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BETA-DECAY OF RENIUM 187 INTO BOUND STATES
FOR STELLAR CONDITIONS.

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Beta-Decay of Renium 187 into Bound States
for Stellar Conditions

by

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Thesis Director's Signature:  

Donald H. Clayton

Houston, Texas
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"Follow the courses of the stars, as if you were going along with them; and consider constantly the changes of the elements into one another... ."

Meditations of Marcus Aurelius
Dedicated
to Better Times
TABLE OF CONTENTS

INTRODUCTION. .............................................. 1

CHAPTER I: Bound State Beta-Decay for First
Forbidden Unique Transitions. ................. 5

CHAPTER II: Energetics and Decay Rate
Calculations ............................................ 15

CHAPTER III: The Temperature Dependent Half-
Life of Re 187 and Its
Consequences. ....................................... 26

REFERENCES. .............................................. 32

ACKNOWLEDGMENTS ................................. 34
INTRODUCTION

In 1964 Donald Clayton invented a new way to measure the age of the atomic nuclei. The method is based on the slow enrichment of the Os$^{187}$ abundance due to the decay of Re$^{187}$. He also discussed the awkward cosmochronological implications of the Renium 187/Osmium 187 abundance ratio.$^1$ This ratio implies considerably older ages for the beginning of galactic nucleosynthesis, hence of the beginning of the galaxy, than commonly accepted.

Approximately 60% of the terrestrial Os 187 is probably the result of the beta-decay of Re 187. Re 187 is an r-process isotope off of the valley of beta-stability which shields Os 187 from r-process contribution except for a very slow beta-decay contribution. Os 187 lies on the valley of beta-stability thereby partaking of s-process synthesis. In the neighborhood of Os 187 the s-process $\sigma$ curve is very flat so that $\sigma(\text{Os 187}) \approx \sigma(\text{Os 186})$. Exact neutron capture cross sections are not yet available but from general neutron capture systematics it seems likely that
\[
\frac{\sigma(\text{Os } 186)}{\sigma(\text{Os } 187)} = 0.4 \pm 0.1 \quad (1)
\]

Therefore the radiogenic Os 187, notated \( \text{Os}_{R}^{187} \), should be given

\[
\frac{\text{Os}_{R}^{187}}{\text{Re } 187} = \frac{\text{Os } 187 - \frac{\sigma(\text{Os } 186)}{\sigma(\text{Os } 187)} \text{ Os } 186}{\text{Re } 187} \quad (2)
\]

If the exponential model of the history of nucleosynthesis is used (which follows from simple and reasonable arguments concerning star formation and disruption) then after time, \( T \), of the nucleosynthesis interval

\[
\frac{\text{Os}_{R}^{187}}{\text{Re } 187} \bigg|_{T} = \left( \frac{\Lambda - \lambda_{187}}{\Lambda} e^{\lambda_{187} T} \frac{1 - \exp(-\Lambda T)}{1 - \exp(-\Lambda + \lambda_{187} T)} \right)^{-1},
\]

(3)

where \( \Lambda \) is a measure of the exponential decrease of nucleosynthesis and \( \lambda_{187} = 1/\tau_{1/2}(187) \ln 2 \) is the production rate of \( \text{Os}_{R}^{187} \) by the beta-decay of Re 187. 3

If \( \frac{\sigma(\text{Os } 186)}{\sigma(\text{Os } 187)} \) is assumed to be 0.4 then from the experimentally determined value of \( \frac{\text{Os } 186}{\text{Os } 187} \),

\[
\frac{\text{Os}_{R}^{187}}{\text{Re } 187} = 0.12 \quad (4)
\]

The latest laboratory measurements of \( \tau_{1/2}(\text{Re } 187) \) have given values \( \approx 4 \times 10^{10} \) years.
These values yield a galactic age, T+(solar system age = 4.6x10^9 years), of 11x10^9 years for "sudden synthesis" ($\Lambda = \infty$) and 18x10^9 years for "uniform synthesis" ($\Lambda = 0$). These ages are considerably older than the respective 8x10^9 years and 14x10^9 years obtained by U^{235}/U^{238} and Th^{232}/U^{238} ratios. The $\Lambda$ which best fits the latter two ratios gives an age of $11.7 \pm 2x10^9$ years while this same value for the exponential decrease gives an age of $\approx 17x10^9$ years when applied to Os^R_{187}/Re 187.

All of these methods do have uncertainties, but the Re 187 chronology does seem to indicate a considerably older galaxy than do other chronologies. This latter group includes the galactic age as determined by the measurements of the Hubble time, Sandage giving it $\approx 13.3x10^9$ years, and by the calculations of Iben et al. showing the age of the oldest globular clusters to be $12.\pm 3.x10^9$ years.

