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A MARTIN BOUNDARY FOR THE HEAT EQUATION

by

James Fenton Hall

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
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Thesis Director's signature:

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§0. Preliminaries

The principal result of this section is a decomposition of supercaloric functions, analogous to the Riesz decomposition of superharmonic functions. This decomposition theorem will be an important tool in obtaining the integral representation of a nonnegative solution of the heat equation by a measure on the "ideal" boundary to be constructed. In addition, many standard facts from the potential theory of the heat equation which are needed later are stated in this section for easier reference.

We are concerned with the heat equation $\Delta u - \frac{\partial u}{\partial t} = 0$ in a subdomain $\Omega$ of $\mathbb{R}^{n+1}$, where we call $x_{n+1}$ the "time" variable, and denote it by $t$, so $\mathbb{R}^{n+1} = \{(x, t) \in \mathbb{R}^n, t \in \mathbb{R}\}$, and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$ Solutions of the heat equation will be called temperatures. We assume the fact that the first initial-boundary problem, which we shall also call the Dirichlet problem, can be solved in a cylinder

$$\{(x, t) \in \mathbb{R}^{n+1}: |x-x_0| < r, |t-t_0| < h\}$$

or a cone $\{(x, t) \in \mathbb{R}^{n+1}: |x-x_0| < \alpha(t_0-t), t_1 < t < t_0\}$. If $C$ is such a set, then by the Riesz representation theorem, for any $(x, t) \in C$, there is a unique measure $d\omega_C^{(x, t)}$ on $\partial C$ such that, for any temperature $u$ on $C$ which is continuous on $\overline{C}$,

$$u(x, t) = \int_{\partial C} u \, d\omega_C^{(x, t)}.$$
d\omega_C^{(x,t)} is called the caloric measure on \partial C for the point 
(x,t), and in fact it is supported on the parabolic boundary 
of the set, which consists of those points (x,t) in \partial C such that 

\{(y,s): |y-x| < \varepsilon, 0 < s-t < \varepsilon\} \cap C \neq \emptyset, \text{ for every } \varepsilon > 0. 

The generalized Dirichlet problem can be solved on arbitrary 
domains by the usual Perron-Wiener-Brelot (PWB) technique 
using upper and lower functions [1,10].

**Definition:** A function u is supercaloric on an open set 
\( \Omega \subset \mathbb{R}^{n+1} \) if 

(i) \( u \) is lower semicontinuous on \( \Omega \) 

(ii) \( u > -\infty \) 

(iii) for any \( (x,t) \in \Omega \) and any open cone \( C \) such that 
\( (x,t) \in C \) and \( \overline{C} \subset \Omega \)

\[ u(x,t) \geq \int_{\partial C} u \, d\omega_C^{(x,t)}. \]

As for Laplace's equation, \( u \in C^2 \) is supercaloric \( \Rightarrow Hu > 0 \), 
where \( H = \Delta - \frac{3}{2t} \). By an approximation argument it is easy 
to show that if \( u \) is supercaloric, then the distribution \( -Hu \) 
is a (positive) measure.

We denote by \( \Gamma(x,t;\xi,\tau) \) the fundamental solution of the 
heat equation with pole at \( (\xi,\tau) \):

\[ \Gamma(x,t;\xi,\tau) = \begin{cases} 
\left(\frac{n}{4\pi}\right)^{-\frac{n}{2}}(t-\tau)^{-\frac{n+2}{2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} & \text{if } t > \tau \\
0 & \text{if } t \leq \tau \end{cases} \]
As a distribution, \( \mathcal{H}(\cdot, \cdot; \xi, \tau) = -\delta_{\xi, \tau} \). The Green function for a domain \( \Omega \) may be defined in two equivalent ways.

First, for \( (\xi, \tau) \in \Omega \), \( G(x, t; \xi, \tau) = \Gamma(x, t; \xi, \tau) - V(\xi, \tau)(x, t) \)
where \( V(\xi, \tau) \) is the PWB solution in \( \Omega \) of the Dirichlet problem for the boundary values of \( \Gamma(\cdot, \cdot; \xi, \tau) \) on \( \partial \Omega \).

Second, \( G(\cdot, \cdot; \xi, \tau) \) is the minimal non-negative supercaloric function on \( \Omega \) of the form \( \Gamma(\cdot, \cdot; \xi, \tau) + V \) where \( V \) is supercaloric on \( \Omega \).

Two features of the potential theory of the heat equation differ sharply from the corresponding results for Laplace's equation.

1. The strong maximum principle is more restricted.
   If a bounded temperature takes on an extremum at a point not in the parabolic boundary, then that temperature is identically equal to that extreme value at every point which can be reached from the extremum point by a curve whose time component is non-increasing \([3]\). Thus, if \( u \geq 0 \) in the region at right and \( u(P) = 0 \), then \( u \) may still be positive in the shaded regions.

2. Related to this first fact is that Harnack's inequality for the heat equation is only one-sided and requires a "time lag." If \( K_1 \) and \( K_2 \) are compact sets in \( \Omega \), if every point in \( K_1 \) may be reached from any point in \( K_2 \) by a curve with a non-increasing time component, and if \( \sup \{ t : \exists (x, t) \in K_1 \} < \inf \{ t : \exists (x, t) \in K_2 \} \), then \( \exists C > 0 \)
such that for any non-negative temperature $u$ on $\Omega$, 

$$\sup_{K_1} u \leq C \inf_{K_2} u.$$ 

In the terminology of the abstract potential theory of Bauer, et al., these problems are connected with the existence of nontrivial absorption sets in the potential theory of the heat equation.

Because of the restriction (1) on the strong maximum principle, we shall be concerned with that part of a domain which can be reached from a fixed reference point by curves whose time component is non-increasing.

The main result of this section is the following decomposition theorem. Although it is rather restricted, it is adequate for our purpose.

**Theorem:** Let $\Omega$ be a domain in $\mathbb{R}^{n+1}$, and let $u$ be a bounded, non-negative supercaloric function on $\Omega$, such that $Hu = 0$ in $\Omega \sim \overline{\Omega}$ where $0$ is a relatively compact open subset of $\Omega$. Then there is a measure $\mu$ on $\Omega$ such that $u = G\mu + h$, where 

$$G\mu(P) = \int G(P,Q) \, d\mu(Q)$$

for $P \in \Omega$ is the Green potential of $\mu$, and $h$ is a temperature on $\Omega$.

**Proof:** Since $u$ is supercaloric, $-Hu$ is a positive measure $\mu$ on $\Omega$, which is supported in $\overline{\Omega} \subset \Omega$. 
1. $H(G_{\mu}) = -\mu$ in $\mathcal{D}'(\Omega)$: Let $\varphi \in C_0^\infty(\Omega)$. Then, writing

$$H^* = \Delta + \frac{\partial}{\partial t},$$

$$(G_{\mu}, H^*\varphi) = \int_\Omega G_{\mu}(x,t)H^*\varphi(x,t) \, dx \, dt$$

$$= \int_\Omega H^*\varphi(x,t) \left( \int_\Omega G(x,t;\xi,\tau) \, d\mu(\xi,\tau) \right) dx \, dt$$

If we can apply Fubini's theorem, then

$$= \int_\Omega \left( \int_\Omega G(x,t;\xi,\tau)H^*\varphi(x,t) \, dx \, dt \right) d\mu(\xi,\tau)$$

Since $HG(\cdot,\cdot;\xi,\tau) = -\delta(\xi,\tau)$ in $\mathcal{D}'(\Omega)$

$$= \int \nu(\xi,\tau) d\mu(\xi,\tau) = (\mu,\varphi), \text{ so } H\mu = \mu.$$ 

To justify applying Fubini, note that

$$\int_\Omega \left( \int_\Omega |G(x,t;\xi,\tau)H^*\varphi(x,t)| \, dx \, dt \right) d\mu(\xi,\tau)$$

$$\leq \left( \sup_{\Omega} |H^*\varphi| \right) \int_\Omega \left[ \int_0^1 G(x,t;\xi,\tau) dx \, dt \right] d\mu(\xi,\tau)$$

$$\leq C_\mu(\Omega) < \infty$$

since $G(\cdot,\cdot;\xi,\tau) \in L^1_{\text{loc}}(\Omega)$ for $(\xi,\tau) \in \Omega$, and its integral over a given compact subset of $\Omega$ is uniformly bounded as long as $(\xi,\tau)$ remains in a compact subset of $\Omega$. $\mu$ is a finite measure since $u$ is bounded: if $\varphi \in C_0^\infty(\Omega)^+$ and $\varphi = 1$ on $0$, then

$$\mu(\Omega) \leq \mu(\varphi) = (u, H^*\varphi) \leq \sup_{\partial} |H^*\varphi| \int_0^1 u \, dx \, dt < \infty.$$
2. Finally, define $h = u - G_\mu$. Then

$$Hh = Hu - HG_\mu = u - \mu = 0 \text{ in } \mathcal{D}'(\Omega).$$

Since $H$ is hypoelliptic, $h \in C^\infty$ and is thus a temperature in $\Omega$

$$u = G_\mu + h.$$  

As for Laplace's equation it is possible to define a balayage process for supercaloric functions, using caloric measure on cones, just as the Poisson integral is used on balls for superharmonic functions. If a supercaloric function $u$ has a caloric minorant $h$, i.e. if there is a temperature $h$ on $\Omega$ such that $h \leq u$ on $\Omega$, then there is a greatest caloric minorant for $u$, viz. $u_\infty$, the balayage of $u$, in the sense that if $w$ is any caloric minorant of $u$ on $\Omega$, then $w \leq u_\infty$. The proof is essentially a repetition of the argument for Laplace's equation [see 4]. Also, the greatest caloric minorant of the Green function or a Green potential is 0, and the greatest caloric minorant of a sum of two supercaloric functions is the sum of their greatest caloric minorants. (For the elliptic versions, see [4]). It follows then that in the decomposition theorem above, $h = u_\infty$.

We shall also need the well-known result that in a cylinder in $\mathbb{R}^{n+1}$, a temperature $u$, continuous on the closed cylinder, may be represented in terms of its boundary values.

If $C = \{(x,t): |x| < a, 0 < t < 1\}$, and $\partial_i C$ is the initial boundary $\{(x,0): |x| < a\}$, $\partial_\ell C$ the lateral boundary $\{(x,t): |x| = a, 0 < t < 1\}$, then for $(x,t) \in C$

$$u(x,t) = \int_{\partial_i C} u(\xi,0)G(x,t;\xi,0)d\xi - \int_{\partial_\ell C} u(\xi,\tau)\frac{\partial G}{\partial V}(s,t;\xi,\tau)d\xi d\tau$$
where \( \frac{\partial}{\partial \nu} \) denotes the outward normal derivative and \( G(x,t;\xi,\tau) \) is the Green function for \( C \). If \( K \) is compact in \( C \), then the derivatives of the kernels are uniformly bounded for \( (\xi,\tau) \in \partial_1 C \cup \partial_C \), so a derivative of \( u \) may be computed by differentiating under the integral sign. In particular, if a family of temperatures \( \{u_\alpha\} \) on \( C \) is uniformly bounded on \( C \), then their derivatives are uniformly bounded on any compact subset of \( C \).

