MARTIN, John Calhoun, 1945-
SUBSTITUTION MINIMAL FLOWS.

Rice University, Ph.D., 1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan
RICE UNIVERSITY

Substitution Minimal Flows

by

John C. Martin

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

Thesis Director's Signature:

William A. Reed

Houston, Texas

April 1971
CONTENTS

Introduction 1

I. Definitions and Preliminaries 6
II. A Basic Lemma 12
III. The Flows $\check{x}_\theta$ and $S^r$ 16
IV. Construction of a Flow Homomorphism 21
V. Calculation of the Structure Transformation Group of $\check{x}_\theta$. 27
VI. Almost Automorphic Substitution Flows 32
VII. Proof that $\check{x}_\theta$ is Point-distal 40
VIII. AI Extensions and AI Flows 41

References 62
Introduction.

One of the simplest ways of generating interesting minimal symbolic flows is by using what is called a substitution. The basic idea is that to each element of a finite set $S$ of symbols, there is associated a finite string, or block, of elements of $S$. By choosing an element of $S$, substituting the block corresponding to it, substituting for each symbol in this block the corresponding block, and continuing this process indefinitely, it is possible to obtain an infinite sequence, which under rather loose restrictions is a recurrent, non-periodic sequence. If we consider the orbit-closure $X$ of this sequence under the left shift $T$, then $T$ defines a minimal flow on the compact metric space $X$.

The most widely-known example was introduced by Morse and was used originally to construct recurrent geodesics on certain surfaces. The sequence he introduced has also arisen in several other contexts. This particular sequence, and the resulting flow, have been widely investigated (see, for example, $[4,5,7,9,10]$.) The flows arising in the more general case, called substitution minimal flows, have also been discussed in some detail; see $[3,7,8]$. There are two natural cases which arise: that in which the blocks determining the
substitution are of the same length, and that in which they are not. The two cases are substantially different, with regard to the structure of the resulting flows, and with regard to the methods of analysis. In this paper, we shall deal only with the first case, although a few of the preliminary results carry over to the more general case. By a substitution minimal flow, therefore, we shall mean the minimal flow arising from a substitution of constant length \( r \).

This paper is an attempt to describe as completely as possible the structure of these flows. The first four sections contain basic definitions, a crucial combinatorial lemma, and the construction of the flows and certain flow homomorphisms. We show in section V that the structure transformation group associated with a substitution minimal flow is \((Z_m \times Z^r, T)\), where \( r \) is the length of the substitution, \( Z^r \) is the \( r \)-adic group, \( m \) is a certain positive integer relatively prime to \( r \), and \( T \) is the homeomorphism corresponding to addition of the group element \((1,1)\). In section VI, conditions on the substitution are obtained which are necessary and sufficient for the flow to be almost automorphic. In VII, we prove that substitution minimal flows are point-distal.

Section VIII is devoted to an analysis of the non-
almost automorphic substitution flows. In [14], Veech showed that the Morse flow could be represented as an AI extension (i.e., an isometric extension of an almost automorphic extension) of the flow on the 2-adic group. We show that for binary substitutions whose associated flows are not almost automorphic, the same conclusion holds (with the 2-adic group replaced by the structure transformation group.) More generally, for a wider class of substitutions, we give a necessary and sufficient condition that this hold.

Minimal flows satisfying the property described above are among the simplest examples of AI flows; in particular, they are AI flows of order two, being AI extensions of equicontinuous flows. We show that any substitution flow which is an AI extension of an equicontinuous flow is in fact an AI extension of its structure transformation group; moreover, the condition mentioned above which characterizes the flows of this type also characterizes the AI flows, for a restricted class of substitutions. Therefore, we are able in some cases to determine those substitutions which give rise to AI flows.

Veech, in [16], proves that every point-distal flow with a residual set of distal points has an almost automorphic extension which is an AI flow. In general,
there are few known examples of flows satisfying these hypotheses which are not themselves AI flows. The flows we consider are shown to satisfy the hypotheses; and thus we construct a class of examples of point-distal flows with a residual set of distal points which are not AI flows. (Shapiro, in [11], has recently constructed examples of a different sort.)

We find it interesting in itself that by going, for example, from binary substitutions to substitutions on three symbols, a great deal of new and interesting behavior appears. Thus, the Morse flow is in many respects not typical at all of the general substitution flow.

Some of our results in the first six sections generalize, or resemble slightly, results of Gottschalk and Keynes contained in [3] and [8], respectively, although our approach to the construction of the flows is different. For necessary background material, and as general references, see [3,4,8].

I would like to extend my sincerest gratitude to my advisor, Professor William A. Veech, for suggesting the basic problems considered in this paper, and for his constant encouragement and valuable advice during its preparation. Leonard Shapiro contributed some helpful suggestions, and I am very grateful to Keith Dennis
for many enlightening discussions. Finally, I wish to thank NASA, the National Science Foundation, and Rice University for financial support during my graduate study.
I. Definitions and Preliminaries

The material in the initial part of this section is well-known; a more complete discussion may be found, for example, in [4].

A **transformation group**, or **flow**, is a triple \((X, G, \pi)\), where \(X\) is a compact Hausdorff space, \(G\) is a topological group, and \(\pi: X \times G \rightarrow X\) is a continuous map satisfying (i) \(\pi(x, e) = x\) for each \(x \in X\), where \(e\) is the identity element of \(G\); (ii) \(\pi(\pi(x, t_1), t_2) = \pi(x, t_1 t_2)\) \((x \in X, t_1, t_2 \in G)\).

If \(\mathcal{I} = (X, G, \pi)\) is a flow, \(X\) is called the phase space of \(\mathcal{I}\), and \(G\) the phase group. We shall be concerned only with the case in which the phase space \(X\) of \(\mathcal{I}\) is metrizable, and there is a homeomorphism \(T\) of \(X\) onto itself which generates the flow, in the sense that \(G\) is the group of integers and \(\pi(x, n) = T^n x\) \((x \in X, n\) an integer\). We shall denote such a flow by \((X, T)\), and we generally omit any further reference to the map \(\pi\).

If \(\mathcal{I} = (X, T_1)\) and \(\mathcal{J} = (Y, T_2)\) are flows as above, we shall use \(T\) to denote both \(T_1\) and \(T_2\); this generally leads to no ambiguity. If \(\mathcal{I} = (X, T)\) and \(\mathcal{J} = (Y, T)\) are flows, a **flow homomorphism** \(f\) from \(\mathcal{I}\) to \(\mathcal{J}\) (written \(f: \mathcal{I} \rightarrow \mathcal{J}\)) is a continuous map \(f\) from \(X\) onto \(Y\) which
satisfies the condition $f(Tx) = Tf(x)$ (x $\in$ X).

A flow (X, T) is **minimal** if \{T^n x: n = \ldots, -1, 0, 1, \ldots\} is dense in X for each x $\in$ X. An equivalent statement is that there be no proper non-empty closed subset M of X which is invariant under T -- i.e., which satisfies T(M) $\subseteq$ M.

If f: x $\rightarrow$ y is a flow homomorphism, and M $\subseteq$ Y is a proper, non-empty, closed, invariant subset of Y, then f$^{-1}$M has the same property in X. An immediate consequence of this is that if the flow x is minimal, then y is also.

A flow x = (X, T) is **equicontinuous** if \{T^n \cdot: n an integer\} is an equicontinuous family of maps. We shall use the fact that if X is a compact abelian topological group and T: X $\rightarrow$ X is group multiplication by an element a which generates a dense cyclic subgroup of X, then (X, T) is an equicontinuous flow.

In the remainder of this section, our notation is taken to some extent from [3].

We denote by Z the set of integers, $Z^+$ the positive integers, and Z(k,m) the set \{n $\in$ Z: k$\leq$n$\leq$m\}.

Let b be a fixed integer greater than 1. We shall be discussing sequences constructed from b distinct symbols, for simplicity taken to be the elements of S = S_b = Z(0, 1). Let $S^n = \{f: Z(0, n-1) \rightarrow S\}$ (n $\geq$ 1).
We call elements of $S^n$ n-blocks, and if $A \in S^n$, we occasionally write $A = a_0a_1\ldots a_{n-1}$, where $a_i = A(i)$.

For $A \in S^n$, $B \in S^m$, we define $AB$ to be the $(n+m)$-block given by $AB(j) = A(j)$ for $j \in \mathbb{Z}(0,n-1)$ and $AB(j) = B(j-n)$ for $j \in \mathbb{Z}(n,n+m-1)$. For $A \in S^n$ and $0 \leq k \leq m \leq n-1$, $A(k,m)$ is the $(m-k+1)$-block defined by $A(k,m)(j) = A(k+j)$ ($j \in \mathbb{Z}(0,m-k)$). If $A \in S^n$ and $B \in S^m$, $(m \leq n)$, we say $B$ appears in $A$ if for some $j$, $A(j, j+m-1) = B$.

Let $r$ be a fixed integer greater than 1. By a substitution $\theta$ of length $r$ over $S$, we shall mean a function $\theta: S \to S^r$ with $\theta(0)(0) = 0$. If $\theta$ is a substitution over $S$ and $A \in S^n$, we denote by $\theta(A)$ the block $\theta(a_0)\ldots \theta(a_{n-1})$, where $a_i = A(i)$. For each $i \in \mathbb{Z}^+$, we define a substitution $\theta^i$ as follows:

(1) $\theta^0 = \theta$; if $\theta^i(p) = A$, $\theta^{i+1}(p) = \theta(A)$

If $\theta$ is a substitution of length $r$ over $S$, $\theta^i$ is a substitution of length $r^i$ over $S$. For a fixed substitution $\theta$ of length $r$, we use the term basic $r^i$-block to denote any one of the $r^i$-blocks $\theta^i(p)$ ($p \in S$).

If $x$ is a sequence over $S$ (a function from the non-negative integers into $S$) or a bisequence over $S$ (a function from $Z$ into $S$), we define the n-block $x(k,k+n-1)$ analogously ($n \in \mathbb{Z}^+$, $k$ in the domain of $x$), and we say the n-block $A$ appears in $x$ if for some $k \in Z$, $x(k,k+n-1) = A$. The sequence (bisequence) $x$ over $S$ is said to be
recurrent if, for \( k \in \mathbb{Z} \), there exists \( m \in \mathbb{Z}^+ \) such that the \( k \)-block \( x(0,k-1) \) (the \( 2k \)-block \( x(-k,k-1) \)) appears in every \( m \)-block of \( x \).

Let \( \theta \) be a substitution of length \( r \) over \( S \). We define a sequence \( x_\theta' \) (later to be extended to a bi-sequence \( x_\theta \)) as follows:

\[
(2) \quad x_\theta'(0, r^{k-1}) = \theta^k(0) \quad (k \in \mathbb{Z}^+) \]

It is easily seen from the definitions of \( \theta \) and \( \theta^k \) that \( x_\theta' \) is well-defined, and that for any fixed \( m \in \mathbb{Z}^+ \), the above formula is equivalent to:

\[
(3) \quad x_\theta'(0, r^{m-1}) = \theta^m(0), \text{ and for } j \in \mathbb{Z}^+, \quad x_\theta'(jr^m, (j+1)r^{m-1}) = \theta^m(x_\theta'(j))
\]

It is also clear that for \( k \in \mathbb{Z}^+ \), we have \( x_\theta' = x_{\theta^k} \).

If the sequence \( x_\theta' \) is a recurrent sequence, \( \theta \) will be called a recurrent substitution. \( \theta \) is said to be non-degenerate if range \( x_\theta' = S \).

**Lemma 1.1.** Let \( \theta \) be a substitution of length \( r \) over \( S \). A necessary and sufficient condition for \( \theta \) to be recurrent and non-degenerate is

\[
(4) \quad \text{If } S_1 \text{ is any non-empty proper subset of } S, \text{ then for some } i \in S_1, \text{ range } \theta(i) \notin S_1.
\]

**Proof.** Suppose \( \theta \) is non-degenerate and (4) is not satisfied. We shall show \( \theta \) is not recurrent. Let \( S_1 = \{i_1, i_2, \ldots, i_p\} \) be a non-empty proper subset of \( S \) with range \( \theta(i_j) \subset S_1 \quad (j = 1, 2, \ldots, p) \). Using (1) and
(3), it is easy to see that for arbitrarily large \( k \in \mathbb{Z}^+ \), there are \( k \)-blocks of \( x_\theta' \) containing only the symbols in \( S_1 \). But some \( n \)-block \( x_\theta'(0,n-1) \) contains every symbol in \( S \), and thus \( x_\theta' \) cannot be recurrent.

Now we suppose (4) holds. To see that \( \theta \) is non-degenerate, we suppose that for some \( i \in S \), \( i \notin \) range \( x_\theta' \). Then letting \( S_1 = \text{range } x_\theta' \) in (4), we obtain a contradiction. To see that \( x_\theta' \) is recurrent, it is enough by (3) to find an integer \( n \in \mathbb{Z}^+ \) so that the symbol 0 appears in any \( n \)-block of \( x_\theta' \). This will be true if, for large enough \( k \), 0 appears in each basic \( r^k \)-block. (Note that since \( \theta(0)(0) = 0 \), if \( 0 \in \text{range } \theta^k(i) \), then \( 0 \in \text{range } \theta^m(i) \) for \( m \geq k \).) Let \( S_1 \) be the set \( \{ i \in S: 0 \notin \text{range } \theta^k(i) \text{ for every } k \in \mathbb{Z}^+ \} \). Then for \( i \in S_1 \) and \( k \in \mathbb{Z}^+ \), range \( \theta^k(i) \subset S_1 \); for otherwise 0 would appear in \( \theta^{k+j}(i) \) for some \( j \in \mathbb{Z}^+ \). Now \( S_1 \neq S \), and if \( S_1 \) is non-empty, we have a contradiction of (4). We conclude that \( S_1 \) is empty, and \( \theta \) is recurrent.  

We shall restrict ourselves not only to recurrent, non-degenerate substitutions, but to those which yield non-periodic sequences. A sequence (bisequence) \( x \) over \( S \) is \textbf{periodic of period} \( p \) if \( x(i+p) = x(i) \) for \( i \geq 0 \) (\( i \in \mathbb{Z} \)), and if \( p \) is the smallest positive integer having this property. \( \theta \) will be called periodic of period \( p(\theta) \) if the sequence \( x_\theta' \) is periodic of period \( p(\theta) \).
For the moment it is not necessary to concern ourselves with an explicit criterion for periodicity, but the following indicates that there are plenty of examples of non-periodic substitutions.

**Lemma 1.2.** If $\theta$ is a non-degenerate periodic substitution of length $r$ over $S$, then either $p(\theta) = b$ ($= \text{card } S$) or $\theta: S \rightarrow S^r$ is not one-to-one.

