MILLER, Terrell Watts, 1943-
A CONSISTENT WORKHARDENING THEORY FOR POROUS LIMESTONE.

Rice University, Ph.D., 1971
Engineering, mechanical

University Microfilms, A XEROX Company, Ann Arbor, Michigan
RICE UNIVERSITY

A CONSISTENT WORKHARDENING THEORY FOR POROUS LIMESTONE

by

TERRELL WATTS MILLER

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Thesis Director's signature:

[Signature]

Houston, Texas

October 1970
I had always wondered why many theses were dedicated to the author's wife.

Now I know.

To

Lynn
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>iii</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>iv</td>
</tr>
<tr>
<td>I. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>II. Theoretical Background and Literature Review</td>
<td>5</td>
</tr>
<tr>
<td>A. Theoretical Background</td>
<td></td>
</tr>
<tr>
<td>B. Review of Related Work</td>
<td>16</td>
</tr>
<tr>
<td>III. Apparatus and Experimental Procedure</td>
<td></td>
</tr>
<tr>
<td>A. Design Philosophy</td>
<td>24</td>
</tr>
<tr>
<td>B. Apparatus</td>
<td>25</td>
</tr>
<tr>
<td>C. Preparation of Samples</td>
<td>26</td>
</tr>
<tr>
<td>D. Test Procedure</td>
<td>27</td>
</tr>
<tr>
<td>E. Experimental Error</td>
<td>29</td>
</tr>
<tr>
<td>IV. Experimental Results</td>
<td></td>
</tr>
<tr>
<td>A. Definition of Tests</td>
<td>33</td>
</tr>
<tr>
<td>B. Curve Interpretation</td>
<td>34</td>
</tr>
<tr>
<td>C. Initial Yield Surface</td>
<td>36</td>
</tr>
<tr>
<td>D. Workhardening Tests</td>
<td>39</td>
</tr>
<tr>
<td>E. Interpretation of Test Results</td>
<td>44</td>
</tr>
<tr>
<td>i) Initial Yield Surface</td>
<td>45</td>
</tr>
<tr>
<td>ii) Hardening Rule</td>
<td>48</td>
</tr>
<tr>
<td>iii) Generalized Yield Condition and Hardening Rule</td>
<td>62</td>
</tr>
<tr>
<td>V. Implications and Application of the Yield Condition and Hardening Rule</td>
<td></td>
</tr>
<tr>
<td>A. Estimation of Wedge Indentation Force-Displacement Relations</td>
<td>69</td>
</tr>
<tr>
<td>VI. Summary and Conclusions</td>
<td>75</td>
</tr>
<tr>
<td>References</td>
<td>78</td>
</tr>
<tr>
<td>Appendix A</td>
<td>81</td>
</tr>
<tr>
<td>Tabulation of Results for the Experimental Verification of the Hardening Rule</td>
<td></td>
</tr>
<tr>
<td>Appendix B</td>
<td>88</td>
</tr>
<tr>
<td>Velocity and Stress Characteristic Relations for Rigid/Plastic Materials Obeying a Compacting Yield Condition Deformed Under Plane-Strain Conditions</td>
<td></td>
</tr>
<tr>
<td>Appendix C</td>
<td>99</td>
</tr>
<tr>
<td>Upper Bound Solution for the Force-Displacement Relation for the Indentation of a Half-Plane of Rigid/Plastic Compacting Material by a Perfectly Rough Wedge</td>
<td></td>
</tr>
<tr>
<td>Tables</td>
<td>106</td>
</tr>
<tr>
<td>Figures</td>
<td>108</td>
</tr>
</tbody>
</table>
ACKNOWLEDGEMENTS

The author wishes to express his appreciation for the advise and assistance of Dr. John B. Cheatham, Jr. throughout the research for and writing of this thesis.

The financial support of the API (Project 67F), the Humble Oil Education Foundation, and Shell Companies Foundation for this work is also greatly appreciated.

The assistance of Michael B. Smith in preparing test samples, performing experiments, and preparing figures as well as the help of Greg Hueni and my wife Lynn in typing the manuscript is also appreciated.
NOMENCLATURE

A, B Parameters in yield functions; points in the half-plane along a velocity discontinuity associated with a wedge indentation.

C, D Points in physical space associated with a wedge indentation test, power of dissipation.

F Force on the wedge, yield function.

J_2, I_2 Stress invariants.

M^w Material constant for clays.

P Pressure, mean normal stress in the x-y plane, point in physical space.

P_c, P_c^(i) Confining pressure, i^th confining pressure.

R, R_I, R_{II} Radii of Mohr's circles.

U_o, U_1 Internal energies of clay, water systems.

V_{AB}, V_{DB}, V_{ND}, V_{DC} Velocities defined in Appendix C.

W, W_o, W^(i) Work per unit volume of material, work per unit volume at the i^th confining pressure.

W_H^(i) Work per unit volume done on material during confining pressure increase to the i^th confining pressure.

\epsilon_0 \epsilon_{ijke} Components of an inverse elastic constant for a generalized Hooke's law.

f, f_1, f_2 Yield functions.

h, l Physical lengths in wedge indentation problems.

k Yield stress in pure shear of a metal, positive constant.

P_o Yield stress of clay in hydrostatic compression.

t Time, depth of wedge penetration.
\( u_1, v_1 \)  
Displacement and velocity components.

\( u, v \)  
Velocities in x and y directions, respectively.

\( u_1, u_2 \)  
Velocities along the first and second slip lines, respectively.

\( v, v_1, v_2 \)  
Volume.

\( v_n, v_{\text{WnPC}} \)  
Velocities defined in Appendix C.

\( x, y, z, \bar{x} \)  
Spatial coordinates.

\( a, \beta \)  
Positive constants, workhardening parameters, first and second slip lines, respectively.

\( a_{ij} \)  
Workhardening parameter.

\( \delta \)  
Distance defined in Appendix C.

\( \partial S \)  
Differential along a characteristic.

\( \varepsilon, \varepsilon, \varepsilon^p \)  
Volumetric strain, volumetric strain at the \( i^{\text{th}} \) confining pressure, plastic volume strain.

\( \varepsilon_{a}, \varepsilon_{a}^{(i)} \)  
Axial strain, axial strain at the \( i^{\text{th}} \) confining pressure.

\( \varepsilon_{H}, \varepsilon_{H}^{(i)} \)  
Volumetric strain occurring during the confining pressure increase to the \( i^{\text{th}} \) confining pressure, effective hydrostatic volume strain.

\( \varepsilon_r \)  
Radial strain.

\( \varepsilon_{ij}, \varepsilon_{ij}^{e}, \varepsilon_{ij}^{p} \)  
Total elastic, and plastic strain components.

\( \Delta \sigma_a, \Delta \sigma_r \)  
Differential axial and radial stresses.

\( \kappa \)  
Workhardening parameter.

\( \lambda \)  
Positive constant, lip angle.

\( \mu \)  
Parameter expressing non-dissipative energy contribution of plastic volume change of clays.

\( \mu \)  
Positive number, workhardening parameter.

\( \phi \)  
Angle of internal friction, fan angle.
ψ

$\sigma_1, \sigma_2, \sigma_3$
Principal stresses.

$\sigma_{\text{max}}, \sigma_{\text{min}}$
Maximum and minimum principal stresses.

$\sigma_x, \sigma_y, \tau_{xy}$
Normal and shear stresses in x-y plane.

$\sigma_a, \sigma_r$
Axial and radial stresses

$\sigma_{ij}, \tau_{ij}$
Components of the stress tensor.

$\sigma_p, \tau_p$
Coordinates of the pole of Mohr's circle.

$\sigma_{AD}, \tau_{AD}$
Stresses on wedge face.

$\theta$
Angle between greatest principal stress and x direction.

$\omega$
Half-wedge angle.

$\xi_1, \xi_2$
First and second slip line angles.
INTRODUCTION

This work is an attempt to explain the plastic behavior of a general type of material that compacts or shows a volume decrease as it undergoes permanent deformation, and still shows an increasing strength with increasing hydrostatic stresses. Essentially three requirements must be satisfied to properly characterize the behavior of any work-hardening body. They are:

(1) an initial yield condition that defines in terms of a general state of stress when a body first begins to undergo permanent deformations;
(2) a flow rule that specifies the strain rates during any loading process causing permanent deformations;
(3) a hardening rule that specifies the changes in the yield condition during loadings.

The formulation of these conditions is essentially complete for most metals [36], [20], [8], [10], [11]; however, metals actually should be classified as a very special material because their yield strengths are effectively independent of hydrostatic stresses and because of their incompressible behavior during plastic flow.

Rocks and soils belong to a more general material class in that they exhibit yield strength dependence upon hydrostatic pressure, as well as show some kind of volume change during plastic deformation. The formulation of a yield condition, a hardening rule, and a flow rule for this type of material is of interest as a more general constitutive
theory of plasticity as well as being a significant improvement over
the present theory of plasticity for these particular materials.

Currently the most prevalently used yield condition for rocks
and soils is the Coulomb yield condition [37], [22], [26], [29]. The
material is assumed to behave as a perfectly plastic material—once
the yield limit is reached unlimited deformation is allowed. A Coulomb
yield condition predicts linearly increasing yield strength of the
material with increasing hydrostatic stresses, and the proper flow rule
associated with the yield condition predicts volumetric expansion
during plastic loading.

The use of a Coulomb yield condition has led to many incon-
sistencies between what is predicted and what is observed experimentally,
some of which are:

(1) Volumetric expansion is not always observed in triaxial
and indentation tests.

(2) The material often shows workhardening properties.

(3) The initial yield point of the material does not always
increase with increasing pressure.

In addition, application of the Coulomb yield condition when calculating
the force-displacement relation for wedge indentation tests predicts
forces much higher than observed experimentally.

Clearly the Coulomb yield condition does not adequately describe
the plastic behavior of some rocks. Mohr type yield conditions that
allow nonlinear strength increases with increasing hydrostatic stresses
have been used to predict force-displacement relation for rocks (Cheatham[1]) and have improved the accuracy of the prediction; however, a flow rule consistent with a Mohr yield condition still predicts volumetric expansion during any deformation process.

Many workers have reported the existence of failure surfaces or limiting yield envelopes for rocks and soils that resemble Coulomb or Mohr yield surfaces (Roscoe and Porooshan[2], Swanson[14], Handin and Hager[24], and Bredthauer[25]). The tests performed in the experimental program reported here indicate that a limiting envelope does exist, but a significant amount of plastic flow occurs at stress levels below the limiting curve. A workhardening theory that incorporates a yield surface that is limited in size by a Coulomb envelope describes the above type of stress-strain behavior more accurately than a perfectly plastic flow theory based solely on a Coulomb or Mohr yield condition.

Several types of triaxial tests were performed on two different types of porous limestones in order to determine a yield condition and hardening rule consistent with the actual material behavior. Once the yield condition and hardening rule that apply to the triaxial tests are determined, the yield condition and hardening rule are generalized to apply for arbitrary stress states and loading paths not necessarily associated with the triaxial tests. The generalized yield function is discussed as a means of analyzing the plastic behavior of other rocks; and a plane-strain yield condition is derived from the generalized yield function and used to analyze wedge indentation tests. The predicted
wedge indentation forces show better agreement with experimental results than predicted forces using a Coulomb yield condition.

The improved accuracy of the predictions of material behavior using the proposed yield condition and hardening rule indicates that the yield condition will be of use to others studying materials that show volume decreases during loading processes.
THEORETICAL BACKGROUND AND LITERATURE REVIEW

This section will contain a brief review of basic plasticity theory as well as a thorough review of workhardening theories and yield conditions generally associated with soil and rock mechanics. Time and thermal effects in plasticity will not be discussed in the review as they are neglected or assumed negligible in the analysis of the rocks studies.

In this discussion, as well as in the remainder of this work, it will be convenient to refer to the state of stress of a point in the body to a point in a stress space. A loading program will refer to a certain path in the stress space. The stress point should ideally be in a nine-dimensional space; however, it is often more convenient and instructive to use sub-spaces when possible.

The concept of a yield function or loading function is used to specify the elastic or plastic character of the material behavior. The yield function

\[ f = f(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) \]

may be expressed as a function of the stresses \( \sigma_{ij} \); plastic strains \( \varepsilon_{ij}^p \), and the stress or strain histories, where here \( \kappa \) is a workhardening parameter defining the stress or strain histories. The value of \( f \) is restricted such that \( f < 0 \).

When \( f < 0 \) the material behaves elastically, and when \( f = 0 \) the material may or may not behave plastically. Now since
\[ f = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} \]

the following conditions may prevail when \( f = 0 \), and when

\[ f = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0, \]

The material is going from a plastic to an elastic state (unloading)

\[ \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0, \]

The loading path is tangent to the surface defined by \( f = 0 \) and no plastic strains occur (neutral loading)

\[ \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0, \]

The material goes from one plastic state to another accompanied by plastic strains (loading).

When the state of stress is such that \( f = 0 \), the surface described by \( f = 0 \) in the stress space is termed the yield surface. The yield surface for a virgin material is called the initial yield surface, and the manner in which the surface changes shape or translates is called the hardening rule. The yield surfaces must be at least piecewise continuous.

Other restrictions are imposed upon \( f \) by the fundamental postulate of plasticity, referred to as Drucker's postulate. Drucker[4] stated that when a stable, workhardening body, in equilibrium with a given set of loads has another set of self-equilibrating external loads slowly applied and removed, positive work must be done by the external loads during application of the loads and non-negative work must be done during the loading cycle. This statement is often paraphrased to say that no useful net energy above any elastic energy may be extracted from a stable, workhardening body and a system of stresses.
Thus from Drucker's postulate we have
\[ W_e = \int_{t_1}^{t_2} (\sigma_{ij} - \sigma_{ij}^*) \dot{\varepsilon}_{ij}^P \, dt \geq 0, \]
where \( W_e \) is the work done by the external loads, \( \sigma_{ij} \) are the stresses caused by the external loads, \( \sigma_{ij}^* \) are the equilibrium stresses and \( \dot{\varepsilon}_{ij}^P \) are the plastic strain rates associated with the deformation. Consequently a Taylor's series expansion of \( W_e \) about \( t=t_1 \) gives,
\[ (\sigma_{ij} - \sigma_{ij}^*) \dot{\varepsilon}_{ij}^P \geq 0, \]
and if \( \sigma_{ij} = \sigma_{ij}^* \), the second term of the expansion also implies that
\[ \dot{\varepsilon}_{ij}^P \varepsilon_{ij}^P \geq 0. \]

Now assuming \( f=0 \) for plastic flow and the state of stress is such that \( f=0 \), the above implies that:

1. The surface \( f=0 \) must be convex.
2. \( \dot{\varepsilon}_{ij}^P \) is parallel to the normal of the smooth segment of the surface.

If the surface is piecewise continuous and the loading point is at a corner of the yield surface, the direction of \( \dot{\varepsilon}_{ij}^P \) lies between the normals of the adjacent regular portions of the surface.

The fact that \( \dot{\varepsilon}_{ij}^P \) is parallel to the normal of the yield surface formalizes the generally used concept of the plastic potential where the plastic strain rates were proportioned to the gradient of the yield function \( \varepsilon_{ij}^P \). Since the gradient of \( f \) defines the normal of the surface \( f=0 \),
\[ \dot{\varepsilon}_{ij}^P = \lambda \frac{\partial f}{\partial \sigma_{ij}} \]
where $\lambda$ is an arbitrary positive constant. Thus by Drucker's postulate the yield function dictates the plastic strains of the material, and a consistent theory requires the plastic strain rates to be derivable from the yield condition.

Inspite of the results of Drucker's postulate, some workers attempt to evoke plasticity theory using different yield functions and plastic potentials (DeJong[5], Barden and Khayatt[6]). This is indicative that either the material behavior cannot be described using plasticity theory or that the yield functions are improperly chosen. Often corners in yield functions are proposed in order to allow a consistent yield function and strain rate field that matches a material's observed behavior (Jenike and Shield[7]).

The plastic strain rates are defined kinematically by

$$
\varepsilon_{ij}^p = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),
$$

where,

$x_i$ are the spatial or Eulerian coordinates of the body

$v_i = v_i(x_j, t)$ are the velocities in terms of the Eulerian coordinates.

Thus the plastic strain rates are independent of the reference state, and normality (or the plastic potential) holds regardless of the magnitude of the strains. Since the plastic strain rates are defined in terms of spatial variables,

$$
\int_{t_0}^{t} \varepsilon_{ij}^p \, dt
$$

is physically meaningless in terms of total plastic strain at a material point. It might be possible to interpret the integral as a
material time integral (analogue of material time derivative) that follows a particle through its motion; and thus the result could be interpreted as a total plastic strain. However, the actual mathematical process of evaluating the integral would be a complex or perhaps an impossible task. See Malvern[27].

If we assume infinitesimal strains, the elastic and plastic parts of a deformation can be superimposed such that

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \]

where,

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

\( u_i \) are components of a displacement,

\( x_i \) are components of the spatial or material coordinates,

\( \varepsilon_{ij}^e \) are components of the elastic strains,

\( \varepsilon_{ij}^p \) are components of the plastic strains.

Assuming linear elasticity holds, then

\[ \varepsilon_{ij}^e = C_{ijkl}^{-1} \sigma_{kl} \]

where,

\[ C_{ijkl}^{-1} \] are components of the inverse elastic constants for a generalized Hooke's law.

Then the plastic strain becomes

\[ \varepsilon_{ij}^p = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - C_{ijkl}^{-1} \sigma_{kl} \]

Note that this definition holds only for infinitesimal strains in a linear elastic/plastic material; but that the plastic strain rates derived from the yield functions are exact regardless of the amount of deformation.
The most commonly used yield functions for metals are the von Mises or maximum distortional energy and the Tresca or maximum shear stress yield functions. They can be expressed in terms of principal stresses as follows:

\[
\frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \right] - k^2 = f \text{ (von Mises)}
\]

\[
\sigma_{\text{max}} - \sigma_{\text{min}} - 2k = f \text{ (Tresca)}
\]

where \( k \) is the yield stress in pure shear. The appropriate yield surface for a von Mises function is a cylinder with radius \( k \) and axis the hydrostatic stress axis where \( \sigma_1 = \sigma_2 = \sigma_3 \); and the Tresca surface is a regular hexagonal prism with axis of symmetry the hydrostatic stress axis.

The above yield functions are for perfectly plastic behavior assuming \( k \) is independent of stress or strain histories. Improved description of material behavior results if the yield function is expressed in a more general form:

\[
f = f(\sigma_{ij}, \varepsilon_{ij}^p, \kappa)
\]

The freedom of assigning the functional dependence upon \( \varepsilon_{ij}^p \) and \( \kappa \) allow the yield surface to change its shape and/or position in space; the manner in which the yield surface is transformed is called a hardening rule. Figures 1-1a, b, c illustrate the three most generally used hardening rules, the sub-space used for Figure 1-1 is the plane perpendicular to the hydrostatic axis in principal stress space (called the \( \pi \) plane).

The isotropic hardening rule where the yield surface grows uniformly is the easiest to apply. In this case the von Mises yield condition can be written,
\begin{align*}
f (\sigma_{ij}, \kappa) &= 1/6 \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] - \kappa^2,
\end{align*}
and \( \kappa \) can be made to grow as a function of the plastic strain or plastic work on the material. The isotropic hardening rule is criticized because it does not explain the Bauschinger effect observed in most metals, however, it can provide a reasonable approximation of some loading programs where no load reversals are present.