§1. Introduction

We wish to obtain a representation of a non-negative temperature \( u \) in a domain \( \mathbb{R}^{n+1} \), analogous to the representation of a temperature \( u \) continuous on a closed cone, or cylinder in terms of the integral of its actual boundary values with respect to the caloric measure on the cone or cylinder. The corresponding problem for harmonic function, generalizing the Poisson integral formula in the ball, was solved by R. S. Martin [8], and it is basically his approach which is followed here. Several characteristic features of the parabolic situation cause difficulties in the adaptation of the original arguments, in particular the restrictions on the strong maximum principle and the Harnack inequality mentioned earlier. The results are accordingly more restricted than their elliptic counterparts.

The general strategy is the same as for the elliptic case. We look at \( u \) on a smaller domain with "nice" boundary and use the Riesz-type decomposition given in the preliminary
section to write \( u \) in the smaller domain as the Green potential of a measure on the boundary of the subdomain.

Taking a sequence of such subdomains expanding to \( \Omega \) isn't quite adequate, because we have no control over the measures obtained. \textit{A priori} their variations need not even be bounded, so they don't obviously converge. This difficulty can be handled by choosing a fixed reference point \( P_0 = (x_0, t_0) \) and writing

\[
    u(x, t) = \int_{\Omega} \frac{G(x, t; \xi, \tau)}{G(x_0, t_0; \xi, \tau)} (G(x_0, t_0; \xi, \tau) d_\mu(\xi, \tau))
\]

where \( \Omega \) is a subdomain, \( G \) is the Green function for \( \Omega \). The measures \( \{G(x_0, t_0; \cdot, \cdot) d_\mu(\cdot, \cdot)\} \) are easily seen to be uniformly bounded. If \( \Omega \) can be embedded in a suitable space, e.g. a compact space, so that any bounded sequence of measures has a weakly convergent subsequence, and so that these normalized Green functions extend to the larger space nicely, then we can obtain the desired representation.

The use of Green functions seems essential, and the behavior of the Green function forces some immediate restrictions on this approach. First, \( G(x, t; \xi, \tau) = 0 \) if \( t < \tau \), so the normalized ratio \( \frac{G(x, t; \xi, \tau)}{G(x_0, t_0; \xi, \tau)} \) only makes sense if \( t_0 > \tau \). This means that rather than representing \( u \) on all the domain, we must fix a time \( t_0 \) and consider \( u \) up to that time, i.e. on \( \Omega \cap \{(x, t): t < t_0\} \). The reference
point must also be chosen with this maximal time coordinate \( t_0 \). If \( \Omega \) is contained in a half-space \( \{ (x,t) : t < t_0 \} \) and has a boundary point \((x_0,t_0)\), this point may be chosen as reference point if \( \partial \Omega \) is "flat" with respect to the time variable near \((x_0,t_0)\), i.e. for some \( \delta > 0\),

\[
\partial \Omega \cap \{(x,t) : |(x,t) - (x_0,t_0)| < \delta \} = \{(x,t_0) : |x - x_0| < \delta \}.
\]

For example, the apex of a cone \( \{(x,t) : |x - x_0| < \alpha (t - t_0), t > 0 \} \) cannot be used as the reference point, because the Green function for the cone with pole at an interior point \((\xi,\tau)\) vanishes at the apex \((x_0,t_0)\).

A second problem with the Green functions is connected with the restricted validity of the strong maximum principle. If \( \Omega \) "branches upward" in the sense that for some \( T \),

\[ \Omega^T = \Omega \cap \{(x,t) : t > T \} \]

is not connected, and if \((x_0,t_0)\) and \((\xi,\tau)\) lie in different components of \( \Omega^T \), then \( G(x_0,t_0;\xi,\tau) = 0 \).

\( G(x_0,t_0;\xi,\tau) \) will be non-zero only if \((\xi,\tau)\) can be reached from \((x_0,t_0)\) by a curve whose time-component is non-increasing.

The use of Green functions consequently can only give information about \( u \) in that part of \( \Omega \) which can be reached from the reference point \((x_0,t_0)\) by curves whose time component is non-increasing. For notational convenience we make the following definition.

**Definition:** If \( \Omega \subset \mathbb{R}^{n+1} \) is open and \((x_0,t_0) \in \overline{\Omega} \), the monotone component of \((x_0,t_0)\) in \( \Omega \) is the set of points \((\xi,\tau)\) in \( \Omega \) such that there is a curve \( \gamma : [0,1] \to \overline{\Omega} \) with \( \gamma(0) = (x_0,t_0) \), \( \gamma(1) = (\xi,\tau) \), \( \gamma([0,1]) \approx \{(x_0,t_0)\} \subset \Omega \), and if
\( \gamma(s) = (\gamma_1(s), \gamma_2(s)) \) where \( \gamma_1(s) \in \mathbb{R}^n \), \( \gamma_2(s) \in \mathbb{R} \), then \( \gamma_2 \) is non-increasing.

To represent temperatures in \( \Omega \) up to time \( t_0 \) we choose a point \( (x_0, t_0) \in \Omega \) and restrict our attention to the monotone component of \( (x_0, t_0) \) in \( \Omega \). If \( \Omega \) has a flat "upper" boundary, \( (x_0, t_0) \) may be chosen there, but as the construction argument will show, we would then be limited to representing temperatures \( u \) which are continuous up to \( (x_0, t_0) \) on the boundary.

The basic construction given here, like the original construction given by Martin, yields a compactification of the space. Sieveking [9] has defined a concept of a Martin boundary in the context of the theory of harmonic spaces (or axiomatic potential theory) which applies to the heat equation, and he constructs the boundary in a special case, \( \mathbb{R}^n \times (-\infty, \gamma) \), where \( -\infty < \gamma \leq \infty \). The approach and techniques used by Sieveking are very different from those employed here, and it is not immediately clear that the boundary we shall construct satisfies his conditions to be a Martin boundary. It is of interest that his axioms for the Martin boundary in general rule out the possibility of a compact boundary. Although our construction does yield a compactification of the basic space \( \Omega \), we shall see in the section on the boundary of Lipschitz domains that one of the "ideal" points seems very artificial, and in fact corresponds to the point at \( \infty \) added to a locally compact space to form its one-point compactification. Nothing is
lost, and the results actually seem more reasonable if this point is omitted, in which case our boundary is also only locally compact. Also, in contrast to Sieveking's approach, we are concerned only with the specific case of the heat equation and the concrete problem of constructing such a boundary for certain domains. We also relate this construction to the boundary behavior problems previously studied by Kemper [7], and Jones and Tu [6]. Although the restrictions mentioned earlier show that this approach still leaves much to be desired so far as generality is concerned, compared to the results for harmonic functions, it does at least generalize the Lipschitz boundary cases previously studied.

§2. The construction of a compactification

To keep notation simple, we assume that $\Omega$ is a domain in $\mathbb{R}^{n+1}$; $t_o = \sup \{ t : \exists x \in \mathbb{R}^n \text{ with } (x,t) \in \Omega \}$; $P_o = (x_o,t_o) \in \partial \Omega$; $\partial \Omega \cap \{(x,t) : |(x,t) - (x_o,t_o)| < \delta \} = \{(x,t_o) : |x-x_o| < \delta \}$ for some $\delta > 0$ (i.e. $\partial \Omega$ is parallel to $\mathbb{R}^n$ near $(x_o,t_o)$); and finally that $\Omega$ is the monotone component of $P_o$ in $\Omega$.

We first note that even though $(x_o,t_o) \notin \Omega$, $G(x_o,t_o;\xi,\tau)$ makes sense for $(\xi,\tau) \in \Omega$. Let $\Omega' = \Omega \cup \{(x,t) : |x-x_o| < \frac{\delta}{2}, 0 \leq t-t_o < \frac{\delta}{2} \}$, i.e. $\Omega'$ is $\Omega$ with a small cylinder over $P_o$ adjoined. Using the characterization of the Green function as the fundamental solution minus a certain PWB solution, and checking that upper (resp. lower) functions on $\Omega'$ restrict to upper (resp. lower) functions for the problem
on $\Omega$, it is easy to see that for $(\xi, \tau) \in \Omega$, the Green function $G'(\cdot; \xi, \tau)$ for $\Omega'$ restricts to the Green function $G(\cdot; \xi, \tau)$ for $\Omega$ on $\Omega$. Thus $G'(\cdot; \xi, \tau)$ gives a continuous extension of $G(\cdot; \xi, \tau)$ to $(x_0, t_0)$.

**Definitions:** A sequence of points $\{Q_m\}_{m=1}^{\infty}$ in $\Omega$ is **fundamental** if

(i) $\{Q_m\}$ leaves every compact subset of $\Omega$

(ii) $\lim_{m \to \infty} \frac{G(P, Q_m)}{G(P_0, Q_m)}$ exists, for every $P \in \Omega$, and the limit exists uniformly on any compact subset of $\Omega$.

Two fundamental sequences $\{Q_m\}, \{Q'_m\}$ are **equivalent** if the limits in (ii) are the same. The equivalence class of $\{Q_m\}$ will be written $[\{Q_m\}]$.

The relationship of equivalence is clearly an equivalence relation of the collection of all fundamental sequences.

**Definition:** $\Delta$ is the set of all equivalence classes of fundamental sequences in $\Omega$. $\mathcal{A} = \Omega \cup \Delta$.

We define $K: \Omega \times \mathcal{A} \to \mathbb{R}^+$ by

$$K(P, Q) = \begin{cases} \frac{G(P, Q)}{G(P_0, Q)} & Q \in \Omega \\ \lim_{m \to \infty} \frac{G(P, Q_m)}{G(P_0, Q_m)} & Q = [\{Q_m\}] \in \Delta \end{cases}$$

By Harnack's convergence theorem, if $Q \in \Delta$, $K(\cdot, Q)$ is a temperature on $\Omega$. 
We use $K$ to define a metric $\rho$ on $\mathcal{Q}$. Let $\lambda$ be the measure 
\[ e^{-|x|^2-t^2} \, dx \, dt, \] (actually, any strictly positive, rapidly 
 decreasing function $\phi(x,t)$ would work as well as $e^{-|x|^2-t^2}$).

