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THE PROPAGATION OF CURVED WAVES IN MIXTURES
WITH SEVERAL TEMPERATURES

by

Robert L. Rankin

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

Thesis Director's signature:

Ray M. Bowen

Houston, Texas

(August, 1971)
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I. INTRODUCTION

In this work the propagation of second and third order curves waves through a mixture of non-linear elastic gases in which non-linear diffusion is taking place is studied. A distinct temperature is associated with each constituent of the mixture. The study treats non-linear conductors as well as non-conductors.

The work uses as a model a special case of the model first theorized by BOWEN and GARCIA [2]. Their thermo-mechanical model is specialized to the case of a mixture of gases without viscosity. This specialization is discussed in Section II. The kinematics of singular surfaces, necessary for the wave propagation calculations, is presented in Section III. This presentation follows the development of TRUESDELL and TOUPIN [5].

Four different cases are then considered. In Sections IV and VI the propagation of second order curved waves through conductors and non-conductors is studied. In Sections V and VII the propagation of third order curved waves through conductors and non-conductors is studied. In each case the propagation condition and the amplitude equation governing the growth and decay of the leading wave are derived.
The theory of singular surfaces for various materials has been developed by several authors [11], [12], [13], [16], and [17]. It is well known that for many models of materials a critical amplitude exists. The critical amplitude is a positive number such that any initial disturbance with a greater amplitude always grows into a shock wave while an initial amplitude less than the critical amplitude results in a decay of the disturbance. For example, in a mixture of gases in which there are chemical reactions but no diffusion [16] this critical amplitude feature may arise when one studies the propagation of second order waves. A critical amplitude may also arise in the study of the propagation of second order waves through an ideal gas mixture in which diffusion is taking place [17]. A common feature of models where a critical amplitude arises is that there is a stabilizing effect on the growth of compressive disturbances in that not all compressive disturbances grow into shock waves.

A feature of most mixture models is that each constituent of the mixture is assumed to have a common temperature. This is not the case in this study and, in fact, one of the purposes of this work is to consider models where each constituent has its own unique temperature and to determine what the effect of this assumption is on the propagation of waves.
In Sections IV and VI, it is shown that for both conductors and non-conductors, the amplitude equations that arise when considering second order waves have the same basic form. A common feature is that both these differential equations have material coefficients that depend on the amplitude of the wave.

The amplitude equations for third order waves in both conductors and non-conductors also have the same forms. However for third order waves the material coefficients no longer depend on the amplitude of the wave. The differential equations obtained here have the same form as those obtained by COLEMAN et al. [11, Part V] for waves of order $N \geq 3$ in materials with memory. For this model the solution to the amplitude equations for third order waves implies that the amplitude of the wave may either grow or decay according to the sign of the material coefficients. For third order waves no critical amplitude feature arises.

It is a common feature in all four cases considered here that because each constituent can have a unique temperature the passage of a wave through the mixture has a separating effect on the temperature distribution behind the wave. In both conductors and non-conductors this tendency to separate the temperatures behind the wave is greater for second order waves than for third order waves. However when comparing conductors and non-conductors it is found
that the passage of a second order wave through a non-conducting mixture will drive the temperatures apart at a greater rate than in a conducting mixture. Similarly the passage of a third order wave has a greater separating effect in a non-conductor than in a conductor.

NOTATION

The direct tensor notation, used for example in [1] and [2], will be employed here whenever possible. A quantity corresponding to a particular component in the mixture is identified by placing a labeling index directly below the symbol or quantity. The constituents will always be labeled by a, b, c, d, or e. Summations over the constituent indices will always be indicated by a summation sign and the range of summation, 1, ..., N, will be omitted unless otherwise indicated.

The tensor notation is the usual direct notation save one exception. If \( \psi \) is a tensor of order \( q \) and \( \phi \) is a tensor of order \( q + 1 \), \( q \geq 3 \), then \( \phi(\psi) \) is a vector defined by

\[
\phi(\psi) = \phi_{ijl \ldots j_q} \psi_{j_1 \ldots j_q} e_i .
\]

The classical notation (e.g. see [5]) is used for the kinematics of singular surfaces. The region into which
a wave travels will be denoted by a plus \( (+) \) superscript, the region behind by a minus \( (-) \) superscript. Hence the region in front of the wave will be denoted by \( p^+ \), the region behind the wave by \( p^- \). A symbol or quantity affixed with a plus superscript shall be evaluated at values of the independent variables in front of the wave while those affixed with a minus superscript shall be evaluated behind the wave.
II. MIXTURES WITH SEVERAL TEMPERATURES

In this section the basic equations for a mixture of \( N \) diffusing bodies \( B \), \( a = 1, \ldots, N \) are discussed. Included in the discussion are the kinematics of motion, the equations of balance for mass, linear momentum, angular momentum, and energy for the constituents of the mixture, and the second axiom of thermodynamics for the mixture. Also discussed are the constitutive relations for a mixture of \( N \) bodies, with the effects of multiple temperatures, conduction, and diffusion included, the restrictions due to the second axiom of thermodynamics, and material-frame indifference and symmetry. For a more detailed discussion of these results see BOWEN and WIESE [1] and BOWEN and GARCIA [2].

Following NOLL [3] a body \( B \) has the structure of a differentiable manifold that is isomorphic to a region in Euclidean three-space and a particle (or element) of \( B \) is denoted by \( X \). A configuration of \( B \) is a homeomorphism \( \chi \) of \( B \) onto a subset of Euclidean space and a motion of \( B \) is a one-parameter family of configurations \( \chi_t \), where \( t \) is the time. The position of the particle \( X \) at time \( t \) is given by

\[
x = \chi_t(X) = \chi(X, t). \tag{2.1}
\]
A reference configuration for \( B \) is a fixed configuration \( \kappa \) such that the position of the particle \( \chi \) in \( \kappa \) is given by

\[
\chi = \kappa(X). \\
\hat{a} \quad \hat{a} \quad \hat{a}
\]

(2.2)

Equations (2.1) and (2.2) imply that

\[
\chi = \chi(\kappa^{-1}(X), t) = \chi_{\kappa}(X, t), \\
\hat{a} \quad \hat{a} \quad \hat{a} \quad \hat{a} \quad \hat{a} \quad \hat{a}
\]

(2.3)

where \( \chi_{\kappa} \) is the deformation function for \( B \). Since \( \chi \) is a homeomorphism each \( \chi_{\kappa} \) has an inverse \( \chi^{-1}_{\kappa} \) and \( \chi_{\kappa} \) and \( \chi^{-1}_{\kappa} \) are \( C^3 \) in their arguments.

The velocity and acceleration of a particle \( \chi \) are given by

\[
\dot{\chi} = \frac{\partial \chi(X, t)}{\partial t}, \\
\hat{a} \quad \hat{a} \quad \hat{a}
\]

(2.4)

and

\[
\ddot{\chi} = \frac{\partial^2 \chi(X, t)}{\partial t^2}, \\
\hat{a} \quad \hat{a} \quad \hat{a}
\]

(2.5)

The deformation gradient and velocity gradient of a particle \( \chi \) are defined by

\[
F = \text{GRAD} \chi_{\kappa}(X, t) \\
\hat{a} \quad \hat{a} \quad \hat{a}
\]

(2.6)
and
\[ L = \nabla \frac{\partial}{\partial t}(x, t) \]  \hspace{1cm} (2.7)

As a result of the smoothness assumptions about \( \frac{x}{\partial \mathbf{a}} \) and \( \frac{1}{\partial \mathbf{a}} \), it follows that

\[ |\det \frac{\partial}{\partial \mathbf{a}} F| > 0. \]  \hspace{1cm} (2.8)

It can also be shown that

\[ L = \frac{\partial}{\partial \mathbf{a}} F \frac{\partial}{\partial \mathbf{a}} F^{-1}. \] \hspace{1cm} (2.9)

If \( \rho \) denotes the density of \( \mathbf{B} \), the density of the mixture is defined by

\[ \rho(x, t) = \frac{\sum}{\partial \mathbf{a}} \rho(x, t). \] \hspace{1cm} (2.10)

The mean velocity of the mixture is given by

\[ \frac{\partial}{\partial \mathbf{a}} \dot{x}(x, t) = \frac{1}{\rho} \frac{\sum}{\partial \mathbf{a}} \rho \frac{\partial}{\partial \mathbf{a}} \dot{x}(x, t), \] \hspace{1cm} (2.11)

and the diffusion velocity of the particle \( X_{a} \) which at time \( t \) is at \( x \) is defined by

\[ u(x, t) = \frac{\partial}{\partial \mathbf{a}} \dot{x}(x, t) - \dot{x}(x, t). \] \hspace{1cm} (2.12)

By use of equations (2.10) and (2.11), it follows from (2.12) that
\[ \sum \rho u = 0. \] (2.13)

The equations of balance for mass, linear momentum, angular momentum, and energy for the \( a^{th} \) constituent are postulated to be

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \ddot{\mathbf{x}}) = 0, \] (2.14)_1

\[ \rho \dddot{\mathbf{x}} = \text{div} \mathbf{T} + \rho \dddot{\mathbf{b}} + \dddot{\mathbf{p}}, \] (2.14)_2

\[ \dddot{\mathbf{M}} = \dddot{\mathbf{T}} - \dddot{\mathbf{T}}^T, \] (2.14)_3

and

\[ \rho \dddot{\varepsilon} = \text{tr}(\mathbf{T}^T \mathbf{L}) - \text{div} \mathbf{q} + \rho \dddot{\mathbf{r}} + \dddot{\varepsilon}. \] (2.14)_4

where \( \dddot{\mathbf{T}} \) denotes the **partial stress** on \( B \), \( \dddot{\mathbf{b}} \) denotes the **partial body force** on \( B \), \( \dddot{\mathbf{p}} \) denotes the **diffusive force** on \( B \) per unit volume, \( \dddot{\mathbf{M}} \) denotes the **body couple** on \( B \) per unit volume, \( \dddot{\varepsilon} \) denotes the **partial internal energy density** of \( B \), \( \dddot{\mathbf{q}} \) denotes the **partial heat supply density** of \( B \), and \( \dddot{\varepsilon} \) denotes the **energy supply** per unit volume per unit time to \( B \) and includes the rate of work of the body couple \( \dddot{\mathbf{M}} \).

Before stating the second axiom of thermodynamics (i.e. the entropy inequality) several terms must be defined.
First the temperature for B is denoted by $\theta$ and is greater than zero and the coldness for B is denoted by $\vartheta$ and is defined by

$$\vartheta = \frac{1}{\theta}.$$  \hspace{1cm} (2.15)

The partial entropy density for B is denoted by $\eta$ and the Massieu Function for B is denoted by $\Lambda$ and is defined by

$$\Lambda = \rho(\eta - \vartheta \varepsilon).$$  \hspace{1cm} (2.16)

The Massieu Function for the mixture $\Lambda$ is thus defined by

$$\Lambda = \sum \Lambda.$$  \hspace{1cm} (2.16)

The coldness gradient $g$ for B is defined by

$$g = \text{grad} \; \vartheta,$$  \hspace{1cm} (2.17)

and the quantity $\hat{\varepsilon}$ is defined by

$$\hat{\varepsilon} = \varepsilon + u \cdot \hat{\mathbf{p}}.$$  \hspace{1cm} (2.18)

In terms of the above quantities, the entropy inequality is
\[
\sum \{ \dot{\Lambda} + \rho \varepsilon \dot{\upsilon} + \text{tr}[\Lambda I + \upsilon T^T] L \} \\
\frac{\partial}{\partial t} + g \cdot q - \upsilon \cdot \dot{p} + \upsilon \cdot \dot{e} \geq 0.
\]

For a discussion of equation (2.19) see BOWEN and GARCIA [2, sec. 4].

The balance equations for mass, linear momentum, angular momentum, and energy for the mixture are postulated to be

\[
\sum \dot{p} = 0, \quad (2.20)_1
\]

\[
\sum \dot{\mathbf{M}} = 0, \quad (2.20)_2
\]

and

\[
\sum \dot{e} = 0. \quad (2.20)_3
\]

These are the equations that guarantee that the balance equations for the constituents are compatible with the balance equations for the mixture.

In this model we want to deal with a mixture of gases that includes the non-linear effects of heat conduction, diffusion, and multiple temperatures. Thus the next topic we must investigate is the constitutive equations for such a theory.
The balance equations and the entropy inequality suggest that constitutive equations be supplied for the quantities $A, \varepsilon, T, \hat{p}, q, \hat{e},$ and $\hat{M}$ for $a = 1, \ldots, N$. Hence we postulate constitutive equations of the form

$$(\dot{A}, \varepsilon, \dot{T}, \dot{\hat{p}}, \dot{q}, \dot{\hat{e}}, \dot{\hat{M}}) = f(\nu, \rho, g, \zeta, \gamma),$$

where

$$\zeta = \text{grad} \rho.$$

The notation in (2.21) is a simplified functional notation that means that each of the dependent variables on the left side of the equation depends on all of the independent variables appearing on the right side. For example

$$A = \Lambda(\nu, \nu, \ldots, \nu, \rho, \ldots, \rho, g, \ldots, g, \zeta, \ldots, \zeta, \gamma, \ldots, \gamma).$$

It is necessary that $f$ and its component functions in (2.21) be of class $C^3$. The choice of the independent variables in (2.21) was influenced by previous mixture theories including those of BOWEN and GARCIA [2, sec. 3], BOWEN and WIESE [1, sec. 2], and MÜLLER [4].

Following BOWEN and GARCIA [2, sec. 4], the restrictions which follow from the entropy inequality are expressed in the following theorem: In order that the entropy inequality be satisfied for every admissible thermodynamic
process for the mixture defined by (2.21), it is necessary and sufficient that for each $a = 1, \ldots, N$,

$$
\Lambda = \Lambda(u, \rho)_{b b}^{a a} \tag{2.23}_1
$$

$$
\varepsilon = \varepsilon(u, \rho) = -\frac{1}{\rho} \frac{\partial \Lambda}{\partial u}_{b b}^{a a}, \tag{2.23}_2
$$

$$
\sum (\dot{x} x \frac{\partial \Lambda}{\partial g} b b) + \sum (\dot{x} x \frac{\partial \Lambda}{\partial g} a a)^T = 0, \tag{2.23}_3
$$

$$
\sum (\dot{x} x \frac{\partial \Lambda}{\partial x} b b) + \sum (\dot{x} x \frac{\partial \Lambda}{\partial x} a a)^T = 0, \tag{2.23}_4
$$

$$
T = -\frac{1}{\rho} \Lambda I - \rho \frac{\partial \Lambda}{\partial \rho} I + \sum (u x \frac{\partial \Lambda}{\partial x} a a), \tag{2.23}_5
$$

and

$$
\sum (q \cdot g - (u \frac{\partial f}{\partial g} a a + \frac{\partial \Lambda}{\partial g} a a) \cdot u + u \varepsilon) \geq 0 \tag{2.23}_6
$$

where

$$
\frac{\partial f}{\partial \rho} a a - \frac{1}{\rho} \left\{ \sum (\frac{\partial \Lambda}{\partial \rho} a a - \frac{\partial \Lambda}{\partial \rho} b b) \right\}, \tag{2.23}_7
$$

for all admissible thermodynamic processes.
For the additional restrictions imposed by material frame-indifference and material symmetry we will quote directly from BOWEN and GARCIA [2, sec. 5].

We might note in passing that material frame-indifference forces the additional restriction that the partial stresses must be symmetric. To see this let $\mathbf{Q}$ be a time dependent orthogonal linear transformation which, as such, obeys the condition.

$$\dot{\mathbf{Q}} \mathbf{Q}^T = - (\dot{\mathbf{Q}} \mathbf{Q}^T)^T.$$  \hspace{1cm} (2.24)

BOWEN and GARCIA [2, sec. 5] show that the energy supply $\dot{\varepsilon}$ transforms according to the following rule:

$$\dot{\varepsilon}(\rho, g, \xi, \dot{\mathbf{x}}) - \text{tr}[T^T(\rho, g, \xi, \dot{\mathbf{x}}) \dot{\mathbf{Q}} \dot{\mathbf{Q}}^T]$$
$$= \dot{\varepsilon}(\rho, g, Q, \dot{Q}, \dot{\mathbf{x}} + \dot{\xi} + \dot{\mathbf{Q}} \mathbf{x}).$$  \hspace{1cm} (2.25)

This must hold for all $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ such that (2.24) is true and for all values of the arguments of $\dot{\varepsilon}$ and $T$. If we take $\mathbf{Q} = \mathbf{I}$ and $\mathbf{x} = \mathbf{0}$, equations (2.24) and (2.25) become

$$\dot{\mathbf{Q}} = - \dot{\mathbf{Q}}^T$$  \hspace{1cm} (2.26)

and

$$\text{tr}(T^T \dot{\mathbf{Q}}) = 0$$  \hspace{1cm} (2.27)
respectively. Equation (2.26) implies that $\dot{Q}$ is skew-symmetric and by use of this fact, equation (2.27) implies that

$$T = T^T. \quad (2.28)$$

Furthermore, we can make use of (2.28) to show that (2.14) reduces to

$$\dot{M} = 0 \quad (2.29)$$

for our model of a mixture of gases with several temperatures.

There are several other important results which can be obtained from the entropy inequality. By differentiating (2.23)$_3$ and (2.23)$_4$ with respect to $\dot{x}$ and evaluating the result at $\dot{x} = 0$, $b = 1, \ldots, N$, we find that

$$\frac{d}{\dot{\xi}} b b b b = 0 ; \quad \frac{d}{\dot{\xi}} b b b b = 0. \quad (2.30)$$

We define an equilibrium state for the mixture to be one where

$$\nu = \nu = \ldots = \nu = \nu, \quad (2.31)_1$$

$$g = g = \ldots = g = 0, \quad (2.31)_2$$
and
\[ \dot{x} = \dot{x} = \ldots = \dot{x} = 0 \quad (2.31)_3 \]

where \( \nu \) is any coldness distribution in the domain of the constitutive functional \( f \). With such an equilibrium state, BOWEN and GARCIA [2, sec. 4] show that \((2.23)_6\) implies that
\[ q(u, \rho, 0, \xi, 0) = 0 , \quad (2.32)_1 \]
\[ e(u, \rho, 0, \xi, 0) = 0 , \quad (2.32)_2 \]
and
\[ f(u, \rho, 0, \xi, 0) = 0 . \quad (2.32)_3 \]
III. KINEMATICS OF SECOND AND THIRD ORDER WAVES

This paper studies the propagation of second and third order waves through the mixture defined by the constitutive equations given in Section II. In this work we regard a wave to be a propagating singular surface. This section concerns the kinematics of such singular surfaces. In particular, we shall derive some general results for second and third order singular surfaces. For a more detailed discussion of singular surface kinematics see TRUESDELL and TOUPIN [5, sec. 177].

A singular surface $S(t)$ is the common boundary between two regions $p^+$ and $p^-$ across which some kinematical quantity $\phi(x,t)$, defined on $p$, suffers a jump discontinuity. The speed of displacement of a singular surface $S(t)$ is denoted by $u = u(x,t)$ and is the normal component of the velocity of the wave.

An important kinematic quantity is the displacement derivative which is the rate of change of a function as seen by an observer moving with the normal velocity $u_n$ of $S(t)$, where $\hat{n}(x,t)$ is the unit normal to the surface $S(t)$ [5, sec. 177]. The displacement derivative is defined for any function $\phi(x,t)$ [5, sec. 179] by
\[
\frac{\delta \phi}{\delta t} = \frac{\partial \phi}{\partial t} + u \cdot \eta \cdot \text{grad } \phi \quad \quad (3.1)
\]

An important kinematical formula for the displacement derivative of \( \eta \) is the HAYES-THOMAS formula,

\[
\frac{\delta n_i}{\delta t} = -a^{\alpha \beta} u;_{\beta} x_{i,\alpha}, \quad (\alpha, \beta = 1, 2) \quad \quad (3.2)
\]

where \( a^{\alpha \beta} \) are the components of the surface metric tensor, the \( x_{i,\alpha} \) are defined by

\[
x_{i,\alpha} = \frac{\partial x_i}{\partial y^\alpha} (y^1, y^2, t), \quad \quad (3.3)
\]

the \( x_i \) (\( i = 1, 2, 3 \)) are spatial cartesian coordinates, the \( y^\alpha \) (\( \alpha = 1, 2 \)) are surface coordinates, and \( u;_{\beta} \) is the surface covariant derivative of \( u \). For a derivation of (3.2) see TRUESDELL and TOUPIN [5, sec. 179] or BOWEN and WANG [18, sec. 3].