One way around this dilemma is the possibility of the beta-decay of thermally populated excited states analogous to the thermally excited alpha-decay situation treated in an article by Perrone and Clayton. However, Clayton has shown that the known nuclear levels of Re 187 are unfavorable for any significant decay contributions below $2x10^8$ deg K. Above that temperature it is doubtful that much matter
destined for reinjection into interstellar space can avoid neutron reactions which will destroy both Re 187 and Os 187. Neither will photon induced beta-decay prove fast enough at low temperatures to be of interest.\textsuperscript{10}

It seems, then, that the only possibility of increasing the Re 187 decay rate, thereby lowering the $\text{Os}_{187}/\text{Re}_{187}$ age to a more acceptable value, will be by the decay into unoccupied bound states of ionized Re 187 atoms inside of stars. Such bound state decay rates should be faster than the laboratory decay rate because of an increase in the number of available electron states and, more significantly, the increase in available neutrino phase space due to the increase in decay energy because of the addition of the large binding energies of the inner electron shells.\textsuperscript{11}

The next two chapters are devoted to deriving and calculating bound state decay rates. Temperature and free electron density-dependent half-lifes along with their cosmochronological and astrophysical implications are set forth in the final chapter.
Chapter I: Bound State Beta-Decay

for a First Forbidden Unique Transition

There are three features which distinguish beta-decay where the negatron (emitted electron) goes into an unoccupied bound atomic state from the more common case of negatron emission into a continuum state. These three features are present whether the transition is allowed (Gamow-Teller or Fermi) or first forbidden unique as in the Re 187 case. In the latter case the nuclear angular momentum change is two units of angular momentum ($\Delta J = 2$), and the nuclear parity changes. Electron capture will also exhibit these features being the more widely investigated analogue of bound state decay. Electron capture and bound state decay exhibit the same formal lepton current terms, except in the former case the wavefunctions are calculated for the $Z$ of the parent nucleus while in the latter for $Z$ of the daughter nucleus. There are also minor differences in energetics.

The above mentioned features for bound state decay are:

I. The end-point energy of the decay, $E_o'$, must take into account the extra energy provided by the binding energy, $b_{\gamma'}$, of the created wavefunction of quantum notation $\gamma'$.
where \( W_0 \approx E_0 \) for the neutral atom, but for an ionized atom \( W_0 \) is the mass difference of the parent system with charge \( Z \) and the daughter system, with \( Z \) electrons and nuclear charge \( Z+1 \), less the rest mass energy of an electron. In the case of electron capture \( b_\gamma \), must be subtracted from \( W_0 \) rather than added.

II. The wavefunction of the created bound state, evaluated for the charge \( Z+1 \) of the daughter nucleus, must be evaluated at the origin for allowed decay and at the moment of nucleon current, \( \langle r \rangle \), for first forbidden unique decay. For electron capture the wavefunction of the bound electron to be captured is evaluated for the charge of the parent nucleus since the wavefunction is already in existence. Continuum decay necessitates the integration over possible free electron states, and the combination of the leptonic phase space availability and the nuclear matrix term is combined in \( f_i(Z,E) \) where \( i \) denotes the degree of "forbiddenness!"

III. The neutrino and negatron cannot freely share the available kinetic transition energy \( E_0 \). The discrete energy of the bound state requires the neutrino to carry off the discrete energy remainder. The electron has energy \(-b_\gamma\), so
the neutrino has energy \( W_o + b_\gamma' = E_o \). It is the neutrino which provides a multi-valued phase space availability to the transition; the electron's phase space is unity having been uniquely given by the quantum state \( \gamma' \).

It has been shown by Bahcall that for allowed decay the ratio of bound state to continuum state decay is given by the ratio of the phase space volumes after these volumes have been corrected for by the density of the created wavefunctions at the nuclear surface.\(^{12}\) If the initial state consisting of a nucleus with charge \( Z-1 \) surrounded by \( N \) atomic electrons is represented by the state vector

\[
|i\rangle = |Z-1, N, \varnothing, k\rangle
\]

(6)

where \( \varnothing \) and \( k \) are the atomic electron and nuclear states, then the final state after negatron emission into an unoccupied state will be

\[
|f\rangle = |Z, N+1, \varnothing', \nu, k'\rangle
\]

(7)

where the prime denotes the new atomic and nuclear states and \( \nu \) is the neutrino state created (actually an anti-neutrino but for brevity, a "neutrino").

By the Golden Rule of time-dependent perturbations, the decay rate is given formally by
\[ \Gamma = \frac{2\pi}{\hbar} \sum_{i,f} |\langle f | H_w | i \rangle|^2 \]  

(7a)

where the weak interaction producing the decay is

\[ H_w = \frac{G_V}{\sqrt{2}} (\psi^\dagger_p (1-x\gamma_5) \psi_n)(\bar{\psi}_e \gamma^\dagger_e (1+\gamma_5) \psi_\nu) + H.C. \]  

(8)

where \( x \equiv C_a/C_V \) is the ratio of the Gamow-Teller to the Fermi coupling factors. \( \psi_p, \psi_n, \psi_e, \psi_\nu \) are the proton, neutron, electron, and neutrino wavefunctions taking part in the weak interaction and the \( \gamma \)'s in Eq. (8) represent \( \gamma \) matrices.

\[ \psi_e^\dagger = \sum_b a_b^\dagger \gamma_b^\dagger + \sum_c a_c^\dagger \gamma_c^\dagger + \text{positron operators} \]  

(9)

where \( a_b^\dagger \) and \( a_c^\dagger \) create electrons in bound and quantum states and \( \gamma \)'s are here the created electron states.

A straightforward evaluation of the decay rate would necessitate the evaluation of

\[ \sum_b \gamma_b^\dagger \langle Z, N+1, \varphi' | a_b^\dagger | Z-1, N, \varphi \rangle . \]  

(10)

Because of the lack of orthogonality between the atomic eigenstates the initial and final nuclear charge potentials of the form
there will be an infinite number of non-vanishing terms.

