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NONSTATIONARY RANDOM RESPONSE OF BILINEAR HYSERETIC SYSTEMS

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Thesis Director's Signature:

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CHAPTER I
INTRODUCTION

1.1 Background

The problem of uncertainty about either properties of materials or exciting forces has led to a probabilistic approach to many engineering problems in recent years. In engineering problems the description of a process as random usually means simply that insufficient data is available to allow one to completely describe its time history. Rather, such a process is characterized by various statistical measures. For example, even though various records of actual excitation by jet noise or earthquake are available, the fact that future excitations by the same type of source cannot be expected to have the same time history gives rise to the probabilistic problem. Analysis of a large number of individual records of such a random process permits determination of the statistical measures which can be expected to also characterize future records.

The theory of the response of deterministic linear systems to random excitation is quite well developed in reference books. Various statistics of response can be calculated from statistics of the excitation. Among the most important of such response statistics are mean levels and mean squared levels. Also, if the excitation has a normal probability distribution, then the response of a linear system is also normal and one can compute such statistics as frequency of zero crossings and peak distribution.

Particularly for nonstationary response there are
other statistics which are not easily computed analytically, even for linear systems. These statistics have been investigated in some instances by empirical and approximate analytical methods. For example in earthquake engineering the most common measure of response is the maximum response of the system due to the transient excitation. Curves of these maxima for a single degree of freedom system plotted against the natural frequency or period of the system are called response spectra and are available for past strong motion earthquakes. Each of these response spectra represents the deterministic response to a simple excitation, but an ensemble average across the spectra for many samples would give a meaningful statistic of the random process. Other statistics which are meaningful for nonstationary vibration, but which are not available as exact analytical solutions, include probability distribution of the maximum response and first passage statistics. These have been studied by Brady, Shinozuka, Crandall et al., Rice and others.

The response of nonlinear systems to random excitation is a much broader and more complicated subject than the response of linear systems. Solution of the appropriate Fokker-Planck equation is the only known analytical method of obtaining exact statistics of random response of a nonlinear oscillator. Unfortunately the most general class of nonlinear second order systems for which even a stationary solution has been obtained excludes hysteretic nonlinearities.

Due to the limited scope of the exact method, various approximate analytical methods for nonlinear hysteretic systems
have been tried. One approach involves finding an approximate equivalence between the nonlinear system and some equivalent linear system, so that the statistics for the linear system can be used as approximations for those of the nonlinear system. The most common such method is the extension of the well known Krylov-Bogoliubov technique to problems with random excitation, as adapted by Caughey. Perturbation techniques have also been used to investigate the random vibrations of nonlinear systems. Predictions of the response of nonlinear systems obtained from these approximate methods can normally be expected to apply only to systems with small nonlinearities.

Another equivalence approach approximates the nonlinear hysteretic system, not by a linear system, but by a nonlinear nonhysteretic system for which statistics of response can be found from the Fokker-Planck equation. These statistics are only for stationary response, however.

Very little information is available on the nonstationary response of hysteretic systems. The empirical work that has been reported deals principally with the response spectra formulation mentioned above for actual earthquakes and pseudo-earthquakes. The concept of equivalent linearisation has been discussed in this context, but only as a means of interpreting the empirical results.

Bilinear hysteresis has been the most widely studied type of hysteretic nonlinearity. This is no doubt due to the inherent simplicity of the system, since the nonlinearity is completely characterized by two parameters—yield level and
slope ratio. At least two fairly extensive analog computer studies of the stationary random response of the bilinear hysteretic oscillator have been conducted. The empirical results have also been compared with the results of some of the approximate analytical methods mentioned above.

1.2 Scope

This study deals with the response of a bilinear hysteretic oscillator which is initially at rest and is then suddenly subjected to a stationary, white noise excitation with normal probability distribution. In particular this study deals with the time interval while the response is building from the zero initial level to the final stationary level.

The basic approach to the problem is empirical—that is, the system is subjected to each member of an ensemble of excitation time histories, and response statistics are computed from averages across the resulting ensemble of response time histories. This empirical approach is distinct from what will be called analytical approaches wherein response statistics are predicted (either exactly or approximately) from knowledge of excitation statistics without the use of individual time histories of excitation or response. The reason for using the empirical approach to the present problem is the lack of even approximate analytical approaches which have been verified for nonstationary response of hysteretic systems.

The specific objectives of the present study are;

(1) By use of a digital computer, to empirically investigate the transient response of the bilinear hysteretic single mass oscillator to random excitation.
(2) To furnish some guidelines as to when the response of such a system can be considered to be essentially stationary so that the more extensive literature dealing with this situation can be applied, and

(3) To investigate the applicability of some approximate analytical techniques based on the concept of equivalent linearisation.

The response statistics studied here are of three distinct types:

(a) Mean squared level
(b) Probability distribution
(c) Extreme value statistics

The determination of these statistics involves ensemble averages of the response at many instants of time.

Systems with two particular values of the slope ratio are given primary emphasis here. One system is nearly elastoplastic and the other is midway between a linear system and an elastoplastic system. The particular slope ratios considered are the same as those in reference 22, so that those results for stationary response apply directly as limiting values of the type (a) and (b) statistics in the present study. Knowledge of these limiting values serves as a guide in interpreting the results of the transient response. Various values of yield level are used and systems with and without viscous damping are both included. The transient response levels of the elastoplastic system are also determined, as another limiting condition. In addition, the transient response levels determined for the hysteretic systems are compared with the transient levels
which can be analytically computed for corresponding linear systems.

1.3 Outline

Chapter II describes the digital computer simulation of the problem. This simulation consists of two major parts. First it was necessary to simulate samples of white noise excitation with a normal probability distribution, by using a sequence of numbers given by a "random number generator". Results are shown to verify that the procedure used does give an excitation process with the desired properties. The second part of the digital simulation consists of the procedure adopted to integrate the equation of motion of the bilinear system in order to obtain time histories of response to the generated time histories of excitation.

Chapter III presents the results of the study of mean squared levels of response. The results are presented in various forms to illustrate the effect of various system parameters on the rate of build up of mean squared response. One form of presentation shows the time required for the response of various systems to reach a specified percentage of the corresponding stationary response levels.

Two types of probability distribution for the response of the bilinear systems are presented in Chapter IV. The first type is simply the probability that the response at some instant of time exceeds a given level. The second distribution is the probability that the maximum response up to some instant of time exceeds a given level. Thus, this chapter includes the empirical values determined for the
statistics of type (b) and (c) mentioned above. The extreme value problem is compared with the problems previously considered by Gumbel and Lieblein. The empirical results are compared with a simplified theoretical Gumbel's distribution.

Chapter V discusses the idea of equivalence between the bilinear hysteretic system and single degree of freedom oscillators. Attempts are made to find coefficients in the linear systems which approximately match the nonstationary mean squared response of the linear and nonlinear systems. Constant coefficient systems are considered with only the damping coefficient chosen for equivalence, and with both damping and stiffness chosen for equivalence. In addition, a variable coefficient linear system is considered wherein both damping and stiffness are functions of the mean squared level of response, so vary with time. Use of the Krylov-Bogoliubov analytical method to determine the coefficients in the constant coefficient system is also discussed.

Chapter VI summarizes the general characteristics empirically determined for the nonstationary response of the bilinear hysteretic oscillators with random excitation.
CHAPTER II
DIGITAL COMPUTER SIMULATION OF PROBLEM

2.1 Description of system

A mechanical system which exhibits a bilinear hysteretic restoring force is shown in fig. 2.1. The equation of motion for this system can be written as:

\[ mx'' + cx' + (k_1 + k_2) \theta(x) = G(t) \]

or

\[ x'' + 2 \beta_0 \omega_0 x' + \omega_0^2 \theta(x) = G(t)/m \]  \hspace{1cm} (2.1)

where \( \omega_0 = \sqrt{(k_1 + k_2)/m} \), small amplitude undamped natural circular frequency

\( \beta_0 = c/2m\omega_0 \), small amplitude fraction of critical damping

\( \theta(x) \) is a nondimensionalised bilinear hysteretic restoring force as shown in fig. 2.2 for \( \alpha = k_2/(k_1 + k_2) \)

\( G(t) \) is the excitation, which is a white stationary Gaussian process, and dots denote mean square derivatives with respect to time of the random process \( x(t) \).

No exact solution for the statistics of the response of such a hysteretic system to random excitation have yet been obtained by analytical techniques. Thus, it is necessary to use some experimental technique. Some of the techniques available are:

(1) A mechanical model
(2) An electrical analog of the system
(3) Numerical integration of equation of motion.

Method 2 was extensively studied by Iwan and Lutes.
for the stationary response of such a bilinear system. Method
3, using a digital computer for numerical integration, was
chosen for the study of nonstationary response. Digital com-
puters are well suited to computation of the numerous aver-
ages needed in such a study, whereas analog computers are
usually suited to computation of time averages such as can be
used in studies of a stationary ergodic process. The white,
stationary and Gaussian excitation, G(t) was simulated by
suitable pseudo-random numbers so it will be referred to as
a digitally simulated white noise.

2.2 Excitation

A pseudo-random number generator has been used to
provide values of the excitation G(t). Strictly speaking, a
sequence of random numbers could only be generated by some
random (nondeterministic) procedure. A digital computer, on the
other hand, can only perform deterministic manipulations, so
that an irregular sequence called pseudo-random numbers is
used as an approximation of a random sequence. Most, if not all,
simple arithmetic schemes for producing pseudo-random numbers,
actually produce a periodic sequence. That is, the sequence
eventually repeats itself. Thus the first problem in choosing
a procedure to produce random numbers is to assure a long
period before the sequence repeats. Number theory discloses
different ways of obtaining long periods with simple arithmetic
procedures, of course the results must be studied statistically
before they can be assumed to be an approximation of a random
sequence. Various techniques have been suggested for gener-
ating pseudo-random sequences. J. Von Neumann's suggested
(19)
midsquare method was replaced by congruence methods due to computer time. The first additive congruence procedure was proposed by Fibonacci using the sequence

\[ U_{n+1} = U_n + U_{n-1} \pmod{m} \]  

(2.2)

where \( m \), the modulus, is typically \( 2^p \) in a binary machine or \( 10^d \) in a decimal machine, where \( p \) is the number of bits per word and \( d \) is the word size of machine. Though this method has a long period of \( 3 \cdot (2^p - 1) \), it fails to have statistically good results. The sequence runs up and down and shows correlation.

A multiplicative congruence procedure taking the form (2.3) was suggested:

\[ U_{n+1} = A U_n + B \pmod{m} \]  

(2.3)

2.3 Power series method

When \( B = 0 \), equation 2.3 is a pure multiplicative procedure and is called a power residue method. The power residue method is in many ways superior to other methods and is entirely satisfactory if used properly. The two main ideas involved are congruences and power residues. The congruences permit working with smaller numbers. The power residue refers to successive powers of a number

\[ U_n = A^n \pmod{m} \]  

(2.4)

with appropriately selected \( A \) and \( m \). Equation 2.4 is the same as equation 2.3 with \( B = 0 \) and \( U_0 = 1 \), as can be shown by induction. Lehmer originally suggested to choose \( m \) to be a large prime \( p \) and \( A \) a primitive root of \( p \). However, it is easier, faster and for most purposes just as good to let \( m \) represent the word size of the computer and to choose \( A \) so as to assure a long period for the sequence of equation 2.4. If the terms of
$A^n(\text{mod } m)$ are multiplied by a constant $a$ which is relative prime to $m$ (meaning that the greatest common divisor of $a$ and $m$ is 1, written $(m,a) = 1$) then the resulting sequence is still of the same length as $A^n(\text{mod } m)$, but with a different starting point $a$ instead of $A$. The choice of $A$ is the main requirement to have a long sequence.

Note that $A$ should be odd, and all odd numbers are in the form of $8j \pm 1$ or $8j \pm 3$. One can observe that $8j \pm 1$ is a divisor of $2^{p-3}$ while $8j \pm 3$ is not and hence to have the maximum order $A$ has to be in the form of $8j \pm 3$. This leads to choosing any odd integer $a$ for the starting point, an integer $A$ of the form $8j \pm 3$ and computing $aA^n(\text{mod } 2^p)$.

The modified form of equation 2.4 will be

$$U_n = aA^n(\text{mod } m) \quad (2.4a)$$

and

$$U_{n+1} = AU_n(\text{mod } m) \quad (2.5)$$

The value of $U_0$ (initial starting number of the sequence) should also be odd and relative prime to $m$, i.e. $(m,U_0) = 1$. If this is not the case then the common factor will be repeated in all terms, and worse, the period will be shortened due to reduced modulus of $m/U_0$ instead of $m$. This will affect computer capacity. Thus one uses one multiplication per number and takes the lower digits of the product for the next number. Note that the calculation of power residues must be done by fixed point integer arithmetic, and division by $m$, retaining only the remainder, is implied by the $(\text{mod } m)$ reduction. The desired output is a sequence of numbers uniformly distributed in the interval $(0,m)$, and an additional division by $m$ converts the number to an appropriate unit interval. It has been shown that
the above procedure will produce $2^{p-2}$ numbers before repeating.

The value of $m$ for the Burroughs B-5500 computer used in this study was $2^{47}$, the value of $A$ was selected by trial and error so as to give the desired statistics for relatively short sequences of numbers and satisfying the $8j+3$ requirement. The value of $A$ selected was 101001001101. The value of $U_0$ can be any odd integer and the product of $A$ and $U_n$ was done in double precision so as not to lose any digits.

2.4 Statistical considerations

The statistical properties desired for these numbers $U_n$ are exactly those that would result if the $U_n$ were obtained by an idealised chance device which selected points from the unit interval independently and with all points $0 \leq x \leq 1$ equally likely. As the sequence of numbers, produced by a computer are completely determined by the starting value, clearly these numbers cannot be random in this sense and hence the output of these numbers should be checked for the desired properties.

Various statistical tests applied to this sequence were

(1) Probability distribution and the first two moments
(2) Correlation

The value of $U_n$ should be uniformly distributed in the unit interval regardless of the number of values. This test was made by dividing the unit interval into subintervals and counting the number of $U_n$ in each subinterval. The cumulative probability distribution was plotted as shown in fig. 2.3 for various sequences and it showed a good agreement with the
desired theoretical distribution. The statistical properties of expected values and standard deviation should be 0.500 and 0.0833, respectively, for uniform random numbers. Table 2.1 summarizes some of the values of the arithmetic mean (AM) and standard deviation (SD) for the sequences used in this study.

It would be desirable to have independent random numbers. However, since congruence procedures use preceding numbers to generate following numbers, this is obviously impossible. The best that can be strived for, in this regard, is lack of correlation between numbers.

Consider the autocorrelation coefficient $C_h$ defined by

$$C_h = \frac{1}{n} \sum_{n=1}^{N} u_n u_{n+h} \tag{2.6}$$

By statistical theory for independent random numbers it can be shown that $C_h$ is approximately normally distributed, with AM = 0.25 and SD = $0.22/\sqrt{N}$ for $h > 0$ and with AM = 1/3 and SD = $0.30/\sqrt{N}$ for $h = 0$. The results obtained from equation 2.5 for different values of $h$ and $N$ showed a good agreement to the normal distribution and their statistical properties of AM and SD are summarized in table 2.2.

From the above results it was confirmed that the numbers produced by the subroutine were uniform in the unit interval, and were sufficiently uncorrelated.

2.5 Normal distribution

A random variable $Y$ with a probability distribution function $F_Y(y) = \text{prob}(Y \leq y)$ can be obtained from a random variable $X$ with a uniform distribution, if the inverse of the function $F_Y$ is known. Since
\[ F_X(x) = \text{prob}(X \leq x) = x \quad \text{for } 0 \leq x \leq 1 \]

for a uniform distribution, taking \( Y = F_Y^{-1}(x) \) where \( F_Y^{-1} \) is the inverse of \( F_Y \) gives

\[ \text{prob}(Y \leq y) = \text{prob}(X \leq F_Y(y)) = F_Y(y) \quad (2.7) \]

If \( Y \) is to have a normal distribution, however, then \( F_Y \) has the form

\[ F_Y(y) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{y - \mu}{\sqrt{2} \sigma} \right) \right) \quad (2.8) \]

where \( \text{erf} \) denotes the error function:

\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2/2) \, dt \quad (2.9) \]

Various approximations for the inverse of the \( F_Y \) have been suggested and compared by such criteria as accuracy, speed, and memory requirements. Leaning towards accuracy as the major criterion, a direct method proposed by Muller was used in this study. This method is based on the fact that if \( X \) and \( Z \) are independent random variables, with \( X \) having a uniform distribution from 0 to \( 2\pi \) and with \( Z \) having the normalized Rayleigh distribution

\[ F_Z(z) = 1 - \exp(-z^2/2) \quad \text{for } 0 \leq z < \infty \]

then \( Y_1 = Z \sin(X) \) and \( Y_2 = Z \cos(X) \) each have a normal distribution with mean zero and standard deviation one, and \( Y_1 \) and \( Y_2 \) are uncorrelated. The inverse of \( F_Z \) is easily shown to be

\[ F_Z^{-1}(s) = \sqrt{-2 \ln(1-s)} \quad \text{for } 0 \leq s < 1 \quad (2.10) \]

To apply the method to the present problem, let \( U_n \) and \( U_{n+1} \) be two random numbers from the uniform distribution, then a corresponding pair \( G_n \) and \( G_{n+1} \) of normal random numbers with zero mean and unit variance is

\[ G_n = \sqrt{-2 \ln U_n} \cos(2\pi U_{n+1}) \]
\[ G_{n+1} = \sqrt{-2 \ln U_n} \sin(2\pi U_{n+1}) \quad (2.11) \]
Assuming that the $U_n$ sequence is independent, this gives a $G_n$ sequence with a normal probability distribution and having
\[
E[ G_n | I=0 ] = \begin{cases} 
1 & \text{if } n=m \\
0 & \text{if } n \neq m 
\end{cases}
\]
and
\[
E[ G_n G_m ] = \begin{cases} 
1 & \text{if } n=m \\
0 & \text{if } n \neq m 
\end{cases} \quad (2.12)
\]

Since normal random variables are independent if and only if they are uncorrelated, equation 2.12 says that the $G_n$ are independent and each has zero mean and variance one. Statistical results obtained for digitally simulated $G_n$ are summarized in table 2.3 and corresponding cumulative distributions are plotted in fig. 2.4.