If \( \phi(x,t) \) is a continuous function of \( x \) in \( p^+ \) and \( p^- \) but suffers a jump discontinuity at \( S(t) \) at some time \( t \), then the \textbf{jump} of \( \phi(x,t) \) at time \( t \) is defined by

\[
[\phi] = \phi(x^-, t) - \phi(x^+, t). \quad \quad (3.4)
\]

We will use the following simplified notation for the quantities \( \phi(x^\pm, t) \):
\[ \phi^+ = \phi(x^+, t) \text{ and } \phi^- = \phi(x^-, t). \]

In equation (3.4), \( \phi^+ \) and \( \phi^- \) denote the limiting values of \( \phi \) as \( S(t) \) is approached from \( p^+ \) and \( p^- \), respectively. Given two quantities \( \phi(x, t) \) and \( \psi(x, t) \), both of which suffer jumps across \( S(t) \), one can prove that

\[ [\phi \psi] = \phi^+[\psi] + \psi^+[\phi] + [\phi][\psi]. \tag{3.5} \]

The jump \([\phi]\) must obey the geometrical condition of compatibility [5, sec. 175]

\[ \left[ \frac{a}{\partial x^k} \right] = \left[ \frac{a}{\partial x^m} \right] n^m n_k + a^{\alpha \beta} x_k; \alpha^{\zeta} a^{\zeta} \beta, \tag{3.6}_1 \]

as well as the kinematical condition of compatibility [5, sec. 180]

\[ \frac{\partial}{\partial t} [\phi] = \left[ \frac{a}{\partial t} \right] + u n^k \left[ \frac{a}{\partial x^k} \right]. \tag{3.6}_2 \]

If \( \phi \) is continuous at \( S(t) \), then \([\phi] = 0 \) and equations (3.6)_1 and (3.6)_2 reduce to

\[ \left[ \frac{a}{\partial x^k} \right] = \left[ \frac{a}{\partial x^m} \right] n^m n_k \tag{3.7}_1 \]

and
\[ \frac{\partial \phi}{\partial t} = - u_n \nabla \frac{\partial \phi}{\partial x} \] \hspace{1cm} (3.7)_2

We now turn to the kinematics of second and third order waves. If \( S(t) \) is a surface across which \( \frac{\partial \chi}{\partial t} \) is continuous but across which \( \frac{\partial \chi}{\partial t} \) is non-zero, then \( S(t) \) is called a second order wave. We define the amplitude \( s \) of a second order wave \( S(t) \) as

\[ s = \frac{\partial \chi}{\partial t} a \] \hspace{1cm} (3.8)_1

If \( s \) is parallel to \( n \), the wave is said to be a longitudinal wave, and if \( s \) is perpendicular to \( n \), the wave is said to be a transverse wave.

Later we will be able to prove that for the materials under consideration here there will be no transverse components of the amplitude, hence we can say that \( s = s_0 \) for longitudinal waves then the amplitude is measured by

\[ s = \frac{\partial \chi}{\partial t} a \cdot n \] \hspace{1cm} (3.8)_2

If we recall that \( \frac{\partial \chi}{\partial t} \) is continuous, and take \( \phi = \chi \) in (3.7)_1 and (3.7)_2, the following relation

\[ u[\nabla \cdot \chi] = - s \cdot \chi \cdot n \] \hspace{1cm} (3.9)_1
is obtained for second order waves in general and

\[ u[\text{grad } \frac{\hat{x}}{a}] = -s \frac{n_\hat{\text{a}} \times n}{a} \tag{3.9}_2 \]

is obtained for second order longitudinal waves. If these two equations are contracted, we obtain

\[ u[\text{div } \frac{\hat{x}}{a}] = -s \frac{n_\hat{\text{a}}}{a} \tag{3.10}_1 \]

and

\[ u[\text{div } \frac{\hat{x}}{a}] = -s \frac{n_\hat{\text{a}}}{a} \tag{3.10}_2 \]

for general and longitudinal waves, respectively.

We also need to obtain relations for the jump in the density and its various derivatives across the wave \( S(t) \). The integral form of the balance of mass equation [16, eqn. 2.14]

\[ [\rho (u - \frac{\hat{x}}{a} \cdot n)] = 0 \tag{3.11}_1 \]

implies that for second order waves

\[ [\rho] = 0. \tag{3.11}_2 \]

That is, the density is continuous across second order waves.
The local form of the balance of mass equation (2.14) supplies us with a method to compute the jumps in \( \frac{\partial \rho}{\partial t} \) and \( \text{grad} \ \rho \). Taking the jump of (2.14), we find that

\[
\left[ \frac{\partial}{\partial t} \right] + \frac{\dot{x}^+}{a} \cdot [\text{grad} \ \rho] = \frac{\rho^s \cdot n}{u}.
\]  

(3.12)

where \( \dot{x}^+ \) and \( \rho^+ \) are the values of the velocity \( \dot{x} \) and the density \( \rho \) in front of the wave \( S(t) \). In deriving (3.12), we made use of (3.10). If we take \( \psi = \rho \) in equations (3.7) and (3.7) and combine the results with (3.12), we obtain

\[
\left[ \frac{\partial}{\partial t} \right] = \frac{\rho^+ \cdot n}{a \cdot u - \dot{x}^+ \cdot n}.
\]  

(3.13)_1

and

\[
u[\text{grad} \ \rho] = \frac{\rho^+(S \cdot n) \cdot n}{a \cdot u - \dot{x}^+ \cdot n},
\]  

(3.13)_2

where we have assumed that \( u \neq \dot{x}^+ \cdot n \). These two equations reduce to

\[
\left[ \frac{\partial}{\partial t} \right] = \frac{\rho^+ s}{a \cdot u - \dot{x}^+ \cdot n}.
\]  

(3.13)_3
and

$$u[\text{grad } \rho] = - \frac{\rho^+ s \eta}{a} \frac{a}{u \cdot \hat{x}^+ \cdot \hat{n}}$$ \hspace{1cm} (3.13)_4$$

for longitudinal waves.

If we take the gradient of the balance of mass equation (2.14)_1, form the jump of that result, and make use of equations (3.7)_2, (3.9), and (3.13), we obtain

$$\frac{\partial \rho}{\partial t} [\text{grad}(\frac{a}{\partial t})] = - \frac{\rho^+ u}{a} \frac{a}{u \cdot \hat{x}^+ \cdot \hat{n}} [\text{grad}(\text{div} \hat{x})]$$

$$\hspace{1cm} \frac{(s \cdot \eta)^2 \eta}{u(u \cdot \hat{x}^+ \cdot \hat{n})^2}$$ \hspace{1cm} (3.13)_5$$

for second order waves in general and

$$\frac{\partial \rho}{\partial t} [\text{grad}(\frac{a}{\partial t})] = - \frac{\rho^+ u}{a} \frac{a}{u \cdot \hat{x}^+ \cdot \hat{n}} [\text{grad}(\text{div} \hat{x})]$$

$$\hspace{1cm} \frac{s^2 \eta}{u(u \cdot \hat{x}^+ \cdot \hat{n})^2}$$ \hspace{1cm} (3.13)_6$$
for second order longitudinal waves.

The displacement derivative of $s$ gives the rate of change of $s$ following $S(t)$ and is hence a measure of the rate of growth and decay of the amplitude of the wave. To obtain the differential equation associated with the wave amplitude we apply the kinematical condition of compatibility (3.6)$_2$ to $\phi = \partial \hat{x}/\partial t$. We then find that

$$\frac{\delta s}{\delta t} = \left[ \frac{\partial^2 \hat{x}}{\partial t^2} \right] + u[\text{grad}\left( \frac{\partial \hat{x}}{\partial t} \right)] n \cdot n.$$  \hspace{1cm} (3.14)

If we let $\phi = \partial \hat{x}/\partial t$ in (3.6)$_2$ and make use of (3.10)$_1$, the result can be combined with (3.14) to yield

$$2 \frac{\partial s}{\partial t} - \frac{\partial s}{\partial t} \frac{\partial u}{\partial t} = \left[ \frac{\partial^2 \hat{x}}{\partial t^2} \right] - u^2[\text{grad}^2 \hat{x}](n, n),$$  \hspace{1cm} (3.15)

where we have used the fact that

$$n \cdot \frac{\partial n}{\partial t} = 0.$$  \hspace{1cm} (3.16)

This result follows from the fact that $n$ is a unit vector.

We now take $\phi = \partial \hat{x}/\partial t$ in the geometric condition of compatibility (3.6)$_1$ and make use of (3.10)$_1$. This leads to
\[
\{ [\text{grad}^2 \dot{x}] (\eta, \eta) \} \cdot \eta = ([\text{grad} (\text{div} \ N \dot{x})] \cdot \eta) \\
- \frac{1}{u} \overline{K}(s, \eta) \quad ,
\]

(3.17)

where \( \overline{K} \) is the mean curvature of \( \overline{S}(t) \) defined by

\[
\overline{K} = a^\alpha_\beta b^\alpha_\beta
\]

(3.18)

and \( b^\alpha_\beta \) are the covariant components of the second fundamental form of \( \overline{S}(t) \). Substituting (3.17) into (3.15), we obtain the desired amplitude equation for second order waves, namely

\[
2 \frac{\dot{a}}{\dot{t}} \cdot n - \frac{\dot{a}}{u} \frac{\delta}{\delta t} = u\overline{K}(s, \eta) + \left[ \frac{\dot{a}}{\dot{t}^2} \right] \cdot n
\]

\[
- u^2 ([\text{grad} (\text{div} \ N \dot{x})] \cdot \eta) \quad .
\]

(3.19)

If we assume that we have longitudinal waves only, then (3.19) reduces to

\[
2 \frac{\dot{a}}{\dot{t}} \cdot n - \frac{\dot{a}}{u} \frac{\delta}{\delta t} = u\overline{K} N + \left[ \frac{\dot{a}}{\dot{t}^2} \right] \cdot n
\]

\[
- u^2 ([\text{grad} (\text{div} \ N \dot{x})] \cdot \eta) \quad ,
\]

(3.20)
where we have again used (3.16) [16, equation 64]. The results (3.19) and (3.20) are general kinematic results for second order waves.

A similar procedure is now followed for third order waves. We define a third order wave as one where \( \dot{\partial x/\partial t} \) are continuous across the surface \( S(t) \), but the jump of \( \ddot{\partial x/\partial t} \) is non-zero. The amplitude of a third order wave is defined by

\[
\frac{\ddot{\partial x}}{\partial t^2} = \left[ \frac{\partial}{\partial t} \right] \left( \frac{\partial x}{\partial x} \right).
\]

(3.21)

Since \( \dot{x} \) and \( \partial \dot{x}/\partial t \) are continuous, we see that \((3.7)\) implies that \( \dot{\partial x/\partial t} \) is also continuous, i.e. we have

\[
[\dot{x}] = 0, \quad \left[ \frac{\partial}{\partial t} \right] = 0, \quad \text{and} \quad [\text{grad } \dot{x}] = 0
\]

(3.22)

for third order waves. Again, we will be able to show that the materials under consideration here only admit longitudinal waves. Thus, for longitudinal waves, the amplitude is measured by

\[
s = \left[ \frac{\partial}{\partial t^2} \right] \cdot n.
\]

(3.23)

If we let \( \phi = \partial x/\partial t \) in \((3.7)\) and use (3.21), we obtain
\[ - \mathbf{u} \left[ \nabla \left( \frac{\mathbf{a}}{\partial t} \right) \right] = \mathbf{s} \otimes \mathbf{n} \] 

(3.24)_1

which reduces to

\[ - \mathbf{u} \left[ \nabla \left( \frac{\mathbf{a}}{\partial t} \right) \right] = \mathbf{s} \cdot \mathbf{n} \otimes \mathbf{n} \] 

(3.24)_2

for longitudinal waves. If these two equations are contracted we obtain

\[ - \mathbf{u} \left[ \nabla \left( \frac{\mathbf{a}}{\partial t} \right) \right] = \mathbf{s} \cdot \mathbf{n} \] 

(3.24)_3

and

\[ - \mathbf{u} \left[ \nabla \left( \frac{\mathbf{a}}{\partial t} \right) \right] = \mathbf{s} \cdot \mathbf{n} \] 

(3.24)_4

If we let \( \phi = \nabla \frac{\mathbf{x}}{\mathbf{a}} \) in (3.7)_2 and use (3.24)_1 we find that

\[ \mathbf{u}^2 \left[ \nabla^2 \frac{\mathbf{x}}{\mathbf{a}} \right] \mathbf{n} = \mathbf{s} \cdot \mathbf{x} \otimes \mathbf{n} \] 

(3.25)_1

and this reduces to

\[ \mathbf{u}^2 \left[ \nabla^2 \frac{\mathbf{x}}{\mathbf{a}} \right] \mathbf{n} = \mathbf{s} \cdot \mathbf{n} \otimes \mathbf{n} \] 

(3.25)_2

for longitudinal waves. Similarly if these two equations are contracted we obtain
\[ u^2 [\text{grad}(\text{div} \hat{x})] \cdot \hat{n} = s \cdot \hat{n} \tag{3.25}_3 \]
and
\[ u^2 [\text{grad}(\text{div} \hat{x})] \cdot \hat{n} = s \cdot \hat{a} \] \tag{3.25}_4

Since \( \hat{x} \) is continuous, equation (3.11)_2 is still valid and we can use these two facts along with (3.22)_3 in the continuity equation (2.14)_1 to show \( \partial \rho / \partial t \) is also continuous. Equation (3.7)_1 and (3.7)_2 and the continuity of \( \partial \rho / \partial t \) then imply that \( \text{grad} \rho \) is also continuous. Thus for third order waves the following equations are valid:

\[ \frac{\partial \rho}{\partial a} \bigg|_{\mathbf{a}} = 0, \quad \left[ \frac{a}{\partial t} \right] = 0, \quad \text{and} \quad \left[ \text{grad} \rho \right] = 0. \tag{3.26} \]

We now obtain expressions for the jumps of the second derivatives of the density \( \rho \). Taking the derivative with respect to time of the continuity equation (2.14)_1 and then forming its jump, we find that the following equation results:

\[ \frac{\partial^2 \rho}{\partial t^2} \bigg|_{\mathbf{a}} + \hat{x}^+ [\text{grad} \left( \frac{a}{\partial t} \right)] + \frac{\partial \rho}{\partial a} \bigg|_{\mathbf{a}} = \frac{\rho^+ s \cdot \hat{n}}{u}. \tag{3.27} \]

where (3.24)_3 has been used. If we now let \( \phi = \partial \rho / \partial t \) in (3.7)_1 and (3.7)_2, it follows from (3.27) that
\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\rho^+ s}{u - x^+ \cdot \hat{\eta}} \]  

and

\[ u[\nabla(\frac{a}{\partial t})] = -\frac{a \rho^+ (s \cdot \hat{\eta})}{u - x^+ \cdot \hat{\eta}} \]  

where we have assumed again that \( u \neq x^+ \cdot \hat{\eta} \). By letting \( \phi = \nabla \rho \) in (3.7)\(_1\) and (3.7)\(_2\) and substituting (3.28)\(_2\) into that result, we obtain

\[ u^2[\nabla^2 \rho] = \frac{\rho^+ s \times \hat{n}}{a \cdot u - x^+ \cdot \hat{\eta}} \]  

These equations reduce to

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\rho^+ s}{u - x^+ \cdot \hat{\eta}} \]  

and

\[ u[\nabla(\frac{a}{\partial t})] = -\frac{a \rho^+ s n}{u - x^+ \cdot \hat{\eta}} \]
\[ u^2 [\nabla^2 \rho] = \frac{\rho^+ s \cdot n \times n}{u - x^+ \cdot n} \]  \hspace{1cm} (3.29)_3

for longitudinal waves.

If we take the time derivative of the continuity equation (2.14) followed by a gradient, form the jump of the result, and make use of (3.7)_, (3.22), and (3.26), we obtain

\[ \frac{\partial^2 \rho}{\partial t^2} = - \frac{\rho^+ u}{u - x^+ \cdot n} \left[ \nabla (\text{div} \left( \frac{\dot{a}}{\partial t} \right)) \right] \]  \hspace{1cm} (3.29)_4

for third order waves in general.

The procedure used to obtain the amplitude equation for third order waves is the same as the procedure used to obtain the amplitude equation for second order waves. Following this procedure we find that the amplitude equation for third order waves in general is [13, equation (2.36)]

\[ 2 \frac{\partial}{\partial t} \cdot n \frac{\partial}{\partial u} \nabla (s \cdot n) = 2 \frac{\partial}{\partial t} \left( \frac{\partial^3 x}{\partial t^3} \right) \cdot n + \frac{\partial^3 x}{\partial t^3} \cdot n \]  \hspace{1cm} (3.30)

\[ u^2 \left( \nabla (\text{div} \left( \frac{\dot{a}}{\partial t} \right)) \right) \cdot n \]
If we let $s = s \cdot n$ in (3.30), form the inner product of the result with $\dot{n}$, and make use of (4.16), we find that

$$
\begin{align*}
2 \frac{\delta s}{\delta t} - \frac{\dot{a}}{u} \frac{\delta u}{\delta t} &= u \dot{K}s + \left[ \frac{\dot{a}}{\delta t} \right] \cdot n \\
\rho \left[ \frac{\delta x}{\delta t} \right]^2 &= -u^2 \left[ \text{grad}(\text{div} \left( \frac{\dot{a}}{\delta t} \right)) \right] \cdot n
\end{align*}
$$

(3.31)

for longitudinal waves.

Any discussion of wave propagation requires that the jump conditions that arise from the integral forms of the balance equations be satisfied at the surface $S(t)$. We have already shown that the jump condition arising from the integral form of the balance of mass (3.11) implies that the density is continuous when the diffusion velocity is continuous.

The jump conditions arising from the integral forms of the balance of linear momentum and the balance of energy are, respectively [17, sec. 2]

$$
-\left[ \rho \dot{x}(u - \dot{x} \cdot n) \right] = [T] \cdot n
$$

(3.32)_1

and

$$
-\left[ \rho (\epsilon + \frac{1}{2} \dot{x}^2)(u - \dot{x} \cdot n) \right] = [T \cdot x] \cdot n - [q] \cdot n.
$$

(3.32)_2

Since $\dot{x}$ and $\rho$ are continuous, equation (3.32)_1 implies that,
for \( a = 1, \ldots, N \),

\[
[T]_{\overline{a} n} = 0. \tag{3.33}
\]

Using the continuity of \( \overline{\rho} \) and \( \overline{\chi} \) again along with (3.33) and the symmetry of \( \overline{T} \), we find that equation (3.32) implies that

\[
\overline{\rho}[\varepsilon](u - \overline{\chi} \cdot n) = [q] \cdot n. \tag{3.34}
\]

To show that (3.33) is indeed satisfied let us look at the equation for the partial stresses (2.23). If we define the hydrostatic partial pressure for the \( a \)th constituent by

\[
\overline{\Pi}(u^a, \rho, g, \xi, \overline{\chi}) = \frac{\Lambda \rho}{\overline{\alpha}} - \frac{a}{\overline{\alpha}} \frac{\partial \rho}{\partial \overline{\alpha}} \tag{3.35}
\]

then (2.23) becomes

\[
\overline{T} = - \overline{\Pi} - \frac{1}{\overline{\alpha}} \sum_{b} (u^b \overline{\chi}^b) \frac{\partial \rho}{\partial \overline{\alpha}}. \tag{3.36}
\]

Equations (2.30) imply that \( \Lambda \) is independent of \( g \) and \( \xi \) when \( \overline{\chi} = 0 \). Thus when \( \overline{\chi} = 0 \) the right hand side of (3.35) depends only on the coldness \( \overline{\alpha} \) and the density \( \rho \). If we make use of (2.12), (2.16)\(_2\), and (2.23)\(_1\), then (3.36) becomes
\[ T = T(v, \rho, g, \xi, 0) = - \pi(v, \rho) I \] 

when \( \dot{\xi} = 0 \). Hence if we consider the propagation of waves in materials where the velocity field \( \dot{\xi} \) is continuous and further only consider waves that propagate into regions where the velocity field \( \dot{\xi} \) is zero, then

\[ [T] = - \pi(v^-, \rho^-) I + \pi(v^+, \rho^+) I. \] 

(3.38)

Since we will only consider waves across which the coldness \( v \) and the density \( \rho \) are continuous, then (3.38) becomes

\[ [T] = - \pi(v^+, \rho^+) I + \pi(v^+, \rho^+) I = 0 \] 

(3.39)

and (3.33) is satisfied.

Similarly, to establish that (3.34) is satisfied, we note that equations (2.23)_1 and (2.23)_2 require that the partial internal energies \( \varepsilon \) be functions of \( v \) and \( \rho \) only.

As a result

\[ [\varepsilon] = \varepsilon(v^+, \rho^+) - \varepsilon(v^+, \rho^+) = 0 \] 

(3.40)

when \( v \) and \( \rho \) are continuous across the wave. In addition when \( g \) is continuous across the wave and
is zero in front of the wave, equation (2.32) implies that

\[ \frac{[q]}{a} = q(v^+, \rho^+, 0, \xi^+, 0) - q(v^+, \rho^+, 0, \xi^+, 0) = 0 \]  \hspace{1cm} (3.41)_1

even though the density gradients \( \xi \) are not continuous.

