_{(0)} + \| \frac{\hat{e}}{\partial t} \|_{(0)+x} \].

Now if we replace \( \| e \|_{(\tau)} \) by \( \| e \|_{W^1_0(\Omega)} \) with the help of Poincare's inequality, then (8.2) is proved upon noting that the left hand side of the above estimate is independent of \( \tilde{u} \epsilon_{l/N} \) while the right hand side is independent of
\[ \tau \in [0,T]. \] Finally (8.4) is an immediate consequence of (8.2) and Lemma 6.1. \[ \text{Q.E.D.} \]

**Corollary 8.1 (W^1(\Omega))-Estimate for Continuous Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP without Interpolation of Coefficients)**

Under assumptions A(i), A(v), and A(iv)', if

\[ \dot{u}_0 \in L^2(\Omega), \ u_0 \in \mathcal{W}^1(\Omega), \]

\[ \frac{\partial^2}{\partial t^2} u, \ \nabla \frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega)), \]

then there exists a positive constant \( C \), independent of \( u \) and the choice of \( \mathcal{M}_N \), such that (8.2) holds with \( \chi = 0 \). The additional assumptions and results specialized to an \( S^{h,0}_{1,k}(\Omega) \)-space are the same as those of Theorem 8.1 except that \( \chi = 0 \).

**Theorem 8.2 (L^2(\Omega))-Estimate for Continuous Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP with Interpolation of Coefficients**}
Under assumptions A(i), A(v), A(iv)', and (5.5), if \( M_N \) is an \( H^{0,0}_1, H^{k,0}_1(\Omega) \)-space, \( u_0, \dot{u}_0 \in W^k(\Omega), (\partial^2 u/\partial t^2) u \in L^2(0,T; W^k(\Omega)) \cap L^2(0,T; L^2(\Omega)) \) with \( k \geq 1 \), and if the elasticity operator for the associated extra elastostatic BVP is 0-regular, then there exists a constant \( C \), independent of \( h \), such that

\[
\|\dddot{u} - u\|_{L^\infty(0,T; L^2(\Omega))} + \|\frac{\partial}{\partial t} (\dddot{u} - u)\|_{L^\infty(0,T; L^2(\Omega))} 
\leq C(h^k + \chi), \quad (8.10)
\]

provided that there is a positive constant \( C_1 \) independent of \( M_N \) such that \( \|\partial^2 \dddot{u} / \partial t^2\|_{L^\infty(0,T; L^\infty(\Omega))} \leq C_1 \), \( \|\dddot{u}_0\|_{L^\infty(\Omega)} \leq C_1 \), and \( \|\nabla \partial^2 \dddot{u} / \partial t\|_{L^\infty(0,T; L^\infty(\Omega))} \leq C_1 \),

where \( \dddot{u} \) is defined by (8.1), and provided that the initial conditions for \( \dddot{u} \) satisfy the estimates

\[
\|\dddot{u}(0) - \dddot{u}_0\|_{W^{-1}(\Omega)} = O(h^k),
\]

\[
\|\frac{\partial}{\partial t} \dddot{u}(0) - (\dddot{u}_0)\|_{L^2(\Omega)} = O(h^k).
\]

(Notice that the initial conditions specified by (5.17) and the equation following it satisfy these conditions.)

Here \( \chi \) is defined by
\[ x = C_1 \| \tilde{\rho} - \rho \|_{L^2(\Omega)} + \| (\tilde{\rho} \tilde{f}) - \rho \tilde{f} \|_{L^2(0,T;L^2(\Omega))} \]

\[ + 2C_1 \sum_{i,j,k,l} \| \tilde{C}_{ijkl} - C_{ijkl} \|_{L^2(\Omega)} \cdot \]

**Proof:** Applying Lemma 8.1 to \( \xi \), we have

\[ \| \xi \|^{2}(\tau) \leq C(\| \xi \|^{2}(0) + \int_{0}^{\tau} \| \frac{\partial \xi}{\partial t} \|^{2} \, dt ) \]

\[ \leq C(\| \xi \|^{2}(0) + \| \tilde{\xi} \|^{2}(0) + \int_{0}^{T} \| \frac{\partial \xi}{\partial t} \|^{2} \, dt). \]

Then with the help of Gronwall inequality, we see that it suffices to prove (8.10) with \( \| \frac{\partial}{\partial t} (U-u) \|_{L^\infty(0,T;L^2(\Omega))} \) on the left hand side only. Subtracting (4.5)\(_1\) from (5.13)\(_1\), using (8.1), and choosing \( v = \tilde{v} = \frac{\partial \xi}{\partial t} \), we obtain

\[ \sum_{i} \langle \tilde{\rho} \frac{\partial^2 e_i}{\partial t^2}, \frac{\partial e_i}{\partial t} \rangle + \sum_{i} \langle (\tilde{\rho} - \rho) \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial e_i}{\partial t} \rangle \]

\[ + \sum_{i} \langle \tilde{\rho} \frac{\partial^2 e_i}{\partial t^2}, \frac{\partial e_i}{\partial t} \rangle \]
\[
+ \sum_{i,j,k,l} \left< \tilde{C}_{ijkl} \frac{\partial^2 \hat{e}}{\partial x_k \partial \tau} \right> \frac{\partial u_k}{\partial x_l} \frac{\partial^2 \hat{e}}{\partial \tau x_j} \\
+ \sum_{i,j,k,l} \left< (\tilde{C}_{ijkl} - C_{ijkl}) \frac{\partial u_k}{\partial x_l} \frac{\partial^2 \hat{e}}{\partial \tau x_j} \right> \\
- \sum_i \left< \rho f_i \right> - \rho f_i \frac{\partial \hat{e}}{\partial \tau} \\
- \lambda_0 \sum_i \left< \vec{e}_i \right> \frac{\partial \hat{e}}{\partial \tau} = 0. \tag{8.11}
\]

This equation is analogous to (8.5), except that a weighted inner product term in $W^1(\Omega)$ is replaced by one in $L^2(\Omega)$. By an argument similar to that of the preceding theorem, we then obtain the following analogue of (8.9):

\[
(\eta_u/4) ||\nabla \hat{e}||^2(\tau) + (\eta_\rho/4) ||\frac{\partial \hat{e}}{\partial \tau}||^2(\tau)
\leq C \left( ||\frac{\partial^2 \hat{e}}{\partial \tau^2}||^2_{L^2 \times L^2} + ||\nabla \hat{e}||^2(0) + ||\frac{\partial \hat{e}}{\partial \tau}||^2(0) + \chi^2 \right) ,
\]
which implies

\[ \left\| \frac{\partial e}{\partial t} \right\| (\tau) \leq C \left\| \frac{\partial^2 e}{\partial t^2} \right\|_{L^2_t L^2_x} \]

\[ + \left\| \nabla e \right\| (0) + \left\| \frac{\partial e}{\partial t} \right\| (0) + \chi. \]

Now applying Lemma 8.1 on \( \frac{\partial e}{\partial t} \) and using the triangle inequality, we get

\[ \left\| \frac{\partial e}{\partial t} \right\| (\tau) \leq C \left\| \frac{\partial^2 e}{\partial t^2} \right\|_{L^2_t L^2_x} \]

\[ + \left\| \frac{\partial e}{\partial t} \right\| (0) + \left\| \nabla e \right\| (0) + \left\| \frac{\partial e}{\partial t} \right\| (0) + \chi, \]

or, equivalently,

\[ \left\| \frac{\partial}{\partial t} (U - \bar{u}) \right\|_{L^\infty_t (0, T; L^2_x (\Omega))} \]

\[ \leq C \left\| \frac{\partial^2}{\partial t^2} (U - \bar{u}) \right\|_{L^2_t L^2_x (\Omega)} \]

\[ + \left\| \frac{\partial}{\partial t} (U - \bar{u}) \right\| (0) + \left\| \nabla (U - \bar{u}) \right\| (0) \]

\[ + \left\| \frac{\partial}{\partial t} (U - \bar{u}) \right\| (0) + \chi. \]
Notice that the time derivatives of \( \ddot{y} \) are the weighted \( W^1(\Omega) \) projections of the time derivatives of \( y \). Thus (8.10) is proved upon using Theorem 7.5 and the assumptions \( u_0, \dot{u}_0 \in W^k(\Omega) \) and \( (\frac{\partial^2}{\partial t^2}) u \in L_2(0,T;\bar{\varphi}^k(\Omega)) \).

Q.E.D.

Corollary 8.2 (\( L_2(\Omega) \)-Estimate for Continuous Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP Without Interpolation of Coefficients)

Under assumptions A(i), A(v), and A(iv)', if \( \mathcal{M}_N \) is an \( S^h_{1,k}(\Omega) \)-space, \( u_0, \dot{u}_0 \in W^k(\Omega) \cap W^1(\Omega), (\frac{\partial^2}{\partial t^2}) u \in L_2(0,T;\bar{\varphi}^k(\Omega)) \) with \( k \geq 1 \), and if the elasticity operator for the associated extra elastostatic BVP is \( 0 \)-regular, then there exists a constant \( C \), independent of \( h \), such that

\[
\| u - \dot{u} \|_{L_\infty(0,T;L_2(\Omega))} + \| \frac{\partial}{\partial t} (u - \dot{u}) \|_{L_\infty(0,T;L_2(\Omega))} \leq C h^k \quad (8.12)
\]

provided that the initial conditions for \( u \) satisfy the
estimates

\[ ||\mathbf{u}(0) - \bar{\mathbf{u}}(0)||_{W^1_0(\Omega)} = O(h^k), \]

\[ ||\frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}})(0)||_{L^2(\Omega)} = O(h^k), \]

where \( \bar{\mathbf{u}} \) is defined by (8.1).

Having treated the cases with homogeneous displacement BC, we now mention the analogous results for the cases with traction-free BC.

**Theorem 8.3 (W^1_{0})-Estimate for Continuous Time Galerkin Approximation of the Traction Free BC Elastodynamic I-BVP with Interpolation of Coefficients)**

Under assumptions A(i), A(v), A(iv), and (5.16), there exists a positive constant \( C \), independent of \( \mathbf{u} \) and the choice of \( \mathcal{M}_N \) such that (8.2) is valid provided \( u_0 \in W^1(\Omega), \forall u_0 \in L^\infty(\Omega), \forall \partial u/\partial t \in L^\infty(0, T; L^\infty(\Omega)), \) and \( \partial^2 u/\partial t^2 \in L^2(0, T; L^\infty(\Omega)). \) If further \( \mathcal{M}_N \) is an \( S^h_{1,k}(\Omega) \)-space, \( u_0, \bar{u}_0 \in W^k(\Omega), (\partial^2 u/\partial t^2) \bar{u} \in L^2(0, T; W^k(\Omega)) \) with \( k > 0 \) and \( \bar{u}(0) \) and \( \partial \bar{u}/\partial t(0) \) are chosen to be the \( \mathcal{M}_N \) interpolates of \( u_0 \) and \( \bar{u}_0 \) respectively, then we have (8.4).
Proof: The proof is very similar to that of Theorem 8.1. The main difference is that instead of (5.15) which is a consequence of the Gårding's inequality, we now use (5.16) which is a consequence of the equivalent form of the Second Korn's Inequality. In I-BVP with traction free BC the Poincaré's inequality is no longer applicable. Hence we have made the extra assumption: \( u_0 \in L_2(\Omega) \), as it is no longer implied by the condition \( \nabla \alpha u/\alpha t \in L_\infty(0,T;L_\infty(\Omega)) \).

Using \( u_0 \in L_2(\Omega) \) and applying Lemma 8.1 to \( \hat{e} \), we can replace the first term \( ||\nabla \hat{e}||^2 \) in (8.9) by \( ||\hat{e}||^2_{W^1(\Omega)} \).

From this point on, the proof is the same as in Theorem 8.1. Q.E.D.

Corollary 8.3 (\( W^1(\Omega) \)-Estimate for Continuous Time Galerkin Approximation of the Traction Free BC Elastodynamic I-BVP Without Interpolation of Coefficients)

Under assumptions A(i), A(v), and A(iv), if

\[
\dot{u}_0 \in L_2(\Omega), \quad u_0 \in W^1(\Omega), \quad \frac{\partial^2}{\partial t^2} u,
\]

\[
\nabla \frac{\partial u}{\partial t} \in L_2(0,T;L_2(\Omega)) ,
\]
then there exists a positive constant $C$, independent of $\chi$ and the choice of $M_N$, such that (8.2) is true with $\chi = 0$. The additional assumptions and results specialized to an $S^h_{1,m}(\Omega)$-space are the same as those of Theorem 8.3 except that $\chi = 0$.

We have also the following analogues of Theorem 8.3 and Corollary 8.3 for the Elastodynamic I-BVP with Traction Free BC.

**Theorem 8.4** \( L_2(\Omega) \)-Estimate for Continuous Time Galerkin Approximation of the Traction Free BC Elastodynamic I-BVP with Interpolation of Coefficients

Under assumptions A(i), A(ii), A(iv), and (5.16), if $M_N$ is an $S^h_{1,k}(\Omega)$-space, $u_0$, $\hat{u}_0 \in W^k(\Omega)$, $(\partial^2/\partial t^2)u \in L_2(0,T;W^k(\Omega)) \cap L_2(0,T;L_\infty(\Omega))$ with $k \geq 1$ and if the elasticity operator for the corresponding extra elastostatic BVP is 0-regular, then there is a constant $C$, independent of $h$, such that (8.10) is true, provided that the initial conditions for $\bar{U}$ satisfy the estimates

$$ ||\bar{U}(0) - \bar{u}(0)||_{W^1(\Omega)} = O(h^k), $$
\[ || \frac{\partial}{\partial t} (U - \tilde{u})(0) ||_{L^2(\Omega)} = O(h^k), \]

and provided that there exists a positive constant \( C_1 \) independent of \( M_N \) such that

\[ || \frac{\partial^2 \tilde{u}}{\partial t^2} ||_{L^\infty(0,T;L^\infty(\Omega))} \leq C_1, \]

\[ || \tilde{u}_0 ||_{L^\infty(\Omega)} \leq C_1, \]

and

\[ || \nabla \frac{\partial \tilde{u}}{\partial t} ||_{L^\infty(0,T;L^\infty(\Omega))} \leq C_1. \]

Here \( \tilde{u} \) is defined by

\[ \sum_{i,j,k,l} C_{ijk\ell} \frac{\partial}{\partial x_k} (u_k(\cdot,t) - \tilde{u}_k(\cdot,t)), \frac{\partial V_i}{\partial x_j} > + \sum_i < u_i(\cdot,t) - \tilde{u}_i(\cdot,t), V_i > = 0 \]

\[ \forall \psi \in M_N, \forall \ t \in [0,T]. \quad (8.13) \]

Corollary 8.4 \( (L^2(\Omega)) \)-Estimate for Continuous Time Galerkin Approximation of the Traction Free BC Elastodynamic I-BVP Without Interpolation of Coefficients)
Under assumptions A(i), A(v), and A(iv), if $\mathcal{N}_N$ is an $S_{1,k}^h(\Omega)$-space, $u_0, \dot{u}_0 \in W^k(\Omega), (\partial^2 / \partial t^2)u \in L_2(0,T;W^k(\Omega))$ with $k \geq 1$, and if the elasticity operator for the corresponding extra elastostatic BVP is 0-regular, then there exists a constant $C$, independent of $h$, such that (8.12) is true, provided that the initial conditions for $\Psi$ satisfy the estimates

$$||\Psi(0) - \tilde{u}(0)||_{W^1(\Omega)} = O(h^k),$$

$$||\frac{\partial}{\partial t} (\Psi - \tilde{u})(0)||_{L_2(\Omega)} = O(h^k),$$

where $\tilde{u}$ is defined by (8.13).

**Discrete Time Galerkin Approximation for the Elastodynamic I-BVP**

As in [13], we shall use the following notation for functions $\phi$ defined at discrete time steps:

$$||\phi||_{L^\Delta t(0,T;X)} = \max ||\phi^{m-1/2}||_{X},$$

$$||\phi||_{L_\Delta t^2(0,T;X)} = \sum ||\phi^{m-1/2}||_{X(\Delta t)},$$
\[ \| \phi \|_{L^2(0,T;X)}^2 = \sum \| \phi^m \|_X^2(\Delta t), \]

\[ \| \phi \|_{L^\infty(0,T;X)}^{\Delta t} = \max \| \phi^m \|_X, \]

where the range of \( m \) is \( \{0, 1, \ldots, M\} \). Unless otherwise specified, \( \tilde{e}, \hat{e}, \hat{e} \) and \( u \), etc. are defined in the same way as in the continuous time case. We need the analogue of Lemma 8.1, also due to Dupont [13], for the discrete time case.

**Lemma 8.2.** Let \( \xi \) be a function on the set \( \Gamma_\Delta = \{ t \in [0,T] : t = j\Delta t, j=0,1,\ldots,M \} \) with values in \( L^2(\Omega) \), and suppose that \( \Delta t < 1 \). Then we have

\[ \| \xi^{j-1/2} \|_{L^2(\Omega)}^2 \leq C \left( \| \xi^{M/2-1/2} \|_{L^2(\Omega)}^2 \right. \]

\[ + \left. \| \Delta_t \xi \|_{L^2(\tilde{\Delta}t,(M-1)\Delta t,j\Delta t;L^2(\Omega))}^2 \right), \]

and

\[ \| \xi^j \|_{L^2(\Omega)}^2 \leq C \left( \| \xi^{M/2-1} \|_{L^2(\Omega)}^2 \right. \]

\[ + \left. \| \Delta_t \xi \|_{L^2(\tilde{\Delta}t,(M-1)\Delta t,j\Delta t;L^2(\Omega))}^2 \right), \]
for any integers $j$ and $M_1$ such that $M_1 > 1$ and $M > j > M_1 + 1$.

For applications later, we state here some results. Let $X$ be a normed linear space and $\phi: [0, T] \rightarrow X$. We presume that $\phi$ possesses the necessary smoothness in the following statements:

\[
||\Delta_t^\phi||_{L^p_\Delta t(0, T; L_2(\Omega))} \\
\leq ||\frac{\partial \phi}{\partial t}||_{L^0(0, T; L_2(\Omega))},
\]

(8.14)

where $p = 2$ or $\infty$, cf. Douglas and Dupont [10, p. 625].

\[
||\Delta_t^{2\phi}||_{L^p_\Delta t(0, T; X)} \\
\leq 2 ||\frac{\partial^{2\phi}}{\partial t^2}||_{L^p(0, T; X)},
\]

(8.15)

where $p = 2$ or $\infty$, cf. Dupont [13, Lemma 7]. It follows from the definition of $\phi^{m,b}$ and (8.14) that

\[
||\phi^{m,b}||_{L^\Delta t(0, T; X)} \leq ||\phi||_{L^\infty(0, T; X)},
\]

(8.16)
\[ \| \Delta_t \phi^{m-1/2}, \|_{L^2_2(0,T;L^2_2(\Omega))} \]
\[ \leq \| \frac{\partial \phi}{\partial t} \|_{L^2(0,T;L^2(\Omega))}, \quad (8.17) \]

and
\[ \| \phi_{t}^{b} \|_{L^2_2(0,T;X)} \leq C \| \phi \|_{L^2(0,T;X)}. \quad (8.18) \]

We need also the following formulas for summation by parts
\[ \sum_{m=1}^{M_2-1} \phi^m \Delta_t \psi^{m+1/2}(\Delta t) = \phi^{M_2-1} \psi^M - \phi \psi^1 \]
\[ - \sum_{m=M_1+1}^{M_2-1} (\Delta_t \phi^{m-1/2}) \psi^m(\Delta t), \quad (8.19) \]
\[ \sum_{m=1}^{M_2-1} \phi^m \Delta_t \psi^{m-1/2}(\Delta t) = \phi^{M_2-1} \psi^M - \phi \psi^1 \]
\[ - \sum_{m=M_1}^{M_2-2} \Delta_t \phi^{m+1/2} \psi^m(\Delta t), \quad (8.20) \]

where \( M_1, M_2 \) are integers such that \( 1 \leq M_1 < M_2 \leq M. \)

Next we present some estimates about the accuracy
of our starting procedures. These results are needed later for error estimates. In the following two lemmas \( \bar{y} \) denotes the solution of (4.5) under assumptions A(i), A(v), and A(iv)' , or the solution of (4.6) under assumptions A(i), A(v), and A(iv).