**Definition:** If $Q_1$ and $Q_2$ are in $\mathcal{Q}$,

\[ \rho(Q_1,Q_2) = \int_{\Omega} \frac{|K(P,Q_1) - K(P,Q_2)|}{1 + |K(P,Q_1) - K(P,Q_2)|} \, d\lambda(P) \]

$\rho$ is clearly non-negative, symmetric, and satisfies the 
triangle inequality because the integrand does, so we need 
only show that $\rho$ is definite. If $\rho(Q_1,Q_2) = 0$, then the 
integrand is 0 a.e. with respect to Lebesgue measure, since 
Lebesgue measure is absolutely continuous with respect to $\lambda$. 
The integrand is continuous in $\Omega \sim \{Q_1,Q_2\}$ and is therefore 0 
in $\Omega \sim \{Q_1,Q_2\}$. If $Q_1 \in \Delta$, then $Q_2$ cannot be in $\Omega$, since the 
temperature $K(\cdot,Q_1)$ would then define a continuation of the 
normalized Green function $K(\cdot,Q_2)$ to $Q_2$, which is impossible, 
since $K(\cdot,Q_2)$ has a singularity at $Q_2$. Then $Q_2 \in \Delta$ and 
$K(P,Q_1) = K(P,Q_2)$ for all $P \in \Omega$, so $Q_1 = Q_2$. If, on the other 
hand, $Q_1,Q_2 \in \Omega$ and if $Q_1 \neq Q_2$, then again $K(\cdot,Q_1)$ would give 
a continuation of $K(\cdot,Q_2)$ to $Q_2$, which is impossible, so 
$Q_1 = Q_2$. Therefore, $\rho$ is a metric on $\mathcal{Q}$. We must now show 
that $(\mathcal{Q},\rho)$ is compact and that $\Omega$ inherits its original 
topology from $(\mathcal{Q},\rho)$.

**Lemma:** If $\{Q_m\}$ is any sequence in $\Omega$ which leaves every 
compact set in $\Omega$, then $\{Q_m\}$ has a subsequence which is 
fundamental.
Proof: Let $Q_m = (\varepsilon_m, \tau_m)$. First, if $\limsup_{m \to \infty} \tau_m = t_0$, then for any compact set $F$, if $T_F = \sup \{ t : (x,t) \in F \}$, then for $\tau_m > T_F$, $G(x,t;\varepsilon_m,\tau_m) = 0$ for all $(x,t) \in F$. Choosing a subsequence $\tau_{m'}$ which increases to $t_0$, we see that for any compact set $F \subset \Omega$, for all sufficiently large $m'$, $G(\cdot,Q_{m'}) = 0$ on $F$, so the sequence $\{ K(\cdot,Q_{m'}) \}$ trivially converges uniformly on any compact $F$, viz. to the constant function $0$. Hence $\{Q_{m'}\}$ is fundamental and $K(\cdot,[[Q_{m'}]]) = 0$.

Otherwise, $\limsup_{m \to \infty} \tau_m < t_0$. In this case, this "time lag" allows the use of Harnack's inequality which shows that the Arzela-Ascoli theorem applies. To obtain a uniform bound on $\{ K(\cdot,Q_m) \}$ on a given compact set $F$, we need a domain containing both $P_0$ and $F$ in which the functions $K(\cdot,Q_m)$ are eventually temperatures.

One way to obtain such a domain is to choose a relatively compact neighborhood $V$ of $F$ in $\Omega$. Since $F$ is compact, we may assume that $V$ has only finitely many components. In each component, choose a point whose time coordinate is larger than the time coordinate of any point in $F$ in that component, and connect that point with $P_0$ by a curve whose time-component is monotone. As observed in the preliminary section, $P_0$ has a neighborhood $W$ to which any function $G(\cdot,Q_m)$ extends continuously, for $Q_m \in \Omega$. For each time-monotone curve chosen, choose a relatively compact neighborhood of that curve in $\Omega U W$. Take $U$ to be the union of $V$ with these finitely many neighborhoods of the curves. $U$ is relatively compact in $\Omega U W$, as a finite union of relatively
compact sets, and since $Q_m$ leaves every compact set in $\Omega$, eventually $Q_m \notin U$, for all sufficiently large $m$.

Now let $N$ be a relatively compact neighborhood of $F$ in $U$ such that $\sup \{t : (x, t) \in N\} < t_o$. Applying Harnack's inequality to $N$ and $P_0$, there is a constant $C > 0$ such that

$$\sup_{P \in N} G(P, Q_m) \leq C G(P_0, Q_m)$$

if $Q_m \notin U$.

In particular, the family $\{K(\cdot, Q_m)\}$ is uniformly bounded on $N \supseteq F$ for all sufficiently large $m$. The bound for $N$ is needed to obtain the bound on the derivatives, because by the argument given in the preliminary section, if a family of temperatures is uniformly bounded in a neighborhood of a point, then the first derivatives of the functions are uniformly bounded in a smaller neighborhood of that point. Thus the uniform bound on the family in $N$ and the compactness of $F$ imply that the first derivatives of the functions are uniformly bounded on $F$, and therefore $\{K(\cdot, Q_m) : m > M_F\}$, where $M_F$ is a constant depending on $F$ such that $m > M_F \Rightarrow Q_m \notin U$, is uniformly equicontinuous on $F$. By the Arzela-Ascoli theorem there is a subsequence $\{Q_m^1\}$ of $\{Q_m\}$ for which $\{K(\cdot, Q_m^1)\}$ converges uniformly on $F$.

Choosing a sequence of relatively compact open sets $F_i$ which increase to $\Omega$, for $F_1$ we obtain a subsequence $\{Q_m^1\}_{m=1}^\infty$ as above; for $F_2$, applying the argument to $\{Q_m^1\}$ to obtain a subsequence $\{Q_m^2\}$ of $\{Q_m^1\}$, and so on. The diagonal sequence $\{Q_m^m\}$ is then a sequence for which $\{K(\cdot, Q_m^m)\}$ converges uniformly on every compact subset of $\Omega$, and is thus fundamental.
Proposition: \((\emptyset, \rho)\) is compact, and \(\rho\) induces the original \(\mathbb{R}^{n+1}\) topology on \(\Omega\).

Proof:

(1) \(\mathbb{R}^{n+1}\)-convergence in \(\Omega \Rightarrow \rho\)-convergence: If \(\{Q_m\} \subset \Omega\) and \(Q_m \rightarrow Q \in \Omega\), then for every \(P \in \Omega \setminus \{Q\}\), it is clear that \(K(P, Q_m) \rightarrow K(P, Q)\). Since the integrand in the integral defining \(\rho\) is bounded by 1, the result follows from the dominated convergence theorem.

(2) If \(\{Q_m\}\) is fundamental and \(Q = \{[Q_m]\} \in \Delta\), then \(Q_m \overset{\rho}{\rightarrow} Q\): since \(K(\cdot, Q_m) \rightarrow K(\cdot, Q)\) pointwise, the dominated convergence theorem again gives the result.

(3) Any sequence \(\{Q_m\} \subset \Delta\) has a \(\rho\)-limit point \(Q \in \Delta\):

For each \(m\), we choose a fundamental sequence \(\{Q_m^{ij}\}\) in \(Q_m\).

Obviously we wish to use a diagonal sequence, but it must be chosen carefully.

(a) Choose a sequence of compact sets \(F_k\) expanding to \(\Omega\).

(b) For each \(m\), by (2) just proved, we may choose \(j(m)\) so that \(\rho(Q_m, j(m), Q_m) < \frac{1}{m}\) and \(Q_m, j(m) \notin F_m\). By construction \(\{Q_m, j(m)\}\) cannot accumulate in \(\Omega\), so by the lemma it has a fundamental subsequence \(\{Q_k\}\). Let \(Q = \{[Q_k]\} \in \Delta\).

Given \(\epsilon > 0\), since \(\tilde{Q}_k \overset{\rho}{\rightarrow} Q\), choose \(N_1\) such that for \(k > N_1\), \(\rho(Q_k, Q) < \frac{\epsilon}{2}\). Since each \(k\) corresponds to some \(m\), choose \(N_2\) such that \(k > N_2\) and \(k-\overset{\rho}{\rightarrow} m \geq \frac{2}{\epsilon}\). Let \(N = \max(N_1, N_2)\). Then \(k > N \Rightarrow \rho(Q_k, Q) < \frac{\epsilon}{2}\) and \(\rho(Q_k, Q_m) < \frac{1}{m} < \frac{\epsilon}{2}\), so \(\rho(Q, Q_m) < \epsilon\) for all such \(m\), and \(Q\) is a \(\rho\)-limit point of \(\{Q_m\}\).
(4) \((\emptyset, \rho)\) is sequentially compact: Let \(\{Q_m\}\) be any sequence \(\in \emptyset\).

**Case 1:** The sequence has infinitely many points in \(\Omega\).

Then either (a) this subsequence accumulates in \(\Omega\), and we are done, by (1)
or (b) the subsequence leaves every compact subset of \(\Omega\), and therefore has a fundamental subsequence, which converges to its equivalence class, by (2).

**Case 2:** The sequence has only finitely many points in \(\Omega\), and hence infinitely many in \(\Delta\), which must accumulate at some point of \(\Delta\), by (3).

Thus \((\emptyset, \rho)\) is compact. We have already shown in (1) one half of the second assertion. There remains only to show that \(\rho\)-convergence \(\Rightarrow R^{n+1}\) convergence.

Suppose \(Q, Q_m \in \Omega\) for all \(m\), and \(Q_m \stackrel{\rho}{\rightarrow} Q\). Then either
(a) \(\{Q_m\}\) accumulates in \(\Omega\) in the \(R^{n+1}\)-topology, or
(b) \(\{Q_m\}\) has a fundamental subsequence \(\{Q_m',\}\).

In (b), \(Q_m' \stackrel{\rho}{\rightarrow} Q' = [\{Q_m'\}] \in \Delta\) so that \(\rho(Q, Q') = 0\), which is impossible.

If \(\tilde{Q}\) is any point of accumulation of \(\{Q_m\}\), then a subsequence \(\{Q_m''\}\) must converge to \(\tilde{Q}\) in \(R^{n+1}\), so by (1) above, \(Q_m'' \stackrel{\rho}{\rightarrow} \tilde{Q}\), and thus \(\rho(Q, \tilde{Q}) = 0\). Therefore \(Q\) is the only point of accumulation, so \(Q_m \to Q\).
§3. Integral Representations of Temperatures in Subsets of \( \Omega \)

The integral representation of a temperature \( u \) by a measure on \( \Delta \) will be obtained from integral representations for \( u \) in subsets of \( \Omega \). For notational simplicity, the generic argument for these representations on subsets is given here, and will be applied in the next section to a sequence of subsets to obtain the final representation on \( \Delta \). The argument is similar to one used by Hunt and Wheeden [5] and is somewhat more transparent than Martin's original argument.