Equations (3.40) and (3.41) imply that (3.34) is satisfied. Later we shall consider non-conductors as well as conductors. For a non-conductor the constitutive functional (2.21) is independent of the coldness gradients \( \xi \), \( b = 1, \ldots, N \). With this restriction the entropy inequality (2.23) then implies that

\[ q(v, \rho, \xi, x) = 0. \]  \hspace{1cm} (3.41)_2

That is, the heat flux vector for non-conductors is zero not only in front of and across the wave as (3.41) implies for conductors, but is also zero behind the wave.

We have just stated several conditions which are sufficient to insure the jump forms of the balance equations are satisfied. We shall achieve these conditions by considering the circumstance where, since time zero, the wave has been propagating into a region where, before the arrival of the wave, each constituent is at rest with equal temperatures, constant densities, zero temperature gradients, zero density gradients, and zero diffusion velocities. That is, we are assuming that
\( u > 0 , \) \hspace{1cm} (3.42)

\[
\begin{align*}
\nu &= \nu = \ldots = \nu = \nu = \text{constant} , \\
1 & \quad 2 & \quad \cdots & \quad N
\end{align*}
\] \hspace{1cm} (3.43)\_1

\[
\rho = \rho^+ = \text{constant} \quad b = 1, \ldots, N , \\
b & \quad b
\] \hspace{1cm} (3.43)\_2

\[
\begin{align*}
g &= g = \ldots = g = g^+ = 0 , \\
1 & \quad 2 & \quad \cdots & \quad N & \quad b
\end{align*}
\] \hspace{1cm} (3.43)\_3

\[
\begin{align*}
\dot{x} &= \dot{x} = \ldots = \dot{x} = \dot{x}^+ = 0 , \\
1 & \quad 2 & \quad \cdots & \quad N & \quad b
\end{align*}
\] \hspace{1cm} (3.43)\_4

and

\[
\begin{align*}
\dot{x} &= \dot{x} = \ldots = \dot{x} = \dot{x}^+ = 0 , \\
1 & \quad 2 & \quad \cdots & \quad N & \quad b
\end{align*}
\] \hspace{1cm} (3.43)\_5

in front of the wave. For notational purposes, from now on any quantity evaluated on the front side of the wave will have a plus superscript and any quantity evaluated on the back side of the wave will have a minus superscript. For example we shall write

\[
\Lambda^+ = \Lambda(\nu, \rho^+, 0, 0, 0) \\
a & \quad a & \quad b
\]

and

\[
( \frac{\partial \Lambda}{\partial \rho} )^+ = \frac{\partial \Lambda}{\partial \rho} (\nu, \rho^+, 0, 0, 0) . \\
\frac{a}{b} & \quad \frac{a}{b}
\]
IV. SECOND ORDER CURVED WAVES IN CONDUCTORS

In this section the propagation condition and the differential equation governing the amplitude are derived for homothermal second order curved waves propagating into the mixture described in Section II.

In addition to the smoothness of the diffusion velocity $\dot{x}$ and the density $\rho$ assumed earlier, it is also assumed that, for $a = 1, \ldots, N$

\[
\frac{\partial}{\partial a} \left[ u \right] = \left[ \frac{a}{\partial t} \right] = 0 .
\]  
(4.1)

Such second order waves are called homothermal waves. Given (4.1), we can use equations (3.7)₁ and 3.7₂ along with the assumption that $u \neq 0$ to show that

\[
\frac{[g]}{\partial a} = \left[ \text{grad } u \right] = 0 .
\]  
(4.2)

In order to obtain the propagation condition for our problem, we must first find a better working form for the momentum equation (2.14)₂. An examination of the equation for the stress tensor (2.23)₅ along with equations (2.23)₃
and \((2.23)_4\) suggests that we introduce a new quantity \(\mathbf{m}\) defined by

\[
\mathbf{m} = m(\nu, \rho, \mathbf{g}, \mathbf{g}, \mathbf{g}, \mathbf{c}) = \Sigma \mathbf{u} \Lambda \frac{\partial}{\partial \mathbf{c}}. \tag{4.3}
\]

It is the introduction of this quantity which allows us to derive a slightly more simple form for equation \((2.23)_5\) and thereby a better working form for the momentum equation.

If we differentiate \((4.3)\) with respect to \(g, \mathbf{g}, \mathbf{g}\), and \(\mathbf{c}\) respectively and use \((2.12), (2.16)_2, \) and \((2.23)_1\), we find that

\[
\frac{\partial \mathbf{m}}{\partial \mathbf{g}} = \Sigma \mathbf{g} \mathbf{c} \frac{\partial}{\partial \mathbf{g}}, \tag{4.4}_1
\]

\[
\frac{\partial \mathbf{m}}{\partial \mathbf{g}} = \Sigma \mathbf{g} \mathbf{c} \frac{\partial}{\partial \mathbf{g}}, \tag{4.4}_2
\]

and

\[
\frac{\partial \mathbf{m}}{\partial \mathbf{g}} = \Sigma \mathbf{g} \mathbf{c} \frac{\partial}{\partial \mathbf{g}} + \Sigma \Lambda \frac{\partial}{\partial \mathbf{g}}. \tag{4.4}_3
\]

If we make use of equations \((2.11)\) and \((2.12)\), we find that

\[
\frac{\partial \mathbf{u}}{\partial \mathbf{c}} = \left( \delta \rho \mathbf{b} \right)^2, \tag{4.5}_1
\]

\[
\frac{\partial \mathbf{u}}{\partial \mathbf{c}} = \left( \delta - \frac{\mathbf{b}}{\rho} \right)^2, \tag{4.5}_2
\]

\[
\frac{\partial \mathbf{u}}{\partial \mathbf{c}} = \left( \delta - \frac{\mathbf{b}}{\rho} \right)^2, \tag{4.5}_3
\]
where $\delta$ is the Kronecker delta in the space of the constituents. If (4.5)_1 is substituted into (4.4)_3 the result is

$$\frac{\partial m}{\partial x} + \frac{\rho}{\delta} \Lambda \mathbf{I} = \Lambda \mathbf{I} + \sum \frac{\partial}{\partial x} \mathbf{u} \bigotimes \frac{\partial \Lambda}{\partial x}.$$  \hspace{1cm} (4.5)_2

A comparison of (4.4)_1 and (4.4)_2 with (2.23)_3 and (2.23)_4 yields

$$\frac{\partial m}{\partial g} + \left( \frac{\partial m}{\partial g} \right)^T = 0.$$  \hspace{1cm} (4.5)_3

and

$$\frac{\partial m}{\partial \mathbf{x}} + \left( \frac{\partial m}{\partial \mathbf{x}} \right)^T = 0.$$  \hspace{1cm} (4.5)_4

By combining (4.5)_2 with (2.23)_5, we find that

$$\mathbf{T} = -\frac{1}{u} \left( \frac{\partial \Lambda }{\partial \rho} \Lambda - \frac{\partial \rho}{\partial \rho} \right) \mathbf{I} + \frac{\partial m}{\partial \mathbf{x}},$$ \hspace{1cm} (4.6)

and it is this equation that is used in simplifying the momentum equation. Earlier it was shown that the stress for this model is symmetric, hence from (4.6) we require that

$$\frac{\partial m}{\partial x} - \left( \frac{\partial m}{\partial x} \right)^T = 0.$$  \hspace{1cm} (4.7)
Since \( \Lambda \) is independent of \( g, \xi, \) and \( \dot{x} \), we can use (4.6) to obtain expressions for the derivatives of \( T \) with respect to \( g, \xi, \) and \( \dot{x} \) in terms of the derivatives of \( m \) only. This calculation results in

\[
\frac{\partial T}{\partial g} = -\frac{1}{\nu} \frac{\partial^2 m}{\partial x \partial g}, \quad (4.8)_1
\]

\[
\frac{\partial T}{\partial \xi} = -\frac{1}{\nu} \frac{\partial^2 m}{\partial x \partial \xi}, \quad (4.8)_2
\]

and

\[
\frac{\partial T}{\partial \dot{x}} = -\frac{1}{\nu} \frac{\partial^2 m}{\partial x \partial \dot{x}}, \quad (4.8)_3
\]

By differentiating (4.5)_3 and (4.5)_4 with respect to \( \dot{x} \), multiplying the result by \( -1/\nu \), using (4.8)_1 and (4.8)_2, and converting to component notation, we find that

\[
\frac{\partial T_{ij}}{\partial g_k} + \frac{\partial T_{ik}}{\partial g_j} = 0 \quad (4.9)_1
\]

and

\[
\frac{\partial T_{ij}}{\partial \xi_k} + \frac{\partial T_{ik}}{\partial \xi_j} = 0 \quad (4.9)_2
\]
In other words, the derivatives $\frac{\partial T_{ij}}{\partial g_k}$ and $\frac{\partial T_{ij}}{\partial k}$ are skew-symmetric in the $j$ and $k$ indices. As a result, it is clearly true that

$$\frac{\partial T_{ij}}{\partial g_k} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0$$

and

$$\frac{\partial T_{ij}}{\partial x_k} \frac{\partial^2 \rho}{\partial x_j \partial x_k} = 0.$$ 

These two equations will now be used to simplify the expression for the divergence of the stress. If we form this divergence, the result is

$$\text{div} \ T = \sum \left( \frac{\partial T}{\partial a} g + \frac{\partial T}{\partial b} \mathcal{E} - \frac{1}{\rho} \frac{\partial^2 m}{\partial x \partial x} (\text{grad} \ \tilde{x}) \right),$$

where (2.17), (2.21), (2.22), (4.8), (4.10), and (4.10) have been used.

With the assumption that the body forces $b$ are zero, the momentum equation becomes

$$\rho \ddot{x} = \text{div} \ T + \ddot{p},$$

where

$$\ddot{p} = \rho \ddot{x}.$$
Substituting (4.11) into (4.12), we obtain the intermediate result

$$
\dot{\rho} \dot{x} = \sum_{a=1}^{b} \frac{\partial T}{\partial u} \dot{a} \dot{b} g - \frac{1}{v} \sum_{a=1}^{b} \frac{\partial T}{\partial x} \dot{a} \dot{b} \left( \text{grad } \dot{x} \right) + \sum_{a=1}^{b} \frac{\partial T}{\partial \rho} \dot{a} \dot{b} \dot{\rho} + \dot{p} \cdot \xi. \tag{4.13}_1
$$

In order to obtain an expression for the third term on the right side of (4.13)_1, we differentiate (2.23)_5 with respect to $\rho$, multiply on the right by $\xi$, and sum the result on $b = 1, \ldots, N$. This calculation results in

$$
\sum_{a=1}^{b} \frac{\partial T}{\partial \rho} \dot{a} \dot{b} \xi = -\frac{1}{v} \sum_{a=1}^{b} \frac{\partial T}{\partial x} \dot{a} \dot{b} \frac{\partial \Lambda}{\partial \rho} - \frac{\partial T}{\partial \rho} \dot{a} \dot{b} \frac{\partial \Lambda}{\partial \rho} \xi
$$

$$
+ \sum_{c=1}^{c} \left( \frac{\partial T}{\partial x} \frac{\partial \Lambda}{\partial \rho} \xi \right) \xi + \frac{1}{v} \frac{\partial \Lambda}{\partial \rho} \xi \cdot \frac{\partial T}{\partial x} \dot{a} \dot{b} \dot{\rho} \frac{\partial \Lambda}{\partial \rho} \xi. \tag{4.13}_2
$$

If we substitute (4.13)_2 into (4.13)_1 we finally arrive at

$$
\dot{\rho} \dot{x} = \sum_{a=1}^{b} \frac{\partial T}{\partial u} \dot{a} \dot{b} g - \frac{1}{v} \sum_{a=1}^{b} \frac{\partial T}{\partial x} \dot{a} \dot{b} \left( \text{grad } \dot{x} \right) + \hat{f} \dot{x}
$$

$$
+ \frac{1}{v} \sum_{a=1}^{b} \frac{\partial \Lambda}{\partial \rho} \frac{\partial \Lambda}{\partial \rho} \dot{a} \dot{b} \dot{\rho} \frac{\partial T}{\partial x} \dot{a} \dot{b} \dot{\rho} \frac{\partial \Lambda}{\partial \rho} \xi, \tag{4.14}
$$
where (2.23)_6 has been used. Equation (4.14) represents the working form of the momentum equation from which the propagation condition and the amplitude equation will be derived.

We can now use the continuity of \( \dot{x} \) and \( \dot{g} \) along with (3.8)_1, (3.11)_2, (3.13)_2, (3.43)_1, (3.43)_2, (3.43)_5, (4.1)_1, (IA.2)_3, and (IIA.6) to show that the jump of (4.14) results in

\[
\frac{u^2 \nu}{\mu} + \Sigma \rho^+ \rho^+ \left( \frac{\partial^2 \Delta}{\partial \rho \partial \rho} \right)^+ (s \cdot n) n. \tag{4.15}
\]

Earlier it was mentioned that we would be able to show that the materials under consideration here would admit longitudinal waves only. It is obvious now from equation (4.15) that indeed \( s \) is proportional to \( \dot{n} \). If we define the quantity \( Q \) by

\[
Q_{ab} = -\frac{1}{u} \rho^+ \rho^+ \left( \frac{\partial^2 \Delta}{\partial \rho \partial \rho} \right)^+, \tag{4.16}
\]

equation (4.15) becomes

\[
\Sigma Q_{ab} - \rho^+ \delta u^2_{ab} s = 0. \tag{4.17}
\]

Equation (4.17) is the propagation condition for second order homothermal waves. The determination of the possible
speeds requires solving for the roots of the characteristic equation \[ \text{det} [Q - \rho^+ \delta u^2] = 0. \] Since \([Q]\) is symmetric and \([\rho^+ \delta]\) is symmetric and positive definite, the speeds squared are necessarily real. We shall assume in addition that these speeds squared are positive. As a result, the equation, \[ \text{det}[Q - \rho^+ \delta u^2] = 0, \] yields real wave speeds. A necessary and sufficient condition to insure the speeds squared are positive is for \([Q]\) to be positive definite. This condition is suggestive of the stability assumptions in thermostatics [5, Section 265]. In fact, it is easy to express \(Q\) in terms of the HELMHOLTZ free energy of the mixture per unit volume \(\psi_1 = \sum_a^N \psi_a\), where \(\psi\) is the HELMHOLTZ free energy of the \(a\)th constituent per unit volume. The quantity \(\psi\) is related to \(\Lambda\) by \(\Lambda = -\sum_a^N \psi_a\).

If we differentiate the latter formula with respect to \(\rho\) and then \(\rho\) and multiply the result by \(-\rho\rho\) we obtain

\[
-\rho\rho \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} = \rho \rho \sum_a^N \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \frac{c}{c} \frac{\partial^2 \psi}{\partial \rho \partial \rho}. \quad (4.18)
\]

If we now evaluate (4.18) in front of the wave and use the fact, mentioned above, that \(\psi_1 = \sum_c^c \frac{\partial^2 \psi}{\partial \rho \partial \rho}\), we find that (4.16) can also be written in the form

\[
Q = \rho^+ \rho^+ \left( \frac{\partial^2 \psi_1}{\partial \rho \partial \rho} \right)^+ . \quad (4.19)
\]
Because the matrix \[ \hat{Q} \] is constant, the squared speeds are also constant.

We shall denote by \( u^2_{(p)} \), \( p = 1, \ldots, N \) the \( N \) eigenvalues associated with the characteristic equation of (4.17), and we shall denote by \( s^{(p)}_a \) the eigenvector associated with the \( p^{th} \) eigenvalue. If we assume that the \( N \) eigenvalues are distinct then the \( N \) eigenvectors are linearly independent.

Before we analyze (4.17) further, we shall first derive the amplitude equation for second order homothermal waves. This derivation is long and involved and will be carried out in several steps. If we look at the general form of the amplitude equation for second order longitudinal waves (3.20) we see that we must somehow eliminate the jump terms in favor of terms involving the amplitude of the wave \( s^a \). The procedure involves finding an expression for the jump of the second time derivative of the velocity \( \dot{x}^a \) in terms of the jump of the gradient of the divergence of \( \dot{x}^a \) and terms involving the amplitude \( s^a \). This expression is then substituted into the amplitude equation (3.20) and put into a form where the propagation condition (4.17) can be used to eliminate the remaining jump term. The result of this calculation is a differential equation for the desired amplitude.

The first step then is to find the expression for the jump of the second time derivative of the velocity \( \dot{x}^a \). Again we make use of the working form of the momentum
equation (4.14) derived earlier. If we write (4.14) in component notation and take its derivative with respect to time, the following long formula results:

\[
\rho \frac{\partial^2 x_i}{\partial t^2} + \frac{\partial a}{\partial t} \frac{\partial a_i}{\partial t} + \rho \frac{\partial a_j}{\partial t} \frac{\partial a_i}{\partial x_j} + \frac{\partial x_i}{\partial t} \frac{\partial}{\partial (\rho \frac{\partial x_i}{\partial t})}
\]

\[
= \sum \left( \frac{\partial T_{ij}}{\partial u} \frac{\partial^2 u}{\partial t \partial x_j} + \frac{\partial T_{ij}}{\partial v} \frac{\partial v}{\partial t} \frac{\partial x_j}{\partial b} - \frac{1}{a} \frac{\partial a_i}{\partial x_j} \frac{\partial x_i}{\partial t} \right)
\]

\[
- \frac{\partial^2 u_i}{\partial x_j \partial t} + \frac{1}{a} \frac{\partial^2 a_i}{\partial x_j \partial t} - \frac{1}{u} \frac{\partial a_i}{\partial x_j} \frac{\partial x_i}{\partial t} + \frac{1}{a} \frac{\partial a_i}{\partial x_j} \frac{\partial x_i}{\partial t} \right]
\]

\[
\frac{\partial^2 u}{\partial c \partial x} + \frac{\partial^2 a_i}{\partial c \partial x} \frac{\partial x_i}{\partial a_j} \frac{\partial x_i}{\partial c} + \frac{\partial^2 a_i}{\partial c \partial x} \frac{\partial x_i}{\partial a_j} \frac{\partial x_i}{\partial c} + \frac{\partial^2 a_i}{\partial c \partial x} \frac{\partial x_i}{\partial a_j} \frac{\partial x_i}{\partial c}
\]

\[
\frac{\partial a_i}{\partial x_j} \frac{\partial x_i}{\partial t} \right] + \frac{\partial a_i}{\partial a_j} \frac{\partial a_j}{\partial t} + \frac{\partial a_i}{\partial b} \frac{\partial a_i}{\partial t} + \frac{\partial a_i}{\partial g} \frac{\partial a_i}{\partial t}
\]

\[
\frac{\partial^2 a_i}{\partial b \partial t} + \frac{\partial^2 a_i}{\partial g \partial t} \frac{\partial x_i}{\partial b} + \frac{\partial^2 a_i}{\partial g \partial t} \frac{\partial x_i}{\partial g} + \frac{\partial^2 a_i}{\partial g \partial t} \frac{\partial x_i}{\partial c}
\]
\[ + \frac{\partial f_i}{\partial t} \frac{\partial^2 \rho}{\partial x_j \partial x_j} + \frac{\partial f_i}{\partial \rho} \frac{\partial^2 \rho}{\partial t \partial x_j} + \frac{\partial f_i}{\partial \rho} \frac{\partial^2 \rho}{\partial x_j \partial t} \]

\[ + \frac{\partial \rho}{\partial t} \sum \left[ \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \nu} \frac{\partial \nu}{\partial t} + \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \rho} \frac{\partial \rho}{\partial t} \frac{\partial \rho}{\partial x_i} \right] \frac{\partial \rho}{\partial x_i} \]

\[ + \frac{1}{\nu} \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial t} \frac{\partial \rho}{\partial x_i} - \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \frac{\partial \rho}{\partial t} \frac{\partial \rho}{\partial x_i} - \frac{1}{\nu} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_i} \]

\[ \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial t} \frac{\partial \rho}{\partial x_i} - \sum \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_i} \right] \}

(4.20)

If we note that (3.37) holds in front of the wave then

\[ T_a^+ = - \frac{\rho^+}{a^+} I_a \]  
(4.21)_1

and \( \pi^+ = \pi^- \). It follows then that

\[ \left( \frac{\partial T}{\partial \nu} \right)^+ = - \left( \frac{\partial \pi}{\partial \nu} \right)^+ I, \]