2.6 White noise

Various methods could be used to generate an approximation to a white noise random process from a sequence of normal random numbers. The procedure used in this study is to take $G(t) = G_n$ for $t_n < t < t_{n+1}$ and $b = t_{n+1} - t_n$ (a constant). The entire normal sequence then generates one very long time history of $G(t)$ and, since the process is "stationary," a set of shorter segments from this time history can be used as an ensemble of $G(t)$ time histories. For the $G(t)$ process described above the autocorrelation function is
\[
E[G(s)G(t)] = \begin{cases} 
1 & \text{if } s \text{ and } t \text{ are in the same time interval} \\
0 & \text{otherwise} 
\end{cases} \quad (2.13)
\]
The autocorrelation function in the $s$-$t$ plane for the process $G(t)$ is shown in fig. 2.5.

For a white noise process with power spectral density of $D/\pi$ per radian for all frequencies, the autocorrelation function is $2D\delta(t-s)$, involving the dirac delta function. One
notes that if \( b \) is sufficiently small as compared to other
time parameters in the problem then \( G(t) \) nearly specifies the
condition of being uncorrelated with itself at distinct values
of time.

As the \( G(t) \) process is not covariant stationary, since
the autocorrelation function depends on both \( s \) and \( t \), one cannot
compute a power spectral density for \( G(t) \) as the Fourier
transform of the stationary autocorrelation function. There
are several approaches to obtain an equivalent power spectral
density. One way is to equate integrals of the autocorrelation
function of the \( G(t) \) process with those for white noise. Using
equation 2.13

\[
\int_{t_i}^{t_i+nb} \int_{t_i}^{t_i+nb} E[G(s)G(t)]dsdt = nb^2 \quad (2.14)
\]

and for white noise

\[
\int_{t_i}^{t_i+nb} \int_{t_i}^{t_i+nb} 2D\delta(t-s)dsdt = 2Dnb \quad (2.15)
\]

Equating equation 2.14 with equation 2.15 this gives

\[
D/\pi = b/2\pi \quad (2.16)
\]

Another approach to determine the equivalent power
spectral density for \( G(t) \) involves computing the levels of
stationary response of a linear system excited by \( G(t) \). For
a linear system described by

\[
x'' + 2\beta_0 \omega_0 x' + \omega_0^2 x = G(t) \quad (2.17)
\]

where \( G(t) \) is white noise having power spectral density \( D/\pi \),
the stationary response is given by

\[
\omega_0^2 E[x^2] = E[x'^2] = D/2\beta_0 \omega_0 \quad (2.18)
\]

Since the \( G(t) \) process is not covariant stationary
one cannot expect the response \( x(t) \) of a linear system to be
covariant stationary. However, one can specify that $E[x^2(t)]$ and $E[x'^2(t)]$ should take on the same values at each discrete time $t_n$. Using this condition the exact analytical solution for the linear system excited by the process $G(t)$ which is constant within any time increment can be obtained.

The solution of equation 2.17 with this excitation can be written as

$$x = x_0 k(t) + x'_0 h(t) + \int_0^t h(t-\tau) d\tau$$

(2.19)

where $x_0$ and $x'_0$ are initial conditions for displacement and velocity respectively, $h(t)$ is the response due to unit velocity impulse and is given by

$$h(t) = \exp(-\beta_0 \omega_0 t) \frac{\sin \omega_0 \sqrt{1-\beta_0^2} t}{\omega_0 \sqrt{1-\beta_0^2}}$$

(2.20)

and, $k(t)$ is the response due to unit displacement impulse and is given by

$$k(t) = \exp(-\beta_0 \omega_0 t) [\cos \omega_0 \sqrt{1-\beta_0^2} t + \frac{\beta_0}{\sqrt{1-\beta_0^2}} \sin \omega_0 \sqrt{1-\beta_0^2} t]$$

(2.21)

Substituting for $h(t)$ and evaluating the integral of equation 2.19 gives

$$x(t) = x_0 k(t-t_n) + x'_0 h(t-t_n) + G[1-k(t-t_n)]/\omega_0^2$$

(2.22)

and differentiating equation 2.22

$$x'(t) = x'_0 k'(t-t_n) + x'' h'(t-t_n) - G k(t-t_n)/\omega_0^2$$

(2.23)

Since $G$ is uncorrelated with itself, one can assume that it is uncorrelated with $x_0$ and $x'_0$. The expected values of $x^2$, $x'^2$ and $xx'$ can then be obtained by using the equations 2.22 and 2.23 as
\[ E[x^2(t)]=E[x_0^2]k^2(t) + E[x_0^*^2]h^2(t) + 2E[x_0x_0^*]k(t)h(t) + \frac{E[G^2][1-k(t)]}{\omega_0^4} \]

\[ E[x^*^2(t)]=E[x_0^2]k'^2(t) + E[x_0^*^2]h'^2(t) + \frac{2E[x_0x_0^*]h'(t)k'(t)}{\omega_0^4} \]

and

\[ E[x(t)x^*(t)]=E[x_0^2]k(t)k'(t) + E[x_0^*^2]h(t)h'(t) + \frac{E[x_0x_0^*]k(t)h'(t) + k'(t)h(t)}{\omega_0^4} \]

\[ -E[G^2][1-k(t)]k'(t)/\omega_0^4 \quad (2.24) \]

where \( t=t-t_n \)

(2.25)

For stationary response we want \( E[x^2], E[x^*^2] \) and \( E[xx^*] \) to be the same at time \( t_n \) and \( t_n+1 \). Imposing this condition on equation 2.24 with \( t-t_n=b \), gives a set of three simultaneous equations with coefficients expressed in terms of the unit impulse functions \( h(t) \) and \( k(t) \) and their derivatives with respect to time

\[ E[x^2][k^2(b)-1] + E[x^*^2]h^2(b) + 2E[xx^*]k(b)h(b) = \]

\[ -E[G^2][1-k(b)]^2/\omega_0^4 \]

\[ E[x^2]k'^2(b) + E[x^*^2][h'^2(b)-1] + 2E[xx^*]k'(b)h'(b) = \]

\[ -E[G^2]k'^2(b)/\omega_0^4 \]

and

\[ E[x^2]k(b)k'(b) + E[x^*^2]h(b)h'(b) + E[xx^*]h(b)k'(b) + k(b)h'(b)-1 = \]

\[ E[G^2]k'(b)[1-k(b)]/\omega_0^4 \quad (2.26) \]

The coefficients in equation 2.26 were expanded in power series and the solution obtained for \( E[x^2] \) and \( E[x^*^2] \) as

\[ E[x^2]=D(1-\omega_0^2b^2/12+9\beta_0^3\omega_0^3b^3(1-\beta_0^2)+O(b^4))/2\beta_0\omega_0^3 \]

and

\[ E[x^*^2]=D(1-\omega_0^2b^2(1+4\beta_0^2)/12+O(b^4))/2\beta_0\omega_0 \quad (2.27) \]

Comparison of equations 2.18 and 2.27 indicates once more that \( G(t) \) is not white noise, but since the correction term is of the second order in \( \omega_0b \) and if \( \omega_0b \) is sufficiently
small then the response to $G(t)$ is approximately the same as if the excitation were white with power spectral density as given by equation 2.16.

The equation of motion for a linear system was integrated by using the linear acceleration method over relatively long periods of time ($T_0$) and the AM of five samples were considered so that the effect of initial conditions would be minimized. Time averages of $x^2$ and $x'^2$ were used as estimates of the stationary expected values of these quantities. Various values of the normalized time increment $\omega_0 b$, the normalized integration time $\omega_0 T_0$, and the damping value $\beta_0$ were used. Table 2.4 compares $b/\pi$ (equation 2.16) with the power spectral density of white noise which would give expected values of $x^2$ and $x'^2$ equal to the observed average values. The comparisons are seen to be quite good, particularly when $\omega_0 b$ is small.

From the above results for the linear system it was concluded that $G(t)$ as defined by $G(t) = G_n$ for $t_n < t < t_{n+1}$ is an acceptable simulation of white noise for this study.

2.7 Integration scheme

The equation of motion for the system considered is

$$x'' + 2\beta_0 \omega_0 x' + \omega_0^2 \Theta(x) = G(t)/m \quad (2.1)$$

The system is assumed to be initially at rest, then suddenly excited by a Gaussian, stationary white noise. Because of the nonlinear and hysteretic character of the restoring force $\Theta(x)$, the solution of equation 2.1 cannot be obtained in closed form. However, an exact stepwise integration is possible due to the piecewise linear character of the resistance deformation relationship and with the following assumptions:
1. Within any time interval the exciting force is constant.

2. The slope of \( \theta(x) \) can change only at the ends of the discrete time intervals of length \( b \).

For simplicity denote \( \theta(x(t_n)), x(t_n) \) and \( x'(t_n) \) by \( \theta_n, x_n \) and \( x_n' \), respectively. It is desired to evaluate the corresponding quantities after an increment of time \( \bar{t} (\bar{t} < b) \).

The displacement \( x(t) \) and spring force \( \theta(t) \) at any time \( t \) can be expressed as

\[
x(t) = x_n + \bar{x}(\bar{t})
\]

and

\[
\theta(t) = \theta_n + J\bar{x}(\bar{t})
\]

where \( J \) is a constant with value 1 if the system is not yielding during the time interval, and \( \alpha \) if the system is yielding. The velocity \( x'(t) \) and acceleration \( x''(t) \) can be obtained by differentiating equation 2.29 with respect to \( t \):

\[
x'(t) = \dot{x}(\bar{t})
\]

and

\[
x''(t) = \ddot{x}(\bar{t})
\]

The equation of motion 2.1 becomes

\[
\dddot{x} + 2\beta_0 \omega_0 \ddot{x} + J\omega_0^2 \dot{x} = G_n - \theta_n \omega_0^2
\]

The initial conditions of this equation are obtained from equations 2.29 and 2.31 as

\[
x(0) = 0 \quad \text{and} \quad \ddot{x}(0) = \ddot{x}_n
\]

A solution for the equation 2.33 with initial conditions (2.34) may now be obtained. One of two expressions will result, depending on whether the value of \( J \) is 1 or \( \alpha \). The displacement and velocity at the end of an increment \( b \) can be written as
\[ x_{n+1} = x_n + x^*_n \frac{\exp(-\beta_0 \omega_0 b) \sin \omega_d b}{\omega_d} \]
\[ (\omega_0^2 - c_n) \{ \exp(-\omega_0 \beta_0 b) [\cos \omega_d b + \beta_0 \omega_0 (\sin \omega_d b) / \omega_d] - 1 \} / \]
\[ (\omega_d^2 + \beta_0^2 \omega_0^2) \]

and
\[ x^*_{n+1} = x^*_n \exp(-\beta_0 \omega_0 b) \{ \cos \omega_d b - \beta_0 \omega_0 (\sin \omega_d b) / \omega_d \} \]
\[ + (G_n - \omega_0^2 c_n) \{ \exp(-\beta_0 \omega_0 b) \sin \omega_d b \} / \omega_d \]

where
\[ \omega_d = \omega_0 \sqrt{1 - \beta_0^2} \]

Equation (2.35)

One way to check the numerical procedure used for computing the response of a bilinear system is to apply it to a linear system for which analytical results are available. This was done for the same linear system for which empirical results have been obtained by the linear acceleration method as given in Table 2.4. The long samples of \( G(t) \) were used and AM of five samples were taken and power spectral density was obtained and compared with those of Table 2.4. The results are in good agreement with each other. The time averages of \( x^2 \) were compared with expression 2.18. It was found for \( \omega_0 b = 0.125 \) and for eight samples of length \( 400 / \omega_0 \) that the AM of the numerically computed response agreed with the analytical prediction within 1.36% while the maximum individual variation was 30.6%.

The inaccuracy arises due to two main factors: (1) the excitation is not white noise, and (2) the statistics of a stationary signal are determined by using averages over some finite time interval called a sampling time. Let \( S^2 \) denote the time average mean squared value for a sampling time of \( T \) seconds. Evaluating \( S^2 \) for a large number of samples gives a random set of \( S^2 \) values. It was shown by Bendat that the expected value of the set of \( S^2 \) values in fact is the true
mean squared value. The variance of the set of $S^2$ gives an indication of the probable error in using one sample value of $S^2$ to estimate the mean squared value of the stationary signal. The variance of $S^2$, above, is about 12.02% of the true mean squared value, hence one can expect to be off as much as 12% when using $T=400/\omega_0$ and one sample time history.
CHAPTER III

MEAN SQUARED RESPONSE

3.1 Linear system

It will be desirable to begin by summarizing some important results for a simple linear oscillator subjected to white noise. It will then be possible to compare the results for nonlinear systems with those for the simple linear systems.

Consider the system described by

\[ x'' + 2\beta_1 \omega_1 x' + \omega_1^2 x = G(t)/m \quad (3.1) \]

Caughey and Stumpf showed that the mean squared levels of transient response of this system with zero initial conditions and with a white excitation having a power spectral density of D/\pi at all frequencies is given by

\[
(\sigma_x/F)^2 = \pi \omega_0^3 [1 - \exp(-2\beta_1 \omega_1 t)\{1 + \beta_1 \omega_1 (\sin 2\omega_d t)/\omega_d
+ 2\beta_1^2 \omega_1^2 (\sin^2 \omega_d t)/\omega_d^2\}/4\beta_1 \omega_1^3)
\]

(3.2)

and

\[
(\sigma_x/\omega_0)^2 = \pi \omega_0 [1 - \exp(-2\beta_1 \omega_1 t)\{1 - \beta_1 \omega_1 (\sin 2\omega_d t)/\omega_d
+ 2\beta_1^2 \omega_1^2 (\sin^2 \omega_d t)/\omega_d^2\}/4\beta_1 \omega_1^3)
\]

(3.3)

where \[ \omega_d = \omega_1 \sqrt{1 - \beta_1^2} \] (3.4)

and

\[ F = \sqrt{(2D/\pi m^2 \omega_0^3)} \] (3.5)

The term F is introduced because it is convenient to characterize the exciting force by a parameter with dimension of length, so that the ratios of response to excitation can be plotted in dimensionless form. The term \( \omega_0 \) will be taken as the small amplitude resonant frequency of the nonlinear system as indicated in the previous chapter. For a constant excitation power spectral density of D/\pi this factor is like a measure of
the effective amplitude of the excitation force divided by the small displacement spring constant $m\omega_0^2$.

One can note that each of the transient mean squared responses has decaying oscillatory functions plus a nonoscillatory portion which approaches exponentially to the stationary response level. The nonoscillatory parts of equation 3.2 and 3.3 are given by

$$\left(\frac{\sigma_{xn}}{F}\right)^2 = \pi \omega_0^3 \frac{1 - \exp(-2\beta_1 \omega_1 t)}{(1 - \beta_1^2)} / 4 \beta_1 \omega_1^3 \tag{3.6}$$

and

$$\left(\frac{\sigma_{x^2}}{F\omega_0}\right)^2 = \pi \omega_0 \frac{1 - \exp(-2\beta_1 \omega_1 t)}{(1 - \beta_1^2)} / 4 \beta_1 \omega_1 \tag{3.7}$$

For reasons discussed in the following paragraphs the empirical results for the nonlinear system are presented in terms of combination time and ensemble averages. The corresponding analytical time average expressions for the linear system of equation 3.1 involve integrals of equations 3.2 and 3.3 with respect to time. These time averages can be evaluated exactly, but are closely approximated by the simpler expressions of 3.6 and 3.7. Sample comparisons of the analytical time averages and the nonoscillatory expressions in 3.6 and 3.7 showed differences of less than 2% of the stationary response levels.