Prager\textsuperscript{8} has proposed a kinematic hardening rule as a proper generalization of the Bauschinger effect observed in uniaxial tension and compression tests. In this case the yield surface is specified to translate as a rigid body in stress space. The appropriate expression for the yield function becomes

\begin{align*}
f (\sigma_{ij}, \dot{\varepsilon}_{ij}^p, \kappa) &= F (\sigma_{ij} - \alpha_{ij}) - \kappa^2,
\end{align*}
where the \( \alpha_{ij} \) are components of a tensor that represents translation of the origin enclosed by the original surface. The \( \alpha_{ij} \) can be functions of either the stress or strain histories. After Shield and Ziegler\textsuperscript{9}, the \( \alpha_{ij} \) may be specified by the rule

\begin{align*}
\dot{\alpha}_{ij} &= C \varepsilon_{ij}^p,
\end{align*}
where \( C \) is a positive constant for linear workhardening. Since from normality

\begin{align*}
\varepsilon_{ij}^p &= \frac{\partial f}{\partial \sigma_{ij}}
\end{align*}
and from the condition that during the plastic flow \( f = 0 \),

\begin{align*}
f &= \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \alpha_{ij}} \dot{\alpha}_{ij}.
\end{align*}
And since

\begin{align*}
\frac{\partial f}{\partial \sigma_{ij}} &= - \frac{\partial f}{\partial \alpha_{ij}},
\end{align*}
It follows that
\[ \lambda = \frac{1}{C} \left( \frac{\partial F}{\partial \sigma_{ij}} \right) \frac{\dot{\sigma}_{ij}}{\left( \frac{\partial F}{\partial \sigma_{kl}} \right) \left( \frac{\partial F}{\partial \sigma_{kl}} \right)} . \]

Thus the \( \alpha_{ij} \) can be found to within an arbitrary constant from the loading function, and the hardening rule is specified, if the \( \dot{\alpha}_{ij} \) are assumed equal to zero for no plastic strain.

It was Prager's intent for the yield surface to translate normal to the loading point of the yield surface. Shield and Ziegler[9] have shown that this is not always the case when some sub-spaces are chosen to represent the yield surface. Ziegler[10] has proposed a modification of Prager's rule in order to provide a consistent translation of the yield surface for any sub-space representations. Ziegler proposed that the translation tensor be defined by,

\[ \dot{\sigma}_{ij} = \mu \left( \sigma_{ij} - \alpha_{ij} \right) \]

By the same process as above

\[ \mu = \frac{\partial F}{\partial \sigma_{ij}} \frac{\dot{\sigma}_{ij}}{\left( \sigma_{kl} - \alpha_{kl} \right) \frac{\partial F}{\partial \sigma_{kl}}} \]

\( \mu \) is some positive quantity that forces the translation tensor increment \( \alpha_{ij} \) to be directed along the line from the instantaneous center of the yield surface to the loading point; and the time derivative is used so that the units will be consistent. Thus the yield surface translates along the direction of the stress vector directed from the instantaneous origin of the translating yield surface. Ziegler has shown this to be invariant under reduction of space dimensions.

Figure 1-1c illustrates the case of a local hardening rule where one part of a yield surface deforms. This perhaps is the most general form of hardening since several parts of the yield surface
could be made to deform such that the isotropic or kinematic hardening rule is obtained. (Hodge[11]). It should be noted that any combination of hardening rules is allowed. The validity of any hardening rule should only be judged by how accurately it explains the phenomena; however, the simplicity of application of the rule influences its usefulness.

The hardening rules discussed above may be applied to any particular yield surface even though the examples of the hardening rules were illustrated using yield conditions normally applied to metals. The yield point of metals, unlike rocks and soils, is essentially independent of the hydrostatic stress state. Rocks and soils, however, show an increase in strength with increasing pressures. Several generalized yield conditions are used for these materials. The three most commonly used are the so-called extended von Mises, extended Tresca, and the Mohr-Coulomb.

Examples:

Extended von Mises:

\[ f = \frac{1}{2} J_2 - aI_1 - k. \]

where, in terms of principal stresses

\[ J_2 = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \right] \]

= Second deviatoric stress invariant

\[ I_1 = \sigma_1 + \sigma_2 + \sigma_3 \]

= First stress invariant

\[ a, k > 0 \]
Here as in the remainder of this work, compressive stresses will be considered positive.

Extended Tresca:

\[ f = \sigma_{\text{max}} - \sigma_{\text{min}} - \alpha I_1 - k \]
\[ \alpha, k > 0 \]

\( \sigma_{\text{max}}, \sigma_{\text{min}} \) are principal stresses

Mohr-Coulomb:

\[ f = \sigma_{\text{max}} - \sigma_{\text{min}} - \alpha (\sigma_{\text{max}} + \sigma_{\text{min}}) - k \]
\[ \alpha, k > 0, \text{ and } \sigma_{\text{max}}, \sigma_{\text{min}} \text{ are principal stresses.} \]

Figure 1-2 shows cross sections of the three different yield conditions on the \( \pi \) plane. The extended von Mises yield condition forms a right regular cone in principal stress space with the \( \sigma_1 = \sigma_2 = \sigma_3 \) line its axis. The extended Tresca and Mohr-Coulomb yield conditions are hexagonal pyramids in principal stress space with cross sections as shown in Figure 1-2.

Each of these yield conditions allow a linear increase in strength with increasing hydrostatic pressure, and reduce to the well known Coulomb yield condition in plane strain:

\[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \frac{1}{2} - \sin \phi \left( \frac{\sigma_x + \sigma_y}{2} + c \cot \phi \right) = 0. \]
\[ c = \text{cohesive strength} \]
\[ \phi = \text{angle of internal friction.} \]

In this case \( \alpha \) and \( k \) would be slightly different for each generalized yield condition. The Coulomb yield condition is a special case of a Mohr yield condition that allows non-linear strength increases with increasing hydrostatic stresses.
Application of normality to each of the yield conditions predicts volumetric increases during plastic deformation for materials that obey either yield condition. This is illustrated using the extended von Mises condition.

Defining
\[ \dot{\varepsilon}^p = \dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p + \dot{\varepsilon}_3^p \]
as the plastic volumetric strain, and computing the volumetric strain from the function,
\[ f = J_2^{1/2} - aI_1 - k = 0 \]
gives the following
\[ \dot{\varepsilon}_1^p = \lambda \frac{\partial f}{\partial \sigma_1} = \lambda \left( \frac{\partial f}{\partial J_2^{1/2}} \frac{\partial J_2^{1/2}}{\partial \sigma_1} + \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_1} \right) \]

Now
\[ \frac{\partial J_2^{1/2}}{\partial \sigma_1} = 6 \frac{1}{J_2^{1/2}} \left( \sigma_1 - \sigma_2 \right) \left( \sigma_1 - \sigma_3 \right) \]
By cyclical permutation of 1, 2, 3, it follows that
\[ \dot{\varepsilon}^p = \dot{\varepsilon}_{ij}^p = -3\lambda \alpha \]
which in a system of positive compressive stresses implies a plastic volumetric increase.

Jenike and Shield[7] and Drucker[12] have discussed the inherent instability of a material that expands while deforming. Even though volumetric expansion has been noted by several experimenters [12], [13], [14], this type of deformation does not normally occur in tests on rocks and soils. This inconsistency between any generalized Coulomb yield condition and the strain behavior of rocks and soils have prompted workers to propose end caps on the cones or pyramids of the generalized Coulomb yield conditions so that application of
Drucker's postulate predicts volumetric decreases during plastic deformations. These proposals will be discussed in detail in the next section.

Review of Related Work

The characterization of rocks as workhardening materials with a yield condition and an associated flow rule that predicts volume compaction during plastic deformation is almost unique to this study. The only other work the author is aware of that treats rock in the manner was an experimental study by Cheatham[21] and it will be discussed later. There are, however, some papers dealing with soils as compacting-workhardening materials having a Coulomb or Mohr type yield surface with an end cap. As stated in the previous section, a yield surface must have an end cap in order that a plastic volume decrease will be predicted.

One of the earliest references to an end cap is given by Drucker, Gibson, and Henkel[5] where a hemispherical end cap to an extended von Mises cone was proposed in order to explain compaction of wet clays at certain stress states. Their workhardening theory proposed that the end cap grew in diameter as the material yielded at stress states on the end cap. The diameter of the end cap would grow such that it always remained in contact with the yield cone. By Drucker, et al's[5] proposal in an axial compression test in a triaxial apparatus where a sample is subjected to increasing axial loads at a constant confining pressure and where the loading path first touches the end cap portion of the yield surface, the following type of deformation would occur:
(1) As the cap increased in size the sample volume would decrease in smaller and smaller increments in proportion to the axial strain as the axial load increased.

(2) Eventually the material would deform incompressibly for an instant until,

(3) Expansion would take place as the loading path touched the yield cone and the material became perfectly plastic.

Even though this work presented a detailed qualitative description of the growth of a yield surface, it is not complete because no quantitative hardening rule was proposed.

Roscoe and Pooroosh and Roscoe, Schofield, and Thurairajh [15]; and Roscoe and Burland[16] have attempted to extend and complete the above work of Drucker. Roscoe and Burland[16] using results of triaxial tests in [2] and [15] propose a yield surface similar to the surface proposed by Drucker, et. al.[5]. The end cap is limited in size by a Coulomb envelope and grows as a function of the volumetric strain such that subsequent yield surfaces are surfaces of uniform volumetric strain. Three different hardening rules are proposed, utilizing different approximations (see [17]) for the amount of dissipative work done on the material as it deforms plastically. The closer the approximations approach the exact expression $W = \int \sigma_{ij} \delta \varepsilon_{ij}^p$ the better their hardening rule predicts the experimental results. It appears that the approximations are chosen more for convenience than for accuracy and that using the exact expression for the plastic work would give just as satisfactory result for the hardening rule.
After establishing the hardening rule and yield surface for the triaxial tests, they generalize it to apply for arbitrary stress states as

\[(M^w)^2 + 6)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + 2(M^w)^2 - 3(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\]

\[-3(M^w)^2 P_0 (\sigma_1+\sigma_2+\sigma_3) = 0\]

where,

- \(M^w\) is a material constant
- \(P_0\) is the hydrostatic yield stress

The above is the yield function for the end cap describing an ellipsoid of revolution; the ellipsoid is limited in size by a Mohr-Coulomb yield envelope.

Palmer[18] has used concepts of internal energy and plastic work in order to predict the shape of, and a hardening rule for the yield surface of a wet clay. He has proposed that the yield surface represents states of equal internal energy of the clay, where the internal energy is given by

\[U_1 = U_0 + \int_{v_0}^{v_1} (p dv^e + \mu dv^p)\]

where,

- \(U_1\) - internal energy at arbitrary state
- \(U_0\) - internal energy at some reference state
- \(v_0\) - volume of material at the reference state
- \(v_1\) - volume at the same arbitrary state above
- \(p\) - mechanical pressure
- \(dv^e\) - elastic volume change
- \(dv^p\) - plastic volume change
- \(\mu\) - a parameter that expresses the non-dissipative energy contribution of plastic volume changes
The shape of the yield surface is given by the dissipation work contribution of the volume changes and shear strains. Using this criteria, normality and experimental results from [15], Palmer states that the shape of the end cap would be approximately a straight line in a two-dimensional stress space. This yield surface does not correspond to the experimentally determined surface reported in [15], therefore the method seems somewhat lacking in that respect. Palmer's work, however, seems to be the first in utilizing the work and energy concepts in workhardening theories for clays.

It is interesting to compare Palmer's and Roscoe's work, where each ignores certain parts of the actual plastic work done on the material. Palmer deletes part of the volume change and Roscoe, somewhat arbitrarily, deletes part of the shear strain contribution of the work. It appears here that a consistent theory to account for the plastic work contribution to the workhardening of a material should consider all of the dissipative work.

Barden and Khayatt[16] have concluded from triaxial tests on sands that sand does work harden. They have derived a plastic potential function to describe the incremental stress-strain relations for the sand. However, it is claimed that the concept in plasticity of coincidence of the plastic potential and yield surface is not applicable for the materials tested. Concerning the concept of failure, Barden and Khayatt discuss the merits of using certain failure envelopes that coincide with the so-called extended von Mises
and Tresca and a Mohr-Coulomb. They conclude that the Mohr-Coulomb criteria is the most applicable failure surface for sands as well as clays.

Jenike and Shield[7] have proposed a yield surface for granular materials that takes into account this material's increase in strength after hydrostatic compaction and its ability to flow with zero volumetric strain. The yield surface is a piecewise linear combination of a Mohr-Coulomb surface with an end cap formed by a flat surface truncating the pyramid. Compaction takes place under hydrostatic stresses, and an increase in hydrostatic stress causes the material to workharden. The workhardening is represented by an isotropic growth of the yield surface. Yielding at stress states on the Mohr-Coulomb surface results in unstable failure of the material causing the surface to isotropically shrink (worksoftening) until the loading point becomes the corner of the Mohr-Coulomb and end cap surfaces. There, because of the freedom allowed in the direction of the strain rate, incompressibly flow is allowed. In their paper no workhardening rule was proposed, and it appears that this work was an attempt to explain the behavior of granular materials using a consistent plasticity approach.

Pariseau[19] has also used a corner in the yield surface to allow increasing material strength with increasing hydrostatic stresses as well as incompressible deformations within a consistent plasticity theory of granular materials. Pariseau uses a modified plane strain Coulomb yield condition using the locus of the maximum shear stress
points at failure (after Hill[20]) with a flat end cap. Pariseau, or Jenike and Shield did not attempt to formulate a general theory of failure for granular materials nor did they propose that these particular yield conditions apply to other types of materials.

Cheatham[21] has reported results of experiments designed to probe the yield surface of a porous limestone. The results definitely indicate the existence of an end cap, and of material work-hardening. No yield condition or hardening rule was proposed to explain this behavior.

Swanson[14] has reported results of triaxial tests on hard, granite-like rocks, generally considered to be brittle materials, and has found them to be ductile at high confining pressures and has proposed a yield condition and workhardening theory to explain their stress-strain behavior. Swanson's yield condition is a Mohr type yield condition that decays exponentially to a von Mises yield condition at high hydrostatic stresses; his failure surface, the surface of stress states at brittle failure of the material, is given by the same type of surface as the initial yield surface only it predicts greater failure stresses at all confining pressures than the initial yield surface.

Swanson's yield condition predicts volume increases during plastic flow, where his experimental results do not always show this, and he proposes a non-associated flow rule to account for this behavior. While his non-associated flow rule (or plastic potential)
gives good agreement with his experimental results, a different yield condition using an end cap might predict the same flow rule within a consistent theory of plastic flow.

In addition to the above papers concerned with the work-hardening properties of rocks, some workers have been primarily concerned with the ductile behavior of rocks. Early workers, most notably Handin and Hager[24], Griggs[38], Heard[39], Handin, Hager, Friedman, and Feather[23], and Bredthauer[25], report that as rocks are subjected to high confining pressures they behave in a ductile manner, with the transition from brittle to ductile behavior occurring at different pressures for different rocks. Also, the results indicated the rocks show increased ultimate strengths when tested at higher confining pressures.

Others have inferred from these data above that a Coulomb or Mohr type yield condition could be applied to characterize the rock's stress-strain behavior [22], [26], [29]. However the tests above did not involve unloading before ultimate failure of the rocks; and it is likely, based on the experimental results reported later in this work and in [14] and [21], that results of the unloading tests would show that the material workhardened before ultimate failure. Thus it is apparent that a Coulomb or Mohr envelope that defines a failure surface would be more appropriate than a yield surface for these materials and studying the plastic behavior of the material at stress states below the failure surface would be more fruitful than studying the material behavior at stress states on the failure surface.
Triaxial tests on two types of porous rocks have been performed by the author that show these materials to be workhardening materials rather than perfectly plastic. It is felt that similarity between the test results obtained by the author and the test results reported in [22], [23], [24], [25] indicate that the materials tested in [22], etc. are also workhardening before reaching an ultimate failure condition that matches a Coulomb yield condition. Following will be the results of the tests performed on the porous rocks as well as the interpretation of the results.
APPARATUS AND EXPERIMENTAL PROCEDURE

Design Philosophy

The object of this experimental program was to determine the initial and subsequent yield surfaces and the hardening rule for some particular porous limestones. Since usable workhardening rules are necessarily simplifications of the actual material behavior and since these rocks tend to be quite inhomogenous, it was felt that a detailed study of the material properties was not as important as the determination of the general behavior patterns of these materials.

Since previous work by Cheatham[21] indicated that any point on the yield surface for porous limestones could be reached using loading paths consisting of hydrostatic stress increments and/or axial stress increases or decreases, the experimental apparatus was designed primarily to

(1) Provide the above type loading paths.

(2) Measure the axial stress-strain behavior of the materials.

It was felt that the most important parameters needed in order to determine the workhardening characteristics of the samples were the dependence of the yield point and workhardening rate on the hydrostatic stress states. During the analysis of the preliminary experimental results, it was found that continuous measurement of the radial strains and loading paths other than axial compression or extension would have been most helpful in determining some segments of the initial and subsequent yield surfaces, consequently certain assumptions concerning these segments were made in order to complete the analysis.
Apparatus

The experimental apparatus is a modified version of the equipment designed by Jones[30] for the testing of rocks subjected to hydrostatic pressures. The pressure vessel itself is unchanged from Jones' design; but the means of applying the confining pressure, loading the specimen, and measuring the axial stress-strain relation for the specimen have been changed extensively.

Figure 2-1 shows a cross-section of the assembled apparatus with a sample inserted ready for testing. The pressure vessel itself was designed to withstand pressures up to 21,000 psi, but the tubing supplying pressure to the vessel and the loading ram is only designed for pressure up to 15,000 psi. As the figure shows the vessel has a hole at one end for a piston used to load the sample. At the other end the vessel has a removable cap for the insertion of test samples. The cap itself has a hole for a piston used to restrain the sample. O-ring seals are used throughout the system.

Axial load is increased or decreased by a hydraulic ram beneath the vessel. Strain gages are attached to a cantilever beam assembly on the loading piston for sample end deflection measurements, and a load cell is placed atop the reaction piston for load measurements. Strain gages are wired as four active arms of a Wheatstone bridge in both the load cell and displacement transducer. Each bridge arrangement has a separate power supply.

Confining pressures are obtained using a standard Enerpac-Test Systems hydraulic pump and are measured using a Heise pressure...
gauge accurate to 50 psi. When the hydrostatic stress state on the sample is increased or decreased, the hydraulic ram is connected to the pressure vessel. When the ram is isolated, a Ruska Instrument Company Metering Pump is used to continuously increase or decrease pressure in the ram pressure chamber. Hydraulic oil is used as the confining fluid.

Initially the load cell was calibrated against a Riehle Testing Machine, and was checked periodically against the pressure gauge. The deflection transducer was calibrated against a standard dial gauge. Output of the load cell and deflection transducer was displayed by an X-Y recorder calibrated such that the curves were axial stress-strain curves for the tests.

Preparation of Samples

Tests were conducted on rock samples nominally one inch long by one-half inch in diameter. The samples were removed from large blocks with a core drill and cut to size; then the faces were ground parallel to each other and perpendicular to the sides on a lapidary stone. Actual diameters of each samples were 0.505" ±0.003" and variance from average diameter for each sample was less than 0.003". The average diameter was determined from measurements at both ends and the center of each sample at mutually perpendicular orientations.