**Lemma 8.3.** If \( Y^0 \) and \( Y^1 \) are defined by (5.17) and (5.21) or (5.22)-(5.23) with \( M_N \) in \( \bar{W}^1(\Omega) \) or \( \bar{W}^1(\Omega) \), and if \( \bar{u} \) is the Hermite interpolant of \( u \) in \( M_N \) at each \( t \in [0,T] \), then there exists a positive constant \( C \), independent of \( \Delta t \) and \( M_N \), such that

\[
|| Y^0 - \bar{u} ||_{\bar{W}^1(\Omega)} + || Y^1 - \bar{u}(\Delta t) ||_{\bar{W}^1(\Omega)}
\]

\[
+ || \Delta_t (Y^1 - \bar{u})^{1/2} ||_{L^2(\Omega)}
\]

\[
\leq C(|| \bar{u}_0 - u_0 ||_{\bar{W}^1(\Omega)} + || \bar{u}_0 - \bar{u}_0 ||_{\bar{W}^1(\Omega)}
\]

\[
+ || (\frac{\partial^2 u}{\partial t^2}(\cdot,0)) - \frac{\partial^2 u}{\partial t^2}(\cdot,0) ||_{\bar{W}^1(\Omega)}
\]

\[
+ (\Delta t)^2 || \frac{\partial^3 \bar{u}}{\partial t^3} ||_{L^\infty(0,T;\bar{W}^1(\Omega))}
\]

(8.21)
provided that \( u_0, \dot{u}_0, a^2u/\Delta t^2(\cdot, 0) \in W^1(\Omega), a^3u/\Delta t^3 \in L^\infty(0, T; W^1(\Omega)) \), and \( \Delta t < 1 \). If furthermore \( M_N \) is an \( S_{1,k}^h(\Omega) \)- or \( S_{1,k}^h(\Omega) \)-space, \( u_0, \dot{u}_0, a^2u/\Delta t^2(\cdot, 0) \in W^k(\Omega) \) with \( k > 0 \), then we have

\[
||u^0-\tilde{u}(0)||_{W^1(\Omega)} + ||u^1-\tilde{u}(\Delta t)||_{W^1(\Omega)}
\]

\[
+ ||\Delta t (U-\tilde{u})^{1/2}||_{L^2(\Omega)} \leq C(h^{k-1} + (\Delta t)^2), \quad (8.22)
\]

where \( C \) is independent of \( \Delta t \) and \( h \).

**Proof:** We shall prove the case when \( y^0 \) and \( y^1 \) are calculated by (5.17) and (5.21); the proof for the other case is simpler. Subtracting (5.17) from (5.21), and dividing the result by \( \Delta t \), we obtain

\[
\sum_{i,j,k,l} C_{ijkl} \frac{\partial}{\partial x_l} \left[ \Delta t (U_k-\tilde{u}_k)^{1/2} \right]
\]

\[
+ \frac{1}{\Delta t} (u_k^1-\tilde{u}_k^0-u_k^0-u_0^k), \quad \frac{\partial v_i}{\partial x_j}
\]

\[
+ \Lambda \sum_i \Delta t (U_i-\tilde{u}_i)^{1/2}
\]
Choosing \( \mathcal{Y} \equiv \Delta_t (\mathcal{U} - \mathcal{E})^{1/2} \) and using (5.16) or (5.15), we have

\[
| | \Delta_t (\mathcal{U} - \mathcal{E})^{1/2} | |_{W^1(\Omega)} \leq C | | 1 \Delta_t (\tilde{u}^1 - \tilde{u}^0 - u^* + u_0) | |_{W^1(\Omega)} .
\]

But since \( \frac{1}{\Delta_t} (\tilde{u}^1 - \tilde{u}^0 - u^* + u_0) \)

\[
= \frac{1}{\Delta_t} (\Delta_t \frac{\partial \mathcal{U}}{\partial t} (x, 0) + \frac{(\Delta_t)^2}{2} \frac{\partial^2 \mathcal{U}}{\partial t^2} (x, 0)
+ \int_0^{\Delta_t} \frac{(\Delta_t - \tau)^2}{2} \frac{\partial^3 \mathcal{U}}{\partial t^3} (\tau) \, d\tau
- [\Delta_t \frac{\partial \mathcal{U}}{\partial t} (x, 0) + \frac{(\Delta_t)^2}{2} \frac{\partial^2 \mathcal{U}}{\partial t^2} (x, 0) ]

= (\dot{u}_0 - \dot{u}_0) + \frac{\Delta t}{2} (( \frac{\partial^2 \mathcal{U}}{\partial t^2} (x, 0) ) - \frac{\partial^2 \mathcal{U}}{\partial t^2} (x, 0))
+ \frac{1}{2 \Delta t} \int_0^{\Delta t} (\Delta t - \tau)^2 \frac{\partial^3 \mathcal{U}}{\partial t^3} (\tau) \, d\tau ,
\]
by using the triangle inequality, the Generalized Minkowsky inequality, and the following inequality

\[ || \frac{\partial^2 \tilde{u}}{\partial t^2} ||_{L^\infty x \mathcal{W}^1_\lambda(\Omega)} \leq || \frac{\partial^2 u}{\partial t^2} ||_{L^\infty x \mathcal{W}^1_\lambda(\Omega)} + || \left( \frac{\partial^2 u}{\partial t^2} \right) - \frac{\partial^2 u}{\partial t^2} ||_{L^\infty x \mathcal{W}^1_\lambda(\Omega)}, \]

we see that \( || \Delta_t (Y - \tilde{u})^{1/2} ||_{\mathcal{W}^1_\lambda(\Omega)} \) is bounded above by the right-hand side of (8.21).

Next choosing \( Y = Y - \tilde{u} \) in (5.17), we have

\[ ||Y^0 - \tilde{u}(0)||_{\mathcal{W}^1_\lambda(\Omega)} \leq C ||u_0 - u_0||_{\mathcal{W}^1_\lambda(\Omega)}. \]

Thus the proof of (8.21) is complete upon noting that

\[ Y^1 - \tilde{u}(\Delta t) = Y^0 - \tilde{u}(0) + (\Delta t) \Delta_t (Y - \tilde{u})^{1/2}, \]

and using the triangle inequality. The proof of (8.22) is now obvious.

Q.E.D.

Lemma 8.4. If \( Y^0 \) and \( Y^1 \) are specified by (5.17) and (5.21) with the coefficient integrals calculated exactly and if \( \tilde{u} \) is the weighted \( \mathcal{W}^1_\lambda(\Omega) \)-projection of \( u \) in \( \mathcal{W}_N \) such that
\[ \sum_{i,j,k,l} C_{ijkl} \frac{\partial}{\partial x_l} (u_k - \bar{u}_k) \frac{\partial V_i}{\partial x_j} + \sum_{i} \lambda_i < u_i - \bar{u}_i, V_i > = 0, \]

\[ \forall \psi \in M_N, \forall t \in [0,T), \quad (8.23) \]

then there exists a positive constant \( C \), independent of \( \Delta t \), and \( M_N \) such that

\[ ||\psi^0 - \bar{\psi}(0)||_{W^1(\Omega)} + ||\psi^1 - \bar{\psi}(\Delta t)||_{W^1(\Omega)} + ||\Delta_t (\psi - \bar{\psi})^{1/2}||_{L^2(\Omega)} \leq C(\Delta t)^2 \quad (8.24) \]

provided that \( \frac{\partial^3 \psi}{\partial t^3} \in L_\infty(0,T;W^1(\Omega)) \).

\textbf{Proof:} From (5.17) and (8.23) it is clear that \( \psi^0 = \bar{\psi}(0) \).

Hence to prove (8.24), it suffices to show \( ||\psi^1 - \bar{\psi}(\Delta t)||_{W^1(\Omega)} \leq C(\Delta t)^3 \). Subtracting (8.23) from (5.21), and choosing \( \psi = \psi^1 - \bar{\psi}(\Delta t) \), we obtain

\[ \sum_{i,j,k,l} C_{ijkl} \frac{\partial}{\partial x_l} [(U^1_k - \bar{u}_k(\Delta t)) + (u_k(\Delta t) - u^k)] \frac{\partial}{\partial x_j} (U^1_i - \bar{u}_i(\Delta t)) \]

\[ + \sum_{i,j,k,l} C_{ijkl} \frac{\partial}{\partial x_l} [(U^1_k - \bar{u}_k(\Delta t)) + (u_k(\Delta t) - u^k)] \frac{\partial}{\partial x_j} (U^1_i - \bar{u}_i(\Delta t)) \]
\[ + \Lambda \sum_i < (U_i^1 - \tilde{u}_i(\Delta t)) \]

\[ + (u_i(\Delta t) - u_i^*), (U_i^1 - \tilde{u}_i(\Delta t)) > = 0. \]

Hence

\[ ||Y^1 - \tilde{u}(\Delta t)||_{W^1(\Omega)} \leq C ||u(\Delta t) - u^*||_{W^1(\Omega)} \]

\[ = C || \int_0^{\Delta t} \frac{(\Delta t - \tau)^2}{2} \frac{\partial^3 u}{\partial t^3}(\tau) d\tau ||_{W^1(\Omega)} \]

\[ \leq C(\Delta t)^3 || \frac{\partial^3 u}{\partial t^3} ||_{L^\infty(0,T;W^1(\Omega))}. \]

\[ \text{Q.E.D.} \]

We now present the error estimates for problems with homogeneous displacement BC.

**Theorem 8.5** \((W^1(\Omega))\)-Estimate for Discrete Time

**Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP with Interpolation of Coefficients**

Under assumptions \(A(i), A(v), A(iv)',\) and (5.15), if \(\Delta t\) is sufficiently small, and \(b > 1/4,\) then there
exists a positive constant $C$, independent of $\psi$, $\Delta t$, and the choice of $\mathcal{M}_N$, such that

$$
||Y - y||_{L^\infty(2\Delta t, T; W^1_\xi(\Omega))} + ||\Delta t (Y - y)||_{L^\infty(\Delta t, T; L_2(\Omega))} \leq C(\inf_{\tilde{y}} ||v_{\tilde{y}} \frac{\partial}{\partial t}(u - \tilde{y})||_{L_2(0, T; L_2(\Omega))})
$$

$$+
||\frac{\partial^2}{\partial t^2} (u - \tilde{y})||_{L_2(0, T; L_2(\Omega))} + ||u_0 - \tilde{u}_0||_{W^1_\xi(\Omega)} + ||\tilde{u}_0 - (\tilde{u}_0)\tilde{y}||_{W^1_\xi(\Omega)}
$$

$$+
||u^0 - \tilde{u}_0||_{W^1_\xi(\Omega)} + ||u^1 - \tilde{u}(\Delta t)||_{W^1_\xi(\Omega)}
$$

$$+
||\Delta t^2 (Y - y)^{1/2}||_{L_2(\Omega)}
$$

$$+
(\Delta t)^2 ||\frac{\partial^4 u}{\partial t^4}||_{L_2(0, T; L_2(\Omega))} + \chi),
$$

(8.25)

provided
\( \forall u, v \in L^\infty_2(\Omega), \frac{\partial u}{\partial t} \in L^\infty_2(0, T; L^\infty_2(\Omega)), \frac{\partial^2 u}{\partial t^2} \in L^2_2(0, T; L^\infty_2(\Omega)), \text{ and } \frac{\partial^4 u}{\partial t^4} \in L^2_2(0, T; L^2_2(\Omega)). \)

In (8.25), \( x \) is defined by

\[
x \equiv \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^\infty_2(0, T; L^\infty_2(\Omega))} + \left\| (\rho \phi', b) - \rho \phi', b \right\|_{L^2_2(\Delta t, T; L^2_2(\Omega))} + \left\| v \frac{\partial u}{\partial t} \right\|_{L^\infty_2(0, T; L^\infty_2(\Omega))} + \left\| \nabla \frac{\partial u}{\partial t} \right\|_{L^\infty_2(0, T; L^\infty_2(\Omega))} + \sum_{i, j, k, \xi} \left\| C_{ijk\xi} - C_{ijk\xi} \right\|. \tag{8.26}
\]

If furthermore, \( U^N \) is an \( S_{1,k}^0(\Omega) \)-space, \( y_0, \dot{y}_0, \frac{\partial^2 y}{\partial t^2}(\cdot, 0) \in W^k(\Omega), \frac{\partial^2 y}{\partial t^2} \in L^2(0, T; W^k(\Omega)) \) with \( k > 0 \), and \( y^0, \dot{y}^1 \) are specified by the starting procedures (5.17) and (5.21) or (5.22)-(5.23), then we have
\[ ||\hat{u}-u||_{L_\infty(0,T;W^1(\Omega))} \]

\[ + ||\Delta^m_t (\hat{u}-u)||_{L_\infty(0,T;L_2(\Omega))} \]

\[ \leq C(h^{k-1+}(\Delta t)^2 + \chi) , \]

(8.27)

where \( C \) is a positive constant independent of \( \Delta t \) and \( h \).

**Proof:** We note first that \( u \) satisfies

\[ \sum_i <\rho \Delta^2_t u^m_i, v_i> + \sum_{i,j,k,l} <C_{ijkl} \frac{\partial u^m_k}{\partial x_l}, \frac{\partial v_i}{\partial x_j}> \]

\[ - \sum_i <\rho f^m_i, b + \xi^m_i, v_i> = 0, \]

\[ \forall \, v \in W^1_0(\Omega), \, 1 \leq m \leq M-1 . \] (8.28)

This result is obtained by averaging (4.5) at times \( t^{m+1}, t^m, t^{m-1} \) with weights \( b, 1-2b, b \) respectively and then by replacing \( u^m_{tt} \) by \( \Delta^2_t u^m - \xi^m \). Subtracting (8.28) from (5.20), and using

\[ \hat{v} = \hat{V} = (\Delta^m_t \hat{e}^{m+1/2} + \Delta^m_t \hat{e}^{m-1/2})/2 \]

\[ = (\hat{e}^{m+1} - \hat{e}^{m-1})/(2\Delta t) , \]
we obtain the following analogue of (8.5),

\[ \sum_{i} < \tilde{\rho} (\Delta_{t} \tilde{e}_{i}^{m+1/2} - \Delta_{t} \tilde{e}_{i}^{m-1/2}) / \Delta t, \]

\[ (\Delta_{t} \tilde{e}_{i}^{m+1/2} + \Delta_{t} \tilde{e}_{i}^{m-1/2}) / 2 > \]

\[ + \sum_{i} \frac{1}{t} \langle \tilde{\rho} - \rho \rangle \Delta_{t} \tilde{u}_{i}^{m} + \rho \Delta_{t}^{2} \tilde{e}_{i} \]

\[ (\Delta_{t} \tilde{e}_{i}^{m+1/2} + \Delta_{t} \tilde{e}_{i}^{m-1/2}) / 2 > \]

\[ + \sum_{i,j,k,\ell} < \tilde{C}_{ijk\ell} \frac{\partial e_{k}^{m,b}}{\partial x_{k}}, \frac{\partial V_{i}}{\partial x_{j}} > \]

\[ + \sum_{i,j,k,\ell} \langle \tilde{C}_{ijk\ell} - C_{ijk\ell} \rangle \frac{\partial u_{k}^{m,b}}{\partial x_{k}} \]

\[ + \tilde{C}_{ijk\ell} \frac{\partial \tilde{e}_{k}^{m,b}}{\partial x_{k}}, \frac{\partial u_{k}^{m,b}}{\partial x_{j}} (\Delta_{t} \tilde{e}_{i}^{m+1/2} + \Delta_{t} \tilde{e}_{i}^{m-1/2}) / 2 > \]

\[ - \sum_{i} \langle \rho f_{i}^{m,b} - \tilde{\rho f}_{i}^{m,b} - \xi_{i}^{m} \rangle \]

\[ (\Delta_{t} \tilde{e}_{i}^{m+1/2} + \Delta_{t} \tilde{e}_{i}^{m-1/2}) / 2 > = 0. \quad (8.29) \]

We rewrite the third term of (8.29) in the form
\[
\frac{1}{(2\Delta t)} \sum_{i,j,k,\ell} \tilde{C}_{ijk\ell} \left( \frac{\partial \varepsilon^m}{\partial x_\ell} - \frac{\partial \varepsilon^{m-1}}{\partial x_\ell} \right) + \left( \frac{b}{2\Delta t} \right) \sum_{i,j,k,\ell} \tilde{C}_{ijk\ell} \left( \frac{\partial \varepsilon^{m+1}}{\partial x_\ell} - \frac{\partial \varepsilon^m}{\partial x_\ell} \right) \\
- \left( \frac{\partial \varepsilon^m}{\partial x_\ell} - \frac{\partial \varepsilon^{m-1}}{\partial x_\ell} \right) \right) \left( \frac{\partial \varepsilon^{m+1}}{\partial x_\ell} - \frac{\partial \varepsilon^m}{\partial x_\ell} \right) \right) > ,
\]

or, equivalently,

\[
\frac{b}{2\Delta t} \left( \| \varepsilon^{m+1} - \varepsilon^m \| \frac{2}{C} - \| \varepsilon^m - \varepsilon^{m-1} \| \frac{2}{C} \right) + \frac{1}{2\Delta t} \left( < \varepsilon^m, \varepsilon^{m+1} > - < \varepsilon^m, \varepsilon^{m-1} > - < \varepsilon^m, \varepsilon^m > \right)
\]

upon using (5.3), the \( \tilde{C} \) norm and \( \tilde{C} \) inner product in this expression are defined by

\[
\langle \mathbf{V}, \mathbf{W} \rangle \equiv \sum_{i,j,k,\ell} \tilde{C}_{ijk\ell} \left( \frac{\partial V_i}{\partial x_j} \frac{\partial W_k}{\partial x_\ell} \right),
\]

\[
\| \mathbf{V} \| \frac{2}{C} \equiv \langle \mathbf{V}, \mathbf{V} \rangle \frac{2}{C}.
\]
We now proceed to derive the analogue of (8.6). First, we rearrange the terms of (8.29) in the same way as that after (8.5). Then we take the discrete sum operation

\[ 2\Delta t \sum_{m=1}^{M_2-1}, \text{with } 2 \leq M_2 \leq M, \]

on the result and bound the right-hand side appropriately, obtaining

\[ b \left\| e^{M_2-1} \right\|_C^2 + \left\langle e^{M_2-1}, e^{M_2-1} \right\rangle_C \]

\[ + \left\| \Delta^{1/2} e^{M_2-1/2} \right\|_2^2 \]

\[ \leq C \left\{ \sum_{m=1}^{M_2 \frac{2}{\Delta t}} \left\| \Delta e^{m-1/2} \right\|_2^2 \right\}^{\Delta t} \]

\[ - \sum_{m=1}^{M_2-1} \sum_{i,j,k,\ell} (\tilde{C}_{ijkl} - C_{ijkl}) \frac{a_{u_k}^{m,b}}{a_{x_k}} \]

\[ + \tilde{C}_{ijkl} \frac{a_{e_k}^{m,b}}{a_{x_k}}, \frac{a_{x_j}}{a_{x_j}} (\Delta_t e_i^{m+1/2} + \Delta_t e_i^{m-1/2})/2 \]

\[ + \left\| \frac{\partial^2 e}{\partial t^2} \right\|_{L^2(0,T;L^2(\Omega))}^2 \]
+ \| \rho - \rho \|_2 \|^2 \frac{\partial^2 u}{\partial t^2} \|_{L_\infty (0,T; L_\infty (\Omega))} \\
+ \| (\rho f, b) - \rho f, b \|^2 \|_{L_2^\Delta t (\Delta t, T; L_2 (\Omega))} \\
+ \| \xi \|^2 \|_{L_2^\Delta t (\Delta t, T; L_2 (\Omega))} \\
+ \| \bar{\nu} \|^2 + \| \bar{\nu} \|^2 + \| \Delta \bar{\nu} \|^2. \quad (8.30)

In deriving the estimate (8.30), we have applied (8.15)
to \( \bar{u}, \bar{v} \), with \( p=2 \) and \( X = L_\infty (\Omega), L_2 (\Omega) \) respectively, and
we have used the telescoping nature of three summations.