(4.21)_2

and we can make use of equations (IA.5), (IIA.6), and (IIA.10) along with (3.5), (4.1), (4.2), and (4.21)_2 to show that the jump of (4.20) yields
\[ \rho^+ \left[ \frac{\partial^2}{\partial t^2} \right] \sigma \left[ \begin{array}{c} \frac{\partial \pi}{\partial b} \\ \frac{\partial \gamma}{\partial b} \end{array} \right] + \left( \frac{\partial a}{\partial \gamma} \right)^+ \left( \frac{\partial a}{\partial \gamma} \right)^- \left[ \text{grad} \left( \frac{\partial b}{\partial t} \right) \right] \]

\[ + \left( \frac{\partial a}{\partial \gamma} \right)^- \left[ \frac{\partial b}{\partial \gamma} \right] + \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \left( \frac{\partial a}{\partial \gamma} \right)^+ \left[ \text{grad} \left( \frac{\partial b}{\partial t} \right) \right] \]

\[ + \frac{\partial}{\partial \gamma} \left( \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \rho} \right)^+ \left[ \frac{\partial b}{\partial \gamma} \right] + \frac{1}{\rho} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^+ \]

\[ \left[ \frac{\partial a}{\partial \gamma} \right] \left[ \text{grad} \left( \frac{\partial b}{\partial \gamma} \right) \right] + \frac{\partial \lambda}{\partial \gamma} \left( \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \rho} \right)^+ \left[ \text{grad} \left( \frac{\partial b}{\partial \gamma} \right) \right] \]

\[ - \Sigma \left( \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \rho} \right)^- \left[ \text{grad} \left( \frac{\partial b}{\partial \gamma} \right) \right] \left[ \frac{\partial \lambda}{\partial \gamma} \right] \]

\[ \text{(4.22)} \]

In deriving (4.22), we made use of the fact that

\[ \left[ \frac{\partial a}{\partial \gamma} \right] \left[ \frac{\partial a}{\partial \gamma} \right] = \frac{\rho^+}{\rho} \left[ \frac{\partial \lambda}{\partial \gamma} \right] \left[ \frac{\partial \lambda}{\partial \gamma} \right] \]

\[ \text{(4.23)} \]

With the use of (3.8)\(_2\), (3.9)\(_2\), (3.13)\(_3\), (3.13)\(_4\), (3.43)\(_5\), and (IIA.13) we can further reduce (4.22) to obtain the intermediate result

\[ \rho^+ \left[ \frac{\partial^2}{\partial t^2} \right] \sigma \left[ \begin{array}{c} \frac{\partial \pi}{\partial b} \\ \frac{\partial \gamma}{\partial b} \end{array} \right] + \left( \frac{\partial a}{\partial \gamma} \right)^+ \left( \frac{\partial a}{\partial \gamma} \right)^- \left[ \text{grad} \left( \frac{\partial b}{\partial t} \right) \right] \]

\[ \text{(4.22)} \]
\[ + \sum_{a,b} \frac{\rho^+}{\gamma} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^* + \frac{\partial \rho}{\partial t} \right)_a + \frac{\partial f}{\partial x} \right)_b \]

\[ + \sum_{b,c} \frac{\rho^+}{\gamma} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^* \cdot \hat{n} \cdot \hat{n} - \frac{\partial^2 \rho}{\partial \rho \partial \rho} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^* \]

\[ - \frac{\rho^+}{\gamma} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^*_b \cdot \hat{n} \cdot \hat{n} - \frac{1}{\gamma} \frac{\partial}{\partial t} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^* \]

(4.24)

and it remains to determine \[\text{grad}(\partial \psi/\partial t)\].

In order to determine \[\text{grad}(\partial \psi/\partial t)\] we must use the energy equation (2.14)_4. If we assume that the partial heat supply term \(r\) is zero and make use of (3.43)_5, (4.21)_1, and (IA.2)_2, the jump of (2.14)_4 results in

\[ \rho^+ \left[ \frac{\partial}{\partial t} \right] = - \frac{\partial}{\partial t} \left[ \text{div} \hat{\chi} \right] - \left[ \text{div} q \right] , \]

(4.25)

and it is from \[\text{div} q\] that we will eventually be able to determine an expression for \[\text{grad}(\partial \psi/\partial t)\]. We must determine expressions for \[\partial \epsilon/\partial t\] and \[\text{div} q\], substitute those results back into (4.25), and solve for \[\text{grad} (\partial \psi/\partial t)\]. If we differentiate \(\epsilon(\nu, \rho)\) with respect to time and form the jump of the result, we find that
\[
\left[ \frac{\partial \epsilon}{\partial t} \right] = \frac{1}{u} \Sigma_{b} \rho b^{+} \left( \frac{\partial a}{\partial \rho} \right) b^{+} s,
\]

(4.26)

where (3.13)_3, (3.43)_5, and (4.1)_2 have been used. We now take the divergence of \( q(v, \rho, g, \xi, \hat{x}) \) and form the jump of the result to obtain

\[
[\text{div} q] = \Sigma_{b} \text{tr}\left( \left( \frac{\partial a}{\partial g} \right)^{-} [\text{grad}^2 v] \right) \left( \frac{\partial q}{\partial b} \right)^{-} \left( \frac{\partial a}{\partial \rho} \right)^{-} \left( \frac{\partial a}{\partial x} \right)^{-} (n \cdot (x \cdot n)) s \),
\]

(4.27)

where (3.5), (3.9)_2, and (4.2) have been used. Substitution of (4.26) and (4.27) into (4.25) leads to

\[
\Sigma_{b} \text{tr}\left( \left( \frac{\partial a}{\partial g} \right)^{-} [\text{grad}^2 v] \right) = \frac{1}{u} \Sigma_{b} \left( - v \left( \frac{\partial \pi}{\partial a} \right)^{+} + n \cdot \left( \frac{\partial a}{\partial x} \right)^{-} n \right) s .
\]

(4.28)_1

In deriving (4.28)_1 we made use of the identity

\[
\rho \rho \frac{\partial a}{\partial \rho} - \pi \delta = v \frac{\partial b}{\partial \nu} .
\]

(4.28)_2
This identity follows directly from (2.23)\textsubscript{2} and (3.35) by differentiation.

By use of (3.7)\textsubscript{2} we find that

\[
[\text{grad}^2 \mathbf{v}] = -\frac{1}{u} [[\text{grad}(\frac{\partial}{\partial t})] \cdot \mathbf{n}) \cdot (\mathbf{n} \times \mathbf{z}).
\] (4.29)

Thus the quantity \([\text{grad} (\frac{\partial \mathbf{v}}{\partial t})]\) can be written in the form

\[
[\text{grad}(\frac{\partial}{\partial t})] = -\varepsilon \frac{u}{d b d} \mathbf{n},
\] (4.30)

where

\[
\varepsilon_{bd c} = \varepsilon \kappa^{-1} \left\{ -\nu \left( \frac{d}{\partial \mathbf{v}} \right)^2 + \eta \cdot \left( \frac{\partial \mathbf{q}}{\partial \mathbf{x}} \right)^2 \mathbf{n} \right\},
\] (4.31)\textsubscript{1}

and

\[
\kappa = \eta \cdot \left( \frac{\partial \mathbf{q}}{\partial \mathbf{g}} \right)^2 \mathbf{n},
\] (4.31)\textsubscript{2}

and we have made the extra assumption that the matrix \([\kappa]\)\textsubscript{ab} has an inverse.

It is simple to show that all the second order derivatives of the coldnesses are discontinuous. For example, if we substitute (4.30) back into (4.29), we find that
\[ u[\text{grad}^2 \mathbf{u}] = \sum_{\mathbf{b}} \mathbf{s}(\mathbf{n} \otimes \mathbf{n}), \quad (4.32) \]

and if we let \( \phi = \frac{\partial u}{\partial t} \) in (3.7) and substitute (4.30) into that result, we find that

\[ \frac{\partial^2 u}{\partial t^2} \mathbf{b} = u \mathbf{S} \mathbf{r} \mathbf{s} \mathbf{d}. \quad (4.33) \]

If we recall that the coldnesses and its first order derivatives are all continuous, the fact that all the second order derivatives of the coldnesses are discontinuous can be interpreted in the following way. Since, before the arrival of the wave, all the coldnesses are the same, the fact that the second order derivatives of the coldness are discontinuous implies that, away from and behind the wave the coldnesses will not necessarily be the same for each constituent. Hence the passage of the wave through the mixture has a separating effect on the coldness distribution.

If we substitute (3.13) and (4.30) into (4.24) we obtain

\[ \rho^+ \frac{\partial^2 \mathbf{x}}{\partial t^2} = \sum_{\mathbf{b}} \mathbf{Q} \left[ \text{grad} \left( \text{div} \mathbf{x} \right) \right] \cdot \mathbf{n} \]

\[ + \sum_{\mathbf{b}} \Delta \mathbf{s} + \sum_{\mathbf{b}, \mathbf{c}} \mathbf{abc}. \quad (4.34) \]
where

\[ \Delta = \eta \cdot \left( \frac{\partial f}{\partial x} \right)^{-1} \eta + \varepsilon \left\{ \left( \frac{\partial a}{\partial u} \right)^{+} - \eta \cdot \left( \frac{\partial a}{\partial g} \right)^{-1} \eta \right\} \tau, \quad (4.35)_1 \]

and

\[ \Pi = \frac{1}{3} \left\{ \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right\}^{+} \cdot \eta - \nu u^2 T \]

\[ \frac{\rho^+}{\rho^+ \frac{\partial}{\partial \rho} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^{+}} \right\}. \quad (4.35)_2 \]

In deriving (4.34) we formed the inner product of (4.24) with \( \eta \) and made use of (4.16).

We can now substitute (4.34) into the amplitude equation for second order longitudinal waves (3.20) to obtain

\[ 2 \rho^+ \frac{\partial}{\partial t} = \varepsilon \left( Q - \rho^+ \delta u^2 \right) \left\{ \text{grad}( \text{div} \, \tilde{x} ) \right\} \cdot \eta \]

\[ + u K \rho^+ s + \varepsilon \Delta s + \varepsilon \pi ss, \quad (4.36) \]

where we have used the fact, mentioned earlier that the wave speed \( u \) is constant. It is important to point out here that the quantities \( \Delta \) and \( \Pi \) both depend on the amplitude of the wave.
You will recall in the discussion of the propagation condition (4.17) that we assumed the N eigenvalues \( u^2_{(p)} \), \( p = 1, \ldots, N \), to be distinct. We can therefore write

\[
s^{(p)}(t) = s^{(p)}(t) r^{(p)} \, ,
\]

where \( r^{(p)} \) is independent of time. If we make use of (4.37) in the propagation condition (4.17), we are led to

\[
\sum_{Q=0}^{\infty} (Q - \rho^+ \delta^{ab} u^2_{p}) r^{(p)} = 0 .
\]

(4.38)

In addition it is also true that the N eigenvectors obey the generalized orthogonality condition,

\[
\sum_{Q=0}^{\infty} \rho^+ r^{(p)} r^{(q)} = 0 ,
\]

(4.39)\_1

and we can always normalize and obtain

\[
\sum_{Q=0}^{\infty} \rho^+ r^{(p)} r^{(p)} = 1 .
\]

(4.39)\_2

Since, after the passage of the first wave, subsequent waves will no longer be propagating into a region where equations (3.43) are true, the analysis here holds only for the leading wave. We can then, without loss of generality, let \( u^2_{(1)} \) denote the squared speed of the leading wave. Hence \( s^{(1)} \) is the amplitude associated with this
speed and for simplicity we will not label this index. Thus for the leading wave we write

\[ s(t) = s(t) r \]

(4.39)\textsubscript{3}

and

\[ \sum_a^b \rho^+ r^2 = 1. \]

(4.39)\textsubscript{4}

Furthermore, if we substitute (4.39)\textsubscript{3} into (4.36), multiply the result by \( r \), and sum on \( a = 1, \ldots, N \), we obtain

\[ 2 \frac{\delta s}{\delta t} = u \bar{K} s + \Delta s + \Pi s^2, \]

(4.40)

where

\[ \Delta = \sum_{a,b} \Delta \]

(4.41)

and

\[ \Pi = \sum_{a,b,c} \Pi \]

(4.42)

In deriving (4.40) we made use of (4.39)\textsubscript{4} and (4.38). The quantities \( \Delta \) and \( \Pi \) are material parameters and, since \( \Delta \) and \( \Pi \) depended on the amplitude, then \( \Delta \) and \( \Pi \) also depend on the amplitude. Note that the differential equation (4.40) governs the amplitude of the leading wave only.

It is quite common in the study of second order
waves for the amplitude equation to be governed by an equation of the form (4.40) (e.g., see [11, part V], [16], and [17]). However, usually when assumptions parallel to equations (3.43) are made the resulting differential equation for the amplitude has constant coefficients. This is not the case here. Both $\Delta$ and $\Pi$ in (4.40) depend upon the amplitude $s$. In those models where $\Delta$ and $\Pi$ are constants a common feature is that if $\Delta < 0$ and $\Pi > 0$, $\Delta$ acts as an attenuation factor and $\Pi$ as a growth factor. In our model however no such statement can be made without knowing how $\Delta$ and $\Pi$ depend on $s$. It may be that $\Delta$ acts as a growth factor and $\Pi$ as an attenuation factor for particular dependencies of $\Delta$ and $\Pi$ on $s$. It can be shown, however, that the principles of material frame indifference and symmetry imply that the response functions for this model are isotropic functions. We can then use the isotropy of the response functions to prove that $\Delta$ is an even function of $s$. Hence, whether we have an expansive or compressive disturbance, if $\Delta < 0$ it will always act as an attenuation factor and if $\Delta > 0$ it will always act as a growth factor.

The major difference in this model is the dependence of $\Delta$ and $\Pi$ on $s$. This difference is brought about by the dependence on the density gradients $\xi_b$, $b = 1, \ldots, N$, in our original constitutive equations (2.21). If we were to linearize around a state of zero density gradients, the
dependence of $\Delta$ and $\Pi$ on $s$ would disappear and their affects would be the same as mentioned earlier.

In general in this model nothing can be said about the signs of $\Delta$ or $\Pi$. However, in certain special cases we can show that $\Delta \leq 0$. If we make the following definitions for the scalars $A$, $B$, $C$, and $D$:

\begin{align*}
A &= n \cdot \left( \frac{\partial \hat{f}}{\partial x} \right) - n, \\
B &= \left( \frac{\partial a}{\partial u} \right)^+ , \\
C &= n \cdot \left( \frac{\partial \hat{f}}{\partial g} \right) - n, \\
D &= \frac{n}{\nu} \cdot \left( \frac{\partial a}{\partial x} \right) - n
\end{align*}

and

\begin{align*}
D &= \frac{n}{\nu} \cdot \left( \frac{\partial a}{\partial x} \right) - n
\end{align*}

then the coefficient $\Delta$ can be written

\begin{align*}
\Delta &= A - \nu \sum_{c,d} (C - B) \kappa^{-1}(D - B).
\end{align*}
If we assume that
\[
\left( \frac{\partial f}{\partial g} \right)^{-b} = \frac{1}{v} \left( \frac{\partial g}{\partial x} \right)^{-a},
\]
(4.45)

that is, \( C = D \), then \( \Delta \) further reduces to
\[
\Delta = A - \sum_{ab} E \kappa^{-1} E
\]
(4.46)

where \( E = C - B \). Since \( \Delta = \sum_{ab} \Delta r \) the assumption
\[\sum_{ab} \Delta \] implies that \( \Delta \) becomes
\[
\Delta = A - \sum_{cd} E \kappa^{-1} E
\]
(4.47)

where \( A = \sum_{a,b} A \) \( r \) \( r \) and \( E = \sum_{b} E \) \( r \). It can be shown
\[2\] that the entropy inequality (2.23) implies that
\[
\sum_{a,b} \left( \frac{\partial f}{\partial x} \right)^{-a} \leq 0
\]
(4.48)

and
\[
\sum_{c,d} \kappa^{-1} w \geq 0
\]
(4.49)

where \( v \) and \( w \), \( a = 1, \ldots, N \) are arbitrary quantities. Equation (4.48) clearly shows that
\[ A \leq 0 . \]  
\[ (4.50) \]

Equation (4.49) also shows that

\[ \sum_{c,d} \kappa^{-1} E > 0 . \]  
\[ (4.51) \]

Since \( \nu > 0 \), (4.50) and (4.51) yield from (4.47) that

\[ \Delta \leq 0 \]  
\[ (4.52) \]

for all values of the amplitude \( s \).

The equation (4.45) is like an ONSAGER relation. In this work we do not adopt (4.45) as a general symmetry condition, we simply point out that if (4.45) is valid then (4.52) follows. Of course, a trivial circumstance where (4.45) holds is when each \( \hat{f} \) is taken to be independent of \( g, b = 1, \ldots, N, \) and, in addition, each \( g \) is taken to be independent of \( \hat{x}, b = 1, \ldots, N. \)

The solution to a differential equation of the form (4.40) is well known, when \( \Delta \) and \( \Pi \) are constants and \( \overline{K} = \overline{K}(t) \). For example DORIA and BOWEN [16, Sec. VI] discuss the general solution to an equation of the form (4.40). In particular they discuss the solutions for cylindrical and spherical waves.
V. THIRD ORDER CURVED WAVES IN CONDUCTORS

We saw in Section IV that an analysis of second order curved waves led to a non-linear differential equation for which we know no solution. The one outstanding feature that led to many of the difficulties was the fact that one of the independent variables appearing in the constitutive equation (2.21), the density gradient, was not continuous and in fact depended on the amplitude of the wave $\tilde{s}$. Clearly then, to eliminate this problem we can look at waves across which the density gradients are continuous.

In Section III the kinematics of third order waves was developed and we found that third order waves (and all higher order waves) have the desired feature that the density gradients be continuous (see equation (3.26)). In this section we shall proceed with an analysis parallel to that in Section IV but now for third order curved waves.

In the same manner as in Section IV the density $\rho_{\alpha}$ and the diffusion velocity $\dot{\tilde{x}}_{\alpha}$ are continuous and equations (4.1) and (4.2) still hold. The physical description of the problem described in Section III is still the same and equations (3.40) and (3.41) also still apply here. For third
order waves, since the density gradient is continuous, we might expect some different results for the jumps of
$$\alpha^2 \frac{\partial^2 u}{\partial t^2}, \text{grad} (\partial u/\partial t), \text{and grad}^2 u.$$ As usual we use the energy equation (2.14) to obtain these jumps. If we again assume that the partial heat supply term $r$ is zero, the energy equation (2.14) becomes
$$\rho \frac{\partial \varepsilon}{\partial t} = \text{tr}(T_L^T L) - \text{div} q + \varepsilon . \quad (5.1)$$

The jump of this equation then results in
$$\rho^+ \left[ \frac{\partial}{\partial t} \varepsilon \right] = - \left[ \text{div} q \right] , \quad (5.2)$$

where we have used the continuity of $\dot{x}$, (3.22) and (IA.2).