This difficulty is avoided by expanding the initial atomic electron state in terms of eigenstates of the final nuclear charge Hamiltonian:

\[ |i\rangle = |Z-1, N, \varnothing, k\rangle = \sum_{\varnothing''} |Z, N, \varnothing''\rangle \langle Z, N, \varnothing''| \]

\[ \times |Z-1, N, \varnothing, k\rangle. \]  

In order to conserve energy, the energy of the final state, \( E(\varnothing', k') \), must equal the energy of the initial state, \( E(\varnothing, k) \), where the \( \varnothing'' \) indicates the expansion of the initial state, \( \varnothing \), in the \( \varnothing'' \) basis of equation (11a).

In this way

\[ |\langle \varepsilon | H_\omega | i \rangle|^2 = \left[ \sum_{\varnothing''} \langle Z, N+1, \varnothing', \nu, k' | H_\omega | Z, N, \varnothing''\rangle \right. \]

\[ \left. \langle Z, N, \varnothing'' | Z-1, N, \varnothing, k \rangle \delta (E(\varnothing'', k) - E(\varnothing', \nu, k')) \right]^{*} \]

\[ \left[ \sum_{\varnothing'''} \langle Z, N+1, \varnothing', \nu, k | H_\omega | Z, N, \varnothing'''\rangle \langle Z, N, \varnothing''' | Z-1, N, \varnothing, k \rangle \right. \]

\[ \left. \delta (E(\varnothing''', k) - E(\varnothing', \nu, k')) \right]. \]  

(11b)
The summations are then brought together and the delta functions consolidated to give

\[
\Gamma = \frac{2\pi}{\hbar} \sum_i \sum \Sigma^* \langle Z,N,\phi'' | Z-1,N,\phi \rangle^* \cdot \langle Z,N,\phi'' | Z-1,N,\phi \rangle
\]

\[
\times \langle Z,N+1,\phi',\nu,k' | H_{\omega} | Z,N,\phi'' \rangle^* \cdot \langle Z,N+1,\phi',\nu,k' | H_{\omega} | Z,N,\phi'' \rangle
\]

\[
\times \delta(\mathcal{E}(\phi'',k) - \mathcal{E}(\phi',\nu,k'))
\]

(12)

where the asterisk on the summation indicates that only those states for which

\[
\mathcal{E}(\phi'',k) = \mathcal{E}(\phi'',k)
\]

are to be included.

If the usual assumptions for allowed decay are made:

(1) The leptonic current is evaluated at the nuclear radius \( R \); and (2) the nucleons are treated non-relativistically, then by inserting the definitions of \( \psi_+^\dagger \) and \( H_{\omega} \) in Eq. (12) the following expression for the bound state decay rate is obtained for initially unoriented nuclei\(^{13} \):

\[
\Gamma_B = \frac{G_F^2 (mc)^5}{2\pi \hbar^7 c} \sum_i \sum^{\Sigma^*} \langle Z,N,\phi'' | Z-1,N,\phi \rangle^* \cdot \langle Z,N,\phi'' | Z-1,N,\phi \rangle
\]

\[
\times \langle Z,N,\phi'' | Z-1,N,\phi \rangle \left( \frac{W_0 + b}{mc^2} \right)^2 \gamma_+^\dagger \gamma_+ \gamma_{\phi''}(R) (1 + \gamma_S) \gamma_{\phi''}(R)
\]

(14)
where $\xi$ is the nuclear matrix element

$$\xi = \langle 1 \rangle^2 = \left( \int \psi_1 \psi_f^* d \right)^2 \text{ for Fermi transitions}$$

(15)

or

$$\xi = x^2 \langle \sigma \rangle = x^2 \left( \int \psi_f \psi_i^* \right)^2$$

(16)

for Gamow-Teller transitions.

It is then a simple matter to show that

$$\frac{\Gamma_B}{\Gamma_C} = \frac{\pi^2}{f_0(Z,W_0)} Y$$

(17)

where $Y$ is the triple sum term in Eq. (14).

It is to be noted that the ratio of bound state to continuum state decay, $\Gamma_B/\Gamma_C$, is independent of the nuclear matrix elements. Fortunately, only one nuclear matrix element appears in first forbidden unique transitions so that $\Gamma_B/\Gamma_C$ will also be independent of nuclear parameters.

Before making the necessary alterations in Eq. (17) to give an expression for $\Gamma_B/\Gamma_C$ when the transition is first forbidden unique, it is first expedient to make the following approximation. The overlap terms

$$\Sigma \langle Z,N,\phi''|Z-1,N,\phi^* \rangle \cdot \langle Z,N,\phi'''|Z-1,N,\phi \rangle$$

$$\phi'',\phi'''$$

(18)
are due to the excitation of the original $N$ electrons when the nuclear charge increases by one proton. It has been calculated that this excitation for Os 187 (the daughter of Re 187) will be $\leq 130$ e.v. But the error in the experimental determination of the neutral atom end-point energy is $\pm 90$ e.v.; it seems reasonable, therefore, to ignore the overlap terms. And as the atom is increasingly ionized the overlap effect should decrease even further. Eqn. (18) is set unity. Leaving out the overlap terms Eq. (17) can be written

$$\frac{\Gamma_B}{\Gamma_c} = \frac{\pi^2}{f_0(Z, w_0)} \sum \frac{W_{\alpha \beta}}{2 \gamma'' m_e c^2} \gamma^+ (R) (1 + \gamma_5) \gamma (R).$$

(19)

In making the necessary changes in Eq. (19) for first forbidden unique transitions it is useful to note that the term

$$\gamma^+ (R) (1 + \gamma_5) \gamma (R)$$

(20)

is the leptonic current term ($\psi_\nu = e^{i k r} \approx 1$) which also appears formally in allowed electron capture transition rates. When the comparison is made between allowed Gamow-Teller and first forbidden unique (which also a Gamow-Teller interaction) leptonic current terms in electron capture rates, it is seen that expression (20) is replaced by