Strictly speaking one should use simple ensemble averages of $x^2(t)$ and $x^2(t)$ to obtain estimates of the nonstationary expected values of these quantities for some particular $t$. The problem with this approach is that very large numbers of samples are required to obtain statistical accuracy, particularly if $x(t)$ is a narrow band process, as in the case of the response of a lightly damped system. The approach used in this study is to estimate $E[x^2(t)]$ and $E[x^2(t)]$ by a
combination of time and ensemble averages. The time average is taken over a time interval of length approximately $\pi/\omega_0$, centered about $t$, and then an ensemble average over the total number of samples is taken. The approximation of $E[x^2(t)]$ and $E[x'^2(t)]$ by such combination averages will be denoted by $\bar{x}^2(t)$ and $\bar{x}'^2(t)$, respectively.

The time averaging, of course, obscures some of the decaying oscillatory behavior of $\sigma_x^2$ and $\sigma_{x'}^2$, but it was felt that this oscillatory behavior was not of primary importance to this study. To check the method of averaging, the linear system with $\beta_0 = 0.10$ was integrated by the same procedure to be used for the bilinear system, with $b = 25/\omega_0$. The responses were calculated using an averaging time $\pi/\omega_0$ and several sample sizes. Figures 3.1 and 3.2 show the mean squared response for displacement and velocity of such a system based on an ensemble of 80 time histories. One can observe that the combination averages are in good agreement with Caughey and Stumpf’s analytical solution. It was concluded, based on empirical investigation of the variability of the estimates obtained using ensembles of different sizes and on the analysis in the following section that 80 samples would be adequate for estimating the mean squared responses, using the combination averages.

3.2 Statistical accuracy

One can obtain analytically some information about how the accuracy of the $\bar{x}^2$ empirical values should be affected by the size of the ensemble used.

Consider the general problem of a number of random variables, $Z_1, Z_2, \ldots, Z_n$ which are mutually independent,
and each of which has the same probability distribution as a random variable \( Z \) for which both mean \( \mu_Z \) and variance \( \sigma_Z^2 \) exist. Let

\[
\bar{Z} = \left( \frac{\sum_{i=1}^{n} Z_i}{n} \right) / n
\]

(3.8)

One can easily show that the mean and standard deviation of the empirical mean, \( \bar{Z} \), are given by

\[
\mu_{\bar{Z}} = E(\bar{Z}) = \mu_Z
\]

(3.9)

and

\[
\sigma_{\bar{Z}} = \sigma_Z / \sqrt{n}
\]

(3.10)

One can further show, by the central limit theorem, that the probability distribution of \( \bar{Z} \) is asymptotically normal as \( n \) tends to infinity. To apply this simple result to the empirical combination average mean squared values of \( x \), let \( \hat{x}^2(t) \) represent the time average of \( x^2 \) from \( t-\pi/2\omega_0 \) to \( t+\pi/2\omega_0 \), then take

\[
Z = \hat{x}^2(t)
\]

(3.11)

for any particular value of \( t \). Then the \( \hat{x}^2(t) \) value for each member of the ensemble gives a sample value, \( Z_i \). The properties of the excitation process, as discussed in chapter II, should assure a response for which the empirical response values, \( Z_i \), simulate samples from independent random variables with identical distributions.

The significance of equations 3.9 and 3.10 can be seen by thinking of using infinitely many ensembles, each containing \( n \) sample functions, to determine values of \( \bar{X}^2 \) for some particular value of \( t \). Each ensemble would give an empirical value of \( \bar{X}^2 \), and these empirical values would show scatter due to statistical irregularity. Equation 3.9 can be considered as saying that the mean of these infinitely many
empirical $\overline{x}^2$ values should be the true value of $E(x^2) = E(x^2)$.

Equation 3.10 tells how widely the empirical values of $\overline{x}^2$ should be scattered, since it gives the SD of these values. In particular, if n is large enough that $\overline{x}$ is approximately normal then one finds that there is 31.7% probability that any particular empirical value of $\overline{x}^2$ will differ from $E(x^2)$ by more than one standard deviation, 4.56% probability that it will differ by more than $2\sigma$ etc.

Unfortunately the variances of the time averages, $\hat{x}^2$, and the combination averages $\overline{x}^2$ are unknown. By assuming that the response displacement $x$ is approximately normal one can, however, obtain the variance of a simple ensemble average of $x^2(t)$ values, i.e. of the arithmetic mean across an ensemble, AM[$x^2(t)$]. The variance of this simple ensemble average is larger than that of the corresponding combination average, so one can obtain a bound on the probable error in this way. The assumption that $x$ is normal is exact for the response of a linear system to a Gaussian process excitation, and empirical results will be presented in chapter IV to show that it is also a fair assumption for the bilinear hysteretic systems.

In order to apply equations 3.9 and 3.10 to a simple ensemble average approximation of $\sigma_{\overline{x}^2}$, let $\overline{z}^* = x^2$. Then taking $x$ as normal with mean zero gives

$$
(\sigma_{\overline{z}^*})^2 = (\sigma_{x^2})^2 = E(x^4) - \sigma_x^4 = 2\sigma_x^4 \tag{3.12}
$$

Since $\overline{x}^2$ is an approximation of $\sigma_x^2$, one can then use equation 3.10 to write

$$
\sigma_{AM(x^2)} = \sqrt{\overline{x}^2} / \sqrt{n} \tag{3.13}
$$
For an ensemble size of $n=80$, as used in this study, equation 3.13 says that the standard deviation of any empirical simple ensemble average, $AM(x^2)$, is approximately 15.8% of that empirical value. Further, an ensemble size as large as 80 usually assures that the probability distribution of the empirical ensemble averages will be closely approximated by a normal distribution to a level several times the standard deviation.

A convenient way to state the results of the above analysis is in terms of confidence intervals. In terms of the present problem, one can say with confidence $C$ that the true mean squared value $\sigma_x^2$ is in a given interval $AM(x^2)-d \leq \sigma_x^2 \leq AM(x^2)+d$, where $AM(x^2)$ is an empirical value, if $C$ is the a priori probability that the inequality would hold for any empirical value $AM(x^2)$.

For $AM(x^2)$ having a normal distribution with standard deviation of $0.158AM(x^2)$ as determined above for an ensemble of size 80, one finds, for example, that $C=0.474$ for $0.90AM(x^2) \leq \sigma_x^2 \leq 1.10AM(x^2)$ and $C=0.90$ for $0.74AM(x^2) \leq \sigma_x^2 \leq 1.26AM(x^2)$. Thus one could only have 47.4% confidence that an empirical simple average value $AM(x^2)$ would lie within 10% of the true unknown value, $\sigma_x^2$, and 90% confidence that the empirical simple average would lie within 26% of the true value.

The width of the confidence intervals indicates that a larger ensemble size would be desirable if simple ensemble averages are to be used to estimate $\sigma_x^2$. Note, however, that doubling the size of the ensemble would only reduce the width of the confidence interval by $1/\sqrt{2}$. Because of the cost of
computer time it seemed more efficient to reduce the probable error by using the combination average (combination of time and ensemble) described above. The width of the confidence interval for these combination averages is not known, but it is definitely less than for the simple ensemble averages. It is also known, from the above, that the width of the confidence interval for the combination averages also varies like $1/\sqrt{n}$ where $n$ is ensemble size.

3.3 Bilinear hysteretic systems

The elastoplastic system ($\alpha=0$) has often been used for nonlinear studies, due to simplicity. The primary interest in this study, however, is directed toward the growth of response statistics from zero initial levels to stationary levels. Since the mean squared displacement response of an elastoplastic system does not reach a stationary level, this system is included here only as a limiting case, rather than as one of primary interest. The two nonlinear systems given primary emphasis in this study are $\alpha=1/2$, a moderately nonlinear system, and $\alpha=1/21$, a nearly elastoplastic system. The mean squared stationary response for such a bilinear system has been reproduced from Iwan and Lutes as shown in figures 3.3 to 3.6.

There are two limiting linear cases of $Y/F=0$ and $Y/F=\infty$ (zero and infinite yield level). These systems are governed by equation 3.1 with $\omega_1^2=\alpha\omega_o^2$ and $\omega_1^2=\omega_o^2$ respectively, and with $\beta_1\omega_1=\beta_0\omega_0$ in both cases. For large $t$, equations 3.2 and 3.3 give the stationary levels of response to white excitation as

$$J(\sigma_{x}/F)^2 = (\sigma_{x}/F\omega_o)^2 = \pi/4\beta_o$$

(3.14)
where \( J = 0 \) for \( Y/F = 0 \) and \( J = 1 \) for \( Y/F = \infty \).

The values of yield level considered in this study were selected so as to cover the points of minimum stationary response, as well as high and low yield levels. For the nearly elastoplastic system figures 3.5 and 3.6 show that for \( Y/F = 2 \) the mean squared velocity response is minimum while for \( Y/F = 5 \) the mean squared displacement response is minimum. These were chosen as two of the yield levels to be considered for transient (nonstationary) response of the system with \( \alpha = 1/2 \). The other two yield levels of \( Y/F = 1 \) and \( Y/F = 15 \) were considered as examples of low and high yield levels for the system. Similarly for \( \alpha = 1/2 \) the response is found to be minimum when \( Y/F = 3 \), \( Y/F = 1 \) and \( Y/F = 15 \) are low and high yield levels, and \( Y/F = 6 \) was selected as an intermediate value between \( Y/F = 3 \) and \( Y/F = 15 \). Note that the spacing of the \( Y/F \) values considered is fairly uniform on a logarithmic scale.

3.4 Mean squared levels of response

The simplest measure of any random variable is the mean level, but it does not indicate anything about the variability of the variables. In fact, in the present study the mean levels of both excitation and response are zero. Another simple measure is the mean squared level of a variable. If the variable has zero mean then the mean squared level coincides with the variance. The standard deviation (square root of variance) measures how much a random variable varies from its mean. For a normal variable knowledge of mean level and mean squared level is sufficient to determine completely all probability density functions for the variable.
The mean squared levels of both the displacement, \( x \), and velocity, \( x' \), of the response of the bilinear system were determined. For stationary response of a linear system the rms velocity is the product of the rms displacement and the undamped natural circular frequency. This simple relationship does not hold for the true expected values of nonstationary response of the linear system, as given in equations 3.2 and 3.3, but it does apply to the nonoscillatory response of equations 3.6 and 3.7. Unfortunately the relationship does not extend to the bilinear system.

In presenting the empirical results for the nonlinear system, the combination averages are plotted at the center of the time interval used in averaging. These points are connected by straight lines, since these averages were only computed for discrete, nonoverlapping time intervals of length \( \pi/\omega_0 \). One could produce smooth curves of combination time and ensemble averages by further computation, but it does not appear that this would add much useful information.

Note that the averaging time used in the combination averages \( (\pi/\omega_0) \) can be considered as a half cycle of narrow band response of the limiting linear system with \( Y=\infty \), provided that the viscous damping is small. One must be very cautious about extending this idea of cycles of response to the bilinear system results, however. This is illustrated by the sample time histories of response shown in fig. 3.7. These time histories show that not only is the predominant frequency of response dependent on the \( Y/F \) ratio, but for some values of \( \alpha \) and \( Y/F \) the response is very broad-band, having no
predominant frequency (as in curve d). For large $Y/F$ (as in curve c) one can continue to think of $\omega_0 t = \pi$ as approximately corresponding to a half cycle of response, and similarly for very small $Y/F$ one may be able to think of $\omega_0 t = \pi/\sqrt{\alpha}$ as corresponding to a half cycle of response, but none of the $Y/F$ values considered here seem to be small enough to allow this latter approximation. For intermediate values of $Y/F$ the concept of cycles of response is not very useful. The studies of the stationary response of the bilinear hysteretic reported in ref. 22 confirm these conclusions.

The complete mean squared response results obtained are plotted in figures B.1 to B.18 of Appendix B for the combination averages versus time for the different values of damping considered. Representative results are shown in figs. 3.8 to 3.14 in a three dimensional plot form which illustrates both how the response builds up with time, and how it is affected by yield level. This also allows easy comparison with the stationary results and with the nonoscillatory response for the limiting linear cases.

Figures 3.8 and 3.9 show the mean squared displacement and velocity response for the moderately nonlinear system, $\alpha=1/2$, and with no viscous damping. Note that for all values of time both displacement and velocity response are smaller for $Y/F=3$ than for any other yield level considered. This agrees with the results for the limiting condition of stationary response shown in figures 3.3 and 3.4 and reproduced at the right hand sides of figures 3.8 and 3.9.

A comparison of the results for $Y/F=1$ with the
limiting linear condition of $Y/F=0$ in figures 3.8 and 3.9, shows that the levels of response are comparable only for $\omega_0 t$ up to about $2\pi$. Beyond that time the linear system response continues to grow, since it is undamped, while the hysteretic energy dissipation for $Y/F=1$ causes a decrease in the rate of growth of response, resulting in an asymptotic approach to the stationary level.

The comparison of $Y/F=15$ with $Y/F=\infty$ in figs. 3.8 and 3.9 shows the same characteristics noted for $Y/F=1$ and 0. The response for $Y/F=15$, however, corresponds closely with the limiting linear case for $\omega_0 t$ up to about $8\pi$ (about four cycles of response). This suggests, in some intuitive sense, that for $\alpha=1/2$, $Y/F=15$ may give a more nearly linear system than does $Y/F=1$. Another way of saying this is that yielding has a more pronounced effect on the system response for $Y/F=1$ than for $Y/F=15$. Other evidence confirms this intuitive conclusion. For example, note that the stationary level of mean squared displacement response (fig. 3.3) for $Y/F=1$ is only about 3.5 times greater than its minimum value for $Y/F=3$, and that the mean squared velocity for $Y/F=1$ is only about 2.5 times greater than its minimum value. For $Y/F=15$, on the other hand, mean squared displacement exceeds five times its minimum value, and mean squared velocity exceeds seven times its minimum value.

One also notes from figs. 3.8 and 3.9 that for $Y/F=3$ and 6 the responses approach nearly stationary levels quite rapidly, while for the lower and higher yield levels the responses continue to grow over the entire time interval shown.
Figures 3.10 and 3.11 give similar plots of response level for the severely nonlinear undamped system, $\alpha=1/21$. The general tendencies seen here are the same as for the moderately nonlinear system, but some differences can also be noted. Particularly for $Y/F=1$ (and to a lesser extend for $Y/F=2$) one notes that the displacement response continues to grow for a considerable amount of time after the velocity response has reached a nearly stationary level. These results, along with those for $\alpha=1/2$, seem to indicate that as a general rule a "low" stationary level is approached more rapidly than is a "high" stationary level. For example, for $\alpha=1/21, Y/F=1$ gives a stationary velocity response which is only about 1.5 times the minimum value (for $Y/F=2$) and this level is approached very rapidly. For the same system, however, the stationary displacement level is nearly four times the minimum level (for $Y/F=5$) and the transient approach to this level is less rapid.

One notes that for $\alpha=1/21$, as for $\alpha=1/2$, the response when $Y/F=15$ is much the same as for the linear system resulting from $Y/F=\infty$, for the range of the time shown in the figures. Further, the differences between the response for $Y/F=1$ and the response for the linear system with $Y/F=0$ are similar in nature for the two slope ratios, but are more pronounced for $\alpha=1/21$, than they were for $\alpha=1/2$.

The results for $\alpha=1/21$ are similar to those for $\alpha=1/2$ in that the $Y/F$ giving the minimum stationary response also gives smaller transient response than does any other yield level considered.
From figure 3.5 one notes that the stationary displacement response of the nearly elastoplastic system with $\beta_0 = 0.05$ has a notably different character than the other situations considered. For this system the stationary displacement response level is nearly monotone decreasing with $Y/F$, being only slightly smaller for $Y/F=15$ than for $Y/F=\infty$. Figure 3.12 shows, however, that the build up of the transient displacement response for a particular $Y/F$ for this system does not seem to be notably different than for the other systems studied.

The limiting linear system with $Y/F=\infty$ in figure 3.12 shows for the first time in the figures presented here, the behavior of the nonoscillatory response of a damped linear system (equation 3.6) as it approaches the stationary response level.

As mentioned earlier, the elastoplastic oscillator is included here only as a limiting case of the bilinear oscillator. The transient response of the elastoplastic system was determined empirically, primarily for comparison with the results for $\alpha=1/21$. The complete results obtained are included in Appendix B (figures B.17 and B.18), and figures 3.13 and 3.14 show a portion of these results in the same form as figs. 3.8 and 3.9 for easy comparison with the response of the nearly elastoplastic system.