Next we use the "summation by parts" formulas (8.19)
and (8.20) with

\[
\phi^m = \sum_{k,l} (\bar{C}_{ijkl} - C_{ijkl}) \frac{\partial u^m_k}{\partial x_l} + \frac{\partial \bar{e}^m}{\partial x_k} + C_{ijkl} \frac{\partial \bar{e}^m_k}{\partial x_l}
\]

and

\[
\psi = \frac{\partial \bar{e}}{\partial x_j},
\]
and obtain, in the same way as (8.7) and (8.6), an upper bound for the terms involving elasticities on the right-hand side of (8.30):

\[ C(|| \tilde{\varepsilon}^0 ||^2_{L^\infty(0,T;L^2(\Omega))} + \sum_{m=1}^{M-1} || \tilde{\varepsilon}^m ||^2(\Delta t) + || \tilde{\varepsilon}^{M-1} ||^2_{L^\infty(0,T;L^2(\Omega))}) \]

\[ + || \tilde{\partial u \over \partial x} ||^2_{L^\infty(0,T;L^2(\Omega))} \]

\[ + \sum_{i,j,k,l} \left| \tilde{C}_{ijkl} - C_{ijkl} \right|^2 \]

\[ + \eta ( || \tilde{\varepsilon}^M ||^2 + || \tilde{\varepsilon}^{M-1} ||^2 ) \]

In deriving this estimate, we have applied (8.16) to \( \partial u_k / \partial x_k, \partial \tilde{\varepsilon}^k / \partial x_k \) with \( X = L^\infty(\Omega), L^2(\Omega) \) respectively, and we have applied (8.17) to \( \partial u_k / \partial x_k, \partial \tilde{\varepsilon}^k / \partial x_k \). Now rewrite the left-hand side of (8.30) in the form

\[ b || \tilde{\varepsilon}^M - \tilde{\varepsilon}^{M-1} ||^2_C + \left( || \tilde{\varepsilon}^M + \tilde{\varepsilon}^{M-1} ||^2_C \right) \]
\[- \frac{1}{4} \left( \frac{\eta u}{Z} \right) \left( \| e^{\frac{M_2}{2}} \|_{W^1(\Omega)}^2 + \| e^{\frac{M_2^{-1}}{2}} \|_{W^1(\Omega)}^2 \right) + \| e^{\frac{M_2}{2}} \|_{L^2(\Omega)}^2 + \| e^{\frac{M_2^{-1}}{2}} \|_{L^2(\Omega)}^2 \] 

and using (5.15), (5.14), the assumption \( b > 1/4 \), and Lemma 8.2, we see that (8.31) can be bounded from below by

\[
\frac{1}{4} \left( \frac{\eta u}{Z} \right) \left( \| e^{\frac{M_2}{2}} + e^{\frac{M_2^{-1}}{2}} \|_{W^1(\Omega)}^2 \right) + \left( b - \frac{1}{4} \right) \left( \frac{\eta u}{Z} \right) \left( \| e^{\frac{M_2}{2}} - e^{\frac{M_2^{-1}}{2}} \|_{L^2(\Omega)}^2 \right) + \frac{\eta \rho}{Z} \left( \| \Delta_t e^{\frac{M_2^{-1}}{2}} \|_{L^2(\Omega)}^2 \right) 
\]

\[
> \left( \frac{\eta u}{2} \right) \min \left( \frac{1}{4}, b - \frac{1}{4} \right) \left( \| e^{\frac{M_2}{2}} + e^{\frac{M_2^{-1}}{2}} \|_{W^1(\Omega)}^2 + \| e^{\frac{M_2}{2}} - e^{\frac{M_2^{-1}}{2}} \|_{W^1(\Omega)}^2 \right)
\]
\[- \frac{1}{2} \| e^2 + e^{M_2 - 1} \|_{L^2(\Omega)}^2 + \frac{\eta_p}{2} \| \Delta e^{M_2 - 1/2} \|_{L^2(\Omega)}^2 \]

\[= \eta_u \min\left( \frac{1}{4}, b - \frac{1}{4} \right) \cdot \left( \| e^2 \|_{W^1(\Omega)}^2 + \| e^{M_2 - 1} \|_{W^1(\Omega)}^2 \right) \]

\[- 2C \left( \| e^{M_2} \|_{L^2(\Omega)}^2 + \| e^{M_2 - 1} \|_{L^2(\Omega)}^2 \right) \]

\[+ \frac{\eta_p}{2} \| \Delta e^{M_2 - 1/2} \|_{L^2(\Omega)}^2 \]

\[\geq \eta_u \min\left( \frac{1}{4}, b - \frac{1}{4} \right) \cdot \left( \| e^2 \|_{W^1(\Omega)}^2 + \| e^{M_2 - 1} \|_{W^1(\Omega)}^2 \right) \]

\[+ \frac{\eta_p}{2} \| \Delta e^{M_2 - 1/2} \|_{L^2(\Omega)}^2 \]
\[ - C(||\dot{\hat{e}}^1||_{L^2(\Omega)} + ||\dot{\hat{e}}^0||_{L^2(\Omega)}^2) + ||\Delta_t \hat{e}||_{L^2(\Delta t, T; L^2(\Theta))}^2 \]

Next we estimate the error due to time discretization. From

\[ \xi^m = \rho \left( - \frac{\partial u^{m,b}}{\partial t^2} + \Delta_t^2 u^m \right) \]

\[ = \frac{\rho}{(\Delta t)^2} \left\{ \int_{t_m}^{t_{m+1}} \frac{(t_{m+1} - \tau)}{3!} \frac{\partial^4 u}{\partial t^4}(\tau) d\tau \right. \]

\[ + \int_{t_m}^{t_{m-1}} \frac{(t_{m-1} - \tau)}{3!} \frac{\partial^4 u}{\partial t^4}(\tau) d\tau \}

\[ - \frac{\rho b}{4} \left\{ \int_{t_m}^{t_{m+1}} (t_{m+1} - \tau) \frac{\partial^4 u}{\partial t^4}(\tau) d\tau \right. \]

\[ + \int_{t_m}^{t_{m-1}} \frac{(t_{m-1} - \tau)}{3!} \frac{\partial^4 u}{\partial t^4}(\tau) d\tau \}, \]

we obtain
\[ ||\xi^m||^2 \leq C(\Delta t)^3 \left( \frac{3 u}{\Delta t^4} \right)^2 L_2(t^{m-1}, t^{m+1}; L_2(\Omega)). \]

Hence we have

\[ ||\xi||_{L_2^0(\Delta t, T; L_2(\Omega))} \leq C(\Delta t)^2 \left( \frac{3 u}{\Delta t^4} \right)^2 L_2(0, T; L_2(\Omega)). \]

By taking \( n \) sufficiently small, say \( n = (n_u/2) \cdot \min(1/4, b-1/4) \), we can rewrite (8.30) in the following form:

\[ \|\hat{e}^M\|_{L^1(\Omega)}^2 + \|\hat{e}^{M^{-1}}\|_{L^1(\Omega)}^2 + \|\Delta t \hat{e}^{M^{-1/2}}\|_{L^2(\Omega)}^2 \]

\[ \leq C \left( \|\nabla \hat{e}\|_{L^\infty L^2}^2 + \|\nabla \hat{e}\|_{L^2 L^2}^2 \right) \]

\[ + \|\frac{\partial \hat{e}}{\partial t}\|_{L^\infty L^2}^2 + \|\nabla \hat{e}^0\|_{L^2 L^2}^2 + \|\nabla \hat{e}^1\|_{L^2 L^2}^2 \]

\[ + \|\Delta t \hat{e}^{1/2}\|_{L^2 L^2}^2 + \|\nabla \hat{e}^2 + \hat{e}^2 + \hat{e}\|_{L^2 L^2}^2 \]

\[ + \|\Delta t \hat{e}^{1/2}\|_{L^2 L^2}^2 \]
provided that $\Delta t$ is sufficiently small. The last two terms on the right-hand side of (8.32) can be disposed of by using Gronwall's inequality. Using the triangle inequality, we have also

$$
||v_e^M ||^2 + ||\Delta_t v_e^M ||^2 
$$

$$
\leq C(||v||_{L^\infty_xL^2}^2 + ||v_t||_{L^\infty_xL^2}^2 + ||v_{tt}||_{L^2_xL^2}^2 + ||v_t^2||_{L^2_xL^2}^2 + ||v_{tt}^1||_{L^2_xL^2}^2 + ||\Delta_t v_{tt}^1||^2 
$$

$$
+ \delta + (\Delta t)^4 ||\frac{\partial^4 u}{\partial t^4}||_{L^2_xL^2}^2 
$$

From this point on the proof of (8.25) is similar to that of (8.2).

So far we have considered the case with $b > 1/4$. 
The proof for the case $b = 1/4$ is similar to that in [13]. Finally, the estimate (8.27) is an easy consequence of (8.25), Lemmas 6.1 and 8.3. Q.E.D.

**Corollary 8.5** $(W^1_0(\Omega))$-Estimate for Discrete Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP Without Interpolation of Coefficients

Under assumptions A(i), A(v), and A(iv)', if $\Delta t$ is sufficiently small, and $b > 1/4$, then there exists a positive constant $C$, independent of $u$, $\Delta t$, and the choice of $M_N$, such that (8.25) is true with $\chi = 0$ provided $u_0 \in W^1_0(\Omega)$, $u_0 \in L_2(\Omega)$, $u/\Delta t^2$, $v/\Delta t^2 \in L_2(0,T;L_2(\Omega))$, and $\Delta^4 u/\Delta t^4 \in L_2(0,T;L_2(\Omega))$. The additional assumptions and results specialized to an $S^h_{1,k}(\Omega)$-space and with $U^0$ and $U^1$ specified by (5.17) and (5.21) or (5.22)-(5.23) are the same as those of Theorem 8.5 except that $\chi = 0$.

**Theorem 8.6** $(L^2(\Omega))$-Estimate for Discrete Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP with Interpolation of Coefficients

Under assumptions A(i), A(v), A(iv)', and (5.15)
if $\mathcal{M}_N$ is an $C_{0,1,k}(\Omega)$-space, $u_0, \dot{u}_0 \in W^k(\Omega) \cap W^{1}(\Omega)$, $(\partial^2/\partial t^2)u \in L_2(0,T;W^k(\Omega)) \cap L_2(0,T;L_\infty(\Omega))$, $\partial^3 u/\partial t^3 \in L_\infty(0,T;W^1(\Omega))$, and $\partial^4 u/\partial t^4 \in L_2(0,T;L_2(\Omega))$ with $k > 0$, and if the elasticity operator for the associated extra elastostatic BVP is 0-regular, then there exists a positive constant $C$, independent of $\Delta t$ and $h$, such that

$$\|u^\Delta - u\|_{L_\infty^\Delta(0,T;L_2(\Omega))} + \|\Delta_t(u^\Delta - u)\|_{L_\infty^\Delta(0,T;L_2(\Omega))} \leq C(h^{k+}(\Delta t)^2 + \chi) \quad (8.33)$$

provided that $y^0, y^1$ are specified by the starting procedure (5.17) and (5.21) with the coefficient integrals calculated exactly, and that there is a positive constant $C_1$ independent of $\mathcal{M}_N$ such that

$$\|\partial^2 s^\Delta/\partial t^2\|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,$$

$$\|\dot{s}^\Delta\|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,$$

$$\|s^\Delta/\partial t\|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,$$
when \( \tilde{y} \) is defined by (8.1). In (8.33), \( x \) is defined by

\[
x \equiv C_1 \| \tilde{\sigma} - \sigma \|_{L^2(\Omega)}
\]

\[+ \| (\rho \tilde{\xi}^x, b) - \rho \tilde{\xi}^x, b \|_{L^2(\Delta t; T; L^2(\Omega))}
\]

\[+ 2C_1 \sum_{i,j,k,l} \| \tilde{C}_{ijkl} - C_{ijkl} \|_{L^2(\Omega)}.
\]

As before we assume \( \Delta t \) is sufficiently small.

**Proof:** The proof of (8.33) is a combination of the proofs for (8.27) and (8.10) with the following exception: Instead of using Lemma 8.3, we use Lemma 8.4 for the initial values \( \tilde{y}^0 \) and \( \tilde{y}^1 \).

Q.E.D.

**Corollary 8.6 (\( L^2(\Omega) \)-Estimate for Discrete Time Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP Without Interpolation of Coefficients)**

Under assumptions A(i), A(v), and A(iv)' if \( M_N \) is an \( S^0_{1,k}(\Omega) \)-space, \( \tilde{u}_0, \dot{u}_0 \in \tilde{W}^k(\Omega) \cap \tilde{W}^1(\Omega), (\partial^2 / \partial t^2) \dot{u} \in L^2(0,T; \tilde{W}^k(\Omega)), (\partial^3 \dot{u} / \partial t^3) \in L^2(0,T; \tilde{W}^1(\Omega)) \) and \( (\partial^4 \ddot{u} / \partial t^4) \in L^2(0,T; \tilde{W}^1(\Omega)) \).
\( L_2(0,T;L_2(\Omega)) \) with \( k > 0 \), and if the elasticity operator for the associated extra elastostatic BVP is \( 0 \)-regular, then there exists a positive constant \( C \), independent of \( \Delta t \) and \( h \) such that

\[
\| u^k - u^1 \|_{L^\infty(0,T;L_2(\Omega))} + \| \Delta_t (u^k - u^1) \|_{L^\infty(0,T;L_2(\Omega))} \leq C(h^k + (\Delta t)^2),
\]

provided that \( u^0, u^1 \) are specified by the starting procedures (5.17) and (5.21) with the coefficient integrals calculated exactly. As before, \( \Delta t \) is assumed to be sufficiently small.

Now we mention the analogous results of Theorem 8.5, Corollary 8.5, Theorem 8.6, and Corollary 8.6 for the cases with traction-free BC.

**Theorem 8.7 (W^1(\Omega)-Estimate for Discrete Time Galerkin Approximation of the Traction-Free BC Elastodynamic I-BVP with Interpolation of Coefficients).**
Under assumptions A(i), A(v), A(iv), and (5.16), if \( u_0 \in W^1(\Omega), \forall u_0 \in L_\infty(\Omega), \forall \partial u / \partial t \in L_\infty(0,T;L_\infty(\Omega)), \partial^2 u / \partial t^2 \in L_2(0,T;L_2(\Omega)), \) and \( \partial^4 u / \partial t^4 \in L_2(0,T;L_2(\Omega)), \) then there exists a positive constant \( C, \) independent of \( u, \Delta t, \) and the choice of \( \mathcal{M}_N, \) such that (8.25) is valid. If furthermore \( \mathcal{M}_N \) is an \( S_{1,k}^h(\Omega)-\)space, \( u_0, \dot{u}_0, \partial^2 u / \partial t^2(\cdot,0) \in W^k(\Omega), \partial^2 u / \partial t^2 \in L_2(0,T;W^k(\Omega)) \) with \( k > 0, \) and \( \bar{u}^0, \bar{u}^1 \) are specified by the starting procedures (5.17) and (5.21), or (5.22)-(5.23), then we have (8.27).

**Corollary 8.7 (\( W^1(\Omega) \)-Estimate for Discrete Time Galerkin Approximation of the Traction-Free BC Elastodynamic I-BVP Without Interpolation of Coefficients)**

Under assumptions A(i), A(v), and A(iv), if \( u_0 \in W^1(\Omega), \dot{u}_0 \in L_2(\Omega), (\partial^2 / \partial t^2)u, \forall (\partial u / \partial t) \in L_2(0,T;L_2(\Omega)), \) and \( \partial^4 u / \partial t^4 \in L_2(0,T;L_2(\Omega)), \) then there exists a positive constant \( C, \) independent of \( u, \Delta t, \) and the choice of \( \mathcal{M}_N, \) such that (8.25) is true with \( \chi = 0. \) The additional assumptions and results specialized to an \( S_{1,k}^h(\Omega)-\)space and with \( \bar{u}^0 \) and \( \bar{u}^1 \) specified by (5.17) and (5.21) or (5.22)-(5.23) are the same as those of Theorem 8.7 except that \( \chi = 0. \)
Theorem 8.8 \((L^2(\Omega))\)-Estimate for Discrete Time Galerkin Approximation of the Traction-Free BC Elastodynamic I-BVP with Interpolation of Coefficients

Under assumptions A(i), A(v), A(iv), and (5.16), if \(M_N\) is an \(S_{1,k}^h(\Omega)\)-space, \(u_0, u'_0 \in W^k(\Omega), (\partial^2 u/\partial t^2) u \in L_2(0,T;W^k(\Omega)) \cap L_2(0,T;L_\infty(\Omega)), \partial^3 u/\partial t^3 \in L_\infty(0,T;W^1(\Omega)),\) and \((\partial^4 u/\partial t^4) u \in L_2(0,T;L_\infty(\Omega))\) with \(k > 0\), and if the elasticity operator for the associated extra elastostatic BVP is 0-regular, then there exists a positive constant \(C\), independent of \(\Delta t\) and \(h\) such that (8.33) is true provided that there is a positive constant \(C_1\) independent of \(M_N\) such that

\[
\left| \left| \frac{\partial^2 u}{\partial t^2} \right| \right|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,
\]

\[
\left| \left| \nu u \right| \right|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,
\]

\[
\left| \left| \nu \frac{\partial u}{\partial t} \right| \right|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,
\]

and \(\tilde{u}\) is defined by (8.13), and that \(u^0, u^1\) are specified by the starting procedure (5.17) and (5.21) with the
coefficient integrals calculated exactly.

Corollary 8.8 (L₂(Ω)-Estimate for Discrete Time Galerkin Approximation of the Traction-Free BC Elastodynamic I-BVP Without Interpolation of Coefficients)

Under assumptions A(i), A(v), and A(iv), if Ψₙ is an Sₗₗ₁,ₖ(Ω)-space, Ψ₀, Ψ₀ ∈ ᴪₗₖ(Ω), (∂²/∂ₜ²)Ψ ∈ L₂(0,T;Ψₗₖ(Ω)), (∂³/∂ₜ³)Ψ ∈ L₂(0,T;Ψₗ⁻¹(Ω)), and (∂⁴/∂ₜ⁴)Ψ ∈ L₂(0,T;L₂(Ω)) with k > 0, and if the elasticity operator for the associated extra elastostatic BVP is O-regular, then there exists a positive constant C, independent of Δt and h, such that (8.34) is true provided that Ψ⁰, Ψ¹ are specified by the starting procedure (5.17) and (5.21) with the coefficient integrals calculated exactly.

Laplace-Modified Galerkin Approximation for Elastodynamic I-BVP with Homogeneous Displacement BC

With only minor modifications in the proofs, we have the following analogous results of Theorem 8.5, Corollary 8.5, Theorem 8.6 and Corollary 8.6.
Theorem 8.9 \((W^1_0(\Omega))-\text{Estimate for Laplace-}
\text{Modified Galerkin Approximation of the}
\text{Homogeneous Displacement BC Elastodynamic}
\text{I-BVP with Interpolation of Coefficients)\n
Under assumptions A(i), A(v), A(iv)', and (5.15),
if \(\forall u_0 \in L_\infty(\Omega), \forall \frac{\partial^2 u}{\partial t^2} \in L_\infty(0,T;L_\infty(\Omega)),\)
\(\frac{\partial^2 u}{\partial t^2} \in L_2(0,T;W^2(\Omega))\), and \(\frac{\partial^4 u}{\partial t^4} \in L_2(0,T;L_2(\Omega))\), then there
exists a positive constant \(C\), independent of \(u, \Delta t,\) and
the choice of \(M_N\), such that (8.25) is true with
\[(\Delta t)^2 \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L_2(0,T;L_2(\Omega))} \]
replaced by
\[(\Delta t)^2 \left( \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L_2(0,T;L_2(\Omega))} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L_2(0,T;W^2(\Omega))} \right). \]

Furthermore if \(M_N\) is an \(S^h_{1,k}(\Omega)\)-space, \(u_0, \dot{u}_0, (\frac{\partial^2 u}{\partial t^2})_0(\cdot,0)\)
\(\in W^k(\Omega)\), and \(\frac{\partial^2 u}{\partial t^2} \in L_2(0,T;W^k(\Omega))\), with \(k > 0\), and \(\Psi^0,\)
\(U^1\) are specified by the starting procedures (5.17) and
(5.21) or (5.22)-(5.23) then we have a positive constant.
C, independent of \(\Delta t\) and \(h\), such that (8.27) is true.