**Proof.** Suppose $p(\theta) \neq b$. Then since $\theta$ is non-degenerate, $p(\theta) > b$; this implies that for some $i \in S$, $i$ appears twice in $x_{\theta'}(0, p(\theta)-1)$. Suppose $x_{\theta'}(j) = x_{\theta'}(m) = i$, where $0 \leq j < m = j+d \leq p(\theta)-1$. Now consider the sets $\{x_{\theta'}(j+n), x_{\theta'}(m+n)\}$ ($n \geq 0$). If each of these sets is a singleton, $x_{\theta'}$ must be periodic of period $d$ or less, which is impossible. Let $n_0 = \min \{n \geq 1: x_{\theta'}(j+n) \neq x_{\theta'}(m+n)\}$, and consider the 2-blocks $x_{\theta'}(j+n_0-1, j+n_0)$ and $x_{\theta'}(m+n_0-1, m+n_0)$, relabeled $uv$ and $uw$, respectively ($v \neq w$). Choose $k$ with $r^k \geq p(\theta)$; then the $2r^k$-blocks $\theta^k(u)\theta^k(v)$ and $\theta^k(u)\theta^k(w)$ both appear in $x_{\theta'}$. But this implies $\theta^k(v) = \theta^k(w)$, and therefore $\theta$ cannot be one-to-one. $\Box$

**Corollary 1.3.** If $b = 2$ (i.e., $S = \{0,1\}$), and $\theta$ is a one-to-one non-degenerate substitution of length $r$ over $S$, then $x_{\theta'}$ is not periodic unless it is of the form $x_{\theta'} = 010101...$

(This is also a special case of Lemma 1 in [7].)
We shall refer to a substitution $\theta$ of length $r$ over $S$ as \textbf{admissible} if $\theta$ is one-to-one, non-degenerate, recurrent, and non-periodic. We observe that if $\theta$ is an admissible substitution of length $r$, then $\theta^k$ is an admissible substitution of length $r^k$ ($k \in \mathbb{Z}^+$). It will frequently be convenient to consider, instead of $\theta$, some power $\theta^k$ of $\theta$.

II. A \textbf{Basic Lemma}

Throughout the remainder of the discussion, $\theta$ will denote a fixed admissible substitution of length $r$ over $S = S_b$.

\textbf{Definition.} Let $A$ be an $s$-block appearing in $x_\theta'$, where $s > 2r^k$. We say $A$ is \textbf{determined up to order} $k$ if, whenever $A = B\theta^k(i_1)\theta^k(i_2)\ldots\theta^k(i_m)C = D\theta^k(j_1)\ldots\theta^k(j_n)E$ (where $m, n \geq 1$; $B = \theta^k(i_0)(t, r^k-1)$, $D = \theta^k(j_0)(u, r^k-1)$, $C = \theta^k(i_{m+1})(0, v)$, $E = \theta^k(j_{n+1})(0, w)$, for some $t, u, v, w$ in $Z(0, r^k-1)$; and the blocks $i_0i_1\ldots i_m$ and $j_0j_1\ldots j_n$ appear in $x_\theta'$), then the blocks $B$ and $D$ are of the same length-- and thus $B = D$, $m = n$, $i_p = j_p$ ($p = 1, 2, \ldots, m$), and $C = E$. Roughly, $A$ is determined up to order $k$ if $A$ appears in $x_\theta'$ in only one manner as a combination of basic $r^k$-blocks and portions of basic $r^k$-blocks.
It is clear that if A is determined up to order k, then it is determined up to order j (j ≤ k); and any block in which A appears is determined up to order k.

**Lemma 2.1.** For k ∈ Z⁺, there exists n ∈ Z⁺ such that any n-block appearing in x₀' is determined up to order k.

**Proof.** Since x₀' = x₀ k' and θ is admissible, we may assume k = 1. Now suppose the lemma is false. Then by recurrence of x₀', no block occurring in x₀' is determined up to order 1. We show

(i) For any t ∈ Z⁺, there is a block Pₜ of length (depending on t) so that x₀'(pₑ,(p+1)ₑ-1) = Pₜ (p = 0, 1, ..., t-1).

Let A₁ = x₀'(0, (t+1)r-1) = θ(i₀,i₁,...,iₜ). Since A₁ is not determined up to order 1, there exist i₀,i₁,...,iₜ₊₁ ∈ S with θ(i₀,i₁,...,iₜ₊₁) = B₁A₁B₁', for some blocks B₁, B₁' of positive length. Let A₂ = θ²(i₀,...,iₜ) = θ(A₁). Since A₂ is not determined up to order 1, there exist i₀², i₁²,...,iₜ⁺¹ ∈ S with θ²(i₀²,...,iₜ⁺¹) = B₂A₂B₂', for some blocks B₂, B₂' of positive length. Now we have θ²(i₀,i₁,...,iₜ₊₁) = θ(B₁)A₂θ(B₁'); but since A₂ is not determined up to order 1, we may assume that if iₚ = iₚ² (p = 0, 1, ..., t+1), then B₂ ≠ θ(B₁). If iₚ = iₚ² (p = 0, 1, ..., t+1), we stop; otherwise, let A₃ = θ(A₂). Since A₃ is not determined up to order 1,
there exist $i_0^3, i_1^3, \ldots, i_{t+1}^3 \in S$ with $\theta^3(i_0^3 \ldots i_{t+1}^3) = B_3 A_3 B_3'$, for some $B_3, B_3'$. Again, if $i_p^3 = i_1^p$ (p=0,\ldots, t+1), we may assume $B_3 \neq \theta^2(B_1)$, and if $i_p^3 = i_2^p$ (p=0, 1,\ldots, t+1), we may assume $B_3 \neq \theta(B_2)$.

We continue this process, stopping at the $a$th stage if $i_p^a = i_p^c$ (p = 0,1,\ldots,t+1) for some $c < a$. Eventually, since $S$ is finite, the process terminates; we have then, for some $a$ and some $c < a$, $\theta^a(j_0 \ldots j_{t+1}) = B_a A_a B_a'$ and $\theta^c(j_0 \ldots j_{t+1}) = B_c A_c B_c'$, where $B_a \neq \theta^{a-c}(B_c)$.

Thus, applying $\theta^{a-c}$ to the latter expression, $\theta^a(j_0 \ldots j_{t+1}) = B_a A_a B_a' = C_a A_a C_a'$ (B_a \neq C_a'). It follows that $A_a Q = P_t A_a$, for some blocks Q, $P_t$ of length less than $r^a$. Since $A_a$ has length $(t+1)r^a$, it is easy to see that this implies $A_a = P_t P_t \ldots P_t P'$ for some $P'$ (where $P_t$ is repeated $t$ times). Since $A_a = x^{(0, (t+1)r^a-1)}$, (i) is proved.

We prove now that under condition (i), $x^\theta$' must be periodic. By considering $\theta^j$ if necessary, we may assume that if any block of the form 0i_1 i_2 \ldots i_s 0 (each $i_k \neq 0$) appears in $x^\theta$' (where we include in blocks of this form the 2-block 00), then it appears in $\theta(0) = x^{(0, r-1)}$. Let $t = 2r^2$, let e be the length of $P_t$ (as in (i)), and choose k so that $r^k \leq te = 2r^2 e < r^{k+1}$, so that $2e < r^{k-1}$. We have $P_t P_t \ldots P_t = \theta^k(0)C$, for some C. (Here $P_t$ is repeated $t$ times.)
Consider all the distinct blocks of the form

\[ 0i_1i_2...i_s0 = A_I \quad (I = (i_1, i_2, ..., i_s); \ i_j \neq 0, \ j = 1, 2, ..., s) \] appearing in \( \theta(0) \). For such a block \( A_I \), let \( A_I = 0i_1i_2...i_s \). (The collection \( \{A_I\} \) will include the 1-block 0 if the 2-block 00 appears in \( \theta(0) \).) Then we may write \( \theta(0) = A_{I_1}A_{I_2}...A_{I_q} \) or \( \theta(0) = A_{I_1}...A_{I_q}B \), where in the second case \( B \) is the initial part of some \( A_I \). Let \( D_1 = \theta^{k-1}(A_{I_1}) \), and let \( D_1 \) be of length \( d_1 \) \((i = 1, 2, ..., q)\). We shall show that there is some block \( E \) so that for \( i \in Z(1,q) \), \( D_1 = EE...E \) (the number of occurrences depending on \( i \)). It will follow then that \( x_\theta' = EEE... \), and hence that \( x_\theta' \) is periodic.

We have \( \theta^k(0) = D_1D_2...D_q \) (or possibly \( D_1D_2...D_q\theta^{k-1}(B) \)). If \( e \mid d_1 \), let \( E^1 = P_t \) and \( e_1 = e \); if not, then since \( 2e < r^{k-1} \leq d_1 \) (for each \( i \)), we have, by looking at the first \( r^{k-1} \)-block of \( D_2 \), which is \( \theta^{k-1}(0) \), that \( P_tP_t = RP_tS \), for some blocks \( R \) and \( S \) of positive length. It is easy to see then that there is a block \( E^1 \) of length \( e_1 \) so that \( P_t = E^1E^1...E^1 \), and \( e_1 \) divides the lengths of \( R \) and \( S \). In either case, we obtain \( D_1 = E^1E^1...E^1 \). If \( e_1 \mid d_2 \), let \( E^2 = E^1 \) and \( e_2 = e_1 \). Otherwise, by looking at the first \( r^{k-1} \)-block of \( D_3 \), we again have \( E^1E^1 = R_1E^1S_1 \), which implies that \( E^1 = E^2E^2...E^2 \) for some \( E^2 \) of length \( e_2 \) dividing the lengths of \( R_1 \) and \( S_1 \). Hence, in either case, \( D_1 = E^2E^2...E^2 \) and
\[ D_2 = E_1^2 E_2^2 \ldots E_2^2. \] We continue this process, and finally we obtain that some \( E^i \) is the desired block \( E \). \( \square \)

III. The Flows \( x_\theta \) and \( \theta^r \).

We begin to discuss the flow generated by the sequence \( x_\theta' \). The first step is to extend \( x_\theta' \) to a recurrent bisquence \( x_\theta \). We note first that for any such extension \( x_\theta \), \( x_\theta(-r^k, -1) \) is a basic \( r^k \)-block for each \( k \in Z^+ \). For, let \( j > k \) be such that \( \theta^j(0) \) is determined up to order \( k \) (using 2.1) and let \( A = x_\theta(-r^k, r^j-1) \). A appears in \( x_\theta' \) by recurrence, and by the way \( j \) was chosen, \( A = x_\theta'(mr^k, nr^k-1) \) for some \( m, n \) (0 \(<= m < n \)). But then \( x_\theta(-r^k,-1) = x_\theta'(mr^k, (m+1)r^k-1) \), which is a basic \( r^k \)-block.

**Lemma 3.1.** (a) There is a recurrent bisquence \( x_\theta \) with \( x_\theta(n) = x_\theta'(n) \) \( (n \geq 0) \).

(b) For any such \( x_\theta \), there exist \( p \in Z^+, j_0 \in S \) such that \( x_\theta(-r^{ip}, -1) = \theta^{ip}(j_0) \) \( (i \geq 0) \).

(c) If \( x_\theta \) and \( y_\theta \) are two such extensions with \( x_\theta(-1) = y_\theta(-1) \), then \( x_\theta = y_\theta \).

Proof. (a) For \( k \in Z^+ \), \( \theta^k(0) = x_\theta'(m_k r^k, (m_k+1)r^k-1) \) for some \( m_k > 0 \), by recurrence. Then \( x_\theta'(m_k r^k, m_k r^k-1) = \theta^k(i_k) \) for some \( i_k \in S \). Choose an infinite
subset \( N_1 \) of \( \mathbb{Z}^+ \) such that \( \theta^k(i_k)(r^k-1) = j_0 \) for some \( j_0 \in S \) (\( k \in N_1 \)). Choose an infinite subset \( N_2 \subset N_1 \) such that \( \theta^k(i_k)(r^k-r,r^k-1) = \theta(j_1) \) for some \( j_1 \in S \) (\( k \in N_2 \)). Similarly, for each \( n \in \mathbb{Z}^+ \), choose an infinite subset \( N_{n+1} \subset N_n \) such that \( \theta^k(i_k)(r^k-r^n,r^k-1) = \theta^n(j_n) \) (some \( j_n \in S \)) for every \( k \in N_{n+1} \) greater than \( j \).

Define \( x_\theta \) as follows: \( x_\theta(-r^n,r^n-1) = \theta^n(j_n0) \) (\( n \geq 0 \)). We check that \( x_\theta \) is recurrent. Consider any block \( A = x_\theta(-p,p-1) \). \( A \) appears in \( x_\theta' \) by construction. Since \( x_\theta' \) is recurrent, there is an integer \( d \) such that \( A \) appears in any \( d \)-block of \( x_\theta' \). Now consider any \( d \)-block \( B = x_\theta(n,n+d-1) \). As before, \( B \) appears in \( x_\theta' \); hence it contains \( A \).

(b) By the opening remarks of this section, we have \( x_\theta(-r^n,-1) = \theta^n(j_n) \) (\( n \geq 0 \)). This means (recalling the definition of \( \theta^n \)) that \( \theta(j_{n+1})(r-1) = j_n \). It follows that in the sequence \( (j_n)_{n=0,1,...} \), each \( j_n \) has only one predecessor in the sequence. This implies that if \( a_1, a_2, ..., a_p \) are the distinct integers occurring in the sequence \( (j_n) \), then this sequence is periodic of period \( p \), and therefore that \( j_{ip} = j_0 \) (\( i \in \mathbb{Z}^+ \)); i.e., \( x_\theta(-r^{ip},-1) = \theta^{ip}(j_0) \) (\( i \geq 0 \)).

(c) Let \( x_\theta, y_\theta \) be recurrent extensions of \( x_\theta' \), and suppose \( x_\theta(-1) = y_\theta(-1) = j_0 \). Let \( \theta^n(j_n) = x_\theta(-r^n,-1) \), \( \theta^n(m_n) = y_\theta(-r^n,-1) \) (\( n \geq 0 \)). As in (b), the sequences
(j_n) and (m_n) are periodic of period p and q, respectively. Suppose p > q. We have j_p = m_q = j_0. By the proof of (b), j_{p-1} = m_{q-1}, j_{p-2} = m_{q-2}, \ldots, j_{p-q} = m_0 = j_0. But then the integer j_0 appears in the sequence (j_n) in two positions less than p apart; this is impossible, since (j_n) is periodic of period p, and there are p distinct integers in the sequence. Thus p = q, and it then follows that j_n = m_n (n \geq 0), which implies x_\theta = y_\theta. □

According to (b) of the preceding lemma, we could originally have defined x_\theta as follows: choose w \in S, p \in \mathbb{Z}^+ so that \theta^p(w)(r^p-1) = w; x_\theta is the bisequence defined by x_\theta(-r^k, r^k-1) = \theta^{kp}(w0) (k \geq 0). This is closer to the definition given in [3]. However, we think it is worth pointing out that with the aid of 2.1, we can deduce the structure of x_\theta from that of x_\theta'.

From now on, using 3.1(b) together with the fact that x_{\theta^k} = x_\theta' and \theta^k is admissible, we may take x_\theta to be a fixed recurrent extension of x_\theta' satisfying

1) x_\theta(-r^k, r^k-1) = \theta^k(w0) for k \geq 0 (some w \in S).