(17)
\[ \langle r \rangle^2 \left[ \frac{g_{\gamma^j 3/2}(R)}{R^2} + \frac{q^2 g_{\gamma^j 1/2}(R)}{9} \right] \] (21)

where \( \langle r \rangle \) is the moment of nucleon current for the first forbidden unique transition, \( q \) is the neutrino momentum in units of \( m_e c^2 \) and \( g_{\gamma^j} \) is the "large" component of the radial part of the bound wavefunction \( \gamma' \) with total angular momentum \( j \) evaluated at the nuclear radius \( R \). The small components do not appear being generally an order of magnitude smaller and having a higher unit of angular momentum which would necessitate multiplying by an extra factor of \( \langle r \rangle^2/R^2 \) because of the \((kr)^\ell\) asymptotic behavior of the wavefunctions. Thus, the formal analogy with electron capture allows the use of the same leptonic current terms.

All that remains to be converted in Eq. (19) is the denominator, \( f_o(Z, \omega_o) \). The general first forbidden phase-space function \( f_1(Z, \omega_o) \) is obtained from \( f_o(Z, \omega_o) \) by the use of a "shape correcting factor" which is used to evaluate the phase space volume at the nuclear radius.\(^{18}\) Again, because of the \((kr)^\ell\) dependence of the leptonic wavefunctions near the origin, the use of only the "large" electron components and the respective neutrino wavefunctions allow the simple substitution
\[ f(Z, w_0) \rightarrow f_1(Z, w_0) \frac{\langle r \rangle^2}{R^2}. \]  

(22)

Using the substitutions in Eqs. (21) and (22) Eq. (19) transforms to

\[
\frac{\Gamma_B}{\Gamma_c} = \frac{\pi^2 \sum_{\gamma'} \frac{w_0 + b}{m_\gamma}}{f_1(Z, w_0)} \left[ g_{\gamma',3/2}^2(R) + \frac{g_{\gamma',1/2}^2(R) q^2 R^2}{9} \right]
\]

(23)

for first forbidden unique transitions.
Chapter II: Energetics and Decay Rate Calculations

The most recent experiments\textsuperscript{19} have given the half-life for continuum state decay of Re 187 as

\[ \tau_{1/2C} = 6.6 \pm 1.3 \times 10^{10} \text{ years.} \]  

(24)

By comparing this value to a geologically determined total half-life\textsuperscript{19a} of

\[ \left( \tau_{1/2} = \right) 4.3 \pm 0.5 \times 10^{10} \text{ years,} \]  

(25)

the decay rate into the unoccupied bound states of the neutral atom has been calculated\textsuperscript{20} to be given:

\[ \frac{\Gamma_{B(\text{neutral atom})}}{\Gamma_C} = 50\% \pm 30\%. \]  

(26)

In the same experiment, the end-point energy of the continuum decay was measured to be 2.62 keV \pm .09 keV.

In the above mentioned experiment the "shielding effect" of the atomic electrons on the continuum states was taken into account by using an "effective charge," \( \bar{Z} \)\textsuperscript{19b} for the continuum states where

\[ \bar{Z} = 0.8 \, Z = 61. \]  

(27)

The energy of the atomic states plays an important role in this decay due to its low end-point energy. A heavy-
atom Thomas–Fermi model approximation for the total atomic binding energy \(^{21}\) gives:

\[
\text{B.E.} = 15.73 \, Z^{7/3} \, \text{ev}.
\]  \(\text{(27)}\)

This expression may be used to give the difference in total atomic binding energy between Re 187 and Os 187 as

\[
\Delta \text{B.E.} = 15.73 \left[ 76^{7/3} - 75^{7/3} \right] \text{ev} = 11.702 \, \text{kev}.
\]  \(\text{(28)}\)

Any systematic errors in the approximation of Eq. (27) will largely cancel out by subtraction in Eq. (28). Thus, for a bare Re 187 nucleus completely stripped of all atomic electrons

\[
W_0 = 2.62 \, \text{kev} - 11.702 \, \text{kev} \approx -9.08 \, \text{kev} \quad \text{(29)}
\]

which is a negative energy. This means that the sum of the masses of a bare Os 187 nucleus and an electron will be larger than the mass of a bare Re 187 nucleus. The positive end-point energy for the neutral atom is only made possible by the binding energy difference between the daughter and parent atoms as shown in Figure 1. However, as will be shown, the bare Re 187 nucleus can decay by inserting the emitted electron into one of the tightly bound inner atomic subshells. The binding energy, \(b\), of some of the inner sub-shells will contribute enough energy to make the transition energetically possible.
In the case of the bare nucleus, the binding energy of the created bound state may be calculated by using the relativistic, one-electron formula,

\[
b_{\gamma'} = \sqrt{\frac{m_e c^2}{1 + \frac{2 Z^2 \alpha^2}{\sqrt{n-j-1/2+{(j+1/2)}^2-Z^2 \alpha^2}}} - m_e c^2}
\]  

(30)

where \( n \) is the principal quantum number, \( j \) the angular momentum, \( \alpha = 1/137 \), and, in the case of the stripped nucleus, \( Z = 76 \).