The actual length of the majority of the samples was 1.000" ±0.010", but later in the testing some samples as long as 1.045" were
used. Parallelness of the ends was carefully checked and no sample was used when the length of the sample measured at different locations disagreed by more than 0.002".

Each sample was measured as described above before each test, and then inserted into thin walled Tygon tubing. The tubing was used to protect the sample from oil as well as keep the sample pore pressure at atmospheric. The reaction piston and load cell are hollow so that the pore pressure remained at atmospheric pressure throughout the test.

**Test Procedure**

Figure 2-2 shows a schematic diagram of the tubing from the two pumps to the pressure vessel and loading ram as well as the wiring from the load cell and deflection transducer to the X-Y recorder. Essentially the same procedure was followed for all types of tests. The sample was placed in the vessel; the vessel was then closed and filled with oil. The desired hydrostatic pressure was then reached with the crossover valve open, then with the pressure vessel and ram systems isolated the desired loading program was completed.

Five basic types of tests were performed on the samples. They were

1. Pure hydrostatic compression
2. Axial compression at given confining pressure
3. Axial extension at a given confining pressure
4. Radial compression where the confining pressure was increased with the crossover valve closed and the axial stress decreased because of the different sample and piston diameters.
During axial compression or extension tests the ink pen of the X-Y recorder was activated after the desired confining pressure was reached and the differential axial stress versus axial strain curves were recorded as the axial load was increased or decreased. Care was taken during these tests to keep the loading rate nearly constant, since it was found that the material properties did depend somewhat on the rate of loading.

As expected increasing the loading rate causes an increase in the apparent yield point and workhardening rate. When the loading was stopped near the yield point, the material would creep; however, the creep rate was negligible compared to the loading rate, even when the ram pressure was changed very slowly. Experience showed that repeatable results were obtained using very slow loading rates.

Radial compression tests are defined here as tests where the confining pressure (actually the radial stresses) is increased while the crossover valve is closed. Meaningful, detailed data were quite difficult to obtain for tests of this type because the piston diameters were larger than the test sample diameter. Thus the load cell reading would change for constant axial stress loading paths, and the yield point in the radial direction could not be detected. Some radial compression tests were performed where the confining pressure was increased in small increments with corrections made in the axial load to approximate a pure radial compression test. Other radial compression tests were performed where the confining pressure was increased without correcting the axial load. In these cases the confining pressure was increased until yield in the axial direction was noted; and the loading path is unknown except for the end point.
Experimental Error

Experimental data for axial compression or extension tests consisted of specimen dimensions before and after each test and the X-Y recorder plot of the differential axial stress versus axial strain at each confining pressure. X-Y recorder plots for hydrostatic tests and radial compression tests, while useful, did not normally show the yield points of the material. Volumetric strains were calculated from dimension changes, and the approximate yield points of axial compression tests, where the axial load was corrected incrementally, were determined by removing the sample at definite points along the loading path and measuring the dimension changes.

Experimental error was primarily caused by the following:

(1) Differences in specimen diameter from sample to sample
(2) Changes in specimen diameter during tests
(3) Differences in specimen length from sample to sample
(4) Changes in confining pressure during each test
(5) O-ring friction
(6) End effects

Errors in differential axial stress as measured by the load cell are caused directly by all but (2) and (6) above. The load cell was calibrated assuming the sample diameters remained constant at .505". Variation of the specimen diameters between samples was less than .5%, and variation of the average sample diameter during most tests was less than 2%. Corrections to the apparent stress were made for the two reported tests where the diameter changes were greater than 2%. 
Changes in the confining pressure not only affected the stress-strain properties of the material but also changed the apparent axial stress because of the difference in the sample and piston diameters. A change of 100 psi in the confining pressure caused an apparent axial stress change of 85 psi; however, the pressure change in most tests was less than 50 psi until the samples were strained 4% to 5%. Where the pressure changed more than 50 psi the change was noted and the output corrected for that test.

The only significant error that could be introduced was caused by O-ring friction. But since the load cell was designed to measure changes in axial load, the effect of the frictional force would be negligible if it were directed in the same way throughout a test. If axial compression tests were conducted after an increase in hydrostatic pressure the frictional force would be directed downward after the hydrostatic increase; and during the loading phase of the compression test the force would still be directed downward. Thus, if the X-Y recorder pen were activated just before the axial compression test was begun, the O-ring friction would not affect the results (assuming the friction were constant). Also if the hydrostatic pressure were decreased before an extension test the O-ring friction would not affect these results either.

Note that the frictional force can be measured when results of tests where the hydrostatic pressure is increased to a certain value and the sample is yielded in extension are compared with the results of tests where the hydrostatic pressure is decreased to the
same value as above and the sample yielded in extension as well. Assuming the yield points of the two different samples are the same, the apparent difference in the yield point as measured on the X-Y recorder output for the different tests would be the effect of the O-ring friction. Tests of the above type were performed and the results indicated the O-ring friction could reduce the apparent total change in axial stress by 500 psi.

The deflection transducer was calibrated assuming the pressure vessel was perfectly stiff and that each sample was 1.000" long so that the deflection was equal to the axial strain. Sample lengths were as long as 1.045" and as short as 0.975", and the resulting inaccuracy in the strain was as much as 4.5% of the total apparent strain. The flexing of the equipment was approximately 0.0002" for every 1000 psi increase in the axial stress. The maximum error occurring from the deformation of the test system was found to be 2.2%, thus the maximum possible error in the axial strain could be as high as 6.7% of the apparent measured axial strain.

Two problems are encountered with end effects,

(1) Stress concentrations occur due to local surface irregularities or due to the loading piston and sample not being aligned properly

(2) Friction at the piston sample interface due to differences in lateral movement of the piston and sample. The end surfaces of the samples were ground flat and parallel in order to minimize stress concentrations. Since the rocks were quite ductile,
even at low confining pressures, it is felt that stress concentrations due to local irregularities are minimal. Frictional end effects seem to be minimal as well since the radial strains were almost uniform along the sample length for tests with axial strains less than 5%. In fact at the higher confining pressures, the radial strains were almost zero.

Even though the above errors are not negligible, they are significantly less than the differences between samples due to inhomogeneous effects. In addition, the object of this experimental study was not to provide a detailed set of stress-strain curves for the materials, but was an attempt to determine the general stress-strain behavior of the material as a function of the confining pressure in order to determine if a workhardening theory of plasticity could be applied to this type of material.
EXPERIMENTAL RESULTS

Several variations of the basic type tests were performed in order to determine the characteristics of the general behavior of the rocks. The tests and the purpose for the tests are given below:

(1) Tests using different samples, each at a unique confining pressure in order to determine the yield point of the rock as a function of confining pressure;

(2) Tests with one sample at several confining pressures in order to verify that the material behavior observed in (1) was not due to anisotropic or inhomogeneous effects;

(3) Tests where the axial load was cycled from extension to compression in order to determine the effect that yielding at one part of the yield surface has on the other parts;

(4) Tests on different samples at unique confining pressures to large axial strains (7%) in order to determine the workhardening characteristics as a function of confining pressure;

(5) Tests on different samples at various confining pressures in order to determine the workhardening rule.

Batesville marble and Cordova Cream limestone, both porous limestones, were used in determining initial yield surfaces. Batesville marble quarried in Batesville, Arkansas is markedly anisotropic with yield strengths at 3000 psi confining pressure of samples cored at different orientations differing by 100%. Tests quoted are from the weakest orientation. Cordova Cream limestone or Austin Chalk is not quite so
anisotropic as the Batesville marble. The anisotropy of both the
Batesville marble and the Cordova Cream limestone is related to the
layers in the rocks resulting as these sedimentary rocks were formed.
These layers are termed bedding planes. Samples of both rocks were
weaker when cored perpendicular to the bedding plane, but the differ-
ence in strengths of the different orientations of the Cordova was
only about 20%.

Curve Interpretation

As stated earlier the yield surfaces for materials may be
determined from the locus of points of the yield stresses for arbi-
trary loading paths. Curves from the X-Y recorder for axial com-
pression or extension give the yield point directly. Yield points
for radial tests are determined by direct measurement of samples
when they were removed from the vessel at certain points along the
loading path.

After yield the slopes of the stress-strain curves for axial
compression or extension tests give additional insight into the
material behavior at the particular confining pressure. Figure 3-1
shows the basic type of curves found during axial compression tests.
Axial extension curves may be interpreted similarly by substituting
negative values for the differential axial stress and axial strain
axes.
Curve A shows a rapid drop in the stress after the material yields and is indicative of brittle behavior. Curve B shows a drop in the stress level after the yield point is reached. This type of behavior indicates an unstable flow process but the rate of yielding in the actual tests is usually sufficiently slow to permit the axial load to be reduced such that the sample can be removed intact. It was found that volume expansion occurred during tests with stress-strain curves similar to curve B; and in addition tests with results similar to curve B indicate that the material shows increasing strength with increasing confining pressure. Thus curves of type B can be associated with yielding on a Coulomb yield surface, and curves of type B will be termed a Coulomb failure.

Curves of type D occur during tests at higher confining pressures than for type B. The rising slope after yield indicates stable flow and is accompanied by a volume decrease during the loading process. Decreasing yield stresses with increasing confining pressures are characteristic of tests with curves of type D and indicate the end cap portion of the yield surface is being probed. The slope of the stress-strain curve will be termed the workhardening rate with steeper slopes implying increased workhardening rates.

Curves of Type C can be interpreted as indicating either perfectly plastic behavior or a transition from compacting to Coulomb failures. In the workhardening theory discussed below, perfectly plastic behavior is not allowed and horizontal stress-strain curves may occur only instantaneously.
Initial Yield Surface

Tests on various samples each at different confining pressures and tests on a single sample at different confining pressure were performed in order to determine the initial yield surfaces for the Cordova limestone and the Batesville marble. Figure 3-2 shows the axial stress-strain behavior of various samples of Cordova limestone with axes perpendicular to the bedding plane when tested at different confining pressures up to 6000 psi. The curves for tests above 6000 psi have essentially the same yield point (0 psi) as the 6000 psi test except the workhardening rate increases with increasing confining pressure. The magnitude of the axial strain at the end of each test has no significance in itself, except that each test was stopped when a constant workhardening rate was established.

Figure 3-3 shows the axial stress-strain behavior of various samples of Batesville marble with axes perpendicular to the bedding plane when tested at different confining pressures. Just as in Figure 3-2 the yield stresses increase with increasing confining pressure up to a given confining pressure then decrease at higher confining pressures. Also on tests where the yield stress is increasing with confining pressure, unstable flow is indicated by the falling stress-strain curve and where stable flow is noted the workhardening rate increases with increasing confining pressure.

Figure 3-4 shows the axial stress-strain behavior of various samples of the Cordova limestone, with axes parallel to the bedding planes, tested at different confining pressures. It can be readily
seen that these results follow the same trends noted in Figures 3-2 and 3-3. However, different samples with axes parallel to the bedding planes did not consistently show the same yield stress when tested at the same confining pressure. In fact the difference between yield stresses for samples with axes parallel to the bedding plane was greater than the difference between the samples cored at different orientations to the bedding planes.

Figure 3-5 shows the maximum differences encountered between the separate samples tested at 2000 and 5000 psi. Figure 3-6 shows the maximum differences for samples tested at 3000 and 6000 psi. Comparison of Figures 3-5 and 3-6 shows that the results shown in Figure 3-4 are not necessarily an illustration of general material behavior.

Results such as these prompted tests on one sample at different confining pressures in order to determine if the material did behave in general as in Figure 3-4. One sample was loaded at one confining pressure until it had apparently yielded, then the confining pressure was changed and the sample was yielded again. Workhardening effects were minimized by keeping the axial strains small. Figure 3-7 shows the results of tests of this type. These stress-strain curves exhibit the same dependence of yield stress on confining pressure as seen in the previous figures. The order of testing, either increasing or decreasing confining pressures, was changed as shown in Figure 3-7a, b in order to show that these results are independent of increasing or decreasing confining pressures as well.
Axial extension tests were also carried out in order to
determine another segment of the initial yield surface. Figure 3-8
shows results of extension tests on different samples of Batesville
marble tested at different confining pressures. The curves show
"increasing" yield stresses with increasing confining pressures and
except for the test at 11,000 psi the samples show unstable failure.
Unstable failure could result from the sample diameter getting smaller
at one point along its length—analagous to necking in conventional
tension tests—or from failure associated with a Coulomb surface.
Inspection of these samples after the tests revealed that about half
of the samples had "necked"; but there was no difference noted in
the yield points of samples that had or had not "necked".

Figure 3-9 shows results of extension tests on samples of
Cordova limestone with axes perpendicular to the bedding plane. In
this case the yield stress for each test "increases" with increasing
confining pressure, and these samples also fail unstably just as the
Batesville marble. The results at 3000 psi shown in Figure 3-8 and
3-9 show an unusual behavior as the axial load is increased after
the samples yielded. This behavior is possibly due to the sample
separating from the piston and therefore the results should not be
regarded as indicating true yield points of the material.

Figure 3-10 shows the results of extension tests on one
sample of Cordova limestone, with axis parallel to the bedding plane,
tested at various confining pressures. The same trend of "increasing"
yield stresses with increasing confining pressures is shown in these
tests. Yielding to large strain in these tests was avoided as the unstable failure at one confining pressure reduces the yield stress of the sample when tested at another pressure. These tests essentially completed the study of the initial yield surface, however, results of initial yield found in the workhardening tests can be useful as well.

**Workhardening Tests**

Various tests were performed in order to determine the general workhardening characteristics of the materials before a detailed study of the workhardening rule was begun. These where the axial load was cycled from compression to extension several times were performed at various confining pressures in order to determine the effect of yielding in compression on the yielding in extension at the same confining pressure and vice versa. In these tests O-ring friction definitely affects the results and must be taken into account. All of the tests reported below were started with compressive loads after raising the confining pressure to the desired level; therefore since the O-ring friction will affect extension tests the yield stress in extension should be about 500 psi lower than indicated on the figures.

Figures 3-11 and 3-12 show results of cyclic loadings of Batesville marble at confining pressures of 8,000 and 12,000 psi. The material appears to have a unique yield point in extension and to workharden in compression. Again a higher workhardening rate is noted at the higher confining pressure.
Figures 3-13, 14 and 15 show results of cyclic loadings of Cordova limestone samples, with axes parallel to the bedding plane, tested at 3000, 5000 and 7000 psi, respectively. These curves also show an apparent unique yield point in extension for each test where the yield point in compression is dependent upon the sample's loading history. Also the workhardening rate increases with increasing confining pressure. The apparent anomaly of workhardening for loading in compression and perfectly plastic behavior for loading in extension will be discussed later, but it is helpful to note that this behavior is explained if the end cap is reached by compressive loading paths and a Coulomb surface is reached by extension loading paths.

Several radial compression tests were performed in an attempt to probe the lower part of the end cap. Results of these tests were inconclusive since it was found that the yield point in a pure radial compression test was not determinable without continuous monitoring of the radial strains; and the results of radial compression tests where no axial load corrections were made gave yield points along the Coulomb yield surface. Some results of these tests are reported below.

Figure 3-16 shows the results of radial compression tests on two samples of Batesville marble where no correction in the axial load was made while the radial stresses (or confining pressure were increased. In the first tests shown in Figure 3-16 the radial stresses were increased after the desired confining pressure was reached, in the second and third tests the axial load was reduced by a set amount before the radial stresses were increased, and in the last test an
axial extension test was performed. In tests [1], [2] and [3] the apparent increase in the axial load is due to resultant force on the piston due to the increased confining pressure. Since the confining pressure had to be rapidly reduced in order to stop the yielding, it appears that yield in these tests was unstable.

As mentioned above, true radial compression tests were also conducted in an attempt to determine the yield surface with this loading path, but the results were quite uncertain due to the limitations of the testing apparatus. Nevertheless, true radial compression tests were performed in order to check the influence of radial loadings on the workhardening in axial compression tests. Figure 3-17 shows the results of axial compression tests at 6000 psi. Test number one was axially compressed after increasing the hydrostatic pressure, tests two and three were radially compressed in 500 psi increments to 8000 and 10,000 psi, respectively, before axial compression at 6000 psi. The axial stress was corrected to 6000 psi after each confining pressure increment. These tests were performed on adjacent samples that were of similar strengths as judged by hydrostatic tests, thus the difference in yield points is not due to inhomogeneous effects. The curves for the samples with a radial loading history are drawn such that the asymptotic behavior matches the asymptotic behavior of the sample with no radial loading history. These curves show that radial compression can influence the material behavior in axial compression tests.
Since radial compression tests or loading on the lower portion of the end cap are stable and cause workhardening and since definitive tests in this region were not possible due to limitations of the experimental apparatus, it was assumed that the behavior in the lower portion of the end cap is similar to the behavior in the upper portion as tested by axial compression tests. Further tests in axial compression on various samples of Cordova were performed in order to determine the workhardening rule.

These tests were performed on different samples of Cordova limestone with axes perpendicular to the bedding plane. The samples perpendicular to the bedding plane were chosen because they showed the most consistent behavior from sample to sample. The Cordova limestone was used to determine the workhardening rule since it was the weakest of the two rocks tested—thus allowing testing to larger axial strains. It is implicitly assumed that the Batesville marble would behave similarly since it behaved similarly in all other tests.

Figures 3-18 through 3-23 show the behavior of different samples tested in axial compression at confining pressures ranging from 3000 to 8000 psi at 1000 psi increments. The samples were loaded to some point and then removed, measured, and retested. Occasionally the samples were unloaded and reloaded without removal. The curves show that the asymptotic behavior of the material at each confining pressure is unique.
Each time the sample was removed between loadings the subsequent loading curves fell below the asymptote of the previous stress-strain curve, however, samples that were loaded again without removal did not yield at the expected level and the workhardening portion of the curve fell on the previous test's asymptote. Thus it appears that the sample was affected by the unloading to atmospheric pressure, or that the material yielded slightly in extension when the axial stress was reduced below the confining pressure in order to overcome the O-ring friction. It will be assumed in the analysis that this effect is inconsequential and will be ignored.

Figure 3-24 shows the behavior of the Cordova limestone at 3000 and 5000 psi confining pressures when samples are subjected to large axial strains. The samples used here are the same as those reported in Figures 3-18 and 3-20. The curves shown in Figure 3-24 are corrected for diameter and confining pressure changes during the tests since both quantities changed significantly during the large deformation tests. Both tests show linear workhardening for axial strains up to 10% and decreasing rates for higher strains. In the 3000 psi test the workhardening rate approached zero at the end of the test and the sample was fractured.

In addition to the above tests, several samples were each tested in axial compression at arbitrary confining pressures in an arbitrary order as a means of establishing the workhardening rule. These results will be presented later after the results of the initial yield surface tests are analyzed.
INTERPRETATION OF TEST RESULTS

As stated earlier a workhardening material behaves elastically or plastically depending upon its state of stress and its loading or strain history. If the state of stress lies inside of the current yield surface (or the value of the loading function $f$ is less than zero), the material will behave elastically. If the state of stress is such that $f=0$, loading, neutral loading, or unloading may take place depending upon the direction of the stress increment. Thus, if a material's initial yield surface and hardening rule are determined, given the state of stress and the stress and strain histories of the material, its behavior can be characterized.