Definition. Let S_{(i)} = S (i \in \mathbb{Z}), with the discrete topology. Let X = \prod_{i=\infty}^{i=-\infty} S_{(i)}, with the product topology. If we define a metric d on X by d(x,y) = (k+1)^{-1} (where k = \min \{j \geq 0: x(-j,j) \neq y(-j,j)\}), then the metric and product topologies coincide, and X is
a compact metric space. Define $T: X \to X$ by $T(x(n)) = x(n+1)$. $T$ is called the **left shift** and is a homeomorphism of $X$ onto itself. Let the **orbit** of $x_\theta$ be the set $O(x_\theta) = \{T^n x_\theta: n \in \mathbb{Z}\} \subset X$, and define $X_\theta = O(x_\theta)^- \subset X$.

Then $T: X_\theta \to X_\theta$ is a homeomorphism onto, and since $x_\theta$ is recurrent, the flow $\pi_\theta \equiv (X_\theta, T)$ is a minimal flow (see [4]).

In general, of course, there is more than one way to extend $x_\theta$ to a recurrent bisequence. It is easily seen, however, that the space $X_\theta$ is independent of the particular choice for $x_\theta$. We note that an element $x \in X_\theta$ is a bisequence over $S$ such that for any $k \in \mathbb{Z}^+$, $x(-k, k-1)$ appears in $x_\theta$. In particular, for $m \in \mathbb{Z}^+$, we choose $k$ (using 2.1) so that any $2k$-block of $x_\theta$ is determined up to order $m$. We then obtain

(2) For $x \in X_\theta$ and $m \in \mathbb{Z}^+$, there is a unique $j_m \in Z(0, r^m-1)$ such that $x(j_m + nr^m, j_m + (n+1)r^m) = A$ is a basic $r^m$-block ($n \in \mathbb{Z}$).

Now suppose $x \in X_\theta$, and let $A$ be an $n$-block appearing in $x$ which is determined up to order $k$, for some $k$. Let $N = \{m \in \mathbb{Z}: x(m, m+n-1) = A\}$, and let $p, q \in N$. Let $j_k$ be as in (2), and let $i_1, i_2$ be the smallest integers so that $j_k + i_1 r^k \geq p+1$, $j_k + i_2 r^k \geq q+1$. Then if $j_k + i_1 r^k - p \neq j_k + i_2 r^k - q$, we have a contradiction of the assumption that $A$ is determined up to order $k$. 
We thus obtain

(3) If \( x \in X_0 \), \( A \) appears in \( x \) and is determined up to order \( k \), and \( x(p,p+n-1) = x(q,q+n-1) = A \), then \( p \equiv q \pmod{r^k} \).

**Definition.** \( Z^r \) will denote the group of \( r \)-adic integers, or the \( r \)-adic group. We view this as the set of all sequences \( z_0z_1z_2 \ldots \), where \( z_i \in \mathbb{Z}(0,r-1) \) for \( i \geq 0 \). Such a sequence represents the formal \( r \)-adic expansion \( z_0 + z_1 r + z_2 r^2 + \ldots \), and the group operation is defined accordingly. We define a metric \( d \) on \( Z^r \) as follows: \( d((a_i),(b_i)) = (k+1)^{-1} \), where \( k = \min \{ j \geq 0 : a_j \neq b_j \} \). \( T : Z^r \to Z^r \) is the homeomorphism of \( Z^r \) onto itself corresponding to addition of the group element \( 1000\ldots \). \( \varrho^r \) is the flow \( (Z^r, T) \). By an "integer" in \( Z^r \), we mean any element of the form \( T^n(000\ldots) \) \((n \in \mathbb{Z})\); a "non-integer" is any element not of this form.

It is easily checked that \( Z^r \) is a compact abelian topological group, and hence that \( \varrho^r \) is a minimal equicontinuous flow.

Frequently in what follows, we shall need to assume that our substitution is \( \varphi^k \), for some \( k > 1 \). It is necessary to observe in this regard that \( \mathbb{Z}^n = \mathbb{Z}^p \times \mathbb{Z}^q \times \ldots \times \mathbb{Z}^s \), where \( p,q,\ldots,s \) are the distinct primes dividing \( n \); in particular, \( Z^r^k = Z^r \), for any positive integer \( k \). \( \varrho^r \) and \( \varrho^r^k \) are isomorphic flows. (Here,
and subsequently, we use \( \cong \) to denote an isomorphism—either a group isomorphism or a flow isomorphism, depending upon the context.

IV. **Construction of a Flow Homomorphism.**

**Lemma 4.1.** There exists a flow homomorphism \( f: X_\theta \to \mathbb{R}^r \).

Proof. We define \( f \) on the orbit of \( x_\theta \) by \( f(T^k x_\theta) = k \in \mathbb{Z}^r \ (k \in \mathbb{Z}) \). Clearly \( f(Tx) = Tf(x) \ (x \in O(x_\theta)) \).

We must therefore show that \( f \) can be extended continuously to \( X_\theta \). The first step is

(i) If \( \lim_{k \to \infty} T^{n(k)} x_\theta \) exists in \( X_\theta \), then \( \lim_{k \to \infty} n(k) \) exists in \( \mathbb{Z}^r \).

For \( m \in \mathbb{Z}^+ \), choose \( k_0 \) such that if \( j, k \geq k_0 \), then \( T^j x_\theta (-s, s-1) = T^k x_\theta (-s, s-1) \), where \( s \) is a positive integer such that any \( s \)-block of \( x_\theta \) is determined up to order \( m \). By III (3), we have that if \( j, k \geq k_0 \), then \( n(j) \equiv n(k) \ (\text{mod } r^m) \), and thus \( b(n(j), n(k)) \leq (m+1)^{-1} \).

This establishes (i).

Now if \( \lim_{k \to \infty} T^{n(k)} x_\theta = y = \lim_{k \to \infty} T^{m(k)} x_\theta \), it follows immediately from (i), by forming the sequence in which \( n(k) \) and \( m(k) \) are alternated, that \( m(k) \) and \( n(k) \) converge to the same limit in \( \mathbb{Z}^r \). Thus for \( x \in X_\theta \), we may
define \( f(x) = \lim_{k \to \infty} n(k) \in \mathbb{Z}^r \), where \( (n(k)) \) is any sequence of integers such that \( \lim_{k \to \infty} T^n(k)x_\theta = x \).

We claim that \( f: X_\theta \to \mathbb{Z}^r \), defined in this way, is continuous. Suppose \( \lim_{n \to \infty} y_n = z \) (\( y_n, z \in X_\theta \)), and let \( m \in \mathbb{Z}^+ \). Choose \( n_0 \) so that for \( n \geq n_0 \), \( y_n(-s,s-1) = z(-s,s-1) \), where any \( s \)-block of \( x_\theta \) is determined up to order \( m \). Then it is easy to see, using the definition of \( f \) and III (3), that \( b(f(y_n), f(z)) \leq (m+1)^{-1} \) (\( n \geq n_0 \)).

Finally, \( f \) is surjective, since its range contains the dense subset \( Z \) of \( \mathbb{Z}^r \) and is closed, being the continuous image of the compact space \( X_\theta \). \( \square \)

We establish some facts about the homomorphism \( f \) which will be helpful later.

1. For \( x \in X_\theta \), \( z = z_0z_1... \in \mathbb{Z}^r \), and \( k \in \mathbb{Z}^+ \), we denote by \( x(z; k+1) \) the \( r^{k+1} \)-block \( x(-(z_0+z_1r^1+...+zkr^k), -(z_0+z_1r^1+z_2r^2^2+...+z_kr^k) + r^{k+1}-1) \).

**Lemma 4.2.** Let \( x \in X_\theta \), \( z = z_0z_1... \in \mathbb{Z}^r \). Then \( f(x) = z \) if and only if \( x(z; k+1) \) is a basic \( r^{k+1} \)-block for each \( k \in \mathbb{Z}^+ \).

**Proof.** It is easily checked that if \( n = z_0+z_1r^1+...+z_kr^k + qr^{k+1} \), then \( T^n x_\theta(z; k+1) \) is a basic \( r^{k+1} \)-block. Now suppose \( f(x) = z \). Choose a sequence \( (n(j)) \) (\( n(j) > 0, j \in \mathbb{Z}^+ \)) such that \( \lim_{j \to \infty} T^n(j)x_\theta = x \). Let \( k \in \mathbb{Z}^+ \); we have \( \lim_{j \to \infty} n(j) = z \in \mathbb{Z}^r \), and thus, for sufficiently large \( j \), \( n(j) = z_0+z_1r^1+...+z_kr^k+q(j)r^{k+1} \) and...
\[ T_n(j) x_\theta (-r^{k+1}, r^{k+1}) = x(-r^{k+1}, r^{k+1}) \]. Using the initial remark, it follows that \( x(z; k+1) \) is a basic \( r^{k+1} \)-block.

On the other hand, suppose \( x \in X_\theta \) has the property of the lemma, and let \( x = \lim_{j \to \infty} T_n(j) x_\theta \). We must show \( \lim_{j \to \infty} n(j) = z \in Z^r \). Let \( k \in \mathbb{Z}^+ \), and choose \( m \in \mathbb{Z}^+ \) so that every \( r^{m+1} \)-block of \( x_\theta \) is determined up to order \( k+1 \). For sufficiently large \( j \), we have \( T_n(j) x_\theta (z; m+1) = x(z; m+1) \), which is a basic \( r^{m+1} \)-block. Now \( T_n(j) x_\theta (-n(j), -n(j) + r^{m+1} - 1) \) is a basic \( r^{m+1} \)-block, and thus for some \( p \in \mathbb{Z} \), \( T_n(j) x_\theta (-n(j) + pr^{m+1}, -n(j) + (p+1)r^{m+1} - 1) = T_n(j) x_\theta (z; m+1) \). But then our assumption on \( m \), together with III(3), implies that \( n(j) = z_0 + z_1 r^1 + \ldots + z_k r^k + q r^{k+1} \), for some \( q \). It follows then that \( \lim_{j \to \infty} n(j) = z \).

**Lemma 4.3.** Let \( 1 \leq n \leq r-2 \), and let \( z = nnn \ldots \in Z^r \).

If \( x, y \in X_\theta \), \( f(x) = f(y) = z \), and \( x(0) = y(0) \), then \( x = y \).

Proof. The proof uses Lemma 4.2, but is otherwise almost identical to that of 3.1(c).

Now, for certain substitutions \( \theta \), it happens that each symbol in \( S \) appears in \( x_\theta \) only in positions differing from each other by integral multiples of some integer greater than 1. This fact turns out to be crucial in determining the equicontinuous structure relation on \( X_\theta \) (Section V). The following remarks and definitions are necessary preliminaries.
Definition. If \( n \in \mathbb{Z}^+ \), and \( n = p_1 p_2 \ldots p_k \), where each \( p_i \) is prime (some may be repeated), we denote by \( n^* \) the product of all the \( p_i \)'s not dividing \( r \). Let \( M = \{ n \in \mathbb{Z}^+: x_\theta(n-1,n) = w0 \} \) (where \( w \) is as in III(1)). Let \( M' = \{ n \in \mathbb{Z}^+: x_\theta(n) = 0 \}; d'_\theta = \text{g.c.d. } M'; d_\theta = \text{g.c.d. } M; m'(\theta) = (d'_\theta)^*; m(\theta) = d_\theta^* \).

Lemma 4.4. \( m(\theta) = m'(\theta) \).

Proof. Choose \( a_1, a_2, \ldots, a_t \in M \) so that \((\text{g.c.d.} \{a_1, a_2, \ldots, a_t\})^* = m(\theta) \); i.e., \( \text{g.c.d.} \{a_1, \ldots, a_t\} = qm(\theta) \), where \( q \) divides some power of \( r \). Choose \( u, v \) so that \( \theta^u(0)(v-1,v) = w0 \). By definition of \( M \), we have \( x_\theta(a_i r^u + v-1, a_i r^u + v) = w0 \) (\( i = 1, 2, \ldots, t \)). We obtain \( m'(\theta) | a_i r^u + v \) and \( m'(\theta) | v \); hence \( a_i r^u = b_i m'(\theta) \) (\( i = 1, 2, \ldots, t \)). But since \( m'(\theta) \) does not divide any power of \( r \), we have \( m'(\theta) | a_i \) (\( i = 1, 2, \ldots, t \)), which implies \( m'(\theta) | m(\theta) \). We certainly have \( m(\theta) | m'(\theta) \), however, and this implies the result. \( \square \)

Definition. For \( x \in X_\theta \), let \( p(x) = \min \{ i \geq 0: x(i) = 0 \} \). Let \( g(x) = -p(x) \mod m(\theta) \). We have \( g: X_\theta \rightarrow Z_m(\theta) \) (the cyclic group of order \( m(\theta) \)). We denote by \((g, f)\) the map from \( X_\theta \) to \( Z_m(\theta)^* Z^r \) given by \((g, f)(x) = (g(x), f(x)) \). Finally, we let \( R^{r, m(\theta)} \) be the flow \((Z_m(\theta)^* Z^r, T)\), where \( T \) corresponds to addition of the group element \((1,1)\).

Lemma 4.5. \((g, f)\) is a flow homomorphism from \( X_\theta \)
to the minimal equicontinuous flow $\mathcal{R}^r, m(\theta)$.

Proof. Since $x_\theta$ is recurrent, there is an integer $n$ with $p(x) \leq n$ ($x \in X_\theta$); it follows easily that $g$ is continuous. The formula $(g,f)(Tx) = T(g,f)(x)$ follows from the fact that $p(Tx) = p(x)-1$. To show $(g,f)$ surjective, it suffices to show that $\mathcal{R}^r, m(\theta)$ is minimal; but this is an immediate consequence, using the Chinese Remainder Theorem, of the fact that $m(\theta)$ and $r$ are relatively prime. \qed

(2) We may assume below, using an appropriate power of $\theta$ if necessary, that 0 appears in each of the $r$-blocks $\theta(i)$ ($i \in S$).

**Definition.** For $i \in S$, let $z(i) = \min \{n \geq 0: \theta(i)(n) = 0\} \pmod{m(\theta)}$. For $i, j \in S$, we say $i$ is equivalent to $j$ ($i \sim j$) if $z(i) = z(j)$. For $p \in Z(0, m(\theta)-1)$, we let $S_p = \{i \in S: z(i) = -p \pmod{m(\theta)}\}$; $S_p^* = \{\theta(i): i \in S_p\}$.

**Lemma 4.6.** (a) For $x \in f^{-1}(z_0 z_1 z_2 \ldots)$, $g(x) = i$ if and only if $x(-z_0, -z_0+r-1) \in S_n^*$, where $n = i - z_0 \pmod{m(\theta)}$.

(b) For $x \in X_\theta$, $g(x) = j$ if and only if $x(0) \in S_p$, where $p = jr \pmod{m(\theta)}$.

(c) If $i, j \in S$, and $i \sim j$, then $\theta^k(i)(n) \sim \theta^k(j)(n)$ ($k \in Z^+, n \in Z(0, r^k - 1)$).

(d) If $\theta^k(i)(n) \sim \theta^k(j)(n)$ for some $n \in Z(0, r^k - 1)$,
then \( i \sim j \).