If we take the derivative of $\varepsilon(u, \rho)$ with respect to time, we form the jump of the result, and use (3.26) and (4.1) we obtain
$$\left[ \frac{\partial}{\partial t} \varepsilon \right] = 0 . \quad (5.3)$$

This fact along with (3.7) and (3.7) imply that
$$\left[ \text{grad} \varepsilon \right] = 0 . \quad (5.4)$$

Similarly if we form the divergence of the heat flux vector $q$, form the jump of the result, and use (3.22), (3.27)
(4.2), and (IA.8)\textsubscript{3}, there results

\[
[\text{div } q] = \varepsilon \text{ tr}\{\left( \frac{\partial}{\partial g}\right)^+ [\text{grad}^2 v]\}. \tag{5.5}
\]

Substituting (5.3) and (5.5) into (5.2), we arrive at

\[
\varepsilon \text{ tr}\{\left( \frac{\partial}{\partial g}\right)^+ [\text{grad}^2 v]\} = 0. \tag{5.6}
\]

Finally, we use the kinematic results (3.7)\textsubscript{1} and (4.29) and the definition (4.31)\textsubscript{2} introduced in Section IV, to show that (5.6) reduces to

\[
[\text{grad } (\varphi_v/\partial t)] = 0, \tag{5.7}_1
\]

where we have again made the assumption that the matrix

\[
[\kappa]
\]

has an inverse. In addition, if we let \( \phi \) be \( \text{grad } v \) \( \text{ab} \) \( \text{b} \) in (3.7)\textsubscript{2} and use (3.7)\textsubscript{1} and (5.7)\textsubscript{1} there results

\[
[\text{grad}^2 v] = 0. \tag{5.7}_2
\]

Similarly if we let \( \phi \) be \( \varphi_v/\partial t \) in (3.7)\textsubscript{2} and use (3.7)\textsubscript{1} and (5.7)\textsubscript{1} again, we find that
\[
\frac{a^2}{b} \left[ \frac{b}{\partial t^2} \right] = 0. \tag{5.73}
\]

Since equation (4.14) is merely another form of the momentum equation it can be used here to derive the propagation condition for third order waves. First it should be pointed out that for third order waves the jump of (4.14) is identically satisfied. If we rewrite equation (4.20) in the following form

\[
\rho \frac{a^2}{\partial t^2} + \frac{a}{\partial t} \frac{\partial a}{\partial t} + \frac{a}{\partial t} \left( \rho \ \text{grad} \ x \right) \times \frac{\partial a}{\partial t}
\]

\[
+ \rho \ \text{grad} \ x \ \frac{\partial a}{\partial t} = \Sigma \left( \frac{a}{\partial t} \right) \ \text{grad} \ u
\]

\[
+ \frac{a}{\partial u} \ \text{grad} \ \left( \frac{b}{\partial t} \right) - \frac{\partial a}{\partial x} \ \frac{\partial m}{\partial x} \ \text{grad} \ \left( \frac{b}{\partial t} \right)
\]

\[
- \frac{1}{u} \ \frac{\partial a}{\partial t} \ \left( \frac{\partial m}{\partial a} \right) \ (\text{grad} \ x) + \frac{1}{u} \ \frac{\partial a}{\partial t} \ \frac{\partial m}{\partial a} \ (\text{grad} \ x)
\]

\[
+ \frac{\partial a}{\partial u} \ \frac{b}{\partial t} + \frac{\partial a}{\partial \rho} \ \frac{b}{\partial t} + \frac{\partial a}{\partial g} \ \text{grad} \ \left( \frac{b}{\partial t} \right)
\]
\[
\begin{align*}
&\frac{\partial \hat{f}}{\partial z} \text{grad}(\frac{\partial \rho}{\partial t}) + \frac{\partial \hat{f}}{\partial \rho} \text{grad}(\frac{\partial \hat{a}}{\partial t}) + \frac{\partial \hat{f}}{\partial \rho} \frac{\partial \hat{b}}{\partial t} + \frac{\partial \hat{f}}{\partial \hat{a}} \text{grad}(\frac{\partial \hat{b}}{\partial t}) \\
&+ \frac{\partial \hat{a}}{\partial \rho} \frac{\partial \hat{2} \Lambda}{\partial \rho} \text{grad} \rho - \frac{1}{\partial u} \text{E} \left( \frac{\partial \hat{c}}{\partial \hat{c}} \text{grad} \left( \frac{\partial \hat{b}}{\partial t} \right) \right) \\
&- \frac{\partial \hat{c}}{\partial t} \left( \frac{\partial \hat{c}}{\partial \hat{c}} \frac{1}{\partial \rho} \text{grad} \rho \right)
\end{align*}
\]

(5.8)

we can then make use of equations (3.21), (3.22), (3.26),
(3.28)\textsubscript{2}, (3.40)\textsubscript{3}, (3.41), (4.1), (4.2), (5.7)\textsubscript{1}, (IA.4)\textsubscript{3}, and
(IIA.6) to show that the jump of (5.8) results in

\[
\begin{align*}
u^2 \rho^+ \Sigma = - \text{E} \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \left( \frac{\partial \hat{c}}{\partial \rho} \right)^+ (\hat{z} \cdot \hat{n}) \hat{n}.
\end{align*}
\]

(5.9)

Here again we see that \( \hat{z} \) is proportional to \( \Sigma \) and we are dealing with longitudinal waves only. If we make use of the definition (4.16) we obtain the propagation condition for third order waves,

\[
\Sigma (Q - \rho^+ \delta \ u^2) \ s = 0.
\]

(5.10)

Note that the propagation conditions for second and third order waves, (4.17) and (5.10), have the same form.
Consequently the quantity $Q_{ab}$ has the same properties as it had earlier. That is, the matrix $[Q_{ab}]$ is symmetric and does not depend on time. Furthermore we shall make the same assumption made in Section IV that the symmetric matrix $[Q_{ab}]$ be positive definite. Thus the squared speeds in (5.10) are positive and we will use the same notation that was introduced in Section IV for the $N$ eigenvalues and eigenvectors associated with the characteristic equation of (5.10). We shall again assume that the $N$ eigenvalues are distinct.

We now turn to the derivation of the amplitude equation for third order waves. It is here that we will see the benefits of having all the independent variables continuous. We begin by taking another time derivative of (5.8) and forming the jump of the result to obtain

$$\rho^+[\frac{\partial^3 \chi}{\partial t^3}] = \Sigma \{ [-(\frac{\partial}{\partial u})^+1 + (\frac{\partial}{\partial \rho})^+] \frac{\partial^2 \varphi}{\partial \rho^2}] [\text{grad}(-\frac{b}{\partial t^2})]$$

$$+ (\frac{\partial}{\partial x})^+ [\frac{b}{\partial t^2}] + (\frac{\partial}{\partial \rho})^+ [\frac{\partial^2 \varphi}{\partial \rho^2}] [\text{grad}(-\frac{b}{\partial t^2})] \} .$$

(5.11)

In deriving (5.11) we made use of (3.22), (3.26), (4.21)$_2$, (5.7)$_1$, (IA.5), (IA.9)$_3$, (IIA.6), and (IIA.10). Since $[\frac{\partial^2 \chi}{\partial t^2}]$ and $[\text{grad}(\frac{\partial^2 \varphi}{\partial t^2})]$ are given by (3.23) and (3.29)$_4$
respectively, it remains to determine \( \text{grad}(\partial^2 v/\partial t^2) \).

Just as we used the energy equation to obtain (4.30), we again use it to obtain an expression for \( \text{grad}(\partial^2 v/\partial t^2) \).

The jump of the time derivative of the energy equation (2.14) is

\[
\rho^+ \left[ \frac{a}{\partial t^2} \right] = - \eta^+ [\text{div}(\frac{a}{\partial t})] - [\text{div}(\frac{a}{\partial t})] + \left[ \frac{a}{\partial t} \right], \tag{5.12}
\]

where we have used (2.7), (3.22), (3.26), (5.3), and (5.4).

If we form the jump of the second time derivative of \( \epsilon(v, \rho) \) and make use of (3.26)\(_2\), (3.29)\(_1\), (3.41)\(_4\), (4.1), and (5.7)\(_3\), we find that

\[
\frac{\partial^2 \epsilon}{\partial t^2} \left[ \frac{a}{\partial t} \right] = \frac{1}{u} \sum_{b} \rho^+ \left( \frac{a}{\partial \rho} \right)_b \frac{\partial \epsilon}{\partial t} \tag{5.13}
\]

The jump in the time derivative of \( \hat{\epsilon}(v, \rho, g \xi \hat{x}) \) is given by

\[
\left[ \frac{a}{\partial t} \right] = 0 \tag{5.14}
\]

where (3.22)\(_2\), (3.26)\(_2\), (4.1)\(_2\), (5.7)\(_1\), and (IA.7)\(_3\) have
been used. Finally if we form \( \text{div}(\partial q/\partial t) \), take its jump, and make use of (3.22), (3.26), (4.1), (4.2), (5.7)_1, (5.7)_2, (IA.8)_1, and (IA.9)_4 we obtain

\[
\frac{\partial q}{\partial t} \bigg|_a \bigg|_b = \sum b \text{ tr} \bigg( \frac{\partial a}{\partial g} \bigg) \bigg[ \text{grad}^2 \bigg( \frac{\partial b}{\partial t} \bigg) \bigg] \\
+ \bigg( \frac{\partial a}{\partial x} \bigg) \bigg[ \text{grad}(\frac{\partial b}{\partial t}) \bigg]. 
\]  
(5.15)

Substituting (5.13), (5.14), and (5.15) into (5.12) and making use of (3.24)_2, (3.24)_4, and (4.28)_2, we arrive at

\[
\sum b \text{ tr} \bigg( \frac{\partial a}{\partial g} \bigg) \bigg[ \text{grad}^2 \bigg( \frac{\partial b}{\partial t} \bigg) \bigg] = \frac{1}{u} \sum b \xi(-\nu \bigg( \frac{\partial \nu}{\partial v} \bigg) +
\]

\[
+ n \cdot (\frac{\partial a}{\partial x} \bigg) \bigg|_b s . 
\]  
(5.16)_1

By use of (3.7)_2 we find that

\[
\text{grad}^2 \bigg( \frac{\partial b}{\partial t} \bigg) = -\frac{1}{u} \bigg( \text{grad} \bigg( \frac{\partial b}{\partial t^2} \bigg) \bigg) \cdot n)(n \otimes n).
\]  
(5.16)_2
Thus if we make use of \((4.31)_1\), \((4.31)_2\), \((5.16)_2\), and the assumed invertibility of the matrix \([\kappa]_{ab}\), \((5.16)_1\) implies that

\[
[\text{grad}(\frac{\partial}{\partial t^2})] = -\varepsilon \Gamma s n d d d .
\]

\((5.17)_1\)

It is easy to show that

\[
[\frac{\partial^3}{\partial t^3}] = u \varepsilon \Gamma s d d d .
\]

\((5.17)_2\)

\[
[\text{grad}^2(\frac{\partial}{\partial t})] = \frac{1}{u} \varepsilon \Gamma s (n \times n) d d .
\]

\((5.17)_3\)

and

\[
[\text{grad}^3(\frac{\partial}{\partial t})] = -\frac{1}{u^2} \varepsilon \Gamma s (n \times n \times n) .
\]

\((5.17)_4\)

where \((3.7)_1\), \((3.7)_2\), and \((5.17)_1\) have been used. We can interpret equations \((5.17)\) just as we did in Section IV by saying that the passage of a wave through the mixture tends to separate the coldnesses behind the wave. Since higher order derivatives are discontinuous for third order waves than for second order waves, the tendency to produce different coldnesses is naturally greater for second order waves.
If we make use of (3.23), (3.29)$_4$, (3.41)$_5$, (4.16),
(4.35)$_1$, and (5.17)$_1$, equation (5.11) reduces to

\[
\rho^+ \left[ \frac{\partial \tilde{a}}{\partial \tilde{t}} \right] \cdot \mathbf{n} = \Sigma Q \left[ \text{grad} \left( \text{div} \left( \frac{\tilde{a}}{\partial \tilde{t}} \right) \right) \right] \cdot \mathbf{n}
\]

\[
+ \Sigma \Delta \quad \frac{\partial \tilde{a}}{\partial \tilde{t}} \cdot \mathbf{n}
\]

(5.18)

We can now substitute (5.18) into (3.31) to obtain the third order amplitude equation

\[
2\rho^+ \frac{\partial \tilde{a}}{\partial \tilde{t}} = \Sigma (Q - \rho^+ \frac{\partial \tilde{a}}{\partial \tilde{t}} \cdot \mathbf{n}) \left[ \text{grad} \left( \text{div} \left( \frac{\tilde{a}}{\partial \tilde{t}} \right) \right) \right] \cdot \mathbf{n}
\]

\[
+ \rho^+ \frac{\partial \tilde{a}}{\partial \tilde{t}} \cdot \mathbf{n}
\]

(5.19)

where we have used the fact that the wave speed $u$ is constant. In contrast to the second order amplitude equation (4.36) where $\Delta$ was a function of the amplitude $s$, in the third order amplitude equation (5.19) $\Delta$ is a constant.

Hence in the study of third order waves we have achieved the desired feature that we obtain an amplitude equation with constant material coefficients.

Since we have again assumed that the $N$ eigenvalues associated with the characteristic equation of (5.10) are distinct, we can substitute (4.39)$_3$ into (5.19), multiply
the result by r, use (4.38) and (4.39)_4, and sum on a = 1, ..., N, to obtain

\[ 2 \frac{\delta s}{\delta t} = u \overline{K} s + \Delta s \]  \hspace{1cm} (5.20)

where \( \Delta \) is defined by (4.41) and is constant. Observe that the differential equation (5.20) governs the growth and decay of the amplitude of the leading wave only.

If we use the fact [5, equation 179.12] that

\[ \frac{\delta a}{\delta t} = -2ua \overline{K} \]  \hspace{1cm} (5.21)

where \( a = \text{det}[a_{\alpha \beta}] \) and \( a_{\alpha \beta} \) is the surface metric tensor introduced earlier, equation (5.20) can be integrated. This leads to

\[ s(t) = s_0 \left[ \frac{a(0)}{a(t)} \right]^{\frac{1}{4}} e^{\frac{\Delta t}{2}} \]  \hspace{1cm} (5.22)

where \( s_0 \) is the initial amplitude of the wave, \( a(0) \) in the initial value of \( a(t) \), and the ratio \( a(0)/a(t) \) is a geometric factor. If we recall that the square root of the determinant of the surface metric tensor is proportional to the elemental area change [19, Section 55] then the ratio \( \sqrt{a(0)/a(t)} \) is simply the ratio of the original elemental area to that at some later time \( t \). For plane, cylindrical, and spherical waves, \( a(t) \) is given by
(1) \( a(t) = 1 \)

(2) \( a(t) = r(t)^2 \)

(3) \( a(t) = r(t)^4 \sin^2 \theta, \ \theta = \text{constant}, \quad (5.23) \)

respectively.

The solution to the amplitude equation (5.22) implies that for an initially compressive wave, \( \Delta < 0 \) implies that the amplitude decays exponentially in time and \( \Delta > 0 \) implies that the amplitude grows exponentially in time. In contrast, for an initially expansive wave, \( \Delta \neq 0 \) implies that the amplitude grows exponentially in time and \( \Delta > 0 \) implies that the amplitude decays exponentially in time.

You will recall that for second order waves nothing could be said about the sign of \( \Delta \) in general. The same is true for third order waves. However, since the material coefficient \( \Delta \) has the same form for both second and third order waves we can consider the same special cases analyzed in Section IV to show that \( \Delta < 0 \). Thus, in those special cases the amplitude cannot grow exponentially in time.
VI. SECOND ORDER CURVED WAVES IN NON-CONDUCTORS

Sections IV and V dealt with the study of the propagation of curved waves through a conducting mixture of gases. It is only natural then to follow with the study of the propagation of curved waves through a non-conducting mixture of gases. The method applied here is to first restrict the equations of Section II to non-conductors. We then follow a development very much similar to that used for conductors to obtain the amplitude equations for second order waves in non-conductors.

By a non-conductor we mean a fluid for which the constitutive function (2.21) is independent of coldness gradients \( g_b \), \( b = 1, \ldots, N \). Thus (2.21) reduces to

\[
(A, \varepsilon, T, \hat{p}, q, \hat{e}, M) = f(\nu, \rho, \xi, \dot{x}) .
\]

The functional \( f \) in (6.1) is still assumed to be of class \( C^3 \).

Let us look at the reduced dissipation inequality (2.23). Since \( g_b \) is no longer one of the independent
variables, equation (2.23) implies that
\[ q = q(u, \rho, \ell, \dot{x}) = 0 \tag{6.2} \]
\[ \dot{\alpha} \dot{a} \dot{b} \dot{b} \dot{b} \dot{b} \]

for all values of the independent variables and that
\[ \Lambda = \Lambda(u, \rho, \ell, \dot{x}) \tag{6.3} \]
\[ a \dot{a} a \dot{b} \dot{b} \dot{b} \dot{b} \]

Equation (6.3) says that the Massieu function for the \( a \)th constituent is independent of the coldnesses of all the constituents except that of the \( a \)th constituent. The Massieu function for the mixture is still \( \Lambda = \Lambda(u, \rho) \) but it should be noted that
\[ \frac{\partial \Lambda(u, \rho, \ell, \dot{x})}{\partial u} \left( a a c c \dot{c} \dot{c} \right) = 0 \tag{6.4}_1 \]

for \( a, b = 1, \ldots, N \) and \( b \neq a \). Thus
\[ \frac{\partial \Lambda(u, \rho)}{\partial u} \left( c c \right) = \frac{\partial \Lambda(u, \rho, \ell, \dot{x})}{\partial u} \left( a a c c c c \right) \tag{6.4}_2 \]

For non-conductors then, equations (2.23) reduce to
\[ \Lambda = \Lambda(u, \rho) , \tag{6.5}_1 \]
\[ b b \]
\[ \varepsilon = \varepsilon(u, \rho) = - \frac{1}{\rho} \frac{\partial \Lambda}{\partial u} \frac{\partial A}{\partial \varepsilon} \frac{\partial A}{\partial u} \frac{\partial A}{\partial \varepsilon} , \]  
(6.5)_{2}

\[ \sum \left( \frac{\partial A}{\partial x} \frac{\partial A}{\partial b} \right) + \sum \left( \frac{\partial A}{\partial x} \frac{\partial A}{\partial b} \right)^T = 0 , \]  
(6.5)_{3}

\[ T = - \frac{1}{u} \left\{ \frac{\partial A}{\partial u} - \rho \frac{\partial A}{\partial \rho} \frac{\partial A}{\partial \varepsilon} \right\} + \sum \left( u \frac{\partial A}{\partial x} \frac{\partial A}{\partial b} \right) \]  
(6.5)_{4}

and

\[ \sum (\varepsilon - \hat{f} \cdot u) u > 0 , \]  
(6.5)_{5}

where

\[ \hat{f} = \hat{p} - \frac{1}{u} \left\{ \sum \frac{\partial A}{\partial u} \frac{\partial A}{\partial \varepsilon} - \frac{\partial A}{\partial \rho} \frac{\partial A}{\partial \varepsilon} \right\} . \]  
(6.5)_{6}

and \( \hat{f}, \hat{e}, \hat{p}, \) and \( T \) are now functions of \( u, \rho, \varepsilon, \) and \( \varepsilon. \)

Note that (2.23)_{3} is identically satisfied since \( g \) is no longer an independent variable.

In addition note that equations (2.28), (2.29), and (2.30)_{2} are not altered by assuming that we have a non-conductor except that \( T \) and \( \hat{M} \) now depend on \( u, \rho, \varepsilon, \) and \( \varepsilon. \) Thus

\[ A \]  
(6.5)_{7}

and \( \Lambda \) depends on \( u, \rho, \varepsilon, \) and \( \varepsilon. \) Thus
\[ T = T^T_a, \quad (6.6)_1 \]

\[ \hat{M} = 0, \quad (6.6)_2 \]

and

\[ \frac{\partial A(u, \rho, z, 0)}{\partial \delta^b} = 0. \quad (6.6)_3 \]

The equilibrium state for a non-conducting mixture of gases is defined to be one where

\[ u = u = \ldots = u = u, \quad (6.7)_1 \]

\[ \begin{array}{c}
1 \\
2 \\
N
\end{array} \]

and

\[ \dot{x} = \dot{x} = \ldots = \dot{x} = 0. \quad (6.7)_2 \]

\[ \begin{array}{c}
1 \\
2 \\
N
\end{array} \]

Hence equations (2.32)_2 and (2.32)_3 become

\[ \hat{e}(u, \rho, z, 0) = 0, \quad (6.8)_1 \]

\[ \hat{a} \]

and

\[ \hat{f}(u, \rho, z, 0) = 0. \quad (6.8)_2 \]

\[ \hat{a} \]

This completes the restrictions imposed by assuming that we have a non-conductor.
We now turn to the study of the propagation of second order waves through such a non-conductor. The physical description of the problem is the same as it was described at the end of Section III except that there are no coldness gradients involved. We assume that in front of the wave

\[ u = u = \ldots = u = v = \text{constant}, \quad (6.9)_1 \]

\[ \rho = \rho^+ = \text{constant} \quad b = 1, \ldots, N, \quad (6.9)_2 \]

\[ \xi = \xi = \ldots = \xi = \xi^+ = 0, \quad (6.9)_3 \]

and

\[ \dot{\xi} = \dot{\xi} = \ldots = \dot{\xi} = \dot{\xi}^+ = 0. \quad (6.9)_4 \]

We can again use the working form of the momentum equation (4.14) to derive the propagation condition for non-conductors. For non-conductors the quantity \( \mathfrak{m} \) is given by

\[ \mathfrak{m} = \mathfrak{m}(u, \rho, \xi, \dot{x}) = \sum_{c} \mathfrak{m}(u, \rho, \xi, \dot{x}) = \sum_{b} \mathfrak{m}(u, \rho, \xi, \dot{x}). \quad (6.10) \]

The results (4.21)_1 as well as (4.21)_2 are valid for non-conductors as well as conductors. It follows from (4.21)_1 that
\[
\begin{align*}
(\frac{\partial T}{\partial \rho})^\pm_b &= - (\frac{\partial \pi}{\partial \rho})^\pm_\mathbb{V}; \\
(\frac{\partial T}{\partial \xi})^\pm_b &= 0. 
\end{align*}
\tag{6.11}
\]

If we form the jump of (4.14) and make use of (3.8)_1, (3.13)_4, (4.21)_2, and (IIA.6) we find that

\[
\rho^+ \mathbb{S} = - \sum_a \left( \frac{\partial A}{\partial \mathbb{V}_a} \right)^+_\mathbb{V} [g] 
\tag{6.12}
\]

To determine \([g]\) we must use the energy equation (2.14)_4.