Particularly for low yield levels the continued growth of the displacement response of the elastoplastic system is obvious in fig. 3.13. When $\omega_0 t = 12\pi$ the elastoplastic system response for $Y/F=1$ is about eight times that for
\( \alpha = 1/21 \), but this ratio decreases to about 2.5 for \( Y/F = 2 \) and is nearly one for \( Y/F = 5 \) and 15. In the limiting case for \( Y/F = 0 \) there is no restoring force and mean squared displacement grows as a cubic function of time, as shown.

The velocity response of the elastoplastic system is seen to be approximately the same as for \( \alpha = 1/21 \) in all situations studied. It appears, in fact, that the mean squared velocity of an elastoplastic system does tend to a stationary value, even though the displacement grows toward infinity.

It is interesting to note that the \( Y/F \) values of 5 and 2 which gave minimum displacement and velocity response, respectively, for \( \alpha = 1/21 \), also give smaller response for \( \alpha = 0 \) than do the other \( Y/F \) values considered.

Figures 3.15 and 3.16 show the displacement and velocity response of the particular system with \( \alpha = 1/21 \) and \( Y/F = 5 \) in a slightly different form. These figures are representative of those in Appendix B and are included here to illustrate the effect of viscous damping on the transient response levels. As might be expected, the figures show that the addition of viscous damping results in a relatively uniform reduction in response for every value of time.

One should note that each of the curves in figs. 3.15 and 3.16 was obtained from combination averages of the response to the same ensemble of 80 excitations. This partially explains why the curves never cross even though they do exhibit irregularities (such as the displacement peaks at \( \omega_0 t = 6.5\pi \) and \( 12.5\pi \)) which are presumably due to the finite size of the ensemble. If the curves for \( \beta_0 = 0.00 \) and \( \beta_0 = 0.01 \),
for example, had been obtained from independent finite ensembles then their statistical irregularities would have been uncorrelated and would probably have caused the curves to cross.

3.5 Time to reach specified response level

One of the aims of this study was to obtain guide lines as to when the response of a system can be considered to be essentially stationary so that the more extensive literature dealing with this situation can be applied. Figs. 3.17 to 3.21 present some of the empirical data on transient response replotted in a form intended to provide maximum information about this matter. The time required for the mean squared displacement response of a given system to first reach some specified fraction \( r \) of its stationary level \( \bar{X}^2_s \) is denoted by \( t_r \), and \( \omega_0 t_r \) has been plotted against the \( Y/F \) value for the system. This has been done for various values of \( r \), and for all the bilinear systems considered. The values of \( r \) considered vary from 0.5 to 0.8. Note that this range of \( r \) corresponds to the rms response being between 70% and 90% of the stationary value. It seems reasonable to assume that the response will be nearly stationary after its rms has reached 90% of the corresponding stationary value.

Comparing figures 3.17 and 3.18 with figure 3.3, one notes that for \( \alpha = 1/2 \) the time \( t_r \) to reach a response level of \( r\bar{X}^2_s \) is nearly proportional to \( \bar{X}^2_s \). This confirms the earlier observation that a "low" stationary level is approached more rapidly than is a "high" level. This conclusion is not strictly supported, however, by the results for \( \alpha = 1/21 \) presented in figures 3.19 to 3.21. For example, figure 3.5 shows that
\( \bar{x}_S^2 \) is always smaller for \( Y/F=2 \) than for \( Y/F=1 \), but for \( \beta_0=0.05 \) and for \( \beta_0=0.00 \) with \( r=0.80 \), \( t_r \) is greater for \( Y/F=2 \) than for \( Y/F=1 \). Similarly the system with \( \beta_0=0.05 \) shows a maximum \( t_r \) when \( Y/F=5 \) whereas \( \bar{x}_S^2 \) is monotone decreasing with \( Y/F \).

The near proportionality between \( t_r \) and \( \bar{x}_S^2 \) for \( \alpha = 1/2 \) suggests that it may be helpful to look at the results in terms of the rate of growth of mean squared response. Let

\[
V(r) = \frac{r \bar{x}_S^2/F^2}{\omega_0 t_r} \tag{3.15}
\]

A value of \( V \) is then the slope of a line from the origin to the point where \( \bar{x}^2 = r \bar{x}_S^2 \) on a plot of \( \bar{x}^2/F^2 \) versus \( \omega_0 t \).

Figure 3.22 shows values of \( V \) determined for some of the situations in figures 3.17 to 3.21. The only values of \( r \) shown are 0.50 and 0.80, but both systems with and without damping are included for both values of \( \alpha \). Note that the results plotted in figure 3.22 are based on empirical studies of both \( \bar{x}_S^2 \) and \( t_r \). The combination of the statistical errors in these two studies probably accounts for the fact that the statistical irregularities are greater in figure 3.22 than in most of the data obtained in this study.

It may be helpful to also note the values of \( t_r \) and \( V \) for the linear equation 3.1. Using the nonoscillatory response of equation 3.6 one finds that

\[
t_r = -\frac{\ln(1-r(1-\beta^2))}{2\beta \omega_0} \tag{3.16}
\]

and

\[
V(r) = \frac{\pi \omega_0^2}{2 \omega_0^2 \ln(1-r(1-\beta_1^2))} \tag{3.17}
\]

Note that the \( V \) values for the linear system are nearly independent of \( \beta_1 \), at least for \( \beta_1 < 0.10 \) or 0.20. On the other hand,
there is a simple inverse proportionality between \( V \) and the stiffness of the linear system.

For the linear system resulting from taking \( Y/F=\infty \) in the bilinear system, equation 3.17 gives \( V(0.80) = 0.70 \) and \( V(0.50) = 1.13 \), for any value of \( \alpha \). The other limiting case of \( Y/F=0 \) so that \( \omega_1^2 = \omega_0^2 \) gives \( V(0.80) = 1.56 \) and \( V(0.50) = 2.26 \) and 23.7 for \( \alpha = 1/2 \) and 1/21, respectively. Values of \( V \) for the two limiting linear cases corresponding to \( Y/F=0 \) and \( Y/F=\infty \) for a damped system are shown in figures 3.18, 3.20 and 3.21.

One notes that the \( V \) values in figure 3.22 for \( \alpha = 1/2 \) are relatively independent of \( Y/F \). Only the line for \( r = 0.50 \) and \( \beta_0 = 0.01 \) shows \( V \) increasing with decreasing \( Y/F \), as one might expect to happen due to the softening spring effect of the nonlinearity. For \( \alpha = 1/2 \), and \( \beta_0 < 0.01 \) it appears that one can use the simple approximation that \( V(0.80) = 1.2 \) and \( V(0.50) = 1.60 \), without making excessively large errors. Using these approximations in equation 3.15 gives

\[
\omega_0 t_{0.80} \approx 0.67 (\bar{x}_s/F)^2 \tag{3.18}
\]

\[
\omega_0 t_{0.50} \approx 0.31 (\bar{x}_s/F)^2 \tag{3.19}
\]

for \( \alpha = 1/2 \) and \( \beta_0 < 0.01 \). Equations 3.18 and 3.19 can be considered as guidelines of the type sought for answering the question of when the bilinear system becomes essentially stationary.

The results in figure 3.22 for \( \alpha = 1/21 \) clearly show an increase in \( V \) resulting from decreasing \( Y/F \) below 5. Presumably this is the previously mentioned softening spring effect, corresponding to reducing the stiffness, \( \omega_1^2 \), in eq. 3.17.
Increasing $\beta_0$ from 0.00 to 0.05 in the nearly elastoplastic system has a distinct effect on the $V$ values, giving a 30.0% to 40.0% reduction in $V$ in most instances. This is in distinction to $\beta_0=0.01$ which does not seem to be consistently different than $\beta_0=0.00$ for either $\alpha$ value considered. Extrapolating from these results it seems reasonable to expect that larger values of $\beta_0$ may also result in reduction in $V$ for the system with $\alpha=1/2$.

It is interesting to note that the $V(r)$ plots for $\alpha=1/21$ in figure 3.22 have the same general shape for all values of $\beta_0$. This is despite the fact that the $\omega_0 t_r$ plots in figure 3.21 for $\beta_0=0.05$ have a distinctly different shape than do figures 3.19 and 3.20 for smaller values of damping.

It is not as easy to give a simple summary of the $t_r$ values for $\alpha=1/21$ as it was to give equations 3.18 and 3.19 for $\alpha=1/2$. For $\beta_0<0.01$ and $5 \leq Y/F \leq 15$ it appears that $V(0.80) = 0.80$ and $V(0.50) = 1.15$, giving

$$\omega_0 t_{0.80} = (\bar{X}_b/F)^2$$  \hspace{1cm} (3.20)

and

$$\omega_0 t_{0.50} = 0.44 (\bar{X}_b/F)^2$$  \hspace{1cm} (3.21)

for $\alpha=1/21, \beta_0 < 0.01$ and $5 \leq Y/F \leq 15$. For smaller values of $Y/F$ it seems to be better to return the form of figures 3.19 to 3.21 and conclude that

$$\omega_0 t_{0.80} = 5\pi$$  \hspace{1cm} (3.22)

and

$$\omega_0 t_{0.50} = 2.5\pi$$  \hspace{1cm} (3.23)

for $\alpha=1/21$ and $1 \leq Y/F \leq 2$. As mentioned earlier, increasing $\beta_0$ to 0.05 results in about a 30.0% to 40.0% reduction in $V$, which gives corresponding increases in equation 3.20 and 3.21 for that situation.
It may be worthwhile to note the relationship between the time it takes $\bar{X}^2$ to reach 50.0% of its stationary value and time it takes to reach 80.0% of its stationary value. This relationship can be represented by the ratio $t_{0.80}/t_{0.50}$. Equations 3.18 and 3.19 give a value of 2.20 for this ratio for $\alpha=1/2$, and equations 3.20 to 3.23 give the ratio as 2.30 for larger $Y/F$ and 2.00 for small $Y/F$. These values agree quite well with the ratio of 2.32 for the linear system of equation 3.16.
CHAPTER IV

PROBABILITY DISTRIBUTIONS

4.1 Instantaneous response

One is often interested in finding the probability distribution of a random variable, since a random variable is completely defined by its probability distribution. If a random variable is normally distributed then one only needs to know the first two moments (or mean and standard deviation) since the probability distribution is completely specified by these two parameters. A random process \( x(t) \) is said to be Gaussian if any finite family of random variables \( x(t_1), x(t_2), x(t_3), \ldots, x(t_j) \), corresponding to a discrete set of \( t \) values, is jointly normal. Hence the Gaussian process is completely defined by its mean function, \( \mu_x(t) = \mathbb{E}[x(t)] \) and its covariance function \( \kappa_{xx}(t_1, t_2) = \mathbb{E}[x(t_1)x(t_2)] \). If a linear system has a Gaussian input then the output is also Gaussian. However, the system considered in this study was a bilinear system and the output of such a system cannot be expected to be Gaussian. The stationary response of similar systems has been shown to be quite different from normal in some instances.

In this study the transient response probability distributions have been obtained from the ensemble of response time histories of the nonlinear system, for the random variables \( x(t) \) and \( x^*(t) \) for the particular values of \( t \). The results have been obtained at an interval of \( \pi/5 \) for values of \( \omega_0 t \) between 0 and 10\( \pi \) and have been denoted by \( P(K, t) \) and \( Q(K, t) \), where

\[
P(K, t) \equiv \text{Prob}(x(t)/s_x(t) > K) \quad (4.1)
\]

and

\[
Q(K, t) \equiv \text{Prob}(x^*(t)/s_{x^*}(t) > K) \quad (4.2)
\]
The values of $x(t)$ and $x'(t)$ have been non-dimensionalised by dividing by the corresponding empirical values of response SD, $s_x(t)$ and $s_{x'}(t)$, respectively:

$$s_x(t) = \sqrt{\langle x^2(t) \rangle}$$  \hspace{1cm} (4.3)

and

$$s_{x'}(t) = \sqrt{\langle x'^2(t) \rangle}$$  \hspace{1cm} (4.4)

The symmetry of the probability distribution has been used in evaluating the results by using the number of members of the ensemble (of size 80) having either $x(t)/s_x(t) > K$ or $x(t)/s_x(t) < -K$. This number divided by 160 is used as the approximation of $P(K,t)$. The velocity distribution, $Q(K,t)$, was approximated in the same manner. Using the symmetry in this way effectively doubles the size of the ensemble. Note that even for the effective ensemble size of 160, one cannot determine very small $P$ or $Q$ values with any degree of precision. An empirical probability level of 2.50% ($P$ or $Q=0.025$), for example, corresponds to only four observed exceedances of the level $K$. Below this probability level one can expect statistical irregularity to make the empirical results nearly meaningless.

The empirical values for $P$ or $Q$, when plotted versus $K$ for various values of $\omega_0 t$, indicated that in most instances the $P(K,t)$ and $Q(K,t)$ vary rather irregularly with $t$. One representative case for $P$ and one for $Q$ have been shown in figures 4.1 and 4.2. Figure 4.1 shows the $P$ curve for transient response of a nearly elastoplastic system, $\alpha=1/21$. The results have been plotted for $\omega_0 t$ equal to $\pi/5, 2\pi, 4\pi, 6\pi, \text{and } 8\pi$. Since the results are plotted on normal paper, and the probability distributions for various values of $\omega_0 t$ are not straight lines,
the results indicate that the probability distribution of the transient response of a bilinear system is not strictly normal, however the deviations from a normal distribution are fairly small for the range of probabilities shown. Figure 4.2 giving the Q curve for the transient response of a moderately non-linear system, once again shows that the probability distribution varies irregularly with time. Because of the noted irregularity of P and Q, it was decided to plot the upper and lower bounds for these curves instead of showing the individual curve for each \( \omega_0 t \). Figures 4.3 to 4.6 show the effect of \( Y/F \) on these bounds of probability distribution for a system with \( \alpha = 1/21 \) and \( \beta_0 = 0.00 \). There are some differences among the figures, but the overall effect of \( Y/F \) on the bounds seems to be quite negligible for the range of K shown. Figures 4.7 and 4.8 show similar bounds for the Q curves for \( Y/F = 1 \) and 5, respectively. The Q results lie in a slightly narrower band than the corresponding P curves, indicating that the probability distribution of transient velocity response is closer to normal than is that of displacement.

The effect of damping on the probability bounds is investigated in figures 4.9 to 4.12. A comparison of figures 4.9 and 4.10 with 4.7 and 4.8 shows that damping seems to have a negligible effect on these Q curves. A similar comparison of figures 4.11 and 4.12 with 4.3 and 4.5 gives basically the same result for the P curves for a system with \( \alpha = 1/21 \). Comparison of figures 4.11 and 4.12 with 4.9 and 4.10, respectively, once again indicates that the probability distribution for velocity is contained in a narrower band than is that of
displacement. Overall, one can note that the effect of damping on the $P$ and $Q$ curves seems to be negligible for the systems considered.

Figures 4.13 and 4.14 show the bounds of $P$ and $Q$ curves for the moderately nonlinear system, $\alpha = 1/2$ with $Y/F = 1$. These results, when compared with figures 4.3 and 4.7 show that the probability distribution is contained in a wider band for this system than for the nearly elastoplastic system.

The overall impact of figures 4.1 to 4.14 seems to be that the response $x(t)$ and $x'(t)$ are normal within the range of parameters studied. This appears, at first observation, to be in contradiction to the previously cited empirical results that the stationary response of such systems is (22) sometimes quite different from normal. Recall, however, that ensemble size limited the results of the present study to probability levels greater than 2.50%. Within this range of probabilities the results in reference 22 also appear to be nearly normal, or at least to fall near the bounds presented in figures 4.3 to 4.14. From the results for the stationary response it appears that one needs to be able to measure probability levels of 0.50% or 0.20% in order to really verify a deviation from the normal distribution for these systems.

4.2 Peak response

Since the maximum or peak response during a particular time history is often used as a measure of the damage to a nonlinear structure, statistics of such extreme values are of considerable interest in a study of this type. In particular, let
\[ y(t) = \max_{0 \leq \tau \leq t} |x(\tau)| \quad (4.5) \]

This maximum absolute value of displacement response for each sample time history was obtained and the AM (denoted by $\bar{y}$) and SD (denoted by $s_y$) of these maxima for an ensemble of responses have been computed. The empirical results for both damped and undamped bilinear hysteretic systems with $\omega_0 t$ equal to $4\pi, 8\pi,$ and $12\pi$ are presented in table 4.1. The values of $\bar{y}$ and $s_y$ are normalised by dividing these quantities by $F$.

The results in table 4.1 show that both $\bar{y}/F$ and $s_y/F$ increase with an increase in the duration of excitation for a fixed system. One also notes that for a given amount of damping and a given slope ratio, the smallest $\bar{y}/F$ and $s_y/F$ usually occur for the $Y/F$ giving minimum $\bar{F}^2/F^2$, as studied in chapter III. The $\bar{y}/F$ and $s_y/F$ values, however, are not proportional to either $\bar{F}^2/F^2$ or $s_x/F$. For $\alpha=1/21$ and $\alpha=0.00$ both $\bar{y}/F$ and $s_y/F$ take on relatively large values when there is a large amount of yielding (small $Y/F$). For $\alpha=1/2$ both $\bar{y}/F$ and $s_y/F$ are approximately constant, the variation in their values being quite small, although their minimum values still correspond to $Y/F=3$ where the mean squared displacement response was minimum. For the undamped elastoplastic system the results are similar to those for the $\alpha=1/21$ system, the greatest differences occurring for large $\omega_0 t$ and small $Y/F$.