**Proof.** (a) We have \( z(0) = 0 \). Now choose \( n \in \mathbb{Z}(r, 2r-1) \) with \( x_\theta(n) = 0 \). Then \( n-r = \min \{ j : x_\theta(r, 2r-1)(j) = 0 \} \pmod{m(\theta)} \), and since \( n = 0 \pmod{m(\theta)} \), we obtain \( z(x_\theta(1)) = -r \pmod{m(\theta)} \). Similarly, \( z(x_\theta(2)) = -2r \pmod{m(\theta)} \); \ldots; \( z(x_\theta(m(\theta))) = 0 \); \( z(x_\theta(m(\theta)+1)) = -r \pmod{m(\theta)} \); \ldots. It is evident, then, that each of the equivalence classes \( S_p \) is non-empty, and that the sequence \( (z(x_\theta^*(n))) \) is periodic of period \( m(\theta) \). It follows that for \( x \in f^{-1}(0) \), \( g(x) = 1 \) if and only if \( x(0, r-1) \in S_1^* \) (\( i \in Z(0, m(\theta)-1) \)). More generally, for \( z = z_0 z_1 \ldots \in \mathbb{Z}^r \) and \( x \in f^{-1}(z) \), then \( x(-z_0, -z_0+r-1) = \theta(i_x) \) (some \( i_x \in S \)), by 4.2, and \( g(x) = z_0 - z(i_x) \pmod{m(\theta)} \); hence, \( g(x) = 1 \) if and only if \( i_x \in S_n \), where \( n = i - z_0 \pmod{m(\theta)} \) --i.e., if and only if \( x(-z_0, -z_0+r-1) \in S_n^* \).

(b) By the periodicity of the sequence \( (z(x_\theta^*(n))) : n \geq 0 \), we have that \( g(x) = \min \{ k \geq 0 : z(x(k)) = 0 \} \pmod{m(\theta)} \). Thus \( g(x) = j \) if and only if \( z(x(m(\theta)-j)) = 0 \); this is true just in case \( z(z(0)) = -jr \pmod{m(\theta)} \).

(c) For \( k = 1 \), this follows from the periodicity of \( (z(x_\theta^*(n))) \) and the definition of \( \sim \); a straightforward induction argument proves the general case.

(d) is proved similarly. \( \Box \)

We conclude this section by proving a lemma which
describes how \( m(\theta) \) can be calculated for a particular substitution \( \theta \).

**Definition.** \( P_\theta \) is the set of 2-blocks appearing in \( x_\theta \); \( p_\theta \) is the number of elements in \( P_\theta \).

**Lemma 4.7.** Suppose \( x_\theta(0,k) \) contains each of the 2r-blocks \( \theta(ij) \) (\( ij \in P_\theta \)). Then \( m(\theta) = (\text{g.c.d.} \{ n:\ 0 \leq n \leq k, \ x_\theta(n) = 0 \})^* \).

**Proof.** We recall that \( m(\theta) = (\text{g.c.d.} M)^* \); if \( M = \{ n_1, n_2, \ldots \} \), and we let \( n_0 = 0 \), it is easy to verify that \( m(\theta) = (\text{g.c.d.} \{ n_{j+1}-n_j : j \geq 0 \})^* \). But we claim that \( \{ n_{j+1}-n_j : n_j \leq k \} = \{ n_{j+1}-n_j : j \geq 0 \} \). To see this, we note that by (2), \( x_\theta(n_j, n_{j+1}) \) appears in some 2r-block \( \theta(pq) \) (\( pq \in P_\theta \)); but \( \theta(pq) \) appears in \( x_\theta(0,k) \).

Finally, we point out that finding the set \( P_\theta \) is a fairly simple matter. We let \( P_1 = \{ ij \in S^2 : ij \) appears in \( \theta(0) \} \). For \( k > 1 \), we let \( P_k = \{ ij \in S^2 : ij \) appears in \( \theta(pq) \) for some \( pq \in P_{k-1} \} \). We have \( P_k \subset P_{k-1} \) since \( S^2 \) is finite, eventually \( P_k = P_{k+1} \), and it is clear that \( P_k = P_\theta \) for this \( k \).

V. **Calculation of the Structure Transformation Group of** \( X_\theta \).

**Definition.** [1] \( \Sigma_\theta \subset X_\theta \times X_\theta \) is the equicontinuous structure relation on \( X_\theta \); i.e., the smallest closed,
invariant equivalence relation \( R \) on \( X_\theta \) such that the
flow \( (X_\theta/R, T) \) is equicontinuous. Let \( X_\theta^* = X_\theta/\Sigma_\theta \);
\( x_\theta^* = (X_\theta^*, T) \). \( x_\theta^* \), which is a homomorphic image of
\( x_\theta \) and hence a minimal equicontinuous flow, is called
the **structure transformation group** of \( x_\theta \). For \( x \in X_\theta \),
we denote by \([x]\) the equivalence class \( x\Sigma_\theta \) of \( x \).

**Lemma 5.1.** [4] There is a unique group structure
on \( X_\theta^* \) making it into a compact abelian topological
group such that the map \( \pi: \mathbb{Z} \to X_\theta^* \) defined by \( \pi(n) = [T^n x_\theta] \) is a group homomorphism. With respect to the
group operation \( \ast \), \([x_\theta]\) is the identity element; and
if \( x = \lim_{k \to \infty} T^n(k)x_\theta \), \( y = \lim_{k \to \infty} T^m(k)x_\theta \), we have \([x] \ast [y] = \lim_{k \to \infty} [T^n(k)+m(k)x_\theta] \).

**Lemma 5.2.** There is a one-to-one correspondence
between (a) continuous characters on the compact abelian
topological group \( X_\theta^* \); and (b) continuous maps \( h: X_\theta \to \Gamma \)
(the compact group of all complex numbers of norm 1)
satisfying \( h(x_\theta) = 1 \) and \( h(Tx) = c_h h(x) \) (for \( x \in X_\theta \) and
some \( c_h \in \Gamma \)).

**Proof.** If \( \pi: X_\theta \to X_\theta^* \) is the projection and \( \sigma \) is
a continuous character on \( X_\theta^* \), it is easy to check that
by letting \( h = h_\sigma = \sigma \circ \pi \) and \( c_h = h(Tx_\theta) \), we obtain a
map \( h_\sigma \) as above.

On the other hand, if \( h: X_\theta \to \Gamma \) is such a map,
and \( \Sigma \) is the equivalence relation on \( X_\theta \) induced by \( h \),
then \((X_\theta^*/\Sigma, T)\) is isomorphic to \((\Gamma_1, T)\), where \(\Gamma_1\) is a closed subgroup of \(\Gamma\) and \(T\) is group multiplication by \(c_h\). Thus \((X_\theta^*/\Sigma, T)\) is equicontinuous, and hence \(\Sigma_\theta \subset \Sigma\).

We may then define \(\sigma_h\) on \(X_\theta^*\) by \(\sigma_h[x] = h(x)\), and it is easy to see that \(\sigma_h\) is a continuous character on \(X_\theta^*\).

Finally, we have \(\sigma(h_{\sigma}) = \sigma; h(\sigma_h) = h\). This implies that the correspondence is one-to-one. \(\square\)

In the remainder of this section we use \(e(a)\) to denote \(\exp(2\pi ia)\), for real numbers \(a\).

**Lemma 5.3.** [6] (a) Let \(x = e(nr^{-k})\) \((k, n \geq 0)\). Then the formula \(\chi_x(1) = x\) can be extended uniquely to a continuous character \(\chi_x\) on \(\mathbb{Z}^r\); and the map taking \(x\) to \(\chi_x\) defines an isomorphism from the group \(U\) of all \((r^j)\)th roots of unity onto the character group of \(\mathbb{Z}^r\).

(b) If \(Z_n\) is the cyclic group of order \(n\), and \(Z_n'\) is its character group, then \(\varphi: Z_n \rightarrow Z_n'\), defined by \(\varphi(m) = \chi_m\) (where \(\chi_m(j) = e(jmn^{-1})\)), is an isomorphism onto.

**Definition.** \(\operatorname{Ker} (g, f)\) is the closed, invariant equivalence relation on \(X_\theta^*\) induced by the map \((g, f)\): \(i_\theta \rightarrow i^{r, m(\theta)}\). Since \(i^{r, m(\theta)}\) is equicontinuous, we have \(\Sigma_\theta \subset \operatorname{Ker}(g, f)\), and we may therefore define \(q: X_\theta^* \rightarrow \mathbb{Z}^{m(\theta)} \times \mathbb{Z}^r\) by \(q([x]) = (g, f)(x)\) \((x \in X_\theta^*)\). It is easily verified that \(q\) is a continuous group homomorphism;
\(q: i_{\theta}^* \rightarrow i^{r, m(\theta)}\) is a flow homomorphism; and if \((\sigma, \chi) \in \)
$Z_m(\theta)' \times U$ is a continuous character on $Z_m(\theta) \times Z^r$, then $(\sigma, \chi) \circ q$ is a continuous character on $X_\theta^*$.

**Lemma 5.4.** If $h$ is any continuous character on $X_\theta^*$, then $h = (\sigma, \chi) \circ q$, for some $(\sigma, \chi) \in Z_m(\theta)' \times U$.

**Proof.** Let $h_1 : X_\theta \to \Gamma$ correspond to $h$ as in 5.2. Suppose $h_1 T x = w h_1(x)$, where $w = e(y)$. By definition of $M'$ (see IV) and by 4.4, we may choose $a(1), a(2), \ldots, a(t) \in M'$ so that $\text{g.c.d.} \{a(i) : i = 1, 2, \ldots, t\} = m(\theta)p$, where $r^k = ps$ (some $k, s \in Z^+$). It is easily checked, using the definition of $M'$, that for $i = 1, 2, \ldots, t$, we have $\lim_{j \to \infty} T a(i) r^j x_\theta = x_\theta$. This implies that $\lim_{j \to \infty} e(y a(i) r^j) = 1$ ($i = 1, 2, \ldots, t$); but it follows that for some integer $j(i)$, $e(y a(i) r^j(i)) = 1$ (each $i$). We choose integers $b(i)$ with $a(0)b(0) + \ldots + a(t)b(t) = m(\theta)p$, and we obtain $s(a(0)b(0) + \ldots + a(t)b(t)) = m(\theta)sp = m(\theta)r^k$. It follows from these two facts that $w = e(nm(\theta)^{-1} r^{-j-k})$, for some $j, n \in Z^+$. If integers $u$ and $v$ are chosen so that $um(\theta) + vr^{j+k} = 1$, it can easily be seen that $h([T x_\theta]) = h_1(T x_\theta) = w = (\sigma, \chi) \circ q([T x_\theta])$, where $\sigma$ and $\chi$ are the characters on $Z_m(\theta)$ and $Z^r$, respectively, determined by the formulae $\sigma(1) = e(nvm(\theta)^{-1})$; $\chi(1) = e(nur^{-j-k})$. Since the continuous characters $h$ and $(\sigma, \chi) \circ q$ then agree on the orbit of $[x_\theta]$, they must be the same. □

**Theorem 5.5.** $q$ is an isomorphism and homeomorphism
from \( X_\theta \) onto \( Z_m(\theta) \times Z^r \), and hence \( x_\theta \) and \( s^r, m(\theta) \) are isomorphic flows.

Proof. \( q \) is a continuous surjective homomorphism, and 5.4, together with the fact that continuous characters on any locally compact abelian topological group separate points, implies \( q \) is injective. \( \Box \)

**Corollary 5.6.** If \( \theta \) is a binary substitution (i.e., \( b = 2 \)), then \( x_\theta \simeq s^r \).

Proof. Since \( x_\theta \) is not periodic one of the 2-blocks 00 and 11 appears in \( x_\theta \). In the first case \( m(\theta) = 1 \), obviously; if \( x_\theta(n,n+1) = 11 \) (\( n > 0 \)), we choose \( k \in Z(0,r-1) \) such that \( \theta(1)(k) = 0 \), and we obtain \( x_\theta(nr + k) = x_\theta((n+1)r + k) = 0 \). Since \( m(\theta) \) divides \( nr + k \) and \( (n+1)r + k \), and is relatively prime to \( r \), we have \( m(\theta) = 1 \), and the result follows from 5.5. \( \Box \)

**Example.** Let \( b = r = 3 \); \( \theta(0) = 012, \theta(1) = 101, \theta(2) = 210 \). We observe that \( \theta \) is admissible, and we find that \( P_\theta = \{01,12,21,10\} \). Using 4.7, \( m(\theta) = (\text{g.c.d.} \{ n : 1 \leq n \leq 14, x_\theta(n) = 0 \})^* = 2 \). By 5.5, we obtain \( x_\theta^* \simeq s^{3,2} \).

Our Theorem 5.5 generalizes a part of Theorem 2.05 in [3]
VI. **Almost Automorphic Substitution Flows.**

**Definition.** [12] If $\xi = (X, T)$ is a minimal flow, a point $x \in X$ is an **almost automorphic point of** $X$ if, whenever $\{n(k): k \in \mathbb{Z}^+\}$ is a sequence of integers so that $\lim_{k \to \infty} T^{n(k)} x = y$ and $\lim_{k \to \infty} T^{-n(k)} y = z$ for some $y, z \in X$, then $x = z$. A point $x$ is an almost automorphic point if and only if $x \Sigma = \{x\}$, where $\Sigma$ is the equicontinuous structure relation on $X$; i.e., if and only if $\pi^{-1} \pi x = \{x\}$, where $\pi: X \to X/\Sigma = X_\pi$ is the projection map. $\xi = (X, T)$ is **almost automorphic** if there is at least one almost automorphic point. If $\xi = (X, T)$ and $\eta = (Y, T)$ are flows and $h: \xi \to \eta$ is a flow homomorphism, $\xi$ is an **almost automorphic extension** of $\eta$ relative to $h$ if $h^{-1} h(x) = \{x\}$ for some $x \in X$.

We shall characterize those admissible substitutions $\theta$ which give rise to almost automorphic flows. From Section V and 5.5, we obtain the following.

**Lemma 6.1.** $\text{Ker}(g, f) = \Sigma_\theta$; hence, $x \in X_\theta$ is an almost automorphic point of $X_\theta$ if and only if $(g, f)^{-1}(g, f)(x) = \{x\}$.

**Definition.** If $w \in S$ is as in III(1), let $S^w$ be the equivalence class of $S$ containing $w$. (See IV).

**Theorem 6.2.** (a) $\xi_\theta$ is an almost automorphic flow if and only if, for some $i \in \mathbb{Z}(0, m(\theta) - 1)$, there are
integers $k$ and $m$ ($k \in \mathbb{Z}^+, m \in \mathbb{Z}(0, r^{k-1})$) such that if $p, q \in S_1$, $\theta^k(p)(m) = \theta^k(q)(m)$.

(b) $x_\theta$ is an almost automorphic point of $X_\theta$ if and only if there exists $k \in \mathbb{Z}^+$ such that for $p \in S_0$, $\theta^k(p)(0) = 0$, and for $p, q \in S^w$, $\theta^k(p)(r^{k-1}) = \theta^k(q)(r^{k-1})$.