If we assume that the partial heat supply term \(r\) is zero and make use of (6.2), the energy equation reduces to

\[
\hat{\rho} \hat{\varepsilon} = \text{tr}(T^T \mathbb{L}) + \hat{\varepsilon}. 
\tag{6.13}
\]

If we form the jump of (6.13) and make use of the fact that, for non-conductors, \(\varepsilon = \varepsilon(\mathbb{V}, \rho)\) along with (3.10)_1, (3.13)_1, (3.41)_5, (4.21)_1, (4.28)_2, and (IA.2)_2 we find that

\[
\left[ \frac{\partial \mathbb{V}}{\partial \mathbb{T}} \right] = \frac{3}{up \mathbb{C}_T} \sum_a \left( \frac{\partial \pi}{\partial \mathbb{V}_a} \right)^+_\mathbb{V} (\mathbb{S} \cdot \mathbb{n}). 
\tag{6.14}
\]
In deriving (6.14) we introduced the specific heat at constant volume for the \( a \)th constituent, \( c_v \), which is defined by

\[
c_v = - \nu \left( \frac{a}{\partial \nu} \right)^a .
\]  

(6.15)

If we let \( \phi = \nu \) in (3.7)\(_1\) and (3.7)\(_2\) and use (6.14) we find that

\[
\left[ g \right] = - \frac{3}{u^2 \rho c_v} \left( \frac{b}{\partial \nu} \right)^a \left( \frac{\partial \pi}{\partial \nu} \right)^a (s \cdot n)_n .
\]  

(6.16)

We should observe here that only the coldness is continuous across the wave for non-conductors whereas the coldness and its first derivatives are continuous for conductors. Hence the passage of a second order wave through a mixture will tend to drive the coldnesses apart at a greater rate in non-conductors than in conductors.

If we substitute (6.16) into (6.12) we are led to

\[
\rho^+ s = \sum \left( \sum \frac{3}{u^2} \left( \frac{a}{\partial \nu} \right)^a \left( \frac{c}{\partial \nu} \right)^b (s \cdot n)_n .
\]  

(6.17)

\[
- \frac{\rho^+ \rho^+}{\rho^+} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^a \left( \frac{c}{\partial \nu} \right)^b (s \cdot n)_n .
\]  

(6.17)
Hence we see that the amplitude $\bar{s}$ is parallel to $\eta$ and, as before, only longitudinal waves are possible. The propagation condition is thus

$$\sum_{ac} \left( \frac{\rho^+}{\rho} u^2 \delta \right) s = 0,$$

where

$$\hat{Q} = - \frac{\rho^+}{u} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi} \right) + \sum_{b} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi} \right) \left( \frac{\partial}{\partial u} \right).$$

Equation (6.19) clearly implies that the matrix $[\hat{Q}]$ is symmetric and does not depend on time. We shall again assume that $[\hat{Q}]$ is positive definite, implying that the squared speeds in (6.18) are positive. We will use the same notation introduced in Section IV and used in Section V for $N$ eigenvalues and eigenvectors associated with the characteristic equation of (6.18).

We follow the same procedure in deriving the amplitude equation for non-conductors as we did for conductors in Section IV. The time derivative of the momentum equation (4.20) is valid for non-conductors as well as conductors. If we form the jump of (4.20), make use of (3.5), (3.8)$_2$, (3.13)$_3$, (3.13)$_4$, (3.13)$_6$, (3.41)$_5$, (4.21)$_1$, (4.21)$_2$, (4.23), (6.11), (6.14), (6.16), (IIA.5), (IIA.6), (IIA.7), (IIA.10), and (IIA.13), after much algebra we arrive at
\[ \rho \frac{\partial^2 x}{\partial t^2} = - \sum_b \left( \frac{\partial a}{\partial u} \right)^* \text{grad} \left( \frac{\partial b}{\partial t} \right) \]

\[ - \sum_b \frac{\rho}{\rho} (\frac{\partial^2 A}{\partial \rho \partial \rho})^* \text{grad} (\text{div} \ \tilde{x}) \]

\[ + \sum_b \bar{\theta} s + \sum_{b,c} \bar{\psi} s s \tilde{n} \quad (6.20) \]

The quantity \( \bar{\theta} \) is a vector defined by \( \bar{\theta} \):

\[ \bar{\theta} = \left( \frac{\partial \tilde{f}}{\partial x} \right)^* \tilde{n} + \sum_{c} \frac{u^3}{\nu^3} \left( \frac{\partial \tilde{f}}{\partial x} \right)^* \left( \frac{\partial \bar{\pi}}{\partial u} \right)^* \left( \frac{\partial b}{\partial v} \right)^* \quad (6.21) \]

and the quantity \( \bar{\psi} \) is a scalar defined by \( \bar{\psi} \):

\[ \bar{\psi} = \frac{1}{u^3} \sum_{d,e} \frac{u}{d} \left( \frac{\partial^2 \pi}{\partial u \partial \rho} \right)^* \left( \frac{\partial a}{\partial \rho} \right)^* \left( \frac{\partial b}{\partial u} \right)^* \left( \frac{\partial c}{\partial u} \right)^* \]

\[ + \sum_{d,e} \frac{b^3}{\rho^3} \left( \frac{\partial a}{\partial u \partial \rho} \right)^* \left( \frac{\partial c}{\partial u \partial \rho} \right)^* \left( \frac{\partial e}{\partial v} \right)^* \]

\[ - u^2 \bar{\psi} \quad \text{abc} \]
\[ - \sum_{d} \frac{\rho^{+} \rho^{+} u^2}{d} \left( \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \rho} \right)^+ \left( \frac{\partial \pi}{\partial u} \right)^+ \]

\[ - \sum_{a, b, c} \frac{\rho^{+} \rho^{+} \rho^{+}}{a b c} \left( \frac{\partial^3 \Lambda}{\partial \rho \partial \rho \partial \rho} \right)^+ \left( \frac{\partial \pi}{\partial u} \right)^+ - 3 \frac{\rho^{+} \rho^{+}}{a b c} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^+ \delta \]

\[ + \frac{\rho^{+} u}{c} \left( \frac{\partial^2 \Lambda}{\partial \rho \partial \rho} \right)^+ \left( \frac{\partial \pi}{\partial u} \right)^+ + \frac{\rho^{+}}{c} \left( \frac{\partial^2 \Lambda}{\partial x \partial \rho} \right)^+ \delta \]

\[ = (6.21)_{2} \]

where \( T \) is defined by (IIA.13).

It remains to determine \([\text{grad} (\partial \nu / \partial t)]\) for a non-conductor.

We again make use of the energy equation (6.13). The jump of (6.13) results in

\[ \left[ \frac{\partial \varepsilon}{\partial t} \right] = \frac{1}{u} \frac{\partial}{\partial \rho^+} s \]

\[ = (6.22)_{1} \]

where \((3.10)_{2}, (3.37), (3.41)_{5}, \) and (IA.2)_{2} have been used.

If we let \( \phi = \varepsilon \) in \((3.7)_{2}\) and use \((3.7)_{1}\) and \((6.22)_{1}\) we find that

\[ [\text{grad} \varepsilon] = - \frac{1}{u} \frac{\partial}{\partial \rho^+} s \]

\[ = (6.22)_{2} \]
These two equations will be needed when we form the jump of the gradient of the energy equation. Performing this operation we obtain

\[\rho^+[\text{grad}(\frac{\partial \epsilon}{\partial t})] = -\pi^+[\text{grad}(\text{div } \vec{x})] \]

\[-\xi(\frac{\partial \xi}{\partial u})^+[(\text{grad } \vec{x}) \cdot \vec{g}] - \xi(\frac{\partial \xi}{\partial \rho})^+[(\text{grad } \vec{x}) \cdot \vec{g}] \]

\[+ \xi(\text{grad } \vec{x})(\frac{\partial \xi}{\partial u})^- + \xi(\text{grad } \vec{x})(\frac{\partial \xi}{\partial \rho})^-[\vec{g}] \quad (6.23)\]

In deriving (6.23) we made use of (3.37), (3.41), (4.21), (6.11), (IA.7), (IIA.6), and the following identity:

\[\xi(\frac{\partial \xi}{\partial \rho})^- = -\frac{\partial \rho^+}{\partial u} \cdot \vec{n} = -\rho^+[\text{grad } \vec{x}) \cdot \text{grad } \epsilon] \quad (6.23)\]

Equation (6.23) follows directly form (3.5), (3.9), (3.13), (6.22), and (6.22). Since \(\epsilon = \epsilon(u, \rho)\) we can form \(\text{grad}(\frac{\partial \epsilon}{\partial \rho})\), take its jump, and multiply by \(\rho^+\) to obtain

\[\rho^+[\text{grad}(\frac{\partial \epsilon}{\partial \rho})] = \rho^+[(\frac{\partial \epsilon}{\partial u})^+[\text{grad}(\frac{\partial \epsilon}{\partial \rho})]]\]

\[\text{grad}(\frac{\partial \epsilon}{\partial \rho})]\]
\[
+ \rho \left( \frac{b}{\partial u \partial t} \right)^+ \frac{\partial u}{\partial x} + \sum_{\gamma} \rho \left( \frac{b}{\partial u \partial \varphi} \right)^+ \left[ \frac{b}{\partial t} \right]_\gamma + \sum_{\gamma} \rho \left( \frac{b}{\partial u \partial \varphi} \right)^+ \left[ \frac{b}{\partial t} \right]_\gamma
\]

\[
+ \sum_{\gamma} \rho \left( \frac{b}{\partial u \partial \varphi} \right)^+ \left[ \text{grad}(\frac{c}{\partial t}) \right]_\gamma + \sum_{\gamma} \rho \left( \frac{b}{\partial u \partial \varphi} \right)^+ \left[ g \frac{c}{\partial t} \right]_\gamma
\]

\[
+ \sum_{\gamma,\delta} \rho \left( \frac{b}{\partial u \partial \varphi} \right)^+ \frac{\partial u}{\partial \gamma} \left[ \frac{c}{\partial t} \right]_\delta
\]

(6.23) \gamma

If we substitute (6.23) \gamma into (6.23) \gamma, solve for

\[\text{grad}(\partial u / \partial t)\], multiply the result by \(- v^2 (\partial u / \partial u)^+\), sum on \(b = 1,\ldots, N\), and use (3.5), (3.9) \gamma, (3.13) \gamma, (3.13) \gamma, (3.13) \gamma, (6.14), and (6.16) we finally arrive at

\[
\sum_{\gamma} \left( \frac{a}{\partial v} \right)^+ \left[ \text{grad}(\frac{b}{\partial u}) \right]_\gamma =
\]

\[
\sum_{\gamma} \frac{v^3}{\rho} \left( \frac{a}{\partial u} \right)^+ \left( \frac{c}{\partial u} \right)^+ \left[ \text{grad}(\text{div} \, x) \right]_\gamma
\]

\[
+ \sum_{\gamma} \frac{\phi}{c} \frac{\partial u}{\partial v} + \sum_{\gamma} \frac{\Omega}{c} \frac{\partial u}{\partial v},
\]

(6.24)

where \(\phi\) and \(\Omega\) are scalars defined by

\[
\text{ac} \quad \text{abc}
\]

\[
\text{abc}
\]
\[ \phi = - \sum_{ac} \frac{u^5}{bd} \left( \frac{\partial \pi}{\partial u} \right)^+ \left( \frac{\partial \varepsilon}{\partial b} \right)^- \left( \frac{\partial \pi}{\partial d} \right)^+ \]

\[ - \sum_{b} \frac{u^2}{b} \left( \frac{\partial \pi}{\partial u} \right)^+ \left( \frac{\partial \varepsilon}{\partial c} \right)^- \cdot \eta \quad (6.25)_1 \]

and

\[ \Omega = \frac{1}{u^3} \sum_{abc} \left\{ \sum_{d} \frac{v^8}{d} \left( \frac{\partial \pi}{\partial u} \right)^+ \left( \frac{\partial^{2} \varepsilon}{\partial d \partial \rho} \right)^+ \left( \frac{\partial \pi}{\partial d} \right)^+ \left( \frac{\partial \pi}{\partial d} \right)^+ \right. \]

\[ + 2 \sum_{d,e} \frac{v^5}{d} \left( \frac{\partial \pi}{\partial d} \right)^+ \left( \frac{\partial^{2} \varepsilon}{\partial e \partial \rho} \right)^+ \left( \frac{\partial \pi}{\partial e} \right)^+ \left( \frac{\partial \pi}{\partial e} \right)^+ \left( \frac{\partial \pi}{\partial e} \right)^+ \delta \]

\[ + 2 \sum_{d,e} \frac{\rho^2}{c} \left( \frac{\partial \pi}{\partial d} \right)^+ \left( \frac{\partial \varepsilon}{\partial e} \right)^+ \delta \]

\[ + \sum_{d,e} \frac{\rho \rho^2}{e} \left( \frac{\partial \pi}{\partial d} \right)^+ \left( \frac{\partial \varepsilon}{\partial e} \right)^+ \delta \]

\[ - \sum_{d,e} \frac{u^5}{d} \left( \frac{\partial \pi}{\partial u} \right)^+ \left( \frac{\partial \pi}{\partial e} \right)^+ \left( \frac{\partial \pi}{\partial e} \right)^+ \delta \]

\[ - \sum_{d,e} \frac{\rho^2}{c} \left( \frac{\partial \pi}{\partial d} \right)^+ \left( \frac{\partial \varepsilon}{\partial e} \right)^+ \delta \delta \} \quad (6.25)_2 \]
Equation (6.24) is the desired expression needed for substitution into (6.20). Upon substitution of (6.24) into (6.20) and forming the inner product of the result with $\mathbf{n}$, we find that

$$\rho^+ \left[ \frac{\partial^2 \mathbf{x}}{\partial t^2} \right] \cdot \mathbf{n} = \sum \frac{\mathbf{Q}}{c} [\text{grad(div } \mathbf{x})] \cdot \mathbf{n}$$

$$+ \sum \mathbf{X} s_s s + \sum \Xi s,$$

(6.26)

where we have used (6.19). The scalars $\Xi$ and $\mathbf{X}$ are defined by

$$\Xi = \frac{\partial}{\partial \mathbf{n}} \mathbf{n} + \Phi$$

(6.27)

and

$$\mathbf{X} = \Psi + \Omega.$$  

(6.27)

Finally, if we substitute (6.26) into (3.20) we obtain,

$$2\rho^+ \frac{\alpha}{\partial t} = \sum (Q - \rho^+ \delta u^2) [\text{grad(div } \mathbf{x})] \cdot \mathbf{n}$$

$$+ u \bar{\kappa} \rho^+ s + \sum \Xi s + \sum \mathbf{X} s_s s,$$

(6.28)

since $u$ is again constant. Since we have again assumed
distinct eigenvalues we can substitute (4.39)_3 into (6.28), multiply the result by \( r \), use (4.38) and (4.39)_4, and sum on \( a = 1, \ldots, N \), to obtain

\[
2 \frac{\delta s}{\delta t} = u \bar{K} s + \Xi s + \chi s^2 \tag{6.29}
\]

where

\[
\Xi = \sum_{a,b} \Xi_{ab} \tag{6.30}_1
\]

and

\[
\chi = \sum_{a,b,c} \chi_{abc} \tag{6.30}_2
\]

Observe that the differential equation (6.29) again governs the amplitude of the leading wave only.

As a final topic of this section, we shall specialize the results of this section to the special case where (6.3) is replaced by

\[
\Lambda = \Lambda(u, \rho) \tag{6.31}
\]

This special case is interesting since it contains the model often used in gas dynamics where each constituent is a perfect gas. When (6.31) is true, equations (6.5)_2, (6.5)_4, and (6.5)_6 reduce to
\[ \varepsilon = \varepsilon(u, \rho) = -\frac{1}{\rho} \frac{\partial A}{\partial u} \cdot \begin{pmatrix} a & a & a \\ a & a & a \end{pmatrix}, \]  
\hspace{2cm} (6.32)_1

\[ \hat{f}(u, \rho, \xi, \dot{x}) = \hat{p}(u, \rho, \xi, \dot{x}) \cdot \begin{pmatrix} a & b & b \\ a & b & b \end{pmatrix}, \]  
\hspace{2cm} (6.32)_2

and

\[ T = -\frac{1}{\rho} \left\{ \begin{pmatrix} A \\ a \end{pmatrix} - \rho \hat{p} \hat{p} \begin{pmatrix} a \\ a \end{pmatrix} \right\} \cdot \begin{pmatrix} a & a \\ a & a \end{pmatrix}. \]  
\hspace{2cm} (6.32)_3

Equation (6.5) is identically satisfied when (6.31) holds. In addition to (6.31), we also assume that

\[ \hat{p} = \hat{p}(u, \rho, \xi) \cdot \begin{pmatrix} a & a & b & b \end{pmatrix}, \]  
\hspace{2cm} (6.33)_1

and

\[ \hat{\varepsilon} = \hat{\varepsilon}(u, \rho, \dot{x}) \cdot \begin{pmatrix} a & a & b & b \end{pmatrix}. \]  
\hspace{2cm} (6.33)_2

In other words, the dependence on density gradients is omitted. If (6.31), (6.33)_1, and (6.33)_2 are true, we call the resulting mixture an ideal non-conductor. This model of a mixture is similar to but not identical to one examined by BOWEN and DORIA [17]. In their model, the ideal mixture was assumed to have a single temperature. Even with this difference, certain of the results below are similar to theirs.
Given (6.31), equation (6.19) reduces to

\[
\hat{Q} = -\frac{\rho}{u}(\frac{a}{\partial \rho})^2 + \frac{3}{\rho} (\frac{\partial \pi}{\partial u}) a a a
\]

(6.34)_1

and

\[
\hat{Q} = 0
\]

(6.34)_2

for \(a \neq b\). If we assume that the coldness \(u\) can be inverted for every \(\rho\), \(a = 1, \ldots, N\), we can change variables from \((\nu, \rho)\) to \((\eta, \rho)\). This results in

\[
\frac{\partial \xi(\eta, \rho)}{\partial \eta} = \frac{\partial \pi(\nu, \rho)}{\partial \rho} + \frac{\partial \pi(\nu, \rho)}{\partial \nu} \frac{\partial \xi(\eta, \rho)}{\partial \rho}.
\]

(6.35)

If we recall the well known thermodynamic relations (e.g., see [16])

\[
\frac{1}{\nu(\eta, \rho)} = \frac{\partial \xi(\eta, \rho)}{\partial \eta} a a a
\]

(6.36)_1

and

\[
\nu(\eta, \rho) = \rho^2 \frac{\partial \xi(\eta, \rho)}{\partial \rho} a a a,
\]

(6.36)_2
differentiate the first with respect to \( \rho \) and make use of the second we find that

\[
\frac{\partial u(\eta, \rho)}{\partial \eta} = - \frac{a^2}{\rho^2} \frac{\partial u}{\partial \eta}, \quad (6.37) \_1
\]

We can now make use of the definition of the specific heat at constant volume (6.15), (6.36)\_1, and the invertibility of \( v \) to show that

\[
\frac{\partial u(\eta, \rho)}{\partial \eta} = - \frac{a}{c_v}. \quad (6.37) \_2
\]

Hence (6.37)\_1 reduces to

\[
\frac{\partial u(\eta, \rho)}{\partial \eta} = - \frac{a^3}{\rho^2} \frac{\partial u}{\partial \eta} \quad (6.38)
\]

It is clear from (3.35) that for an ideal gas

\[
\frac{\partial \pi(u, \rho)}{\partial \rho} = - \frac{a^2}{\rho} \frac{\partial \Lambda(u, \rho)}{\partial \rho} \quad (6.39)
\]

If we substitute (6.38) and (6.39) into (6.35) and evaluate the result in front of the wave we find that
\[ \rho^+ \left( \frac{\partial \pi}{\partial \rho} \right)^+ = - \frac{\rho^2}{u} \left( \frac{\partial^2 \rho}{\partial \rho^2} \right)^+ + \frac{3}{\rho c_v} \left( \frac{\partial \pi}{\partial u} \right)^2. \]  

(6.40)_1

If we compare (6.40)_1 and (6.34)_1 we see that

\[ \hat{Q} = \rho^+ \left( \frac{\partial}{\partial \rho} \right)^+ \frac{\partial \pi}{aa} \]

(6.40)_2

where \((\eta^+, \rho^+)_a\) are now the independent variables. This result is similar to one obtained by BOWEN and DORIA [17].