The binding energies for subshells into which an electron from the bare nucleus can be emitted \((b_{\gamma'} > 9.08 \text{ kev})\) are:

<table>
<thead>
<tr>
<th>( K(n=1,j=1/2) )</th>
<th>( L_I(n=2,j=1/2,\ell=0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b(\text{kev}) )</td>
<td>85.83</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( L_{II}(n=2,j=1/2,\ell=1) )</th>
<th>( L_{III}(n=2,j=3/2,\ell=1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b(\text{kev}) )</td>
<td>21.55</td>
</tr>
</tbody>
</table>

(31)

Decay of the bare Re 187 nucleus will not be allowed energetically to proceed into any other bound state (or continuum state) even if allowed by angular momentum conservation.

The large component of the radial wave equation for \( Z = 76 \) has been calculated for a one-electron, relativistic system:
\[ g_{\gamma}^2, (R) \left( \frac{1.71}{1.71} \right)^2 \left( \frac{.67}{.67} \right)^2 \times 10^{-6} \]

where the evaluation used a value of \( R = 3.088_1 \times 10^{-3} x A^{1/3} \) (\( A = 187 \)) in units of \( \frac{\hbar}{m_e c} \).

In order to determine \( \Gamma_B / \Gamma_c \), where the bound-state decay rate comes from the bare Re 187 nucleus, \( f_1(Z, E_o) \) must be evaluated with, as discussed before, the effective charge, \( Z = 61 \), of the neutral atom and \( E_o = 2.62 \) kev. There is a simple relationship between \( f_1(Z, E_o) \) and \( f_o(Z, E_o) \) given by \(^{24}\)

\[ f_1(Z, E_o) = f_o(Z, E_o) \left[ a(Z) \left( \frac{E_o}{m_e c^2} \right)^2 - 1 \right] + b(Z) \left( \frac{E_o}{m_e c} - 1 \right) \]

(33)

where \( a(61) = 4.6 \times 10^{-2} \) and \( b(61) = -3.8 \times 10^{-2} \). \( f_o(Z, E_o) \) may be evaluated by noting that whenever \( \frac{E_o}{m_e c^2} \ll 1 \)

\[ f_o(0, E_o) = .216 \left( \frac{E_o}{m_e c} - 1 \right)^{7/2} \]

(34)

for the phase-space factor of allowed decay with the Coulomb effect ignored by setting \( Z = 0 \). \(^{25}\) Using a formula giving the ratio of the phase-space factors with and without the Coulomb effect: \(^{26}\)

\[ \frac{f_o(61, 2.62 \text{ kev})}{f_o(0, 2.62 \text{ kev})} = 34.2 \]

(35)
Eq. (33) finally yields

\[ f_1(61, 2.62 \text{ kev}) = 2.05 \times 10^{-8}. \]  

(36)

Equation (23) may now be used with

\[ \omega_0 = 2.62 - 11.712 = -9.08 \text{ kev}, R = 3.0881 \times 10^{-3} \times 187^{1/3}, \]

\[ f_1(61, 2.62 \text{ kev}) \] given by Eq. (36), the \( g^2(R) \) terms of Eq. (32), and the binding energies of Eq. (31) to give

Bare Re 187 Nucleus

\[
\begin{array}{cccc}
\gamma' = k & L_I & L_{II} & L_{III} \\
\Gamma_{\gamma'}/\Gamma_c & 21,550. & 2.37 & 1.24 & 184. \\
\end{array}
\]  

(37)

The bare nucleus is seen to have a much faster beta-decay rate than the neutral atom.

The decay into the K shell is large because of the large binding energy contribution. The \( L_I \) and \( L_{II} \) subshells have considerably smaller binding energies, and being \( j=1/2 \) angular momentum states the neutrino is required to carry off the unit of orbital angular momentum. For decay into \( L_{III} \) the neutrino state is s-wave and the decay proceeds much faster than for \( L_I \) and \( L_{II} \) because of their extra \( \frac{g^2 R^2}{m} \) factor appearing in Eq. (23).
When the atom is only partially ionized several complications arise. There is no longer a simple formula for $b_{\gamma'}$, or $g_{\gamma'}(R)$ when the parent system includes atomic electrons. Even $\omega_o$, the total mass difference between the parent system with charge $Z$ and the daughter system with charge $Z+1$ plus an electron, is no longer given by exact solution. Several assumptions are made to allow approximate solutions for these necessary values.

For the square of the radial wavefunction at the nuclear radius, the non-relativistic, hydrogenic rule

$$\psi_n^2 \propto \frac{Z_{\text{eff}}^3}{n^3}$$

(38)

is used where $Z_{\text{eff}}$ is the effective charge of the atom ionized down to the "ith level," and $n$ is the principal quantum number. The state of ionization is $i = L_{\text{II}}$, for example, when the $L_{\text{II}}$ subshell and all lower subshells are filled but all higher subshells are empty. $i = L_{\text{III}}+1$ likewise represents the ion with the $L_{\text{III}}$ and all lower subshells filled and one electron in the next higher subshell, $M_{\text{I}}$.

For the atom in ionization state $i$, a created wavefunction in any higher-lying $j = 3/2$, $\ell = 1$ quantum state, $r_{\text{III}}$, has a squared radial value at the radius given by the approximation
\[
g^2(i_{III}, r_{III}) \frac{(R)}{g^2(o_{III}, L_{III}) \frac{(R)}{n_r}} = \frac{2^3}{3} \left( \frac{Z_i}{76} \right)^3
\]  

(39)

where \( n_r \) is the principal quantum number of the \( r_{III} \) subshell, and \( g(i_{III}, r) \) refers to the created wavefunction in subshell \( r \) for the ionization level \( i \) (\( g(o_{III}, L_{III}) \) is the \( L_{III} \) wavefunction for the bare nucleus, \( i = 0 \)).