**Remark:** From Sobolev's lemma,

\[
\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;W^2(\Omega))
\]

implies

\[
\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^\infty(\Omega)) .
\]

**Proof:** It can be checked easily that \(u\) satisfies

\[
\sum_i <\rho \Delta_t^2 u_i^m, v_i> + \sum_{i,j} <\frac{\partial}{\partial x_j} (u_i^{m+1} - 2u_i^m + u_i^{m-1}), \frac{\partial v_i}{\partial x_j}> \\
+ \sum_{i,j,k,l} <C_{ijkl} \frac{\partial u_k^m}{\partial x_l}, \frac{\partial v_i}{\partial x_j}> \\
- \sum_i <\rho f_i^m + \xi_i^m, v_i> = 0,
\]

\[\forall \; v \in W^1(\Omega), \; 1 \leq m \leq M-1, \tag{8.35}\]

where
\[ \xi_i^m = \rho \left( \Delta_t u_i^m - \frac{\partial u_i^m}{\partial t} \right) \]

\[- b \nu^2 (u_i^{m+1} - 2u_i^m + u_i^{m-1}) \]  

(8.36)

Subtracting (8.35) from (5.26) and using the same test function as in the proof of Theorem 8.5, we have the analogue of (8.29):

\[ \sum_i < \tilde{\rho} \left( \Delta_t \tilde{e}_i^{m+1/2} - \Delta_t \tilde{e}_i^{m-1/2} \right) / \Delta t, \]

\[ (\Delta_t \tilde{e}_i^{m+1/2} + \Delta_t \tilde{e}_i^{m-1/2})/2 > \]

\[ + \sum_i < (\tilde{\rho} - \rho) \Delta_t u_i^m + \tilde{\rho} \Delta_t e_i \]

\[ (\Delta_t \tilde{e}_i^{m+1/2} + \Delta_t \tilde{e}_i^{m-1/2})/2 > \]

\[ + \left\{ \frac{b}{2\Delta t} \sum_{i,j} < \frac{a}{\partial x_j} (\tilde{e}_i^{m+1} - \tilde{e}_i^m) - (\tilde{e}_i^m - \tilde{e}_i^{m-1}) > \right\} \]

\[ \frac{a}{\partial x_j} \left( (\tilde{e}_i^{m+1} - \tilde{e}_i^m) + (\tilde{e}_i^m - \tilde{e}_i^{m-1}) \right) > \]

\[ + \sum_{i,j,k,l} < \zeta_{ijkl} \frac{\partial e_k}{\partial x_l}, \frac{\partial e_i^{m+1}}{\partial x_j}, \frac{\partial e_i^m}{\partial x_j}, - \frac{\partial e_i^{m-1}}{\partial x_j} > \} \]
\[
+ \left\{ \sum_{i,j,k,l} \left[ \bar{C}_{ijkl} - C_{ijkl} \right] \frac{\partial u_k}{\partial x_l} \right. \\
+ \bar{C}_{ijkl} \frac{\partial \bar{e}}{\partial x_k} - \frac{\partial}{\partial x_j} \left( \Delta_t \bar{e}_i^{m+1/2} + \Delta_t \bar{e}_i^{m-1/2} \right) / 2 > \\
+ b \sum_{i,j} \left( \bar{e}_i^{m+1} - 2 \bar{e}_i^{m} + \bar{e}_i^{m-1} \right), \\
\frac{\partial}{\partial x_j} \left( \Delta_t \bar{e}_i^{m+1/2} + \Delta_t \bar{e}_i^{m-1/2} \right) / 2 > \} \\
- \sum_{i} \left( (\rho \bar{f}_i^{m}) - \rho \bar{f}_i^{m} - \xi_i^{m} \right) \\
\left( \Delta_t \bar{e}_i^{m+1/2} + \Delta_t \bar{e}_i^{m-1/2} \right) / 2 > = 0 . \quad (8.37)
\]

As in the proof of Theorem 8.5, the terms in the first braces are telescoping in nature when \( m \) is summed from \( 1 \) to \( M_2 - 1 \), with \( 2 \leq M_2 \leq M \), the terms in the second braces can be summed by parts, and the remaining terms can be treated in the same way as those in the proof of Theorem 8.5. The left-hand side of the analogue of (8.30) is

\[
b \| e^M_2 - \bar{e}^{M_2-1} \|_{W^1(\Omega)}^2
\]
\[ + \left( \| \hat{e}_e^2 + \hat{e}_e^{M_2-1} \|_C^2 - \| \hat{e}_e^{M_2-1} \|_C^2 \right) / 4 \]

\[ + \| \rho^{1/2} \Delta_T \hat{e}_e^M_2 \|_2 \]

which can be bounded from below by,

\[ (b - 3c_u/8) \| \hat{e}_e^{M_2-1} \|_{W^1(\Omega)}^2 \]

\[ + (\eta_u/8) \| \hat{e}_e + \hat{e}_e^{M_2-1} \|_{W^1(\Omega)}^2 \]

\[ - (\lambda_0/4) \| \hat{e}_e + \hat{e}_e^{M_2-1} \|_{L_2(\Omega)}^2 \]

\[ + (\eta_p/2) \| \Delta_T \hat{e}_e^M_2 \|_2^2 \]

\[ \geq 2 \min (b - 3c_u/8, \eta_u/8) \]

\[ \cdot \left( \| \hat{e}_e^{M_2} \|_{W^1(\Omega)}^2 + \| \hat{e}_e^{M_2-1} \|_{W^1(\Omega)}^2 \right) \]

\[ - C \left( \| \hat{e}_e^{M_2} \|_2^2 + \| \hat{e}_e^{M_2-1} \|_2^2 \right) \]

\[ + (\eta_p/2) \| \Delta_T \hat{e}_e^M_2 \|_2^2 \]
\[ \geq 2 \min \left( b - 3 C_u / 8, \eta_u / 8 \right) \\
\cdot \left( \| \hat{e}^2 \|_{W^1_0(n)}^2 + \| \hat{e}^{M_2-1} \|_{W^1_0(n)}^2 \right) \\
- C \left( \| \hat{e}^1 \|_{L_2(n)}^2 + \| \hat{e}^0 \|_{L_2(n)}^2 \right) \\
+ \| \Delta_t \hat{e} \|_{L^2_2(\Delta t; T; L_2(n))}^2 \\
+ \frac{\eta_p}{2} \| \Delta_t e_i \|_{L^2_2(n)}^2, \tag{8.39} \]

upon using (5.15), (5.14), and Lemma 8.2.

In order to bound the error due to time discretization, we rewrite (8.36) as

\[ \xi^m_i = \rho \left( \int_{t^m}^{t^{m+1}} \frac{(t^{m+1} - \tau)}{3!} \frac{\partial^4 u_i}{\partial t^4} (\tau) d\tau \right. \\
+ \int_{t^m}^{t^{m-1}} \frac{(t^{m-1} - \tau)}{3!} \frac{\partial^4 u_i}{\partial t^4} (\tau) d\tau \right) \\
- b \nu^2 \int_{t^{m-1}}^{t^{m+1}} (\Delta t - |\tau - t^m|) \frac{\partial^2 u_i}{\partial t^2} (\tau) d\tau. \]
Then, we use the Generalized Minkowsky inequality and the Cauchy-Schwartz inequality to show

\[ \| \xi^m \|^2 \leq C(\Delta t)^3 \left( \| \frac{\partial^4 u}{\partial t^4} \|_{L_2(t^{m-1}, t^{m+1}; L_2(\Omega))} \right)^2 
+ \| \frac{\partial^2 u}{\partial t^2} \|_{L_2(t^{m-1}, t^{m+1}; W^2(\Omega))} \). \]

Hence we have

\[ \| \xi \|_{L_2^\Delta t(\Delta t, T; L_2(\Omega))} \]

\[ \leq C(\Delta t)^2 \left( \| \frac{\partial^4 u}{\partial t^4} \|_{L_2(0, T; L_2(\Omega))} \right)^2 
+ \| \frac{\partial^2 u}{\partial t^2} \|_{L_2(0, T; W^2(\Omega))} \). \]

From this point on the proof is the same as that of Theorem 8.5. Q.E.D.

**Corollary 8.9** (\(W^1(\Omega)\)-Estimate for Laplace-Modified Galerkin Approximation of the Homogeneous Displacement BC Elastodynamic I-BVP without Interpolation of Coefficients)
Under assumptions A(i), A(v), and A(iv)', if
\( u_0 \in W^1(\Omega), \quad \dot{u}_0 \in L^2(\Omega), \quad a^2 y/\dot{a}t^2 \in L^2(0,T;W^2(\Omega)), \quad \forall \ a y/\dot{a}t \in L^2(0,T;L^2(\Omega)), \quad \text{and} \quad a^2 y/\dot{a}t^4 \in L^2(0,T;L^2(\Omega)), \) then there exists a positive constant \( C \), independent of \( y, \Delta t \), and the choice of \( M_N \), such that (8.25) is true with \( \chi = 0 \) and

\[
(\Delta t)^2 \frac{a^4 u}{a t^4} \in L^2(0,T;L^2(\Omega))
\]

replaced by

\[
(\Delta t)^2 \left( \frac{a^4 u}{a t^4} \right) \in L^2(0,T;L^2(\Omega))
\]

\[
+ \frac{a^2 u}{a t^2} \in L^2(0,T;W^2(\Omega)).
\]

Furthermore if \( M_N \) is an \( S_{1,k}^0(\Omega) \)-space, \( u_0, \dot{u}_0, a^2 y/\dot{a}t^2(\cdot,0) \in W^k(\Omega) \), \( a^2 y/\dot{a}t^2 \in L^2(0,T;W^k(\Omega)) \) with \( k > 0 \), and if \( y^0 \), \( y^1 \) are specified by the starting procedures (5.17) and (5.21), (5.22)-(5.23) then we have a positive constant \( C \), independent of \( \Delta t \) and \( h \), such that (8.27) is true with \( \chi = 0 \).
Theorem 8.10 \((L_2(\Omega))\)-Estimate for Laplace- 
Modified Galerkin Approximation of the 
Homogeneous Displacement BC Elastodynamic 
I-BVP with Interpolation of Coefficients)

Under assumptions A(i), A(v), A(iv)', and (5.15), if \( u\) is an \( S_{1,k}^0(\Omega) \) space, \( u_0, \dot{u}_0 \in W^1_0(\Omega) \cap W^1(\Omega), (\partial^2/\partial t^2)u \in L_2(0,T;W^k(\Omega)) \cap L_2(0,T;L_\infty(\Omega)), (\partial^3/\partial t^3)u \in L_\infty(0,T;W^1(\Omega)), \) and \( (\partial^4/\partial t^4)u \in L_2(0,T;L_2(\Omega)) \) with \( k > 0 \), and if the elasticity operator for the associated extra elastostatic BVP is \( 0 \)-regular, then there exists a positive constant \( C \), independent of \( \Delta t \) and \( h \), such that (8.33) is true, provided that there is a positive constant \( C_1 \) independent of \( H_N \) such that the solution \( \tilde{u} \) defined by (8.1) satisfies the estimates

\[
||v\tilde{u}||_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1
\]

\[
||\partial_t^{3/2}\tilde{u}||_{L_\infty(0,T;L_\infty(\Omega))} \leq C_1,
\]

and

\[
||\partial_t^{2}\tilde{u}||_{L_2(0,T;W^2(\Omega))} \leq C_1,
\]
and provided that \( Y^0, Y^1 \) are specified by the starting procedure (5.17) and (5.21) with the coefficient integrals calculated exactly.

**Proof:** It can be checked that \( y \) satisfies

\[
\sum_i \left< \rho \Delta^2 u^m_i, v_i \right> + b \sum_{i,j} \left< \frac{\partial}{\partial x_j} (\ddot{u}^{m+1}_i - 2\ddot{u}_i^m + \ddot{u}_i^{m-1}), \frac{\partial v_i}{\partial x_j} \right> + \sum_{i,j,k,l} C_{ijkl} \left< \frac{\partial u^m_k}{\partial x_k}, \frac{\partial v_i}{\partial x_l} \right> - \sum_i \left< \rho \dot{f}_i^m + \xi^m_i, v_i \right> = 0,
\]

\( \forall \, \gamma \in \mathcal{W}^1_{\infty}(\Omega), \, 1 \leq m \leq M-1, \) \hspace{1cm} (8.40)

where

\[
\xi^m_i = \rho (\Delta^2 u^m_i - \frac{\partial^2 u^m_i}{\partial t^2}) - b v^2 (\ddot{u}^{m+1}_i - 2\ddot{u}_i^m + \ddot{u}_i^{m-1}) \hspace{1cm} (8.41)
\]
Hence

\[ \| \xi \|_{L^2_2(\Delta t; T; L^2_2(\Omega))} \]

\[ \leq C(\Delta t)^2 \left( \| \frac{\partial^4 u}{\partial t^4} \|_{L^2_2(0,T; L^2_2(\Omega))} \right) \]

\[ + \| \frac{\partial^2 u}{\partial t^2} \|_{L^2_2(0,T; W^2_1(\Omega))} \].

From this point on the same techniques of deriving optimal \( L^2_2(\Omega) \) error estimates for the transient problems together with the arguments in the proof of Theorem 8.9 can be used to prove this theorem. Q.E.D.

Corollary 8.10 \( (L^2_2(\Omega))-Estimate\) for Laplace-

**Modified Galerkin Approximation of the**

**Homogeneous Displacement BC Elastodynamic**

**I-BVP without Interpolation of Coefficients)**

Under assumptions A(i), A(iv) and A(iv)', if \( M_\infty \) is an \( S^h_{1,k}(\Omega) \)-space, \( u_0, \dot{u}_0 \in W^k_1(\Omega) \cap \bar{W}^1_0(\Omega), (\partial^2 / \partial t^2) u \in L^2_2(0,T; W^k_1(\Omega)), \partial^3 u / \partial t^3 \in L^\infty_\omega (0,T; W^1_0(\Omega)) \) and \( \partial^4 u / \partial t^4 \in L^2_2(0,T; L^2_2(\Omega)) \) with \( k > 0 \), and if the elasticity operator for the associated extra BVP is 0-regular, then there
exists a positive constant $C$, independent of $\Delta t$ and $h$ such that (8.34) is true provided that there is a positive constant $C_1$ independent of $\mathcal{M}_N$ such that the solution $\tilde{\mathbf{u}}$ defined by (8.1) satisfies the estimates

$$\left| \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} \right|_{L^2(0,T;W^2(\Omega))} \leq C_1,$$

and that $\mathbf{u}^0$, $\mathbf{u}^1$ are specified by the starting procedure (5.17) and (5.21) with the coefficient integrals calculated exactly.

**Alternating Direction Galerkin Methods on Rectangles for Elastodynamic I-BVP with Homogeneous Displacement BC**

We mention first some preliminary results which are useful in our error analysis later. Suppose $\mathbf{u} \in W^1(\Omega)$ and $\rho \in W^1(\Omega)$. Then we have

$$\frac{1}{\nu_p} \frac{\partial \mathbf{v}_k}{\partial x_\ell} = \frac{\partial}{\partial x_\ell} \left( \frac{\nu_k}{\nu_p} \mathbf{v}_k \right) + \frac{1}{2} \rho^{3/2} \frac{\partial \rho}{\partial x_\ell} \mathbf{v}_k \quad (8.42)$$

and
\[ \left\| \frac{v}{\sqrt{\rho}} \right\|^2 = \sum_{i,j} \frac{1}{\rho} \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \]

\[ - \sum_{i,j} \frac{1}{\rho} \frac{\partial v_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} v_i \]

\[ + \frac{1}{4} \sum_{i,j} \frac{1}{\rho} \frac{\partial \rho}{\partial x_j} v_i \frac{\partial \rho}{\partial x_j} v_i \]  \hspace{1cm} (8.43)

From (8.42), (8.43), and the assumption \( \|v\|_{L^\infty(\Omega)} < \infty \), it follows that given any (small) positive constant \( \eta \), there exists a constant \( C \) such that

\[ \left( \frac{1}{C_\rho} - \eta \right) \|v\|^2 - C \|v\|^2 \leq \left\| \frac{v}{\sqrt{\rho}} \right\|^2 \]

\[ \leq \left( \frac{1}{C_\rho} + \eta \right) \|v\|^2 + C \|v\|^2 \]  \hspace{1cm} (8.44)

We present also some estimates about the accuracy of our starting procedure.

**Lemma 8.5.** Under assumptions A(i), A(v), A(iv)', and \( \Delta t < 1 \), if \( \tilde{u} \) is the Hermite interpolate of \( u \) in \( \mathcal{M}_N \) at each \( t \in [0,T] \), and if \( u_0 \in W^2(\Omega) \), \( \partial u / \partial t \in L^\infty(0,T;W^1(\Omega)) \), \( \partial^2 u / \partial t^2 \in L^2(0,T;L^2(\Omega)) \), then there exists a positive constant \( C \), independent of \( u, \Delta t, \) and \( \mathcal{M}_N \) such that for \( n=2 \)
\[ ||\bar{\mathbf{u}}^0 - \bar{\mathbf{u}}_0 ||_{W^1(\Omega)} + ||\mathbf{u}^1 - \bar{\mathbf{u}}(\Delta t) ||_{W^1(\Omega)} \]

\[ + ||\Delta_t (\mathbf{u} - \bar{\mathbf{u}})^{1/2} ||_{L_2(\Omega)} \]

\[ + (\Delta t)^2 \left| \frac{\partial^2}{\partial x_1 \partial x_2} \right| \Delta_t (\mathbf{u} - \bar{\mathbf{u}})^{1/2} ||_{L_2(\Omega)} \]

\[ \leq C (||\frac{\partial}{\partial t} (\mathbf{u} - \bar{\mathbf{u}}) ||_{L_\infty(0,T;W^1(\Omega))}) \]

\[ + (\Delta t)^2 \left| \frac{\partial^3}{\partial t^3} \frac{\partial^2}{\partial x_1 \partial x_2} \right| \mathbf{u} - \bar{\mathbf{u}} ||_{L_2(0,T;L_2(\Omega))} \]

\[ + (\Delta t)^2 \left| \frac{\partial^2}{\partial t^2} \right| \mathbf{u} - \bar{\mathbf{u}} ||_{L_2(0,T;L_2(\Omega))}, \quad (8.45) \]

is true, and for \( n = 3 \), (8.45) is still true if the two differential symbols \( \frac{\partial^2}{\partial x_1 \partial x_2} \) are replaced by

\[ \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_3} + \frac{\partial^2}{\partial x_3 \partial x_1} + (\Delta t)^2 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}, \]

provided that \( \bar{\mathbf{u}}^0 = \bar{\mathbf{u}}_0 \) and \( \mathbf{u}^1 \) is the Galerkin approximation of \( u(\Delta t) \) defined by (5.32) for \( n = 2 \) and by (5.33) for \( n = 3 \). If furthermore \( \mathcal{M}_N \) is an \( \mathcal{S}_h^{1,0}(\Omega) \)-space, \( (\partial/\partial t) u \in L_\infty(0,T;W^k(\Omega)) \) with \( k \) being a positive integer, then we
have for $n = 2$

$$||y^0 - \tilde{y}_0||_{W^1(\Omega)} + ||y^1 - \tilde{y}(\Delta t)||_{W^1(\Omega)}$$

$$+ ||\Delta_t (y - \tilde{y})^{1/2}||_{L^2(\Omega)}$$

$$+ (\Delta t)^2 |\frac{\partial^2}{\partial x_1 \partial x_2} \Delta_t (y - \tilde{y})^{1/2}|_{L^2(\Omega)}$$

$$\leq C(h^{k-1} + (\Delta t)^2), \quad (8.46)$$

and for $n=3$, (8.46) with the same adjustment made on $\frac{\partial^2}{\partial x_1 \partial x_2}$, where $C$ is independent of $\Delta t$ and $h$.