With the assumptions (6.31) and (6.33) the material coefficient \(X\) defined by (6.27)_2 reduces to

\[ X = \psi + \Omega \]

(6.41)_1

\(aaa\) \(aaa\) \(aaa\)

and

\[ X = 0 \]

(6.41)_2

\(abc\)

for \(a \neq b \neq c\). The quantities \(\psi\) and \(\Omega\) in (6.41)_1 are defined by

\[ u^3 \psi = \frac{6}{\rho^2 + c_v^2} (\frac{\partial^2 \pi}{\partial \rho^2} + \frac{\partial \pi}{\partial u})^+ \left( \frac{\partial}{\partial u} \right)^+ + \frac{3}{\rho c_v} \left( \frac{\partial^2 \pi}{\partial u \partial \rho} \right)^+ \left( \frac{\partial}{\partial u} \right)^+ \]

(6.41)_2

\(aaa\) \(aaa\)
\[
\begin{align*}
\frac{\rho + u^2}{a} & = \left( \frac{3}{a} \right) \pi + \left( \frac{3}{a} \right) \pi + \left( \frac{3}{a} \right) \Lambda - \frac{a}{c_v} \left( \frac{\partial a}{\partial \rho} \right) \pi + \left( \frac{a}{a} \right) \pi + \frac{a}{a} \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \Lambda \\
& - \frac{3}{a} \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \frac{a}{c_v} \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi \tag{6.42}_1
\end{align*}
\]

and

\[
\begin{align*}
\frac{u^3}{\Omega} & = \frac{u}{2} \left( \frac{a}{a} \right) \pi + \left( \frac{2}{a} \right) \pi + \frac{2}{c_v} \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi + \left( \frac{a}{a} \right) \pi \tag{6.42}_2
\end{align*}
\]

It is easy to show that the following identities are true for an ideal gas:

\[
\left( \frac{a}{a} \right) \pi = - \frac{\rho + \frac{a^2}{\rho}}{a} \pi \tag{6.43}_1
\]
\[ \frac{a^2 \pi}{a^2} + \frac{\rho^+}{a^2} \left( \frac{a^3}{a^2} \right) + \frac{\rho^+}{a^2} \left( \frac{a^2}{a^2} \right) + \frac{\rho}{a^2} \left( \frac{a^3}{a^2} \right)^+ + \frac{\rho}{a^2} \left( \frac{a^2}{a^2} \right)^+ , \quad (6.43)_2 \]

and

\[ \frac{\partial}{\partial \rho} \left( \frac{\rho^2}{a^2} \right) + \frac{\rho^3}{a^2} \left( \frac{\rho}{a^2} \right)^+ + \frac{\rho^3}{a^2} \left( \frac{\rho}{a^2} \right)^+ + \frac{\rho^3}{a^2} \left( \frac{\rho}{a^2} \right)^+ \quad (6.43)_3 \]

Equations (6.43) follow from (3.35) written for an ideal gas. These three identities can be used to show that

(6.42)_1 simplifies to

\[ u^3 \nabla = \frac{6}{\rho^2 c^2} \left( \frac{a^2}{a^2} \right)^+ + \frac{a}{a^2} \left( \frac{a^2}{a^2} \right)^+ + \frac{2a}{a^2} \left( \frac{a^2}{c^2} \right)^+ + \frac{a}{a^2} \left( \frac{a^2}{a^2} \right)^+ + \frac{a}{a^2} \left( \frac{a^2}{a^2} \right)^+ \quad (6.44) \]

If we write (4.28)_2 for an ideal gas we can establish that

\[ \frac{a^2 \epsilon}{a^2} \left( \frac{a^2}{a^2} \right)^+ = \frac{2}{a^2} \left( \frac{a^2}{a^2} \right)^+ + \frac{u^2}{a^2} \left( \frac{a^2}{a^2} \right)^+ \quad (6.45)_1 \]

and
\[ \frac{\partial^2 \varepsilon}{\partial \rho^2} = \frac{1}{2} \left( \frac{\partial^2 \pi}{\partial \rho^2} \right)^+ + \frac{\nu}{2} \left( \frac{\partial^2 \pi}{\partial \nu \partial \rho} \right)^+ \]

\[ - \frac{1}{\rho^3} \left( \frac{\partial \pi}{\partial \rho} + \nu \left( \frac{\partial \pi}{\partial \nu} \right)^+ \right) \quad (6.45)_2 \]

By use of equations (6.45) we see that (6.42)_2 reduces to

\[ u^3 \frac{\partial \Omega}{\partial u} = - \frac{6}{\rho^2} \frac{\partial c_v}{\partial \rho} \frac{\partial^2 \pi}{\partial \rho^2} a^3 + \frac{5}{\rho^2} \frac{\partial c_v}{\partial \rho} \frac{\partial^2 \pi}{\partial \rho^2} a^3 \]

\[ + \frac{2 \nu}{\rho^2} \frac{\partial c_v}{\partial \rho} a^2 \left( \frac{\partial^2 \pi}{\partial \rho^2} \right)^+ a^2 + \frac{\nu}{c_v} \frac{\partial^2 \pi}{\partial \nu \partial \rho} a^3 \frac{\partial \pi}{\partial \nu} \quad (6.46) \]

In arriving at (6.46) we used the fact that

\[ \frac{\partial^2 \varepsilon}{\partial \nu^2} = - \frac{1}{\nu^2} \frac{\partial c_v}{\partial \nu} a^2 + \frac{2 c_v}{\nu^3} a \quad (6.47) \]

which follows from (6.15) by differentiation. If we substitute (6.44) and (6.46) into (6.41) we obtain
\[
\frac{u^3}{aa} = \frac{\partial}{\partial \rho} \left( \frac{\partial^2}{\partial \rho^2} \frac{a}{a} \right) + \frac{3u}{a} \frac{\partial}{\partial u} \left( \frac{\partial^2}{\partial u \partial \rho} \right) + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2
\]

\[
+ \frac{5u^5}{\rho^2 c^2} \left( \frac{\partial}{\partial u} \right)^3 + \frac{3u^6}{\rho^2 c^2} \left( \frac{\partial}{\partial u} \right)^4 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2
\]

\[
- \frac{6u}{\rho^2 c^3 \partial u} \left( \frac{\partial}{\partial u} \right)^3 ,
\]

(6.48)

where \((u, \rho^+)\) are the independent variables.

If we make another change of variables from \((u, \rho)\) to \((\eta, \rho)\), we find that

\[
\frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) = \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2
\]

(6.49)

where (6.35) has been used. If we perform the indicated operations and make use of (6.38) we arrive at

\[
\frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) = \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \frac{3u^3}{c^2 \partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2 + \frac{\partial}{\partial u} \left( \frac{\partial}{\partial u} \right)^2
\]
\[
+ \frac{5u^5}{\rho^{2+}} \left( \frac{\partial \pi}{\partial \rho} \right)^{3+} + \frac{3u^6}{\rho^{2+}} \left( \frac{\partial^2 \pi}{\partial \rho^2} \right)^{2+} \left( \frac{\partial \pi}{\partial \rho} \right)^{2+} \\
- \frac{6}{\rho^{2+}} \left( \frac{\partial c_v}{\partial \rho} \right)^{2+} \left( \frac{\partial \pi}{\partial \rho} \right)^{3+} 
\] 

(6.50)

In deriving (6.50) we used the identity

\[
\frac{ac_v}{a} = - \frac{2u^2}{\rho^{2+}} \left( \frac{\partial \pi}{\partial \rho} \right)^{2+} - \frac{u^3}{\rho^{2+}} \left( \frac{\partial^2 \pi}{\partial \rho^2} \right)^{2+} 
\] 

(6.51)

which again follows from (6.15). If we compare (6.48) and (6.50) we see that (6.48) reduces to

\[
u^3 X = \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial a}{\partial \rho} \right)^{2+} 
\] 

(6.52)

where \((\eta^+, \rho^+)\) are the independent variables. The result (6.52) is again similar to another one of the results of Bowen and Doria [17].

We now turn back to the discussion of the amplitude equation (6.29). Observe that the second order amplitude equation for non-conductors (6.29) has exactly the same form as the second order amplitude equation for conductors (4.40). The only difference is that the material coefficients are
different. Because of this similarity many of the results concerning \( \varepsilon \) and \( X \) are the same as those for \( \Delta \) and \( \Pi \).

For example, as was the case for \( \Delta \) and \( \Pi \) both \( \varepsilon \) and \( X \) depend on the amplitude \( s \) of the wave. Also, the statements at the end of Section IV concerning when \( \Delta \) and \( \Pi \) are and are not constants are true for \( \varepsilon \) and \( X \) as well. Similarly the result that the material coefficient \( \Delta \) of \( s \) in the amplitude equation (4.40) was an even function of \( s \) carries over to the material constant \( \varepsilon \) in (6.29). Hence, whether we have an expansive or compressive disturbance, if \( \varepsilon < 0 \) it will always act as an attenuation factor and if \( \varepsilon > 0 \) it will always act as a growth factor.

Again, the major difference in this model is the dependence of the material coefficients \( \varepsilon \) and \( X \) on \( s \). Both \( \varepsilon \) and \( X \) depend on \( s \) through the density gradients \( \xi_b \), \( b=1,\ldots,N \). If we linearize around a state of zero density gradients, the dependence of \( \varepsilon \) and \( X \) on \( s \) would again disappear.

An important difference between conductors and non-conductors arises here. It can be shown that for non-conductors the material coefficient \( \varepsilon \) is less or equal to zero for all \( s \) without making any specializing assumptions.

If we define scalars \( \bar{A} \), \( \bar{B} \), \( \bar{C} \) and \( \bar{D} \), and \( \bar{E} \) by
\[
\begin{align*}
\text{ab} & \quad \text{ab} & \quad \text{ab} \\
\text{ab} & \quad \text{ab} & \quad \text{ab}
\end{align*}
\]
\[ \bar{\mathbf{A}} = n \cdot (\frac{\partial f}{\partial x})^{-n}, \]  
(6.53)_{1}

\[ \bar{\mathbf{B}} = \frac{u^2}{u^2 + (\frac{\partial f}{\partial u})^{-n} \cdot n}, \]  
(6.53)_{2}

\[ \bar{\mathbf{C}} = v (\frac{\partial u}{\partial u})^{+}, \]  
(6.53)_{3}

\[ \bar{\mathbf{D}} = \frac{u^2}{u^2 + (\frac{\partial u}{\partial u})^{-n} \cdot n}, \]  
(6.53)_{4}

and

\[ \bar{\mathbf{E}} = \frac{\epsilon^{4}}{u^2 + (\frac{\partial \epsilon}{\partial u})^{-n}}, \]  
(6.53)_{5}

the quantity \( \varepsilon \) defined by equation (6.27)_{1} can be written as

\[ \varepsilon = \bar{\mathbf{A}} + \sum_{ab} \bar{\mathbf{B}} \bar{\mathbf{C}} - \frac{1}{\nu} \sum_{ab} \bar{\mathbf{C}} \bar{\mathbf{B}} - \frac{1}{\nu_c} \sum_{c,d} \bar{\mathbf{C}} \bar{\mathbf{E}} \bar{\mathbf{C}}. \]  
(6.54)

The entropy inequality implies that
\[
\sum_{a,b} \left( \sin \left( \frac{\pi}{a} \right) - \sin \left( \frac{\pi}{b} \right) \right) - \sum_{a,b} \left( \frac{\pi}{a} \right)^{-1} - \sum_{a,b} \left( \frac{\pi}{b} \right)^{-1} \leq 0.
\]

(6.55)

Since \(a\) and \(b\) are arbitrary we let
\[a = \frac{\pi}{u} \sum_{b} c \quad \text{and} \quad b = \frac{\pi}{u} \sum_{a} \]

(6.56)

whereupon (6.55) reduces to
\[
\sum_{a,b} \left( A + \sum_{b} B \sum_{c} C + \frac{1}{u} \sum_{c} D \right) - \sum_{a,b} \left( \frac{\pi}{a} \right)^{-1} - \sum_{a,b} \left( \frac{\pi}{b} \right)^{-1} \leq 0.
\]

(6.57)

If we compare the expression the expression in brackets in (6.57) with (6.54) we find that
\[
\Xi = \sum_{a,b} \Xi \leq 0.
\]

(6.58)

Hence we can then say that for non-conductors for both expansive and compressive disturbances \(\Xi\) will always act as an attenuation factor.
VII. THIRD ORDER CURVED WAVES IN NON-CONDUCTORS

Here again, as in Section V, we turn to the study of the third order curved waves in non-conductors since third order waves have the desired feature that the density gradients are continuous.

The physical description of the problem is the same, but as before, we might expect some different results for the jumps of $\frac{\partial u}{\partial t}$ and $g$. The jump of (4.14) results in

$$[g] = 0$$  \hspace{1cm} (7.1)_1

when we use (3.22), (3.26), and (IA.2)$_3$. If we make use of (3.7)$_1$, (3.7)$_2$, and (7.1)$_1$ it follows that

$$[\frac{\partial u}{\partial t}] = 0 .$$  \hspace{1cm} (7.1)$_2$

A similar analysis shows that the jump in the energy equation for non-conductors (6.13) results in

$$[\frac{\partial \varepsilon}{\partial t}] = 0 .$$  \hspace{1cm} (7.2)$_1$

In deriving (7.2)$_1$ we made use of (3.22), (2.41)$_5$, and
(IA.2)\textsubscript{2}. Again use of (3.7)\textsubscript{1}, (3.7)\textsubscript{2}, and (7.2)\textsubscript{1} implies that

\[
[\text{grad } \varepsilon] = 0.
\]  

(7.2)\textsubscript{2}

The first step in obtaining the propagation condition for third order waves is to form the jump of (4.20). The result is

\[
\rho^+ s = - \sum_a \left( \frac{a}{b} \right)^+ \left[ \text{grad} \left( \frac{b}{\partial t} \right) \right] + \sum_a \left( \frac{a}{b} \right)^+ \left( \frac{2 a}{\partial \rho \partial \rho} \right)^+ (s \cdot n) n,
\]

(7.3)

where (3.21), (3.22), (3.28)\textsubscript{2}, (4.21)\textsubscript{2}, (7.1)\textsubscript{1}, (7.1)\textsubscript{2}, (IA.4)\textsubscript{3}, and (IIA.6) have been used.

To obtain an expression for $[\text{grad}(\partial \varepsilon/\partial t)]$ the first step is to take the gradient of the energy equation (6.13), form its jump, and use (3.22), (3.25)\textsubscript{1}, (3.26), (4.21)\textsubscript{1}, (6.11), (7.1), (7.2), and (IA.8)\textsubscript{3} to obtain

\[
\rho^+[\text{grad} \left( \frac{b}{\partial t} \right)] = - \frac{\pi^+}{b} (s \cdot n) n.
\]

(7.4)

Recalling that $\varepsilon = \varepsilon(\nu, \rho)$, we form $[\text{grad}(\partial \varepsilon/\partial t)]$, substitute
the result into (7.4), make use of (3.26), (3.28)\(_b\) (4.28)\(_b\), and (7.1), and solve for \(\text{grad}(\partial \mathbf{u}/\partial t)\) to obtain

\[
[\text{grad}(\frac{\partial \mathbf{u}}{\partial t})] = - \frac{3}{u^2} \sum \left( \frac{c}{\partial \mathbf{u}} \right)^{(s \cdot \eta)} \eta . \tag{7.5}_1
\]

Equations (3.7)\(_1\), (3.7)\(_2\), and (7.5)\(_1\) can be used to show that

\[
[\frac{\partial^2 \mathbf{u}}{\partial t^2}] = \frac{3}{u^2} \sum \left( \frac{c}{\partial \mathbf{u}} \right)^{(s \cdot \eta)} \eta . \tag{7.5}_2
\]

and

\[
[\text{grad}^2 \mathbf{u}] = \frac{3}{u^2} \sum \left( \frac{c}{\partial \mathbf{u}} \right)^{(s \cdot \eta)} (\eta \otimes \eta) . \tag{7.5}_3
\]

We should observe again that the passage of a wave tends to separate the coldnesses behind the wave. This tendency to drive the coldnesses apart is greater for a non-conductor than a conductor. Also, for non-conductors, second order waves tend to drive the coldnesses apart at a greater rate than third order waves. If we substitute (7.5)\(_1\) into (7.3) and make use of (6.19) we obtain

\[
\rho \frac{\partial \mathbf{S}}{\partial a} = \frac{1}{u^2} \sum \hat{Q} (\mathbf{S} \cdot \eta) \eta . \tag{7.6}
\]
Again we see that \( \hat{s} \) is parallel to \( \hat{n} \) and only longitudinal waves are permitted. Hence for third order waves the propagation condition becomes

\[
\sum \left( \hat{Q} - \rho^+ \delta u^2 \right) \frac{\partial \rho}{\partial t^2} s = 0.
\]

(7.7)

As was true for conductors, for non-conductors the propagation conditions for second and third order waves have the same form. Since \( \hat{Q} \) has the same form for both second and third order waves the comments below equation (6.19) also apply here.

In the usual manner, if we take the time derivative of (5.8) and form its jump we find that

\[
\rho^+ \left[ \frac{\partial \hat{a}}{\partial t} \right] = - \sum \left( \frac{\partial}{\partial t^2} \right)^+ [\text{grad}(b)] - [\text{grad}(b)]
\]

\[
+ \sum \left( \frac{\partial}{\partial t} \right)^+ \frac{\partial \hat{a}}{\partial t} \frac{\partial}{\partial \rho} (\frac{\partial^2 A}{\partial \rho \partial \rho})^+ \text{[grad(div}(b)]).}
\]

(7.8)

In deriving (7.8) we made use of (3.22), (3.26), (3.29)_4, (IA.10)_2 (4.21), (7.5), (IA.5), (IA.9), (IIA.6), and (IIA.10). To obtain an expression for [\text{grad}(\partial^2 \nu / \partial t^2)] we take a time derivative and a gradient of the energy equation (6.13) and
form the jump of the result. Making use of (3.22), (3.24), (3.26), (4,21), (7.1), (7.2), and (IA.7) we obtain

\[ \rho^+ [\text{grad}(\frac{b}{\partial t^2})] = - \sum \pi^+ \delta [\text{grad}(\text{div}(\frac{c}{\partial t}))] \]

\[ - \sum \{ \sum \frac{3}{d} \rho^+ \left( \frac{\partial \pi}{\partial u} \right)^+ (\frac{d}{\partial u})^+ \cdot \frac{\partial \pi}{\partial t^2} \cdot \frac{\partial n}{\partial b} \} \]

(7.9)_1

Since \( \varepsilon = \varepsilon(u, \rho) \) we can form

\[ \rho^+ [\text{grad}(\frac{b}{\partial t^2})] = \rho^+ \left( \frac{b}{\partial u} \right)^+ [\text{grad}(\frac{b}{\partial t^2})] \]

\[ + \sum \rho^+ \rho^+ \left( \frac{b}{\partial \rho} \right)^+ [\text{grad}(\text{div}(\frac{c}{\partial t}))] \]

(7.9)_2

where we made use of (3.26), (3.29), and (7.1). By substituting (7.9)_2 into (7.9)_1 and making use of (4.28)_2 and (IA.8) we find that

\[ [\text{grad}(\frac{b}{\partial t^2})] = \sum \frac{u^2}{c} \rho^+ \left( \frac{\partial \pi}{\partial u} \right)^+ [\text{grad}(\text{div}(\frac{c}{\partial t}))] \]

\[ + \sum \frac{u^5}{c,d,e} \left( \frac{\partial \pi}{\partial u} \right)^+ (\frac{b}{\partial u})^+ (\frac{e}{\partial d})^+ (\frac{c}{\partial d})^+ \]

(7.10)
This equation can then be substituted into (7.8) to obtain

\[ \rho^+ \left( \frac{\alpha}{\frac{\partial^3 x}{\partial t^3}} \right) \cdot n = \sum_c \hat{Q} \left[ \text{grad} (\text{div} \left( \frac{\frac{c}{\partial t}}{\partial t} \right)) \right] \cdot n \]

\[ + \sum_c \hat{\xi} \cdot s \cdot \delta \cdot c \quad (7.11) \]

where (6.19) and (6.27) have been used. This is the desired form of the equation needed for substitution into (3.31). This leads to

\[ 2 \rho^+ \frac{\alpha}{\frac{\partial^3 x}{\partial t^3}} = \sum_c (\hat{Q} - \rho^+ \delta u^2) \left[ \text{grad} (\text{div} \left( \frac{\frac{c}{\partial t}}{\partial t} \right)) \right] \]

\[ + \sum_c \hat{\xi} \cdot s \cdot \delta \cdot c \quad (7.12) \]

where \( \hat{\xi} \) is defined by (7.14) and \( u \) is constant. We again have a contrast between the second and third order theory for non-conductors in that the material coefficient in (7.12) is no longer a function of the amplitude \( s \). We again obtain an amplitude equation with constant coefficients.

Our assumption concerning the \( N \) eigenvalues associated with the characteristic equation of (7.7) along with
(4.39)\textsubscript{3} can be used to simplify (7.12). This leads to

\begin{equation}
2 \frac{\delta s}{\delta t} = uKs + \hat{\Xi s}
\end{equation}

(7.13)

where we have multiplied (7.12) by \( r \), used (4.39)\textsubscript{a} and (4.39)\textsubscript{4}, and summed on \( a = 1, \ldots, N \).