For a given damping and yield level the values of $\bar{y}/F$ and $s_y/F$ are smallest for the moderately nonlinear system, larger for the severely nonlinear system, and maximum for the elastoplastic system. One can also note that the increase in $\bar{y}/F$ and $s_y/F$ is more during $0 \leq \omega_0 t < 4\pi$, than it is during any
other time interval of this length.

Note that \( \bar{y} \) represents the mean of 80 independent random variables, all having the same probability distribution. Thus one can use equation 3.10 to obtain the standard deviation of \( \bar{y} \) as

\[
\sigma_{\bar{y}} = \frac{s_y}{\sqrt{80}} = 0.11s_y
\]  

(4.6)

where \( s_y \) has been replaced by its estimator \( s_y \).

For the cases shown in table 4.1 one finds that \( s_y \) is never more than 70.0% of \( \bar{y} \), and is more commonly less than 50.0% of \( \bar{y} \). Using the central limit theorem to justify an assumption that \( \bar{y} \) is approximately normal, and using the worst case of \( s_y \) equal to 0.70\( \bar{y} \), leads to the conclusion that one can have 90.0% confidence that \( 0.87\bar{y} < \text{E}(y) < 1.13\bar{y} \) for an empirical value of \( \bar{y} \) from an ensemble of size 80.

Empirical values of \( \bar{y} \) and \( s_y \) have also been obtained using averages across only some members of an ensemble of 80 time histories. Figures 4.15 to 4.18 show how empirical values of the standard deviation of peak response varied with respect to the size of the ensemble used in computing them, for some representative cases. Both \( \bar{y} \) and \( s_y \) were nondimensionalised by dividing by the yield level. The particular cases included in these figures were chosen to represent instances when \( \bar{y} \) and \( s_y \) seemed to converge less rapidly with increasing ensemble size than for the other situations studied. The width of a 90.0% confidence interval for \( \bar{y} \) for various ensemble sizes has also been included in figures 4.15 to 4.18. Note that this width has been plotted simply as a symmetric band about the empirical value of \( \bar{y} \) for an ensemble size of 80. Nearly
all the empirical values in the figures are seen to fall within this band, and the general rate of convergence of \( \bar{y} \) seems to correspond well to the width of the confidence band. It is not as easy to obtain an analytical description of a confidence interval for the quantity \( s_y \), and it has not been attempted in this study. Note, though, that the empirical results indicate that \( s_y \) converges at least as rapidly as \( \bar{y} \) for the situations shown.

The ratio of maximum response to yield level is called ductility, and is considered as an important parameter in many design problems. For example, ductility is recognised as a specific requirement in design of earthquake resistant structures. This has led to a number of theoretical and experimental investigations of yielding type structures. For example, (29) Liu obtained the maximum response of nonlinear elastoplastic and stiffness degrading models using a digitally simulated stochastic process representing strong ground motion caused by earthquakes.

Mean values of ductility \( (\bar{y}/Y) \) for the present study can easily be computed from the empirical data given in table 4.1. Note, though, that the variation of these ductility values is much greater than that of the \( \bar{y} \) values. For example, for the systems with \( \alpha=1/2 \) and with \( \omega_0 t=4\pi \) one finds that \( \bar{y}/F=6 \) in all cases, but the corresponding ductility ranges from 0.40 to 6.0. Note that a mean ductility value, \( \bar{y}/Y \), smaller than one does not necessarily mean that all members of the ensemble behaved elastically, but such could be the case.

Another normalization of \( \bar{y} \) is represented by the
plots of $\bar{y}(t)/s_X(t)$ versus $\omega_0 t$ in figures 4.19 to 4.24, where $s_X(t)$ is the SD of the response at the particular time $t$. The physical significance of this ratio $\bar{y}/s_X$ relates to the question, "How much bigger than the SD of response is the peak response apt to be?" Of course, both the numerator and the denominator of the quantity $\bar{y}/s_X$ had to be determined empirically in this study. The general characteristics of $s_X$ (or $s_X^2 = \overline{r^2}$), however, have already been discussed in chapter III.

One can note from figures 4.19 to 4.24 that for all the values of $Y/F$ and damping studied, $\bar{y}/s_X$ shows a tendency to increase with increasing $\omega_0 t$ beyond about $2\pi$. This was certainly to be expected for the large values of $\omega_0 t$ where $s_X$ reaches essentially a constant level, while $\bar{y}$ continues to grow, since it corresponds to the peak values of ever larger samples of the random response. An exception to this rule, though, is the elastoplastic system, for which both $s_X$ and $\bar{y}$ continue to grow indefinitely.

For the elastoplastic system, figure 4.19 shows that for essentially all values of $\omega_0 t$ the ratio $\bar{y}/s_X$ is larger for $Y/F=5$ than for the other yield levels considered. This is the $Y/F$ value which was shown in chapter III to give the minimum mean squared response, $s_X^2$. The ratio $\bar{y}/s_X$, however, is not inversely proportional to $s_X$ since $\bar{y}$ is not a constant, as was shown in table 4.1. Figures 4.20 to 4.22 show the plots for the nearly elastoplastic system. These plots reveal characteristics similar to those noted for the elastoplastic system. One notes that the addition of damping reduces $\bar{y}/s_X$ in a quite irregular manner. Plots for the moderately
nonlinear system, as shown in figures 4.23 and 4.24, indicate that this system has higher $\bar{Y}/s_x$ values than do the elasto-plastic systems. The variation with $Y/F$ is similar to that noted for $\alpha=0$ and $\alpha=1/21$.

Similar to the above plots of $\bar{Y}/s_x$, figures 4.25 to 4.30 show the variations of $s_y/s_x$ with respect to time $\omega_0 \cdot t$. Figure 4.25 shows the undamped elastoplastic case. The plot indicates that $s_y$ grows less rapidly than does $s_x$. There is no definite trend of variation with respect to $Y/F$, but all the curves are contained within a fairly narrow band. Figures 4.26 to 4.28 show that the results are similar for the nearly elastoplastic system. Damping has no apparent effect on the variations of $s_y/s_x$. The minimum value of $s_y/s_x$ for the undamped system is for $Y/F=5$, where the mean squared response ($\bar{x}^2$ or $s_x^2$) is also smaller than for other $Y/F$ values. The $s_y/s_x$ ratio varies irregularly with respect to $Y/F$, but it decreases fairly consistently as $\omega_0 \cdot t$ increases, and it appears that it may reach some nearly constant value as $\omega_0 \cdot t$ becomes large. Figures 4.29 and 4.30 show similar plots of $s_y/s_x$ versus $\omega_0 \cdot t$ for the moderately nonlinear system, $\alpha=1/2$. The results have rather irregular variations, as did those for $\alpha=1/21$. One can note, however, that the curves for $\alpha=1/2$ are not contained within as narrow a band as were those for $\alpha=1/21$ or $\alpha=0$.

The probability distribution of peak response, $y$, was estimated empirically for each system considered for $\omega_0 \cdot t$ equal to integer multiples of $\pi$ from $\pi$ up to $10\pi$, and also for the largest value of $\omega_0 \cdot t$ for which the sample responses were
computed for that system. The estimate of the probability that y exceeds some value is the fraction of the 80 samples having y greater than that value. Note that in this case four exceedances gives a probability estimate of 5.00% so that statistical irregularity can be expected to be very large for smaller probability levels (and also for probability levels greater than 95.00%).

Figures 4.31 to 4.36 show some representative empirical probability distributions in the form of

\[ R(K,t) = \text{Prob}(y(t)/s_x(t) > K) \] (4.7)

Note that y has been normalised, not by its own SD, but by the empirical SD of the instantaneous response displacement, x(t). Thus, the R(K,t) value for K=3, for example, corresponds to the probability that the peak \(|x|\) up to time t exceeds three times the rms of x at time t. The figures show that this probability for K=3 varies from much less than 5.00% for \(\omega_0 t = \pi\) for all systems up to about 82.00% for \(\omega_0 t = 12\pi\) for the particular system with \(\alpha = 0, Y/F = 1\) and \(\beta_0 = 0.00\). The fact that R(K,t) increases as t increases should, of course, be expected. This corresponds to the growth of \(\bar{Y}(t)/s_x(t)\) with t, as given in table 4.1 and figures 4.19 to 4.24.

Figures 4.31 to 4.33 show the effect of yield level on the value of R for a nearly elastoplastic system, \(\alpha = 1/21\). Increasing \(Y/F\) shifts the R curves toward the left for all \(\omega_0 t\), i.e. the value of R increases with increased yielding (low yield level). Recall that R is related to the growth of \(\bar{Y}(t)/s_x(t)\), as noted in table 4.1 the effect of yielding is to increase \(\bar{Y}\), hence the increase in R with yielding is as expected.
The values of $R$ for $K=3$ and for $\omega_0 t=10\pi$ are 4.0%, 21.0% and 52.0%, respectively, for $Y/F=5, 2$ and 1.

Figure 4.34 (when compared with figure 4.31) shows that for all values of $\omega_0 t$ considered the $R$ curves shift toward the left when damping is added to the system. This decrease in $R$ due to damping decreases with increase in $\omega_0 t$ and $K$. The corresponding curves for 1.00% damping were also obtained, but they are not presented here since they were consistent with those shown. Note that figures 4.19 to 4.24 above showed that $\overline{Y}/s_X$ was reduced by the addition of damping to the system. Thus the shift of the $R$ curves toward the left was to be expected.

Figures 4.35 and 4.36 show the effect of the slope ratio $\alpha$, on the values of $R$. The $R$ curves are seen to shift to the left as the second slope increases for all $\omega_0 t$. The effect of $\alpha$ is small for $\omega_0 t=\pi$ and $2\pi$. This is reasonable since many of the sample response time histories had not yet reached the yield level at these times. This latter point is illustrated by the fact that $\overline{Y}/Y$ was in the range of 0.246 to 0.287 for $\omega_0 t=\pi$ and 0.391 to 0.623 for $\omega_0 t=2\pi$.

Some theoretical simplified Gumbel distribution curves are also included in figures 4.31 to 4.36. This extreme value distribution will be discussed in the following section. Note that figures 4.31 to 4.36 are plotted on normal probability paper. The plots are not straight lines, because, of course, the distribution of $Y$ is not normal. Plots of other forms will be used in the following section to compare the empirical results of this study with the analytical results presented there.
4.3 Extreme value theory

The statistical theory of extreme values has been found to have wide applicability in many fields. One method of analysing such problems, which was used by Kimball and others, is based on the idea of maximum likelihood. A simpler approach is the method of moments which was developed by Gumbel for application to gust load problem. Gumbel's theory has also been used by Hou, for example, to fit the distribution of maximum ground acceleration and maximum structural response of a linear system due to random exciting forces.

The classical statistics problem studied by Gumbel involves observation of the values of \( m \) independent random variables, all having the same probability distribution. Let \( y \) denote the largest of these \( m \) observed values. For any probability distribution of the original random variables meeting the condition that their distribution function tends to unity at least as fast as an exponential function, Gumbel demonstrated that the probability distribution of \( y \) tends to the form

\[
F_y(\xi) = \text{Prob}[y \leq \xi] = \exp\{-\exp\left[-(\xi - u)/q\right]\} \quad (4.8)
\]

as \( m \) tends to infinity. This distribution is called the asymptotic distribution of the largest values. One way to see the significance of this result is to consider \( N \) independent samples, each sample consisting of \( m \) independent observations, and determine the extreme value, \( y \), for each of the \( N \) samples. The above says that if \( m \) is sufficiently large then the probability distribution of the \( y \) values (\( N \) in number) should
approach equation 4.8 for large N.

Rather than using the above form directly it is often convenient to define a new random variable by

\[ z = (y-u)/q \]  \hspace{1cm} (4.9)

Then equation 4.8 gives

\[ F_z(z) = \exp[-\exp(q-z)] \]  \hspace{1cm} (4.10)

Note that the distribution of \( Z \) involves no unknown parameters. However, one must determine values for \( q \) and \( u \) before observed values of \( y \) can be converted into observed values of \( Z \).

From equation 4.10 one can show that

\[ E(Z) = y = 0.577216 \ldots \] (Euler's constant) \hspace{1cm} (4.11)

and \( \sigma_z = \pi/\sqrt{6} = 1.28255 \ldots \) \hspace{1cm} (4.12)

thus \( E(y) = u + \gamma q \) \hspace{1cm} (4.13)

and \( \sigma_y = q\pi/\sqrt{6} \) \hspace{1cm} (4.14)

Probably the simplest way to choose \( q \) and \( u \) is to equate \( E(y) \) and \( \sigma_y \) to the corresponding empirical values, \( \bar{y} \) and \( s_y \), giving

\[ q = \sqrt{6} s_y / \pi \]  \hspace{1cm} (4.15)

\[ u = \bar{y} - \gamma q \]  \hspace{1cm} (4.16)

or \[ z = \gamma + (y-\bar{y}) / s_y \sqrt{6} \]  \hspace{1cm} (4.17)

This is an application of the general method of moments, wherein population moments are approximated by empirical moments. Lieblein has called these results simplified Gumbel estimators, and that notation will also be used here.

Gumbel actually used a more complicated procedure for choosing \( q \) and \( u \) than that presented above, although both approaches are based on the method of moments. Gumbel's approach involves noting that for a given number of samples,
N, the empirical data will give certain discrete values of the function $F_z(z)$. These particular F values depend only on the number of samples, although the corresponding z values, of course, depend on the empirical data. For Gumbel's method of plotting the data these N values of F will always be given by

$$F_j = j/(N+1) \quad \text{for } j=1,2,\ldots,N \quad (4.18)$$

The division by $N+1$ rather than $N$ is primarily so that all the empirical values can be plotted for $F<1$, since $F=1$ should often correspond to infinite $K$ and is not included on most probability papers. For a distribution described by equation 4.10, one can determine the z value corresponding to each of the discrete F values:

$$z_j = -\ln[-\ln(j/(N+1))] \quad (4.19)$$

Gumbel's method consists of using the empirical mean and standard deviation of Z to approximate the arithmetic mean and standard deviation of these $z_j$ values, for the given $N$:

$$AM(z) = \frac{1}{N} \sum_{j=1}^{N} z_j \quad (4.20)$$

and

$$SD(z) = \frac{1}{N} \left( \sum_{j=1}^{N} (z_j-AM(z))^2 \right) \quad (4.21)$$

Gumbel tabulated values of $AM(z)$ and $SD(z)$ for selected values of $N$ between 15 and 1000. The two quantities approach $E(z)$ and $\sigma_z$, respectively, both from below, as $N$ becomes large. The values of $q$ and $u$ resulting from this method can be written as

$$q = s_y/SD(z) \quad (4.22)$$

and

$$u = \bar{y} - q \cdot AM(z) \quad (4.23)$$

keeping in mind that $SD(z)$ and $AM(z)$ are not empirical averages, but rather are numbers depending on $N$. These results
are called original Gumbel estimators. 

Campion has shown that the results of Gumbel's original method can be closely approximated by using the following equations

\[
AM(z) = 0.57722 - \exp(-1.057 - 0.549 \ln N - 0.0226 \ln^2 N)
\]  
\[\text{(4.24)}\]

\[
SD(z) = 1.28255 - \exp(0.4560 - 0.613 \ln N - 0.010 \ln^2 N)
\]  
\[\text{(4.25)}\]

These equations require less computation than equations 4.19 to 4.21, but one should keep in mind that each of these quantities need only be computed once for a given number of samples \( N \).

Lieblein has presented a method for choosing \( q \) and \( u \) in equation 4.8 without using the empirical mean, \( \bar{y} \), and \( SD, s_y \). Rather than being a method of moments this is an order statistic approach. Let the maxima of the samples be ordered and denoted by

\[
Y_1 < Y_2 < Y_3 < \cdots < Y_N
\]  
\[\text{(4.26)}\]

The basic idea of what Lieblein has done is to take

\[
u = \sum_{i=1}^{N} a_i y_i
\]  
\[\text{(4.27)}\]

and

\[
q = \sum_{i=1}^{N} b_i y_i
\]  
\[\text{(4.28)}\]

A function \( L_p \) is defined by

\[
L_p = u + z_p q
\]  
\[\text{(4.29)}\]

where \( z_p \) is the \( z \) value in equation 4.10 corresponding to a probability level \( P \):

\[
P_z(z_p) = P
\]  
\[\text{(4.30)}\]

The \( a_i \) and \( b_i \) coefficients in equations 4.27 and 4.28 are
then chosen to make $L_p$ a minimum variance unbiased estimator of the $\xi_p$ value corresponding to any probability $P$ in eq.4.8:

$$F_y(\xi_p) = P \quad (4.31)$$

By minimum variance unbiased estimator it is meant that

$$E(L_p) = \xi_p \quad (4.32)$$

and

$$\text{MV}=\mathbb{E}[(L_p-\xi_p)^2] = \min_{(a_i,b_i)} \mathbb{E}[(L_p-\xi_p)^2] \quad (4.33)$$

Note that these expressions say that the mean squared error in using $L_p$ to estimate $\xi_p$ is smaller than for any other unbiased estimator of the form $L_p$.