Let \( U \) be a relatively compact open subdomain of \( \Omega \), such that \( \partial U \) is regular for the Dirichlet problem in \( \Omega \sim \bar{U} \). Suppose \( u \geq 0 \) is a temperature on \( \Omega \). Let \( \tilde{u} \) be the PWB (generalized) solution of the Dirichlet problem in \( \Omega \sim \bar{U} \) for the boundary values \( u \) on \( \partial U \), 0 on \( \partial \Omega \).

Define \( u^* \) on \( \Omega \) by
\[
    u^*(P) = \begin{cases} 
        u(P) & P \in \bar{U} \\
        \tilde{u}(P) & P \in \Omega \sim \bar{U} 
    \end{cases}
\]

\( u^* \) is clearly a temperature in \( U \cup (\Omega \sim \bar{U}) \), but it is not supercaloric in general because it is not lower semicontinuous. Let \( T_0 = \inf \{ t: (x,t) \in U \} \). Then for \( t > T_0 \), since \( \partial U \) is regular, if \( P = (x,t) \in \partial U \), then
\[
    \lim_{Q \to P, Q \in \Omega \sim \bar{U}} u^*(Q) = u(P) = u^*(P)
\]
However, since all the boundary values on
\[ \partial(\Omega \sim \overline{U}) \cup \{(x,t) : t < T_0\} = \partial \Omega \cap \{(x,t) : t < T_0\} \] are 0, \( \bar{u}(x,t) = 0 \) for \( t < T_0 \). Unless \( u(x,T_0) = 0 \) for all \( (x,T_0) \in \partial U \), \( u^* \) will not be lower semicontinuous there, since then
\[
\lim_{(y,s) \to (x,T_0)} u^*(y,s) = \lim_{\delta \to 0} \left( \inf_{|(y,s)-(x,T_0)| < \delta} u^*(y,s) \right)
\]
\[
= 0 < u(x,T_0).
\]

[N.B. The notation "\( \lim \)" will indicate that the lim inf is taken over undeleted neighborhoods of \( (x,T_0) \), so that \( u^*(x,T_0) \) is itself a candidate for the infimum. This convention seems rather standard in potential theory, but differs from the usual calculus definition, which uses deleted neighborhoods of \( (x,T_0) \).]

Since \( u \) itself is an upper function for the PWB problem in \( \Omega \sim \overline{U} \), \( u^* \leq u \) in \( \Omega \sim \overline{U} \), so \( u^* \leq u \) in \( \Omega \). We define \( \hat{u}^*(P) = \lim_{Q \to P} u^*(Q) \), the lower semicontinuous regularization of \( u^* \), and write \( w = \hat{u}^* \). Then \( w \) is supercaloric. By its definition it is lower semicontinuous, and clearly \( w \geq 0 \).

Let \( C_\alpha \) denote the cone \( \{(x,t) : |x| < -\alpha t, -1 < t < 0\} \), for \( 0 < \alpha < \infty \). For any \( (x',t') \in \Omega \), let \( \overline{C}_{\alpha,h}(x',t') = (x',t') + hC_\alpha \), the cone with vertex \( (x',t') \), height \( h \), opening \( \alpha \), with axis parallel to the \( t \)-axis. It is sufficient to show that for any \( (x,t) \in \Omega \), there is a \( \delta > 0 \) such that if
\((x,t) \in C_{\alpha,h}(x',t'), \quad \overline{C_{\alpha,h}(x',t')} \subset \Omega\), for \(h < \delta\), then

\[
\int_{\partial C_{\alpha,h}(x',t')} w \, dw(x,t) \leq w(x,t)
\]

where \(dw(x,t)\) is the caloric measure on \(\partial C_{\alpha,h}(x',t')\) for the point \((x,t)\).

Clearly we need only check the points in \(\partial U\). Let \((x,t) \in \partial U\). If \(t > T_0\), since \(\partial U\) is regular, then \(u^x\) is continuous at \((x,t)\), and \(u^x = u(x,t)\). If \(C_{\alpha,h}\) is a cone in \(\Omega\) containing \((x,t)\), then

\[
\int_{\partial C_{\alpha,h}} w \, dw(x,t) \leq \int_{\partial C_{\alpha,h}} u \, dw(x,t) = u(x,t) = w(x,t).
\]

If \(t = T_0\), then \(w(x,T_0) = 0\). If

\((x,T_0) \in C_{\alpha,h}(x',t'), \quad \overline{C_{\alpha,h}(x',t')} \subset \Omega\), then the caloric measure \(dw(x,T_0)\) is concentrated on the set \(\{(x,t): t < T_0\}\)

(i.e. \(w(x,T_0)\) \([\{(x,t): t < T_0\}\] is concentrated on the set \(\{(x,t): t < T_0\}\)

\(\partial C_{\alpha,h} \cap \{(x,t): t < T_0\}\) so \(w(x,T_0) = 0 = \int_{\partial C_{\alpha,h}} w \, dw(x,T_0)\).

This shows that \(w\) is supercaloric.

By the Riesz-type decomposition stated in §0, we may write \(w = G_\mu + h\) where \(h\) is a temperature on \(\Omega\), and \(\mu = -Hw \in \mathcal{D}'(\Omega)\). Clearly, the measure \(\mu\) is supported in \(\partial U\).

**Claim:** \(h \equiv 0\) in \(\Omega\).

If \((\xi, \tau) \in \Omega\), let \(T(\xi, \tau) = \{(x,t) \in \Omega: t > \tau\ \text{and there is a path from} \ (x,t) \ \text{to} \ (\xi, \tau) \ \text{whose time-component is non-increasing}\}\). \(T(\xi, \tau)\) is clearly open. Since \(U\) is compact,
there is a finite collection of points \((\xi_j, \tau_j), \ j = 1, \ldots, N\)
such that

\[ U \subset \bigcup_{j=1}^{N} T(\xi_j, \tau_j) \quad \text{and} \quad \tau_j < \inf \{ t : (x,t) \in U \cap T(\xi_j, \tau_j) \} \]

for each \( j = 1, \ldots, N \). Let \( g(x,t) = \sum_{j=1}^{N} G(x,t; \xi_j, \tau_j) \). Since \( G(x,t; \xi_j, \tau_j) \geq C_j > 0 \) for \( (x,t) \in \overline{U} \cap T(\xi_j, \tau_j) \), there is a constant \( C > 0 \) such that \( g \geq C \) on \( \overline{U} \). Choose \( A > 0 \) so large that

\[ \inf_{U} Ag \geq \sup_{U} u \]

Then \( Ag \) is an upper function for the PWB solution \( \tilde{u} \) in \( \Omega \sim \overline{U} \), since \( \lim inf_{(x,t) \to \partial \Omega} Ag(x,t) \geq 0 \), and the choice of the constant \( A \).

In §0 it was pointed out that \( h \) is the greatest caloric minorant of \( w \). Since \( w \leq Ag \), \( h \) is a caloric minorant of \( Ag \), but since the greatest caloric minorant of each Green function is 0, and since the greatest caloric minorant of the sum is the sum of such minorants, the greatest caloric minorant of \( g \), and therefore \( Ag \), is 0. Hence \( h \leq 0 \). But since \( 0 \leq w \), 0 is a caloric minorant of \( w \), so \( 0 \leq h \).

Thus we have shown that \( w = G_\Omega \) on \( \Omega \), i.e.

\[ w(x,t) = \int_{\overline{\Omega}} G(x,t; \xi, \tau) d\mu(\xi, \tau) \]

for all \( (x,t) \in \Omega \).
§4. The Representation on $\Delta$

To obtain an integral representation on the boundary $\Delta$ constructed in §2, we use the preceding construction on an increasing sequence of relatively compact subdomains $\Omega_m$ such that $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$, and $\partial \Omega_m$ is regular for the Dirichlet problem in $\Omega \sim \Omega_m$. It is easy to construct such a sequence. For each $m$ we obtain a function $u_m$, corresponding to the regularized function $w$ of the preceding section, and a measure $\mu_m = -H u_m$, supported on $\partial \Omega_m$, such that

$$u_m(x,t) = \int_{\Omega} G(x,t;\xi,\tau) \ d\mu_m(\xi,\tau).$$

Since $\Omega \subset \Omega$ we may regard $\{u_m\}$ as a sequence of measures on the compact space $\Omega$. If the total variations $||\mu_m||$ were uniformly bounded, we could use Alaoglu's theorem to obtain a weakly convergent subsequence. However, we have essentially no information about the measures $\mu_m$, other than that they are compactly supported. To obtain a bounded sequence of measures, we rewrite

$$u_m(x,t) = \int_{\Omega} \frac{G(x,t;\xi,\tau)}{G(x_0,t_0;\xi,\tau)} G(x_0,t_0;\xi,\tau) \ d\mu_m(\xi,\tau)$$

and define a measure $d\nu_m = G(x_0,t_0;\cdot,\cdot) d\mu_m$. Then

$$||\nu_m|| = \int_{\Omega} d\nu_m = \int_{\Omega} G(P_0,Q) d\mu_m(Q) = u_m(P_0)$$

which makes sense because the functions $u_m$ can be extended
continuously to a neighborhood of $P_0$ in a similar fashion to the extension of the Green functions. Since $u$ is an upper function for the generalized Dirichlet problem which $u_m$ solves in $\Omega \sim \overline{\Omega}_m$, $u_m \leq u$ in $\Omega \sim \overline{\Omega}_m$. At this point we need a further assumption on $u$, that it can be continuously extended to $P_0$. This is no significant loss of generality, since the case of interest is the representation of a temperature on a given domain up to a certain time, although the temperature is still defined for later times. In this case $u$ will be continuous across the top boundary of the truncated domain.

Assuming $u$ is continuous at $P_0$, we obtain

$$\|\nu_m\| = u_m(P_0) \leq u(P_0)$$

so that the measures $\nu_m$ are uniformly bounded. Since the space of bounded, signed measures on $\mathcal{A}$ is the dual of $C(\mathcal{A})$, Alaoglu's theorem implies that $\{\nu_m\}$ has a weakly convergent subsequence. [It is sufficient to consider sequences, because since $\mathcal{A}$ is a compact metric space, $C(\mathcal{A})$ is separable, and then a closed ball in $C(\mathcal{A})^*$ inherits a metric topology from the weak-$\star$ topology on $C(\mathcal{A})^*$. See [2], pp. 426, 437.]

Thus we may assume that a subsequence of the $\nu_m$ converges weakly to a bounded measure $\mu$ on $\mathcal{A}$, and passing to this subsequence, we may assume that $\{\nu_m\}$ converges weakly to $\mu$. 
(1) \( \| \mu \| \leq u(P_0) \) is clear, since \( \| \nu_m \| = \int_\Omega 1 \, d\nu_m \to \int_\Omega 1 \, d\mu = \| \mu \| \).