Proof. (a) First we prove sufficiency; suppose the condition in (a) holds. We may replace $\theta$ by an appropriate power of $\theta$, so as to assume $k = 1$. We recall that if $\theta(p)(0) = \theta(q)(0)$, then $\theta^2(p)(0, r-1) = \theta^2(q)(0, r-1)$; similarly, if $\theta(p)(r-1) = \theta(q)(r-1)$, then $\theta^2(p)(r^2-r, r^2-1) = \theta^2(q)(r^2-r, r^2-1)$. Therefore, by taking an even higher power of $\theta$ if necessary, we may assume $1 \leq m \leq r-2$. Let $z = \ldots \in \mathbb{Z}^r$; $f^{-1}(z) = \{x_1, x_2, \ldots, x_n\}$. We may assume by re-ordering this set that $\{x_1, x_2, \ldots, x_u\} = \{x \in f^{-1}(z) : x(-m, -m+r-1) \in S_1^\ast\}$ (see definition in IV), where $i$ is as in the condition in (a). Then using 4.6, we obtain $\{x_1, x_2, \ldots, x_u\} = (g, f)^{-1}(g, f)\{x_1, x_2, \ldots, x_u\}$. The assumption on $i$ implies that for $1 \leq s \leq t \leq u$, $x_s(0) = x_t(0)$; but this implies, using 4.3, that $x_1 = x_2 = \ldots = x_u$, or that $u = 1$. We have then that $(g, f)^{-1}(g, f)(x_1) = \{x_1\}$, and $x_1$ is an almost automorphic point of $X_\theta$.

For the converse, we assume that for $i \in \mathbb{Z}(0, m(\theta)-1)$, $k \in \mathbb{Z}^+$, and $m \in \mathbb{Z}(0, r^{k-1})$, there are integers $p, q \in S_1$ with $\theta^k(p)(m) \neq \theta^k(q)(m)$. Let $(j, z) = (j, z_0z_1\ldots) \in \ldots$.
$Z_m(\theta)^x Z^r$. We shall prove $(g,f)^{-1}(j,z)$ contains two distinct points. It will be sufficient to prove:

(i) If $(g,f)(T^s x_\theta) = (j, z_0 z_1 \ldots z_t w_{t+1} w_{t+2} \ldots)$, then there exists $u \in \mathbb{Z}$ such that $(g,f)(T^u x_\theta) = (j, z_0 z_1 \ldots z_t w_{t+1} w_{t+2} \ldots)$ and $T^u x_\theta(0) \neq T^s x_\theta(0)$.

(For then, if $s(a) (a \in \mathbb{Z}^+) is any sequence of integers such that $x = \lim_{a \to \infty} T^s(a) x_\theta \in (g,f)^{-1}(j,z)$, we may find a sequence $u(a) (a \in \mathbb{Z}^+)$ such that

$\lim_{a \to \infty} (g,f)(T^u(a) x_\theta) = (j,z)$ and $T^u(a) x_\theta(0) \neq T^s(a) x_\theta(0)$.

By choosing a subsequence, we may assume $\lim_{a \to \infty} T^u(a) x_\theta = y$, and thus $x,y \in (g,f)^{-1}(j,z)$, but $x(0) \neq y(0)$.)

Let $(g,f)T^s x_\theta = (j, z_0 z_1 \ldots z_t w_{t+1} w_{t+2} \ldots) \in Z_m(\theta)^x Z^r$

Consider all integers of the form $u(n) = s + nm(\theta)r^{t+1}$ $(n \in \mathbb{Z})$. We have $(g,f)(T^u(n) x_\theta) = (j, z_0 \ldots z_t w_{t+1} \ldots)$.

It will evidently suffice, then, to prove

(ii) For $s \in \mathbb{Z}$, $t \in \mathbb{Z}^+$, there exists $n \in \mathbb{Z}$ such that $x_\theta(s) \neq x_\theta(s + nm(\theta)r^{t+1})$.

In proving (ii), we may assume as usual that $t=0$ and $0 \leq s \leq r-1$. But since $\{x_\theta(nm(\theta)r, (nm(\theta)+1)r-1); n \in \mathbb{Z}\} = S_0^*$ (by 4.6), the statement follows from our assumption.

(b) We prove first the sufficiency of the condition; we may assume $k = 1$, as usual. Let $y \in (g,f)^{-1}(0,0)$. Then $y(-r^j, r^j - 1) = \theta^j(i_j', i_j) (j \in \mathbb{Z}^+, \text{some } i_j', i_j \in S)$. This implies $\theta(i_{j+1}')(r-1) = i_j'$ and
\[ \theta(i_{j+1})(0) = i_j \quad (j \in Z^+) \]. Now \( i_{1}' \in S^w \) and \( i_1 \in S_0 \) by 4.6, and an induction argument shows \( i_{j}' \in S^w \) and \( i_j \in S_0 \) for each \( j \). By hypothesis \( i_{j}' = w \) and \( i_j = 0 \) for each \( j \), which implies \( y = x_\theta \). We conclude that \( x_\theta \) is an almost automorphic point.

Now suppose the condition fails to hold; then for \( k \in Z^+ \), either \( \theta^k(p)(0) \neq 0 \) for some \( p \in S_0 \), or \( \theta^k(p)(r^{k-1}) \neq \theta^k(q)(r^{k-1}) \) for some \( p, q \in S^w \). If the former holds for arbitrarily large \( k \), then it clearly holds for every \( k \); similarly for the latter. We assume, then, that for each \( k \in Z^+ \), there exists \( p \in S_0 \) with \( \theta^k(p)(0) \neq 0 \). The proof for the other case is almost identical.

Let \( m \in Z^+ \); as in (a), it is sufficient to prove that for some \( n \), \( (g,f)(T^n x_\theta) = (0, 00...0w_m1w_{m+1}w_{m+2}...) \) and \( T^n x_\theta(0) \neq 0 \). This will be true if we can find \( q \in Z \) with \( x_\theta(qm(\theta)r^m) \neq 0 \) (let \( n = qm(\theta)r^m \)). But now, exactly as in (a), we may assume \( m = 1 \), and since \( \{x_\theta(qm(\theta)r^r, (qm(\theta) + 1)r-1) : q \in Z \} = S_0^x \), we have \( x_\theta(qm(\theta)r) \neq 0 \) for some \( q \), by our assumption. \( \square \)

**Corollary 6.3.** If \( m(\theta) > \frac{1}{2}b \), then \( x_\theta \) is almost automorphic.

**Proof.** This is immediate, since for some \( i \), \( S_i \) contains only one element. \( \square \)

**Definition.** An admissible substitution \( \theta \) of length
r will be called simple if, for \(i, j \in S, i \neq j\), we have 
\[\theta(i)(m) \neq \theta(j)(m) \quad (m \in Z(0, r-1)).\]
We note that if \(\theta\) is a simple substitution, then \(\theta^k\) is also \((k \in Z^+).\)

**Lemma 6.4.** If \(\theta\) is a simple substitution, then for \(i, j \in Z(0, m(\theta)-1), \text{ card } S_i = \text{ card } S_j.\)

**Proof.** We may assume that for \(i \in S, \text{ range } \theta(i) = S.\) Suppose \(\text{ card } S_i = p > q = \text{ card } S_j.\) Choose \(n \in S_i, s \in Z(0, r-1), \) with \(\theta(n)(s) \in S_j.\) Then by 4.6, \(\theta(m)(s) \in S_j \quad (m \in S_i).\) Since \(p > q, \) we obtain \(\theta(m_1)(s) = \theta(m_2)(s)\) for some \(m_1, m_2 \in S_i\) with \(m_1 \neq m_2,\) contradicting the assumption that \(\theta\) is simple. \(\square\)

**Corollary 6.5.** If \(\theta\) is simple, \(x_\theta\) is not almost automorphic.

**Proof.** From the proof of 4.6, since \(x_\theta\) is not periodic, we obtain \(m(\theta) \leq b-1: \) thus each \(S_i\) has at least two elements by 6.4, and we have the result from Theorem 6.2. \(\square\)

**Corollary 6.6.** If \(b = 2, x_\theta\) is almost automorphic if and only if \(\theta\) is not simple.

We have from Theorem 6.2 a fairly simple criterion, in terms of the blocks defining the substitution, for \(x_\theta\) to be an almost automorphic flow. Unfortunately, if \(\theta\) is neither a binary substitution nor a simple substitution, it is sometimes not immediately obvious whether or not the condition is satisfied. The fol-
lowing remarks prescribe a method for determining this.

We assume for the remainder of this section that \( m(\theta) = 1 \). The generalization to \( m(\theta) \) is not difficult, but the notation becomes even more tedious. Define

\[
A(i_1) = \{ \theta(j)(i_1) : j \in S \} \quad (i_1 \in Z(0,r-1))
\]

\[
A(i_1, i_2) = \{ \theta(j)(i_2) : j \in A(i_1) \} \quad (i_1, i_2 \in Z(0,r-1))
\]

\[
A(i_1, i_2, \ldots, i_k) = \{ \theta(j)(i_k) : j \in A(i_1, \ldots, i_{k-1}) \}
\]

\( (k > 1, i_1, i_2, \ldots, i_k \in Z(0,r-1)) \)

It is easy to verify that

\[
(1) \quad A(i_1, \ldots, i_k) = \{ \theta^k(j)(i_1 r^{k-1} + i_2 r^{k-2} + \ldots + i_k) : j \in S \}.
\]

Thus the condition in 6.2 is satisfied if and only if

\[
(2) \quad \text{There exist } k \in Z^+ \text{ and integers } i_1, \ldots, i_k \in Z(0,r-1) \text{ such that } A(i_1, \ldots, i_k) \text{ contains only one point.}
\]

Now suppose that for some \( k \in Z^+ \), the following condition holds:

\[
(3) \quad \text{Each of the sets } A(i_1, \ldots, i_k) \text{ is equal to some set } A(j_1, \ldots, j_m) \text{ for some } m < k \text{ and some } m \text{-tuple } (j_1, \ldots, j_m).
\]

Then clearly, for any \( p > k \), (3) is satisfied for \( p \). That is, the process need not be continued past \( k \).

If \( k \) satisfies (3), then \( Z_\theta \) is almost automorphic if and only if there are integers \( i_1, \ldots, i_k \) such that \( A(i_1, \ldots, i_k) \) contains only a single point. Since there are only \( 2^b \) distinct subsets of \( S \), the process will
terminate, either by reaching \( k \) satisfying (2), or by reaching \( k \) satisfying (3).

We point out that all this work is not generally necessary in practice. One can see that if \( A(i_1, \ldots, i_k) = S \), or if \( A(i_1, \ldots, i_k) \) contains any set \( A(j_1, \ldots, j_m) \) \((m \leq k)\), then we may disregard \( A(i_1, \ldots, i_k) \).

We work an example to illustrate the procedure. In the accompanying diagram (Figure 1), an underlined set is one at which we may stop.

**Example.** Let \( b = 4 \), \( r = 2 \); \( \theta(0) = 01 \), \( \theta(1) = 13 \), \( \theta(2) = 20 \), \( \theta(3) = 02 \). It may be checked that \( x_\theta(23, 24) = 00 \); hence \( m(\theta) = 1 \).

It appears from the diagram that each set of the form \( A(i_1, \ldots, i_6) \) has two distinct elements, and each has occurred at a previous stage: we may conclude, therefore, that \( x_\theta \) is not almost automorphic.

**Remark.** It follows from 1.10 and 1.15(3) of [3] that in the flows \( x_\theta \), the almost automorphic points are precisely the regularly almost periodic points. Thus Theorem 4 of [7] contains the case \( b = 2 \) in our Theorem 6.2.
Figure 1

\[ A(0) = \{0,1,2\} \]
\[ A(1) = \{0,1,2,3\} \]

\[ A(0,0) = \{0,1,2\} \quad A(0,1) = \{0,1,3\} \]

\[ A(0,1,0) = \{0,1\} \quad A(0,1,1) = \{1,2,3\} \]

\[ A(0,1,0,0) = \{0,1\} \quad A(0,1,1,0) = \{0,1,2\} \]
\[ A(0,1,0,1) = \{1,3\} \quad A(0,1,1,1) = \{0,2,3\} \]

\[ A(0,1,0,1,0) = \{0,1\} \quad A(0,1,1,1,1) = \{0,1,2\} \]
\[ A(0,1,0,1,1) = \{2,3\} \quad A(0,1,1,1,0) = \{0,2\} \]

\[ A(0,1,0,1,1,0) = \{0,2\} \quad A(0,1,1,1,0,0) = \{0,2\} \]
VII. **Proof that** $\mathcal{X}_\theta$ **is Point-distal.**

**Definition.** [16] Let $\mathcal{X} = (X, T)$ be a minimal flow. Two points $x, y \in X$ are **proximal** if $\inf \{d(T^n x, T^n y) : n \in \mathbb{Z}\} = 0$. $x \in X$ is a **distal point** of $X$ if $x$ is proximal only to itself. $x$ is **distal** if every point of $X$ is a distal point, and **point-distal** if there exists at least one distal point of $X$.

**Theorem 7.1.** If $\theta$ is an admissible substitution, $\mathcal{X}_\theta$ is a point-distal flow.

**Proof.** We observe first that if $x, y \in X$ are proximal, then there exists a sequence of integers $(n(k))$ with $\lim_{k \to \infty} T^n(k)x = \lim_{k \to \infty} T^n(k)y$. From this it follows that $f(x) = f(y)$. Therefore, if $\mathcal{X}_\theta$ is almost automorphic, any almost automorphic point is a distal point. In general, it is sufficient to prove that for some $z \in Z^r$, no two distinct points of $f^{-1}(z)$ are proximal.

For $k \in Z^+$, define $D(i,k) = \{\theta^k(j)(i) : j \in S\}$ ($i \in Z(0, r^k-1)$); $d = \min \{\text{card } D(i,k) : k \in Z^+, \ i \in Z(0, r^k-1)\}$. Choose $k \in Z^+$, $i \in Z(0, r^k-1)$ with $\text{card } D(i,k) = d$. Then by choosing an appropriate power of $\theta$, we may assume $k = 1$ and moreover that $1 \leq i \leq r-2$.

Let $z = iii\ldots \in Z^r$, and suppose $x, y \in f^{-1}(z)$ ($x \neq y$). Then for $k \geq 0$, we have $x(z; k+1) = \theta^{k+1}(j_k)$; $y(z; k+1) = \theta^{k+1}(n_k)$, for some $j_k, n_k \in S$. This implies
\( \theta(j_{k+1})(i) = j_k \) and \( \theta(n_{k+1})(i) = n_k \) \( (k \geq 0) \). By an argument almost identical to that in 3.1, we see that if for some \( k \), \( j_k = n_k \), then \( j_k = n_k \) for every \( k \), and thus \( x = y \), since \( z \) is a non-integer. We conclude that \( \theta(j_k)(i) \neq \theta(n_k)(i) \) \( (k \in \mathbb{Z}^+) \). Now if \( x \) and \( y \) are proximal, then \( x(n) = y(n) \) for some \( n \), and since \( z \) is a non-integer, this implies that for some \( k \in \mathbb{Z}^+ \), \( m \in \mathbb{Z}(0, r^{k+1}-1) \), we have \( \theta^{k+1}(j_k)(m) = \theta^{k+1}(n_k)(m) \).