**Proof:** We shall prove the case when $n = 2$; the proof for the case $n = 3$ is similar. Let $\tilde{e}^m = \tilde{y}^m - y^m$, $\tilde{e}^m = y^m - \tilde{y}^m$, $\tilde{e}^m = y^m - u^m$. Clearly $\tilde{e}^0 = 0$. We note that $u^* = u^1 - u_0 + \xi$ with

$$\xi = \int_0^{\Delta t} (\Delta t - \tau)^2 \frac{\partial^3 u}{\partial t^3} (\tau) d\tau.$$ 

Choosing

$$V = \Delta_t \tilde{e}^{1/2} = \frac{1}{\Delta t} (\tilde{e}^{1} - \tilde{e}^0) \quad \text{in (5.32),}$$
we have

\[ < (\Delta t)^2 \left( \Delta_t \varepsilon^1_{-1/2} + \Delta_t \varepsilon^{-1}_{-1/2} \right) + \xi, \Delta_t \varepsilon^1_{-1/2} > \]

\[ + \langle v \left( \varepsilon^1_{-1} - \varepsilon^0_{+1} + \varepsilon^1_{-0} + \varepsilon^0_{+0} \right), v \varepsilon^1_{-1} - \varepsilon^0_{+0} \rangle \]

\[ + (\Delta t)^4 < (\Delta t) \frac{\partial^2}{\partial x_1 \partial x_2} \left( \Delta_t \varepsilon^1_{-1/2} + \Delta_t \varepsilon^{-1}_{-1/2} \right) \]

\[ + \frac{\partial^2}{\partial x_1 \partial x_2} \xi, \frac{\partial^2}{\partial x_1 \partial x_2} \Delta_t \varepsilon^1_{-1/2} > = 0, \]

which implies

\[ (\Delta t) \| \Delta_t \varepsilon^1_{-1/2} \|^2 + \| v \left( \varepsilon^1_{-1} - \varepsilon^0_{+0} \right) \|^2 \]

\[ + (\Delta t)^5 \| \frac{\partial^2}{\partial x_1 \partial x_2} \Delta_t \varepsilon^1_{-1/2} \|^2 \]

\[ \leq (\Delta t) \| \Delta_t \varepsilon^{-1}_{-1/2} \|^2 + (\Delta t)^2 \| v \left( \Delta_t \varepsilon^1_{-1/2} \right) \|^2 \]

\[ + (\Delta t)^5 \| \frac{\partial^2}{\partial x_1 \partial x_2} \Delta_t \varepsilon^{-1}_{-1/2} \|^2. \]

(8.45) is proved upon using (8.14) and the assumption \( \Delta t < 1 \). The proof of (8.46) is obvious. \( \text{Q.E.D.} \)
Theorem 8.11. \((W^1(\Omega))\)-Estimate for Alternating Direction Galerkin Method on Rectangles for Elastodynamic I-BVP with Homogeneous Displacement BC Without Interpolation of Coefficients)

Under assumptions A(i), A(v) and A(iv)', on a rectangular domain, if \(\mathcal{M}_N\) is a subspace of \(\tilde{W}^1(\Omega)\) with a tensor-product basis, then there exists a positive constant \(C\), independent of \(u\), \(\Delta t\), and the choice of \(\mathcal{M}_N\), such that for \(n = 2\),

\[
||| \Delta_t (u-u) |||_{L^2(\Delta_t, T; L^2(\Omega))}^2 + \frac{\partial^2}{\partial x_1 \partial x_2} \Delta_t (u-u) |||_{L^2(\Delta_t, T; L^2(\Omega))}^2 \leq C \left( \inf_{\tilde{u}} \left( ||v^{\frac{2}{\partial t}} (u-\tilde{u})||_{L^2(0, T; L^2(\Omega))} + \frac{\partial^2}{\partial t^2} (u-\tilde{u}) |||_{L^2(0, T; L^2(\Omega))} \right) \right)
\]
\[ + (\Delta t)^2 \left| \frac{\partial^3}{\partial t \partial x_1 \partial x_2} (u - \bar{u}) \right|_{L^\infty(0,T;L^2(\Omega))} \]
\[ + \left\| u_0 - \bar{u}_0 \right\|_{W^1(\Omega)} + \left\| u_0 - (\bar{u}_0) \right\|_{W^1(\Omega)} \]
\[ + \left\| y^0 - \bar{u}_0 \right\|_{W^1(\Omega)} + \left\| y^1 - \bar{u}(\Delta t) \right\|_{W^1(\Omega)} \]
\[ + \left\| \Delta_t (y - \bar{u})^{1/2} \right\|_{L^2(\Omega)} \]
\[ + (\Delta t)^2 \left\| \frac{\partial^2}{\partial x_1 \partial x_2} \Delta_t (y - \bar{u})^{1/2} \right\|_{L^2(\Omega)} \}
\[ + (\Delta t)^2 \left\| \frac{\partial^4 u}{\partial t^4} \right\|_{L^2(0,T;L^2(\Omega))} \]
\[ + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(0,T;W^2(\Omega))} \]
\[ + \left\| \frac{\partial^4 u}{\partial x_1 \partial x_2 \partial t^2} \right\|_{L^2(0,T;L^2(\Omega))} \}, \quad (8.47) \]

is true, and for \( n = 3 \), (8.47) is still true if the four differentiation symbols \( \partial^2 / \partial x_1 \partial x_2 \) are replaced by

\[ \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_3} + \frac{\partial^2}{\partial x_3 \partial x_1} + (\Delta t)^2 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}, \]
provided that \( ||v_0||_{L^\infty(\Omega)} < \infty, u_0 \in W^1(\Omega), \dot{u}_0 \in L^2(\Omega), \)
\( v \in u/\partial t \in L^2(0,T;L^2(\Omega)), \) \( a^2 u/\partial t^2 \in L^2(0,T;W^2(\Omega)), \) and \( a^4 u/\partial t^4 \in L^2(0,T;L^2(\Omega)). \) In addition, we assume for \( n=2, \)
\[
\frac{a^3 u}{\partial t \partial x_1 \partial x_2} \in L^\infty(0,T;L^2(\Omega)),
\]
and for \( n = 3, \)
\[
\frac{a^3 u}{\partial t \partial x_1 \partial x_2}, \frac{a^3 u}{\partial t \partial x_3 \partial x_1}, \frac{a^3 u}{\partial t \partial x_2 \partial x_3}, \frac{a^4 u}{\partial t \partial x_1 \partial x_2 \partial x_3}
\]
\( \in L^\infty(0,T;L^2(\Omega)). \)

If furthermore \( M_N \) is an \( S_{1,k}(\Omega) \)-space, \( u_0 \in W^2(\Omega) \cap W^k(\Omega), \)
\( \dot{u}_0 \in W^k(\Omega), a^2 u/\partial t^2 \in L^2(0,T;W^k(\Omega)) \) with \( k > 0, \) and \( \dot{y}^0 \)
\( = \tilde{u}_0, \) while \( y^1 \) is defined by (5.32) for \( n = 2, \) or (5.33)
for \( n = 3, \) then we have for \( n = 2 \) or 3
\[
||U-u||_{L^\infty(t,0,T;W^1)}
\]
\[+ ||A_t(U-u)||_{L^\infty(t,0,T;L^2)}
\]
\[\leq C(h^{k-1} + (\Delta t)^2), \tag{8.48}\]
where $C$ is a positive constant independent of $\Delta t$ and $h$.

**Proof:** We shall prove the case when $n = 2$ only; the proof for the case $n = 3$ is similar. It can be checked easily that $v$ satisfies

$$
\sum_i < \Delta^2 u_i, v_i >
$$

$$
+ b(\Delta t)^2 \sum_{i,j} \frac{\partial}{\partial x_j} (\Delta^2 u_i^m), \frac{\partial v_i}{\partial x_j} >
$$

$$
+ b^2 (\Delta t)^4 \sum_i \frac{\partial^2}{\partial x_1 \partial x_2} (\Delta^2 u_i^m), \frac{\partial^2 v_i}{\partial x_1 \partial x_2} >
$$

$$
+ \sum_{i,j,k}\langle C_{ijk\ell} \frac{\partial u_k^m}{\partial x_j}, \frac{\partial v_i}{\partial x_k} (\frac{v_i}{\rho}) \rangle - \sum_i < f_i^m, v_i >
$$

$$
= \sum_i < \xi_i^m, v_i >
$$

$$
+ b^2 (\Delta t)^2 \sum_i \frac{\partial^2}{\partial x_1 \partial x_2} \frac{\omega_i^m}{\rho}, \frac{\partial^2 v_i}{\partial x_1 \partial x_2} >,
$$

$$
\forall \gamma \in \mathcal{M}_N, \quad 1 \leq m \leq M - 1,
$$

(8.49)
where

\[ \hat{e}_i^m = (\Delta_t \hat{u}_i^m - \frac{\partial \hat{u}_i^m}{\partial t}) \]

\[ - b v^2 (u_i^{m+1} - 2u_i^m + u_i^{m-1}), \quad (8.50) \]

\[ \hat{u}_i^m = u_i^{m+1} - 2u_i^m + u_i^{m-1}. \quad (8.51) \]

Subtracting (8.49) from (5.30) and using the same test function as in the proof of Theorem 8.5, we have the analogue of (8.37),

\[ \sum_i < (\Delta_t \hat{e}_i^{m+1} - \Delta_t \hat{e}_i^{m-1}) / \Delta t, \]

\[ (\Delta_t \hat{e}_i^{m+1} + \Delta_t \hat{e}_i^{m-1}) / 2 > \]

\[ + \sum_i < \Delta_t \hat{e}_i^{-1}, (\Delta_t \hat{e}_i^{m+1/2} + \Delta_t \hat{e}_i^{m-1/2}) / 2 > \]

\[ + \frac{b}{2 \Delta t} \sum_{i,j} \frac{\partial}{\partial x_j} \left\{ (\hat{e}_i^{m+1} - \hat{e}_i^m) - (\hat{e}_i^m - \hat{e}_i^{m-1}) \right\}, \]

\[ \frac{\partial}{\partial x_j} \left\{ (\hat{e}_i^{m+1} - \hat{e}_i^m) + (\hat{e}_i^m - \hat{e}_i^{m-1}) \right\} > \]
\[ + \frac{b^2(\Delta t)^4}{2} < \frac{\partial^2}{\partial x_1 \partial x_2}(\Delta t_e^{m+1/2} - \Delta t_e^{m-1/2}), \]

\[ \frac{\partial^2}{\partial x_1 \partial x_2}(\Delta t_e^{m+1/2} + \Delta t_e^{m-1/2}) > \]

\[ + \frac{1}{2\Delta t} \sum_{i,j,k,\lambda} C_{ijk\lambda} \frac{\partial e_k^m}{\partial x_\lambda} \frac{a}{\partial x_j}(\hat{e}_i^{m+1/p}) \]

\[- \frac{\partial}{\partial x_j}(\hat{e}_i^{m-1/p}) > \]

\[ + \sum_{i,j,k,\lambda} C_{ijk\lambda} \frac{\partial e_k^m}{\partial x_\lambda} \frac{a}{\partial x_j}(\frac{\Delta t_e^{m+1/2} + \Delta t_e^{m-1/2}}{2p}) > \]

\[ + b \sum_{i,j} \frac{\partial}{\partial x_j}(\hat{e}_i^{m+1} - 2\hat{e}_i^{m} + \hat{e}_i^{m-1}), \]

\[ \frac{\partial}{\partial x_j}(\Delta t_e^{m+1/2} + \Delta t_e^{m-1/2})/2 > \]

\[ + b^2(\Delta t)^4 < \frac{\partial^2}{\partial x_1 \partial x_2}(\Delta t_e^{m+1/2} - \Delta t_e^{m-1/2}), \]

\[ \frac{\partial^2}{\partial x_1 \partial x_2}(\Delta t_e^{m+1/2} + \Delta t_e^{m-1/2})/2 > \]

\[ = \sum_{i} \hat{\xi}_i^m, (\Delta t_e^{m+1/2} + \Delta t_e^{m-1/2})/2 > \]
\[ + b^2 (\Delta t)^2 \leq \frac{a^2}{\partial x_1 \partial x_2} \omega^m_i, \]

\[ \frac{a^2}{\partial x_1 \partial x_2} (\Delta t e_i^{m+1/2} + \Delta t e_i^{m-1/2})/2 > . \]  

(8.52)

At this point we should like to comment that in the case \( \rho \) is uniform, (say \( \rho = 1 \)) (8.47) can be proved as in [11]. Indeed, comparing (8.52) with (8.37), we see that only those three terms involving a factor \( b^2 \) need to be considered. The first term has the telescoping property when summed, so that it can be treated in the usual way. The second term can be treated by using the Cauchy-Schwartz inequality, and, when summed later, also by using the Gronwall inequality. The third term can be treated in the same way as the second term but we need also the following estimates:

\[ \left| \frac{\partial^2 \omega}{\partial x_1 \partial x_2} \right|_{L^2(0,T;L^2(\Omega))} \]

\[ \leq (\Delta t)^2 \left| \frac{\partial^4 u}{\partial t \partial x_1 \partial x_2} \right|_{L^2(0,T;L^2(\Omega))} , \]

which is a consequence of
\[ \omega^m_i = \int_{t^{m-1}}^{t^m+1} (\Delta t - |\tau - t^m|) \frac{\partial^2 u_i}{\partial t^2}(\tau) d\tau, \]

and

\[ \left\| \frac{\partial^2}{\partial x_1 \partial x_2} \Delta t^{-1/2} \right\|_{L_2(\Omega)} \leq \left\| \frac{\partial^3 e}{\partial t \partial x_1 \partial x_2} \right\|_{L_2(0,T;L_2(\Omega))}, \]

which follows from (8.14).

Now we can come back to the case when \( \rho \) is not uniform. The difficulty in this general case is mainly due to the fact that we can no longer make an assumption as strong as (4.9). To overcome this difficulty we treat the terms involving both \( \rho \) and \( C_{ijkl} \) in (8.52) in the following way:

First, an upper bound for the second term involving \( \rho \) in (8.52) can be found easily by using assumption A(\( \nu \)). Then that term can be summed with respect to time and treated by the now familiar techniques, i.e. be using the Cauchy-Schwartz inequality, Gronwall inequality, (8.44), Lemma 8.2, etc.

The first term involving \( \rho \) in (8.52) is somewhat
more difficult to handle. Applying (8.42) twice, we have

\[
\frac{\partial \hat{e}_k}{\partial x_k} \frac{\partial}{\partial x_j} (\hat{e}_i^{m+1}/\rho) = \rho^{1/2} \left( \frac{\partial}{\partial x_k} (\hat{e}_k/\sqrt{\rho}) - \frac{1}{2} \rho^{-3/2} \frac{\partial}{\partial x_k} (\hat{e}_k^{m+1}) \right)
\]

\[
- \left( \frac{\partial}{\partial x_j} (\hat{e}_i^{m+1}/\sqrt{\rho}) - \frac{1}{2} \rho^{-2} \frac{\partial}{\partial x_j} (\hat{e}_i^{m+1}) \right)
\]

\[
= \frac{\partial}{\partial x_k} (\hat{e}_k/\sqrt{\rho}) \frac{\partial}{\partial x_j} (\hat{e}_i^{m+1}/\sqrt{\rho})
\]

\[
+ \frac{1}{2} \rho^{-3/2} \left( \frac{\partial}{\partial x_k} e_k \frac{\partial}{\partial x_j} (\hat{e}_i^{m+1}/\sqrt{\rho}) \right)
\]

\[
- \frac{\partial}{\partial x_j} \hat{e}_i^{m+1} \frac{\partial}{\partial x_k} (\hat{e}_k/\sqrt{\rho}) - \frac{\rho^{-3}}{4} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} (\hat{e}_k^{m+1} \hat{e}_i^{m+1})
\]

Using this result, we can decompose that term into three parts, of which two can be treated in the same way as before, while the remaining one is telescoping in nature when \( m \) is summed from 1 to \( M_2 - 1 \) with \( 2 \leq M_2 \leq M \). Thus the analogue of (8.38) is

\[
b \| e^{M_2 - \hat{e}}^{M_2 - 1} \|_2 + \| \rho^{-1/2} (\hat{e}^{M_2} + \hat{e}^{M_2 - 1}) \|_C^2
\]

\[
- \| \rho^{-1/2} (\hat{e}^{M_2} - \hat{e}^{M_2 - 1}) \|_C^2 / 4 + \| \rho^{1/2} A_{\hat{e}} e^{M_2 - 1/2} \|_C^2
eq
\]
As in (8.39) we can bound this expression from below by

\[ 2 \min \left( b - \frac{C_u}{4 \eta_p}, \frac{\eta_u}{4 C_p} \right) \left( \| \varepsilon^M_T \|^2_{L^2(\Omega)} + \| \varepsilon^{M-1}_T \|^2_{L^2(\Omega)} \right) \]

\[ - C \| \varepsilon^1 \|^2_{L^2(\Omega)} + \| \varepsilon^0 \|^2_{L^2(\Omega)} \]

\[ + \| \Delta_t \varepsilon \|^2_{L^2(\Delta_t, T; L^2(\Omega))} + \eta \| \Delta_t \varepsilon^{M-1/2} \|^2_{L^2(\Omega)}, \]

upon using (5.15). 1.

From this point on the proof is the same as that of Theorem 8.5. The proof of (8.48) is obvious upon using Lemma 8.5.

Q.E.D.
9. A PRIORI ESTIMATE FOR GALERKIN APPROXIMATION
   FOR THERMOELASTIC INITIAL-BOUNDARY
   VALUE PROBLEMS

In this section we present the error bounds for
the approximating solutions based on (5.34) in the
continuous-time cases and those based on (5.37) and (5.38)
in the discrete-time cases. For simplicity, we shall
state and prove the error bounds under assumptions A(i),
A(ii), A(iii), A(iv), A(v) only and without interpolation
of coefficients. However, our results can be modified,
as those in the preceding section, when A(iv) is replaced
by A(iv)', and when the coefficients are interpolated.
For problems with homogeneous displacement BC (and on
rectangular domains), we can formulate Laplace-Modifed
(and Alternating-Direction) Galerkin methods for the
mechanical sub-problem and derive their error-estimates.
Again, we omit the detail of these results.