The experiments reported here using a triaxial testing apparatus give results that are applicable to only one portion of the yield surfaces of the materials tested. That part of the yield surface is the part cut by a plane containing the lines $\sigma_2 = \sigma_3$, and the $\sigma_1$ axis; therefore the best stress space representation for the test results is two dimensional cartesian space with axes $\sigma_1$ and $\sqrt{2}\sigma_3$.

If the material is anisotropic, the labeling of the stress coordinate axes is not independent of the material orientation, thus $\sigma_1$, $\sigma_2$, or $\sigma_3$ must refer to specific directions in the physical space. For example, in the materials tested the principal stress axes must be labeled to refer to some specific direction in relation to the rock's bedding plane for each yield surface reported. Even though the materials tested are anisotropic comparison of the results of similar
tests (compare Figures 3-2, 4, 7; Figures 3-9, 10; and Figures 3-13, 14, 15, 18 through 23) on the samples of Cordova limestone with axes at different orientations with respect to the bedding plane, show, that for the Cordova limestone at least, the dependence of the yield strengths and workhardening rates on the confining pressure is independent of sample orientation. Since the general stress-strain behavior of the Cordova limestone is independent of the sample orientation, the general stress-strain behavior should not be due to material anisotropy. Thus the material behavior will be treated within an isotropic theory.

Figures 3-5 and 3-6 show that different samples of Cordova limestone with the same bedding plane orientation do not necessarily show the same yield stress when tested at the same confining pressure. Figures 3-7a, b show, however, that the dependence of yield strength on confining pressure seen in Figures 3-2 and 4 is not due to inhomogeneous effects. Thus the following analysis will also assume that the material is homogeneous as well as isotropic.

**Initial Yield Surface**

The initial yield surface was experimentally determined by plotting the locus of points of the yield stresses, as determined from the differential, axial stress versus axial strain curves. Ideally the samples should be yielded using loading paths that would produce yield at all points of the yield surface; but as stated above, equipment limitations did not allow the determination of yield points using a true radial compression path. Figure 3-25 shows the locus of
yield points of virgin samples of Batesville marble as determined from Figures 3-3 and 3-8. The yield point in both extension and compression is determined by the point of juncture of the straight line extensions of the workhardening and elastic portions of the test curves.

Figure 3-26 shows the locus of yield points for virgin samples of Cordova limestone with axes perpendicular to the bedding planes. The yield points are determined in the same manner as above from the differential axial stress-axial strain curves in Figures 3-2 and 3-9. The dashed line represents an expected value based on the behavior of the material in extension and compression and the fact that yield surfaces must remain convex. Figure 3-27 shows the locus of yield points for virgin samples of Cordova limestone with axes parallel to the bedding plane. The yield points are taken from Figures 3-4, 3-7 and 3-10.

Inspection of Figures 3-25, 26 and 27 reveals the strong similarity between the three different yield surfaces. The interesting aspects of the figures are:

(1) A closing off of the yield surface at some confining pressure.

(2) An apparent increase in strength in extension with increasing confining pressure, up to a particular pressure before closing off the surface.

The closing off of a yield surface is the most significant of the experimental results concerning the initial yield surface, proving experimentally the existence of an end cap for these two rocks.
It is interesting to compare the actual strains of the samples with the strains that are predicted by Drucker's postulate. In Figure 3-28 the actual measured strains are represented by the arrows on the initial yield surface of Figure 3-25 for Batesville marble. While normality is not strictly followed, the prediction of positive or negative volumetric strain holds in the majority of cases and the normals to the surface would not be a bad approximation to the actual strains. Table 3-1 tabulates the measured strains for several cases.

Figure 3-29 shows the measured strains for Cordova limestone samples with axes perpendicular to the bedding plane superimposed on the experimentally determined initial yield surface. Table 3-2 tabulates the measured strains. In this case there is even better agreement between predicted and measured strains.

Thus it appears that the concept of an initial yield surface is valid for the material, and that the initial strains predicted from plasticity theory are consistent with the measured strains. The sections of the yield surface for the extension tests and for the low pressure compression tests appear to be Coulomb-type yield surfaces. The stress-strain behavior in these regions confirms this concept as the differential axial stress versus axial strain curves exhibit unstable behavior.

Unstable behavior in these regions implies the lack of workhardening, and perhaps it implies worksoftening. However, the stable stress-strain curves in the end cap region allow increasing
material strengths in this region. Compressive tests to large strains at low confining pressures (Figure 3-24) indicate that the material workhardens to some limiting value before failure. The experimental results of others have shown that most rocks exhibit the above type of behavior, with increasing ultimate strengths with increasing confining pressures. (Handin, Hager, Friedman and Feather[23]; Handin and Hager [24]; and Bredthauer[25]).

Radial compression tests without axial load corrections also indicate the existence of a limiting strength in extension for this material. Results of analysis of Figure 3-15a, b show that the Batesville marble yielded unstably with the yield points located on the continuation of the line drawn through the yield points in extension for Figure 3-25. These results together with the tests reported in [23], [24], and [25] and the results for axial compression tests to large strains Figure 3-24 indicate that the material workhardens to a limiting curve given by a Coulomb yield condition.

**Hardening Rule**

The hardening rule specifying the growth of the yield surface is necessary in order to complete the description of the material behavior. In general, it is thought that the yield surface can best be described by separating it into two separate parts defined by:

(1) A Coulomb yield envelope that explains the unstable material behavior in compression at low pressures and the extension behavior at all pressures.
end cap section that grows as the material is strained, where the end cap is limited in growth by Coulomb envelope.

Figure 3-32 illustrates the concept of limiting Coulomb envelope and a growing end cap.

Samples of Cordova limestone with axes perpendicular to the bedding plane were used because these samples were the weakest and the least inhomogeneous of the samples tested. Weak samples were desirable so that larger axial strains could be reached without exceeding the design pressure in the tubing to the loading ram; and homogeneous samples were desired so that comparison between tests at different confining pressures would not be influenced by different sample strengths.

The results of the tests for initial yield surface and workhardening behavior for both of the Cordova limestone samples and the Batesville marble samples indicate that the general behavior of all the samples tested followed the same pattern—the general behavior of the samples meaning:

(1) Yield stress dependence upon confining pressure;
(2) Workhardening rate dependence upon confining pressure;
(3) Unstable, Coulomb failure at low confining pressures and stable, workhardening failure at higher confining pressures for axial compression tests;
(4) Unstable, Coulomb failure for all confining pressures for extension tests.
The strong similarity in these results indicate that the same hardening rule would also be valid for samples other than the Cordova limestone with axes perpendicular to the bedding planes.

Axial compression tests were used to verify the hardening rule because data from radial compression tests were very unreliable. Results of the radial compression tests shown in Figure 3-17 show that radial compression tests do affect the workhardening in axial compression. Analysis of the tests shown in Figure 3-17 indicates the hardening rule applies for these tests; however the results can only be considered approximate because of uncertainties in the true stress-strain behavior and the influence of the material anisotropy.

Figures 3-18 through 23 show the behavior of individual samples tested at different confining pressures. The results show that the material has a unique workhardening rate for each confining pressure as well as a unique yield point. It was noted that if one sample were tested at different confining pressures, the material workhardened at the rate unique for that particular pressure.

Refering to Figure 3-30, the amount of workhardening or the growth of the yield surface can be experimentally determined (assuming axial compression tests only) by,

1. Loading the material to a set differential axial stress at a particular confining pressure, and then unloading the sample,

2. Changing the confining pressure,

3. Loading the sample until yield is reached at the new confining pressure.
The value of the differential axial stress \( \Delta \sigma_a \) before unloading the sample at the first confining pressure gives one point on a subsequent yield surface caused by a loading path described in (1) above and shown on Figure 3-30a. Another point on the subsequent yield surface is given by the yield point of the stress-strain curve at the other confining pressure (assuming no yield during change in confining pressure) shown in Figure 3-30b. More points on the subsequent yield surface caused by loading the material the set amount in (1) above can be found by repeating these tests for different samples, using different confining pressures each time for the test in (3) above.

Now referring to Figure 3-31 and defining \( \varepsilon_a \) as the axial strain at any one confining pressure and \( \Delta \sigma_a \) as the differential axial stress associated with \( \varepsilon_a \) through the unique stress-strain curve for that confining pressure shown in Figure 3-31a. Let \( \varepsilon_a \) and \( \Delta \sigma_a \) be the axial strain and differential axial stress related to each other as shown in Figure 3-31b.

If one sample were loaded a set amount at one confining pressure and unloaded, define the resultant permanent axial strain as \( \varepsilon_{a_0}^{(1)} \). If the sample were immediately loaded again the new yield stress for the second loading would be \( \Delta \sigma_{a_0} \). If another sample had been loaded to some differential axial stress at some other confining pressure and then unloaded with a resultant permanent axial strain \( \varepsilon_{a_0} \) such that \( \varepsilon_{a_0}^{(2)} = \varepsilon_{a_0}^{(2)} \), and this same sample were then loaded at
the first confining pressure, the yield stress for this test \( \Delta \sigma_{y0}^{(1)} \) would not necessarily equal \( \Delta \sigma_{y0}^{(1)} \). \( \Delta \sigma_{y2}^{(1)} \) would correspond to some axial stress \( \epsilon_{a2}^{(1)} \) as shown in Figure 3-31a, if the material had been loaded from the virgin state such that \( \Delta \sigma_{a} = \Delta \sigma_{y2}^{(1)} \).

Define \( \epsilon_{a2}^{(1)} \) as the equivalent axial strain at one confining pressure due to the actual axial strain \( \epsilon_{a2}^{(2)} \) for a loading at some other confining pressure. If the equivalent axial strain at one confining pressure could be determined for any axial strain at some other confining pressure, the growth of the yield surface at any confining pressure can be determined in terms of the growth at any other confining pressure; and the hardening rule would be specified.

In the next section, a hardening rule will be proposed and compared with the experimental results. The rule will be used to predict the equivalent axial strain at particular confining pressures from the stress-strain history of the material.

Figure 3-32 shows two different loading paths that cause the material to workharden to the same subsequent yield surface. Loading paths of type one cause permanent strains only during the axial compression part of the path, whereas loading paths of type two, in addition, cause some permanent strain as the confining pressure is increased to the desired level. Loading path three is an arbitrary path taking the material from the end point of path two to the end point of path one such that only elastic strains result. If the material were further assumed to be rigid/workhardening, no strains
would occur along path three. Since the elastic strains are small in comparison to the plastic strains, it will be assumed that the material is rigid/workhardening.

Even though the stress states at the end of path one and a path combining paths two and three are the same, the strains would not necessarily be the same. It is reasonable to expect, however, that a loading path OABA0 would produce the same amount of plastic work/unit volume as the loading path OACDBAO, since path three would produce no work. Therefore if the work is given by

\[ W = \int \sigma_{ij}^d \varepsilon_{ij}, \]

then the work done along path one is given by

\[ W = \int \Delta \sigma_{ij} \varepsilon_{ij} \varepsilon + P_c \varepsilon \]

\[ W = \int \Delta \sigma_{ij} \varepsilon_{ij} \varepsilon + P_c \varepsilon \]

where,

\[ P_c = \sigma_2 = \sigma_3 \] is the confining pressure

\[ \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \] is the volumetric strain

\[ = \varepsilon_a + 2\varepsilon_r \]

\[ \Delta \sigma_a = \sigma_1 - \sigma_3 \] is the differential axial stress

The work done on the material along path two is then,

\[ W = \int P \varepsilon \varepsilon + \int \Delta \sigma_{ij} \varepsilon_{ij} \varepsilon + P_c (\varepsilon - \varepsilon_H), \]

\[ W = \int P \varepsilon \varepsilon + \int \Delta \sigma_{ij} \varepsilon_{ij} \varepsilon + P_c (\varepsilon - \varepsilon_H), \]

where,

\[ \varepsilon_H \]

\[ \int P \varepsilon = W_H, \] is the work done during hydrostatic compression

\[ \varepsilon_H \]

\[ \int P \varepsilon = W_H, \] is the work done during hydrostatic compression

\[ \varepsilon_H \] is the volumetric strain during hydrostatic compression
The initial yield surface, in addition to being a surface of zero work, is a surface of zero volumetric strain. If the subsequent surface is also a surface of uniform volumetric strain, then, since

\[ W = W \text{ and } \varepsilon = \varepsilon \]

\[ (1) \quad (2) \]

\[ (1) \quad (2) \]

\[ \varepsilon_a \]

\[ \int \Delta \sigma_a d\varepsilon_a + P_c (\varepsilon - \varepsilon_H) + \int \Delta \sigma_a d\varepsilon_a \]

\[ (1) \quad (1) \quad (1)(2) \quad (2) \quad (2)(2) \]

\[ (2) \quad (2) \]

Therefore, if the volumetric strain during the hydrostatic compression and the hydrostatic work are known, the equivalent axial strain at one confining pressure can be found in terms of the axial strain at another confining pressure since the \( \Delta \sigma_a \) versus \( \varepsilon_a \) curve at each pressure is known.

If the equivalent axial strain at one confining pressure is desired after several tests at other confining pressures, equation 3-1 must be modified to account for the sum of the work done and the total volumetric strain during the tests at the different confining pressures. If we desire the equivalent axial strain at the \( i^{th} \) confining pressure resulting from loading at the \( j^{th} \) confining pressure, equation 3-1 one becomes

\[ \varepsilon_a \]

\[ \int \Delta \sigma_a d\varepsilon_a + P_c (\varepsilon - \varepsilon_H) + W_H = \]

\[ (i)(i) \quad (i)(i)(i) \quad (i) \]

\[ (3-2) \]

\[ \varepsilon_a \]

\[ \int \Delta \sigma_a d\varepsilon_a + P_c (\varepsilon - \varepsilon_H) + W_H = \]

\[ (j)(j) \quad (j)(j)(j) \quad (j) \]
where,

\[ \Delta \sigma \] is the differential axial stress at the \( i^{th} \) confining (\( i \)) pressure.

\[ \varepsilon_a \] is the axial strain at the \( i^{th} \) confining pressure (\( i \)) associated with \( \Delta \sigma \).

\[ P^c \] is the \( i^{th} \) confining pressure (\( i \)).

\[ \varepsilon \] is the volumetric strain associated with the loading at (\( i \)) the \( i^{th} \) confining pressure.

\[ \varepsilon_H \] is the volumetric strain occurring during the increase (\( i \)) of the confining pressure to the \( i^{th} \) pressure.

\[ W_H \] is the work done on the sample during the pressure (\( i \)) increase.

Since the volumetric strain of the equivalent path \( \varepsilon \) must equal the sum of the volumetric strains for the loadings at the \( j^{th} \) confining pressures

\[ \varepsilon = \sum_{i=1}^{n} \varepsilon_i. \]

Equation 3-2, expressing the hardening rule, was used to predict the equivalent axial strain at six different confining pressures using results of a series of axial compression tests performed on six different samples of Cordova limestone. As mentioned earlier, the tests were performed on samples with axes perpendicular to the bedding planes because these samples were the most homogeneous and the weakest of the samples tested. The tests were conducted at various confining pressures ranging from 3000 to 8000 psi, with the confining pressure of the initial test on each sample covering the above range at 1000 psi increments. Each sample was tested at an additional three or four different confining pressures with the axial...
strain averaging about 1.5% for each test. Thus the total axial strain resulting from the axial compression tests for a sample at the end of the series of tests was about 6 to 7.5%. The confining pressures of the additional tests were varied at random except that one of the tests was performed at the initial confining pressure.

Calculation of the equivalent axial strain at one pressure using equation 3-2 requires knowledge of

1. The differential axial stress versus axial strain curves for axial compression tests at other confining pressures,

2. The volumetric strain \( \varepsilon_{(j)} \) that occurred during each axial compression test,

3. The volumetric strain \( \varepsilon_{(j)} \) and work on the sample \( W_{(j)} \) that occurred during any change in the confining pressure,

4. The characteristics of the differential axial stress versus axial strain curve at the particular confining pressure of equivalent axial strain.

It is obvious from equation 3-2 that items (2) and (3) above can be substituted directly into equation 3-2. Item (1), the stress-strain curves at the \( j^{th} \) confining pressures in equation 3-2 is needed for the calculation of the work done by the differential axial load on the sample \( \int_{o}^{\varepsilon_{(j)}} \Delta \sigma \, d\varepsilon_{(j)} \). Item (4) above is needed for the calculation of the equivalent axial strain at the \( i^{th} \) confining pressure from the value of \( \int_{o}^{\varepsilon_{(i)}} \Delta \sigma \, d\varepsilon_{(i)} \) given by equation 3-2. The integrals of the
stress-strain curves at the \( j^{th} \) confining pressures were determined using a planimeter; and the equivalent axial strain at the \( i^{th} \) confining pressure \( \varepsilon_a^{(i)} \) was determined by trial and error from the unique stress-strain curve of the \( i^{th} \) confining pressure with a planimeter.

The value of the volumetric strain occurring for the axial compression test at the \( j^{th} \) confining pressure \( \varepsilon^{(j)} \) (Item (2) above) could be accurately determined by removing the sample after each test and measuring the dimension changes, however, removing the samples between tests resulted in lower yield points that if the samples were immediately reloaded at the same confining pressure (See Figures 3-18 to 23). Thus it was felt that more error would be introduced into the experiments by removing the samples for accurate dimension measurements than if the volumetric strain was approximated for each test. The volumetric strain for each test was approximated from the total volumetric strain in the same proportion as the axial strain for each test was proportional to the total axial strain.

\[
\frac{\varepsilon^{(j)}}{\sum \varepsilon^{(j)}} = \frac{\varepsilon_a^{(j)}}{\text{Total } \varepsilon_a \text{ for each sample}}
\]

Values of the volumetric strains during hydrostatic pressure changes \( \varepsilon_H^{(j)} \) and the work done during the pressure change \( W_H^{(j)} \) were also determined empirically. Experience showed that during hydrostatic tests the axial strain was approximately one-half the volumetric strain instead of one-third as would be expected. Therefore, during hydrostatic pressure changes in the series of tests, the axial strain
was recorded and the volumetric strain during the pressure change
\( \varepsilon_H \) was determined as twice the recorded axial strain for that test.
\( (j) \)
The work done during the hydrostatic compression \( W_H \) was determined
\( (j) \) assuming the material was perfectly rigid until yield and then
workhardened linearly until the desired confining pressure was reached.
Thus
\[
W_H = \frac{1}{2} \left( \frac{P_C - \sigma_{Ho}}{(j)} \right) \varepsilon_H (j)
\]
\( \sigma_{Ho} \) is the hydrostatic yield stress of the virgin material.

The relation of the hydrostatic volume strain being twice
the axial strain during a hydrostatic test was shown to be due to the
material anisotropy. Results of hydrostatic tests to the same confining
pressures on samples of Cordova limestone with axes parallel to the
bedding planes showed that the volumetric strain was equal to twice
the radial strain—where the radial strain measurements were taken
perpendicular to the bedding planes. Also, in these tests, the axial
strain equaled the radial strain—where the radial strain measurements
were taken parallel to the bedding planes.