The quantity \( \hat{\Xi} \) in (7.13) is defined by

\begin{equation}
\hat{\Xi} = \sum_{a,b} \{n \cdot \left( \frac{\partial f}{\partial x} \right)^{\hat{a}} \}
\end{equation}

\begin{equation}
- \sum_{c,d} \frac{u^5}{c, d} \left[ \left( \frac{a}{\partial u} \right)^{\hat{c}} \left( \frac{c}{\partial u} \right)^{\hat{d}} \left( \frac{b}{\partial u} \right)^{\hat{a}} \right] \n \end{equation}

(7.14)

The material coefficient \( \hat{\Xi} \) is constant and the differential equation (7.13) governs the amplitude of the leading wave only.

Observe that the third order amplitude equation for non-conductors (7.13) has the same form as the third order amplitude equation for conductors (5.21). Hence the solution has the same form, namely

\begin{equation}
s(t) = s_0 \left[ \frac{a(0)}{a(t)} \right] \frac{1}{4} \hat{\Xi t} e^{\frac{\hat{\Xi t}}{2}}
\end{equation}

(7.15)

where \( s_0 \), \( a(0) \), and \( a(t) \) are defined in Section V.
Since the differential equations (5.20) and (7.13) have the same form and thus the same solution the discussion concerning the solution at the end of Section V applies in exactly the same way here. The only difference is the form of the material coefficients. However, if we compare (7.14) with (6.57) we see that \( \hat{\varepsilon} \) is a special case of \( \varepsilon_{ab} \) where \( b = b' = 0 \). Hence it is also true for third order waves in non-conductors that the material coefficient \( \hat{\varepsilon} \leq 0 \). Hence for third order waves the amplitude decays exponentially in time in all cases.
VIII. SUMMARY OF RESULTS

Many of the important results and concepts in this work are scattered throughout Sections IV through VII. Hence, in this section we attempt to unify these results for the benefit of the reader.

In each of the previous four sections the first thing that is done is to derive the propagation condition for each case considered. For conductors the propagation conditions are the same for second and third order waves (cf. (4.17) and (5.10)). For non-conductors the propagation conditions are also the same for second and third order waves (cf. (6.18) and (7.7)). The propagation condition is simply the equation that determines the \( N \) wave speeds possible in each case. In addition, the propagation condition determines the "direction" of \((s, \ldots, s)\) in the space \( \mathbb{R}^N \). Physically, of course, we require that the wave speeds be real and positive. In each case the propagation condition also shows us that the \( N \) wave speeds are constant.

In each of Sections IV through VII the assumption was made that the wave propagates into a region at rest with given values for the coldness, the density, the coldness gradient, the density gradient, and the velocity of each
constituent. Since, after the passage of the leading or fastest wave, subsequent waves will no longer be traveling into such a region, the analysis in each section is valid only for the wave associated with the fastest wave speed. Hence, of the N possible wave speeds we choose the one associated with the leading wave and study the affect that wave has on the mixture.

After the derivation of the propagation condition the next step in each case is to derive the amplitude equation that governs the growth and decay of the amplitude of the leading wave. Before discussing those equations though, an important byproduct of those calculations should be pointed out. In each case it turns out that the passage of the leading wave through the mixture has an effect on the coldness distribution behind the wave. The passage of the wave tends to separate the coldnesses behind the wave. In both conductors and non-conductors this separation is greater with the passage of second order waves than with third order waves. However, when comparing the same order waves this tendency to separate the coldnesses is greater in non-conductors than in conductors. Thus heat conduction can be said to have a stabilizing effect on this process.

a. Second Order Waves

When considering second order waves the amplitude
equation that arises has the same form for both conductors and non-conductors. A critical feature in both cases is that the material coefficients all depend on the amplitude of the wave. Thus equations (4.40) and (6.29) are non-linear ordinary differential equations. This non-linearity is a consequence of the fact the original constitutive equation (2.21) depended upon density gradients which, when considering second order waves are discontinuous. However, if we were to consider the case where the density gradient is sufficiently close to its initial value in front of the wave (i.e. when it is close to zero), the material coefficients become constants. In that case equations (4.40) and (6.29) can be solved.

In general we cannot solve (4.40) and (6.29). However, some important information can be obtained about the material coefficients $\Delta$ and $\varepsilon$ in those equations which tells us something about the growth and decay of the amplitude. For example it can be shown that both $\Delta$, defined by (4.41), and $\varepsilon$ defined by (6.30) are even functions of the amplitudes. Hence, whether we have an expansive or compressive disturbance, if $\Delta$(or $\varepsilon$) < 0 it will always act as an attenuation factor and if $\Delta$(or $\varepsilon$) > 0 it will always act as a growth factor.

These signs of $\Delta$ and $\varepsilon$ also have an important bearing on the behavior of the amplitude. For conductors the sign of the material coefficient can be obtained when a
certain symmetry condition, similar to an ONSAGER relation, exists between the quantities $C$ and $D$ defined by equation \((4.43)_{3}\) and \((4.43)_{4}\) respectively. In that special case \(\Delta \leq 0\) for all $s$ and it will always act as a decay factor.

For non-conductors the sign of the material coefficient can be obtained without making any assumptions. That is, $\Xi \leq 0$ for all $s$ and $\Xi$ always acts as a decay factor. In conductors the diffusive force, the pressure, and the heat flux contribute to the attenuation factor whereas in non-conductors, the diffusive force, the pressure, and the energy supply contribute to the attenuation factor.

Of course in both \((4.40)\) and \((6.29)\) the appearance of the mean curvature $\overline{K}$ implies that the curvature of the wavefront is an attenuation factor for the amplitude.

An important special case is studied in non-conductors. If we assume that we have an ideal mixture then many of the results obtained in Section VI are similar but not identical to those obtained by BOWEN and DORIA [17]. The difference is that in their model the mixture was assumed to have a single temperature.

b. Third Order Waves

The non-constancy of the material coefficients in the differential equations that arose in the study of second
order waves is a consequence of the fact that the density gradients are discontinuous when studying second order waves. We study third order waves because the density gradients are continuous and the differential equations that arise, (5.20) and (7.13), therefore have constant coefficients. The differential equations for third order waves again have the same form for both conductors and non-conductors, and the solution in each case is given.

For conductors the material coefficient has the same form as it had in the study of second order waves. Hence \( \Delta \) has the same properties as it had in the study of second order waves. For non-conductors however, the material coefficient is a special case of the one obtained for second order waves. Nevertheless it has the same properties as it had in the study of second order waves.
BIBLIOGRAPHY


APPENDIX I

In this appendix we derive some results from the equilibrium equations (2.32). These results are true for both conductors and non-conductors.

Since we have assumed that the wave propagates into a region at rest, then the diffusion velocity $u$ is zero in front of the wave. If we make use of this fact, equations (2.32) and (2.18) imply that

$$\hat{e}(u, \rho, 0, \xi, 0) = \hat{e}(u, \rho, 0, \xi, 0) = 0. \quad (IA.1)$$

Equations (2.32) and (IA.1) must hold for all values of $u$, $\rho$, and $\xi$. In particular they must hold for those values of $u$, $\rho$, and $\xi$ in front of and behind the wave. Thus

$$\hat{e}^\pm = \hat{e}(u^\pm, \rho^\pm, 0, \xi^\pm, 0) = 0, \quad (IA.2)_1$$

$$\hat{e}^\pm = \hat{e}(u^\pm, \rho^\pm, 0, \xi^\pm, 0) = 0, \quad (IA.2)_2$$

$$\hat{f}^\pm = \hat{f}(u^\pm, \rho^\pm, 0, \xi^\pm, 0) = 0, \quad (IA.2)_3$$

IA.1
and

\[ q^\pm = q(\dot{\psi}^\pm, \rho^\pm, 0, \varepsilon^\pm, 0) = 0. \]  

(IA.2)

It should be noted that since \( v \) and \( \rho \) are continuous across the wave for both second and third order waves, then \( v^- = v^+ \) and \( \rho^+ = \rho^- \). Since all waves propagate into the region defined by equations (3.41) then \( \varepsilon^+ = 0 \) for both second and third order waves. This fact along with (3.13)_4 and (3.26)_3 implies that

\[ \varepsilon^- = \frac{1}{u^2} \frac{\rho^+}{\rho^-} \frac{\sin}{n} \]  

(IA.3)

for second order waves and

\[ \varepsilon^- = 0 \]  

(IA.3)_2

for third order waves. It should also be observed that equations (IA.2) and (IA.3) are true for non-conductors as well as conductors.

It follows from equation (2.32)_3 that

\[ \frac{\partial \varepsilon(\dot{\psi}, \rho, 0, \varepsilon, 0)}{\partial \dot{\psi}} \bigg|_{a}^{b} \bigg|_{b}^{b} \neq 0, \]  

(IA.4)

but
\[ \frac{\partial f(u, \rho, 0, \xi, \Theta)}{\partial \rho} = 0, \quad (\text{IA.4})_2 \]

and

\[ \frac{\partial f(u, \rho, 0, \xi, \Theta)}{\partial \xi} = 0. \quad (\text{IA.4})_3 \]

Again these equations must hold for all values of \( u, \rho \), and \( \xi \) including \( u^\pm, \rho^\pm \), and \( \xi^\pm \). Hence,

\[ (\frac{\partial f}{\partial \rho})^\pm = 0 \quad \text{and} \quad (\frac{\partial f}{\partial \xi})^\pm = 0. \quad (\text{IA.5}) \]

A similar argument for \( \hat{e}, \hat{\varepsilon}, \) and \( q \) shows that

\[ (\frac{\partial \hat{e}}{\partial \rho})^\pm = 0 \quad \text{and} \quad (\frac{\partial \hat{e}}{\partial \xi})^\pm = 0, \quad (\text{IA.6}) \]

\[ (\frac{\partial \hat{\varepsilon}}{\partial \rho})^\pm = 0 \quad \text{and} \quad (\frac{\partial \hat{\varepsilon}}{\partial \xi})^\pm = 0, \quad (\text{IA.7}) \]

and
\[
\left( \frac{\partial q}{\partial \rho} \right)^\pm = 0, \quad \text{and} \quad \left( \frac{\partial q}{\partial \xi^a} \right)^\pm = 0. \quad (\text{IA.8})
\]

We can extend these results even further by the same argument to show that

\[
\left( \frac{a^2}{\partial \xi^a \partial \xi^b} \right)^\pm = 0, \quad \left( \frac{a^2}{\partial \xi^a \partial \xi^b} \right)^\pm = 0, \quad \left( \frac{a^2}{\partial \xi^a \partial \xi^b} \right)^\pm = 0,
\]

and

\[
\left( \frac{\partial q}{\partial \xi^a \partial \xi^b} \right)^\pm = 0. \quad (\text{IA.9})
\]

When considering second order waves, equation (IA.3)_1 implies that when the minus sign is employed in all the above equations the resulting quantity depends on the amplitude \(s\). Similarly equation (IA.3)_2 implies that when considering third order waves there is no dependence on \(s\) in any of the above results.

Equations (IA.5) through (IA.9) are true for non-conductors as well as conductors.

The principles of material frame-indifference and material symmetry imply that \(\varepsilon\) is an isotropic scalar.
Hence $\frac{\partial \hat{\tau}}{\partial \hat{x}}$ is an odd order isotropic tensor which, when evaluated on $(v, p^+, 0, 0, 0)$ gives

$$
\frac{\partial \hat{\tau}}{\partial \hat{x}} (\frac{a}{b})^\pm = 0
$$

(IA.10) \_1

for third order waves only. A similar result can be obtained for $\frac{\partial \hat{f}}{\partial \hat{u}}$:

$$
\frac{\partial \hat{f}}{\partial \hat{u}} (\frac{a}{b})^\pm = 0
$$

(IA.10) \_2

for third order waves only.
APPENDIX II

The principles of material frame indifference and material symmetry imply that, for all orthogonal linear transformations \( Q \),

\[
Q m(u, \rho, \Omega, \xi) = m(\nu, \rho, Q g, Q \xi, Q u),
\]

where \( u = u \cdot u \). Hence \( m \) is an isotropic vector function of the two scalars \( \nu \) and \( \rho \) and of the vectors \( g, \xi \), and \( u \). We also know that \( m \) must satisfy the differential equation (4.5) which, in component notation is

\[
\frac{\partial m_{11}}{\partial \xi} + \frac{\partial m_{12}}{\partial \xi} + \frac{\partial m_{21}}{\partial \xi} = 0, \quad a = 1, \ldots, N.
\]

The solution to (IIA.2) is known to be [4, section 8]

\[
m_{11}(\nu, \rho, g, \xi) = m_{11}^{0}(\nu, \rho, g, \xi) + \sum_{a_{1}} \omega_{123}^{a_{1}}(\nu, \rho, g, \xi) \frac{\partial^{2}}{\partial \xi^{2}}
\]

\[
+ \sum_{a_{1}, a_{2}} \omega_{123}^{a_{1}, a_{2}}(\nu, \rho, g, \xi) \frac{\partial^{2}}{\partial \xi^{2}}
\]

\[
IIA.1
\]
\[ + \sum_{a_1, a_2, a_3} \Omega_{i_1 i_2 i_3 i_4}^{(v, \rho, g, \lambda)} a_1 a_2 a_3 \]
\[ + \ldots + \sum_{a_1, \ldots, a_N} \Omega_{i_1 i_2 \ldots i_{N+1}}^{(v, \rho, g, \lambda)} c c c a_1 a_2 a_N \]

where all the \( \Omega \)-tensors are skew-symmetric with respect to the permutation of any of the \( i_k \) indices. Since \( m \) is an isotropic function the \( \Omega \)-tensors are also isotropic functions.

If we take the derivative of (IIA.3) with respect to \( \dot{x} \) and \( \ddot{x} \) and evaluate the result at \( (v, \rho^+, 0, \lambda, 0) \) we obtain

\[ \frac{\partial^2 m_{i_1}^{(v, \rho^+, 0, \lambda, 0)}}{\partial x^a \partial x^b} = \frac{\partial^2 m_{i_1}^{(v, \rho^+, 0, 0)}}{\partial x^a \partial x^b} \]

\[ + \sum_{a_1} \frac{\partial^2 \Omega_{i_1 i_2}^{(v, \rho^+, 0, 0)}}{\partial x^a \partial x^b} a_1 \]

\[ + \ldots + \sum_{a_1, \ldots, a_N} \frac{\partial^2 \Omega_{i_1 \ldots i_{N+1}}^{(v, \rho^+, 0, 0)}}{\partial x^a \partial x^b} a_1 a_N \]

(IIA.4)
We can now use the fact that all odd order isotropic tensor functions of vectors are zero when evaluated at zero. Since all the derivatives appearing on the right side of (IIA.4) are isotropic tensors, when we evaluate them at the indicated values, all the odd ordered tensors are zero.

We can also make use of the known representations for even order isotropic tensors [5]. This fact along with the skew-symmetry of the $i_k$ indicies of the $\Omega$-tensors and the symmetry of the $j$ and $k$ indicies can be used to prove that all the even order tensors on the right side of (IIA.4) are also zero. We are left with

$$\frac{\partial^2 m}{\partial x_j \partial x_k} (\nu, \rho^+, 0, \xi, 0) = 0. \tag{IIA.5}$$

Since this must hold for all values of $\xi$, we let $\xi = \xi^+$ and obtain

$$\left( \frac{\partial^2 m}{\partial x \partial x} \right)^{c} = 0. \tag{IIA.6}$$

If we take the derivative of (IIA.4) with respect to $\rho, \nu, g^k, \text{and } \xi^k$ respectively and evaluate the results at $(\nu, \rho^+, 0, \xi, 0)$ we obtain some more useful results. For example, the tensorial character of (IIA.4) is the same as
that of \( \dot{a}^3 m_{\alpha \beta \gamma} / \dot{a}^\alpha j \dot{a}^\beta \dot{a}^\gamma \) and \( \dot{a}^3 m_{\alpha \beta \gamma} / \dot{a}^\alpha j \dot{a}^\beta \dot{a}^\gamma \). Hence, by the same reasoning that led to (IIA.5) we find that

\[
\left( \frac{\dot{a}^3 m}{\dot{a}^\alpha \dot{a}^\beta \dot{a}^\gamma \dot{a}^\rho} \right)^2 = 0
\]

(IIA.7) \(_1\)

and

\[
\left( \frac{\dot{a}^3 m}{\dot{a}^\alpha \dot{a}^\beta \dot{a}^\gamma \dot{a}^\rho} \right)^2 = 0 .
\]

(IIA.7) \(_2\)

If we form the derivative of (IIA.4) with respect to \( g^\alpha \), we obtain

\[
\frac{\dot{a}^3 m_{\alpha \beta \gamma}}{\dot{a}^\beta j \dot{a}^\gamma \dot{a}^\rho} = \frac{\dot{a}^3 m_{\alpha \beta \gamma}}{\dot{a}^\beta j \dot{a}^\gamma \dot{a}^\rho}
\]

(IIA.8)
The derivatives on the right side of (IIA.8) are again either even or odd order isotropic tensors with the same symmetry properties as mentioned earlier. Thus when evaluated on the indicated values we obtain

\[
\frac{\partial^3 m_i}{\partial x^a \partial x^b \partial x^c} (\nu, \rho^+, 0, z^j, 0) = 0.
\]

(IIA.9)

Hence, if we let \( \frac{z^j}{d} = \frac{\partial}{\partial \tilde{z}^j} \),

\[
\left( \frac{\partial^3 m}{\partial x^a \partial x^b \partial x^c} \right)^{\pm} = 0
\]

(IIA.10)

Finally, if we form the derivative of (IIA.4) with respect to \( z^m \) we obtain

\[
\frac{\partial^3 m_i}{\partial x^a \partial x^b \partial x^c} (\nu, \rho^+, 0, z^j, 0) \sum_{a_1} \frac{\partial^2 n_{i1n}}{\partial x^a_1 \partial x^b_1} (\nu, \rho^+, 0, 0, 0) = \sum_{a_1} \frac{\partial^2 n_{i1n}}{\partial x^a_1 \partial x^b_1} (\nu, \rho^+, 0, 0, 0)
\]

\[
\sum_{a_1} \frac{\partial^2 n_{i1n}}{\partial x^a_1 \partial x^b_1} (\nu, \rho^+, 0, 0, 0)
\]
Again we use the isotropy and symmetry of the derivatives appearing on the right side of (IIA.11) to show that

\[
\left( \frac{\partial^3 m}{\partial x^a \partial x^b \partial x^c} \right)_{x=0} = 0.
\]  

(IIA.10)_3

We can also obtain a simplified expression for \( \partial^3 m / \partial x^a \partial x^b \partial x^c \) evaluated behind the wave. If we form the derivative of (4.3) with respect to \( \hat{x} \), \( \tilde{x} \), and \( \check{x} \) and evaluate the result at \( \hat{x} = 0, d = 1, \ldots, N \), we find that in component notation

\[
\left( \frac{\partial^3 m}{\partial x^a \partial x^b \partial x^c} \right)_{x=0} = \delta_{ij} \left( \frac{\partial^2 \Lambda}{\partial x^a \partial x^b} \right)_{x=0}.
\]
\[ + \delta_{ik} \left( \frac{\partial^2 \Lambda}{\partial x^i \partial x^k} \right)_{x=0} + \delta_{ik} \left( \frac{\partial^2 \Lambda}{\partial x^j \partial x^l} \right)_{x=0} \]. \] (IIA.11)_1

If we multiply (IIA.11) by \( n_j n_k n_l \) we obtain

\[ \left( \frac{\partial^3 m}{\partial x^i \partial x^k \partial x^l} \right)_{x=0} n_j n_k n_l = \left\{ \frac{\partial^2 \Lambda}{\partial x^j \partial x^l} \right\} n_j n_k n_l \]

\[ + \frac{\partial^2 \Lambda}{\partial x^j \partial x^k} n_j n_l + \frac{\partial^2 \Lambda}{\partial x^a \partial x^b} n_j n_k \right\} n_i. \] (IIA.11)_2

We can now evaluate (IIA.11)_2 on \((u, v^+, 0, \xi^-, 0)\) to obtain, in direct notation

\[ \left( \frac{\partial^3 m}{\partial x^i \partial x^k \partial x^l} \right) \sim (n \sim \Xi \sim \bar{\Xi} \sim n) = \mathcal{T} n, \] (IIA.12)

where \( \mathcal{T} \) is a scalar defined by

\[ \mathcal{T} = \text{tr}\{( \frac{\partial^2 \Lambda}{\partial x^a \partial x^b} )^{-1} + ( \frac{\partial^2 \Lambda}{\partial x^a \partial x^c} )^{-1} + ( \frac{\partial^2 \Lambda}{\partial x^a \partial x^a} )^{-1} \} \right\} (n \sim \Xi \sim n) \}. \]

(IIA.13)