\( Z_i \) is assumed to be given by

\[
Z_i = 76 - \text{Number of electrons in } i \text{ level and lower subshells}
\]  

(40)

until \( Z_i = 61 \), after which point it remains constant in order to give agreement with the \( Z = 61 \) value for the neutral atom in its effect upon low-energy continuum states.

The total energy release

\[
E(i, r) = W_o(i) + b(i, r)
\]  

(41)

may be approximated in the following way. Equation (28) gave the difference in total binding energies of Re 187 and Os 187 as 11.702 kev. By adding up the differences of the commonly tabulated one-electron ionization energies, determined by x-ray techniques, and taking into account the sub-shell degeneracies, the difference in total one-electron
ionization energies between Re 187 and Os 187 is 9.78 kev. The latter value is smaller than $\Delta \text{B.E.}$ by a factor of $1.211$. This is to be expected because the two numbers measure different quantities. A measurement of the binding energy may be envisioned as progressively removing electrons from the full electron compliment all the way to the bare nucleus. The one-electron ionization measures the separation energy difference for single electrons and sums over these values. In the latter method all other electrons are present except for the one being measured so that the total of the single-electron separation energies is less than the true total binding energy. In order to approximate the electronic energy differences between Os and Re ions for progressive degrees of ionization, the one-electron ionization values are used after being corrected by a scale factor of 1.211:

$$E_{i,r} = 1.211(D_{i} + \frac{Z^2}{r^2} \sum d_{i,r}) \text{kev} - 9.08 \text{ kev}$$

(42)

where $D_{i}$ is the total one-electron ionization energy difference for the $i$ level and lower electrons of Re 187 and Os 187. Correction by the scale factor gives the approximation

$$w^o_{(i)} \text{ kev} = 1.211 D_{(i)} \text{ kev} - 9.08 \text{ kev} \ .$$

(43)
\( d(i,r) \) is the one-electron ionization value for subshell \( r \) of Os 187 (into which the electron is assumed to be emitted) multiplied by a factor of \( \frac{Z_r^2}{Z_i^2} \) so that

\[
b(i,r) = 1.211 \frac{Z_r^2}{Z_i^2} d(i,r) \tag{43a}
\]

where the factor \( \frac{Z_r^2}{Z_i^2} \) comes from non-relativistic, hydrogenic binding energy behavior. This factor approximates the effect of the absent electrons between the ionization level \( i \) and the created wavefunction in subshell \( r \).

Equation (23) may now be used to give the ratio of decay into a \( j = 3/2, \ell = 1 \) subshell, \( r_{III} \), of the ion \( i \) to the decay from the bare nucleus into \( L_{III} \):

\[
\frac{\Gamma(i,r_{III})_{j=3/2}}{\Gamma(o,L_{III})} = \frac{2^3 Z_i^3}{n_r^3 (76)^3} \left( \frac{E(i,r)}{b(o,L_{III})-9.08} \right)^2
\]

where \( E(i,r) \) is given by (42) and \( b(o,L_{III})-9.08 = E(o,L_{III}) \).

Similarly, for decay into \( j = 1/2, \ell = 0 \) subshells

\[
\frac{\Gamma(i,r)_{j=1/2,\ell=0}}{\Gamma(o,k)} = \frac{1^3 Z_i^3}{n_r^3 (76)^3} \left( \frac{E(i,r)}{b(o,k)-9.08} \right)^4 \tag{45}
\]

where now the energy term appears to the fourth power because of the extra \( q^2 \) appearing in the s-wave electron lepton term.
of Eq. (23). $b_{0, L_{III}}$ and $b(o, k)$ are given in (31). After
the $L_I$ subshell is filled all further bound state and con-
tinuum state decay into $j = 1/2$ electron wavefunctions may
be ignored because of the extra $q^2 R^2$ factor for these terms
in Eq. (23). In effect, this means that the electron, being
non-relativistic and having a comparatively large (compared
to $E_o$) rest mass energy, carries off the unit of orbital
angular momentum much easier than does the neutrino. The
reason for the high decay rates into $K$ and $L_I$ subshells is
their larger binding energies which, in turn, give larger
transition energies.

Table 1 lists values of $E(i, r')$, $Z_i$, and $Z_r$. $E(i, r)$
was approximated by Eq. (42) using the one-electron ioniza-
tion energies. Table 2 lists some of the total bound state
decay rates which are used in the next chapter to give total
half-lifes. The total decay from ionization level $i$ is
also given where

$$\Gamma_i = \sum_{r > i} \Gamma(i, r) \rho(i, r) \cdot (46)$$

$\rho(r, i)$ is a factor equal to unity except for cases of $(i, r)$
when the $i$th level includes electrons in the same subshell $r$
into which the electron is to be emitted. For example,
when $i = L_{II+1}$ (the $L_{II}$
subshell being filled and one electron
in the $L_{III}$ subshell) there is a degeneracy of 4
possible states for the one electron in the \( L_{III} \) subshell, but only one of the four possible quantum states of \( L_{III} \) is the state into which the emitted electron may be placed. Therefore, in this case

\[
\rho(L_{II} + 1, L_{III}) = \frac{3}{4} . \tag{47}
\]

Similarly,

\[
\rho(L_{II} + 3, L_{III}) = \frac{1}{4} . \tag{48}
\]