Lieblein used the method of Lagrange multipliers to obtain expressions for the $a_i$ and $b_i$, and evaluated these expressions for $N=2$ to 6. He also showed that for the minimum variance of the estimator $L$ for a probability level $P$ and a sample size $N$, $\text{MV}(N)$ was proportional to $q^2$ times a quadratic function of $z_p$. The coefficients in this quadratic were also evaluated for $N=2$ to 6.

For more than six samples the computational effort to determine the $a_i$ and $b_i$ directly becomes excessive, so a simpler approximate method was presented. When the total number of samples can be broken up into $N_1$ groups each containing $N_2$ samples ($N=N_1N_2$), the approximate method consists of finding the $a_i$ and $b_i$ for $N_2$ samples then evaluating

$$u_j = \sum_{i=1}^{N_2} a_i y_i \quad (4.34)$$

and

$$q_j = \sum_{i=1}^{N_2} b_i y_i \quad (4.35)$$

for each of the $N_1$ groups ($j=1,2,\ldots,N_1$). These subestimates are then averaged:
\[ u = \frac{1}{N_1} \sum_{j=1}^{N_1} u_j \quad (4.36) \]

and

\[ q = \frac{1}{N_1} \sum_{j=1}^{N_1} q_j \quad (4.37) \]

This approximate method does not give minimum mean squared error, as in equation 4.33, but the mean squared error is small.

In fact, it can easily be shown that for this method,

\[ \mathbb{E}[(L_p - \hat{\xi}_p)^2] = \frac{\text{MV}(N_2)}{N_1} \quad (4.38) \]

Confidence intervals for Lieblein's estimator are easily obtained. If \( N_1 \) is greater than about 10 in equations 4.36 and 4.37, then \( u \) and \( q \) are approximately normal, by the central limit theorem. The estimator \( L_p \) is also approximately normal since it is a linear combination of \( u \) and \( q \) (equation 4.29). Thus one can have approximately 95.0% confidence, for example, that the correct value, \( \hat{\xi}_p \), lies within plus or minus two standard deviations of \( L_p \) of the observed value of \( L_p \).

Using equation 4.38 this gives the 95.0% confidence limits as

\[ L_p \pm 2\left[ \frac{\text{MV}(N_2)}{N_1} \right]^{1/2}. \]

Gumbel obtained confidence limits by assuming that his estimator had an extreme value distribution of the form of equation 4.8. He further assumed that the scale parameter \( q \) for his estimator \( \hat{\xi}_p \) was the same as for \( \gamma \) (at least for the largest value of \( P \) determined, i.e. \( N/(N+1) \)).

Lieblein investigated the mean and variance of both the original and simplified Gumbel estimators. He showed that the variance of \( \xi_p \) involves a quadratic function of \( z_p \) for both of these methods, so the scale factor \( q \) for \( \xi_p \) must depend on the level \( p \). From values of the variance of \( \xi_p \) one can
determine confidence intervals, using Gumbel's assumption that \( \xi_p \) has the form of equation 4.8.

Lieblein used a semiempirical approach to obtain values for the mean and variance of the Gumbel estimators for certain situations. These results indicate that the simplified Gumbel estimator is more accurate than the original Gumbel estimator, having both smaller bias and smaller variance and, consequently, narrower confidence intervals. Lieblein's results indicate that his own order statistics method gives a more accurate estimator than does either Gumbel method. For the theoretical limiting case of \( N = \infty \) and \( P = 1.00 \) the ratio of the mean squared error for the simplified Gumbel method over that for Lieblein's method is 1.389. For other situations the ratio is nearer to 1.00.

4.4 Comparison of empirical results with extreme value theory

There is obviously some similarity between the extreme value statistic \( y \) studied by Gumbel and Lieblein and the statistics denoted by the same letter in equation 4.5 for the present empirical results. In both cases \( y \) is the maximum of a particular sample. However in equation 4.5 the sample is a single time history of the random process \( |x| \), whereas it is a collection of \( m \) independent random variables in Gumbel's problem. There seems to be sufficient similarity between the problems to warrant a comparison between the empirical probability distribution curves obtained for \( y \) in this study, and the theoretical results of Gumbel and Lieblein.

Since the empirical values of \( \bar{Y} \) and \( s_Y \) have already been obtained, it is easy to determine the parameters \( q \) and \( u \)
for the two Gumbel estimates from equations 4.15, 4.16, 4.22 and 4.23. For 80 samples Gumbel gives the other terms needed in the original Gumbel method as \( AM(z)=0.5569 \) and \( SD(z)=1.1938 \).

Comparing the parameters for Lieblein's method involves somewhat more effort. The technique of equations 4.34 to 4.37 was applied by dividing the observed values of \( y \) for each situation into 16 groups, each containing 5 samples. The \( a_i \) and \( b_i \) coefficients for these groups of sizes 5 had been given by Liebléin.

For some representative cases, figures 4.37 to 4.41 show the empirical probability distribution of \( y \) on a graph paper where equation 4.8 plots a straight line. The empirical values of \( y \) have been normalised by their own empirical SD, \( s_y \), and the occurrence of \( j \) exceedances of a given level has been considered as corresponding to a probability of \( j/(N+1) = j/81 \) of exceeding that level, as suggested by Gumbel. The straight lines resulting from choosing \( u \) and \( q \) by the simplified Gumbel method, and the Lieblein method are both included in the figures for comparison with the experimental results. The results of the original Gumbel method are not included since they were similar to those for the simplified Gumbel method.

Figures 4.37 to 4.41 also include approximate 95.0% confidence intervals for the simplified Gumbel method and Lieblein method. Using the assumption that the Lieblein estimator is approximately normal, as described in the preceding section, and the minimum variance MV(5) given by Lieblein, one
finds the 95% confidence limits for $\xi_p$ as $L_p \pm \Delta$ with

$$\Delta = \sqrt{0.08332 \, z_p^2 + 0.0339 \, z_p + 0.1157} \, q \quad (4.39)$$

The variance of the Gumbel estimator for 80 samples was not available, so it had to be estimated. The semi-empirical method used by Lieblein for smaller numbers of samples was used for this purpose.

The variance of the simplified Gumbel estimator, as given by Lieblein, is

$$(\sigma_{\xi_p}^2)^2 = q [ \, s^2 / 6n + 6(z_p-\gamma)^2 \, \sigma_s^2 / \pi^2 + 
2\sqrt{6}(z_p-\gamma) \, \text{cov}(\bar{z}, s) / \pi ] \quad (4.40)$$

where $s$ and $\bar{z}$ are defined by

$$\bar{z} = [ \sum_{i=1}^{N} z_i ] / N \quad (4.41)$$

and

$$s^2 = [ \sum_{i=1}^{N} (z_i - \bar{z})^2 ] / N \quad (4.42)$$

for independent random variables $z_1$ to $z_N$, all having the probability distribution of equation 4.9. The empirical part of the method has to do with determining $\sigma_s^2$ and $\text{cov}(\bar{z}, s)$. This was done in the present instance for $N=80$ by using 12,000 numbers from the $Z$ distribution, divided into 150 groups of 80 each. After determining an empirical $\bar{z}_j$ and $s_j$ value for each of the 150 groups, by equations 4.41 and 4.42 the unknown values were estimated by

$$\sigma_s^2 = [ \sum_{j=1}^{150} (s_j - \bar{s})^2 ] / 150 \quad (4.43)$$

$$\text{cov}(\bar{z}, s) = [ \sum_{j=1}^{150} (s_j - \bar{s}) (\bar{z}_j - \bar{z}) ] / 150 \quad (4.44)$$

with

$$\bar{s} = [ \sum_{j=1}^{150} s_j ] / 150 \quad (4.45)$$
and \[ \bar{Z} \neq \frac{150}{\sum_{j=1}^{150}} \bar{Z}_j / 150 \] (4.46)

The results were that for \( N = 80, \sigma_s^2 = 0.02022 \) and \( \text{cov}(\bar{Z}, s) = 0.1116 \).

Using Gumbel's assumption that this estimator has a distribution of the form of equation 4.8, and writing an approximate 95.0% confidence interval as \( \xi_p \pm \Delta \) gives

\[ \Delta = 3.06685 \sigma_{\xi_p} \] (4.47)

The above results for the variance of \( \xi_p \) then give

\[ \Delta = 3.06685 \sqrt{6} q (1.229 \ z_p^2 + 0.3937 \ z_p + 1.4194) / \pi \] (4.48)

for 80 samples. Recall that \( z_p \) can be evaluated by

\[ z_p = -\ln(-\ln P) \] (4.49)

In the case shown, all the empirical data for \( F_y(\xi) \) fall within both the 95.0% confidence bands just described. This indicates that for these cases equation 4.8 is at least a fair approximation of the true probability distribution of the \( y \) variable in this study.

The figures presented include a range of yield level values for \( \alpha = 1.21 \) and \( \beta_0 = 0.00 \) (figures 4.37 to 4.39), a similar case with \( \beta_0 = 0.05 \) (figure 4.40) and one case for \( \alpha = 1/2 \) (fig. 4.41). None of these parameter variations seems to have any significant effect on the goodness of fit of the theoretical lines.

One should note that the examples shown in figures 4.37 to 4.41 all correspond to fairly large values of \( \omega_0 t \) such that the system response is nearly stationary. Table 4.2 presents the \( u \) and \( q \) values for each of the theoretical methods for the near stationary response of all the systems studied.
From the above it appears that the simplified Gumbel method fits the data about as well as the other theoretical methods presented, and it certainly is easiest to use, particularly if only the estimator (not the confidence limits) is desired. The simplified Gumbel estimator was also computed from \( \bar{y} \) and \( s_y \) for all the systems shown in figures 4.31 to 4.36. The resulting estimates are compared with the data in those figures (which are on normal probability paper). There is good agreement between the data and the estimator.

There are various other methods of analyzing extreme value data. Davenport, for example, has derived an asymptotic probability distribution for the peak value \( y \) of a sample from the stationary Gaussian process \( x \). This distribution has the form

\[
F_y(\xi) = \exp[-\lambda T \exp(-\xi^2)/2] 
\]

(4.50)

where

- \( T \) is the total time of one sample, and
- \( \lambda \) denotes the average number of zero crossings per unit time, so for lightly damped system \( \lambda T \) is approximately the number of oscillations during the sample.

In Davenport's theory the value of \( \lambda T \) should be large, and if \( \lambda T \) is of moderate size then some deviation of the true distribution from the given form should be expected. Malhotra and Penzien used this approach to predict the extreme value statistics of a linear system subjected to white noise. In the present study the maximum value of \( \lambda T \) considered was 16, which is very small as compared to values used by others.
(The value of $\lambda T$ used by Davenport and Malhotra was of the order of 1000) For this value of $\lambda T$ the deviation expected is as much as 100.0\% of the true probability and hence it was decided that it would not be worth while to study this approach.
CHAPTER V

EQUIVALENT LINEAR SYSTEMS

5.1 Introduction

The idea of seeking an equivalence between linear (23) and nonlinear systems was first presented by Jacobsen in 1930. He equated the work done at the end of every cycle by a hysteretic force and by the "equivalent" viscous damping. The majority of work on equivalent linearization has been done for systems subjected to periodic excitation. (10,20,24) Limited work has been done with earthquake-like excitation (20,23) and random excitation. (10,33)

There are two basic reasons for seeking equivalence of response between nonlinear systems and linear systems. Both are based on the fact that the statistics of response of a simple linear oscillator to random excitation are well known. First, the physical significance of experimental results for nonlinear systems may become more obvious if a linear system can be found which has approximately the same response to the same excitation. Second, if an analytical method is available for choosing the parameters of the equivalent linear system without first knowing the dynamic response of the nonlinear system, then the method can be used to predict the nonlinear system response.

The results of various equivalent linearization schemes for the stationary response of the bilinear hysteretic systems considered in this study have been summarized in reference 33. The following quotations give some of the conclusions of that study.
1. A particular "equivalent" linear system can generally only be expected to match a limited number of response statistics of a particular nonlinear system with a particular type of excitation. Greatly different "equivalent" linear systems are found when different response statistics or excitations are considered.

2. Equivalent viscous damping in a linear system with stiffness equal to the initial tangent stiffness of a yielding system can be chosen to match only one statistic of response of the linear and nonlinear system. Considerably higher levels of equivalent viscous damping may be found, for example, if equivalence is defined in terms of velocity response than if it is defined in terms of displacement response. Also the results based on displacement response to random excitation are inconsistent with those for harmonic excitation.

3. A linear system in which both stiffness and damping are chosen for equivalence can match two statistics of the response of a nonlinear system. The values of equivalent viscous damping obtained by matching both random displacement and velocity response levels in such a system are consistent with those for harmonic excitation.

It is obvious that if two systems do not have the same levels of stationary response, then their levels of transient response must also diverge. For that reason the only linear systems considered for equivalence to a particular hysteretic system in the present study are ones which match at least one statistic of stationary response of the hysteretic system. If a particular linear system matches the stationary displacement response \( \overline{x_0^2} \) but not the stationary velocity response \( \overline{x_0^*^2} \) of a hysteretic system, for example, then the transient \( \overline{x^2} \) values will be compared for the two systems, but \( \overline{x^{k2}} \) values will not be compared.
5.2 Equivalent constant coefficient linear system

One of the simplest equivalent linear systems is one which has the same resonant frequency as the small amplitude resonant frequency of the nonlinear system. (This is the system of conclusion 2 from reference 33) Such a linear system is governed by

\[ x'' + 2\beta_{eq}\omega_0 x' + \omega_0^2 x = G(t)/m \]  

(5.1)

The damping coefficient can be chosen so as to match some measure of the response of equation 5.1 with the corresponding measure for some particular nonlinear system.

Matching displacement: Hudson used the system of equation 5.1 as an equivalent linear system for nonlinear hysteretic systems excited by harmonic and earthquake-like excitations. He matched the maximum transient displacement of linear and nonlinear systems, both subjected to the earthquake-like excitation. Using thirty seconds of strong earthquake-like excitation, he obtained \( \beta_{eq} = 0.04 \) for a system with a strongly nonlinear curved hysteretic loop. For a given amplitude of harmonic exciting force Hudson also noted that the amplitude of resonant response \( A \) is approximately the same for equation 5.1 and equation 2.1 if

\[ \beta_{eq} = \frac{1}{\pi} (1 - \alpha) \frac{Y}{A} (1 - \frac{Y}{A}) \text{ for } A \gg Y \]  

(5.2)

The maximum damping to be expected is then 16% for the elasto-plastic system, and for the moderately nonlinear system and nearly elastoplastic system of this study the maximum damping values are 8% and 15% respectively.

One way presented in reference 33 for choosing \( \beta_{eq} \) is to match the stationary mean squared displacement level
of equation 5.1 and 2.1. Letting $\beta_{eqX}$ denote the $\beta_{eq}$ chosen in this way, one obtains

$$\beta_{eqX} = \frac{\pi F^2}{4x_S^2}$$

(5.3)

Values of $\frac{x_S^2}{F^2}$ obtained from the analog computer results in ref. 22 for the systems with $\alpha=1/2$ and $1/21$ were used to evaluate $\beta_{eqX}$. For the particular systems considered in this study table 5.1 shows the difference between the resulting equivalent viscous damping and actual viscous damping, $\beta_{eqX} - \beta_0$. This is the additional damping due to the hysteretic nonlinearity.

One notes from table 5.1 that additional damping due to yielding is relatively small in all cases. The maximum values are of the same order of magnitude as the value found by Hudson for earthquake-like excitation of his system.

Curve A in figures 5.1 to 5.8 shows the transient mean squared displacement response obtained by using the analytical solution for the linear system with parameters $\omega_0$ and $\beta_{eqX}$. Curve D in these figures represents the empirical mean squared displacement response of the corresponding nonlinear system. At all times considered the response of the nonlinear system is much higher than that of the "equivalent" linear system, particularly for the nearly elastoplastic system. It appears that this method of matching mean squared response cannot be used to predict the transient response of the bilinear system, except possibly when the system is nearly linear.

**Matching velocity:** An alternate choice of equivalent viscous damping is to match the stationary mean squared velocity
responses of equations 5.1 and 2.1. Denoting the $\beta_{eq}$ values obtained in this way as $\beta_{eqx}$, expression 2.18 gives

$$\beta_{eqx} = (\pi/4) \left( \frac{F^2 \omega_o^2}{K_g^2} \right)$$

(5.4)

The stationary results for the systems with $\alpha=1/2$ and $\alpha=1/21$ were used to evaluate $\beta_{eqx}$, and table 5.2 shows the resulting difference, $\beta_{eqx} - \beta_0$ (the additional damping due to yielding).