Let \( \tilde{u} \in M_N \) be defined by (8.1) with \( \lambda_0 = 0 \) when
the homogeneous displacement BC is used and by (8.13) when
the traction-free BC is used. Similarly, let \( \tilde{\varepsilon} \in M_{N_\theta} \) be
defined by
\[ \sum_{i,j} K_{ij} \frac{\partial}{\partial x_j} (\theta(\cdot, t) - \bar{\theta}(\cdot, t)), \frac{\partial}{\partial x_i} \phi > = 0 \]

\[ \forall \phi \in \mathcal{M}_{N_0}, \forall t \in [0,T], \quad (9.1) \]

when the homogeneous temperature BC is used, and by

\[ \sum_{i,j} K_{ij} \frac{\partial}{\partial x_j} (\theta(\cdot, t) - \bar{\theta}(\cdot, t)), \frac{\partial}{\partial x_i} \phi > + < \theta(\cdot, t) - \bar{\theta}(\cdot, t), \phi > = 0, \]

\[ \forall \phi \in \mathcal{M}_{N_0}, \forall t \in [0,T], \quad (9.2) \]

when the heat flux BC is used. We define

\[ \bar{e} = \bar{u} - u, \quad \hat{e} = U - \bar{u}, \quad e = U - u = \hat{e} + \bar{e}, \]

\[ \bar{e} = \bar{\theta} - \theta, \quad \hat{e} = \Theta - \bar{\theta}, \quad e = \Theta - \theta = \hat{e} + \bar{e}. \]

In deriving our error estimates, we shall use the techniques of Douglas and Dupont [10], Dupont, Fairweather, and Johnson [14], Price and Varga [52, p. 62], and Wheeler [54] for the thermal part of our problem.
Continuous-Time Galerkin Approximations
for the Thermoelastic I-BVP

Theorem 9.1. (Error-Estimate for Continuous
Time Galerkin Approximation of Homogeneous
Dirichlet or/and Zero Flux BC Thermoelastic)
I-BVP without Interpolation Coefficients)

Under assumptions $A(i)-A(v)$, and $m_{ij} \in L_\infty(\Omega)$, if
$u_0 \in W^1_0(\Omega)$, $\dot{u}_0 \in L_2(\Omega)$, $(\varepsilon^2/\partial t^2) u$, $v \varepsilon \partial u/\partial t \in L_2(0,T;L_2(\Omega))$,
$\varepsilon_0 \in L_2(\Omega)$, $\varepsilon \in L_2(0,T;W^1_0(\Omega))$, $\varepsilon \partial \varepsilon/\partial t \in L_2(0,T;L_2(\Omega))$, then

$W^1_0(\Omega)$ Error Estimate

There exists a positive constant $C$ independent of
$(u, \varepsilon)$ and the choice of $M_N$, such that

$$
\|u - u\|_{L_\infty(0,T;W^1_0(\Omega))} + \|\varepsilon \partial u/\partial t\|_{L_\infty(0,T;L_2(\Omega))}
$$

$$
+ \|\varepsilon - \varepsilon\|_{L_\infty(0,T;L_2(\Omega))} + \|v(\varepsilon - \varepsilon)\|_{L_2(0,T;L_2(\Omega))}
$$

$$
\leq C\|v \varepsilon \partial (u - \dot{u})\|_{L_2(0,T;L_2(\Omega))}
$$
\[ + \| \frac{\partial^2}{\partial t^2} (u - \tilde{u}) \|_{L^2(0,T;L^2(\Omega))} \\
+ \| v(\delta - \tilde{\delta}) \|_{L^2(0,T;L^2(\Omega))} + \| \delta \frac{\partial}{\partial t} (\delta - \tilde{\delta}) \|_{L^2(0,T;L^2(\Omega))} \\
+ \| \frac{\partial}{\partial t} (u - \tilde{u}) \|_{L^\infty(0,T;W^1_\infty(\Omega))} + \| u - \tilde{u} \|_{L^\infty(0,T;W^1_\infty(\Omega))} + \| \delta - \tilde{\delta} \|_{L^\infty(0,T;L^2(\Omega))} \]
\]

(9.3)

If furthermore \( \mathcal{M}_N \) is an \( S_{1,k}(\Omega) \)-space, \( u_0, \dot{u}_0 \in W^k(\Omega), (a^2/\partial t^2)u \in L^2(0,T;W^k(\Omega)), \theta_0 \in W^k(\Omega), \theta \in L^2(0,T;L^2(\Omega)), \) with \( k > 0 \), then we have

\[ \| u - \tilde{u} \|_{L^\infty(0,T;W^1_\infty(\Omega))} + \| \frac{\partial}{\partial t} (u - \tilde{u}) \|_{L^\infty(0,T;L^2(\Omega))} \]

\[ + \| \theta - \tilde{\theta} \|_{L^\infty(0,T;L^2(\Omega))} + \| v(\delta - \tilde{\delta}) \|_{L^2(0,T;L^2(\Omega))} \]

\[ \leq C h^{k-1} , \]

(9.4)

where \( C \) is a positive constant independent of \( h \).

**L_2(\Omega)-Error Estimate**

When either the temperature or the displacement satisfies the homogeneous Dirichlet BC, and furthermore
\[ \forall m_{ij} \in L_\infty(\Omega), \text{ there exists a positive constant } C \text{ independent of } (u, \theta) \text{ and the choice of } M_N \text{ such that} \]

\[ \| u - \bar{u} \|_{L_\infty(0,T;L_2(\Omega))} + \| \frac{\partial}{\partial t} (u - \bar{u}) \|_{L_\infty(0,T;L_2(\Omega))} \]

\[ + \| \theta - \bar{\theta} \|_{L_\infty(0,T;L_2(\Omega))} \]

\[ \leq C \left\{ \| \frac{\partial^2}{\partial t^2} (u - \bar{u}) \|_{L_2(0,T;L_2(\Omega))} + \| \frac{\partial}{\partial t} (\theta - \bar{\theta}) \|_{L_2(0,T;L_2(\Omega))} \right\} \]

\[ + \| \frac{\partial}{\partial t} (u - \bar{u}) \|_{L_2(\Omega)} + \| u - \bar{u} \|_{L_2(\Omega)} \]

\[ + \| \theta - \bar{\theta} \|_{L_2(\Omega)} \]

\[ + \| \theta - \bar{\theta} \|_{L_2(\Omega)} \]

\[ + \| u - \bar{u} \|_{W^1_1(\Omega)} \leq h^k \] \quad (9.5)

If furthermore \( M_N \times M_N \) is an \( S_{1,k}(\Omega) \)-space, \( u_0, \bar{u}_0 \in W^k(\Omega), (\partial^2/\partial t^2)u \in L_2(0,T;W^k(\Omega)), \theta_0 \in W^k(\Omega), \theta \in L_2(0,T;L_2(\Omega)), \) with \( k > 0 \), and if the operators used in defining \( \bar{u} \) and \( \theta \) are \( 0 \)-regular, then we have

\[ \| \frac{\partial}{\partial t} (u - \bar{u}) \|_{L_\infty(0,T;L_2(\Omega))} \]

\[ + \| \theta - \bar{\theta} \|_{L_\infty(0,T;L_2(\Omega))} \leq C h^k, \] \quad (9.6)
where C is a positive constant independent of h.

**Proof:** Subtracting (4.7)$_1$ from (5.34)$_1$, and choosing \( y = \dot{y} = \dot{\varepsilon} / \dot{t} \), we obtain the analogue of (8.5)

\[
\sum_{l} \langle \rho \frac{\partial r^e_i}{\partial t} \rangle + \sum_{l} \langle \rho \frac{\partial e^e_i}{\partial t} \rangle + \sum_{l} \langle \rho \frac{\partial e^e_i}{\partial t} \rangle 
\]

\[
+ \sum_{i,j,k,\lambda} \langle C_{ijk\lambda} \frac{\partial e_k}{\partial x_\lambda} + \frac{\partial e_k}{\partial x_\lambda} \frac{\partial e^e_i}{\partial t} \rangle 
\]

\[
- \langle \sum_{i,j} \varepsilon_{ij} \frac{\partial^2 e^e_i}{\partial t \partial x_j} \rangle + \langle \sum_{i,j} \varepsilon_{ij} \frac{\partial^2 e^e_i}{\partial t \partial x_j} \rangle = 0.
\]

(9.7)

Subtracting (4.7)$_2$ from (5.34)$_2$, and choosing \( \phi = \phi = \dot{\varepsilon} \), we have

\[
\langle \rho c_D \frac{\partial \dot{\varepsilon}}{\partial t} \rangle + \langle \rho c_D \frac{\partial \dot{\varepsilon}}{\partial t} \rangle + \sum_{i,j} \langle \dot{\varepsilon}_{ij} \frac{\partial \dot{\varepsilon}}{\partial x_j} \rangle + \sum_{i,j} \langle \dot{\varepsilon}_{ij} \frac{\partial \dot{\varepsilon}}{\partial x_j} \rangle 
\]

\[
+ \sum_{i,j} \langle K_{ij} \frac{\partial \dot{\varepsilon}}{\partial x_j} \rangle + \sum_{i,j} \langle \dot{\varepsilon}_{ij} \frac{\partial \dot{\varepsilon}}{\partial x_j} \rangle = 0.
\]

(9.8)
Now multiplying (9.7) by $\theta_C$, and adding the result to (9.8), we have

$$
\frac{\theta_C}{2} \frac{\partial}{\partial t} \left| \begin{bmatrix} \varepsilon \end{bmatrix} \right| \left| \begin{bmatrix} \varepsilon \end{bmatrix} \right| + \frac{\theta_C}{2} \frac{\partial}{\partial t} \sum_{i,j,k,l} \mathcal{C}_{ijkl} \frac{\partial \varepsilon_k}{\partial x_i} \frac{\partial \varepsilon}{\partial x_j}
$$

$$
+ \frac{1}{2} \frac{\partial}{\partial t} \left| \begin{bmatrix} \rho \varepsilon_{CD} \end{bmatrix} \right|^{1/2} \left| \begin{bmatrix} \varepsilon \end{bmatrix} \right|^{2} + \sum_{i,j} \mathcal{K}_{ij} \frac{\varepsilon_i}{\partial x_i} \frac{\varepsilon_j}{\partial x_j}
$$

$$
= \{- \theta_C \sum_{i,j,k,l} \mathcal{C}_{ijkl} \frac{\partial \varepsilon_k}{\partial x_i} \frac{\partial \varepsilon}{\partial x_j}
$$

$$
- \sum_{i,j} \mathcal{K}_{ij} \frac{\varepsilon_i}{\partial x_i} \frac{\varepsilon_j}{\partial x_j}\} - \theta_C \rho \frac{\partial \varepsilon}{\partial t} \frac{\partial \varepsilon}{\partial t}
$$

$$
- \rho \frac{\partial \varepsilon_{CD}}{\partial t} \frac{\varepsilon}{\partial t} + \theta_C \sum_{i,j} \mathcal{m}_{ij} \frac{\varepsilon_i}{\partial t} \frac{\varepsilon_j}{\partial t}
$$

$$
- \theta_C \sum_{i,j} \mathcal{m}_{ij} \frac{\partial \varepsilon_i}{\partial t} \frac{\varepsilon_j}{\partial t}\}.
$$

From the definitions of $\tilde{u}$ and $\tilde{\theta}$, the terms in the braces of (9.9) are of zeroth order (with respect to spatial variables) or actually zero depending on whether the subproblems have homogeneous flux BC or homogeneous
Dirichlet BC. Integrating (9.9) from $t = 0$ to $t = \tau$, using (4.10), (4.11), or (4.13), assumptions A(i) and A(ii), and the Cauchy-Schwartz inequality, we obtain

\[
\left\| \frac{\partial \hat{e}}{\partial t} \right\|^2(\tau) + \left\| \nabla \hat{e} \right\|^2(\tau) + \left\| \hat{e} \right\|^2(\tau) + \int_0^\tau \left\| \nabla e \right\|^2 \, dt
\]

\[
\leq C \left( \left\| \frac{\partial \hat{e}}{\partial t} \right\|^2(0) + \left\| \nabla \hat{e} \right\|^2(0) + \left\| \hat{e} \right\|^2(0)
\right)
\]

\[
+ \left\| \hat{e} \right\|^2_{L^2} + \left\| \hat{e} \right\|^2_{L^2} + \left\| \frac{\partial \hat{e}}{\partial t} \right\|^2_{L^2} + \int_0^\tau \left\| \frac{\partial \hat{e}}{\partial t} \right\|^2 \, dt + \int_0^\tau \left\| \hat{e} \right\|^2 \, dt
\]

\[
+ \theta C \sum_{i,j} \int_0^\tau \left< m_{ij} e_i, \frac{\partial^2 e_i}{\partial t \partial x_j} \right> \, dt
\]

\[- \theta C \sum_{i,j} \int_0^\tau \left< m_{ij} e_i, \frac{\partial e_i}{\partial t} \right> \, dt.
\]

(9.10)

The second last term in (9.10) can be integrated by parts with respect to time, and we see that the result is bounded from above by
\[ \theta C \sum_{i,j} < m_{ij} \varepsilon, \frac{\partial e_i}{\partial x_j} > |_0^\tau - \theta C \sum_{i,j} \int_0^\tau < m_{ij} \frac{\partial \varepsilon}{\partial t}, \frac{\partial e_i}{\partial x_j} > dt \]

\[ \leq n \| \varepsilon \|^2(\tau) + C \{ \| \hat{\varepsilon} \|^2(0) + \| \overline{\varepsilon} \|^2_{L_\infty} + \int_0^\tau \| \varepsilon \|^2 dt \}. \] (9.11)

The last term in (9.10) can be bounded from above by

\[ C \{ \| \frac{\partial \hat{\varepsilon}}{\partial t} \|^2_{L_2(L_2)} + \int_0^\tau \| \varepsilon \|^2 dt \}. \] (9.12)

Substituting (9.11) and (9.12) into (9.10), taking \( n \) sufficiently small and applying Gronwall inequality and Lemma 8.1, we obtain

\[ \| \frac{\partial \hat{\varepsilon}}{\partial t} \|^2(\tau) + \| \hat{\varepsilon} \|^2_{W^1(\Omega)}(\tau) + \| \hat{\varepsilon} \|^2(\tau) + \int_0^\tau \| \varepsilon \|^2 dt \]

\[ \leq C \{ \| \frac{\partial \hat{\varepsilon}}{\partial t} \|^2(0) + \| \hat{\varepsilon} \|^2_{W^1(\Omega)}(0) + \| \hat{\varepsilon} \|^2(0) + \| \frac{\partial \varepsilon}{\partial t} \|^2(0) + \| \varepsilon \|^2(0) + \| \overline{\varepsilon} \|^2(0) + \| \frac{\partial \varepsilon}{\partial t} \|^2_{L_2(L_2)} \].
\[ + \left\| \frac{\partial \hat{e}}{\partial t} \right\|_{L_2^2}^2 + \left\| \frac{\partial \hat{v}}{\partial t} \right\|_{L_2^2}^2 \right\}. \] (9.13)

Now if we use the triangle inequality and Lemma 8.1 and note that the left-hand side of the result is independent of \( \tau \in [0, T] \), then we obtain (9.3). The proof of (9.4) is now obvious upon using Corollary 7.1 or Theorem 7.5, and their analogues for the steady-state heat conduction problem.

When either the temperature or the displacement satisfies the Dirichlet BC, the last term in (9.10) can be integrated by parts with respect to space, we see that the result is bounded from above by

\[ \theta C \sum_{i,j} \int_0^\tau \left\langle \frac{\partial e_i}{\partial t}, \frac{\partial}{\partial x_j} (m_{ij} \hat{e}) \right\rangle \, dt \]

\[ \leq h \int_0^\tau \| \hat{v} \|_{L_2^2}^2 \, dt + C \left\| \frac{\partial \hat{e}}{\partial t} \right\|_{L_2^2}^2 + C \int_0^\tau \| \hat{e} \|_{L_2^2}^2 \, dt. \] (9.14)

Then the usual \( L_2^2(\Omega) \)-analysis leads to (9.5) and (9.6).

Q.E.D.
Discrete Time Galerkin Approximations
for the Coupled Thermoelastic I-BVP

We first give results about the accuracy of the initial values \( \theta^0, \theta^1 \).

**Lemma 9.1.** Let \( \theta^0, \theta^1, \theta^2 \in M_{N_\theta} \) be the Hermite interpolate of \( \theta_0, \theta^*, \theta^{**} \) respectively, with

\[
\theta^* \equiv \theta(x,0) + \Delta t \frac{\partial \theta}{\partial t}(x,0), \quad \theta^{**} \equiv \theta(x,0) + 2\Delta t \frac{\partial \theta}{\partial t}(x,0).
\]

Let \( \tilde{\theta}(\cdot, t) \in M_{N_\theta} \) be defined by (9.1) or (9.2). If \( M_{N_\theta} \) is an \( S^1_{k, \Omega} \)-space and if \( \theta_0, (\partial \theta/\partial t)(\cdot, t) \in W^k(\Omega), \partial^2 \theta/\partial t^2 \in L_\infty(0, T; W^1(\Omega)), \theta \in L^2(0, T; W^k(\Omega)) \) with \( k > 0 \), then there is a positive constant \( C \) independent of \( h \) and \( \Delta t \) such that

\[
||\theta^0 - \tilde{\theta}||_{W^1(\Omega)} + ||\theta^1 - \tilde{\theta}(\Delta t)||_{W^1(\Omega)} + ||\theta^2 - \tilde{\theta}(2\Delta t)||_{W^1(\Omega)} \leq C(h^{k-1} + (\Delta t)^2).
\]

(9.15)

If furthermore the heat conduction operator used in defining \( \tilde{\theta} \) is 0-regular, then we have

\[
||\theta^0 - \tilde{\theta}||_{L^2(\Omega)} + ||\theta^1 - \tilde{\theta}(\Delta t)||_{L^2(\Omega)} + ||\theta^2 - \tilde{\theta}(2\Delta t)||_{L^2(\Omega)} \leq C(h^{k} + (\Delta t)^2),
\]

(9.16)
where $C$ is a positive constant independent of $h$ and $\Delta t$.

**Proof:** From the usual $W^1(\Omega)$- or $L^2(\Omega)$-analysis for scalar elliptic equations, it is known that $||\Theta^0 - \tilde{\Theta}_0||$ and $||\Theta^1 - \tilde{\Theta}(\Delta t)||$ are both bounded from above by $C h^{k-1}$ or $C h^k$. From the definition of $S_{1,k}^h(\Omega)$-space, we have also

$$
||\Theta^1 - \Theta^*|| \leq C h^k ||\Theta^*||_{W^k(\Omega)} \leq C h^k (||\Theta_0||_{W^k(\Omega)} + \Delta t ||\frac{\partial \Theta}{\partial t} (\cdot, t)||_{W^k(\Omega)}).
$$

Now since

$$
||\Theta^* - \Theta^1|| = || \int_0^{\Delta t} (\Delta t - \tau) \frac{\partial^2 \Theta}{\partial t^2} (\tau) d\tau ||
$$

$$
\leq C(\Delta t)^2 ||\frac{\partial^2 \Theta}{\partial t^2}||_{L^\infty \times L^2},
$$

upon using the triangle inequality, we see that $\Theta^1$ satisfies the following estimate

$$
||\Theta^1 - \tilde{\Theta}(\Delta t)|| \leq C((\Delta t)^2 + h^{k-1}) \text{ or } C((\Delta t)^2 + h^k).
$$

A similar result holds for $\Theta^2$. Thus (9.16) is proved.