As a consistency check to Eq. (23), approximations (39) and (44) the total decay into the unoccupied bound states of the neutral atom is calculated. Such decay proceeds with \( \tilde{Z}_i = 61, b_\gamma, \approx 0 \), into \( j = 3/2, l = 1 \) states of principal quantum numbers \( n = 6, 7, 8, \ldots \). \( E_{(i,r)} \) is set equal to 2.62 kev since \( b_\gamma, \approx 0 \) for these high-lying states. The result is

\[
\frac{\Gamma_B \text{(neutral atom)}}{\Gamma_c} = 58\% . \tag{50}
\]

This compares extremely well to the experimentally determined value of 50% in (26). Such an agreement gives a measure of confidence to the approximations and assumptions which have been made.
Chapter 3: The Half-Life of Re 187 and Its Consequences

For a particular stellar region of temperature $T$ and free electron density $n_e$ it is necessary to determine the fraction of an ensemble of Re 187 ions in each degree of ionization in order to determine the total half-life for the entire ensemble. The total decay rate is simply the sum of the products of the decay rate for each degree of ionization and the fraction of the ensemble at that degree of ionization. It is assumed that each degree of ionization is in its lowest energy state.

The Saha equation,

$$
\frac{n_{i+1}n_e}{n_i} = \frac{G_{i+1}}{G_i} g_e \frac{(2\pi m_kT)^{3/2}}{\hbar^3} e^{-\chi_i/kT},
$$

is used to determine the relative fractions in the different degrees of ionization $i$ (where $i$ now refers to the number of electrons missing from the full atom). In Eq. (51) $n_i$ is the number density of the $i$-times ionized Re 187 atom, $G_i$ is the partition function of the $i$-times ionized atom, $g_e = 2$ is the partition function of the free electron, and $\chi_i$ is the ionization energy required to ionize the ground state one more time giving an $i+1$-times ionized atom. In the same way
\[
\frac{n_{i+2}}{n_i}, \frac{n_{i+3}}{n_i+1}, \frac{n_{i+3}}{n_i+2}, \text{ etc. are determined which are then converted to the ratios}
\]

\[
\frac{n_{i+1}}{n_i}, \frac{n_{i+2}}{n_i}, \frac{n_{i+3}}{n_i} \ldots \frac{n_{i+j}}{n_i}.
\] (52)

The fraction of the ensemble in each degree of ionization is then given by

\[
f_m = S_m \left( \frac{n_m}{n_i} \right) ^{N \sum_{j=0}^{\infty} S_{i+j} \frac{n_{i+j}}{n_i}}^{-1}
\] (53)

where \(S_{i+j}\) is the degeneracy of the \(i+j\)-times ionized ground state, and \(N\) is the number of electrons remaining in the initial \(i\)-times ionized atom. The assumption is made in Eq. (53) that the ratios of the partition functions for nearby degrees of ionization are equal to the ratios of the ground state degeneracies. The partition functions themselves are prohibitively difficult to calculate.

Table 3 gives values of \(S_i\) and \(\chi_i\) for various levels of ionization. The quantity \(\chi_i\) is determined by

\[
\chi_i = 1.211 \times d_i
\] (54)

where 1.211 is the same scaling factor used in chapter 2, and \(d_i\) is the one-electron ionization value discussed in chapter 2. But now \(d_i\) for Re 187 must be used since the
concern is with the parent ionization. The quantity $d_i$ refers to the value of the highest occupied or partially occupied subshell. The factor of 1.211 enters for the same reason as before.

The reference value of $i$, on which the other relative fractions are based, does not have to be the initially non-ionized atom. At fairly high temperatures most of the outer electrons are quickly ionized so $i$ can be some convenient intermediate degree of ionization.

In determining $S_m$, the assumption that the ion is in its ground state means that there is a possible degeneracy determined by the existence of a partially filled subshell of the ion. If there is no partially filled subshell $S = 1$, otherwise $S > 1$. For example, for the ion of level $L_{II} + 1$ the single electron in the $L_{III}$ subshell with $j = 3/2$ has a degeneracy of $2j + 1$ so that $S = 4$.

For temperatures of interest ($\geq 10^6$ deg K) $\frac{n_{i+j}}{n_i}$ turns out to be a dominant number only for $1-5$ consecutive values of $i+j$. Since the largest fractions are in a few nearby states it is assumed that the ratios of the partition functions, $G_{i+j}$, are equal to the ratios of the ground state degeneracies, $S_{i+j}$, for these dominating degrees of ionization. For the other states with possibly differing
partition functions, the relative fractions will be negligibly small. Thus, the partition function dependence of Eq. (51) turns out to be of little importance.

It is now possible to write the temperature-dependent, electron density-dependent half-life of Re 187:

\[
\tau_{1/2}(T, n_e) = \left[ \sum_{i} \frac{f_i(T, n_e)}{\tau_{1/2}(i)} \right]^{-1}
\]  

where \( \tau_{1/2}(i) = 1/\Gamma(i) \) (with \( \Gamma(i) \) as given in Table 2).

Figure 2 shows the half-life of Re 187 for certain values of \( n_e \) over a range of \( T \). Conditions giving half-lifes \( \geq 10^{10} \) years, not being of astrophysical importance, are not included. For electron densities of \( 10^{20} \) and \( 10^{18} \) electrons per cubic centimeter, the Re 187 half-life is longer than \( 10^{10} \) years at temperatures below \( 8 \times 10^6 \) and \( 6 \times 10^6 \) deg. K., respectively.

The "ledge" structure appearing in Fig. 2 is due to the prohibition of bound state decay into the \( M_{III}, L_{III}, \) and \( K \) subshells when these subshells are fully occupied. Only when the ionization of the atom extends into one of these subshells does the overall decay rate increase markedly.