It is easy to see from this that \( \text{card } D(ir^{k+1}+m-1,k+2) \leq d-1 \), contradicting the definition of \( d \). Therefore, \( x \) and \( y \) cannot be proximal. \( \Box \)

VIII. *AI Extensions and AI Flows.*

**Definition.** [2,14] Suppose \( \mathcal{I} = (X,T) \) and \( \mathcal{J} = (Y,T) \) are flows, and \( \pi: \mathcal{I} \to \mathcal{J} \) is a flow homomorphism. Let \( K = \{ (x,z) \in X \times X : \pi x = \pi z \} \). \( \mathcal{I} \) is an **isometric extension of** \( \mathcal{J} \) **relative to** \( \pi \) if there is a continuous function \( R \) from \( K \) to the non-negative reals satisfying

(i) For \( y \in Y \), \( R \) restricted to \( \pi^{-1}(y) \times \pi^{-1}(y) \) defines a metric on \( \pi^{-1}(y) \).

(ii) \( R(Tx,Tz) = R(x,z) \) \( ((x,z) \in K) \).

\( \mathcal{I} \) is a **proper** isometric extension of \( \mathcal{J} \) if \( \pi^{-1}(y) \) contains at least two points, for each \( y \in Y \).
If \( \mathcal{X} \), \( \mathcal{Y} \), and \( \mathcal{G} \) are flows, and \( \pi_1 : \mathcal{X} \rightarrow \mathcal{Y} \), \( \pi_2 : \mathcal{Y} \rightarrow \mathcal{G} \) are flow homomorphisms, \( \mathcal{X} \) is an **AI extension of** \( \mathcal{G} \) **relative to** \( \pi = \pi_2 \circ \pi_1 \) if \( \mathcal{Y} \) is an almost automorphic extension of \( \mathcal{G} \) relative to \( \pi_2 \) and \( \mathcal{X} \) is an isometric extension of \( \mathcal{Y} \) relative to \( \pi_1 \). \( \mathcal{X} \) is a **proper** AI extension of \( \mathcal{G} \) if \( \mathcal{X} \) is a proper isometric extension of \( \mathcal{Y} \).

A flow \( \mathcal{X} \) is an **AI flow of order** \( \alpha \) if for some ordinal \( \alpha \), and for no smaller ordinal, there exists an inverse system \( \{ \mathcal{X}_\beta ; \pi_\beta \gamma (\gamma \leq \beta) \}_\beta < \alpha \) satisfying

(iii) \( \mathcal{X}_0 \) is the flow on the one-point space; \( \mathcal{X}_\alpha = \mathcal{X} \).

(iv) If \( \beta + 1 < \alpha \), \( \mathcal{X}_{\beta+1} \) is a proper AI extension of \( \mathcal{X}_\beta \); if \( \beta + 1 = \alpha \), \( \mathcal{X}_{\beta+1} \) is an AI extension of \( \mathcal{X}_\beta \).

(v) If \( \beta \leq \alpha \) is a limit ordinal, \( \mathcal{X}_\beta = \lim_{\gamma < \beta} \mathcal{X}_\gamma \).

We call an admissible substitution \( \theta \) of length \( r \) an **S-substitution** if for \( p, q \in S \), \( (p \neq q) \), \( \theta(p)(0) \neq \theta(q)(0) \) and \( \theta(p)(r-1) \neq \theta(q)(r-1) \). The class of S-substitutions contains the class of simple substitutions: and if \( \theta \) is an S-substitution, then so is \( \theta^k \), for \( k \in \mathbb{Z}^+ \).

(1) Let \( \theta \) be an admissible substitution of length \( r \), and let \( i \in \mathcal{Z}(0, m(\theta) - 1), \) \( k \in \mathbb{Z}^+, \) \( j \in \mathcal{Z}(0, r^k - 1) \).
Define \( S(i,j,k) = \{ \theta^k(p)(j) : p \in S_i \} \)

\[ c = \min \{ \text{card } S(i,j,k) \} \]

\[ \mathcal{E} = \{ S(i,j,k) : \text{card } S(i,j,k) = c \} \]

Finally, for \( S' \subset S \), \( k \geq 1 \), \( j \in \mathcal{Z}(0, r^k - 2) \), let
\[ P(S', j, k) = \{ \theta^k(p)(j, j+1) : p \in S' \}. \]

We denote by (A) the following condition:

(A) \( \mathcal{S} \) is a partition of \( S \) into (disjoint) sets \( U_1, U_2, \ldots, U_e \) (where \( b = ce \)); and \( \mathcal{S} = \{ P(U_s, j, k) : s = 1, 2, \ldots, e; k \geq 1; j \in Z(0, r^k - 2) \} \) is a partition of the set \( P_0 \).

**Lemma 8.1.** If \( \theta \) is an \( S \)-substitution, there exists \( k \in Z^+ \) such that \( \theta^k(i)(0) = \theta^k(i)(r^k - 1) = i \) (\( i \in S \)).

Proof. We shall find, for each \( i \in S \), integers \( k(i) \) and \( m(i) \) such that \( \theta^k(i)(0) = \theta^m(i)(0) = i \). Then it is clear that letting \( k = \text{l.c.m.} \{ k(i), m(i) : i \in S \} \) will suffice, since, for example, if \( \theta^k(i)(0) = i \), then \( \theta^j(i)(0) = i \) for any \( j \) which is a multiple of \( k(i) \).

Let \( \theta(i)(0) = n(i, 1) \); if \( n(i, 1) = i \), let \( k(i) = 1 \). If not, let \( n(i, 2) = \theta(n(i, 1))(0) \). If \( n(i, 2) = i \), let \( k(i) = 2 \). Otherwise note that \( n(i, 2) \neq n(i, 1) \) (since \( n(i, 1) \neq i \), and thus \( \theta(n(i, 1))(0) \neq \theta(i)(0) \)), and let \( n(i, 3) = \theta(n(i, 2))(0) \). Continue this process, observing that if \( n(i, j) \neq i \) (\( j = 1, 2, \ldots, m \)), then \( n(i, m) \notin \{ n(i, j) : j = 1, 2, \ldots, m-1 \} \) for the same reason as above. The process must terminate. Let \( k(i) = \min \{ j : n(i, j) = i \} \). It is easily verified that \( \theta^k(i)(0) = i \).

**Hypothesis.** We assume for the remainder of the
discussion that \( \theta \) is an S-substitution of length \( r \).

Using 8.1, we also assume \( \theta(i)(0) = \theta(i)(r-1) = i \) (\( i \in S \)).

**Lemma 8.2.** The map \( \sigma: f^{-1}(0) \to P_\theta \) given by \( \sigma(x) = x(-1,0) \in P_\theta \) is a bijection from \( f^{-1}(0) \) onto \( P_\theta \).

**Proof.** Let \( uv \in P_\theta \). Define \( x_{uv} \) by \( x_{uv}(-r^k,r^k-1) = \theta^k(uv) \) (\( k \in \mathbb{Z}^+ \)). First we note that \( x_{uv} \in X_\theta \), since the \( 2r^k \)-block \( \theta^k(uv) \) appears in \( x_\theta \) for each \( k \in \mathbb{Z}^+ \).

Moreover, \( \sigma(x_{uv}) = uv \). That \( \sigma \) is one-to-one follows from 4.2 and the fact that \( \theta \) is an S-substitution. \( \square \)

**Definition.** For \( uv \in P_\theta \), we let \( [x_{uv}] = \sigma^{-1}(uv) \); i.e., \( x_{uv} \) is the unique point in \( f^{-1}(0) \) with \( x_{uv}(-1,0) = uv \).

**Lemma 8.3.** Suppose \( \phi: \mathcal{X}_\theta \to \mathcal{Y} \), \( \psi: \mathcal{Y} \to \mathcal{R}^{r,m(\theta)} \) are flow homomorphisms, and \( \mathcal{X}_\theta \) is an AI extension of \( \mathcal{R}^{r,m(\theta)} \) relative to \( \psi \circ \phi \). Then \( \min \{ \text{card } (\psi \circ \phi)^{-1}(i,z): (i,z) \in \mathbb{Z}_{m(\theta)} \times \mathbb{Z}^r \} = c \) (see (1)); hence, the map \( \phi \) is c-to-one.

**Proof.** We may clearly assume \( \psi \circ \phi = (g,f) \), since there is a flow isomorphism \( I: \mathcal{R}^{r,m(\theta)} \to \mathcal{R}^{r,m(\theta)} \) with \( I(\psi \circ \phi(x_\theta)) = (0,0) \).

We shall find a point \( (i,z) \in \mathbb{Z}_{m(\theta)} \times \mathbb{Z}^r \) such that

(i) \( \text{Card } (g,f)^{-1}(i,z) = c \)

(ii) If \( x,y \in (g,f)^{-1}(i,z) \) (\( x \neq y \)), then \( x(n) \neq y(n) \) (\( n \in \mathbb{Z} \)).

Choose \( m, j, k \) so that \( \text{card } S(m,j,k) = c \). We may assume by choosing a higher power of \( \theta \) that \( k = 1 \), \( 0 < j < r-1 \), and that if \( C = \{ \theta(p)(j): p \in S \} \), then
card $C = \min \{\text{card } \{\theta^k(p)(j') : p \in S\} : k' \geq 1, j' \in \mathbb{Z}(0, r^{k'} - 1)\}$.

We now claim that for $q \in S(m, j, 1)$, there exists $x \in f^{-1}(jjj\ldots)$ with $x(0) = q$. We observe first that
\[
\{\theta(p)(j) : p \in C\} = C \quad \text{(otherwise card } \{\theta^2(p)(jr+j) : p \in S\} < \text{card } C, \text{ which is impossible.)}
\]
For $q \in S(m, j, 1)$, define $x$ as follows.

(iii) $x(0) = q; x(-j, -j+r-1) = \theta(q_1)$, where $q_1$ is the unique element of $C$ with $\theta(q_1)(j) = q$; similarly,

for $t \geq 1$, $x(-j(1+r+\ldots+r^t), -j(1+r+\ldots+r^t)+r^{t+1}-1) = \theta^{t+1}(q_{t+1})$, where $q_{t+1}$ is the unique element of $C$ with $\theta(q_{t+1})(j) = q_t$.

This clearly defines a point $x \in X_\emptyset$, and by 4.2, $x \in f^{-1}(jjj\ldots)$. We obtain, therefore,

(iv) Card \{x \in f^{-1}(jjj\ldots) : x(0) \in S(m, j, 1)\} = c.

Now by 4.6, there is an integer $n \in \mathbb{Z}(0, m(\emptyset) - 1)$ such that $z(p) = n$ ($p \in S(m, j, 1)$). Furthermore, if $n = -ir \pmod{m(\emptyset)}$, then 4.6 yields \{x(0) : x \in (g,f)^{-1}(i,jj\ldots)\} = S(m, j, 1). This, together with (iv), gives (i), where $z = jjj\ldots \in Z^r$. (ii) follows for this point $(i, z)$, by an argument similar to the proof of 7.1.

It is easy to see from (i) and (ii) that $c$ must be the minimum of all the numbers card $(g,f)^{-1}(i', z')$, since $x_\emptyset$ is minimal.

To show $\varphi$ is c-to-one, we show first that $\varphi$ is $n$-to-one, for some $n$. Suppose $\varphi^{-1}(y_1) = \{x_1, \ldots, x_{n_1}\}$,
\( \varphi^{-1}(y_2) = \{w_1, w_2, \ldots, w_{n_2}\} \) for some \( y_1, y_2 \in Y \), with \( n_1 < n_2 \). Choose a sequence of integers \((m(k))\) so that 
\[ \lim_{k \to \infty} T^{m(k)} w_1 = x_1. \]  
By taking subsequences, we may assume \( \lim_{k \to \infty} T^{m(k)} w_i \) exists \((i = 1, 2, \ldots, n_2)\). Now for \( 1 \leq i, j \leq n_2 \), \( R(w_i, w_j) = R(T^{m(k)} w_i, T^{m(k)} w_j) \) \((k \in \mathbb{Z}^+)\), and therefore if \( i \neq j \), \( \lim_{k \to \infty} T^{m(k)} w_i \neq \lim_{k \to \infty} T^{m(k)} w_j \).
But \( \varphi(\lim_{k \to \infty} T^{m(k)} w_i) = \lim_{k \to \infty} T^{m(k)} \varphi(w_i) = \lim_{k \to \infty} T^{m(k)} \varphi(w_1) = y_1 \). Thus \( \text{card } \varphi^{-1}(y_1) \geq n_2 \), which is a contradiction, and \( n_1 = n_2 \).

Since \( \varphi \) is n-to-one, \( \psi \) is one-to-one at some point, and \( c = \min \{ \text{card}(\psi \circ \varphi)^{-1}(i, z) : (i, z) \in Z_{m(\theta)} \times \mathbb{Z}^r \} \), it follows that \( \varphi \) is c-to-one. \( \Box \)

**Theorem 8.4.** \( x_\theta \) is an AI extension of \( x_\theta^* \) if and only if condition (A) holds.

**Proof.** Suppose \( \varphi : x_\theta \to \mathcal{G}, \psi : \mathcal{G} \to \mathcal{G}_{r, m(\theta)} \) are flow homomorphisms, \( \psi \circ \varphi = (g,f) \), and \( x_\theta \) is an AI extension of \( \mathcal{G}_{r, m(\theta)} \) relative to \( \psi \circ \varphi \). We shall prove first:

(i) For \( k \geq 1 \) and \( j \in Z(0, r^k - 2) \), if \( P(U, j, k) = \{u_1 v_1, u_2 v_2, \ldots, u_c v_c\} \) for some \( U \in \mathcal{E} \), then \( \varphi(x_{u_1 v_1}) = \varphi(x_{u_2 v_2}) = \ldots = \varphi(x_{u_c v_c}). \)

We may assume \( U = S(i, m, 1) \) for some \( i \in Z(0, m(\theta) - 1) \), and \( 0 < m < r - 1 \); moreover, we may assume \( k = 1 \).