The proof of (9.15) is now obvious. **Q.E.D.**
Theorem 9.2 (Error-Estimate for Discrete Time Galerkin Approximation of Homogeneous Dirichlet or/and Zero Flux BC Thermoelastic I-BVP Without Interpolation of Coefficients)

Under assumptions $A(i) - A(v)$, and $m_{ij} \in L_\infty(\Omega)$, if $b > \frac{1}{4}$ in the Galerkin procedure, and if $u_0 \in W^1(\Omega)$, $\dot{u}_0 \in L_2(\Omega)$, $$(\partial^2/\partial t^2)u$, $v(\partial u/\partial t)$, $v(\partial^3 u/\partial t^3)$, $\partial^4 u/\partial t^4 \in L_2(0,T;L_2(\Omega))$, $\theta_0 \in L_2(\Omega)$, $\dot{\theta}$, $\partial^2 \theta/\partial t^2 \in L_2(0,T;W^1(\Omega))$, $\partial \theta/\partial t$, $\partial^3 \theta/\partial t^3 \in L_2(0,T;L_2(\Omega))$, then

$W^1(\Omega)$ Error Estimate

There exists a positive constant $C$ independent of $(u, \theta)$, $\Delta t$, and the choice of $M_N \times M_\theta$ such that

$$\left|\left|\frac{u-u_t}{\Delta t}\right|\right|_{L_\infty(2\Delta t, T; W^1(\Omega))} + \left|\left|\frac{\Delta_t (u-u)}{\Delta t}\right|\right|_{L_\infty(\Delta t, T; L_2(\Omega))}$$

$$+ \left|\left|\frac{\theta-\theta}{\Delta t}\right|\right|_{L_\infty(2\Delta t, T; L_2(\Omega))} + \left|\left|\nabla (\theta-\theta)^b\right|\right|_{L_2^2(0,T; L_2(\Omega))}$$

$$\leq C \left|\left|\nabla \left. \frac{\partial}{\partial t} (u-u)\right|\right|_{L_2(0,T; L_2(\Omega))}$$

$$+ \left|\left|\frac{\partial^2}{\partial t^2} (u-u)\right|\right|_{L_2(0,T; L_2(\Omega))}$$
\[ + \left| \frac{\partial}{\partial t} (\theta - \tilde{\theta}) \right|_{L_2(0,T;L_2(\Omega))} + \left| \nabla (\theta - \tilde{\theta}) \right|_{L_2(0,T;L_2(\Omega))} + \left( \Delta t \right)^2 \left( \left| \frac{\partial^4 u}{\partial t^4} \right|_{L_2(0,T;L_2(\Omega))} + \left| \nabla \frac{\partial^3 u}{\partial t^3} \right|_{L_2(0,T;L_2(\Omega))} \right) \]

\[ + \left| \frac{\partial^3 \theta}{\partial t^3} \right|_{L_2(0,T;L_2(\Omega))} + \left| \nabla \frac{\partial^2 \theta}{\partial t^2} \right|_{L_2(0,T;L_2(\Omega))} \]

\[ + \left| \Delta_t (U - \bar{u}) \right|_{L_2(\Omega)}^{1/2} + \left| u^0 - \bar{u}_0 \right|_{W^1(\Omega)} \]

\[ + \left| u^1 - \bar{u}(\Delta t) \right|_{W^1(\Omega)} + \left| \theta^0 - \bar{\theta}_0 \right|_{L_2(\Omega)} \]

\[ + \left| \theta^1 - \bar{\theta}(\Delta t) \right|_{L_2(\Omega)} + \left( \left| \frac{\partial}{\partial t} (u - \bar{u}) \right|_{L_2(\Omega)}(0) \right) \]

\[ + \left| u - \bar{u} \right|_{L_2(\Omega)}(0) + \left| \theta - \tilde{\theta} \right|_{L_2(\Omega)(0))} \right) . \]  

If furthermore \( N \times N \) is an \( S_{1,k}(\Omega) \)-space, \( u_0, \bar{u}_0 \)
\( \frac{\partial^2 u}{\partial t^2} \in W^k(\Omega), \frac{\partial^2 u}{\partial t^2} \in L_2(0,T;W^k(\Omega)), \theta_0, (\theta_0/\partial t)(\cdot,0) \)
\( \in W^k(\Omega), \theta_0/\partial t \in L_2(0,T;W^k(\Omega)), \frac{\partial^2 \theta}{\partial t^2} \in L_\infty(0,T;L_2(\Omega)) \)
with \( k > 0 \), and if \( U^0, \bar{u}^1 \) are specified by (5.17) and
(5.21), and \( \theta^0, \theta^1 \) are described in Lemma 9.1, then we have
\[ ||y-y||_{L^\Delta t(0,T;H^1(\Omega))} + ||\Delta t(y-y)||_{L^\Delta t(0,T;L^2(\Omega))} + ||\varepsilon-\varepsilon||_{L^\Delta t(0,T;L^2(\Omega))} + ||\nabla(\varepsilon-\varepsilon)\cdot b||_{L^2(0,T;L^2(\Omega))} \leq C(h^{k-1} + (\Delta t)^2) , \]  

where \( C \) is a positive constant independent of \( \Delta t \) and \( h \).

**\( L^2(\Omega) \) Error Estimate**

When either the temperature or the displacement satisfies the homogeneous Dirichlet BC, and furthermore \( \nabla m_{ij} \in L_\infty(\Omega) \), there exists a positive constant of \( (u, \theta) \), \( \Delta t \), and the choice of \( M_N \) such that

\[ ||y-y||_{L^\infty(2\Delta t,T;L^2(\Omega))} + ||\Delta t(y-y)||_{L^\Delta t(2\Delta t,T;L^2(\Omega))} + ||\varepsilon-\varepsilon||_{L^\Delta t(2\Delta t,T;L^2(\Omega))} \leq \text{right-hand side of (9.17) with the first and third terms deleted.} \]  

If furthermore the operators used in defining \( \ddot{u} \) and \( \ddot{\theta} \)
are 0-regular, and if we make the same additional assumptions as those after (9.17), then we have

\[ \| Y - u \|_{L_\infty(0,T;L_2(\Omega))} + \| \Delta t (Y - u) \|_{L_\infty(0,T;L_2(\Omega))} \]

\[ + \| \theta - \theta_0 \|_{L_\infty(0,T;L_2(\Omega))} \leq C(h^{k+}(\Delta t)^2) \]  \hspace{1cm} (9.20)

where \( C \) is a positive constant independent of \( \Delta t \) and \( h \).

**Proof:** It can be checked that \((u, \theta)\) satisfies the analogue of (8.28):

\[ \sum_{i,j,k,l} \langle \rho \Delta t^2 u_{i}^m, v_i \rangle + \sum_{i,j,k,l} \langle C_{ijkl} \frac{\partial u_k^m}{\partial x_k}, \frac{\partial v_i}{\partial x_j} \rangle \]

\[ - \sum_{i,j} \langle \theta_{ij}^m, \frac{\partial}{\partial x_j} v_i \rangle - \sum_{i} \langle \theta_{i}^m, v_i \rangle = 0, \]

\[ < \rho c_D \frac{\theta^{m+1} - \theta^{m-1}}{2\Delta t}, \phi > + \sum_{i,j} \langle K_{ij} \frac{\partial}{\partial x_j} (\theta_{i}^m + \omega^m), \frac{\partial}{\partial x_i} \phi \rangle \]

\[ + \frac{\theta_C}{2} \sum_{i,j} \langle m_{ij} \frac{\partial}{\partial x_j} (\Delta_t u_i^{m+1/2} - \Delta_t u_i^{m-1/2}), \phi \rangle \]

\[ - < \rho c_D r^m + \epsilon^m, \phi > = 0, \forall \phi \in W^1(\Omega) \text{ or } W^1(\Omega), \]

\[ \phi \in W^1(\Omega) \text{ or } W^1(\Omega), \quad 1 \leq m \leq M-1, \]  \hspace{1cm} (9.21)
where
\[ \xi^m = \rho \left( \frac{\Delta_t^2 u^m_{i,j}}{\Delta t^2} - \Delta_t^2 u^m \right), \quad \omega^m = \varepsilon^m - \varepsilon^m \]

\[ \zeta = \rho c_p \left( \frac{\varepsilon^{m+1}_i - \varepsilon^{m-1}_i}{2\Delta t} - \frac{\partial \varepsilon^m}{\partial t} \right) \]

\[ + \varepsilon C \sum_{i,j} m_{ij} \frac{\partial}{\partial x_j} \left( \frac{u^{m+1}_i - u^{m-1}_i}{2\Delta t} - \frac{\partial u^m_i}{\partial t} \right). \]

Subtracting \((9.21)\) from \((5.37)\), and choosing \(\phi = \varepsilon^m \)

and
\[ y = Y \equiv \left( \Delta_t^{m+1/2+\Delta_t^{m-1/2}} \right) / 2 = \left( \varepsilon^{m+1} - \varepsilon^{m-1} \right) / (2\Delta t), \]

we have the analogue of \((8.29)\)

\[ \sum_i < \rho \left( \Delta_t^{m+1/2+\Delta_t^{m-1/2}} \right) / \Delta t, \left( \Delta_t^{m+1/2+\Delta_t^{m-1/2}} \right) / 2 > \]

\[ + \sum_i < \rho \Delta_t^{2m+1/2+\Delta_t^{m-1/2}} , \left( \Delta_t^{m+1/2+\Delta_t^{m-1/2}} \right) / 2 > \]

\[ + \frac{b}{2\Delta t} (||\varepsilon^{m+1} - \varepsilon^m||_C^2 - ||\varepsilon^{m-1} - \varepsilon^m||_C^2) \]

\[ + \frac{1}{2\Delta_t} (\varepsilon^m, \varepsilon^{m+1}_C - \varepsilon^{m-1}_C, \varepsilon^m >_C) \]
\[+ \sum_{i,j,k,l} \langle C_{ijkl} \frac{\partial \hat{e}_m}{\partial x_k}, \frac{\partial}{\partial x_j} (\Delta_t \hat{e}_m^{1/2} + \Delta_t \hat{e}_m^{-1/2})/2 \rangle \]

\[- \sum_{i,j} \langle \hat{e}_m^b, \frac{\partial}{\partial x_j} (\Delta_t \hat{e}_m^{1/2} + \Delta_t \hat{e}_m^{-1/2})/2 \rangle \]

\[+ \langle \xi^m, (\Delta_t \hat{e}_m^{1/2} + \Delta_t \hat{e}_m^{-1/2})/2 \rangle \]

\[- \sum_{i,j} \langle \hat{e}_m^b, \frac{\partial}{\partial x_j} (\Delta_t \hat{e}_m^{1/2} + \Delta_t \hat{e}_m^{-1/2})/2 \rangle = 0, \quad (9.22)\]

\[\frac{b}{(2\Delta t)} \rho c_D \{ (\hat{e}_m^{1/2} - \hat{e}_m^b) + (\hat{e}_m - \hat{e}_m^{-1/2}) \}, \]

\[\{ (\hat{e}_m^{1/2} - \hat{e}_m^b) - (\hat{e}_m - \hat{e}_m^{-1/2}) \} \]

\[+ \frac{1}{2\Delta t} \rho c_D \hat{e}_m^{1/2}, \hat{e}_m \] - \[\frac{1}{2\Delta t} \rho c_D \hat{e}_m, \hat{e}_m^{-1/2} \]

\[+ \frac{1}{2} \rho c_D (\Delta_t \hat{e}_m^{1/2} + \Delta_t \hat{e}_m^{-1/2}), \hat{e}_m^b \]

\[+ \sum_{i,j} \langle K_{ij} \frac{\partial}{\partial x_j} \hat{e}_m^b, \frac{\partial}{\partial x_i} \hat{e}_m^b \rangle \]

\[+ \sum_{i,j} \langle K_{ij} \frac{\partial}{\partial x_j} (\hat{e}_m^b + \omega^m), \frac{\partial}{\partial x_i} \hat{e}_m^b \rangle \]
\[
\begin{align*}
+ \frac{\theta C}{2} & \sum_{i,j} <m_{ij} \frac{\partial}{\partial x_j} (\Delta_t e_i^{m+1/2} + \Delta_t e_i^{m-1/2}), \dot{e}_m, b > \\
+ <\zeta^m, \dot{e}_m, b > \\
+ \frac{\theta C}{2} & \sum_{i,j} <m_{ij} \frac{\partial}{\partial x_j} (\Delta_t \dot{e}_i^{m+1/2} + \Delta_t \dot{e}_i^{m-1/2}), \dot{e}_m, b > = 0, \\
\end{align*}
\]

(9.23)

where we have used the notation

\[
< \mathcal{V}, \mathcal{W} > \equiv \sum_{i,j,k,\ell} C_{ijk\ell} \frac{\partial V_i}{\partial x_j} \frac{\partial W_k}{\partial x_\ell}, \\
\\
||\mathcal{V}||^2_C = < \mathcal{V}, \mathcal{V} >_C.
\]

Multiplying (9.22) by \(\theta C\), and adding the result to (9.23), we have

\[
\begin{align*}
\frac{\theta C}{2\Delta t} (||\rho^{1/2} \Delta_t \dot{e}_m^{m+1/2}||^2 - ||\rho^{1/2} \Delta_t \dot{e}_m^{m-1/2}||^2 ) \\
+ \frac{b\theta C}{2\Delta t} (||\dot{e}_m^{m+1} - \dot{e}_m^{m}||_C^2 - ||\dot{e}_m^{m} - \dot{e}_m^{m-1}||_C^2 ) \\
+ \frac{\theta C}{2\Delta t} (<\dot{e}_m, \dot{e}_m^{m+1} >_C - <\dot{e}_m^{m-1}, \dot{e}_m >_C ) \\
+ \frac{b}{2\Delta t} (||\rho c_D^{1/2} (\dot{e}_m^{m+1} - \dot{e}_m^{m})||^2 - ||\rho c_D^{1/2} (\dot{e}_m^{m} - \dot{e}_m^{m-1})||^2 )
\end{align*}
\]
\[
+ \frac{1}{2\Delta t} \left( \langle \rho c_D \dot{\varepsilon}^{m+1}, \dot{\varepsilon}^m \rangle - \langle \rho c_D \dot{\varepsilon}^m, \dot{\varepsilon}^{m-1} \rangle \right) \\
+ \sum_{i,j} \langle K_{ij} \frac{\partial}{\partial x_j} \dot{\varepsilon}^m, b, \frac{\partial}{\partial x_i} \dot{\varepsilon}^m, b \rangle \\
= \{ \frac{\theta_C}{2} \sum_{i,j,k,l} \langle C_{ijkl} \frac{\partial \dot{\varepsilon}^m, b}{\partial x_k}, \frac{\partial}{\partial x_j} (\Delta_t \dot{\varepsilon}^{m+1/2} + \Delta_t \dot{\varepsilon}^{m-1/2}) \rangle \}
- \sum_{i,j} \langle K_{ij} \frac{\partial}{\partial x_j} \dot{\varepsilon}^m, b, \frac{\partial}{\partial x_i} \dot{\varepsilon}^m, b \rangle \}
- \theta_C \sum_i \langle \rho \Delta_t^{2-}, (\Delta_t \dot{\varepsilon}^{m+1/2} + \Delta_t \dot{\varepsilon}^{m-1/2})/2 \rangle/2 \rangle \\
- \frac{1}{2} \langle \rho c_D (\Delta_t \dot{\varepsilon}^{m+1/2} + \Delta_t \dot{\varepsilon}^{m-1/2}), \dot{\varepsilon}^m, b \rangle \\
- \theta_C \langle \dot{\varepsilon}^m, (\Delta_t \dot{\varepsilon}^{m+1/2} + \Delta_t \dot{\varepsilon}^{m-1/2})/2 \rangle/2 \rangle - \langle \dot{\varepsilon}^m, \dot{\varepsilon}^m, b \rangle \\
- \sum_{i,j} \langle K_{ij} \frac{\partial \omega^m}{\partial x_j}, \frac{\partial}{\partial x_i} \dot{\varepsilon}^m, b \rangle \\
+ \theta_C \sum_{i,j} \langle m_{ij} \dot{\varepsilon}^m, b, \frac{\partial}{\partial x_j} (\Delta_t \dot{\varepsilon}^{m+1/2} + \Delta_t \dot{\varepsilon}^{m-1/2})/2 \rangle \}
- \theta_C \sum_{i,j} \langle m_{ij} \frac{\partial}{\partial x_j} \frac{\Delta_t \dot{\varepsilon}^{m+1/2} + \Delta_t \dot{\varepsilon}^{m-1/2}}{2} \rangle, \dot{\varepsilon}^m, b \rangle .
\]

(9.24)
As remarked in the proof of Theorem 9.1, the terms in the braces of (9.24) are of zeroth order or actually zero. Taking the discrete sum operation

\[ (\Delta t) \sum_{m=1}^{M_2-1} , \text{with } 2 \leq M_2 \leq M, \]

on (9.24), and using the same technique as in the proof of Theorem 8.5, we have the following analogue of (9.10):

\[
\left| \left| \Delta_t \hat{e}_{e}^{M_2-1/2} \right| \right|^2 + \left( \left| \left| \hat{e}_{e}^{M_2} \right| \right|^2 + \left| \left| \hat{e}_{e}^{M_2-1} \right| \right|^2 \right) + \left( \left| \left| \hat{e}^m_{b} \right| \right|^2 \right) + \left( \left| \left| \hat{e}^m_{e} \right| \right|^2 \right)
\]

\[
+ \left( \left| \left| \hat{e}^m_{e} \right| \right|^2 \right) + \sum_{m=1}^{M_2-1} \left( \left| \left| \hat{e}^m_{e} \right| \right|^2 \right)
\]

\[
\leq C \left( \left| \left| \Delta_t \hat{e}_{e}^{1/2} \right| \right|^2 + \left( \left| \left| \hat{e}_{e}^1 \right| \right|^2 + \left| \left| \hat{e}_{e}^0 \right| \right|^2 \right) + \left( \left| \left| \hat{e}^1 \right| \right|^2 + \left| \left| \hat{e}^0 \right| \right|^2 \right)
\]

\[
+ \left( \left| \left| \hat{e}^1_{e} \right| \right|^2 \right) + \left( \left| \left| \hat{e}^0_{e} \right| \right|^2 \right) + \left( \left| \left| \hat{e}^1_{e} \right| \right|^2 \right)
\]

\[
+ \left( \left| \left| \hat{e}^0_{e} \right| \right|^2 \right)
\]

\[
+ \left( \left| \left| \hat{e}_{e}^{M_2-1/2} \right| \right|^2 \right) + \left( \left| \left| \hat{e}_{e}^{M_2-1} \right| \right|^2 \right) + \sum_{m=1}^{M_2-1} \left( \left| \left| \hat{e}^m_{e} \right| \right|^2 \right) + \sum_{m=1}^{M_2-1} \left( \left| \left| \hat{e}^m_{e} \right| \right|^2 \right)
\]
\[ + \frac{1}{L_2} \left( \frac{2}{\Delta t} \right) \left\{ \frac{1}{L_2} \sum_{m=1}^{M_2} \sum_{i,j} \langle m \rangle_{i,j} \frac{e^m_i}{\Delta t} \right\} + \frac{1}{L_2} \sum_{m=1}^{M_2} \sum_{i,j} \langle m \rangle_{i,j} \frac{e^m_i}{\Delta t} \]  

\[ + \frac{\Theta C}{L^2} \sum_{m=1}^{M_2} \sum_{i,j} \langle m \rangle_{i,j} \frac{\partial e^m_i}{\partial x_j} \left( \frac{\Delta t^m}{2} + \frac{\Delta t^{m-1}}{2} \right) > \Delta t \]

\[ - \frac{\Theta C}{L^2} \sum_{m=1}^{M_2} \sum_{i,j} \langle m \rangle_{i,j} \frac{\partial e^m_i}{\partial x_j} \left( \frac{\Delta t^m}{2} + \frac{\Delta t^{m-1}}{2} \right) > \Delta t, \]

\[ (9.25) \]

provided that \( \Delta t \) is sufficiently small.

The second last term in (9.25) can be summed by parts, and we see that the result is bounded from above by

\[ n \left( \left| \vec{v}_{e}^M \right|^2 + \left| \vec{v}_{e}^{M-1} \right|^2 \right) \]

\[ + C \left( \left| \vec{v}_{e}^1 \right|^2 + \left| \vec{v}_{e}^0 \right|^2 \right) + \left| \vec{e} \right|^2 \left( \Delta t^m \right) \frac{1}{L_2} \sum_{i,j} \left( \left| \vec{v}_{e} \right|^2 + \left| \vec{v}_{e}^{m-1} \right|^2 \right) \Delta t \]

\[ + \sum_{m=2}^{M_2} \left( \left| \vec{v}_{e} \right|^2 + \left| \vec{v}_{e}^{m-1} \right|^2 \right) \Delta t \]  

\[ (9.26) \]

The last term in (9.25) can be bounded from above by

\[ C \left( \left| \Delta t \vec{v}_{e} \right|^2 \right) + \sum_{m=2}^{M_2} \left| \vec{e} \right|^2 \Delta t \]
+ \Delta t(\|\hat{e}^M_2\|^2 + \|\hat{e}^1\|^2 + \|\hat{e}^0\|^0).