(2) \( \text{supp } u \subseteq \Delta \):

For any \( E \) relatively compact in \( \Omega \), there is an integer \( M(E) \) such that for \( m > M(E) \), \( E \subset \Omega_m \), so \( \nu_m(E) = 0 \). Then if \( P \in \Omega \cap \text{supp } \mu \), then every neighborhood of \( P \) must have positive \( \mu \)-measure. But if \( V \) is any relatively compact neighborhood of \( P \), then \( \mu(V) = 0 \):

If \( \varphi \) is continuous and \( \geq 0 \) in \( \Omega \), with compact support in \( \Omega \), and such that \( \varphi \equiv 1 \) on \( V \), then \( \varphi \) extends continuously to \( \Omega \), and \( \mu(V) \leq \mu(\varphi) = \lim_{m \to \infty} \nu_m(\varphi) \), let \( \nu_m(\varphi) = 0 \) for \( m > M(\text{supp } \varphi) \).

Thus \( (\text{supp } \mu) \cap \Omega = \emptyset \), so \( \text{supp } \mu \subseteq \Delta = \emptyset \sim \Omega \).

Finally, since \( \nu_m \to \mu \) weakly means that for every continuous \( f \) on \( \emptyset \), \( \lim_{m \to \infty} \int \emptyset f \, d\nu_m = \int \emptyset f \, d\mu = \int \emptyset f \, d\mu \), and since the functions \( K(P, \cdot) \) has a singularity at \( P \), we need a slight modification to obtain the representation.

\( K(P, \cdot) \) is continuous on \( \emptyset \sim \{P\} \) because \( \emptyset \) was constructed so that this function would extend continuously to the boundary \( \Delta \). To handle the singularity at \( P \), let \( B(P,r) = \{Q \in \mathbb{R}^{n+1} : |P-Q| < r\} \), and suppose \( B(P,r) \subset \Omega \). By Urysohn's lemma there is a continuous function \( \varphi \) such that \( \varphi \equiv 0 \) in \( B(P, \frac{r}{4}) \), and \( \varphi \equiv 1 \) on \( \emptyset \sim B(P, \frac{r}{2}) \) then \( \varphi(Q)K(P,Q) \) is a continuous function of \( Q \) in \( \emptyset \), and for \( m > M(B(P, \frac{r}{2})) \), \( \varphi \equiv 1 \) on \( \text{supp } \nu_m \), so that
\[ u(P) = \lim_{m \to \infty} u_m(P) = \lim_{m \to \infty} \int_{\mathbb{S}} K(P, Q)d\nu_m(Q) \]
\[ = \lim_{m \to \infty} \int_{\mathbb{S}} \varphi(Q)K(P, Q)d\nu_m(Q) \]
\[ = \int_{\mathbb{S}} \varphi(Q)K(P, Q)d\mu(Q) = \int_{\Delta} K(P, Q)d\mu(Q). \]

This shows that if \( u \) is any non-negative temperature on \( \Omega \), which is continuous up to the reference point \( P_0 \), then there is a measure \( \mu \) on \( \Delta \) such that
\[ u(P) = \int_{\Delta} K(P, Q)d\mu(Q). \]

\( \S 5. \) \( \Delta \) and the Topological Boundary

If \( \Delta \) is a reasonable notion of an "ideal boundary," then we should expect that if \( \Omega \) has a sufficiently smooth topological boundary \( \partial \Omega \) in \( \mathbb{R}^{n+1} \), the preceding construction of \( \Delta \) should essentially yield \( \partial \Omega \). More accurately, the cases which this construction attempts to generalize represent a temperature \( u \) in terms of its values on the parabolic boundary \( \partial_P \Omega \), and \( \Delta \) should be more closely related to this part of \( \partial \Omega \) rather than the entire boundary. Kemper [7] previously investigated analogous representations of non-negative temperatures in domains in \( \mathbb{R}^2 \) whose boundaries are curves satisfying uniform Lipschitz (or Hölder) conditions with exponent \( \frac{1}{2} \), and his estimates for the boundary behavior of temperatures make it relatively simple to identify \( \Delta \) in this case.
More precisely, let \( \Omega = \{(x,t) : \eta_1(t) < x < \eta_2(t), \ t > 0\} \)
where \( \eta_1 \) and \( \eta_2 \) are functions such that for any \( T > 0 \), \( \eta_1 \) and \( \eta_2 \) satisfy a uniform Lipschitz condition with exponent \( \frac{1}{2} \) on \([0,T] \): i.e. for every \( T > 0 \), there exists \( C = C(T) > 0 \) such that

\[
|\eta_i(s) - \eta_i(t)| \leq C|s-t|^{\frac{1}{2}} \quad \text{for all } s,t: 0 \leq s,t \leq T.
\]

In this situation every point \((\eta_i(t),t), \ t \geq 0\), is a regular boundary point for the Dirichlet problem \([7]\).

For any \( T > 0 \), let \( \Omega_T = \{(x,t) : (x,t) \in \Omega: t < T\} \). We shall show that the nontrivial part of \( \Delta_T \), the boundary constructed by the earlier process applied to \( \Omega_T \) and a reference point \((x_0,T)\), where \( \eta_1(T) < x_0 < \eta_2(T) \), is homeomorphic to \( \partial_p(\Omega_T) \),

which in this case is \( \partial_p \Omega = \{(\eta_i(t),t): 0 \leq t < T, \ i = 1 \text{ or } i = 2\} \cup \{(x,0): \eta_1(0) < x < \eta_2(0)\} \). By the "nontrivial" part of \( \Delta_T \) we mean \( \Delta_p^T = \Delta_{\sim} [Q,\omega] \), where \( Q,\omega \) is the equivalence class in \( \Delta_T \) characterized by \( K(P,\omega) = 0 \) for all \( P \in \Omega_T \).

To state Kemper's estimates requires additional notation. First, let \( \mu \) and \( d \) be two constants such that

0 < \( \mu < \omega \), and \( d > 2C \) where \( C \) is the Lipschitz constant for \( \eta_1 \) and \( \eta_2 \). If \((y_o,s_o) = (\eta_1(s_o),s_o), i = 1 \text{ or } 2\), and

0 \leq s_o < T, define

\[
\psi((y_o,s_o),r) = \{(x,t) : (x,t) \in \partial_\Omega: |x-y_o| < rd, |t-s_o| < r^2\}
\]

and \( \psi((y_o,s_o),r) = \partial_\Omega \cap \psi((y_o,s_o),r) \).
For \( s_0 = 0, \eta_1(0) < y_0 < \eta_2(0) \), use the same definitions with \( d \) replaced by \( 1 \). We also define

\[
A((y_0, s_0), r) = (y_0 + (-1)^{1-1} rd, s_0 + r^2 (1+u))
\]

if \( y_0 = \eta_1(s_0), 0 < s_0 < T \)

\[
A((y_0, 0), r) = (y_0, r^2 (1+u)) \quad \text{for} \quad \eta_1(0) < y_0 < \eta_2(0).
\]

Kemper's estimate is the following lemma.

**Lemma 1:** There is a constant \( C > 0 \) such that for \((y_0, s_0) \in \partial_P \Omega_T\) and all sufficiently small \( r \), then

(i) if \( N \subset \mathcal{V}((y_0, s_0), \frac{r}{4}) \) is a neighborhood of \((y_0, s_0)\) in \( \Omega_T \) such that \( \Omega_T - N \) is bounded by curves satisfying a Lipschitz condition of exponent \( \frac{1}{2} \)

(ii) \( u \) is a non-negative temperature in \( \Omega_T - N \) which is continuous on \( \overline{\Omega_T - N} \), and \( u = 0 \) on \( (\partial_P \Omega_T) - N \)

then for any \((x, t) \in \Omega_T - \mathcal{V}((y_0, s_0), (1+u)^{\frac{1}{2}} \frac{r}{4})\)

\[
u(x, t) \leq C \nu((A(y_0, s_0), r)) \nu(x, t) ((\nu((y_0, s_0), r))
\]

where \( \nu(x, t) \) is the caloric measure on \( \partial_P \Omega \) with respect to \((x, t)\).

**Proof:** See [7].

To show that \( \partial_P \Omega_T \) is essentially \( \Delta_P^T \) we shall show that there is a 1-1 correspondence between fundamental sequences in \( \Omega_T \) and points in \( \partial_P \Omega \).
Lemma 2: Suppose \( \{Q_m\} \) is a fundamental sequence in \( \Omega_T \), and \( \lim_{m \to \infty} K(\cdot, Q_m) \neq 0 \). Then \( \{Q_m\} \) has a unique accumulation point in \( \partial_p \Omega_T \).

Proof:

Since \( \Omega_T \) is bounded and \( \{Q_m\} \) leaves every compact subset of \( \Omega_T \), \( \{Q_m\} \) has at least one accumulation point, and all accumulation points must belong to \( \partial \Omega \). If \( Q_m = (\varepsilon_m, \tau_m) \), then \( \limsup_{m \to \infty} \tau_m < T \); otherwise \( K(\cdot, \{Q_m\}) \equiv 0 \), so any accumulation point must be in \( \partial_p \Omega \).

Suppose \( Q \) and \( Q' \) are distinct accumulation points of \( \{Q_m\} \), and \( Q = (y_0, s_0) \), \( Q' = (y'_0, s'_0) \). Let \( N \) and \( N' \) be neighborhoods of \( Q \) and \( Q' \) respectively, \( N \subset \mathcal{V}(Q, \frac{r}{4}) \), \( N' \subset \mathcal{V}(Q', \frac{r}{4}) \) where \( r \) is sufficiently small that lemma 1 holds at both points, and \( \mathcal{V}(Q, r) \cap \mathcal{V}(Q', r) = \emptyset \).

Then for any \( Q_m \in N \), \( K(\cdot, Q_m) \) satisfies the hypothesis of lemma 1, so that for \( P \in \Omega_T \sim \mathcal{V}(Q, (1+\mu)^{\frac{1}{2}} \frac{r}{4}) \)

\[
K(P, Q_m) \leq C K(A(Q, r), Q_m) w^P(\phi(Q, r)) \\
\leq C w^P(\phi(Q, r)) \quad \text{by Harnack's inequality.}
\]

[Note: The letter "C" simply stands for a constant, and its value may be different each time it appears.]