As noted in conclusion 2 from reference 33, one can observe that matching velocities results in considerably larger values of additional damping due to yielding, then does matching displacements. The mean squared velocity response for the linear system with parameters $\beta_{eqx}$ and $\omega_o$ was calculated using the analytical solution and is plotted as curve A in figures 5.9 to 5.16. Curve D again represents the empirical values for the nonlinear systems. As for the displacement response, the transient response of the nonlinear system is much higher than that of the "equivalent" linear system. However, the values predicted are much better than for displacement response.

Matching displacement and velocity: Both the above methods only consider the damping effect of the hysteretic nonlinearity. Another approach is to consider a linear system in which both resonant frequency and damping factor are chosen for equivalence, in order to account for the softening spring effect as well as the damping effect of the hysteretic nonlinearity. Consider a linear system given by

$$x'' + 2\beta_{eq} \omega \omega_{eq} x' + \omega_{eq}^2 x = G(t)/m$$

(5.5)

Jennings has previously used this system in
studying the equivalent viscous damping for systems with harmonic excitation. His approach gives maximum damping values of approximately $11.8\%$ for $\alpha=1/2$ and $58\%$ for $\alpha=1/21$.

A simple way to choose $\beta_{eq}$ and $\omega_{eq}$ for random vibration is to match both the mean squared displacement and mean squared velocity of stationary response. Equation 2.18 gives these values of $\beta_{eq}$ and $\omega_{eq}$ as

$$
\omega_{eq} = \omega_0 \sqrt{\frac{\bar{x}_S^2}{\bar{x}_S^2 + \bar{v}_S^2}} \frac{F^2}{\bar{v}_S^2}
$$

(5.6)

and

$$
\beta_{eq} = \frac{\pi}{4} \frac{F^2 \omega_0^2}{\bar{x}_S^2} \frac{\omega_0}{\omega_{eq}}
$$

(5.7)

These are the results referred to in conclusion 3 from ref. 33 above.

The experimental values of reference 22 were used to evaluate $\omega_{eq}/\omega_0$ and $\beta_{eq}$ and the values obtained are given in tables 5.3 and 5.4. As should be expected, $\omega_{eq}$ decreases from approximately $\omega_0$ to approximately $\omega_0 \sqrt{\alpha}$ as $Y/F$ decreases from large to small values.

If the nonlinear system had no energy dissipation other than viscous damping then one should expect $\beta_{eq} \omega_{eq} = \beta_0 \omega_0$ and so the softening spring effect increases the effective viscous damping to $\beta_0 \omega_0 / \omega_{eq}$. For this reason the additional damping due to hysteretic energy dissipation is tabulated as $\beta_{eq} - \beta_0 \omega_0 / \omega_{eq}$ in table 5.4. The variation of this quantity has similar trends as in table 5.2, however, the magnitudes are significantly greater than those given by matching velocity only.
Using the values of $\beta_{eq}$ and $\omega_{eq}$ given by tables 5.3 and 5.4, the analytical solution of the mean squared displacement and velocity response for the linear system of equation 5.3 were calculated and plotted as curve B in figures 5.1 to 5.16. For the moderately nonlinear system, $\alpha=1/2$, the equivalent system in some cases underpredicts the transient mean squared displacement response, as in $Y/F=6$ and 15, and can be off as much as 50%. For lower yield levels ($Y/F=1$ and 2), though, the agreement is somewhat better. For the nearly elastoplastic system the "equivalent" linear system overpredicts the transient displacement response in all cases. For $Y/F=15$, though, this error is small.

Comparing curves B and D in figures 5.9 to 5.16, for mean squared velocity response, one finds that for $\alpha=1/21$ the response of the "equivalent" linear system builds up less rapidly than that of the bilinear system, while for $\alpha=1/2$ it varies irregularly, in some case exceeding that of the nonlinear system. The prediction of transient velocity response by using this approach may be off as much as 50%.

An overall comparison of curves A, B and D, shows that matching both displacement and velocity response (curve B) gives a better "equivalent" linear system for predicting transient response, then does matching only one statistic. This agrees with the conclusions of reference 33 for stationary response. It appears that these two parameter equivalent linear systems may serve to adequately predict transient response levels of the bilinear system with $\alpha=1/2$ in many situations where great accuracy is not required. For $\alpha=1/21$,
on the other hand, the results are less satisfactory, except when the yield level is so high that the bilinear system is almost linear in behavior \((Y/F=15)\). Note that for low yield levels, with \(a=1/21\), the two parameter linear system simultaneously overpredicts the transient displacement response and underpredicts the transient velocity response.

5.3 Krylov-Bogoliubov method

The above methods for choosing the parameters of the equivalent linear systems require prior knowledge of the stationary random response levels of the system. Thus those methods cannot be used to predict response statistics about which nothing is known. However, an extension of the method of Krylov-Bogoliubov can be used to choose the parameters in the linear system of equation 5.5 without first knowing the response of the nonlinear system. (A method, suggested by Liu for choosing \(\beta_{eq}\) based only on prior knowledge about periodic response, will not be considered in this study since it was shown in reference 33 that the method does not predict the correct levels of stationary response of the systems considered here.)

Caughey applied the Krylov-Bogoliubov method to the study of the stationary response of the bilinear hysteretic oscillator. In this method the parameters \(\omega_{eq}\) and \(\beta_{eq}\) are chosen to minimize the mean squared difference between equation 2.1 and 5.5. Certain assumptions of small nonlinearity are used in carrying out this process of minimization.

It was shown in reference 22 that the Krylov-Bogoliubov method predicts the rms displacement and velocity
response levels of the bilinear system with $\alpha=1/2$ within about 15%. Thus the values of $\omega_{eq}/\omega_0$ and $\beta_{eq}=\beta_0\omega_0/\omega_{eq}$ from this method are similar to those given in tables 5.3 and 5.4, for $\alpha=1/2$. Curve E in figures 5.1 to 5.4 and 5.9 to 5.12 shows the transient response of this "equivalent" linear system. Curve E is of course, similar to curve B in all cases. Thus, for the moderately nonlinear system one can derive the equivalence parameters and predict both transient and stationary response levels with a fair degree of accuracy without any prior empirical investigations.

The Krylov-Bogoliubov method does not accurately predict the levels of stationary response of the nearly elastoplastic system. Thus the transient response of that system was not studied by this method.

5.4 Equivalent variable coefficient linear system

The parameters of the linear systems described in the preceding section were chosen to match levels of stationary response of the linear system and nonlinear system. For very small $\omega_0 t$ (before yielding begins) each of the nonlinear systems acts as a linear system with stiffness $\omega_0^2$ and damping $\beta_0$. The equivalent linear systems presented, though, have some other stiffness (for equation 5.5) and damping based on the stationary response, which will occur later (when yielding is occurring). An effect of this inconsistency was noted earlier. Namely, the reduced stiffness and increased damping of the equivalent linear system (equation 5.5) for $\alpha=1/21$ and small $Y/F$, caused this linear system to have a displacement response which was larger and a velocity response that was smaller.
than the corresponding measures for the bilinear system, for small $\omega_0 t$.

A more logical, but less simple, approach is to let $\omega_{eq}$ and $\beta_{eq}$ in equation 5.5 vary with time. The parameters at a given time $t$ can then be chosen at the levels appropriate to the amount of yielding taking place at that time. A precise solution of such a problem with variable coefficients which are functions of the response level would be very difficult. A simplified approach based on this idea was used, though, in the present study.

The approach used here involves keeping $\omega_{eq}$ and $\beta_{eq}$ constant within a given fixed interval of $\omega_0 t$, then instantaneously changing the parameters before the next time interval. The values assigned to $\omega_{eq}$ and $\beta_{eq}$ for a particular time interval are based on the level of $s_x/Y$ at the beginning of that time interval. The interval of $\omega_0 t$ used was $\pi/25$. Use of a small interval reduces the error due to changes in $s_x/Y$ during the interval.

The equivalent damping and stiffness used, as functions of $s_x/Y$, are shown in figures 5.17 and 5.18. These are actually the equivalence parameters previously described which match both stationary displacement and velocity levels of equations 2.1 and 5.5. These two figures are reproduced from reference 33, and are based on empirical results. Note that it is being assumed that the equivalent stiffness and damping of a system depend only on $s_x/Y$, and not on whether that $s_x/Y$ corresponds to stationary response or transient response. In order to put figures 5.17 and 5.18 into the computer each curve was
approximated by a polynomial. The degree of these polynomials varied from 9 to 11, and the coefficients were chosen on the basis of minimum mean squared error.

Consider a linear system described by equation 5.5 with random initial conditions $x_o$ and $x_o^*$. The response at time $t$ is described by

$$E[x^2(t)] = E[x_o^2] k^2(t) + E[x_o^*2] h^2(t)$$

$$+ 2 E[x_o x_o^*] k(t) h(t) + D[1 - \exp(-2\beta_{eq} \omega_{eq} t)]$$

$$\{(1 + 2 \beta_{eq}^2 \sin^2 \varphi_{eq} t)/(1 - \beta_{eq}^2) +$$

$$\beta_{eq} (\sin 2\varphi_{eq} t)/\sqrt{1 - \beta_{eq}^2}/2\beta_{eq} \omega_{eq}^3 \}$$

(5.8)

$$E[x^*2(t)] = E[x_o^2] k^{'2}(t) + E[x_o^*2] h^{'2}(t)$$

$$+ 2 E[x_o x_o^*] k'(t) h'(t) + D[1 - \exp(-2\beta_{eq} \omega_{eq} t)]$$

$$\{(1 + 2\beta_{eq}^2 \sin^2 \varphi_{eq} t)/(1 - \beta_{eq}^2) -$$

$$\beta_{eq} (\sin 2\varphi_{eq} t)/\sqrt{1 - \beta_{eq}^2}/2\beta_{eq} \omega_{eq} \}$$

(5.9)

and

$$E[x(t)x^*(t)] = E[x_o^2] k(t) k'(t) + E[x_o^*2] h(t) h'(t)$$

$$+ 2 E[x_o x_o^*] [ k'(t) h(t) + k(t) h'(t)]$$

$$+ D \exp(-2\beta_{eq} \omega_{eq} t)(\sin^2 \varphi_{eq} t)/\varphi_{eq}^2$$

(5.10)

where

$$\varphi_{eq}^2 = \omega_{eq} \sqrt{1 - \beta_{eq}^2}$$

(5.11)
and the free vibration response functions h and k are given
by equations 2.20 and 2.21 with \( \omega_0 \) and \( \beta_0 \) replaced by \( \omega_{eq} \)
and \( \beta_{eq} \). By treating t as elapsed time since the beginning
of a particular time interval, equations 5.8 to 5.10 allow
computation of the expected response at the end of the time
interval from expected conditions at the beginning of the
interval. This then gives expected initial conditions for the
next time interval (where \( \omega_{eq} \) and \( \beta_{eq} \) have been modified).
Thus, a straightforward procedure yields values of mean squared
displacement and velocity response at discrete values of time.

Curves C in figures 5.1 to 5.16 show the mean squared
response levels predicted by the method just described. One
noted that these results are generally better than those
obtained from the constant coefficient linear systems. The
method sometimes overpredicts and sometimes underpredicts the
response, but the variation is generally less than 20% of the
empirical results. It appears that this method does allow one
to predict the transient mean squared levels of response with
reasonable accuracy. The method does require, however, prior
knowledge of \( \omega_{eq} \) and \( \beta_{eq} \) as functions of \( s_x/y \), such as might
be obtained from empirical studies of stationary response.

5.5 Damped systems

The above comparisons of linear and nonlinear transient response have included only nonlinear systems with no
viscous damping. It was shown in reference 22 that the addition
of damping, for one thing, can result in response which is more
nearly normal in probability distribution. Thus one would
generally expect an equivalent linearization scheme to work
better when viscous damping is present than when it is not (since a linear system has a normal response). Figures 5.19 to 5.22 show a few selected results for systems containing viscous damping.

A comparison of figures 5.19 and 5.20 with 5.2 and 5.10 for $\alpha=1/2$ and $Y/F=3$, shows that the equivalent linearization methods do work slightly better when $\beta_0=0.01$ than when $\beta_0=0.00$ in this system, particularly for displacement response. This was found to also be true for other systems with $\alpha=1/2$. This trend does not appear, however, in the displacement response of the nearly elastoplastic system. Figs. 5.21 and 5.22, when compared with figure 5.6, show that making $\beta_0=0.01$ has little effect on the accuracy of equivalent linearization for $\alpha=1/21$ and $Y/F=2$, but increasing $\beta_0$ to 0.05 significantly reduces accuracy. This unexplained reduction in accuracy for $\beta_0=0.05$ was also noted in the displacement response for $\alpha=1/21$ and $Y/F=5$ as shown in figure 5.22. The velocity response, in each case, did show increased accuracy with increased damping.
CHAPTER VI
SUMMARY AND CONCLUSIONS

A digital computer was used to empirically investigate the transient response of the bilinear hysteretic single mass oscillator with zero initial conditions, subjected to an approximately (digitally simulated) white, stationary and Gaussian excitation. The second slope parameter of the bilinear restoring force curve (a) was chosen to have values of 1/2 and 1/21 because of the availability of the stationary response results for these situations. These stationary results give limiting values for the statistics of transient response. The slope parameter equal to 1/21 represents a nearly elastoplastic system, while a=1/2 is a moderately nonlinear system. The elastoplastic system (a=0) was also studied as a limiting case of the bilinear system. The responses which were determined in this study were mean squared levels of displacement and velocity, probability distribution of displacement and velocity response, and probability distribution of peak displacement up to time t. The empirical values for the two types of probability distribution were compared with normal and Gumbel distributions, respectively. The coefficients in the Gumbel distribution and the corresponding confidence limits were evaluated by three different methods suggested by Gumbel (16) and Lieblein (28).

The empirical curves showing the build-up of mean squared displacement and velocity response for the bilinear system were compared with corresponding analytical results for single mass linear oscillators. Attempts were made to evaluate the parameters in the linear system to make them
"equivalent" to the nonlinear systems. Both constant coefficient and variable coefficient linear systems were considered, with the parameters chosen based on prior knowledge of the stationary response levels of the nonlinear systems. The only technique suggested for finding such an "equivalent" linear system for some particular nonlinear system without any prior knowledge about the response of the nonlinear system was based on the Krylov-Bogoliubov method.

The following paragraphs summarize the results of the above studies.

1. The stationary, white, Gaussian excitation can be digitally simulated by a step-function-type process which changes value only at uniformly spaced discrete instants of time. This process was analytically shown to approximate white noise, both on the basis of autocorrelation function and on the basis of the response of a linear oscillator to the two excitations.

2. For a system with specified slope ratio ($\alpha$) and damping ($\beta_0$) the variations of transient mean squared response due to changes in yield level were found to be consistent with the variations of stationary response for that system. Specifically if the stationary response was smaller for some yield level $Y_1$ than for some other yield level $Y_2$, then for all values of time considered the transient response was smaller for $Y_1$ than for $Y_2$. The transient levels were not proportional to the stationary levels, however.
3. For small values of time, the mean squared levels of displacement response of the elastoplastic system (α=0) were almost the same as those for α=1/21. However, the response for α=0 continued to grow for larger values of $\omega_0 t$, when the response for α=1/21 approached stationarity. The mean squared velocity response of the two systems was approximately the same for all situations studied.

4. The rate of growth of mean squared response was found to be relatively independent of yield level and damping for the moderately nonlinear system (α=1/2). For this system the time to reach 80.0% of the stationary mean squared displacement response, at which time the response was considered to be approximately stationary, was given by $\omega_0 t_{0.80} = 0.67 \overline{x}_0^2/F^2$. Similarly, for α=1/21 the time to reach 80.0% of the stationary mean squared displacement response for small damping ($\beta_0 \leq 0.01$) and high yield level ($Y/F \geq 5$) was given by $\omega_0 t_{0.80} = \overline{x}_0^2/F^2$. This formula did not apply, however, for low yield levels ($Y/F=1$ and 2) in the nearly elastoplastic system, where the rate of growth of response was larger. In these latter situations it was found that $\omega_0 t_{0.80} = 5\pi$. The ratio of the time to reach 80.0% of stationary response over the time to reach 50.0% of the stationary response, for the nonlinear systems, was found to be about the same as for linear systems (≈2.3).
5. The empirical curves of probability distribution of the response at various instants of time were found to fall within a fairly narrow band about the normal distribution, for the range of probabilities determined in this study. Significant deviation from the normal distribution can be neither proved nor disproved on the basis of these results.