Figures 3-33 to 38 show the results of the application of
equation 3-2 to the tests on the six different Cordova limestone
samples. The first curve in each figure shows the initial stress-
strain behavior at the different confining pressures. Each sample
was strained a sufficient amount to establish the workhardening rate
characteristic of that pressure and then unloaded. The line defined
by alternating short and long dashes indicates the asymptotic behavior
of the stress-strain curve unique at that particular pressure. The solid stress-strain curves labeled [1], [2], [3], etc. are the stress-strain curves for other samples previously tested at other various confining pressures. The initial point of these solid curves is the predicted equivalent axial strain from equation 3-2. See Appendix A for tabulated values of the work done by the differential axial stress 
\[ \varepsilon_a \int \Delta \sigma d\varepsilon_a \] at the \( j^{th} \) confining pressure, the volumetric strain \( \varepsilon \) at the \( j^{th} \) confining pressure, the volumetric strain occurring during the hydrostatic compression \( \varepsilon_H \), and the work done during the pressure increase \( W_H \) for the tests used to calculate the equivalent axial strain at the \( i^{th} \) confining pressure.

It can be seen from the figures that the prediction of equivalent axial strain for each test is quite good as judged by the actual yield point and workhardening behavior for each test in comparison to the predicted yield point and workhardening behavior given by the asymptote. However, one inconsistency was noted. The value of the volumetric strain used in the computation of the equivalent axial strain in equation 3-2,

\[ \varepsilon = \sum_{i} \varepsilon_{(i)} \]

was not always consistent with the predicted equivalent axial strain. The differences between the predicted volumetric strain and the volumetric strain associated with the predicted equivalent axial strain are the greatest when the equivalent axial strain for tests at 3000, 4000 or 6000 psi is predicted from tests where the initial axial
compression test was at 6000, 7000, or 8000 psi, and when the equivalent axial strain for tests at 6000, 7000, or 8000 psi is predicted from previous tests at 3000, 4000, or 5000 psi.

Since the tests where the first axial compression loading was at 6000 psi or above involved plastic flow during increases in the hydrostatic stress, it was thought that the behavior during the hydrostatic test introduced the inconsistency. When different samples of the Cordova limestone with axes perpendicular to the bedding planes were loaded to the same yield surface with either a pure hydrostatic loading or an axial compression loading at a confining pressure below the hydrostatic yield stress, it was found that the volumetric strain and the work done on the material in the pure hydrostatic loading was approximately twice as large as the volumetric strain and work done on the material during the axial compression test. Apparently this behavior is due to the anisotropy. The same types of tests were performed on samples with axes parallel to the bedding plane with approximately reversed results.

Values of $\varepsilon_H^1$ and $W_H^1$ defined to be one-half $\varepsilon_H$ and $W_H$ and termed the effective volume strain for hydrostatic stress increases and the effective hydrostatic work, respectively, were then used in equation 3-2 to compute the equivalent axial strains for the same tests as above. The predictions for the equivalent axial strains using equation 3-2 and the effective volume strain $\varepsilon_H^1$ and effective hydrostatic work $W_H^1$ are shown as dashed lines in Figures 3-33 to 38. No dashed lines are shown when there was no change in the predicted
equivalent axial strains. The results show that, in general, use of the effective volume strain and effective hydrostatic work improved the predictions of the equivalent axial strain. Also, the inconsistencies between the computed volume strain and the volume strain associated with the predicted effective strain were reduced.

Since the predictions of the equivalent axial strains using equation 3-2 were quite good, the workhardening rule expressed by equation 3-2 appears to be the proper specification of the growth of the yield surface. Thus the plastic work done during any loading and volume strain that occurred during the loading are the proper measures of the change of the yield surface.

Equation 3-2 was used to compute the equivalent axial strain at 4000, 5000, 6000, 7000, and 8000 psi as a function of the axial strain of a compression test at 3000 psi. Figure 3-39 shows subsequent yield surfaces resulting from axial strain increments of 2.5% during the 3000 psi confining pressure test. The application of equation 3-32 assumed the material was rigid/workhardening and the values of the effective hydrostatic work, effective hydrostatic volumetric strain, the initial yield points, and the workhardening rates were the same as used to verify the hardening rule.

Even though the behavior in the radial compression region of the end cap was assumed to be the same as in the axial compression region, some tests with results shown in Figure 3-17 were performed to verify that at least radial compression tests would result in
workhardening for normal compression tests. The plastic work for radial compression tests can be written,

\[ W = \int_0^{\varepsilon_h} P \varepsilon + 2 \int_0^{\varepsilon_r} \Delta \sigma_r d \varepsilon_r + P_c \varepsilon \]

where in this case,

\[ P_c = \sigma_1 = \sigma_a \]
\[ \Delta \sigma_r = \sigma_3 - P_c \]

The second integral must be approximated because \( \varepsilon_r \) cannot be determined continuously. The predicted equivalent axial strain for the tests shown in Figure 3-17, where samples were radially compressed to 8000 and 10,000 psi before being axially compressed at 6000 psi, is approximately twice the actual equivalent axial strain. (This discrepancy could be due to the same anisotropic effects present in the hydrostatic test compared to the normal compression tests, and if the same type of "correction" is used the predicted and measured equivalent axial strains are quite similar.)

Therefore, approximately at least, the workhardening rule can be applied for all regions of the end cap in order to predict the growth of the yield surface. In the next section both the initial yield condition and hardening rule will be generalized to apply for arbitrary states of stress and stress-strain histories.

**Generalized Yield Condition and Hardening Rule**

The results of the triaxial tests have shown that some type of end cap to the yield surface exists for these porous limestones. The initial yield surface and hardening rules determined for the
special cases where the stresses $\sigma_2 = \sigma_3$, and $\sigma_1$ varied independently must be generalized to apply for arbitrary loadings in order for these results to be useful for other purposes besides triaxial tests. The generalization can be made using certain symmetry arguments reflecting the known results; however, more test points using either torsion or true three-dimensional loading apparatus would be most useful for more accurate generalizations.

The shape of the initial yield surface and the material behavior when loaded along certain paths in triaxial tests can give insight into the material's generalized yield condition. The initial yield surfaces of the Batesville marble and the two different orientations of the Cordova limestone in Figures 3-25, 26, and 27 and the material behavior along similar loading paths indicate that

1. Yielding of the material depends upon the difference of the maximum and minimum principal stresses,
2. Yielding is independent of the intermediate principal stress,
3. The magnitude of the difference of the maximum and minimum principal stresses depends upon the minimum principal stress.

Similar loading paths that indicate the above dependence are shown in Figure 3-40, and defined as

1. Axial compression with constant confining pressure and radial compression with constant axial stress,
2. Axial extension with constant confining pressure and a test where the confining pressure is reduced with constant axial stress.
Figures 3-25, 26 and 27 show that the material behavior is the same along the similar loading paths. At low confining pressures, axial and radial compression tests, as well as extension and its corresponding similar test, will produce unstable (or Coulomb) failure at yield. At higher confining pressures, both tests of type (2) will still cause unstable (or Coulomb) failure, while the tests of type (1) show workhardening results.

As stated in the previous section, the general behavior of the material is assumed to be such that the end cap grows in size as work is done on the material, or as the material's volume decreases; but the extent of the end cap is limited by a Coulomb limit surface. A yield function describing the Coulomb limit surface that is consistent with the material behavior along the similar loadings paths and reduces to a Coulomb yield condition in plane strain is,

\[ f_1 = \sigma_{\text{max}} - \sigma_{\text{min}} - A\sigma_{\text{min}} - B \]

\[ A, B \text{ are positive constants} \]

The end cap yield function must be consistent with the material behavior along the similar loadings paths and it must "close off" the Coulomb limit surface such that yielding can occur during hydrostatic stress increases. It must also be sufficiently general in order to allow the end cap to grow in size as the material is yielded. A function that is consistent with the above requirements is

\[ f_2 = \sigma_{\text{max}} - \sigma_{\text{min}} + a\sigma_{\text{min}} - \beta \]  \hspace{1cm} (3-3)

\[ a, \beta \text{ are positive functions of the plastic work or of the volumetric strain.} \]
Both \( f_1 \) and \( f_2 \) above describe surfaces that form straight lines when they intersect the plane formed by the \( \sigma_1 \) axis and the hydrostatic axis. Figure 3-40 shows the image of \( f_1 \) and \( f_2 \) in the triaxial stress space. The surface formed by \( f_1=0 \) is similar to the Mohr-Coulomb yield surface described in the first section. Since the yield function for the end cap \( f_2 \) is the most important aspect of this research the remainder of the discussion will concern \( f_2 \).

The surface formed by \( f_2=0 \) in equation 3-3 is a right hexagonal pyramid with a cross section parallel to the \( \pi \) plane as shown in Figure 3-41. The other yield function shown in the Figure 3-41 is a generalization of the von Mises yield condition; and the surface formed by \( f=0 \) in this case is a right regular cone. If anisotropy caused the symmetries leading to equation 3-3, this von Mises type yield function would be useful.

The most elementary form for \( \alpha \) and \( \beta \), assuming \( \alpha \) and \( \beta \) are functions of the volumetric strain, is

\[
\alpha = \alpha_0 + \alpha_1 \varepsilon \\
\beta = \beta_0 + \beta_1 \varepsilon
\]

(3-4)

where \( \alpha_0 \) and \( \beta_0 \) are material constants given by the initial yield surface.

\[
\alpha_0 = - \frac{\partial (\Delta \sigma)}{\partial \sigma_{\text{min}}} \\
\beta_0 = \alpha_0 \sigma_{\text{Ho}}
\]

\( \sigma_{\text{Ho}} \) is the initial hydrostatic yield stress.

\( \alpha_1 \) and \( \beta_1 \) may be specified to cause the surface to grow such that the growth reflects the workhardening behavior of the material.
Figure 3-42 shows how different values of $\alpha_0$, $\alpha_1$, $\beta_0$ and $\beta_1$ affect the hardening rule. Rules of the type shown in Figure 3-42b or c seem to give the best description of the Cordova limestone results. For the yield surface and hardening rule for Figure 3-37 values of $\alpha_0 = .93$ and $\beta_0 = 4650$ and $\beta_1 = 1.35 \times 10^5$ give the results shown in Figure 3-43.

The hardening rule in Figure 3-42 is a local hardening rule where one part of the yield surface is specified to move in a prescribed manner. In this case the end cap is the hardening part and the Coulomb part remains stationary. Specifying the hardening rule completes the description of the material behavior.

It is evident that the workhardening features of the material behavior reported here are significantly different from previous workhardening [14] or perfectly plastic material behavior normally assumed for rocks, [22], [26], [29]. Thus it would be reasonable to expect that an analysis of a problem using the yield condition and hardening rule porposed here should differ somewhat from an analysis of the same problem using a Coulomb yield condition. Some of the implications of the yield condition and hardening rule will be discussed in the next chapter.
IMPLICATIONS AND APPLICATION OF
THE YIELD CONDITION AND HARDENING RULE

The present study of the workhardening of porous limestones was primarily motivated by many inconsistencies between experimental evidence and theoretical predictions that exist in the analysis of the indentation of these rocks by wedges and cones. These experiments have been treated theoretically as wedges indenting a half-plane (or half-space for cones) of rigid/perfectly plastic material obeying a Coulomb yield condition [22], [26], [29]. The experimentally measured quantities are the indentation force and the penetration depth of the wedge below the undisturbed rock surface. Accurate theoretical calculation of these quantities requires knowledge of the geometry of the flow of the material adjacent to the wedge.

Since a Coulomb yield condition requires material expansion during any deformation process, the predicted heights of the lips around the craters and the predicted volume of deformed material are large. In some cases, for example for a wedge with an included angle of 60° indenting a typical Coulomb material (angle of internal friction of 30°), material as far away as 10 crater widths should be affected by the wedge; and the predicted lip heights are on the order of the depth of penetration.

In some cases for certain rocks, particularly the stronger marbles, these predictions are quite reasonable when tested at low confining pressures, and the calculated force-displacement relations agree well with experimentally measured values (Gnirk and Musselman[26], and Gnirk and Cheatham[22]). In some of these cases, the predicted
forces of indentation can be made even more accurate by assuming a parabolic or Mohr yield envelope (Cheatham[1]). However, the predictions of the force-displacement relations, lip heights, and the volume of deformed material are much larger than measured for weaker, porous limestones even when tested at low confining pressures.

The author has observed cases where little or no lip is formed during the indentation of Cordova limestone, indicating material compaction during the indentation process. In addition it has also been observed that lip heights decrease in the stronger rocks as they are tested at higher confining pressures. Consistent with this observation Gnirk and Musselman[26] report that the volume of deformed material adjacent to the crater decreases as the confining pressure of the test is increased. This indicates that even the stronger rocks must have an end cap to their yield surfaces and that a yield condition similar to the one for the Cordova limestone might apply. For convenience a material that obeys the yield condition and hardening rule proposed above will be termed a workhardening-compacting material.

Therefore, it seems that predictions of the force-displacement relations as well as the deformation geometry for the wedge indentation of these material would improve if a more realistic yield condition and hardening rule were used. Finite-element calculation techniques appear to be the only approach available at this time for a complete solution to the wedge indentation problem for a workhardening-compacting material. While the finite-element method should provide accurate results, its application to this problem is beyond the scope of this paper. However, an approximate method shown below should give
insight to the problem of wedge indentation into a workhardening-compacting material.

Method of Estimating the Force-Displacement Relation for Wedge Indentation of Workhardening-Compacting Materials

The approximate method is an attempt to estimate the effect of taking material compaction into account during a wedge indentation test. Since an exact analysis using a workhardening-compacting material would be beyond the scope of this investigation, the wedge indentation problem is solved assuming the material were rigid/perfectly plastic obeying a compacting yield condition. In this case a rigid/perfectly plastic material obeying a compacting yield condition would have a yield condition given by equation 3-3 with \( \alpha \) and \( \beta \) constant. Two different values of \( \alpha \) and \( \beta \) will be used to find two solutions—one using the initial yield condition as a low estimate to the force-displacement relation and the other will use a yield condition that matches a subsequent yield surface determined by the amount of volumetric strain during the indentation process. The second yield condition is used to find a high estimate to the force-displacement relation.

The wedge indentation problem is a plane-strain problem, therefore the generalized yield condition must be written in terms of the proper plane-strain variables. Recall the general yield condition, equation 3-3:

\[
f = \sigma_{\text{max}} - \sigma_{\text{min}} + \alpha \sigma_{\text{min}} - \beta
\]
Now if the z direction is the direction of zero strain, \( \varepsilon_z = 0 \). The normal stress for the z direction cannot be either the minimum or the maximum principal stress; since assuming \( \sigma_z \) were either the maximum or minimum principal stress; applying normality to determine \( \varepsilon_z \) and setting the expression equal to zero gives inconsistent results. Therefore the x-y plane contains the maximum and minimum principal stresses and hence from the Mohr's circle representation of the state of stress on an element in the x-y plane we have (Figure 4-1),

\[
\sigma_{\text{max}} = \frac{\sigma_x + \sigma_y}{2} + \left( \frac{\sigma_x - \sigma_y}{2} + \tau_{xy} \right)^{1/2}
\]

\[
\sigma_{\text{min}} = \frac{\sigma_x + \sigma_y}{2} - \left( \frac{\sigma_x - \sigma_y}{2} + \tau_{xy} \right)^{1/2}
\]  

(4-1)

Substitution of equation 4-1 into equation 3-3 gives the plane-strain yield condition as

\[
f = \left( \frac{\sigma_x - \sigma_y}{2} + \tau_{xy} \right) + \frac{\sigma_x + \sigma_y}{2 - \alpha} - \frac{\beta}{2 - \alpha}
\]  

(4-2)

For equation 4-2 just as for equation 3-3, the hydrostatic yield stress \( \sigma_H \) is determined for the case

\[
\sigma_x = \sigma_y = \sigma_H, \text{ and } \tau_{xy} = 0
\]

Hence

\[
\sigma_H = \frac{\beta}{\alpha}
\]

Analogous to the angle of internal friction in the Coulomb yield condition, we define the angle the compaction envelope makes with the normal stress axis as the compaction angle \( \psi \), where

\[
\sin \psi = \frac{\alpha}{2 - \alpha}
\]
Equation 4-2 becomes

\[ f = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \frac{\tau_{xy}^2}{2} + \sin \psi \left( \frac{\sigma_x + \sigma_y}{2} - \sigma_H \right) \]  

(4-3)

Figure 4-1 shows the yield envelope of equation 4-3 in plane-strain stress space.

Now if the work per unit volume during a loading process is given by

\[ dW = \sigma_{ij} \, d\varepsilon_{ij} \]

where,  \[ d\varepsilon_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \lambda > 0 \]

and,  \[ d\varepsilon = d\varepsilon_{ii} \]

then \( dW \) can be found in terms of \( \sigma_H, \sin \psi, \) and \( d\varepsilon. \)

From equation (4-3) we have

\[ \frac{\partial f}{\partial \sigma_x} = \frac{\left( \frac{\sigma_x - \sigma_y}{2} \right)}{2\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \frac{\tau_{xy}^2}{2}} + \sin \frac{\psi}{2} \]

\[ \frac{\partial f}{\partial \sigma_y} = \frac{\left( \frac{\sigma_y - \sigma_x}{2} \right)}{2\left( \frac{\sigma_y - \sigma_x}{2} \right)^2 + \frac{\tau_{xy}^2}{2}} + \sin \frac{\psi}{2} \]

\[ \frac{\partial f}{\partial \tau_{xy}} = \frac{\tau_{xy}}{2\left( \frac{\sigma_x - \sigma_y}{2} + \frac{\tau_{xy}}{2} \right)} \]

and therefore

\[ \sigma_{ij} \frac{\partial f}{\partial \sigma_{ij}} = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 + \sin \psi \left( \frac{\sigma_x + \sigma_y}{2} \right) \]  

(4-4)
The yield function \( f \) equals zero for any plastic flow to occur, thus from equations (4-4) and (4-3) we have

\[
\sigma_{ij} \frac{\partial f}{\partial \sigma_{ij}} = \sin \psi \sigma_H
\]

\( \sigma_H \) is the hydrostatic yield stress

Also since

\[
d\epsilon = d\epsilon_{ii} = \lambda \frac{\partial f}{\partial \sigma_{ii}} = \lambda \sin \psi
\]

The work per unit volume becomes

\[
dW = \lambda \sigma_H \, d\epsilon.
\]

Thus for any loading process on a compacting material with a constant hydrostatic yield stress \( \sigma_H \), the work is proportional to the volumetric strain. Note also that volumetric strain is independent of \( \sigma_H \) and depends only on the compaction angle \( \psi \).

Consider a case where we have a workhardening-compacting material that workhardens in such a way that \( \psi \) (or \( \alpha \)) remains constant. Let \( \sigma_H \) be the instantaneous hydrostatic yield stress of the material. Let the hydrostatic yield stress of the virgin material be given by \( \sigma_{H0} \), and let the hydrostatic yield stress of the material that is strained the most during a loading process be \( \sigma_{H1} \). If the material undergoes a uniform volume strain during this loading process, it follows that if

\[
dW_0 = \sigma_{H0} d\epsilon
\]

\[
dW_1 = \sigma_{H1} d\epsilon
\]
then
\[ dW = dW_0 + dW \]

Thus the work done on a workhardening-compacting material that workhardens with constant compaction angle \( \psi \) is bound above and below by the work done on a perfectly plastic-compacting material with the hydrostatic yield stress of \( \sigma_{Ho} \) and \( \sigma_{H1} \), respectively.