By the same argument as in 8.3, we find \( n \in Z(0, m(\theta) - 1) \) such that \((g,f)^{-1}(n, z) = \{x \in f^{-1}(z) : x(0) \in U\}, \) where \( z = \text{mmm} \ldots \in \mathbb{Z}^r \). Since \( \text{card } U = c \),
4.3 implies \( \text{card } (g, f)^{-1}(n, z) \leq c \); but then by 8.3, \( \text{card } (g, f)^{-1}(n, z) = c \). If \( U = \{a_1, a_2, \ldots, a_c\} \), we let \( (g, f)^{-1}(n, z) = \{x_1, x_2, \ldots, x_c\} \), where \( x_t(0) = a_t \) for \( t = 1, 2, \ldots, c \). Since \( \varphi \) is c-to-one, we obtain

\[
(ii) \quad \varphi(x_1) = \varphi(x_2) = \ldots = \varphi(x_c).
\]

Now by 4.2, we have

\[
(iii) \quad x_t(z; k+1) = \theta^{k+1}(d_t, k) \quad \text{for some } d_t, k \in S \quad (k \geq 0, \ t = 1, 2, \ldots, c). \quad \text{Thus } d_t, 0 = a_t \quad (t = 1, 2, \ldots, c).
\]

Each of the sequences \( \{d_t, k: k = 0, 1, \ldots\} \) is periodic, of period \( \lambda_t \); letting \( \lambda = 1. \text{c.m.} \{\lambda_1, \lambda_2, \ldots, \lambda_c\} \), we have

\[
(iv) \quad d_t, q\lambda = a_t \quad (q \geq 0, \ t = 1, 2, \ldots, c).
\]

It follows from the fact that \( \theta(p)(0) = \theta(p)(r-1) = p \) (\( p \in S \)) that \( P(S', u, v) = P(S', (u+1)r^w-1, v(w+1)) \) for any \( v, w \geq 1, \ u \in Z(0, r^v-2), \ S' \subset S \). Thus in particular:

\[
(v) \quad P(u, j, 1) = P(U, (j+1)r^{q_\lambda}-1, q_\lambda + 1) \quad (q \geq 0).
\]

We define a sequence of integers \( n(q) \) by \( n(q) = -m(1+r+r^2+\ldots+r^{q_\lambda})+(j+1)r^{q_\lambda} \). Then from (iv) and (v) it follows that

\[
(vi) \quad \{T^n(q)x_t(-1, 0): t = 1, 2, \ldots, c\} = P(U, j, 1) \quad (q \geq 0).
\]

Furthermore, by definition of \( n(q) \),

\[
(vii) \quad T^n(q)x_t(-r^{q_\lambda}, -1) \quad \text{and } T^n(q)x_t(0, r^{q_\lambda}-1) \quad \text{are basic } r^{q_\lambda}-\text{blocks} \quad (q \geq 0, \ t = 1, 2, \ldots, c).
\]

We may assume, by choosing a subsequence, that \( \lim_{q \to \infty} T^n(q)x_t \) exists for each \( t \). Now (vii) implies,
using 4.2, that \( f(\lim_{q \to \infty} T^n(q)x_t) = 0 \) \( (t = 1, 2, \ldots, c) \),
and using (vi), we obtain \( \{ \lim_{q \to \infty} T^n(q)x_t : t = 1, 2, \ldots, c \} \)
\( = \{ x_{u_1}v_{1}, x_{u_2}v_{2}, \ldots, x_{u_c}v_{c} \} \). But (ii) and the fact that
\( \phi \) is a continuous flow homomorphism imply that
\( \phi(\lim_{q \to \infty} T^n(q)x_1) = \phi(\lim_{q \to \infty} T^n(q)x_2) = \ldots = \phi(\lim_{q \to \infty} T^n(q)x_c) \),
and the proof of (i) is therefore complete.

This establishes half of (A), since if two distinct members of \( \mathcal{A} \) have a non-empty intersection, then
\( \text{card } \phi^{-1}(y) > c \) for some \( y \in Y \) by (i), contradicting
8.3.

For the second half of (A), suppose \( U_1, U_2 \in \mathcal{S} \)
and \( U_1 \cap U_2 \neq \emptyset \). Write \( U_1 = \{ p_1, p_2, \ldots, p_c \}, U_2 = \{ q_1, q_2, \ldots, q_c \} \). Then \( P_1 = \{ \theta(p_i)(0, 1) : i = 1, 2, \ldots, c \} \) and
\( P_2 = \{ \theta(q_i)(Q_1) : i = 1, 2, \ldots, c \} \) are elements of \( \mathcal{A} \), and
\( P_1 \cap P_2 \neq \emptyset \), so that \( P_1 = P_2 \). But then \( \{ \theta(p_i)(0) : i = 1, 2, \ldots, c \} = \{ \theta(q_i)(0) : i = 1, 2, \ldots, c \} \), and these
two sets are simply \( U_1 \) and \( U_2 \), respectively. This com-
pletes one direction of the proof.

Now suppose condition (A) holds. We shall first
construct the flow \( \mathfrak{Y} \) and the map \( \phi: X_{\theta} \to \mathfrak{Y} \).

Define a relation \( \Lambda \) on \( X_{\theta} \) as follows: \( (x, y) \in \Lambda \) if:
(a) \( f(x) = f(y) \) \( (= z = z_0z_1z_2 \ldots \in \mathbb{Z}^r) \)
(b) For \( z \) a non-integer, \( x(z; k+1) = \theta^{k+1}(p_x, k), \)
\( y(z; k+1) = \theta^{k+1}(p_y, k) \), where \( p_x, k, p_y, k \) are
elements of some \( U_k \in \mathcal{S} \) \( (k \geq 0) \).
(c) For \( z = n \in \mathbb{Z}, T^{-n}x(-1,0), T^{-n}y(-1,0) \in P \), for some \( P \in \mathfrak{P} \).

It is clear that \( \Lambda \) is a \( T \)-invariant equivalence relation; and from 4.6, we obtain that \( \Lambda \subset \ker(f,g) \). We observe that

(viii) If \( U \in \mathcal{E} \), then \( \{ \theta^k(p)(m) : p \in U \} \in \mathcal{E} \)

\((k \geq 1, m \in \mathbb{Z}(0,r^k-1))\).

(Otherwise some \( S(i,j,k) \) would have fewer than \( c \) points.)

It follows from (viii) and the definition of \( \mathfrak{P} \) that

(ix) If \( P_1 = \{ u_1v_1, u_2v_2, \ldots, u_cv_c \} \in \mathfrak{P} \), then

\( \{ u_1, u_2, \ldots, u_c \}, \{ v_1, v_2, \ldots, v_c \} \in \mathcal{E} \).

We now assert:

(x) If \( (x,y) \in \Lambda \) and \( f(x) = f(y) = 0 \), then for \( k \in \mathbb{Z}^+ \) and \( n \in \mathbb{Z} \), \( x(nr^k, (n+1)r^k-1) = \theta^k(p_x,n,k) \), and

\( y(nr^k, (n+1)r^k-1) = \theta^k(p_y,n,k) \), where, for some \( U_n,k \in \mathcal{E} \),

\( P_x,n,k, P_y,n,k \in U_n,k \).

This follows from (viii) and (ix), together with the fact that \( \theta(p)(0) = \theta(p)(r-1) = p \) \((p \in S)\).

We prove now that \( \Lambda \) is closed. Suppose \( (x_m,y_m) \in \Lambda \)

\((m \in \mathbb{Z}^+)\), and \( \lim_{m \to \infty} (x_m,y_m) = (x,y) \). (a) is satisfied immediately for \((x,y)\), since \( f \) is continuous. Let

\( f(x_m) = f(y_m) = z_m = z_0^mz_1^mz_2^m \ldots \), and \( f(x) = f(y) = z = z_0z_1z_2 \ldots \). We first consider the case in which \( z \) is a non-integer. Let \( k \in \mathbb{Z}^+ \); for sufficiently large \( m, z_0^mz_1^m \ldots z_k^m = z_0^0z_1^0 \ldots z_k^0 \), and \( x_m(z; k+1) = \ldots
\(x(z; k+1), y_m(z; k+1) = y(z; k+1)\). For such \(m\), if \(z^m\) is a non-integer, (b) follows from the equivalence of \(x_m\) and \(y_m\); if \(z^m\) is an integer, it follows from (x).

Now suppose \(z\) is an integer; we may assume, then, that \(z = 0\). For sufficiently large \(m\), \(x_m(-1,0) = x(-1,0)\) and \(y_m(-1,0) = y(-1,0)\). Since \(x_m\) and \(y_m\) are equivalent, it is clear that either \(x(-1,0), y(-1,0) \in P\) for some \(P \in \Phi\), or \(x_m = x, y_m = y\) by 8.2.

We note that an equivalence class contained in \(f^{-1}(0)\) contains exactly \(c\) elements; if \(f(x)\) is a non-integer and \((x,y) \in \Lambda\), then for each \(k\) there are only \(c\) choices for the \(r^{k+1}\)-block \(y(z; k+1)\), and therefore the equivalence class of \(x\) contains at most \(c\) elements.

(xi) If \((x,y) \in \Lambda (x \neq y)\), then \(x(n) \neq y(n) (n \in \mathbb{Z})\). For \(f(x)\) a non-integer, this follows from (viii); for \(f(x)\) an integer, it follows from (viii) and (ix).

(xi) implies that each equivalence class has exactly \(c\) elements, since each equivalence class in \(f^{-1}(0)\) has exactly \(c\) elements.

We define \(\mathcal{Y} = (Y, \mathcal{T}) = (X_\theta/\Lambda, \mathcal{T})\), with the quotient topology on \(Y\). Let \(\varphi: X_\theta \to Y\) be the natural projection. Finally, it makes sense to define \(\psi: Y \to Z_m(\theta)^x Z^r\) by the formula \(\psi \varphi = (g,f)\). \(\varphi\) and \(\psi\) are flow homomorphisms, the map \(\varphi\) is \(c\)-to-one, and by (i) of 8.3, \(\mathcal{Y}\) is an almost automorphic extension of \(\mathcal{Y}^{r,m(\theta)}\). If
$y \in Y$, and $x_1, x_2 \in \varphi^{-1}(y)$, define $R(x_1, x_2) = 0$ if $x_1 = x_2$, and $R(x_1, x_2) = 1$ otherwise. Obviously, $R(Tx_1, Tx_2) = R(x_1, x_2)$, and $R$ is a metric on each set $\varphi^{-1}(y)$. Finally, from (xi) it follows easily that $R$ is continuous. This completes the proof. □

We note that when $\theta$ is a simple substitution, (A) assumes a simpler form. We recall that in that case, $m(\theta) \mid b$; and it is easy to see that each set $S(i,j,k)$ is simply $S_m$, for some $m \in Z(0,m(\theta)-1)$, so that the first part of (A) is automatically satisfied. We obtain the following.

**Corollary 8.5.** If $\theta$ is simple, then $x_\theta$ is an AI extension of $\mathcal{A}^{r,m(\theta)}$ if and only if the collection

$\{P(S_1,j,k)\}$ is a partition of $P_\theta$.

**Corollary 8.6.** If $\theta$ is simple and $m(\theta) = \frac{1}{2}b$, then $x_\theta$ is an AI extension of $\mathcal{A}^{r,m(\theta)}$.

**Proof.** We show that the condition in 8.5 always holds. Suppose $P(S_1,j,k) = \{uv, w_1z_1\}$, $P(S_m,m,p) = \{uv, w_2z_2\}$. Then $u \neq w_q$ ($q = 1,2$); but $z(u) = z(w_q)$, by 4.6 ($q = 1,2$). This means $u, w_1, w_2 \in S_t$, for some $t \in Z(0,m(\theta)-1)$. But then by hypothesis, $w_1 = w_2$. Likewise, $z_1 = z_2$. □

**Corollary 8.7.** If $\theta$ is a binary substitution, then $x_\theta$ is either an almost automorphic flow or an AI extension of $\mathcal{A}^r$. 
Proof. This is immediate from 5.6, 6.6, and 8.6. □

Theorem 8.8. If a flow \( x = (X, T) \) is an AI extension of an equicontinuous flow, then \( x \) is an AI extension of \( x^* \).

Proof. Suppose \( \varphi: x \to (Y, T) \) and \( \psi: y \to (Z, T) \) are flow homomorphisms, \( R \) is equicontinuous, and \( x \) is an AI extension of \( R \) relative to \( \psi \circ \varphi \). Let \( \pi: x \to x^* \) be the projection map. Since \( R \) is equicontinuous, there is a flow homomorphism \( \rho: x^* \to R \) with \( \rho \circ \pi = \psi \circ \varphi \). Define an equivalence relation \( \Lambda \) on \( X \) by letting \( \Lambda = \text{Ker } \varphi \cap \text{Ker } \pi \). \( \Lambda \) is a closed, invariant equivalence relation, and we may define \( \mathbb{W} = (W, T) = (X/\Lambda, T) \). Let \( h: x \to \mathbb{W} \) be the projection; then there is a flow homomorphism \( k: \mathbb{W} \to x^* \) such that \( k \circ h = \pi \).

We show that if \( \varphi(x) \) is an almost automorphic point of \( Y \), then \( f(x) \) is an almost automorphic point of \( W \). Suppose \( k(h(x)) = k(h(u)) \) (some \( u \in X \)). Then \( \psi(\varphi(x)) = \rho(\pi(x)) = \rho(\pi(u)) = \psi(\varphi(u)) \); but since \( \varphi(x) \) is almost automorphic, \( \varphi(x) = \varphi(u) \). This implies \( (x, u) \in \Lambda \), and thus \( h(x) = h(u) \).

Since \( x \) is an isometric extension of \( y \) relative to \( \varphi \), it follows easily that \( x \) is an isometric extension of \( \mathbb{W} \) relative to \( h \). □

We return now to the definition of an AI flow. AI flows of order 1 are precisely the equicontinuous flows;
and AI flows of order 2 are those which are either almost automorphic flows or AI extensions of equicontinuous flows. For any admissible substitution \( \theta \), \( x_\theta \) is not equicontinuous. Therefore, VI and the first part of VIII may be interpreted as a partial characterization of those substitutions \( \theta \) for which \( x_\theta \) is an AI flow of order 2. For a restricted class of substitutions, this is sufficient to characterize the AI flows. More precisely:

**Theorem 8.9.** If \( \theta \) is simple and \( r \) and \( b \) are both prime, \( x_\theta \) is an AI flow if and only if \( x_\theta \) is an AI extension of \( \mathfrak{g}_r \).

Proof. Suppose \( x_\theta \) is an AI flow of order \( \alpha \); this means that for \( \beta < \alpha \), there is a flow \( \mathfrak{g}_\beta \), and there are flow homomorphisms \( \rho_\beta : x_{\beta+1} \to \mathfrak{g}_\beta \), \( \tau_\beta : \mathfrak{g}_\beta \to x_\beta \) such that

(i) \( \tau_\beta \circ \rho_\beta = \pi_{\beta+1, \beta} \) (\( \pi_{\beta+1, \beta} \) as in the definition)

(ii) \( x_{\beta+1} \) is an AI extension of \( x_\beta \) relative to \( \tau_\beta \rho_\beta \).

(Here it might be the case that for certain \( \beta \), \( \mathfrak{g}_\beta = x_\beta \); it might also be possible that \( x_\alpha = \mathfrak{g}_{\alpha-1} \).) We claim:

(iii) For some \( \gamma < \alpha \), \( x_\gamma = \mathfrak{g}_r \).

To prove this, we observe first that an almost automorphic extension of a minimal flow on a finite phase space is isomorphic to the flow itself. Thus \( \mathfrak{g}_0 = x_0 \); furthermore, we have
(iv) If \( \mathfrak{x} = (X, T) \) and \( \mathfrak{y} = (Y, T) \) are minimal flows, where \( Y \) is finite and \( \mathfrak{x} \) is an isometric extension of \( \mathfrak{y} \), then \( \mathfrak{x} \) is equicontinuous.

We note that if \( Y \) is a single point, (iv) is well-known (see, for example, [2]); we then prove that if \( \mathfrak{x} \) is an isometric extension of \( (Z_p, T) \) for some prime \( p \), then in fact \( \mathfrak{x} \) is an isometric extension of the trivial flow. Suppose \( f: \mathfrak{x} \to (Z_p, T) \) is a flow homomorphism, and \( \mathfrak{x} \) is an isometric extension of \( (Z_p, T) \) relative to \( f \). Let \( X_i = f^{-1}(i) \) \( (i \in Z_p) \). Then we have a continuous function \( R: \bigcup [X_i \times X_i: i \in Z_p] \to \mathbb{R}^+ \) as in the definition. Since \( X_i \times X_i \) is compact for each \( i \), \( R \) is bounded; let \( c = \sup R \). Define \( R_1: X \times X \to \mathbb{R}^+ \) by \( R_1(x,y) = R(x,y) \) if \( f(x) = f(y) \), and \( R_1(x,y) = c \) otherwise. Since \( R \) is continuous, it is easy to see that \( R_1 \) is continuous. Furthermore, \( R_1(x,y) \leq R_1(x,z) + R_1(z,y) \) (if \( f(x) = f(z) = f(y) \), this is immediate; if \( f(x) \neq f(z) \) or \( f(z) \neq f(y) \), it is true since the right-hand side is at least \( c \)). Finally, \( R_1(Tx,Ty) = R_1(x,y) \) \( (x,y \in X) \). Thus (iv) is proved.