The right side is independent of \( m \), so letting \( m \to \infty \) we get

\[
K(P, \{Q_m\}) \leq C w^P(\phi(Q, r)) \quad \text{for} \quad P \in \Omega_T \sim \mathcal{V}(Q, (1+\mu)^{\frac{1}{2}} \frac{r}{4}).
\]
Since $\partial_P^\Omega$ is regular for the Dirichlet problem,
\[ w^P(\bar{\phi}(Q,r)) \to 0 \quad \text{as} \quad P \to P' \in \partial_P^\Omega \sim \bar{\phi}(Q,r) \]
Thus $K(P',[[Q_m]]) = \lim_{P\to P'} K(P,[[Q_m]]) = 0$ for $P' \in \partial_P^\Omega \sim \bar{\phi}(Q,r)$

Similarly, for $Q_m \in N'$ we have

\[
K(P,Q_m) \leq C K(A(Q',r),Q_m) \ w^P(\bar{\phi}(Q',r)) \\
\text{for} \ P \in \partial_P^\Omega \sim \bar{\phi}(Q',(1+\mu)^{\frac{1}{2}} \frac{r}{4}) \\
\leq C \ w^P(\bar{\phi}(Q',r))
\]

and taking the limit

\[
K(P,[[Q_m]]) \leq C \ w^P(\bar{\phi}(Q',r)) \quad \text{for} \ P \in \partial_P^\Omega \sim \bar{\phi}(Q',r) .
\]

so that $K(P',[[Q_m]]) = \lim_{P\to P'} K(P,[[Q_m]]) = 0$ for $P' \in \partial_P^\Omega \sim \bar{\phi}(Q',r)$. For $r$ sufficiently small

$\bar{\phi}(Q,r) \cap \bar{\phi}(Q',r) = \emptyset$, and in this case we have shown that

$K(\cdot,[[Q_m]]) = 0$ on $\Omega_T$, which is a contradiction.

Thus $\{Q_m\}$ has a unique accumulation point in $\partial_P^\Omega_T$. //

By this lemma we are able to associate to each non-trivial (i.e., corresponding to a limit function which is not identically 0 on $\Omega_T$) fundamental sequence in $\Omega_T$ a unique point in the parabolic boundary of $\Omega_T$. This association will define a mapping from $\Delta_P^T$ into $\partial_P^\Omega_T$ if the association is actually determined by the equivalence class of the sequence. But this point is easy: if $\{Q_m\}$ and $\{Q'_m\}$ are equivalent, non-trivial fundamental sequences, then the...
sequence $Q_1, Q'_1, Q_2, Q'_2, \ldots$ formed by interlacing the sequences must also be fundamental, and hence by lemma 2 has a unique accumulation point in $\partial_p \Omega_T$. We thus have a mapping

$$\Lambda : \Delta P^T \rightarrow \partial_p \Omega_T,$$

which we may extend by the identity mapping on $\Omega_T$ to a mapping

$$\Lambda : \mathfrak{B}_P^T \rightarrow \Omega_T \cup \partial_p \Omega_T$$

where $\mathfrak{B}_P^T = \Omega_T \cup \Delta_P^T$, with the $\rho$-topology.

**Proposition:** $\Lambda$ is a homeomorphism.

**Proof:**

$\Lambda$ is continuous: the continuity on $\Omega_T$ is obvious.

Let $Q \in \Delta_P^T$.

**Case 1:** $\{Q_m\} \subseteq \Omega_T$ and $Q_m \not\rightarrow Q$. We must show that

$\Lambda(Q_m) = Q_m \rightarrow \Lambda(Q)$ in $\mathbb{R}^{n+1}$. Suppose not. Then for some $\varepsilon > 0$, there is a fundamental subsequence $\{Q_{m'}\}$ of $\{Q_m\}$ such that $|Q_{m'} - \Lambda(Q)| \geq \varepsilon$.

If $Q' = \{Q_{m'}\}$, then by lemma 2, $Q_{m'} \rightarrow Q'$ in $\mathbb{R}^{n+1}$. But $Q_{m'} \rightarrow Q' = Q = Q'$, so $Q_{m'} \rightarrow \Lambda(Q') = \Lambda(Q)$ in $\mathbb{R}^{n+1}$ a contradiction. Thus $Q_{m'} \rightarrow \Lambda(Q)$.

**Case 2:** $\{Q_m\} \subseteq \Delta_P^T$ and $Q_m \not\rightarrow Q$.

By an argument used in showing that $(\mathfrak{B}_T, \rho)$ was compact, we may choose a sequence $\{R_m\} \subseteq \Omega_T$ such that

(i) $R_m$ leaves every compact subset of $\Omega_T$

(ii) $\rho(Q_m, R_m) < \frac{1}{m}$

(iii) $|\Lambda(Q_m) - R_m| < \frac{1}{m}$
Then \( R_m \to Q \), since
\[
\rho(Q, R_m) \leq \rho(Q, Q_m) + \rho(Q_m, R_m) \quad \text{and the right side} \to 0.
\]
Now, a repetition of the argument just given in Case 1 shows that \( \Lambda(R_m) = R_m \to \Lambda(Q) \), so
\[
|\Lambda(Q) - \Lambda(Q_m)| \leq |\Lambda(Q) - R_m| + |R_m - \Lambda(Q_m)|
\]
and the right side \( \to 0 \).

The general case is an obvious consequence of cases 1 and 2.
To show that \( \Lambda \) is bijective, we first notice that lemma 1 actually gives a much stronger convergence result for non-trivial fundamental sequences than the definition alone does.

By considering \( \Omega_{T+1} = \Omega \cap \{(x,t): t < T+1\} \), we may apply lemma 1 to the functions \( K(\cdot, Q) \) considered as functions on \( \Omega_{T+1} \), for \( Q \in \Omega_T \) and obtain that the convergence of these normalized Green functions \( K(\cdot, Q_m) \), for non-trivial fundamental sequences in \( \Omega_T \), is uniform up to the top boundary of \( \Omega_T \). In particular, the convergence is uniform in a neighborhood of \( (x_0, T) \), so that if \( \{(\varepsilon_m, \tau_m)\} \) is fundamental and \( \lim_{m \to \infty} \tau_m < T \), and if \( Q = \{(\varepsilon_m, \tau_m)\} \), then
\[
K((x_0, T), Q) = 1
\]
and \( K(\cdot, Q) \) is continuous at \( (x_0, T) \). Thus \( K(\cdot, Q) \not\equiv 0 \) on \( \Omega_T \).

\( \Lambda \) is 1-1: By applying lemma 1 for a sequence of \( r_k \)'s converging to 0, we see that if \( \{Q_m\} \subset \Omega_T \) is fundamental and corresponds to \( Q \in \Delta_T \), then \( K(\cdot, Q) \) is a non-negative temperature in \( \Omega_T \), continuous at every point \( P \in \partial_P \Omega_T \sim \{\Lambda(Q)\} \), and for each such \( P \), \( K(P, Q) = 0 \).
By the argument above $K(P_0, Q) = 1$, so $K(\cdot, Q)$ is a kernel function for $\Lambda(Q)$ with respect to $(x_0, T) = P_0$, in Kemper's terminology. Kemper has shown that in domains $\Omega$ as described here, the kernel function at a given boundary point $P \in \partial_\Omega$ with respect to $P_0$ is unique [7].

Thus, if $\{Q'_m\}$ is another fundamental sequence, corresponding to $Q' \in \Delta^T_P$, and $\Lambda(Q) = \Lambda(Q')$, then $K(\cdot, Q)$ and $K(\cdot, Q')$ are both kernel functions for $\Lambda(Q)$ with respect to $P_0$, so $K(\cdot, Q) = K(\cdot, Q')$, therefore $Q = Q'$.

$\Lambda$ is onto: If $Q \in \partial_\Omega T$, then there is a fundamental sequence $\{Q_m\} \subset \Omega_T$ such that $Q_m \to Q$ in $\mathbb{R}^{n+1}$. If $Q' \neq Q_m$, since $K(\cdot, Q')$ is a kernel function at $Q$, and

$\Lambda^{-1}$ is continuous: Again, only $\partial_\Omega T$ needs to be checked.

Case 1: $\{Q_m\} \subset \Omega_T$, $Q \in \partial_\Omega T$, $Q = \Lambda(R)$, $Q_m \to Q$ in $\mathbb{R}^{n+1}$.

If $Q_m \not\to R$, then for some $\varepsilon > 0$ we can find a fundamental subsequence $\{Q_{m'}\}$ of $\{Q_m\}$ such that $\rho(Q_{m'}, R) \geq \varepsilon$. If $S = \{Q_{m'}\}$, then $Q_m \not\rho S$, and by continuity of $\Lambda$, $Q_{m'} \to \Lambda(S)$, so $\Lambda(S) = Q = \Lambda(R)$. $\Lambda^{-1} \not\to S = R$, a contradiction.

Case 2: $\{Q_m\} \subset \partial_\Omega T$, $Q_m \to Q = \Lambda(R) \in \partial_\Omega T$.

Let $Q_m = \Lambda(R_m)$, since $\Lambda$ is bijective. We must show that $R_m \not\rho R$. Suppose not. Then for some $\varepsilon > 0$, there is a subsequence $\{R_{m'}\}$ such that $\rho(R, R_{m'}) \geq \varepsilon$, for all $m'$. 

We may choose a sequence \( \{S_m''\} \) in \( \Omega_T \) satisfying

(i) \( \{S_m''\} \) is fundamental; \( \{m''\} \) is a subsequence of \( \{m'\} \);

(ii) \( \rho(S_m'', R_m'') < \frac{1}{m''} \)

(iii) \( |S_m'' - Q_m''| < \frac{1}{m''} \)

Let \( S = \lfloor \{S_m''\} \rfloor \). Then \( S \neq R \), since

\[ \varepsilon \leq \rho(R, R_m'') \leq \rho(R, S) + \rho(S, S_m'') + \rho(S_m'', R_m'') \]

and the last two terms on the right \( \to 0 \).

(iii) \( S_m'' \to Q \) in \( \mathbb{R}^{n+1} \), so \( K(\cdot, S) \) is a kernel function at \( Q \), so \( S \neq Q\. Therefore \( \Lambda(S) \in \partial \Omega_T \), and \( S_m'' \to \Lambda(S) \) in \( \mathbb{R}^{n+1} \). Now

\[ |\Lambda(S) - \Lambda(R)| \leq |\Lambda(S) - S_m''| + |S_m'' - Q_m''| + |Q_m'' - Q| \]

and the right side \( \to 0 \), so \( \Lambda(S) = \Lambda(R) \) and \( S = R \), a contradiction.

Again, the general case is now obvious, and this completes the proof that \( \Lambda \) is a homeomorphism. //
§6. Examples

The following examples give concrete illustrations of the construction in situations in which the Green function may be explicitly computed. The representation results are not new, but these computations show that these standard results can be obtained from the construction.