Appendix B shows that if the material were rigid/plastic obeying a compacting yield condition, the method of characteristics may be used to find the force-displacement relation and the deformation geometry associated with wedge indentation problems. Therefore, if \( \sigma_{Ho} \) and \( \sigma_{H1} \) were known, the workhardening force-displacement relation could be bound. \( \sigma_{Ho} \) is known from the initial yield condition. An approximate value for \( \sigma_{H1} \) can be found using the hardening rule requiring the yield surface to grow as a function of the volumetric strain and the volume strain can be approximated from the predicted deformation geometry of the problem.

In Appendix C a method is presented for determining the force-displacement relations and the flow geometry for the wedge indentation problem. Figure 4-2 compares previous upper and lower "bounds" using a Coulomb yield condition [28] and experimental results for various wedges indenting Cordova limestone at a confining pressure of 5000 psi with the upper and lower estimates calculated by the above method. The figure indicates the effect of compaction on the force-displacement relation.
The upper and lower "bounds" found using the Coulomb yield condition are calculated by assuming that the wedge is either perfectly rough (material shearing at wedge surface) or perfectly smooth. Both of the estimates found using the compacting yield condition are calculated assuming a perfectly rough wedge-material interface. The upper estimate using the compacting condition is only as accurate as the assumption of uniform volume strain during the wedge indentation process. However, the most significant aspect of Figure 4-2 is the trend of the compacting yield condition to accurately predict the force-displacement relations' dependence upon the wedge angle.

Thus it appears that the use of a compacting yield condition can improve the accuracy for the prediction of the force-displacement relation for the wedge indentation problem. In addition, the compacting condition predicts much smaller lip angles as well as predicting that less material will be affected adjacent to the wedge, which would explain the observations by Gnirk and Musselman[26]. The closer agreement of the theoretical to the experimental results indicate that the workhardening-compaction yield condition can explain the material behavior better than a Coulomb yield condition. Further study with the finite-element method using the actual stress-strain behavior of a workhardening-compacting material would be most useful for a better analysis of the wedge indentation problem.
SUMMARY AND CONCLUSIONS

Inconsistencies between the actual stress-strain behavior of some rocks, particularly porous, relatively weak limestones, and the predicted behavior of these rocks assuming they were rigid/plastic materials obeying a Coulomb yield condition prompted the present study to determine a yield condition that predicted the material behavior more accurately. The Coulomb yield condition is generally used to describe the behavior of rocks because it allows for increasing material strength with increasing hydrostatic stresses. While the ultimate strength of porous limestones does increase with increasing hydrostatic stresses, permanent strains occur at stress levels below the yield stresses allowed by a Coulomb yield condition. In addition, the initial strains determined from experiments are associated with volume decrease instead of a volume increase predicted by a Coulomb yield condition.

Triaxial tests were performed on two types of porous limestones in order to find detailed information about the stress-strain behavior of these materials. Results of these tests show that a yield condition that combines a Coulomb yield condition with an end cap provides an adequate description of the materials' stress-strain behavior. The end cap is allowed to grow in a definite manner and is limited in size by the Coulomb condition. The yield condition can be expressed as follows

\[ f_1 = \sigma_{\text{max}} - \sigma_{\text{min}} - A\sigma_{\text{min}} - B \]

\[ f_2 = \sigma_{\text{max}} - \sigma_{\text{min}} + A\sigma_{\text{min}} - \beta \]
where 

\[ A\alpha_{\text{min}} + B < \beta - A\alpha_{\text{min}} \Rightarrow f_1 \] is the governing yield function

\[ A\alpha_{\text{min}} + B > \beta - A\alpha_{\text{min}} \Rightarrow f_2 \] is the governing yield function,

and where \( A \) and \( B \) are material constants and \( \alpha \) and \( \beta \) are workhardening parameters associated with the material. It was found that the growth of the yield surface depended upon the amount of volumetric strain or the amount of plastic work per unit volume done on the material.

The implications of the new yield condition are shown to be consistent with experimental observations in triaxial tests and wedge indentation tests. An approximation of the yield condition was used to show how a compacting yield condition affects theoretical predictions of the force-displacement relation for wedge indentation tests. The consistency between the predicted and observed stress-strain behavior using a yield condition that combines a Coulomb surface and an end cap indicates that this type yield condition is the proper type of yield condition needed for a more accurate description of the behavior of some rocks.

In addition, this yield condition could be generalized to apply to any material that shows a volume change during permanent deformation processes. It is quite similar to yield conditions proposed by workers in soil mechanics (Roscoe, et al., [2], [15], [16], Drucker, Gibson and Henkel [13], and Barden and Khayatt [6]); and results of tests noted by Gnirk and Musselman [26], and Miller and Cheatham [29] show that even stronger, denser rocks show volume decreases if the hydrostatic stress state is high enough.
It is recognized that the proposed yield condition is limited by the many assumptions and approximations used during the analysis of the experimental results. The major assumptions of material isotropy and homogeneity are justified because the general behavior of the material was the same regardless of sample location or orientation. However, further study into yield conditions and hardening rules for anisotropic material would be desirable in order to help determine a more general yield condition.

A means of continuously measuring the radial strains of the material during triaxial tests is necessary for any meaningful results from tests on anisotropic materials. Continuous readings of radial strains would also be helpful in defining the stress-strain behavior during radial compression tests on any material.

This study has been an attempt to formulate a yield condition and hardening rule that consistently predicts the stress-strain behavior of materials that show a volume decrease while undergoing a permanent deformation. The proposed yield condition and hardening rule have accomplished this purpose, and their use should advance knowledge about the plastic behavior of these materials.
REFERENCES


28. Private communication of unpublished work by P. F. Gnirk, South Dakota School of Mines, Rapid City, South Dakota.


APPENDIX A

EXPERIMENTAL VERIFICATION OF THE HARDENING RULE

The primary purpose of this appendix is to present the numerical results that are needed to verify the hardening rule for the end cap of the yield surface for Cordova limestone samples with axes perpendicular to the bedding planes. The reader is referred to the Hardening Rule section of the text (page 48) for details. Experimentally, the end cap can only be reached by stress paths consisting of either axial or radial compression tests. Axial compression tests are tests where the axial load on the samples is increased while the confining pressure is kept constant, and radial compression tests are tests where the confining pressure is increased while the axial stress is kept constant. Limitations of equipment prevented detailed stress-strain data during radial compression tests, therefore, only axial compression tests were used to verify the hardening rule. Assuming that the hardening in the radial compression tests was the same as in the axial compression tests, the hardening rule is verified if the yield point of a sample used in an axial compression test at an arbitrary confining pressure can be predicted from the sample's stress-strain history.

Since all the samples tested had the same stress-strain curves at each confining pressure, the yield point in any test can be associated with a particular axial strain through the stress-strain curve. Thus the hardening rule is also verified if it can predict
that particular strain (defined as equivalent axial strain in the thesis body) from a sample's stress-strain history. The hardening rule was used to predict the equivalent axial strain.

The proposed hardening rule stated that the subsequent yield surfaces were also surfaces of uniform plastic work as well as surfaces of uniform volumetric strain, thus the work done on the samples and the volumetric strain occurring along any two loading paths terminating on the same yield surface is the same. These assumptions lead to equation 3-2 repeated below.

\[
\varepsilon_a^{(1)} + \int \Delta \sigma_a d \varepsilon_a^* + P_c (\varepsilon - \varepsilon_H^2) + W_H = \varepsilon_a^{(2)} + \int \Delta \sigma_a d \varepsilon_a^* + P_c (\varepsilon - \varepsilon_H^2) + W_H
\]

where,

- \( P_c \) is the confining pressure for the \( i^{th} \) test. \( (i) \)
- \( \Delta \sigma_a \) is the differential axial stress \( (\sigma_a - P_c) \) at the \( i^{th} \) confining pressure. \( (i) \)
- \( \varepsilon_a \) is the axial strain at the \( i^{th} \) confining pressure. \( (i) \)
- \( \varepsilon_H^i \) is the volumetric strain occurring during pressure \( (i) \) increases to the \( i^{th} \) confining pressure.
- \( \varepsilon \) is the volumetric strain occurring during the loading. \( (i) \)
- \( W_H \) is the work done during the pressure increase to the \( i^{th} \) confining pressure.

The equivalent axial strain at one confining pressure from the loading history at another confining pressure \( \varepsilon_a \) is determined from \( 1b \) when \( (i) \).
all the terms on the right hand side are known and if $\sigma_H$ and $W_H$ are material constants, $\varepsilon_a$ can easily be determined from the value of

$$\varepsilon_a^{(i)}$$

the work done by the differential axial stress $\int_0^1 \Delta \sigma_a d \varepsilon_a$ since the $\Delta \varepsilon_a$ versus $\varepsilon_a$ curve is known.

If the equivalent axial strain at one confining pressure is desired after several tests at other confining pressures, equation 1a

$$\varepsilon_a^{(i)}$$

becomes

$$\int_0^1 \Delta \sigma_a d \varepsilon_a = -P_o (\varepsilon - \varepsilon_H) - W_H$$

where all the terms are defined the same as in equation 1a. The reader is referred to the text immediately following equation 3-2 for the details of the evaluation of the above equation.

Following are tabulated values of terms on the right hand side of equation 2a used to determine the work done by the differential axial stress $\int_0^1 \Delta \sigma_a d \varepsilon_a$ at the $i^{th}$ pressure. The equivalent axial strain is then determined from $\int_0^1 \Delta \sigma_a d \varepsilon_a$. The solid curves in Figures 3-33 to 38 are the test curves that were recorded at each confining pressure. The curves were then started at the value of the equivalent axial strain calculated from 2a.
Tabulated Values of the Equivalent Work at one Confining-Pressure from the Work at Other Confining Pressures

<table>
<thead>
<tr>
<th>I 3000 psi</th>
<th>Sample No.</th>
<th>21-7,4</th>
<th>21-2,3</th>
<th>21-7,9</th>
<th>21-7,7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ε&lt;sub&gt;j&lt;/sub&gt;</td>
<td>96.0</td>
<td>84.0</td>
<td>228.8</td>
<td>26.9</td>
</tr>
<tr>
<td></td>
<td>Σ (Δσ&lt;sub&gt;j&lt;/sub&gt; δ&lt;sub&gt;j&lt;/sub&gt;)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Σ (ΔP&lt;sub&gt;j&lt;/sub&gt; ε&lt;sub&gt;j&lt;/sub&gt; - P&lt;sub&gt;j&lt;/sub&gt; ε&lt;sub&gt;jH&lt;/sub&gt;)</td>
<td>90.0</td>
<td>36.0</td>
<td>-33.0</td>
<td>9.0</td>
</tr>
<tr>
<td></td>
<td>P&lt;sub&gt;j&lt;/sub&gt; ε&lt;sub&gt;jH&lt;/sub&gt; - W&lt;sub&gt;H&lt;/sub&gt;</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(i)(i)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>W&lt;sub&gt;H&lt;/sub&gt;</td>
<td>-</td>
<td>-</td>
<td>430.0</td>
<td>55.0</td>
</tr>
<tr>
<td></td>
<td>(j)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Equivalent</td>
<td>186.0</td>
<td>120.0</td>
<td>625.3</td>
<td>90.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>II 4000 psi</th>
<th>Sample No.</th>
<th>21-2,3</th>
<th>21-7,4</th>
<th>21-7,11</th>
<th>21-7,7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equivalent Work</td>
<td>181.6</td>
<td>228.9</td>
<td>38.8</td>
<td>124.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>III 5000 psi</th>
<th>Sample No.</th>
<th>21-7,11</th>
<th>21-7,4</th>
<th>21-2,3</th>
<th>21-7,10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equivalent Work</td>
<td>69.5</td>
<td>3.5</td>
<td>22.5</td>
<td>259.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>IV 6000 psi</th>
<th>Sample No.</th>
<th>21-7,7</th>
<th>21-7,11</th>
<th>21-7,10</th>
<th>21-7,9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equivalent Work</td>
<td>125.0</td>
<td>98.8</td>
<td>34.7</td>
<td>139.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>V 7000 psi</th>
<th>Sample No.</th>
<th>21-7,10</th>
<th>21-7,4</th>
<th>21-7,11</th>
<th>21-7,9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equivalent Work</td>
<td>187.2</td>
<td>1.0</td>
<td>179.4</td>
<td>46.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>VI 8000 psi</th>
<th>Sample No.</th>
<th>21-7,9</th>
<th>21-7,7</th>
<th>21-7,10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equivalent Work</td>
<td>111.4</td>
<td>210.7</td>
<td>28.7</td>
</tr>
</tbody>
</table>
As noted in the text following equation 3-2, the application of equation 1a lead to some inconsistencies between the predicted volumetric strain for the test at the $i^{th}$ confining pressure and the volumetric strain associated with the equivalent axial strain. It was determined that tests involving yielding during hydrostatic stress increases caused twice as much volumetric strain as were caused by axial compression tests that hardened the material the same amount. These effects were shown to be caused by the material anisotropy. Therefore, in order to account for the anisotropy, values of effective volumetric strain for hydrostatic stress increases $\varepsilon_H^{(i)}$ and effective hydrostatic work $W_{H}^{(i)}$ equal to one-half the measured $\varepsilon_H$ and $W_H$ were then used in 2a to find the equivalent axial strains as before. The results are shown as dashed lines in Figures 3-33 to 38. Following are tabulated values of the terms on the right hand side of equation 2a that were changed by using $\varepsilon_H^{(i)}$ and $W_{H}^{(i)}$.

**Tabulated Values of the Equivalent Work at One Confining Pressure from the Work at Other Confining Pressures Using $\varepsilon_H^{(i)}$ and $W_{H}^{(i)}$**

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>21-7.7</th>
<th>21-7.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma \int \Delta \sigma , d\varepsilon_p$</td>
<td>26.9</td>
<td>228.8</td>
</tr>
<tr>
<td>$\int_{j} (j)(j)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma (\Delta P \cdot \varepsilon - P \cdot \varepsilon_H)$</td>
<td>24.0</td>
<td>77.0</td>
</tr>
<tr>
<td>$\int_{j} (j)(j)(j)(j)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P \cdot \varepsilon_H^{(i)} - W_H^{(i)}$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\int_{j} (j)(i)(i)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_H^{(i)}$</td>
<td>27.5</td>
<td>215.0</td>
</tr>
<tr>
<td>Equivalent $\int \Delta \sigma , d\varepsilon_p$</td>
<td>78.4</td>
<td>520.8</td>
</tr>
<tr>
<td>$\int_{(i)(i)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sample No.</td>
<td>Equivalent Work</td>
</tr>
<tr>
<td>--------</td>
<td>--------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>II 4000 psi</td>
<td>21-7,7</td>
<td>117.4</td>
</tr>
<tr>
<td>III 5000 psi</td>
<td>21-7,10</td>
<td>235.2</td>
</tr>
<tr>
<td>IV 6000 psi</td>
<td>21-7,11 21-7,10 21-7,9</td>
<td>96.3 32.2 124.5</td>
</tr>
<tr>
<td>V 7000 psi</td>
<td>21-7,4 21-7,11 21-7,9</td>
<td>14.0 155.4 39.5</td>
</tr>
<tr>
<td>VI 8000 psi</td>
<td>21-7,7 21-7,10</td>
<td>186.2 18.3</td>
</tr>
<tr>
<td>Sample I.D. Number</td>
<td>Total ε_a (%)</td>
<td>Total ε (%)</td>
</tr>
<tr>
<td>-------------------</td>
<td>---------------</td>
<td>-------------</td>
</tr>
<tr>
<td>21-7,4</td>
<td>7.6</td>
<td>6.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21-2,3</td>
<td>9.7</td>
<td>7.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21-7,11</td>
<td>7.5</td>
<td>6.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21-7,7</td>
<td>8.0</td>
<td>6.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21-7,10</td>
<td>9.1</td>
<td>12.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21-7,9</td>
<td>8.9</td>
<td>11.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX B

VELOCITY AND STRESS CHARACTERISTIC RELATIONS
FOR RIGID/PLASTIC MATERIALS OBEYING A COMPACTING
YIELD CONDITION DEFORMED UNDER PLANE-STRAIN CONDITIONS

The graphical method developed by Prager[31] for the construction of slip lines and the hodograph for problems of plane plastic flow of metals greatly simplifies the solutions of many problems. DeJong[5] has shown that Prager's method can be adopted for materials obeying a Coulomb yield condition as well. It is the purpose of this appendix to show that the method will also apply for a rigid/plastic compacting material and to develop the expressions for compacting materials necessary for the application for Prager's method. Appendix C will apply the method to the solution of the problem of finding the force-displacement relation for the indentation of a half-plane of compacting material by a wedge.

The plane-strain yield condition for a rigid/plastic compacting material is given by equation 4-3 as

\[ f = \left( \frac{\sigma_x - \sigma_y}{2} + \tau_{xy} \right)^{1/2} + \sin \psi \left( \frac{\sigma_x + \sigma_y - \sigma_H}{2} \right) \]  

(1b)

where,

\( \sigma_H \) is the hydrostatic yield stress

\( \psi \) is the compaction angle.

The Mohr envelope for equation 1b is shown in Figure 4-1. Equation 1b together with the plane-strain equilibrium equations, neglecting body forces,
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{2b}
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]

form a hyperbolic set of equations. Thus a set of two real characteristics exists, with the slopes of the characteristics in the x-y plane given by

\[
\frac{dy}{dx} = -\cot \left[ \theta \pm \left( \frac{\pi}{4} - \psi/2 \right) \right] \tag{3b}
\]

where,

\[\theta = \text{angle between greatest principal stress and x directions.}\]

The relation of the characteristic directions to the x-y, and principal stress directions in the physical plane are defined in Figure 1b. The \( \alpha \) characteristic, or first slip line, direction is defined to make an angle of \( \pi/4 - \psi/2 \) with the least principal stress direction, and the positive \( \beta \) characteristic, or second slip line, direction is defined such that a counterclockwise rotation of the \( \alpha \) slip-line by \( \pi/2 + \psi/2 \) coincides with the \( \beta \) slip line. These definitions are arbitrary, but the definition must be consistent throughout the discussion in order to remain meaningful.