Substituting (9.26) and (9.27) into (9.25), choosing \( n \) to be sufficiently small, applying the Gronwall inequality and Lemma 8.2, and using (8.14)-(8.18) and the fact that \( \Delta t \) is sufficiently small, we obtain

\[
\|\hat{e}^M_2\|_{W^1(\Omega)}^2 + \|\Delta_t \hat{e}^M_2\|_{L^2(\Omega)}^2 + \|\hat{e}^0\|_{W^1(\Omega)}^2 \leq C\left( \|\hat{e}^1\|_{W^1(\Omega)}^2 + \|\hat{e}^0\|_{W^1(\Omega)}^2 \right) + (\|\hat{e}^1\|^2 + \|\hat{e}^0\|^2) + \|\frac{\partial}{\partial t} \bar{e}\|^2(0) + \|\bar{e}\|^2(0)
\]

\[
+ \|\frac{\partial \Delta_t^2}{\partial x^2} \frac{\partial}{\partial t^2} \bar{e}\|^2_{L^2_x L^2_t} + \|\nabla \bar{e}\|^2_{L^2_x L^2_t} + \\|\nabla \bar{e}\|^2_{L^2_x L^2_t} + \|\frac{\partial}{\partial t} \bar{e}\|^2_{L^2_x L^2_t} + \|\frac{\partial}{\partial t} \bar{e}\|^2_{L^2_x L^2_t} + \|\frac{\partial}{\partial t} \bar{e}\|^2_{L^2_x L^2_t}.
\]

Now we estimate the error due to time discretization.
We have already shown in the proof of Theorem 8.5 that

\[ \| \xi \|_{L_2^2} \leq C(\Delta t)^2 \| \frac{\partial^4 u}{\partial t^4} \|_{L_2^2} \cdot \]

Similarly from

\[ \zeta^m = \rho \frac{c_D}{2\Delta t} \left\{ \int_{t_m}^{t_{m+1}} \frac{(t_{m+1}-\tau)^2}{3!} \frac{\partial^3 \theta}{\partial t^3} (\tau) d\tau \right. \]

\[ - \int_{t_m}^{t_{m-1}} \frac{(t_{m-1}-\tau)^2}{3!} \frac{\partial^3 \theta}{\partial t^3} (\tau) d\tau \left\} + \frac{\theta}{2\Delta t} \sum_{i,j} \frac{\partial}{\partial x_j} \left\{ \int_{t_m}^{t_{m+1}} \frac{(t_{m+1}-\tau)^2}{3!} \frac{\partial^3 u_i}{\partial t^3} (\tau) d\tau \right. \]

\[ - \int_{t_m}^{t_{m-1}} \frac{(t_{m-1}-\tau)^2}{3!} \frac{\partial^3 u_i}{\partial t^3} (\tau) d\tau \right\} , \]

and

\[ \omega^m = b(\Delta t)^2 \Delta t^2 \theta^m = b \int_{t_{m-1}}^{t_{m+1}} (\Delta t - |\tau - t^m|) \frac{\partial^2 \theta}{\partial t^2} (\tau) d\tau , \]

we have
\[ || \tau ||_{L^2_t x L^2} \leq C(\Delta t)^2 \left( || \frac{\partial}{\partial t} \theta ||_{L^2_t x L^2} + || v \frac{\partial}{\partial t} \frac{\partial}{\partial x} \theta ||_{L^2_t x L^2} \right), \]

and

\[ || v \theta ||_{L^2_t x L^2} \leq C(\Delta t)^2 \left( || v \frac{\partial^2}{\partial t^2} \theta ||_{L^2_t x L^2} \right). \]

As before, we can use the triangle inequality and complete the proof of (9.17). The proof of (9.18) is now obvious upon using Corollary 7.1 or Theorem 7.5, their analogues for the steady-state heat conduction problem, Lemma 8.4, and Lemma 9.1.

When either the temperature or the displacement satisfies the homogeneous Dirichlet BC, and furthermore \( \forall m_{ij} \in L^\infty(\Omega) \), the last term in (9.25) can be integrated by parts with respect to space, and we see that the result is bounded from above by

\[ \frac{\theta_C}{2} \sum_{m=1}^{M_2-1} \sum_{i,j} m_{ij} \left( \Delta t \frac{\partial}{\partial t} \frac{\partial}{\partial x} \theta \right)_{i}^{m+1/2} + \Delta t \frac{\partial}{\partial x} \theta_{i}^{m-1/2}, \frac{\partial}{\partial x} \frac{\partial}{\partial x} \theta_{j}^{m-1/2} > \Delta t \]

\[ \leq n \sum_{m=1}^{M_2-1} || v \frac{\partial}{\partial t} \frac{\partial}{\partial x} \theta ||_{L^2_t x L^2} + C \left( || \Delta t \frac{\partial}{\partial x} \theta ||_{L^2_t x L^2} \right)^2 \]
\[ M_2^{-1} + C \sum_{m=1}^{\infty} ||c^m, b||^2 \Delta t. \] (9.29)

This remark shows that (9.19) is valid in this case.

Finally, Corollary 7.2 or Theorem 7.5, their analogues for temperature, Lemma 8.4, and Lemma 9.1 imply (9.20).

Q.E.D.

Decoupled Discrete Time Galerkin Approximations for the Coupled Thermoelastic I-BVP

Theorem 9.3 (Error Estimate for Decoupled Discrete-Time Galerkin Approximation of Thermoelastic I-BVP Without Interpolation of Coefficients)

Under assumptions A(i)-A(v) with homogeneous displacement BC or homogeneous temperature BC on both, if

\[ m_{ij} \in L_\infty(\Omega), \forall m_{ij} \in L_\infty(\Omega), u_0, \bar{u}_0 \in L_2(\Omega), \]
\[ (\partial^2/\partial t^2)u, \partial^3 u/\partial t^3, \partial^4 u/\partial t^4 \in L_2(0,T;L_2(\Omega)), \theta_0 \in L_2(\Omega), \]
\[ \partial \theta/\partial t, \partial^2 \theta/\partial t^2 \in L_2(0,T;W^1(\Omega)), \partial \theta/\partial t, \partial^3 \theta/\partial t^3 \in L_2(0,T;L_2(\Omega)) \]

and if \( b \geq 1/4 \), then there exists a positive constant \( C \) independent of \( (u, \theta) \), \( \Delta t \), and the choice of \( M_N \times M_\theta \) such that
\[
||v-u||_{L^2_\infty(\Omega)} + ||\Delta_t(\nu-u)||_{L^2_\infty(\Delta t; L^2_2(\Omega))} + ||v(\nu-u)||_{L^2_\infty(\Delta t; L^2_2(\Omega))} \\
+ ||v(\nu-u)||_{L^2_\infty(3\Delta t; L^2_2(\Omega))} + ||v(\nu-u)||_{L^2_\infty(\Delta t; L^2_2(\Omega))} \\
\leq C( ||\nu-u||_{L^2_\infty(0,T; W^{1,2}(\Omega))} + ||\frac{\partial^2}{\partial t^2}(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} \\
+ ||v(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} + ||\frac{\partial}{\partial t}(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} \\
+ (\Delta t)^2( ||\frac{\partial^4}{\partial t^4}(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} + ||v(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} \\
+ ||\frac{\partial^3}{\partial t^3}(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} + ||\frac{\partial}{\partial t}(\nu-u)||_{L^2(0,T; L^2_2(\Omega))} \\
+ (||\Delta_t(\nu-u)||_{L^2(\Omega)}^{1/2} ||\nu||_{L^2(\Omega)} + ||\nu^0-\nu_0||_{W^{1,2}(\Omega)} \\
+ ||(\nu-u)(\Delta t)||_{W^{1,2}(\Omega)} + ||\nu^0-\nu_0||_{W^{1,2}(\Omega)} \\
+ ||\nu-\nu(\Delta t)||_{W^{1,2}(\Omega)} + ||\theta-\tilde{\theta}(\Delta t)||_{L^2_2(\Omega)} \\
+ ||\theta^2-\tilde{\theta}(2\Delta t)||_{L^2_2(\Omega)} + (||\frac{\partial}{\partial t}(\nu-u)||_{L^2_2(\Omega)}(0) \\
+ ||\nu-\nu||_{L^2_2(\Omega)}(0) + ||\theta-\theta||_{L^2_2(\Omega)}(0) ) ), \quad (9.30)
\]
and

$$||U-u||_{L^\Delta t(2\Delta t,T;L_2(\Omega))} + ||\Delta t(U-u)||_{L^\Delta t(\Delta t,T;L_2(\Omega))} + ||\Theta-\Theta||_{L^\Delta t(3\Delta t,T;L_2(\Omega))}$$

right-hand side of (9.30) with the first and third terms deleted. (9.31)

If furthermore $\mathcal{M}_N \times \mathcal{M}_N$ is an $S_{1,k}(\Omega)$-space, $U_0$, $U_0'$, $U^2_{/\Delta t} \in W^k(\Omega)$, $U^2_{/\Delta t} \in L_2(0,T;W^k(\Omega))$, $\Theta_0$, $(\Theta_0/\Delta t)(\cdot,0) 
\in W^k(\Omega)$, $\Theta_0/\Delta t \in L_2(0,T;W^k(\Omega))$, $\Theta^2_{/\Delta t} \in L_{\infty}(0,T;W^1(\Omega))$

with $k > 0$, and if $U_0^0$, $U_0^1$ are specified by (5.17) and (5.21), $\Theta_0^0$, $\Theta_0^1$, $\Theta_0^2$ are as described in Lemma 9.1, then

$W^1(\Omega)$ Error Estimate

There exists a positive constant $C$ independent of $\Delta t$ and $h$ such that

$$||U-u||_{L^\Delta t(0,T;W^1(\Omega))} + ||\Delta t(U-u)||_{L^\Delta t(0,T;L_2(\Omega))} + ||\Theta-\Theta||_{L^\Delta t(0,T;L_2(\Omega))} + ||\Theta(\Theta'')_{,b}||_{L^2(0,T;L_2(\Omega))}$$

$\leq C(h^{k-1} + (\Delta t)^2).$ (9.32)
L_2(\Omega) Error Estimate

If the operators used in defining \( \tilde{u} \) and \( \tilde{e} \) are 0-regular, then there exists a positive constant \( C \) independent of \( \Delta t \) and \( h \) such that

\[
||u-u||_{L_\infty(0,T;L_2(\Omega))} + ||\Delta_t (u-u)||_{L_\infty(0,T;L_2(\Omega))}
\]

\[
+ ||\theta-\theta||_{L_\infty(0,T;L_2(\Omega))}
\]

\[
\leq C(h^k + (\Delta t)^2).
\] (9.33)

Proof: We observe first that \((u, \theta)\) satisfies (9.21) and

\[
< \rho c_D \frac{\theta^{m+1} - \theta^{m-1}}{2\Delta t}, \phi > + \sum_{i,j} K_{ij} \frac{\partial}{\partial x_j} (\theta^m, b + \omega^m), \frac{\partial}{\partial x_i} \phi >
\]

\[
+ \frac{\theta C}{2} \sum_{i,j} \sum_{m} \frac{\partial}{\partial x_j} (3\Delta t u_{i}^{m-1/2} - \Delta t u_{i}^{m-3/2}), \phi >
\]

\[- < \rho c_D r^m + \zeta^m, \phi > = 0,
\]

\[\nabla \phi \in W^1(\Omega) \text{ or } W^1(\Omega), \ 2 \leq m \leq M-1, \] (9.32)

where
\[ \omega^m = \varepsilon^m - \dot{\varepsilon}^m, b, \]

\[ \zeta = \rho C_D \left( \frac{\dot{\varepsilon}^m_{-1} - \dot{\varepsilon}^m_{-2}}{2\Delta t} - \frac{\partial \varepsilon^m}{\partial t} \right) \]

\[ + \theta_C \sum_{i,j} m_{ij} \frac{a}{\partial x_j} \left( \frac{3\Delta t u_{i}^{m-1/2} - \Delta t u_{i}^{m-3/2}}{2} - \frac{\partial u_{i}^{m}}{\partial t} \right). \]

Subtracting (9.32) from (5.38), using \( \phi = \psi \equiv \varepsilon^m, b \), and integrating the coupling terms by parts with respect to space, we have

\[ \frac{b}{2\Delta t} \left( \langle (\rho C_D)^{1/2} (\varepsilon^m_{-1} - \varepsilon^m_{-2}) \rangle^{2} - \langle (\rho C_D)^{1/2} (\varepsilon^m_{-2} - \varepsilon^m_{-1}) \rangle^{2} \right) \]

\[ + \frac{1}{2\Delta t} < \rho C_D \varepsilon^m_{-1}, \varepsilon^m > - \frac{1}{2\Delta t} < \rho C_D \varepsilon^m, \varepsilon^m_{-1} > \]

\[ + \frac{1}{2} < \rho C_D (\Delta_t \varepsilon^m_{-1/2} + \Delta_t \varepsilon^m_{-3/2}), \varepsilon^m, b > \]

\[ - \sum_{i,j} < K_{ij} \frac{a}{\partial x_j} \omega^m, \frac{a}{\partial x_i} \varepsilon^m, b > \]

\[ + \sum_{i,j} < K_{ij} \frac{a}{\partial x_j} \varepsilon^m, b, \frac{a}{\partial x_i} \varepsilon^m, b > \]

\[ - \{ \sum_{i,j} < K_{ij} \frac{a}{\partial x_j} \varepsilon^m, b, \frac{a}{\partial x_i} \varepsilon^m, b > \} \]
\[ \frac{\theta C}{2} \sum_{i,j} \left( \Delta t \hat{e}_i^{m-1/2} - \Delta t \hat{e}_i^{m-3/2} \right) \frac{\partial}{\partial x_j} (m_{ij} \hat{e}_i^m, b) > \\
+ \left< \zeta^m, \hat{e}_i^m, b \right> \]

\[ - \frac{\theta C}{2} \sum_{i,j} \left< \Delta t \hat{e}_i^{m-1/2} - \Delta t \hat{e}_i^{m-3/2} \right> \frac{\partial}{\partial x_j} (m_{ij} \hat{e}_i^m, b) = 0, \quad 2 \leq m \leq M-1. \quad (9.33) \]

As remarked before the term in the braces of (9.33) is zeroth order or actually zero. Let \( 2 \leq M_2 \leq M \). Then we can take the discrete sum operation

\[ \sum_{m=2}^{M_2-1} (\Delta t) \]

on (9.33) and proceed in the same way as before obtaining

\[ \left< \zeta^m \right> \]

\[ \leq C \left< \left\| \frac{\partial}{\partial t} (\theta - \tilde{\theta}) \right\|_{L_2 \times L_2}^2 + \left\| \frac{\partial}{\partial t} (u - \tilde{u}) \right\|_{L_2 \times L_2}^2 \right. \]

\[ + \left( \Delta t \right)^4 \left( \left\| \frac{\partial^3 \theta}{\partial t^3} \right\|_{L_2 \times L_2}^2 + \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L_2 \times L_2}^2 \right. \]
\[ + \frac{1}{2} \left| \left| \dddot{\varepsilon}^2 \right| \right|^2 + \left| \left| \dot{\varepsilon}^1 \right| \right|^2 \]

\[ + \sum_{m=2}^{M_1-1} \left| \Delta_t \dot{\varepsilon}^{m-1/2} \right|^2 \Delta t + \left| \left| \Delta_t \dddot{\varepsilon}^{1/2} \right| \right|^2 \Delta t \} . \quad (9.34) \]

In deriving this estimate we have made use of

\[ \omega^m = b(\Delta t)^2 a^2 t^m = b \int_{t^{m-1}}^{t^{m+1}} (\Delta t - |\tau - t^m|) \frac{\partial^2 \theta}{\partial t^2}(\tau) d\tau, \]

\[ \ell^m = \frac{\rho C_D}{2\Delta t} \left\{ \int_{t^m}^{t^{m+1}} \frac{(t^{m+1} - \tau)^2}{31} \frac{\partial^3 \dot{\theta}}{\partial t^3}(\tau) d\tau \right. \]

\[ - \left. \int_{t^m}^{t^{m-1}} \frac{(t^{m-1} - \tau)^2}{31} \frac{\partial^3 \dot{\theta}}{\partial t^3}(\tau) d\tau \right\} \]

\[ + \frac{\theta C}{2\Delta t} \sum_{i,j} \dot{m}_{ij} \frac{\partial}{\partial x_j} \int_{t^{m-2\Delta t}}^{t^{m-2\Delta t}} \frac{(t^{m-2\Delta t} - \tau)^2}{2} \frac{\partial^3 u_i}{\partial t^3}(\tau) d\tau \]

\[ - \int_{t^m}^{t^{m-\Delta t}} \frac{(t^{m-\Delta t} - \tau)^2}{2} \frac{\partial^3 u_i}{\partial t^3}(\tau) d\tau \} . \]

Next we replace the fifth term in (9.22) by a zeroth order one or simply zero, integrate the second term involving \( m_{ij} \) by parts with respect to space, take the
discrete sum operation

\[
(M_2^{-1/2}) \sum_{m=1}^{M_2-1} (\Delta t) \sum_{m=1}^{M_2-1} M_2^{-1/2} || e^m ||^2_{W^1(\Omega)} + || e^M ||^2_{W^1(\Omega)}
\]

on (9.22), and sum the first term involving \( m_{ij} \) there by parts to obtain

\[
\leq C \{ || \frac{\partial^2}{\partial t^2} (\tilde{\nu} - \tilde{v}) ||^2_{L^2(\tilde{\nu} - \tilde{v})} + || \frac{\partial}{\partial t} (\theta - \tilde{\theta}) ||^2_{L^2(\theta - \tilde{\theta})} + (\Delta t)^4 || \frac{\partial^4}{\partial t^4} u ||^2_{L^2(\tilde{\nu} - \tilde{v})}
\]

\[
+ (\Delta t)^4 || \frac{\partial^4}{\partial t^4} u ||^2_{L^2(\tilde{\nu} - \tilde{v})}
\]

\[
+ \{ || \tilde{e}^0 ||^2_{W^1(\Omega)} + || \tilde{e}^1 ||^2_{W^1(\Omega)} + || M_2^{-1/2} \Delta t \tilde{e}^1 ||^2 + || \theta - \tilde{\theta} ||^2(0) \} + \sum_{m=2}^{M_2-1} || v_{e^m, b} ||^2 \Delta t + C \sum_{m=2}^{M_2-1} || \hat{e}^m ||^2 \Delta t. \quad (9.35)
\]

Adding (9.34) and (9.35), taking \( \eta \) to be sufficiently small, and applying the Gronwall inequality, the triangle inequality, and Lemma 8.2, we obtain (9.30) and (9.31).
The proof of (9.32) ((9.33)) is now obvious upon using Corollary 7.1 (Corollary 7.2) or Theorem 7.5, their analogues for the steady-state heat conduction problem, Lemma 8.4, and Lemma 9.1

Q.E.D.
BIBLIOGRAPHY


34. Lions, J. L., Problèmes aux Limites dans les équations aux Derivees Partielles, Lecture Notes, Université de Montreal, Montreal, 1962.


46. Schultz, M. H., $L^2$ error bounds for the Rayleigh-

47. Schultz, M. H., $L^\infty$-multivariate approximation theory,

48. Strang, G., Approximation in the finite element

49. Strang, G., The finite element method and approxima-
tion theory, Numerical Solution of Partial Differential
Equations II, (SYNAPDE), ed. by Hubbard, Academic

50. Tong, P., and Pian, T. H. H., The convergence of the
finite-element method in solving linear elastic
problems, International Journal of Solids and Struc-
tures 3, 865-879 (1967).

51. Truesdell, C., and Noll, W., The Non-Linear Field
Theories, in Flügge's Handbuch der Physik, Band III/3,

52. Varga, R. S., Functional Analysis and Approximation
Theory in Numerical Analysis, SIAM Regional Conference
Series in Applied Mathematics, SIAM, Philadelphia,
Penn., 1971.

53. Wang, C.-C., and Truesdell, C., Nonlinear Elasticity,

54. Wheeler, M. F., A priori $L^2$ error estimates for
Galerkin approximations to parabolic partial differential

55. Zienkiewicz, D. C., The Finite Element Method in