Example 1: \( \Omega = \{(x,t): 0 < x < \pi, t < 0\} \); \( P_o = (x_o, 0) \), \( 0 < x_o < \pi \);

Let \((\xi, \tau) \in \Omega\), and suppose \( T < \tau \). If \( \Omega^T = \Omega \cap \{(x,t): t > T\} \), and \( G_T(x, t; \xi, \tau) \) is the Green function for \( \Omega^T \) with pole \((\xi, \tau)\), while \( G \) denotes the Green function of \( \Omega \), then

\[
G(x, t; \xi, \tau) = G_T(x, t; \xi, \tau) \quad \text{for} \quad (x, t) \in \Omega^T. \quad \text{Both are 0 for} \quad t \leq \tau.
\]

Thus, by separation of variables, we obtain

\[
G(x, t; \xi, \tau) = \begin{cases} 
\sum_{m=1}^{\infty} e^{-m^2(t-\tau)} \sin mx \sin m\xi & t > \tau \\
0 & t \leq \tau
\end{cases}
\]

since the right side is the well-known expansion of \( G_T \).

Suppose \( \{(\xi_m, \tau_m)\}_{m=1}^{\infty} \) leaves every compact subset of \( \Omega \).

Then either

(i) for some \( T > -\infty \), \( \tau_m > T \), all \( m \)

(ii) \( \lim \inf_{m \to \infty} \tau_m = -\infty \).

In the first case, the argument for \( \Omega_T \) shows that the sequence is fundamental if and only if it converges to a point in \( \partial \Omega \)
(or \( \tau_m \to 0 \)). If \( \lim_{m \to \infty} \tau_m = -\infty \), then there is a subsequence \( \tau'_m \to -\infty \). In this case, rewrite the Green function

\[
G(x, t; \xi, \tau) = e^{-(t-\tau)} \sin x \sin \xi \\
\quad \left[ 1 + \sum_{m=2}^{\infty} e^{(1-m^2)(t-\tau)} \frac{\sin mx}{\sin x} \frac{\sin mx}{\sin \xi} \right]^{-1}
\]

Since \( |\sin mx| \leq m \sin x \), \( 0 \leq x \leq \pi \)

the series on the right is dominated by \( \sum_{m=2}^{\infty} m^2 e^{(1-m^2)(t-\tau)} \) which is uniformly convergent

for \( \tau < t \), and \( -0 \) as \( \tau \to -\infty \). Then

\[
\frac{G(x, t; \xi, \tau)}{G(x_0, 0; \xi, \tau)} = \frac{e^{-(t-\tau)} \sin x \sin \xi [1 + O(1)]}{e^\tau \sin x_0 \sin \xi [1 + O(1)]}
\]

As \( \tau \to -\infty \to \frac{1}{\sin x_0} (e^{-t} \sin x) \).

Clearly, the entire sequence \( \{K(\cdot, \cdot; \xi_m, \tau_m)\} \) will converge only if \( \tau_m \to -\infty \), and the behavior of \( [\xi_m] \) is irrelevant in this case.

The boundary \( \Delta_\varphi \) of \( \Omega \) thus consists of the lateral boundaries of \( \Omega \) plus an "ideal" point corresponding to the limit function

\[
ce^{-t} \sin x
\]

The representation obtained, with the kernel functions for lateral boundary points, is given in [6].
Example 2: (Half-space) \( \Omega = \{(x,t) \in \mathbb{R}^{n+1}: t < t_0 \} \);
\( P_0 = (x_0, t_0) \). The Green function for \( \Omega \) is just the fundamental solution: if \( t < t_0 \)
\[
G(x,t; \xi, \tau) = \begin{cases} 
\frac{-n}{4\pi^2} \frac{n}{2} \frac{|x - \xi|^2}{e^{4(t-\tau)}} & t > \tau \\
0 & t \leq \tau 
\end{cases}
\]
so
\[
\frac{G(x,t; \xi, \tau)}{G(x_0,t_0; \xi, \tau)} = \left( \frac{t - \tau}{t_0 - \tau} \right)^{-\frac{1}{2}} \exp \left( -\frac{|x - \xi|^2}{4(t-\tau)} + \frac{|x_0 - \xi|^2}{4(t_0 - \tau)} \right)
\]
The exponent can be rewritten
\[
-\frac{|x - \xi|^2}{4(t-\tau)} + \frac{|x_0 - \xi|^2}{4(t_0 - \tau)} = \frac{1}{4(t-\tau)(t_0 - \tau)} \left[ |\xi|^2(t-t_0) + 2x \cdot \xi(t-\tau) \right.
\]
\[
- 2x_0 \cdot \xi(t_0 - \tau) + (t-\tau)|x_0|^2 - (t_0 - \tau)|x|^2
\]
It is easy to see that if \( \tau \) remains bounded, \( \tau \geq T > -\infty \), then as \( |\xi| \to \infty \), the first term in the sum dominates that sum, so the term in brackets \( \to -\infty \), and thus the ratio goes to 0.
Suppose that \( \tau \to -\infty \) and \( \frac{\xi}{2\tau} \to y \in \mathbb{R}^n \). Then
\[
|\xi|^2 \left( \frac{t-t_0}{4(t-\tau)(t_0 - \tau)} \right) \to |y|^2(t-t_0)
\]
\[
\frac{2x \cdot \xi(t-\tau) - 2x_0 \cdot \xi(t_0 - \tau)}{4(t-\tau)(t_0 - \tau)} \to y \cdot (x-x_0)
\]
\[
\frac{(t-\tau)|x_0|^2 - (t_0 - \tau)|x|^2}{4(t-\tau)(t_0 - \tau)} \to 0
\]
Thus, if $(\xi_m, \tau_m)$ leaves every compact subset of $\Omega$ in such a way that \( \frac{\xi_m}{2\tau_m} \) converges to some $y \in \mathbb{R}^n$, then the ratio of Green functions does converge, to $e^{\frac{1}{2}(t-t_0) + y \cdot (x-x_0)}$. Moreover, this is the only method in which the ratio can converge:

Let $C_\alpha = \{(x,t): |x| < -\alpha t, \; t < 0\}$. The case in which $\tau \geq T > -\infty$ was previously considered, so we consider a sequence $(\xi_m, \tau_m)$ which leaves every compact subset of $\Omega$, and such that $\lim \inf_{m \to \infty} \tau_m = -\infty$. Then either

(i) there is a cone $C_\alpha$ and an integer $M$ such that 

$$(\xi_m, \tau_m) \in C_\alpha \; \text{for} \; m > M$$

or

(ii) for every $\alpha > 0$, and every $M > 0$, there is a point $(\xi_m, \tau_m)$ with $m \geq M$ such that $(\xi_m, \tau_m) \notin C_\alpha$.

In case (i), if $\xi_m, i$ is the $i$-th coordinate of $\xi_m \in \mathbb{R}^n$, then each of the sequences $\left\{\frac{\xi_m, i}{\tau_m}\right\}_{m=1}^\infty$ is bounded, for $m > M$, for $i = 1,2,\ldots,n$. Thus a subsequence $\{m'\}$ of $\{m \in \mathbb{Z}: m > M\}$ may be chosen so that each sequence $\frac{\xi_{m'}, i}{\tau_{m'}}$ converges, and consequently, $\frac{\xi_{m'}}{\tau_{m'}}$ converges in $\mathbb{R}^n$, to a point $y$.

If $(\xi_m, \tau_m)$ is a fundamental sequence in this case, then the entire sequence must satisfy $\frac{\xi_m}{\tau_m} \to y$: if not, then at least one of the sequences $\frac{\xi_{m'}, i}{\tau_{m'}}$ must have a subsequence
bounded away from \( y^*_i \), and since it is itself a
bounded subsequence, it would have a further subsequence
\[
\frac{\varepsilon^{m''}_{m''}}{\tau_{m''}}
\]
converging to a different limit \( y^{**}_i \), and as before,
\[
\{m''\}
\]
may be chosen so that
\[
\frac{\varepsilon^{m''}}{\tau_{m''}} \to y' \in \mathbb{R}^n \sim \{y\}.
\]
But the
limit of the ratio of Green functions, over \( m'' \), is then
e \left| y' \right|^2 (t-t_0) + y' \cdot (x-x_0),
and the sequence \((\varepsilon_m, \tau_m)\) cannot
be fundamental, since it has two subsequences with distinct
limiting functions.

Thus, if \((\varepsilon_m, \tau_m)\) is fundamental and satisfies case (i),
then
\[
\frac{\varepsilon_m}{\tau_m} \to y \text{ for some } y \in \mathbb{R}^n.
\]

If \((\varepsilon_m, \tau_m)\) satisfies case (ii) then we may choose a
subsequence \(\{m'\}\) so that
\[
\frac{\varepsilon_m}{\tau_m} \to 0.
\]
In this case, the
first term in the exponent again dominates the other terms,
and the limit function must be identically 0.

Therefore, the only fundamental sequences with non-
trivial limits are sequences \((\varepsilon_m', \tau_m')\) such that
\[
\lim_{m \to \infty} \tau_m = -\infty
\]
and
\[
\lim_{m \to \infty} \frac{\varepsilon_m}{2 \tau_m} = y \in \mathbb{R}^n \text{ exists.}
\]

The representation in this case takes the form
\[
u(x,t) = \int_{\mathbb{R}^n} e^{-\frac{1}{2} \left| y \right|^2 (t-t_0) + y \cdot (x-x_0)} d\mu(y),
\]
or, absorbing some factors into the measure to get the standard form

\[ = \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2} + y \cdot x} \, d\tilde{\mu}(y), \]

where \( d\tilde{\mu}(y) = e^{-t_0|y|^2 - x_0 \cdot y} \, d\mu(y). \)

**Example 3:** For a less trivial and somewhat more interesting region, we consider \( \Omega = \{ (x, t) : -1 < t < 1, x < 0, 0 < t < 1 \} \) with \( P_0 = (x_0, 1). \) To find the Green function of \( \Omega, \) it is necessary to consider several cases.