Clayton suggested the possibility of measuring a depleted Re 187/Os 187 ratio in the solar wind by a measure-
ment of the abundances deposited on the lunar surface. However, it appears that the solar convection zone does not present a favorable environment for a Re 187 half-life of less than $10^{10}$ years. Therefore, it seems unlikely that a Re 187/Os 187 depletion exists in the solar wind.

When a comparison is made between the Re 187 half-life of Fig. 2 and stellar models in the 1-9 solar mass range and the 15 and higher solar mass range, it appears that only stars of mass $7 \, M_\odot \lesssim M < 15 \, M_\odot$ are promising contributors of depleted Re 187 abundances. Figure 3 shows the evolution of the central temperature and density for 5 and 9 solar mass stars. The $9 \, M_\odot$ star presents an environment for which the Re 187 decays in approximately $10^8$ years which is about the age of the star itself. Larger stars ($\gtrsim 15 \, M_\odot$) have lives of $\approx 10^6$ years but do not present environments for Re 187 decay in less than $10^8$ years.

Because of the $T \propto \rho^{1/3}$ relationship in a star, the weak density sensitivity of the Re 187 decay means that the most favorable environment for Re 187 depletion will be found at the center of the star.

The smaller stars cannot create stable environments where the Re 187 decays faster than $5 \times 10^9$ years. Therefore, it seems that the 5% of the total produced Re 187 which may
have undergone enhanced beta-decay (as suggested by Clayton to give an age to the Re 187/Os 187 ratio consistent with the galactic age) did so in stars of around 7–9 solar masses.
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ACKNOWLEDGMENTS

It gives me great pleasure to acknowledge the encouragement and assistance of Professor Donald D. Clayton. Indebtedness to the National Science Foundation and to Rice University is acknowledged for financial support.
Table 1: \( \tilde{Z}_i, \tilde{Z}_r, \) and the transition energy, \( E_{(i,r)} \) are shown for the ion level \( i \) and created electron state in subshell \( r \).
<table>
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<th>(r)</th>
<th>76</th>
<th>74</th>
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<th>70</th>
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<td>(Z_r)</td>
<td>K</td>
<td>(L_I)</td>
<td>(L_{II})</td>
<td>(L_{III})</td>
<td>(M_{III})</td>
<td>(N_{III})</td>
<td>(O_{III})</td>
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<tr>
<td>76</td>
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<td>12.47*</td>
<td>12.47*</td>
<td>10.953*</td>
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<tr>
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<td>(M_{III})</td>
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<td></td>
</tr>
<tr>
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<tr>
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</tr>
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<td>1.556</td>
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</table>

* evaluated by the relativistic hydrogenic binding energy formula of Eq. (21) rather than Eq. (24)
Table 2: The ratio of bound state decay from level $i$ to continuum state decay of the neutral atom is given for selected values of $(i,r)$. $\rho(i,r) = 1$ unless otherwise given in parentheses.
Created State in Subshell \( r \)

\[
\begin{array}{cccccccc}
\ i & K & L_I & L_{II} & L_{III} & M_{III} & N_{III} & \Gamma_i / \Gamma_c \\
\hline
\text{B.N.+1} & (1/2)22,864 & 1.067 & 0.0048 & 122 & & & 11,555 \\
\text{K} & 1.55 & 0.0055 & 183.2 & .11 & & & 185 \\
\text{K+1} & \frac{1}{2} & 2.02 & 0.0055 & 193.7 & .428 & & 195 \\
\hline
\end{array}
\]

Filled to level \( i \):

\[
\begin{array}{cccc}
L_{II} \quad \rho(i,r) \frac{\Gamma(i,r)}{\Gamma_c} \\
L_{II+1} \quad \frac{3}{4} & 169.7 & 2.82 & 130 \\
L_{II+2} \quad \frac{3}{2} & 165.8 & 3.32 & .017 & 86.9 \\
L_{III} \quad \frac{3}{5} & \frac{3}{5} & 3.55 & .069 & 3.6 \\
M_{III} \quad \frac{1}{3} & \frac{1}{3} & .272 & .272 \\
\end{array}
\]
Table 3. The degeneracy of ionization level \( i \) is given by the notation \( s_i \). The ionization energy at different levels of ionization is approximated by \( \chi_i \) for Re 187 and is derived from Eq. (54).
<table>
<thead>
<tr>
<th>i</th>
<th>B.N. +1</th>
<th>K</th>
<th>K +1</th>
<th>L İ</th>
<th>L İ +1</th>
<th>L İI</th>
<th>L İI +1</th>
<th>L İI +2</th>
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<td>s</td>
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<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>χ</td>
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<td>15.17</td>
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<td>2.87</td>
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<td>2.87</td>
</tr>
</tbody>
</table>
Figure 1. Total energy differences for parent and daughter systems is shown for the neutral atom and the totally bare nucleus.
Figure 2. The half-life for an ensemble of Re 187 atoms at different free-electron densities and temperatures is graphed. Half-lifes $\gtrsim 10^{10}$ years are neglected. The scale is linear between enumerated half-life graph lines. The Saha Equation is used as if the electrons are nondegenerate throughout the range shown.
Figure 3. The evolutionary track of the central temperature and density for 5 $M_\odot$ and 9 $M_\odot$ stars are shown. The Re 187 half-life contours are superimposed showing the possibility of Re 187 depletion in 9 $M_\odot$ stars due to a Re 187 half-life of $\approx 10^8$ years.