Now we recall that since \( \theta \) is simple, \( m(\theta) \mid b \); thus, by hypothesis, \( m(\theta) = 1 \). Now by (iv), \( \mathfrak{x}_1 \) is equicontinuous; but then, by 5.5, \( \mathfrak{x}_1 \) is a homomorphic image of \( \mathfrak{y}^r \). Since \( r \) is prime, \( \mathfrak{x}_1 \) is either a flow with phase space \( Z_r^{k(1)} \), for some \( k(1) > 0 \), or \( \mathfrak{x}_1 = \mathfrak{y}^r \), in
which case (iii) holds. If \( x_1 = \langle Z, k(1), T \rangle \), then
\( y_1 = x_1 \) and \( x_2 \) is equicontinuous by (iv). Thus either
\( x_2 = \langle Z, k(2), T \rangle \) (some \( k(2) > k(1) \)) or \( x_2 = \mathcal{G}^T \). Continuing in this manner, we obtain that either \( x_k = \mathcal{G}^T \), for some integer \( k \), or \( x_w = \lim_{i \to \infty} \langle Z, k(i), T \rangle = \mathcal{G}^T \), where \( w \) is the first infinite ordinal. In either case, the proof of (iii) is complete.

Now \( x_\gamma = \mathcal{G}^T \), and we may assume \( \pi_{\alpha, \gamma} \) is simply the homomorphism \( f \). It follows that \( \alpha = \gamma + n \), for some positive integer \( n \). We consider first the case in which \( x_\alpha \) is a proper AI extension of \( x_{\alpha-1} \); i.e., the map \( \rho_{\alpha-1} \) is not one-to-one.

Suppose \( \rho_{\gamma+j} \) is \( k_j \)-to-one \((j = 0, 1, \ldots, n-1)\), where each \( k_j > 1 \); we claim

(v) There is a residual set \( A \subset X_\gamma \) such that for \( x \in A \), \( \text{card } \pi_{\alpha, \gamma}^{-1}(x) = k_0 k_1 \ldots k_{n-1} \).

To prove this, we use the fact, which is proved in [16], that for any \( \beta < \alpha \), the set \( X_{\beta} \) of all points which have one-point inverse images under \( \tau_\beta \) is a residual subset of \( X_\beta \) (and hence \( \pi_{\beta+1, \beta}^{-1}(X_\beta) \) is a residual subset of \( X_{\beta+1} \).) Now the intersection of two residual sets is again residual. Thus \( X_\gamma \) is residual in \( X_\gamma \); \( \pi_{\gamma+1, \gamma}^{-1}(X_\gamma) \) is residual in \( X_{\gamma+1} \); \( \pi_{\gamma+2, \gamma+1}^{-1}(X_{\gamma+1} \cap \pi_{\gamma+1, \gamma}^{-1}(X_\gamma)) \) is residual in \( X_{\gamma+2} \); continuing this process, we obtain that the set \( A \) of
points \( x \) in \( X_\gamma \) having the property that any point \( z \in \pi_{\gamma+j,\gamma}^{-1}(x) \) is in \( X_{\gamma+j} \) (\( j = 1, 2, \ldots, n \)) is a residual subset of \( X_\gamma \). But it is clear that \( A \) satisfies (v).

Now we recall that if \( z \in Z^\mathbb{R} \) is a non-integer, then \( \text{card } f^{-1}(z) = b \). But \( f = \pi_{\alpha,\gamma} \); and the non-integers constitute a residual set in \( Z^\mathbb{R} \), so that there is at least one non-integer in \( A \). From (v), we have \( b = k_0 k_1 \ldots k_{n-1} \), and since \( b \) is prime, we conclude that \( n = 1 \). Therefore \( \bar{x}_\theta \) is an AI extension of \( \mathbb{R}^\mathbb{R} \).

We prove now that the remaining case is impossible; i.e., \( \bar{x}_\theta \) cannot be an almost automorphic extension of \( \bar{x}_{\alpha-1} \) relative to \( \pi_{\alpha,\alpha-1} \). Let \( \pi = \pi_{\alpha,\alpha-1} \); let \( B = \{ x \in X_{\theta} : \pi^{-1}(x) = \{x\} \} \), and suppose \( B \) is non-empty. Since \( B \) is then residual, choose \( x \in B \) with \( f(x) \) a non-integer. Now let \( y \in X_{\theta} \) be any point in \( X_{\theta} \) such that \( f(y) \) is a non-integer, and suppose \( \pi(y) = \pi(z) \) for some \( z \neq y \). Choose a sequence of integers \( n(k) \) such that \( \lim_{k \to \infty} T^n(k)x = x \) and \( \lim_{k \to \infty} T^n(k)z \) exists. Then \( \pi(\lim_{k \to \infty} T^n(k)z) = \pi(x) \), which implies \( \lim_{k \to \infty} T^n(k)y = \lim_{k \to \infty} T^n(k)x = x \). But it is easy to see that \( y(m) \neq z(m) \) (\( m \in \mathbb{Z} \)), and this is impossible. Thus \( y \in B \). A similar argument shows that if \( f(y) \) is an integer, then \( y \in B \). We conclude that \( B = X_{\theta} \); i.e., \( \bar{x}_\theta = \bar{x}_{\alpha-1} \), which contradicts the fact that \( \bar{x}_\theta \) is an AI flow of order \( \alpha \).
Finally, we point out that for a simple substitution \( \theta \), since for any non-integer \( z \in \mathbb{Z}^r \) we have \( x(n) \neq y(n) \ (x, y \in f^{-1}(z), x \neq y, n \in \mathbb{Z}) \), then any point in \( f^{-1}(z) \) (\( z \) a non-integer) is a distal point of \( X_\theta \). Therefore, 8.9 yields a wide class of examples of substitutions \( \theta \) for which \( \mathcal{F}_{\theta} \) is not an AI flow, although it is a point-distal flow with a residual set of distal points.

**Examples.** (1) Let \( b = r = 3; \theta(0) = 010, \theta(1) = 121, \theta(2) = 202 \). Since \( \theta \) is simple, and thus \( m(\theta) \mid b \), we have \( m(\theta) = 1 \). Since \( \theta(1)(0) = \theta(1)(r-1) = 1 \ (1 = 0, 1, 2) \), every set \( P(S_1, j, k) \) is either \( P(S_0, 0, 1) = \{01, 12, 20\} \) or \( P(S_0, 1, 1) = \{10, 21, 02\} \). We conclude that the condition in 8.5 is satisfied, and \( \mathcal{F}_{\theta} \) is an AI extension of \( g^3 \).

(2) Let \( b = 6; r = 5; \theta(0) = 04130, \theta(1) = 13251, \theta(2) = 25042, \theta(3) = 30423, \theta(4) = 42304, \theta(5) = 51515 \). It can be checked without much difficulty that \( \theta \) is admissible and simple, and that \( m(\theta) = 2 \). \( S_0 = \{0, 1, 2\} \), \( S_1 = \{3, 4, 5\} \). We have \( P(S_0, 0, 1) = \{04, 13, 25\} \), and \( P(S_1, 1, 1) = \{04, 23, 15\} \). Thus (A) fails to hold, and \( \mathcal{F}_{\theta} \) is not an AI extension of any equicontinuous flow.

In the above example, we could merely have checked that \( p_\theta = \text{card} \ P_\theta = 11 \), which is not a multiple of \( c=3 \). That \( c \) must divide \( p_\theta \) is an obvious necessary condition.
for (A) to hold; that it is not sufficient is shown by the following example.

(3) Let $b = r = 3$; $\theta(0) = 011$, $\theta(1) = 202$, $\theta(2) = 120$. Again $m(\theta) = 1$. Although the smallest integer $k$ satisfying 8.1 is 6, we note that $P(S_0, 0, 1) \cap P(S_0, 1, 1) = \{20\}$, and (A) cannot hold. Since $\theta$ satisfies all the hypotheses of 8.9, $x_\theta$ is not an AI flow.

(4) Let $b = 6$, $r = 5$; $\theta(0) = 03140$, $\theta(1) = 14031$, $\theta(2) = 25252$, $\theta(3) = 31523$, $\theta(4) = 40404$, $\theta(5) = 52315$. The following assertions may easily be verified:

(i) $\theta$ is admissible and simple.

(ii) $m(\theta) = 2$; $S_0 = \{0, 1, 2\}$, $S_1 = \{3, 4, 5\}$.

(iv) $x_\theta$ is an AI extension of $g^5, 2$.

(5) Let $b = r = 4$; $\theta(0) = 0130$, $\theta(1) = 1201$, $\theta(2) = 2132$, $\theta(3) = 3203$. We observe that $\theta$ is an $S$-substitution; $m(\theta) = 1$, since $x_\theta(3) = x_\theta(10) = 0$. It is easy to verify that $c = 2$, and $\mathcal{G} = \{\{0, 3\}, \{1, 2\}\}$; $\mathcal{S} = \{\{01, 32\}, \{03, 20\}, \{21, 12\}, \{13, 20\}\}$. Thus (A) is satisfied, and $x_\theta$ is an AI extension of $g^4 \approx g^2$.

(6) Let $b = 4$; $r = 5$; $\theta(0) = 01320$, $\theta(1) = 12031$, $\theta(2) = 21322$, $\theta(3) = 32033$. $\theta$ is an $S$-substitution, $m(\theta) = 1$, $c = 2$. $\mathcal{G} = \{1, 2\}, \{0, 3\}, \{2, 3\}\}$. Therefore (A) fails to hold, and $x_\theta$ is not an AI extension of an equicontinuous flow.
This is an example to show that condition (A) is not in general necessary for \( \mathbf{x}_\theta \) to be an AI flow. We first establish a general condition sufficient to obtain a flow homomorphism from one substitution flow to another.

Suppose \( \theta \) is an admissible substitution of length \( r \), and suppose there is an equivalence relation \( \Lambda \) on \( S = S_b \) such that if \( (i,j) \in \Lambda \), then \( (\theta(i)(p),\theta(j)(p)) \in \Lambda \) \( (p \in Z(0,r-1)) \). If \( \text{card } S/\Lambda = d \), we define a new substitution \( \Theta \) of length \( r \) over \( S_d \) in the following manner. Let \( S/\Lambda = \{ U_0, U_1, \ldots, U_{d-1} \} \), where \( 0 \in U_0 \); and for each \( j \), choose \( i_j \in U_j \). Let \( \rho : S_b \to S_d \) be defined by \( \rho(m) = n \), where \( m \in U_n \). Then for \( j \in Z(0,d-1), \ p \in Z(0, r-1) \), let \( \Theta(j)(p) = \rho(\Theta(i_j)(p)) \).

Lemma. (Notation as above). \( x_{\Theta'} \) is recurrent (although possibly periodic), and there is a flow homomorphism \( g \) from \( \mathbf{x}_\theta \) to \( \mathbf{x}_{\Theta'} \).

Proof. It is easily checked, using the property which \( \Lambda \) possesses, that the sequence \( x_{\Theta'} \) is obtained from \( x_{\Theta} \) by the formula \( x_{\Theta'}(n) = \rho(x_{\Theta'}(n)) \) \( (n \geq 0) \), and it follows that \( x_{\Theta'} \) is recurrent. We thus define \( g : X_\theta \to X_{\Theta} \) by letting \( g(x)(n) = \rho(x(n)) \) \( (x \in X_\theta, \ n \in Z) \). \( g \) is then obviously a continuous surjection and hence defines a flow homomorphism. \( \square \)

Now we let \( b = 4, \ r = 7; \ \theta(0) = 0032110, \ \theta(1) = \)
1123001, $\theta(2) = 2200232$, $\theta(3) = 3311323$. We observe that $\theta$ is simple and $m(\theta) = 1$; $r$ is prime but $b$ is not. $P(S_0, 2, 1) \cap P(S_0, 5, 1) = \{32, 23\}$, and thus (A) fails to hold. The equivalence relation on $S$ generated by the set $\{(0, 1), (2, 3)\}$ satisfies the hypotheses above, and we have therefore a flow homomorphism $g$: $\pi_\theta \rightarrow \pi_\Theta$, where $\Theta$ is the binary substitution defined by $\Theta(0) = 0011000$, $\Theta(1) = 1100111$. We shall show, moreover, that $\pi_\Theta$ is an AI extension of $\pi_\Theta$, and it will follow from 8.7 that $\pi_\Theta$ is an AI flow of order 3.

Let $f$, $f'$ be the usual homomorphisms from $\pi_\Theta$ and $\pi_\Theta$, respectively, to $G^7$. Then $f = f' \circ g$. Define an equivalence relation $\Gamma$ on $X_\Theta$ as follows. $(x, y) \in \Gamma$ if:

(i) $f(x) = f(y)$

(ii) If $f(x) = f(y) = n (x \neq y)$, $[T^{-n}x(-1, 0)$, $T^{-n}y(-1, 0)] = P([0, 1], j, k)$ or $P([2, 3], j, k)$ for some $k \in Z^+$, $j \in Z(0, r^k-2)$

(iii) If $f(x) = f(y)$ is a non-integer, $[x(0), y(0)]$ $= [0, 1]$ or $[2, 3]$.

Since $\{[\theta^k(0)(j), \theta(1)(j)]: k \geq 1, j \in Z(0, r^k-1)\} = \{[\theta^k(2)(j), \theta^k(3)(j)]: k \geq 1, j \in Z(0, r^k-1)\} = \{[0, 1], [2, 3]\}$, it is easy to check that $\Gamma$ is well-defined, closed, and invariant, and if $(x, y) \in \Gamma (x \neq y)$, then $x(n) \neq y(n)$ $(n \in Z)$. Letting $p: X_\Theta \rightarrow Y = X_\Theta/\Gamma$ be the natural projection, and defining a function $R$
on $U \cup \{p^{-1}(y) \times p^{-1}(y) : y \in Y\}$ by $R(x, z) = 0$ if $x = z$
and $R(x, z) = 1$ otherwise, an argument almost identical to that in 8.4 shows that $R$ is continuous. Thus $\mathcal{X}_\Theta$ is
an isometric extension of $\mathcal{Y} = (Y, T)$ relative to $p$.

We have in addition that if $(x, y) \in \Gamma$, then $g(x) = g(y)$. Let $h : Y \to X_\Theta$ be the map defined by $h \circ p = g$.
It is easy to check that if $x \in (f')^{-1}(z)$, where $z \in Z^7$
is a non-integer, then $h^{-1}(x)$ is a single point. Thus,$\mathcal{Y}$ is an almost automorphic extension of $\mathcal{X}_\Theta$ relative to
$h$, and therefore $\mathcal{X}_\Theta$ is an $AI$ extension of $\mathcal{X}_\Theta$. 
REFERENCES


14. ______, "Strict ergodicity in zero dimensional