For \( \tau < 0, \) then \( G(x, t; \xi, \tau) = \begin{cases} 
0 & t \leq \tau \\
\frac{(4\pi)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{e^{\frac{(x-\xi)^2}{4(t-\tau)}}} & t < \tau \leq 0
\end{cases} \)

For \( t > 0, \) we extend \( G(\cdot, 0; \xi, \tau) \) as an odd function of \( x, \) and solve the resulting initial value problem in \( \{ (x, t) : 0 < t < 1 \} \). The solution is \( 0 \) for \( x = 0, 0 < t < 1 \) and agrees with \( G(\cdot, 0; \xi, \tau) \) for \( x < 0, \) so it is the Green function of \( \Omega \) in \( \{ (x, t) : t > 0 \} \cap \Omega. \) Explicitly,

\[ G(x, t; \xi, \tau) = (4\pi)^{-\frac{1}{2}} t^{-\frac{1}{2}} (-\tau)^{-\frac{1}{2}} \left[ -\int_{0}^{\infty} e^{-\frac{(x-\eta)^2}{4t}} e^{-\frac{(\eta+\xi)^2}{4\tau}} \, d\eta \\
+ \int_{-\infty}^{0} e^{-\frac{(x-\eta)^2}{4t}} e^{-\frac{(\eta-\xi)^2}{4\tau}} \, d\eta \right] \]
Expanding the exponents and replacing $\eta$ by $\frac{\eta}{2}$ makes the two integrals on the right

$$- \int_0^\infty \exp \left[ \eta^2 \left( -\frac{1}{\xi} + \frac{1}{\tau} \right) + \eta \left( \frac{x}{\xi} + \frac{\xi}{\tau} \right) + \left( \frac{x^2}{4\xi} - \frac{x^2}{4\tau} \right) \right] d\eta$$

and

$$\int_{-\infty}^0 \exp \left[ \eta^2 \left( -\frac{1}{\xi} + \frac{1}{\tau} \right) + \eta \left( \frac{x}{\xi} - \frac{\xi}{\tau} \right) + \left( \frac{x^2}{4\xi} - \frac{x^2}{4\tau} \right) \right] d\eta$$

respectively.

Replacing $\eta$ by $-\eta$ in the second integral, and factoring out the term independent of $\eta$ in both integrals gives an integral of the form

$$\int_0^\infty e^{-\alpha \eta^2 - \beta \eta} d\eta$$

which can be rewritten as

$$e^\gamma \int_0^\infty e^{-\left(\alpha \eta + \gamma\right)^2} d\eta$$

where $\gamma = \frac{\beta}{2\alpha}$.

The obvious change of coordinates gives

$$\frac{e^{\left(\beta/2\alpha\right)^2}}{\alpha} \int_\beta/2\alpha^{\infty} e^{-\theta^2} d\theta$$

Defining $F(u) = e^{u^2} \int_u^{\infty} e^{-\theta^2} d\theta$, then

$$G(x,t;\xi,\tau) = (4\pi)^{-\frac{1}{2}} t^{-\frac{1}{2}} (-\tau)^{-\frac{1}{2}} \frac{e^{\frac{x^2}{4\tau} - \frac{x^2}{4\tau}}}{\left( \frac{1}{\xi} - \frac{1}{\tau} \right)^{\frac{1}{2}}} \left[ F \left( \frac{x}{\xi} - \frac{\xi}{\tau} \right) \right]$$

$$- F \left( \frac{-x}{2(\frac{1}{\xi} - \frac{1}{\tau})^{\frac{1}{2}}} \right)$$

for $t > 0$. 
\[ \tau \geq 0: \, G(x, t; \xi, \tau) = \Gamma(x, t; \varepsilon, \tau) - \Gamma(x, t; -\varepsilon, \tau) \]

where \[ \Gamma(x, t; \xi, \tau) = \begin{cases} 
(4\pi)^{-\frac{1}{2}} (t-\tau)^{-\frac{1}{2}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} & t > \tau \\
0 & t \leq \tau 
\end{cases} \]

is the fundamental solution.

Limits of \( K(x, t; \xi, \tau) = \frac{G(x, t; \xi, \tau)}{G(x_0, 1; \xi, \tau)} \):

**Case 1:** \( \tau_m > 0 \), \( \xi_m \to 0 \)

\[
K(x, t; \xi, \tau) = \left(\frac{1}{(1-\tau)^{\frac{1}{2}}} \left[ e^{\frac{(x-\xi)^2}{4(1-\tau)}} - e^{\frac{(x+\xi)^2}{4(1-\tau)}} \right] \right) \quad (\tau < t < 1)
\]

Since the exponential is even with respect to the space variable, by the Mean Value Theorem, the difference

\[
\frac{(x-\xi)^2}{e^{4(t-\tau)}} - \frac{(x+\xi)^2}{e^{4(t-\tau)}} = \frac{(x-\theta)^2}{e^{4(t-\tau)}} \frac{(x-\theta)}{2(t-\tau)} \left( -2\xi \right)
\]

for some \( \xi < \theta < -\xi \), so

\[
K(x, t; \xi, \tau) = \frac{(1-\tau)^{\frac{1}{2}}}{(t-\tau)^{\frac{1}{2}}} \frac{e^{-\frac{(x-\theta)^2}{4(t-\tau)}}}{e^{-\frac{(x_0-\eta)^2}{4(1-\tau)}}} \frac{(x-\theta)}{2(t-\tau)} \frac{2(1-\tau)}{(x_0-\eta)} \cdot \frac{(-2\xi)}{(-2\xi)}
\]

As \( \xi \to 0 \) and \( \tau \to s \), then \( \theta \) and \( \eta \to 0 \), and the expression has
the limit
\[ \lim_{s \to 0} \left( \frac{1-s}{t-s} \right)^{3/2} \cdot \frac{\exp(-\frac{x^2}{4(t-s)})}{\exp(-\frac{x_o^2}{4(1-s)})} \cdot \frac{x}{x_o} \]

in particular, for \( s = 0 \), the limit is
\[ \frac{\exp(-\frac{x^2}{4})}{x_o} t^{-3/2} \exp(-\frac{x_o^2}{4t}) \]

**Case 2:** For \( 0 < \tau < T < 1 \), as \( \xi \to -\infty \) we obtain an asymptotic estimate

\[ K(x,t;\xi,\tau) = \left( \frac{1-\tau}{t-\tau} \right)^{3/2} \cdot \frac{\exp(-\frac{x^2}{4(t-\tau)})}{\exp(-\frac{x_o^2}{4(1-\tau)})} \cdot \frac{\exp(-\frac{\xi^2}{4(1-\tau)})}{\exp(-\frac{\xi^2}{4(t-\tau)})} \]

\[ \left[ \sinh \frac{x\xi}{2(t-\tau)} \right] \cdot \left[ \sinh \frac{x_o\xi}{2(1-\tau)} \right] \]

The third factor can be rewritten

\[ \exp \left( -\frac{\xi^2}{4} \frac{(1-t)(1-\tau)}{(t-\tau)(1-\tau)} \right) \]

and this term \( \to 0 \) as \( \xi \to -\infty \), and clearly dominates all the other factors.

**Case 3:** If \( \tau_m < 0 \) and \( \tau_m \to 0 \), with \( \xi_m \geq c > -\infty \), then the argument of the function \( F \) introduced earlier has the form

\[ \frac{\left( \frac{\xi}{\tau} \pm \frac{X}{t} \right)}{2\sqrt{\frac{1}{t} - \frac{1}{\tau}}} \sim \frac{\xi}{\sqrt{-\tau}} \quad \text{as } \tau \to 0 . \]
By the Mean Value Theorem

\[ F \left( \frac{\frac{x}{T} - \frac{\varepsilon}{T}}{2 \left( \frac{\frac{1}{T} - \frac{1}{\tau}}{\tau} \right)^{\frac{1}{2}}} \right) - F \left( \frac{-\frac{x}{T} - \frac{\varepsilon}{T}}{2 \left( \frac{\frac{1}{T} - \frac{1}{\tau}}{\tau} \right)^{\frac{1}{2}}} \right) = F'(\theta) \left[ \frac{\frac{x}{T}}{2 \left( \frac{\frac{1}{T} - \frac{1}{\tau}}{\tau} \right)^{\frac{1}{2}}} \right] \]

for some \( \theta, \frac{x}{T} - \frac{\varepsilon}{T} < \theta < \frac{-\frac{x}{T} - \frac{\varepsilon}{T}}{2 \left( \frac{\frac{1}{T} - \frac{1}{\tau}}{\tau} \right)^{\frac{1}{2}}} \), so that

\[ K(x, t; \varepsilon, \tau) = \frac{e^{-\frac{x}{4t}}}{\frac{1}{2}} \left( 1 - \frac{1}{\tau} \right)^{\frac{1}{2}} \frac{F'(\theta) \cdot \frac{x}{T} \cdot (1 - \frac{1}{\tau})^{\frac{1}{2}}}{F'(\eta) \cdot x_0 \cdot (1 - \frac{1}{\tau})^{\frac{1}{2}}} \]

If \( \frac{\varepsilon}{\sqrt{-\tau}} \) is unbounded, we need asymptotic estimates for \( F \).

L'Hospital's Rule gives that \( uF(u) \to \frac{1}{2} \) and \( u^2 F'(u) \to -\frac{1}{2} \), so writing

\[ \frac{F'(\theta)}{F'(\eta)} = \frac{2^2 F'(\theta)}{\eta^2 F'(\eta) \cdot \left( \frac{\theta}{\eta} \right)^2} \]

and checking that \( \frac{\theta}{\eta} \to 1 \) as \( \tau \to 0 \), the limit function as \( \tau \to 0 \) is still

\[ \frac{x_0^2}{4} \cdot \frac{e}{x_o^3/2} \cdot e^{-\frac{x}{4t}} \cdot e^{...} \]
This argument determines all possible limits for fundamental sequences \((\varepsilon_m, \tau_m)\) with \(\tau_m < 0\) and \(\tau_m \to 0\), since either \(\frac{\varepsilon_m}{\sqrt{-\tau_m}}\) is unbounded, or has a finite limit point, and one of the above arguments applies.

**Case 4:** \(\tau_m < 0, \varepsilon_m \to -\infty\).

The ratio defining \(K(x,t;\xi,\tau)\) contains a factor
\[
\frac{\varepsilon^2}{e^{2\tau^2}} \left[ \frac{1}{(\eta - \tau)} - \frac{1}{(1 - \frac{1}{\tau})} \right]
\]
and the quantity in brackets is always \(< 0\); this term, with \(\varepsilon^2\) in the exponent, clearly dominates all other terms, so the limit function is identically 0, for \(t > 0\), and therefore on \(\Omega\) by the maximum principle.

**Case 5:** \(\tau_m < T < 0, \varepsilon_m \to +\infty\):

The argument for a strip \([x,t): -1 < t < 0\) shows that the limit function vanishes for \(t < 0\), but for \(t > 0\), the expression involving \(F\) must be used. The asymptotic estimate is still valid, since as \(\xi \to +\infty\), while \(\tau\) is bounded away from 0, the argument of \(F\) still \(-\infty\). The expression will have a limit only if \(\tau\) converges, to some \(s\), \(-1 \leq s < 0\), giving the limit
\[
\frac{x^2}{e} \frac{x}{t^{3/2}} e^{-\frac{x^2}{4t}} \frac{(1 - \frac{1}{s})^{3/2}}{(1 - \frac{1}{s})^{3/2}} \text{ for } t > 0. 
\]

The case for boundary points \((x, -1)\) is straightforward.
References