6. The empirical curves of the probability distribution of the maximum absolute response prior to a given time, \( y(t) \), were found to agree quite well with the Gumbel distribution, especially for large values of time \( t \). The agreement was acceptable, with the empirical curves falling within 95% confidence bands, for each of the methods used to evaluate the parameters in the Gumbel distribution. The so-called simplified Gumbel method involving only the empirical mean and standard deviation of peak response, is certainly the easiest way to evaluate these parameters.

7. The mean \( \bar{y} \) and standard deviation \( s_y \) of the peak response grow with increasing time. For the moderately nonlinear system, \( \alpha = 1/2, \bar{y}/F \) and \( s_y/F \) were nearly independent of yield level \( (Y/F) \) and damping \( (\beta_0) \) for any given value of time. For \( \alpha = 0 \) and \( 1/2 \), \( \bar{y}/F \) and \( s_y/F \) varied more noticeably with changes in \( Y/F \) and \( \beta_0 \), being largest for the parameters which made rms response \( (s_x/F) \) largest and smallest when \( s_x/F \) was smallest. The values of \( \bar{y}/F \) and \( s_y/F \) were found to be largest for the elastoplastic system (\( \alpha = 0 \)) and smallest for the moderately nonlinear
system \((a=1/2)\). The ratio \(\bar{y}/s_x\) was found to be the largest for the \(Y/F\) which makes \(s_x/F\) smallest, for a particular value of \(\alpha\) and \(\beta_o\). Addition of damping reduces \(\bar{y}/s_x\) in a quite irregular manner. The ratio \(s_y/s_x\) for a given system decreases somewhat with increasing \(\omega_0 t\).

8. It was found that neglecting the softening spring effect of the hysteretic system results in predictions of mean squared response which grow much too slowly. This was observed both when the damping in an "equivalent" constant coefficient linear system was chosen to match stationary displacement response, and when it was chosen to match stationary velocity response, although the discrepancy was smaller for velocity response.

9. Choosing both damping and stiffness of the constant coefficient linear system to simultaneously match stationary levels of displacement and velocity gave significantly better predictions of transient response than did matching only one statistic. For \(a = 1/2\) this method gave quite accurate predictions of velocity response, but the predictions of mean squared displacement were sometimes off as much as 50\% (for high yield level). For the nearly elastoplastic system this method predicts displacement growth which is too rapid and velocity growth which is too slow.

10. One can predict the mean squared displacement and
velocity response by the Krylov-Bogoliubov method with almost the same accuracy as in the preceding paragraph without prior knowledge about stationary response, for the moderately nonlinear system, (\( \alpha = 1/2 \)). This method will not give an accurate prediction for the nearly elastoplastic system as it does not predict accurately the stationary levels of response in that case.

The best "equivalent" linear system was one with variable damping and stiffness coefficients. Prior knowledge about the stationary response of the system was used to vary the coefficients as functions of the amount of yielding taking place. This method gave acceptable predictions of mean squared response levels for all systems considered.
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APPENDIX B

EMPIRICAL MEAN SQUARED RESPONSES
FIG. B-1 MEAN SQUARED DISPLACEMENT RESPONSE

FIG. B-2 MEAN SQUARED DISPLACEMENT RESPONSE
FIG. B-3 MEAN SQUARED DISPLACEMENT RESPONSE

FIG. B-4 MEAN SQUARED DISPLACEMENT RESPONSE
FIG. B-5 MEAN SQUARED VELOCITY RESPONSE

FIG. B-6 MEAN SQUARED VELOCITY RESPONSE
FIG. B-7 MEAN SQUARED VELOCITY RESPONSE

FIG. B-8 MEAN SQUARED VELOCITY RESPONSE
FIG. B-9 MEAN SQUARE DISPLACEMENT RESPONSE

FIG. B-10 MEAN SQUARE DISPLACEMENT RESPONSE
FIG. B-11 MEAN SQUARED DISPLACEMENT RESPONSE

FIG. B-12 MEAN SQUARED DISPLACEMENT RESPONSE
FIG. B-13 MEAN SQUARED VELOCITY RESPONSE

FIG. B-14 MEAN SQUARED VELOCITY RESPONSE
FIG. B-15 MEAN SQUARED VELOCITY RESPONSE

FIG. B-16 MEAN SQUARED VELOCITY RESPONSE
FIG. B-17 MEAN SQUARED DISPLACEMENT RESPONSE

FIG. B-18 MEAN SQUARED VELOCITY RESPONSE
FIG. 2.1 BILINEAR HYSTERETIC OSCILLATOR

FIG. 2.2 BILINEAR HYSTERETIC RESTORING FORCE
FIG. 2.4 CUMULATIVE PROBABILITY DISTRIBUTION—Normal numbers
FIG. 2.5 AUTOCORRELATION FUNCTION OF DIGITAL WHITE NOISE
FIG. 3.1 MEAN SQUARED DISPLACEMENT RESPONSE OF A LINEAR SYSTEM -- $\beta_0 = 0.10$

FIG. 3.2 MEAN SQUARED VELOCITY RESPONSE OF A LINEAR SYSTEM -- $\beta_0 = 0.10$
FIG. 3.4 STATIONARY MEAN SQUARED VELOCITY RESPONSE
FIG. 3.6 STATIONARY MEAN SQUARED VELOCITY RESPONSE
FIG. 3.7 SAMPLE TIME HISTORY OF TRANSIENT DISPLACEMENT RESPONSE
FIG. 3.10 MEAN SQUARED DISPLACEMENT RESPONSE
\[ \alpha = \frac{1}{2} \quad \beta_0 = 0.0 \]
FIG. 3.12 MEAN SQUARED DISPLACEMENT RESPONSE
FIG. 3.14 MEAN SQUARED VELOCITY RESPONSE
FIG. 3.15 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

FIG. 3.16 TRANSIENT MEAN SQUARED VELOCITY RESPONSE
FIG. 3.17 TIME TO REACH SPECIFIED RESPONSE LEVEL

FIG. 3.18 TIME TO REACH SPECIFIED RESPONSE LEVEL
**FIG. 3.19 TIME TO REACH SPECIFIED RESPONSE LEVEL**

**FIG. 3.20 TIME TO REACH SPECIFIED RESPONSE LEVEL**
FIG. 3.21 TIME TO REACH SPECIFIED RESPONSE LEVEL
FIG. 3.22 RATE OF GROWTH OF MEAN SQUARED RESPONSE
FIG. 4.3 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION

FIG. 4.4 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION

FIG. 4.5 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION

FIG. 4.6 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION
FIG. 4.7 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION

FIG. 4.8 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION

FIG. 4.9 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION

FIG. 4.10 BOUNDS OF TRANSIENT PROBABILITY DISTRIBUTION
Fig. 4.11 Bounds of transient probability distribution

Fig. 4.12 Bounds of transient probability distribution

Fig. 4.13 Bounds of transient probability distribution

Fig. 4.14 Bounds of transient probability distribution
FIG. 4.16 CONVERGENCE OF MEAN AND STANDARD DEVIATION OF PEAK DISPLACEMENT
$\alpha = \frac{1}{21}$  \hspace{1cm}  $\beta_0 = 0.01$  \hspace{1cm}  $\gamma/F = 5.0$

FIG. 4.17 CONVERGENCE OF MEAN AND STANDARD DEVIATION OF PEAK DISPLACEMENT.
FIG. 4.19 MEAN OF PEAK DISPLACEMENT RESPONSE

FIG. 4.20 MEAN OF PEAK DISPLACEMENT RESPONSE
FIG. 4.21 MEAN OF PEAK DISPLACEMENT RESPONSE

FIG. 4.22 MEAN OF PEAK DISPLACEMENT RESPONSE
\[\alpha = \frac{1}{2} \quad \beta_0 = 0.0\]

\[\frac{\bar{y}}{\sigma_x}\]

\[0 \quad 4\pi \quad \omega_0 t \quad 8\pi \quad 12\pi\]

FIG. 4.23 MEAN OF PEAK DISPLACEMENT RESPONSE

\[\alpha = \frac{1}{2} \quad \beta_0 = 0.01\]

\[\frac{\bar{y}}{\sigma_x}\]

\[0 \quad 4\pi \quad \omega_0 t \quad 8\pi \quad 12\pi\]

FIG. 4.24 MEAN OF PEAK DISPLACEMENT RESPONSE
FIG. 4.25 VARIANCE OF PEAK DISPLACEMENT RESPONSE

FIG. 4.26 VARIANCE OF PEAK DISPLACEMENT RESPONSE
FIG. 4.27 VARIANCE OF PEAK DISPLACEMENT RESPONSE

FIG. 4.28 VARIANCE OF PEAK DISPLACEMENT RESPONSE
\[ \alpha = \frac{1}{2} \quad \beta_0 = 0.0 \]

\[ \frac{s_y}{s_x} \]

\[ \gamma/F \]

\[ 1 \quad 3 \quad 6 \quad 15 \]

0 \quad 4\pi \quad 8\pi \quad \omega_0 t \quad 12\pi \quad 16\pi

**FIG. 4.29 VARIANCE OF PEAK DISPLACEMENT RESPONSE**

\[ \alpha = \frac{1}{2} \quad \beta_0 = 0.01 \]

\[ \frac{s_y}{s_x} \]

\[ \gamma/F \]

\[ 1 \quad 3 \quad 6 \quad 15 \]

0 \quad 4\pi \quad 8\pi \quad \omega_0 t

**FIG. 4.30 VARIANCE OF PEAK DISPLACEMENT RESPONSE**
\( \alpha = \frac{1}{21} \quad \beta_0 = 0.0 \quad \gamma/F = 1.0 \)

**FIG. 4.31** PROBABILITY DISTRIBUTION OF PEAK DISPLACEMENT RESPONSE
FIG. 4.32 PROBABILITY DISTRIBUTION OF PEAK DISPLACEMENT RESPONSE
\[ \alpha = \frac{1}{21} \quad \beta = 0.0 \quad \gamma/F = 5.0 \]

\[ \omega_0 t = 2\pi \quad 4\pi \quad 6\pi \quad 8\pi \quad 14\pi \]

\[ R(K, t) \% \]

FIG. 4.33 PROBABILITY DISTRIBUTION OF PEAK DISPLACEMENT RESPONSE
$\alpha = \frac{1}{21}$   \hspace{1cm} $\beta_0 = 0.05$   \hspace{1cm} $Y/F = 1.0$

*SIMPLIFIED GUMBEL DISTRIBUTION*

**FIG. 4.34 PROBABILITY DISTRIBUTION OF PEAK DISPLACEMENT RESPONSE**
$\alpha = \frac{1}{2} \quad \beta = 0.0 \quad \gamma = 1.0$

$\omega_0 t = \pi, 2\pi, 4\pi, 6\pi, 8\pi$

$R(K,t)$

$99, 99.9, 99.99$

$k = 0, 1, 2, 3, 4, 5$

Fig 4.35 Probability distribution of peak displacement response
FIG 4.36 PROBABILITY DISTRIBUTION OF PEAK DISPLACEMENT RESPONSE
\[ \alpha = \frac{1}{21} \quad \beta_0 = 0.0 \quad \gamma / F = 1.0 \]

**EMPIRICAL DATA**

95% CONFIDENCE INTERVAL

LIÉBLEIN

GUMBEL

**FIG. 4.37** COMPARISON OF EMPIRICAL DISTRIBUTIONS OF PEAK RESPONSE WITH RESULTS OF LIÉBLEIN AND SIMPLIFIED GUMBEL FOR \( \omega_0 t = 12\pi \)
$\alpha = \frac{1}{21}$ \hspace{1cm} $\beta_0 = 0.0$ \hspace{1cm} $\gamma / F = 2.0$

- EMPIRICAL DATA

---

**FIG. 4.38** COMPARISON OF EMPIRICAL DISTRIBUTIONS OF PEAK RESPONSE WITH RESULTS OF LIEBLEIN AND SIMPLIFIED GUMBEL FOR $\omega_0 t = 12\pi$
\[ \alpha = \frac{1}{21} \quad \beta = 0.0 \quad \gamma/f = 5.0 \]

**EMPIRICAL DATA**

**FIG. 4.39** COMPARISON OF EMPIRICAL DISTRIBUTION OF PEAK RESPONSE WITH RESULTS OF LIEBLEIN AND SIMPLIFIED GUMBEL FOR \( \omega_0 t = 14\pi \)
Fig. 4.40 Comparison of empirical distribution of peak response with results of Lieblein and simplified Gumbel for $\omega_0 t = 14\pi$. 

$a = \frac{1}{21}, \quad \beta_0 = 0.05, \quad y/F = 5.0$ 

- Empirical data

95% confidence interval
$\alpha = \frac{1}{2}$  \hspace{1cm} $\beta_0 = 0.0$  \hspace{1cm} $\gamma / \beta = 1.0$

**EMPIRICAL DATA**

**Fig. 4.41** Comparison of empirical distribution of peak response with results of Lieblein and simplified Gumbel for $\omega_0 t = 24\pi$
FIG. 5.1 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

FIG. 5.2 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE
FIG. 5.3 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

FIG. 5.4 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE
\[ \alpha = \frac{1}{2\pi} \quad \beta_0 = 0.0 \quad \gamma/F = 1.0 \]

Figure 5.5 Transient Mean Squared Displacement Response

\[ \alpha = \frac{1}{2\pi} \quad \beta_0 = 0.0 \quad \gamma/F = 2.0 \]

Figure 5.6 Transient Mean Squared Displacement Response
FIG. 5.7 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

FIG. 5.8 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE
FIG. 5.9 TRANSIENT MEAN SQUARED VELOCITY RESPONSE

FIG. 5.10 TRANSIENT MEAN SQUARED VELOCITY RESPONSE
FIG. 5.11 TRANSIENT MEAN SQUARED VELOCITY RESPONSE

FIG. 5.12 TRANSIENT MEAN SQUARED VELOCITY RESPONSE
FIG. 5.13 TRANSIENT MEAN SQUARED VELOCITY RESPONSE

FIG. 5.14 TRANSIENT MEAN SQUARED VELOCITY RESPONSE
FIG. 5.15 TRANSIENT MEAN SQUARED VELOCITY RESPONSE

FIG. 5.16 TRANSIENT MEAN SQUARED VELOCITY RESPONSE
FIG. 5.17 EQUIVALENT STIFFNESS

FIG. 5.18 EQUIVALENT DAMPING
FIG. 5.19 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

FIG. 5.20 TRANSIENT MEAN SQUARED VELOCITY RESPONSE
FIG. 5.21 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

\[ \alpha = \frac{1}{21}, \quad \beta_0 = 0.01, \quad \gamma/F = 2.0 \]

\[ \frac{\langle X^2 \rangle}{F^2} \]

\(0\) \(4\pi\) \(\omega_0 t\) \(8\pi\) \(12\pi\)

FIG. 5.22 TRANSIENT MEAN SQUARED DISPLACEMENT RESPONSE

\[ \alpha = \frac{1}{21}, \quad \beta_0 = 0.05, \quad \gamma/F = 2.0 \]

\[ \frac{\langle X^2 \rangle}{F^2} \]

\(0\) \(4\pi\) \(\omega_0 t\) \(8\pi\) \(12\pi\)
### TABLE 2.1

**STATISTICAL PROPERTIES OF UNIFORM RANDOM NUMBERS**

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<thead>
<tr>
<th>Length of Sequence</th>
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<th>SD</th>
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### TABLE 2.2

**STATISTICAL PROPERTIES OF AUTOCORRELATION SEQUENCES**

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### TABLE 2.3

**STATISTICAL PROPERTIES OF NORMAL RANDOM NUMBERS**

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### TABLE 2.4

**POWER SPECTRAL DENSITY BY LINEAR ACCELERATION METHOD**

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<tr>
<th>( \beta_0 )</th>
<th>( \omega_0 b )</th>
<th>( \omega_0 T_0 )</th>
<th>White noise power spectral density from rms values</th>
<th>( b/\pi )</th>
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<td>Disp.</td>
<td>Velocity</td>
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<td>400</td>
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<td>0.003162</td>
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<td>400</td>
<td>0.003567</td>
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<tr>
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<td>slope ratio ( \alpha )</td>
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<td>( \beta_0 )</td>
<td>( \omega_0 t = 4\pi )</td>
<td>( \omega_0 t = 8\pi )</td>
</tr>
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<td>-----------------</td>
<td>--------</td>
<td>------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td></td>
<td>( \bar{Y}/F )</td>
<td>( s_y/F )</td>
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### TABLE 5.1

**ADDITIONAL DAMPING DUE TO HYSTERESIS—MATCHING DISPLACEMENT**

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### TABLE 5.2

**ADDITIONAL DAMPING DUE TO HYSTERESIS—MATCHING VELOCITY**

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### TABLE 5.3

EQUIVALENT FREQUENCY----MATCHING DISPLACEMENT AND VELOCITY

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### TABLE 5.4

ADDITIONAL DAMPING DUE TO HYSTERESIS—MATCHING DISPLACEMENT AND VELOCITY

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