If \( \phi \) is defined as above and

\[P = \frac{\sigma_x + \sigma_y}{2} = \text{Mean normal stress in the x-y plane.}\]

and then from Figure 4-1

\[\sigma_x = P + (\sigma_H - P) \sin \psi \cos 2\theta\]
\[\sigma_y = P - (\sigma_H - P) \sin \psi \cos 2\theta\]
\[\tau_{xy} = (\sigma_H - P) \sin \psi \sin 2\theta\]
the equilibrium equations can be written in terms of \( P \) and \( \theta \). These equations can then be transformed into equations along the slip lines. The equilibrium equations in terms of \( P \) and \( \theta \) become,

\[
\frac{3P}{\partial x} \left( 1 - \sin \psi \cos 2\theta \right) + 2 \left( P - \sigma_H \right) \sin \psi \sin 2\theta \frac{\partial \theta}{\partial x} \\
- \frac{3P}{\partial y} \sin \psi \sin 2\theta + 2 \left( \sigma_H - P \right) \sin \psi \cos 2\theta \frac{\partial \theta}{\partial y} = 0 \\
- \frac{3P}{\partial x} \sin \psi \sin 2\theta + 2 \left( \sigma_H - P \right) \sin \psi \cos 2\theta \frac{\partial \theta}{\partial x} \\
+ \frac{3P}{\partial y} \left( 1 + \sin \psi \cos 2\theta \right) + 2 \left( \sigma_H - P \right) \sin \psi \sin 2\theta \frac{\partial \theta}{\partial y} = 0
\]

(4b)

If \( \theta = 3\pi/4 - \psi/2 \), the \( \alpha \) line is parallel to the \( x \) axis and \( \partial x = \partial S_\alpha \) and \( \partial y = \partial y^* \) where \( \partial S_\alpha \) is the differential along the \( \alpha \) characteristic and \( \partial y^* \) is some differential related to \( \partial S_\alpha \). Substitution of \( \theta = 3\pi/4 - \psi/2 \) into equations 4b, multiplication of the first equation by \( \cos \psi \) and second by \( -\sin \psi \), and addition of the two gives the equilibrium equation along the \( \alpha \) line. Also, if \( \theta = 3\pi/4 + \psi/2 \), \( \partial y = \partial S_\alpha \), similar manipulations will give the equilibrium equations along the \( \beta \) line.

\[
\cos \psi \frac{3P}{\partial S_\alpha} + 2 \left( \sigma_H - P \right) \sin \psi \frac{\partial \theta}{\partial S_\alpha} = 0 \quad (\alpha \ line) \\
- \cos \psi \frac{3P}{\partial S_\beta} + 2 \left( \sigma_H - P \right) \sin \psi \frac{\partial \theta}{\partial S_\beta} = 0 \quad (\beta \ line)
\]

(5b)

Equations 5b can be integrated along the slip lines to find the stresses if the slip lines are known. Slip lines are associated with kinematically admissible velocity fields, thus integration of the 5b along the slip lines will give upper bound solutions.
The relation of the velocity field to the slip lines is discussed below. The strain rates for a compacting material are given by the normality condition of Drucker's postulate or the plastic potential,

\[ \dot{\varepsilon}_x = \frac{\partial u}{\partial x} = \lambda \frac{\partial f}{\partial \sigma_x} \]

\[ \dot{\varepsilon}_y = \frac{\partial v}{\partial y} = \lambda \frac{\partial f}{\partial \sigma_y} \]

\[ \dot{\varepsilon}_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \lambda / 2 \frac{\partial f}{\partial \tau_{xy}} \]  

where \( \lambda \) is some positive constant, \( f \) is the yield function, and \( u \) and \( v \) are the particle velocities in the \( x \) and \( y \) directions respectively. Equations 6b and the relations for the derivatives of \( u \) and \( v \) along the characteristics.

\[ \frac{\partial u}{\partial S} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial S} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial S} \]

\[ \frac{\partial v}{\partial S} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial S} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial S} \]

where \( \partial S \) is the differential along a characteristic, are also hyperbolic, and the directions of the velocity characteristics coincide with the stress characteristics.

**Inextensibility of Slip Lines**

Once the characteristic directions are known, the flow rule can be applied in order to determine the velocity dependence along the slip lines. Just as in metals, the slip lines for compacting materials are inextensible. Equations 1b and 4b give
\[ \ddot{e}_x = \frac{\partial u}{\partial x} = \lambda/2 \{\sin \psi + \cos 2\theta\} \]
\[ \ddot{e}_y = \frac{\partial v}{\partial y} = \lambda/2 \{\sin \psi - \cos 2\theta\} \]  (7b)
\[ \ddot{e}_{xy} = 1/2 (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \lambda/2 \sin 2\theta \]

if the \(\alpha\) slip line is coaxial with the \(x\) axis, Figure 1b shows that 
\(\theta = 3\pi/4 - \psi/2\), and thus
\[ \frac{\partial u}{\partial x} = 0. \]

If the \(\alpha\) slip line is coaxial with the \(y\) axis, Figure 1b shows that 
\(\theta = \pi/4 - \psi/2\), and thus
\[ \frac{\partial v}{\partial y} = 0. \]

The above relations hold as well when the \(\beta\) slip line is coaxial with 
either the \(x\) or \(y\) directions. Since \(\partial u/\partial x\) and \(\partial v/\partial y\) are zero along 
the slip lines, the slip lines are inextensible.

**Velocity Relations Along Slip Lines**

Refering to Figure 2b, let \(u_1\) and \(u_2\) be the tangential 
velocities along the \(\alpha\) and \(\beta\) slip lines respectively. Now the angle 
between \(u\) and \(u_1\) from Figure 1b is \(3\pi/4 - \psi/2 - \theta\), and the angle 
between \(v\) and \(u_2\) is \(3\pi/4 + \psi/2 - \theta\). Therefore \(u_1\) and \(u_2\) can be 
written in terms of \(u\) and \(v\) and vice versa.

\[ u_1 = u \cos (3\pi/4 - \psi/2 - \theta) + v \sin (3\pi/4 - \psi/2 - \theta) \]  (8b)
\[ u_2 = -u \sin (3\pi/4 + \psi/2 - \theta) + v \cos (3\pi/4 + \psi/2 - \theta) \]
\[ u = \frac{1}{\cos \psi} \{u_1 \cos (3\pi/4 + \psi/2 - \theta) - u_2 \sin (3\pi/4 - \psi/2 - \theta)\} \]  (7b)
\[ v = \frac{1}{\cos \psi} \{u_1 \sin (3\pi/4 + \psi/2 - \theta) + u_2 \cos (3\pi/4 - \psi/2 - \theta)\} \]
When \( \theta = \frac{3\pi}{4} - \frac{\psi}{2} \), \( du = 0 \); and when \( \theta = \frac{3\pi}{4} + \frac{\psi}{2} \), \( dv = 0 \), therefore equations 7b gives

\[
\begin{align*}
du_1 + (u_1 \tan \psi + u_2 \sec \psi) \, d\theta &= 0 \text{ along } \alpha \text{ lines} \\
\frac{du_2}{\sec \psi} - (u_2 \tan \psi + u_1 \sec \psi) \, d\theta &= 0 \text{ along } \beta \text{ lines}
\end{align*}
\] (10b)

thus we have the velocity relations along slip lines. Equations 8b, the velocity boundary conditions, and the compaction relation given by normality

\[
\dot{\varepsilon} = \dot{\varepsilon}_x + \dot{\varepsilon}_y = \lambda \sin \psi
\]

= Volumetric strain rate

are sufficient to determine the velocity field when the slip lines are known.

**Velocity Relations For Some Specific Slip Line Patterns**

Slip line patterns where either one or both slip lines are straight often occur and the associated velocity and stress distributions are simpler to determine. When both \( \alpha \) and \( \beta \) slip lines are straight, equations 5b show that the stresses are constant in these regions and equations 10b show that the velocities are constant along these lines.

When one family of slip lines is straight and joined at a common point, the other is a set of logarithmic spirals. Figure 3b shows a typical case where the \( \alpha \)-lines are straight. If the \( \alpha \)-lines are the straight lines, equations 10b give
\[ du_1 = 0 \text{ and thus } u_1 = u_1(\phi) \text{ where } \phi \text{ is defined in Figure 3b} \]

\[ u_2(\phi) = e^{\phi} \tan \psi \sec \psi \int_0^\phi u_1(t) e^{-t}\tan \psi \, dt + A_2 e^{\phi} \tan \psi \]

\[ A_2 = \text{some constant for each } \beta \text{ line} \]

If the \( \beta \)-lines are straight and the \( \alpha \)-lines are logarithmic spirals, equations 10b give,

\[ du_2 = 0 \text{ and thus } u_2 = u_2(\phi) \]

\[ u_1(\phi) = -\sec \psi e^{-\phi} \tan \psi \int_0^\phi u_2(t)e^{t}\tan \psi \, dt + A_1 e^{\phi} \tan \psi \]

\[ A_1 = \text{some constant for each } \alpha \text{ line} \]

It is assumed everywhere above that in all cases \( u_1 \) and \( u_2 \) must be consistent with the compaction condition \( dc = \lambda \sin \psi \).

**Velocity Discontinuities**

In the above analysis it has been tacitly assumed that the stress and velocity fields are continuous. However there is no reason to expect continuous velocity fields everywhere. The implications of velocity discontinuities and the jump conditions will be discussed in this section. In problems of flow of rigid/perfectly plastic continua, there are normally regions of rigid material as well as plastic material. Equations 10b imply that the boundary separating the rigid and plastic regions must be a characteristic. Assuming that there is some line not a characteristic separating the two regions and applying equations 10b leads to rigid regions on both sides of the line violating the original assumption. However, letting the line separating rigid and plastic regions be a characteristic does not lead
to any inconsistency. It should also be noted that the jump in velocity normal to this line is not zero. If the characteristic is a line of discontinuity, the jump conditions can be easily established.

Considering the case where the $\alpha$ line is the line of discontinuity, align the $x$-$y$ axes such that the positive $x$ axis is directed in the positive $\alpha$ direction and is tangent to the $\alpha$ line at some point $P$. At $P$ we have, from Figure 1b and equations 7b.

$$\phi = 3\pi/4 - \psi/2$$
$$\dot{e}_x = 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} = 0$$
$$\dot{e}_y = \lambda \sin \psi$$
$$\dot{\gamma}_{xy} = -\lambda \cos \psi$$
$$\lambda = \text{some positive, arbitrary constant}$$

Since the $x$ axis is directed along the $\alpha$ line, $\frac{\partial v}{\partial x}$ is small in comparison to $\frac{\partial u}{\partial y}$, therefore

$$\frac{\delta v}{\delta u} = -\tan \phi$$

Thus the jump condition for a $\alpha$ line as a line of discontinuity is such that the velocity vector is directed at an angle $\psi$ from the $\alpha$ line into the rigid region. Similar arguments produce the same jump condition when the line of discontinuity is a $\beta$ line.

A special case that often occurs is one where the slip line that is not a discontinuity is a straight line. Figure 4b shows the case where the $\alpha$ lines are straight lines, the velocity vector is directed into the rigid region perpendicular to the $\alpha$ line. If the $\alpha$ lines are joined at a common point, and the $\beta$ line is a line of discontinuity, the $\beta$ line must be a logarithmic spiral. In this case equations 10b and 11b give
\[ u_1 = 0 \]
\[ u_2(\psi) = A_1 e^{\psi \tan \psi} \]
\[ A_1 = \text{constant} \]

See Figure 3b.

In order for a velocity field to be compatible with a given slip line field, the velocity field must generate a positive power of dissipation - i.e. the field must be such that the material is deforming in a stable configuration. Defining the power of dissipation as the rate work is done on the material by the loads, we have

\[ D = \sigma_{ij} \dot{\varepsilon}_{ij} \]

Where \( \dot{\varepsilon}_{ij} \) are defined by equations 6b. Using equations 6b and 1b, the power of dissipation is given by,

\[ D = \lambda \sin \psi \sigma, \lambda > 0 \]

Since all quantities are positive the material is deforming stably.

Figure 5b shows the velocities of two adjacent points on a slip line and the mapping in the hodograph plane. Since the slip lines are inextensible, the velocity vectors at adjacent points on a slip line have equal projections on the line segment between the points. Therefore, just as Prager has shown for metals, the lines \( P_1 P_2 \) and \( P_1^* P_2^* \) are orthogonal and any corresponding elements of a slip line in the physical plane and their image in the hodograph are orthogonal.

Orthogonality of the slip lines in the physical plane and their images in the hodograph together with the jump conditions for velocity discontinuities allow the construction of the complete hodograph from the slip line field. In the next section it will be shown that the tangent to the image of a slip line in stress space is
parallel to the tangent of the other slip line in the physical space, and thus Frager's[31] method for the mapping of the slip lines in stress space applies as well for compacting materials.

**Mapping of Characteristics in Stress Space**

The mapping of the characteristics, or slip lines, in stress space utilizes the concept of the pole of Mohr's circle. Now the characteristic directions at a point P as defined in Figure 1b are parallel to the lines connecting the pole of Mohr's circle defining the state of stress at P and the points of tangency of the circle and the yield envelope. It is sufficient to show that Frager's technique will also apply to compacting materials if the pole of Mohr's circle moves in the direction of one slip line as the state of stress is determined for points along the other slip line.

The equilibrium equations along the slip lines (equations 5b) can be written in terms of the coordinates of the pole of the Mohr's circle. Referring to Figure 6b where the coordinates of the pole, \( \sigma_p \), and \( \tau_p \), are given by,

\[
\sigma_p = P + (\sigma_H - P)\sin \psi \cos 2 \theta
\]
\[
\tau_p = (P - \sigma_H)\sin \psi \sin 2 \theta
\]
\[
P = \frac{\sigma_x + \sigma_y}{2}
\]
\[
\sigma_H = \text{Hydrostatic yield stress}
\]
\[
\theta = \text{Angle between x axis and direction of maximum principal stress}
\]
\[
\psi = \text{Compaction angle}
\]
If $\xi_1$ and $\xi_2$ are defined as in Figure 6b

$$\xi_1 = \frac{3\pi}{4} - \frac{\psi}{2} - \theta$$
$$\xi_2 = \frac{5\pi}{4} + \frac{\psi}{2} - \theta$$

and $\frac{\partial \sigma_p}{\partial s_a}, \frac{\partial \tau_p}{\partial s_a}, \frac{\partial \sigma_p}{\partial s_\beta},$ and $\frac{\partial \tau_p}{\partial s_\beta}$ are found in terms of $\frac{\partial P}{\partial s_a}$ etc. and $\xi_1$ and $\xi_2$ are substituted for $\theta$, equations 5b can be expressed as follows

$$\frac{\partial \sigma_p}{\partial s_a} \sin \xi_2 - \frac{\partial \tau_p}{\partial s_a} \cos \xi_2 = 0 \quad (\text{a line})$$
$$\frac{\partial \sigma_p}{\partial s_\beta} \sin \xi_1 - \frac{\partial \tau_p}{\partial s_\beta} \cos \xi_1 = 0 \quad (\text{b line})$$

These equations give the trajectory of the pole in stress space as the state of stress is measured at different points along a slip line.

Since

$$\frac{\partial \tau_p}{\partial s_a} / \frac{\partial \sigma_p}{\partial s_a} = \tan \xi_2$$
$$\frac{\partial \tau_p}{\partial s_\beta} / \frac{\partial \sigma_p}{\partial s_\beta} = \tan \xi_1$$

the pole moves in the direction of the second slip line for different stress states along the first slip line, and conversely. Thus Prager's method applies as well for compacting materials, and the slip lines can be graphically constructed from boundary conditions. Note that all the relations mentioned above reduce the special case for metals where $\psi$ goes to 0.
APPENDIX C

UPPER BOUND SOLUTION FOR THE FORCE-DISPLACEMENT
RELATION FOR THE INDENTATION OF A HALF-PLANE OF
RIGID/PLASTIC COMPACTING MATERIAL BY A PERFECTLY ROUGH WEDGE

Solutions to wedge indentation problems for perfectly plastic materials are obtained using the methods of limit analysis developed by Drucker, Prager, and Greenberg[32] for continuous media. Limit analysis is concerned with the question of whether or not certain loads are capable of producing deformations in a rigid/plastic material. Lower bound loads except for the greatest lower bound loads are loads that are not capable of producing deformations where upper bound loads are capable of producing deformations.

A lower bound load is further specified by the following theorem:

If an equilibrium distribution of stress can be found which balances the applied loads and which nowhere violates the yield condition, then the load cannot exceed the limit load.

And an upper bound load is specified by the following theorem:

Flow will occur if there is any compatible pattern of plastic deformation for which the rate of work done by the external forces equals or exceeds the rate of internal dissipation.

Naturally, the greatest lower bound load equals the least upper bound load which is the limit load that causes plastic deformation.

In the analysis of the wedge indentation problem, the characteristics are constructed with a compatible pattern of plastic strain rates (kinematically admissible velocity field) and hence the calculated force causing indentation is an upper bound. Lower bound loads may be calculated using discontinuous stress fields, but this is not done here since the calculated lower bounds do not necessarily
add to this discussion. The loads determined from the slip line field are the least upper bound loads if the slip lines can be extended into the rigid region without violating the yield condition there. This has been done for indentation of metals by a flat punch by Bishop[33].

The slip lines have not been extended into the region in this appendix, but since Bishop has extended the lines into the rigid region for a similar problem with materials, and since the same mapping technique for the construction of slip lines applies for compacting materials, it is reasonable to expect that the extension can be done.

In order to determine the solution for the force-displacement relation for the wedge indentation problem, the slip line field must be constructed and then the stresses on the wedge face determined from them. This is done following Hill, Lee and Tupper[34] and Shield[35] who have solved the wedge indentation problem for metals and soils, respectively. Figure 1c shows the characteristics as constructed using Frager's[31] mapping technique, for the indentation of a rigid/plastic compacting material by a perfectly rough wedge; note that since the slip line field is geometrically similar at each state of the indentation process the entire history of the deformation can be represented by one drawing with the depth of penetration of the wedge representing time.

Since the wedge is perfectly rough—i.e. the material is failing in shear at the wedge face and no slippage may occur along the
wedge face, the fan of shearing material extends from the wedge face AD to the constant state stress field BCD. The region BCD is adjacent to the surface that is either a free surface or subjected to some confining pressure.

As the wedge is pressed into the compacting material, the material compacts as it is deformed and a lip may or may not form at each side of the wedge. It is assumed that the top of the displaced material remains in a straight line and once the lip angle \( \lambda \) is determined the force on the wedge can be determined. Since \( \lambda \) is determined from \( \phi \) and \( \omega \), the fan angle and half-wedge angle respectively by

\[
\lambda = \omega - \phi + \pi/4 - \psi/2,
\]

determination of \( \phi \) specifies \( \lambda \) as well. The fan angle is determined by the geometry of the slip line field and velocity field.

Using geometry, the lengths, \( h \), \( l \), and \( b \) shown in Figure lc are related to each other and the depth of penetration of the wedge, \( t \) as follows:

\[
h = l \sin \lambda
\]

= height of the lip above the undisturbed surface

\[
l = 2t/(\sec(\pi/4 + \psi/2)e^{\phi \tan \psi} \cos \omega - 2 \sin \lambda)
\]

= length of the lip

\[
b = (1/2)\sin \lambda \{\sec(\pi/4 + \psi/2)\sin \omega e^{\phi \tan \psi} + 2 \cos \lambda\}
\]

= length defined in Figure lc.

The quantities \( h \), \( l \), and \( b \) are related as well through the velocity field for the problem. Assuming that the wedge is pushed downward at unit velocity, the depth of penetration can be directly associated with the time of penetration.
The line ABC is a line of velocity discontinuity with the region below ABC rigid. Thus the velocity of the material above ABC is directed at an angle $\psi$ into the rigid region. ABC is a $\beta$ characteristic, thus $u_1 = 0$ (where $u_1$ is the velocity along the $\alpha$ characteristic) everywhere in the region ABCD. Continuity required the normal velocities of the material and the wedge face along AD to be equal.

The wedge moves downward at unit velocity, and the normal velocity of the wedge face is,

$$v_n = \sin \omega = \text{normal velocity of the wedge face} \quad (2c)$$

Since AD is an $\alpha$ characteristic,

$$V_{AD} = v_n = \text{velocity of the material along AD} \quad (3c)$$

and there along AD

$$u_2 = v_n \cos \omega = \text{velocity along the $\beta$ characteristic.}$$

In the region ABD, the velocity is decreasing and is given as a function of the fan angle from equations 12b.

$$V = V_{AD} e^{-\phi \tan \psi} = \text{the velocity of the material in the fan region at some angle $\phi$.}$$

and hence,

$$V_{DB} = V_{AD} e^{-\phi \tan \psi} = \text{the velocity of the material along DB.} \quad (5c)$$

BCD is a constant state field and moves as a rigid region

$$V_{DC} = V_{DB} = \text{the velocity of the material along DC.} \quad (6c)$$

The normal component of $V_{DC}$ along DC is given by

$$V_{nDC} = V_{DC} \cos (\pi/4 + \psi/2) \quad (7c)$$

and from equations 2c, 3c, 4c, 5c, 6c, and 7c

$$V_{nDC} = v_n \cos (\pi/4 + \psi/2) e^{-\phi \tan \psi} \quad (8c)$$
The component of the velocity of the wedge normal to DC is given by
\[ v_{WnDC} = \cos \lambda. \]  \hfill (9c)

Since \( t = 0 \) at the beginning of the test, the distance between A and DC is given by,
\[ \delta = t(v_{WnDC} + V_{nDC}) = t(\cos \lambda + \sin \omega \cos (\pi/4 + \psi/2)e^{-\phi \tan \psi}) \]  \hfill (10c)

From Figure 1c the same distance is given by
\[ \delta = t \cos \lambda + b \]  \hfill (11c)

Equations 1c can be substituted into 11c, and then equating 10c and 11c give an expression for the determination of \( \phi \).
\[ \sin \lambda = \frac{\sin \omega \cos \omega}{\left(\sin \omega \sec(\pi/4 + \psi/2)e^{\phi \tan \psi} + \sin \omega \cos (\pi/4 + \psi/2)e^{-\phi \tan \psi} + 2 \cos \lambda\right)} \]  \hfill (12c)

Equation 12c must be solved by trial and error, and once \( \lambda \), or \( \phi \), is found Prager's[31] mapping technique may be used to determine the stress on the wedge interface.

**Stress State on the Wedge Face**

In Appendix B it was shown that Prager's mapping method may be modified for use with compacting materials. His methods allows the determination of stress states anywhere in the slip line field of a plastic material from the boundary conditions, using the concept of the one-to-one mapping of characteristics in the physical plane into the stress plane.

The slip line through some point P is mapped into a curve traced by the pole for the state of stress at P as Mohr's circle of
stress "rolls" without slipping along the upper or lower limit line of the yield envelope. The circle rolls along the upper line for different stress states along a lines and the lower limit line for stress states along β lines.

Assuming the wedge is pressed into the material at some confining pressure $P_c$ the pole of Mohr's circle for the region BCE is shown in Figure 2c. $R_1$ or the radius of Mohr's circle for the region BCE is

$$R_1 = \frac{(\sigma_H - P_c) \sin \psi}{1 + \sin \psi},$$  \hspace{1cm} (13c)

where $P_c$ is the confining pressure of the test, $\sigma_H$ is the hydrostatic yield stress and $\psi$ is the compaction angle. The radius of Mohr's circle decreases as it "rolls" without slipping along the limit lines of the yield envelope. If $\theta$ is prescribed as the angular rotation of the circle during the rolling process, the radius of Mohr's circle after rotating through the angle $\theta$ is

$$R = R_1 e^{-\theta \tan \psi}.$$ \hspace{1cm} (14c)

From Figures 1c and 2c the angular rotation for this problem is given by

$$\theta = 2\phi$$

and therefore $R_{II}$, the radius of Mohr's circle for the state of stress along AD is,

$$R_{II} = R_1 e^{-2\phi \tan \psi}.$$ \hspace{1cm} (15c)

The pole of Mohr's circle for AD is located such that a line parallel to the wedge face through the pole intersects the point of tangency.
of the yield envelope and the circle, since AD is an a characteristic. Thus the state of stress along AD is given by

\[
\sigma_{AD} = \sigma_H + R_{II} \left( \sin \psi - \csc \psi \right) = \text{normal stress along AD} 
\]

\[
\tau_{AD} = R_{II} \cos \psi = \text{shear stress along AD} 
\]

(16c)

**Force-Displacement Relation**

The only experimentally measurable quantity for experimental wedge indentation tests is the force on the wedge versus the depth of penetration. From Figure 1c, the length of the face of the wedge AD is given by,

\[
AD = \left( \frac{1}{2} \right) \sec(\pi/4 + \psi/2) e^{\lambda \tan \psi} 
\]

where l is defined in equation 1c.

The force on the wedge is given by twice the vertical component of the stress vector on AE which is

\[
F = 2 \left( \sigma_{AD} \sin \omega + \tau_{AD} \cos \omega \right) \sec(\pi/4 + \psi/2) e^{\lambda \tan \psi} 
\]

or using equation 1c

\[
F/t = \frac{2 \left( \sigma_{AD} \sin \omega + \tau_{AD} \cos \omega \right) \sec(\pi/4 + \psi/2) e^{\lambda \tan \psi}}{\left\{ \sec(\pi/4 + \psi/2) \cos \omega e^{\lambda \tan \psi} - 2 \sin \lambda \right\}} 
\]

(17c)

\[
\lambda = \omega - \phi + \pi/4 - \psi/2 
\]

\[
t = \text{depth of penetration of wedge below the undisturbed surface.} 
\]

Equation 17c indicates that F over t should be a constant for any one test. Prediction of F/t requires the values of \( \sigma_H \), the hydrostatic yield stress of the material, \( \psi \) the compaction angle for the material, and \( \omega \) the half-wedge angle.
<table>
<thead>
<tr>
<th>Test Type</th>
<th>Confining Pressure</th>
<th>ε(%)</th>
<th>ε₁(%)</th>
<th>ε₃(%)</th>
<th>√2ε₃(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compression</td>
<td>500 psi</td>
<td>-1.7</td>
<td>+1.6</td>
<td>-1.7</td>
<td>-2.3</td>
</tr>
<tr>
<td></td>
<td>1000 psi</td>
<td>-0.7</td>
<td>+1.7</td>
<td>-1.2</td>
<td>-1.7</td>
</tr>
<tr>
<td></td>
<td>2000 psi</td>
<td>-0.3</td>
<td>+2.0</td>
<td>-1.2</td>
<td>-1.7</td>
</tr>
<tr>
<td></td>
<td>2500 psi</td>
<td>-0.1</td>
<td>+1.9</td>
<td>-0.8</td>
<td>-1.1</td>
</tr>
<tr>
<td></td>
<td>3000 psi</td>
<td>-0.8</td>
<td>+1.9</td>
<td>-1.4</td>
<td>-2.0</td>
</tr>
<tr>
<td></td>
<td>4000 psi</td>
<td>+0.5</td>
<td>+1.7</td>
<td>-0.6</td>
<td>-0.9</td>
</tr>
<tr>
<td></td>
<td>5000 psi</td>
<td>+0.5</td>
<td>+1.1</td>
<td>-0.3</td>
<td>-0.4</td>
</tr>
<tr>
<td></td>
<td>7000 psi</td>
<td>+1.1</td>
<td>+2.6</td>
<td>-0.8</td>
<td>-1.1</td>
</tr>
<tr>
<td></td>
<td>8000 psi</td>
<td>+1.3</td>
<td>+2.3</td>
<td>-0.5</td>
<td>-0.7</td>
</tr>
<tr>
<td></td>
<td>9000 psi</td>
<td>+0.8</td>
<td>+0.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>11,000 psi</td>
<td>+1.0</td>
<td>+0.7</td>
<td>+0.1</td>
<td>+0.1</td>
</tr>
<tr>
<td>Extension</td>
<td>3000 psi</td>
<td>-0.1</td>
<td>-0.2</td>
<td>+0.1</td>
<td>+0.1</td>
</tr>
<tr>
<td></td>
<td>5000 psi</td>
<td>-0.9</td>
<td>-1.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>7000 psi</td>
<td>-0.9</td>
<td>-1.3</td>
<td>+0.2</td>
<td>+0.3</td>
</tr>
<tr>
<td></td>
<td>9000 psi</td>
<td>+0.3</td>
<td>-0.9</td>
<td>+0.6</td>
<td>+0.9</td>
</tr>
<tr>
<td></td>
<td>11,000 psi</td>
<td>+0.4</td>
<td>-1.2</td>
<td>+0.8</td>
<td>+1.1</td>
</tr>
<tr>
<td>Test Type</td>
<td>Confining Pressure</td>
<td>ε1(%)</td>
<td>ε1(%)</td>
<td>ε3(%)</td>
<td>$\sqrt{2}\varepsilon_3(%)$</td>
</tr>
<tr>
<td>------------</td>
<td>-------------------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>Compression</td>
<td>500 psi</td>
<td>-0.1</td>
<td>+1.4</td>
<td>-0.8</td>
<td>-1.1</td>
</tr>
<tr>
<td></td>
<td>1000 psi</td>
<td>+0.7</td>
<td>+1.7</td>
<td>-0.5</td>
<td>-0.7</td>
</tr>
<tr>
<td></td>
<td>2000 psi</td>
<td>+1.5</td>
<td>+2.0</td>
<td>-0.3</td>
<td>-0.4</td>
</tr>
<tr>
<td></td>
<td>3000 psi</td>
<td>+1.5</td>
<td>+1.5</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>4000 psi</td>
<td>+1.9</td>
<td>+1.7</td>
<td>+0.1</td>
<td>+0.1</td>
</tr>
<tr>
<td></td>
<td>5000 psi</td>
<td>+1.3</td>
<td>+1.5</td>
<td>-0.1</td>
<td>-0.1</td>
</tr>
<tr>
<td>Extension</td>
<td>3000 psi</td>
<td></td>
<td></td>
<td></td>
<td>samples destroyed</td>
</tr>
<tr>
<td></td>
<td>4000 psi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5000 psi</td>
<td>+1.2</td>
<td>-0.4</td>
<td>+0.8</td>
<td>+1.1</td>
</tr>
</tbody>
</table>
FIGURE 1-1a
Isotropic Hardening for von Mises Yield Condition

FIGURE 1-1b
Kinematic Hardening for von Mises Yield Condition

FIGURE 1-1c
Local Hardening for Tresca Yield Condition
FIGURE 2-1
Schematic Diagram for Triaxial Testing
Apparatus with Sample Inserted for Tests
Figure 2-2
Schematic Diagram of Experimental Apparatus
FIGURE 3-1
Typical Stress-Strain Curves for
Axial Compression Tests at Various
Confining Pressures
FIGURE 3-2
Axial Compression Tests on Cordova Limestone Samples
with Axes Perpendicular to the Bedding Planes
FIGURE 3-3
Batesville Marble Compression Tests for the Initial Yield Surface
FIGURE 3-4
Cordova Limestone Compression Tests for the Initial Yield Surface, Samples' Axes Parallel to the Bedding Planes
Axial Compression Tests Showing Inhomogeneity in Cordova Limestone Samples with Axes Parallel to Bedding Planes
FIGURE 3-6
Axial Compression Tests Showing Inhomogeneity in Cordova Limestone Samples with Axes Parallel to Bedding Planes
FIGURE 3-7
Axial Compression Tests of One Sample of Cordova
Limestone with Axis Parallel to the Bedding Planes
Tested at Different Confining Pressures
FIGURE 3-8
Results of Extension Tests on Batesville Marble
FIGURE 3-9
Results of Extension Tests on Cordova Limestone
Samples with Axes Perpendicular to Bedding Plane
FIGURE 3-11
Cyclic Loading Tests at 8000 psi on Batesville Marble
FIGURE 3-12
Cyclic Loading Tests at 12,000 psi on Batesville Marble
FIGURE 3-13
Cyclic Loading Test at 3000 psi on Cordova Limestone
Sample with Axis Parallel to Bedding Plane
FIGURE 3-15
Cyclic Loading Test at 7000 psi on Cordova Limestone Sample with Axis Parallel to Bedding Plane.
FIGURE 3-16
Radial Compression Tests on Batesville Marble
FIGURE 3-17
Effects of Radial Loading History on Axial Compression Tests at 6000 psi on Cordova Limestone Samples with Axes Perpendicular to Bedding Plane

Test | Radial Load History
---|---------------------
1   | None
2   | Loaded to 8,000 psi
3   | Loaded to 10,000 psi
FIGURE 3-18
Axial Compression Test at 3000 psi on Cordova Limestone
Sample with Axis Perpendicular to Bedding Planes
FIGURE 3-19
Axial Compression Test at 4000 psi on Cordova Limestone
Sample with Axis Perpendicular to Bedding Planes
$\Delta \sigma_a \text{ (psi)}$

$P_c = 5000 \text{ psi}$

FIGURE 3-20
Axial Compression Test at 5000 psi on Cordova Limestone
Sample with Axis Perpendicular to Bedding Planes
**FIGURE 3-21**
Axial Compression Test at 6000 psi on Cordova Limestone Sample with Axis Perpendicular to Bedding Planes

$P_c = 6000$ psi

**Δσ_a (psi)**

0  1.0  2.0  3.0  4.0  5.0  6.0  7.0

$\varepsilon_a (%)$
Figure 3-23
Axial Compression Test at 8000 psi on Cordova Limestone
Sample with Axis Perpendicular to Bedding Planes

\( \Delta \sigma_a \) (psi)

\( \varepsilon_a \) (%)

\( P_c = 8000 \text{ psi} \)
Figure 3-24
Axial Compression Tests to Large Strains at 3000 and 5000 psi on Different Samples of Cordova Limestone with Axes Perpendicular to Bedding Planes

\[ \Delta \sigma_g \ (\text{psi}) \]

- \( P_c = 5000 \) psi
- \( P_c = 3000 \) psi
FIGURE 3-25
Initial Yield Surface for Batesville Marble
FIGURE 3-26
Initial Yield Surface for Cordova Limestone Samples
with Axes Perpendicular to Bedding Planes
FIGURE 3-27
Initial Yield Surface for Cordova Limestone
Samples with Axes Parallel to Bedding Plane
FIGURE 3-28
Plastic Strains Associated with the Initial Yield Surface for Batesville Marble
FIGURE 3-29
Plastic Strains Associated with the Initial Yield Surface for Cordova Limestone Samples with Axes Perpendicular to the Bedding Plane
FIGURE 3-30
(a) Differential Axial Stress Versus Axial Strain Curve at Some Particular Confining Pressure (Loading and Unloading); (b) Differential Axial Stress Versus Axial Strain Curve at Some Other Confining Pressure (Loading Only); (c) Changes in the Yield Surface Due Straining the Material at $P_{c1}$ as Detected by Axial Compression Test at Some Other Confining Pressure
Idealized Differential Axial Stress-Axial Strain Curves for Axial Compression Tests at Two Different Confining Pressures
FIGURE 3-32
Examples of Two Loading Paths Giving the Same Amount of Workhardening
FIGURE 3-33a

Workhardening Rule Tests at 3000 psi on Cordova Samples with Axes Perpendicular to Bedding Planes. Curves (2), (3), and (4) should be Tangent to Asymptote of Curve (1) for Optimum Results.
FIGURE 3-33b.
Workhardening Rule Test at 3000 psi on Cordova Limestone Sample with Axis Perpendicular to Bedding Plane. Curve should be Tangent to Asymptote for Optimum Results.
FIGURE 3-34
Prediction of Equivalent Axial Strain at 4000 psi for Samples Previously Tested at Other
Confining Pressures. Samples are of Cordova Limestone with Axes Perpendicular
to Bedding Planes. Curves should be Tangent to Asymptote for Optimum Results.
FIGURE 3-35a
Prediction of Equivalent Axial Strain at 5000 psi for Samples Previously Tested at Other Confining Pressures. Samples are of Cordova Limestone with Axes Perpendicular to Bedding Planes. Curves should be Tangent to Asymptote for Optimum Results.
FIGURE 3-35b
Continuation of Predicted Axial Strains in 3-35a

\[ \Delta \sigma_a \text{ (PSI)} \]

Sample Number

1.7, 10

Test

5, 6
FIGURE 3-36
Prediction of Equivalent Axial Strain at 6000 psi for Samples Previously Tested at Other Confining Pressures. Samples are of Cordova Limestone with Axes Perpendicular to Bedding Planes. Curves should be Tangent to Asymptote for Optimum Results.
FIGURE 3-37
Prediction of Equivalent Axial Strain at 7000 psi for Samples Previously Tested at Other Confining Pressures. Samples are of Cordova Limestone with Axes Perpendicular to Bedding Planes. Curves should be Tangent to Asymptote for Optimum Results.
FIGURE 3-38
Prediction of Equivalent Axial Strain at 8000 psi for Samples Previously Loaded at Other Confining Pressures. Samples are Cordova Limestone with Axes Perpendicular to Bedding Planes. Curves should be Tangent to Asymptote for Optimum Results.
FIGURE 3-39
Initial and Subsequent Yield Surfaces for Cordova Limestone
Samples Perpendicular to Bedding Plane Caused by
2.5% Axial Strain Increments at 3000 psi
FIGURE 3-40
Similar Loading Paths Assuming Yield is Independent of Intermediate Principal Stress and the Difference of the Maximum and Minimum Principal Stresses is a Function of Minimum Principal Stress
\[ \sigma_{\text{MAX}} - \sigma_{\text{MIN}} + \alpha \sigma_{\text{MIN}} = \beta \]

\[ J_2^{1/2} + \alpha l_1 = \beta \]

FIGURE 3-41
Cross Sections of Two Possible Yield Functions for End Caps
\[ \sigma_1 \]
\[ \sqrt{2} \sigma_3 \]

(a) \[ \alpha_0 > 0 \quad \alpha_1 < 0 \]
\[ \beta = \beta_0 \]

(b) \[ \alpha = \alpha_0 \]
\[ \beta_0 < 0 \quad \beta_1 > 0 \]

(c) \[ \alpha_0 > 0 \quad \alpha_1 < 0 \]
\[ \beta_0 < 0 \quad \beta_1 > 0 \]

(d) \[ \alpha_0 > 0 \quad \alpha_1 < 0 \]
\[ \beta_0 > 0 \quad \beta_1 < 0 \]
\[ \beta < \alpha_1 \]

FIGURE 3-42
Examples of Different Workhardening Rules
FIGURE 3-43
Initial and Subsequent Yield Surfaces for Cordova Limestone Samples with Axes Perpendicular to Bedding Planes. Subsequent Yield Surfaces Given by the Hardening Rule $\alpha = .93; \beta_0 = 4650, \beta_1 = 1.35 \times 10^5$. 
FIGURE 4-2
Comparison of Predicted and Experimental Values of Force Over Depth of Penetration Versus Wedge Angle for Wedge Indentation of the Half-Plane at 5000 psi
FIGURE 1b
Definition of Characteristic Directions

FIGURE 2b
Velocity Components in X, Y, α and β Directions
FIGURE 3b
Typical Set of Slip Lines where One Family is Straight and the Other Logarithmic Spirals

(3b)

FIGURE 4b
Jump Condition Across Slip Line that is a Line of Discontinuity and the Other Slip Line is Straight

Plastic Region

\[ \pi/2 + \psi \]

Velocity Discontinuity

Rigid Region

(4b)
FIGURE 5b
Mapping of Velocities at Two Adjacent Points in the Physical Plane in the Hodograph Plane
FIGURE 1c
Slip Line Field for Rough Wedge Indentation into Compacting Material
FIGURE 2c
Mapping in Stress Space of $\beta$ Characteristics in Figure 1c